Identifying routines in the discourse of undergraduate students when defining

Aurora Fernández-León, José María Gavilán-Izquierdo, Alfonso J. González-Regaña, Verónica Martín-Molina, Rocío Toscano

Affiliation (same for all five authors): Departamento de Didáctica de las Matemáticas, Facultad de Ciencias de la Educación, Universidad de Sevilla, Spain

Postal address (same for all five authors): Departamento de Didáctica de las Matemáticas, Facultad de Ciencias de la Educación, Universidad de Sevilla, c/ Pirotecnia s/n, 41013, Sevilla, Spain

Email address and telephone number of the corresponding author (A. Fernández-León): aurorafl@us.es , (+34) 955420550

Email addresses of the rest of the authors: gavilan@us.es (J. M. Gavilán-Izquierdo), agonzalez@us.es (A. J. González-Regaña), veronicamartin@us.es (V. Martín-Molina), rtoscano@us.es (R. Toscano),

ORCID: 0000-0002-6780-093X (A. Fernández-León), 0000-0002-3369-5377 (J. M. Gavilán-Izquierdo), 0000-0002-3176-7402 (A. J. González-Regaña), 0000-0002-6359-5246 (V. Martín-Molina), 0000-0003-3396-268X (R. Toscano)

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ABSTRACT

In this paper, we study how undergraduate students define 3D geometrical solids. With this aim, we have identified the routines that are present in the discourse of the students when describing and defining these solids. These routines are one of the properties that characterise the mathematical discourse in the theory of commognition (Sfard 2008). Our results show three different types of routines. The first type is related to the process of describing the solids, the second one to the process of defining the solids and the rest of the routines have a transversal nature. All of them together give us a global vision of the mathematical practice of defining of these undergraduate students. For instance, it seems that some of these students do not have a clear idea of what a definition is. Moreover, there are also differences between the discourse of students when defining 2D figures and the discourse of students when defining 3D solids.

Keywords: theory of commognition, defining, routines, undergraduate students.

INTRODUCTION

Research in mathematics education at university level has gained more relevance and has been the focus of many studies in recent years (Biza et al. 2016). One point of interest at university level is the mathematical practices. In particular, some authors point out the importance, at this level, of the mathematical practices of defining, modelling or proving in the teaching and learning of mathematics (Inglis and Alcock 2012; Martín-Molina et al. 2018a; Ouvrier-Buffet 2011; Viirmann and Nardi 2017, 2019; Weber and Mejia-Ramos 2013).

Several researchers remark on the importance of the defining process versus other mathematical practices. For instance, Freudenthal (1973) states that "establishing a definition can be an essential feat, more essential than finding a proposition or a proof" (p. 134). Tall (1991), among others, characterises the transition from elementary mathematical thinking to advanced mathematical thinking by highlighting the role of mathematical definitions. In particular, he points out that, in this significant transition, students change "from describing to defining, [and] from convincing to proving in a logical manner based on those definitions" (p. 20). For De Villiers (1998), "the construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc." (p. 249). More recently, Ouvrier-Buffet (2011) has focused on how university students define when involved in proving activities. She considers definitions as "temporary statements giving us information about the stages of concept formation at the dialectic interplay with proofs" (p. 166).

Furthermore, in recent decades, some researchers have considered sociocultural perspectives as the theoretical frameworks in their studies (Lave and Wenger 1991; Lerman 2001). From these perspectives, learning and any cognitive activity can be seen as social processes and not purely as biological processes. Specifically, Heyd-Metzuyanim et al. (2013) highlight the importance of human communication in mathematics education and, consequently, the importance of discourse as the main focus of study.

In this work, we use the sociocultural theory of commognition (Sfard 2008) as our theoretical framework. This theory has proved useful for studying the teaching and learning of mathematics at university level. For instance, Nardi et al. (2014) review several commognitive studies about the discursive shifts that occur when students begin to study Calculus, Tabach and Nachlieli (2015) focus on the mathematical definition of a function, Biza (2017) studies students' discourses about the tangent line and Viirman and Nardi (2017) study the evolution of biology students' mathematical discourses when modelling.

Several authors have researched the practice of defining from the commognitive approach. For instance, Sánchez and García (2014) study commognitive conflicts between sociomathematical and mathematical norms when students define in 2D geometry. Also, in 2D geometry, Escudero et al. (2014) characterise students' changes in the mathematical discourse (mathematical learning). In addition, in a previous study, we began to explore what routines may appear when undergraduate students define in 3D geometry and if defining in 3D for students is similar to defining in 2D or if it has some special characteristics (Martín-Molina et al. 2018b). In the last three studies, the participants were mainly undergraduate students that were studying a bachelor's degree in Primary Education to become teachers of children aged 6-12 years. Research on the discourse of defining of pre-service teachers is of particular importance to determine whether they understand that defining is not a mere description of characteristics.

In this study, we focus on the mathematical practice of defining, which we consider as a process where the definition is the final product. In particular, we analyse how undergraduate students that are studying to be Primary Education teachers describe 3D geometrical solids and how they construct mathematical definitions of them. The fact that these students are pre-service teachers implies that our results provide information not only about their present knowledge (which may be useful for their university teachers) but also about the knowledge that will become the base for their future teaching.

CONCEPTUAL FRAMEWORK

This section is divided in two parts. The first one is devoted to describe how we understand the mathematical practice of defining. In the second one, we present Sfard's (2008) theory of commognition, which we use in this study to analyse the data and obtain our results.

The mathematical practice of defining

The mathematical practice of defining appears in different mathematical situations like proving a theorem, generalising a concept, solving a problem, etc., which has led to different ways of considering it. In this study, we have considered the mathematical practice of defining as the focus, not as a process that arises in other situations.

Several authors (Dreyfus 1991; Tall 1991) point out the necessary transition in mathematical learning from more informal processes (as discovering or describing) to more formal mathematical processes (as defining or proving). Taking this into account, we consider defining as a process that consists of several steps: first the description of the object to define; then the formulation of preliminary definitions of it; and finally the selection of the *best* definition among the ones constructed, which would constitute the *formal* mathematical definition of the object (Martín-Molina et al. 2018b). Finally, we would like to remark that we agree with Rasmussen et al. (2005) on the consideration of students' mathematical practices (like symbolising, algorithmatising and defining) as social or cultural practices. For this reason, we have adopted a sociocultural approach to study the practice of defining among undergraduate students.

The theory of commognition

Among the sociocultural frameworks, we have adopted Sfard's (2008) theory of commognition, also called commognitive framework. This theory has become widely used in recent years (Presmeg 2016; Tabach and Nachlieli 2016) and has been employed at all educational levels, from pre-K to university. For instance, see Lavie et al. (2019) for pre-K level, Caspi and Sfard (2012) for elementary level, Emre-Akdoğan et al. (2018) for secondary level, and Thoma and Nardi (2018) for university level.

The term *commognition* (a combination of the words *communication* and *cognition*) was introduced by Sfard (2008) as a way of showing that there is no difference between thinking and communicating because thinking is a way of communicating with oneself (intrapersonal communication). For Sfard (2008), discourses are "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (p. 93) and constitute the main object of study in her framework. Sfard (2008) considers that mathematics is a particular type of discourse characterised by four properties: word use, visual mediators, narratives and routines.

The first property, word use, refers to the use of both mathematical words (i.e. polygon, prism, parallel, etc.) and ordinary words that are used with mathematical meaning (i.e. the prism is *leaning* instead of *oblique*). Secondly, visual mediators are the visual objects that participants of the mathematical discourse use in the process of communicating (i.e. mathematical formulae, graphs, drawings, diagrams, etc.). Narratives are “any sequence of utterances, spoken or written, framed as a description of objects, of relations between objects, or of activities with or by objects” (Sfard 2008, p. 223). These narratives are subject to endorsement or rejection by the participants in the communication (called *discursants*). Examples of endorsed narratives are definitions and theorems that are accepted by the mathematical community. Finally, routines are *repetitive patterns* that are characteristic of the mathematical discourse and can be inferred by observing if there are regularities in the use of the other three properties of the discourse, for instance, when discursants define mathematical objects, prove theorems, etc.

For Sfard (2008), learning can be viewed as “improving participation in historically established forms of activity” (p. 301), that is, as a process of participation in communities of practice (Lave and Wenger 1991). Therefore, learning mathematics is changing one’s discourse, and can be measured by the change of any of the previous four properties. The main source of learning mathematics is commognitive conflicts. A commognitive conflict “is defined as the phenomenon that occurs when seemingly conflicting narratives come from different discourses—from discourses that differ in their use of words, in the rules of substantiation, and so on” (Sfard 2007, p. 575).

On the other hand, the relevant role of routines has been highlighted by several authors. For instance, Lavie et al. (2019) state that “investigating learning is tantamount to answering the question of how routines emerge and how they later evolve” (p. 10) and that, through the study of routines, we can investigate “learning not just of individuals, but also of collectives" (p. 11). Other authors like Ní Ríordáin and Flanagan (2019) point out the important role of routines when they study the bilingual undergraduate students' language use in relation to functions. For Ioannou (2018), routines are crucial when studying students' difficulties in learning mathematics.

In this paper, we aim to characterise, through the identification of routines, the discourse of undergraduate students when they describe and define 3D geometrical objects. We are also interested in seeing if, for these students, the process of defining 3D objects has some characteristics that distinguish it from the process of defining 2D objects that some researchers have studied (Escudero et al. 2014; Sánchez and García 2014).

METHODOLOGY

We show below the methodology we have followed to obtain the results of the next section. In particular, we present the participants of this study, how we collected the data and how we later analysed it. We also give details about our research instrument.

Participants and context

The participants of this study were undergraduate students of a big public university of Spain. These students were enrolled in a bachelor's degree in Primary Education to be future teachers of children from 6 to 12 years old. This degree included only two courses related to mathematics, both compulsory. One of them, in the first year of the degree, focused on mathematical content and the other one, in the second year, on pedagogical mathematical content. The participants of this study were students that were taking the course in mathematical content. This subject had two parts, one theoretical (one lesson of two hours per week) and one practical (one lesson of one hour per week). In the theoretical lessons, the students learned about numbers, geometry, functions, probability and statistics. In the practical lessons, the students, organised in small groups, worked on the same mathematical content through problem solving.

In one of these practical lessons, the researchers presented the research instrument to a class of students and asked for volunteers to participate in the study. There were 12 groups of 3-4 students that wished to participate, with a total of 45 students. We called the groups G1, …, G12 and the students of each group S1, S2, S3 and S4. Since we always specify the group when discussing students' discourse, we do not specify it in the notation we use to refer to the students.

Research instrument

The research instrument was a worksheet with questions that the students had to answer. It was designed to promote the appearance of mathematical discourse among the students, that we would later analyse using Sfard's (2008) theory of commognition. Therefore, the worksheet had to generate a rich discourse in two ways, vocal and written. With this aim, we included open questions that promote the elaboration of narratives and the appearance of patterns that allow us to identify routines of students when they define.

In particular, the worksheet is composed of a brief explanation presenting our research to the students, a picture of three solids drawn with GeoGebra (see Figure 1) and nine questions about describing and defining the objects. The three solids were chosen because they have some common characteristics (like the fact that all of them are solids with six faces) and each of them have some characteristic that differentiated it from the other solids (e.g., the first solid is the only regular one). In the worksheet, we always used both words *characteristics* and *properties* (and used them as synonyms) because the students came from diverse backgrounds and, thus, we did not know with which word they were familiar.

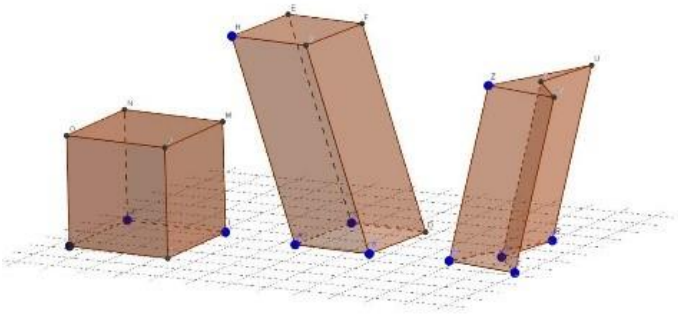

Fig. 1 The three solids presented in the research instrument (Gavilán-Izquierdo et al. in press)

In order to design the questions of the worksheet (and their order), we have taken into account the three steps of the defining process that were presented in the conceptual framework. That means that we consider that the process begins with the description of the object that we want to define, continues with the construction of several mathematical definitions and finishes with the selection of the *best* definition. In particular, the first four questions of the worksheet were related to the description of the solids through the identification and comparison of their characteristics. The next two questions asked the students to construct two definitions for each of the solids. In the next two questions, we requested that the students reflect on the definitions that they had just constructed in order to give a definition that was valid for two or three of the solids. Finally, in question nine, we asked the students to select the *best* definition among the ones they had constructed.

Data collection

The data were collected during a one-hour practical lesson. We provided each group of students with a copy of the worksheet and asked them to answer the questions by first debating among the members of the group and later writing down their consensus. We audio recorded these discussions (which we later transcribed) and collected their written answers.

Analysis

In a previous study (Gavilán-Izquierdo et al. in press), we analysed all our data (the transcripts and the written answers) in order to identify two of Sfard's (2008) properties: word use and narratives. In the present study, we have analysed these two properties in search of regularities in their use, which have allowed us to infer repetitive patterns (routines) in the discourse of students. These patterns inform us about the way students describe and define. In order to identify these patterns, each researcher analysed the data individually and then the whole team of researchers met to compare their findings. The patterns that had been identified by all the researchers were accepted and those that had been proposed by only some of the researchers were discussed by the whole team until they were accepted or rejected. We only considered visual mediators when they appeared explicitly in the discourse of the students. Most of the students did not refer to other visual mediators apart from the figures in the worksheet.

Next, we show part of the discussion of group G7 when they were constructing the definition of the first solid. We include an excerpt of the transcript and an excerpt of their written answers. In the transcript, the mathematical words are in bold and the narratives are in italics:

219: S1: *first the name*

220: S2: a **cube**, right?

221: S3: yes

222: S2: *a **cube** is… a **solid***

223: S4: *which are all **prisms** because they are formed by [sic] several **polygons***

224: S1: ***polygons** of 6 **faces***

225: S2: *of 6 **faces**… that are **equal*** [writing]

226: S4: *with a* ***square basis***

227: S2: [repeats while writing]

228: S1: *it is a* ***hexahedron****, that is the first thing*

We now show in Table 1 the answer they wrote after the discussion above. It is remarkable that the word *hexahedron* does not appear, despite the narrative of line 228. Each of the written answers shown as examples in this paper will be given together with their translations into English.

A cube is a solid of 6 equal faces with a quadrangular base whose angles are right.	CUERPO 1: Un cubo es un cuerpo geométrico de 6 caras iguales de base cuadrangular cuyos ángulos son rectos.

Table 1 Excerpt from G7 (on the right) and its translation into English (on the left)

From these two excerpts (transcript and written answer) and some others of similar nature, we inferred the existence of a repetitive pattern in the construction of a definition. This pattern consists on first giving a name for the solid, followed by a description of the characteristics of that solid. We named this routine *Defining is labelling and describing the characteristics of a solid*.

RESULTS

In this section, we show the different types of routines that we have identified in the students' discourse. We organise these routines in three blocks attending to certain characteristics that some of these routines have in common. In the first block, we include the routines related to the description of the properties or characteristics of the elements of the solids. In the second block, the routines describe how students construct the definitions of the solids. Lastly, the third block includes routines of a transversal nature, in the sense that such routines may appear during any step of the process of defining the solids. For each routine, we have given a name and a description, followed by some implications that we have obtained from its existence. We also include representative examples from the student's discourse (translated into English) that, with others of similar nature, have allowed us to infer these routines. We deem these examples important because they help the reader see pieces of our data from which we have obtained our results.

Routines when students describe

In this block, we have included three routines that inform us about how the students describe the properties or characteristics of the elements of the solids and how they identify characteristics that are common to two or more solids.

Counting when describing (DES1). This type of routine is observed when the students try to identify characteristics of the elements of a solid (vertices, faces, edges, etc.) by counting how many times such elements appear (routine found in all groups). For this routine, we have differentiated two ways of counting: counting elements one by one, which we call *additive counting* (DES1a) (found in G1, G3, G4, G5, G6, G7, G9, G10, G11, G12), or counting elements by using a multiplicative strategy, called *multiplicative counting* (DES1b) (G2, G3, G7, G8, G9, G11).

A representative example of the routine of additive counting appears when the students of group G1 describe the edges of the solids:

> 52: S2: Edges.
>
> 53: S1: No, 8 vertices and, edges, they have 1, 2, 3, 4, 5, 6, 7, 8, 9… I think that this edge is already counted. Just a moment, I made a mistake.
>
> 54: S2: There are also 12. All the solids have 12 edges.

An example of the routine of multiplicative counting appears when the students of group G7 describe angles:

69: S3: And how many right angles? 1, 2, 3, 4…

70: S4: There are three times four, aren't there?

71: S3: Yes.

Routine DES1 informs us about how the students describe the properties and characteristics of the solids given in the worksheet. Although all the groups add some adjectives that describe the elements of the solids, we have not considered this relevant because, in this case, they merely do what the question requests. We highlight that all the groups describe the solids by first counting how many vertices, edges and faces they have. Although we have distinguished two different ways of counting such elements of the solids (*additive counting* and *multiplicative counting)*, in many cases the students do not specify how they count the number of elements of the solids, so we cannot say if they are using additive, multiplicative or other type of counting. In any case, it is interesting that, in a question that asks about describing elements, all the groups decide to begin their descriptions by counting these elements.

Defining an element before describing it (DES2). This routine is identified when the students describe an element of a solid (vertex, face, edge, etc.) and have the need to define that element first (G7, G9, G11).

Now we show an example of this routine that appears in the following excerpt from the discussion of G9:

24: S4: These, the edges.

25: S2: These, these because they connect two of these, two vertices.

26: S3: So, how many does it have? 10.

27: S2: 12, right?

28: S3: 4.

29: S1: What?

30: S2: 12 edges.

31: S1: Yes, what connects two vertices.

This routine seems to show us the need that some students have of giving a definition of certain elements of the solids before using them to describe such solids. This need may reveal students' lack of knowledge about the meaning of mathematical words, which impedes their description of mathematical objects.

Searching common characteristics of the solids (DES3). This routine is observed when the students try to find characteristics that two or more solids have in common by considering the number of elements (vertices, faces, edges, etc.) that these solids share and paying attention to the shape of the lateral faces or the basis of the solids (routine found in all groups).

Next, we provide in Table 2 an example of this routine that appears in the written answer of group G2 when asked the question "among the properties or characteristics of question 1, can you identify any property that the three solids have in common?":

All [solids] have 12 edges, 8 vertices and 6 faces, and all the bases have 4 sides and 4 vertices.	Todos tienen 12 aristas, 8 vértices y 6 caras, y todas las bases tienen 4 lados y 4 vértices.

Table 2 Excerpt from G2 (on the right) and its translation into English (on the left)

This routine is related to the first one, DES1, in the sense that students focus again on counting (this time, the common number of elements of the solids) before doing anything else.

Taking into account the routines of this block, we deduce that the groups of students use a very similar discourse when describing solids. This may indicate that their small repertoire of words and routines related to describing 3D objects makes them go back to routines that are more familiar to them (e.g., counting elements). For instance, almost none of the students mention the parallelism or perpendicularity of the faces, the convexity or concavity of the solids or the proper word for "oblique" (instead they use words like "leaning"). Many of them also employ the word "regular" incorrectly because they use it to mean polygons or solids that are familiar to them.

Routines when students define

In this block of routines, we have included seven routines that inform us about how students construct, analyse and compare the definitions of the solids. The first three routines show the different structures that the students employ to construct a definition for each solid: DEF1a, DEF1b and DEF1c. The fourth one, DEF2, describes how the students construct a new definition of a solid by using one they have previously constructed. Furthermore, the fifth routine, DEF3, describes how the students give a common definition for several solids. Finally, routines DEF4 and DEF5 show how the students choose a definition among several ones that they have previously constructed.

Defining is labelling and describing the characteristics of a solid (DEF1a). This routine is inferred when the students define a solid by giving a label (a signifier in the theory of commognition) for the solid and describing its mathematical characteristics (G1, G2, G5, G6, G7, G8, G9, G10, G11, G12).

A representative example of this routine can be seen when group G2 defines the first solid:

61: S1: A cube, a solid with volume, right? That occupies a space.

62: S3: But that is true for all of them, isn't it?

63: S1: Yes, but now we have to say that one is regular, another one has … everything that we said before. […]

Another example of this routine appears when group G7 writes down the following definition (see Table 3):

A cube is a geometric solid of 6 equal faces with a quadrangular base whose angles are right.	CUERPO 1: Un cubo es un cuerpo geométrico de 6 caras iguales de base cuadrangular cuyos ángulos son rectos.

Table 3 Excerpt from G7 (on the right) and its translation into English (on the left)

Defining is labelling (DEF1b). This routine is observed when the students construct the definition of a solid by giving only a label for it (G4, G5, G9).

Next, we present in Table 4 an example of this routine that appears in the following excerpt from the written answers of group G5 when asked to define the first solid. In this case, the label that appears is "regular parallelogram":

Regular parallelogram.	CUERPO 1: Paralegramo regular

Table 4 Excerpt from G5 (on the right) and its translation into English (on the left)

Defining is giving a list of characteristics of the solid (DEF1c). This routine is inferred from certain situations in which the students define a solid by giving only a list of characteristics of the solid (G2, G3, G4, G5, G6, G10, G11).

An example of this routine appears in the excerpt of Table 5, found in the written answers of group G10 when asked to give another definition for the first solid:

All the angles are right and its height is the same in all its sides.	CUERPO 1: Todos sus ángulos son rectas y su altura es igual en todos sus lados

Table 5 Excerpt from G10 (on the right) and its translation into English (on the left)

These three routines (DEF1a, DEF1b and DEF1c) give us information about how students construct definitions when asked for them. We would like to highlight that sometimes one group uses different routines when trying to define the solids, as is seen in Table 9. Indeed, group G5 uses these three routines when answering the questions of the worksheet and other six groups use two of them. The rest of the groups show in their discourse a single routine when defining, mostly *Defining is labelling and describing the characteristics of a solid.* Furthermore, we would like to remark that three groups only give one definition for each solid (they do not answer Question 6, in which we ask for a second definition). In addition, other two groups only give two definitions for either one or two of the solids. Sometimes, the students state explicitly that they do not know how to construct another definition for each of them. This lack of homogeneity when defining may be linked to the fact that the students may have not received explicit instruction about how to define in mathematics.

We present now the fourth and fifth routines of the second block (DEF2, DEF3), which inform us about how the participants give another definition for the solids that they have previously defined.

Giving a definition by removing characteristics from another one (DEF2). This routine is inferred from certain situations where the students, in order to give a definition for a solid, remove some characteristics given in a former definition of it (G10, G11).

For instance, group G11 defines the first solid the first time (when answering Question 5) as seen in Table 6:

It is a geometrical solid composed by 6 faces, 8 vertices and 12 edges, that is, it is a cube. All the edges have the same measure, because the faces are regular squares and all the vertices form angles of 90 degrees.	CUERPO 1: Es un cuerpo geométrico formado por 6 caras, 8 vértices y 12 aristas, es decir, es un cubo. Todas las aristas tienen la misma medida, ya que, las caras son cuadrados regulares. Y todos los vértices forman ángulos de 90.

Table 6 Excerpt from G11 (on the right) and its translation into English (on the left)

Later, answering Question 6, the students of this group (G11) construct a new definition for the same solid by removing some characteristics from the definition above. This new definition appears in the written answers of group G11 as in Table 7:

It is a geometrical solid composed by 6 faces, 8 vertices and 12 edges.	CUERPO 1: Es un cuerpo geométrico formado por 6 caras regulares, 8 vértices y 12 aristas.

Table 7 Excerpt from G11 (on the right) and its translation into English (on the left)

This routine may inform us about which characteristics the students consider more relevant than others to define. We highlight that this routine is similar to a routine that mathematicians sometimes use to construct new definitions when they do research (Martín-Molina et al. 2018a).

Giving a common definition by choosing common labels or characteristics (DEF3). This routine is inferred when the students, in order to give a definition that is valid for several solids, select the characteristics of these solids that appear in both of their former definitions (G1, G2, G4, G5, G6, G8, G9, G10, G11).

A representative example of this routine can be observed in group G4 when, after having defined the second solid as a *"*rectangular prism*"* and the third solid as an "irregular prism*"*, they state that:

> 53: S2: For two solids we can do it, can't we? We can say that the second solid and the third solid are prisms.

We highlight that this routine is compatible with routines DEF1a, DEF1b and DEF1c. Indeed, when students employ routine DEF3, they choose common labels or characteristics to construct a definition that will have one of the structures described in routines DEF1b, DEF1b or DEF1c.

Routines DEF2 and DEF3 seem to be related to the *minimality* of a definition, that is, to whether there is "no superfluous unnecessary conditions or information in the definition" (Zaslavsky and Shir 2005, p. 320). However, we cannot conclusively state this relation from the available data we have.

The last routines of the second block are DEF4 and DEF5, which show students' criteria for selecting one of the two definitions of a solid that they had previously constructed.

Choosing the most "complete" definition (DEF4). This routine is observed when the students, in order to choose a definition for a solid between the two that they had previously constructed, select the definition with the largest number of characteristics (G2, G5, G10, G11).

A representative example of this routine may be observed in the following excerpt from group G2, where they justify their choice of definition:

> 255: S2: I would choose the definitions that we have written in question six.
>
> 256: S1: But I will not write it again.

257: S3: Question six.

258: S2: Why?

259: S3: Because it is more complete.

260: S2: Because it is describing with all the characteristics that the figure has.

Choosing the most "accurate" definition (DEF5). This routine is observed when the students, in order to choose a definition for a solid between the two that they had previously constructed, select the definition that they consider the most *accurate*, understanding by *accurate* the definition given by labelling (G4).

As before, a representative example of this routine can be observed when the students of G4 justify which definition they have chosen from the two they had previously constructed (Table 8):

Definition: Cube Justification: This is the definition that best fits with the figure. (IN ALL THE SAME JUSTIFICATION).	Definición: cubo. Justificación: Es la definición que más se corresponde con la figura. (EN TODAS LO MISMO JUSTIFICACIÓN)

Table 8 Excerpt from G4 (on the right) and its translation into English (on the left)

We point out that most of the groups that give two definitions for each solid choose the one with the greatest number of characteristics. Only group G4 chooses the shortest definition between the two they had constructed, specifically, the definition constructed by giving only one word, which we refer to as a label. Therefore, we can conclude that most groups seem to prefer again definitions that are not minimal. We remark that, according to Borasi (1992), this means that the students' definitions do not have the commonly accepted requirement of *essentiality*. This may indicate that, despite many years of geometry classes, students do not seem to have adopted the routines that are common among mathematicians when defining.

As a consequence of the variety of routines of this second block and the variety of groups where these routines are identified, we deduce that the discourse on describing is more or less homogeneous, but the discourse on defining is quite heterogeneous.

Transversal routines

In the third block, we include two routines of transversal nature, that is, these routines may appear both when the students describe and define the solids: T1 and T2.

Resorting to 2D to solve 3D problems (T1). In this routine the students turn to their knowledge of 2D geometry as a way to bolster their (scant) knowledge of 3D geometry. This routine reflects a return to simpler problems when facing an unknown situation (G1, G2, G3, G5).

For instance, group G5 uses this routine when its students try to define the second solid. Specifically, when they seek a label for this solid:

161: S2: The solid 2 is a rectangle that has the two bases…

162: S1: It's not a rectangle.

163: S2: Is it?

164: S1: That is, this is not called a rectangle …it is a prism …

[…]

173: S1: The thing is that [student S2] is extrapolating what she has studied in class about faces to prisms, more or less.

174: S4: We have studied in class what a trapezium is and what a trapezoid is but not what a prism is.

We hypothesise that the appearance of this routine may be due to the Spanish curriculum, which gives more emphasis to 2D geometry than to 3D geometry.

Searching for information in external sources (T2). This routine appears when the students ask the teacher for help, search for information in their class notes or on the internet. We highlight that this routine is not exclusive of mathematics but has a social nature (G1, G2, G3, G5, G6, G7, G9, G11).

For example, the students of group G7 use this routine several times:

> 28: S3: [asking the teacher] But…, do we have to define in the first part, like..., what is a vertex, what is a face…? Then, what do we have to say, like…, how many faces they have, how many vertices there are?
>
> […]
>
> 59: S2: Look that up on the internet.

Finally, we summarise in Table 9 which groups use each routine.

ROUTINE CODE AND NAME		GROUPS
DES1. Counting when describing	DES1a. Additive counting	G1, G3, G4, G5, G6, G7, G9, G10, G11, G12
	DES1b. Multiplicative counting	G2, G3, G7, G8, G9, G11
DES2. Defining an element before describing it		G7, G9, G11
DES3. Searching common characteristics of the solids		ALL GROUPS
DEF1a. Defining is labelling and describing the characteristics of a solid		G1, G2, G5, G6, G7, G8, G9, G10, G11, G12
DEF1b. Defining is labelling		G4, G5, G9
DEF1c. Defining is giving a list of characteristics of the solid		G2, G3, G4, G5, G6, G10, G11
DEF2. Giving a definition by removing characteristics from another one		G10, G11

DEF3. Giving a common definition by choosing common labels or characteristics	G1, G2, G4, G5, G6, G8, G9, G10, G11
DEF4. Choosing the most “complete” definition	G2, G5, G10, G11
DEF5. Choosing the most “accurate” definition	G4
T1. Resorting to 2D to solve 3D problems	G1, G2, G3, G5
T2. Searching for information in external sources	G1, G2, G3, G5, G6, G7, G9, G11

Table 9 Table of routines and the groups which use each of them

CONCLUSIONS AND DISCUSSION

In this study, we have made a contribution to the existing literature on the defining process by using the theory of commognition. In particular, this theory has allowed us to identify routines that inform us about the characteristics of the mathematical process of defining of undergraduate students that are also pre-service teachers.

These routines have been organised in three blocks attending to their nature. In the first block, we have included three routines that appear when the students describe the solids; in the second block, seven routines closely linked to the defining of the solids; and, lastly, in the third block, we have included two routines of a transversal nature.

This study informs us about some characteristics of the process of defining that may have remained hidden otherwise. The existence of these characteristics may have implications for the teaching and learning of 3D geometry. For example, the appearance of the routine *Counting when describing* contrasts with the findings of researchers that studied (also through the lens of the theory of commognition) how students define 2D quadrilaterals. In particular, Gavilán-Izquierdo et al. (2014) found that, when the participants of their study (also pre-service teachers) described a square, rhombus or rectangle, the characteristics that they included in their descriptions were mainly qualitative ones, like the parallelism of their sides or whether the sides were equal. Indeed, such pre-service teachers did not mention the number of elements of the quadrilaterals. Regarding the second routine, *Defining an element before describing it*, these authors also found that the participants of their study did not seem to have the need to define an element before describing it, thus contrasting again with the results obtained in the present paper on the process of defining in 3D geometry. We hypothesise that these differences may appear because Spanish students are usually more familiar with quadrilaterals (and their elements) than with solids (and their elements), since 2D geometry is usually studied more in depth than 3D geometry in Spanish schools. These differences in the discourse of students when defining in 2D or 3D reveal the existence of two discourses with different characteristics (the discourse of defining in 2D and the discourse of defining in 3D), and the complex relationship between them. Specifically, we consider that the discourse of defining in 3D is not a mere generalisation of the discourse of defining in 2D. For example, there exist routines in the 2D discourse that are not extendable to the 3D discourse (for instance, the routine of checking if two straight lines are parallel or not is more involved in 3D geometry than in 2D geometry). This finding is similar to the one presented by Ioannou (2018), who points out that group theory is not a mere generalisation of set theory.

The first three routines of the second block, *Defining is labelling and describing the characteristics of a solid*, *Defining is labelling* and *Defining is giving a list of characteristics of the solid*, show that the participants of this study have different ways of defining the solids of the worksheet. In fact, the students of the same group sometimes proposed different definitions (with different characteristics) for the same solid. In general, the students first used the routine of *Defining is labelling and describing the characteristics of a solid* and, when asked to construct a second definition, some groups stated that they did not know another one and some others considered that they had to give another definition with a different structure, thus using one of the other two routines.

This could mean that they consider the first definition the correct one and the second one is one that they feel forced to give, with mixed results. Therefore, either the students do not seem to have clear criteria for constructing a definition or what seems to be a sociomathematical norm (different mathematical questions need different answers) has more importance for them than their own criteria. We consider sociomathematical norms to be normative aspects of "mathematics discussions specific to students' mathematical activity" (Yackel and Cobb 1996, p. 461).

The lack of common criteria we have observed among the students when constructing a mathematical definition also occurs among mathematicians, since there does not seem to be an agreed upon *definition* of a mathematical definition. For example, in mathematics education, according to Borasi (1992), mathematical definitions have the following requirements: precision in terminology, isolation of the concept, essentiality, noncontradiction, and noncircularity, while Zaslavsky and Shir (2005) state that definitions must be noncontradicting, unambiguous, invariant under change of representation and hierarchical. On the other hand, Tabach and Nachlieli (2015) point out that "in the mathematical community, *mathematical definitions* are precise definitions that contain necessary and sufficient conditions to help us determine whether or not a word applies to certain examples" (p. 167). These last authors, citing Copi (1972), say that the definitions should be minimal and non-circular.

The study of the students' routines when defining has allowed us to infer what seems to be a commognitive conflict between students' discourse and mathematicians' discourse. Authors like Sánchez and García (2014) identified commognitive conflicts between sociomathematical and mathematical norms in the discourse of students when they construct definitions in 2D geometry. Our study complements theirs, since our results permit us to identify a commognitive conflict that is similar to one of theirs. In particular, a routine that students use, *Choosing the most "complete" definition,* could be considered a sociomathematical norm. There seems to be a commognitive conflict among our participants between this sociomathematical norm and the mathematical norm that states that a definition must be minimal.

Another source of commognitive conflicts for students is the confusion between the routines used when describing and the routines used when defining. In particular, some students considered that there is no difference between defining and describing, what led them to employ routines normally used when describing (for instance, giving a list of characteristics of a solid) to define objects. Indeed, when we asked the students to give a definition for each of the solids, some of them explicitly stated that they had already done that in a previous question, in which they were asked to describe the elements of the solid.

Lastly, continuing this study with other students and other questions could produce a more complete vision of how students describe and define mathematical objects. For instance, we could obtain more information about the differences that exist when students define 3D objects instead of 2D objects. All this information could also be valuable to teachers of mathematics and of mathematics education in the sense that knowing how students describe and define could influence their teaching.

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