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Pre-service mathematics teachers' discourse: Differences between defining in task situations involving prototypical and non-prototypical solids

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ABSTRACT

The literature has highlighted the significant role of definitions and defining in mathematics learning and teaching. Furthermore, non-prototypical figures are particularly important when teaching geometry, but teachers and pre-service teachers still have problems defining them. For these reasons, we investigated whether there were differences in the way that pre-service mathematics teachers constructed and selected definitions for prototypical and non-prototypical solids. In particular, the commognitive framework was employed to investigate the differences in the discourse of 33 pre-service secondary-school teachers when constructing and selecting definitions in task situations that involved prototypical and non-prototypical solids. Moreover, we studied if some commognitive conflicts appeared in task situations involving non-prototypical solids but not in similar task situations involving prototypical solids. The findings show some differences between the pre-service teachers' discourses in both types of task situations. Additionally, some commognitive conflicts appeared only in task situations with non-prototypical solids. Lastly, we classified those commognitive conflicts.

1. Introduction

Definitions have a major role in mathematics and its teaching and learning at all educational levels (e.g., Avcu, 2023; Freudenthal, 1973; Molitoris Miller, 2018; Tabach & Nachlieli, 2015; Zaslavsky & Shir, 2005).

Many voices in mathematics education also attribute geometry a prominent role in learning mathematical reasoning (e.g., Clements & Sarama, 2011; Sinclair & Bruce, 2015; Sinclair et al., 2011). In such learning, defining geometric objects and 3D geometry have a noticeable place (National Council of Teachers of Mathematics [NCTM], 2000).

Consequently, researchers have called to pay more attention to the development of "teachers' competency and comfort with

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geometric strands of mathematics" (Sinclair et al., 2011, p. 136), which involves defining geometrical 3D objects. Avcu (2023) stated that there should be more research on how teachers construct definitions and on the characteristics of those definitions because "their knowledge of definitions may largely influence their choice of pedagogical approaches to teaching mathematical definitions" (p. 744). Moreover, various researchers (e.g., Fujita & Jones, 2007; Molitoris Miller, 2018) have highlighted various problems or difficulties that pre-service mathematics teachers (PSTs) have with mathematical definitions. Specifically, one particular problem that teachers may encounter when defining or classifying geometric figures is that they rely excessively on prototypical figures (Clements & Sarama, 2011; Fujita, 2012; Fujita & Jones, 2007). A prototypical figure is one that has some tacit characteristics that are not stated in the formal definition (Fischbein & Nachlieli, 1998), like a square that is always drawn with its sides parallel to the edges of the paper.

Therefore, aware of the need for further research on the way that PSTs construct definitions of 3D objects and on the characteristics of those definitions, in this paper we specifically study the differences between how PSTs define and select definitions for prototypical solids and how they do it for non-prototypical solids.

2. Literature review

2.1. Mathematical definitions and defining

Researchers in mathematics education and mathematics teachers organizations point out that defining and mathematical definitions are topics that need to be part of teacher training programs (e.g., Conference Board of the Mathematical Sciences [CBMS], 2012; Leikin & Zazkis, 2010; Molitoris Miller, 2018; National Council of Teachers of Mathematics [NCTM], 2020). Indeed, Molitoris Miller (2018) stated that PSTs should learn about defining so that they can "unpack the form and purpose of mathematical definitions" (p. 143) and Avcu (2023) added that they should "engage in dialogical interactions with their students when creating and critiquing mathematical definitions" (p. 745).

Due to the importance of mathematical definitions and defining in geometry, mathematics educators have conducted research on how students, teachers and PSTs define and on the mathematical definitions they construct or select (Avcu, 2023; Fernández-León et al., 2021; Fujita, 2012; Fujita & Jones, 2007; Gavilán Izquierdo et al., 2014; González-Regaña et al., 2021; Molitoris Miller, 2018; Sánchez & García, 2014; Sinclair & Bruce, 2015; Tsamir et al., 2015; Ulusoy, 2021; Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008). This research includes studies on defining and definitions of 3D objects (Bruce & Hawes, 2015; Downton & Livy, 2022; Martín-Molina et al., 2023). For example, Zaslavsky and Shir (2005) found that one of the features that are central when defining both geometric and analytic objects is whether a definition is procedural or structural. Procedural definitions comprise of procedure through which the focal project emerges, whereas structural definitions describe the properties of an object. Furthermore, Zazkis and Leikin (2008) studied how secondary school PSTs evaluated definitions that had been "presented to them in order to investigate their conceptions of a mathematical definition" (p. 146) and found that asking them to evaluate definitions generated an abundant source of data that would not have been available if they had asked the PSTs about what a mathematical definition is. Fernández-León et al. (2021) and González-Regaña et al. (2021) analyzed pre-service primary school teachers' discourse when they constructed definitions of 3D solids and identified difficulties that arose when defining. For instance, they found that some PSTs considered that defining a mathematical object is describing its mathematical properties. Lastly, Tsamir et al. (2015) investigated how teachers defined both 2D and 3D objects and found that they had little difficulty with 2D polygons, but struggled more with cylinders (3D geometric objects).

2.2. Prototypical and non-prototypical examples

As mentioned in the introduction, the learning of a geometric concept is often influenced by prototypical examples, which are those that have "the 'longest' list of attributes – all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics" (Hershkowitz, 1990, p. 82). Hershkowitz (1990) highlighted that these examples can become obstacles for both students and teachers. Indeed, when "a certain figure is strongly different from the prototype (though still corresponding to the constraints of the definition)" (Fischbein & Nachlieli, 1998, p. 1197), difficulties may arise when identifying that figure as an instance of a class. For example, Bernabeu and Llinares (2017) found that some students do not consider non-convex quadrilaterals to be quadrilaterals because, for them, every quadrilateral must be convex. This means that some students see convex quadrilaterals as prototypical examples and non-convex quadrilaterals as non-prototypical ones (Fischbein & Nachlieli, 1998).

These obstacles and misconceptions also exist among mathematics teachers and PSTs (Clements & Sarama, 2011; Fujita, 2012; Fujita & Jones, 2007; Ulusoy, 2021). For instance, Ulusoy (2021) found that many pre-service early childhood and elementary school mathematics teachers wrote inappropriate definitions for triangles and that many of their drawings of triangles and non-triangles could be considered prototypical. Fujita (2012) identified an excessive use of prototypical examples when several PSTs defined and classified quadrilaterals.

These and other studies suggest that both prototypical and non-prototypical examples should be considered when training PSTs about definitions and defining. However, despite the relevant role of the generation of 3D geometric definitions for the teaching and learning of mathematics and the distinctive features that defining may have in the three-dimensional space (Tsamir et al., 2015), there are few studies that focus on the differences that may exist between how PSTs construct definitions for prototypical and non-prototypical three-dimensional solids, which led us to conduct this research.

3. Theoretical framework

In this paper, the commognitive framework (Sfard, 2008) was employed to study the mathematical practice of defining. This sociocultural framework has been recognized by the international mathematics education community as a conceptual and analytical framework that considers the specificity of mathematics when studying learning and teaching (e.g., Knox & Kontorovich, 2023). This framework considers mathematical practices as sociocultural activities and defining as “a matter of human decision about the use of words” (Sfard, 2008, p. 57). However, this conceptualization of defining as a sociocultural activity is often lost in classrooms because mathematical definitions appear as objects in a textbook instead of being constructed through a process. Therefore, it is useful to employ the commognitive framework to study PSTs’ discursive activities when defining because it permits a fine-grained analysis of how the PSTs answer questions on defining related to prototypical and non-prototypical solids.

3.1. Mathematics as a discourse

The term commognition was introduced by Sfard (2008) as a portmanteau of the terms communication and cognition. In the commognitive framework, thinking is seen as a particular type of *interpersonal* communication, as communicating with oneself, and mathematics is conceptualized as “a discourse about mathematical objects, such as numbers, functions, sets, and geometrical shapes” (Sfard, 2008, p. 129). The notion of mathematical object is elusive (Sfard, 2008), but the commognitive framework proposes a way to operationalize it by defining a mathematical object as a pair <a signifier, all realizations of that signifier> (Lavie et al., 2019). Signifiers “are words or symbols that function as nouns in utterances of discourse participants” (Sfard, 2008, p. 154), while a realization is a “perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about [a signifier]” (Sfard, 2008, p. 154). For instance, one can use the word “circle” as a signifier and realize it in a picture on a piece of paper or as the algebraic equation of a circle. In that case, the mathematical object *circle* would be the word circle and all its possible realizations. We could also consider the equation or the picture on a piece of paper as signifiers and the rest as realizations of it.

3.2. Characteristics of mathematical discourse

In the commognitive framework, mathematics is seen as a specific type of discourse distinguished by four interrelated characteristics: keywords, visual mediators, narratives, and routines (Sfard, 2023). In particular, *keywords* refer to mathematical words (edge, cube, vertex, etc.). *Visual mediators* refer to the visible objects used by the participants in the discourse as part of their communication (for example, the diagram of a function, the expression of an algebraic equation or the drawing of a polyhedron). Furthermore, *narratives* are expressions (of written or spoken language) about objects, relationships between objects, or activities with or of the objects of the discourse. Narratives can be accepted or rejected by the participants in the discourse. If accepted, the narratives are called *endorsed narratives* (e.g., definitions, mathematical propositions, theorems). For example, a particular community may propose “polyhedron of six faces that are squares” as a definition for a cube. Finally, *routines* are repeated patterns in the participants’ discourse, which are inferred by identifying regularities in any of the other three discursive characteristics (for example, in how objects are defined or in how theorems are proved). Several researchers have pointed out the important role of routines (e.g., Fernández-León et al., 2021; Ioannou, 2018; Ní Ríordáin & Flanagan, 2020).

3.3. Routines as task-procedure pairs

According to Lavie et al. (2019), the study of routines does not only involve how and when a routine is performed, but also how an activity is interpreted by those performing it and what previous experiences they have in that type of activity. Therefore, those researchers define routines as a *task-procedure* pair. The *task* captures information about how the performer views a certain *task situation*, which is regarded as “any setting in which a person considers herself bound to act – to do something” (Lavie et al., 2019, p. 159). The *procedure* refers to the actions this performer undertakes to tackle the task situation (Lavie et al., 2019), as interpreted by the researcher (Nachlieli & Tabach, 2022). For example, a task situation arises when a group of students is asked to give a definition of a geometric concept of which a figure is an example. For mathematicians, the task in that task situation would be to provide a narrative that they consider a valid definition for the figure. A procedure that they could apply is the following two-step procedure: (1) searching for a signifier of a mathematical object of which the figure is a realization; (2) producing a mathematical definition, i.e., a narrative that includes a new signifier and some mathematical properties.

When learners face a new task situation, they turn to their *precedents*, which are those past situations that, for them, are similar to the new task situation and, therefore, offer a way of proceeding (Lavie et al., 2019). The search for precedents is normally confined to a *precedent-search-space*, which only includes previous situations that the performers regard as germane to their present one (Nachlieli & Tabach, 2022). For instance, when a student has to generate a geometric definition, their precedent-search-space may include all the previous situations in which they had to define or handle definitions. Additionally, Kontorovich (2021) introduced the notion of *precedent pocket* to “refer to a set of precedents on which a person draws in their precedent-search-space; a set that prescribes distinctive actions that are matched to the precedent identifiers that this person discerned from an assigned task situation” (p. 4). For example, if a student has to give a definition for a cube, they could replicate a memorized definition of it, while, if that particular student had to give a definition for a non-prototypical figure, they could list its properties. That student’s actions would be based on two different precedent pockets (Kontorovich, 2021) depending on the figure involved in the particular task situation.

Nachlieli and Tabach (2022) distinguished between two kinds of tasks: *object-level and meta-level tasks*. The first are tasks where the mathematical objects are mentioned explicitly and the latter are those "concerned with what the solution to a question should include based upon the meta-rules for doing mathematics that the class has endorsed" (Nachlieli & Tabach, 2022, p. 4). For instance, in a task situation in which students have to decide which definition they would choose for a particular figure and justify their choice, commognitive researchers could consider that the object-level task is to choose (and justify the choice of) definitions, while the meta-level task would be to decide which criteria to employ when making that choice and justifying it.

3.4. Commognitive conflicts as learning opportunities

The commognitive framework defines learning as a change in the interlocutors' discourse, which can be identified from variations produced in one or more of the four abovementioned discourse characteristics. A source of mathematical learning are *commognitive conflicts*, which arise when participants of incommensurable discourses strive to communicate across such discourses (Sfard, 2023). *Incommensurable discourses* are those with different *meta-rules*, i.e., rules that define patterns of activity of the participants in the discourse when trying to produce and substantiate *object-level narratives* (Sfard, 2008). These last narratives are those that concern mathematical objects (Sfard, 2008). Commognitive conflicts can be sometimes identified when different interlocutors use the same words in different ways or endorse contradictory narratives. For learning to occur, it is not enough that a commognitive conflict exists, it is also necessary for a participant to be *dialogically engaged*, that is, "alert to the possibility of incommensurability between her own and her interlocutor's discourses" (Sfard, 2020, p. 94) and strive to determine the rules of her interlocutor's discourse, making an effort to individualize that discourse (Sfard, 2020).

Several authors have delved into the notion of commognitive conflict (Gavilán-Izquierdo et al., 2022; González-Regaña et al., 2021; Ioannou, 2018; Knox & Kontorovich, 2023; Martín-Molina et al., 2023; Nachlieli & Heyd-Metzuyanim, 2022; Sánchez & García, 2014). In particular, focusing on the interlocutors' awareness of the commognitive conflicts, Knox and Kontorovich (2023) expanded the notion by distinguishing between *communicational clashes* (if the interlocutors are aware of the disagreements, although maybe not of the reasons for them) and *discursive gaps* (if the interlocutors are not aware of the disagreements). Additionally, Nachlieli and Heyd-Metzuyanim (2022) studied the progression in the explicitness of commognitive conflicts between Delivery Pedagogical Discourse (DPD) and Explorative Pedagogical Discourse (EPD). These authors associated the participation in a particular Pedagogical Discourse with particular teaching practices and distinguished between DPD, which "values actions that assume a delivery model of teaching" (Nachlieli & Heyd-Metzuyanim, 2022, p. 350) and EPD, which highlights the "externalization of thinking processes" (p. 348), group discussions and that students construct their own mathematical narratives and face meaningful problems.

Specifically, Nachlieli and Heyd-Metzuyanim (2022) identified four levels of conflict to characterize the progression in the explicitness of commognitive conflicts between DPD and EPD: *commognitive conflicts inaccessible to one of the parties (CCI),* in which despite the fact that narratives aligned with the DPD are explicit in the participants' discourse, the conflict is inaccessible to those participants unfamiliar with implicit EPD narratives; *explicit commognitive conflicts not reflected upon (ECCNR),* where despite there being explicit conflicting narratives in the discourse, "there is no meta-level discourse about" (p. 355) them; *explicit commognitive conflicts (ECC),* in which the conflicting narratives and the meta-rules that lead to them are made explicit and discussed; and *commognitive conflicts reflected upon (CCRU),* which add to the previous one reflections of the participants that usually recognize a movement or change in their discourse from DPD to EPD.

In this study, we hypothesize that those levels of conflict identified for the pedagogical discourse could be extrapolated to the mathematical discourse, which could inform us about students' and PSTs' learning of mathematics and thus give valuable information for the teaching of mathematics. We propose the following description of the conflicts at each level. A *CCI* is a conflict that is inaccessible to some of the interlocutors, who are not familiar enough with one of the discourses to realize that there are conflicting narratives or why they are in conflict. An *ECCNR* is a conflict where the misaligned narratives are accessible to the interlocutors, but not the underlying meta-rules that led to the conflict. In contrast, in an *ECC* and a *CCRU*, both conflicting narratives and the meta-rules that may have led to them are known to the interlocutors. The difference is that the interlocutors in a *CCRU* also position themselves "in relation to these conflicting narratives" (Nachlieli & Heyd-Metzuyanim, 2022, p. 355), that is, the interlocutors explicitly state which of the conflicting narratives they endorse.

4. Research questions

The aim of the paper, which was to study the differences between how PSTs define prototypical and non-prototypical three-dimensional solids, can be rewritten in the terms of the theoretical framework as the following three research questions:

(RQ1) What differences exist between PSTs' discourse when they define and select definitions in task situations involving prototypical solids versus those involving non-prototypical solids?

(RQ2) What commognitive conflicts emerge when defining or selecting definitions in task situations involving non-prototypical solids that do not appear in task situations involving prototypical solids?

(RQ3) What type of commognitive conflicts emerge as answer to RQ2?

In the last question, we distinguished between communicational clashes and discourse gaps (Knox & Kontorovich, 2023) and between (our adaptation of) the four levels of explicitness of a conflict (Nachlieli & Heyd-Metzuyanim, 2022).

5. Methods

5.1. Participants and context

The participants in this study were 33 students enrolled in a master in Secondary Education, which qualifies them to teach mathematics in secondary schools (12–18 years old). These pre-service secondary school teachers had a mathematics or science undergraduate degree: 12 were mathematicians, 12 engineers, 7 architects and 3 physicists (a PST was both mathematician and physicist). As part of that master, they were taking an 8-week course about teaching innovation and educational research in mathematics. Each week, they had a two-hour session on teaching innovation in mathematics and a two-hour session on educational research in mathematics. In the latter sessions, they learned about different theoretical frameworks, research methods in mathematics education, prominent studies in the field, etc.

In one of the course sessions on educational research, the lecturer presented the data collection instrument to the PSTs and asked them to complete it as a course activity. In the later sessions, the PSTs had to analyze their answers through the lens of a theoretical framework. At the end of the course, the PSTs had to submit a short report, which included the theoretical constructs used, the findings, and conclusions.

Some of the researchers of this study attended that data collection session and asked for the PSTs' consent for their data to be used in this study. Of the 46 PSTs enrolled in the course, 33 provided a written consent. Those 33 PSTs were organized into 9 working groups of 3 or 4 members, called here M1, M2,…, M9 ("M" for the master that they were studying), and the members of each group were referred to as S1, S2, S3 and S4, regardless of the group they belonged to.

5.2. Worksheet

The data collection instrument, inspired by one employed in previous studies (Fernández-León et al., 2021; González-Regaña et al., 2021), was designed to promote the discussion among the PSTs and to facilitate the identification of patterns in the PSTs' discursive activity when defining and selecting definitions of geometric solids.

The data collection instrument was a worksheet that included fifteen questions concerning three geometric solids: a cube; a quadrangular, oblique, and convex prism; and a quadrangular, an oblique, and a concave prism (see A, B and C in Fig. 1). These solids were provided to PSTs in two different ways: five groups were given the solids constructed in GeoGebra (Fig. 1), while four groups received a physical model of them.[6]

These three solids were chosen because they have some properties in common (e.g., all are quadrangular prisms) and other distinctive properties. For instance, the cube is a regular polyhedron (its faces are congruent regular polygons which are arranged the same way around each vertex), while the other two solids are irregular, and the solid C is concave while the solids A and B are convex. We recall that a polygon or solid is convex if, given two points in the polygon or solid, the segment that joins them is completely contained in the polygon or solid. A polygon or solid is concave if it is not convex. Moreover, in the worksheet, we wished to present both prototypical and non-prototypical solids. We hypothesized that the solid A would be prototypical for most PSTs because it was regular (and thus convex) and that the solid C would be a non-prototypical example of prism for most PSTs because it was concave. We also hypothesized that the solid B could be prototypical for some PSTs (because it was convex) and non-prototypical for others (because it was oblique).

The fifteen questions of the worksheet were structured in five sections. Each section had three questions that were almost identical, only differing in the solid to which they referred (see Table 1 for the five questions concerning solid C). The three questions in the first section (1.A, 1.B and 1.C) asked to construct a definition. The questions in the second section (2.A, 2.B and 2.C) asked for the evaluation of examples. This would be an *identification task* which required the PSTs to provide arguments for or against the solids being examples of a particular concept (Kontorovich, 2018). In the third section, Questions 3.A, 3.B and 3.C addressed the description of the solids. The fourth section (Questions 4.A, 4.B and 4.C) dealt with constructing a second definition for the solids and, in the fifth section (Questions 5.A, 5.B and 5.C), the PSTs were asked to select a definition to talk to a mathematician and a definition to teach in Secondary Education. In Questions 5.A, 5.B and 5.C, the PSTs had to select one definition among the five: the two definitions they had previously constructed and the additional three proposed in the worksheet. Those three additional definitions were selected by the researchers from the Internet with the purpose of presenting different types of definitions.

The participants of the study could perform different routines to answer these fifteen questions. In fact, particular task situations could be interpreted differently by the participants and, therefore, the object-level and meta-level tasks addressed by the participants in a concrete task situation could be different. In addition, the procedures implemented to accomplish each task could also be varied.

5.3. Data collection

During the data collection session, the worksheet and an audio-recorder were given to each of the nine groups. Furthermore, five of the groups were given a laptop with a GeoGebra file with the three solids (Fig. 1), while the other four groups received a physical model of each of the solids. Then, the PSTs were asked to discuss the proposed questions in order to reach an agreement, which they had to

[6] These two different ways were aimed at investigating PSTs' discourse when those two types of visual mediators coexist in mathematics classrooms. This will be the research focus of future works.

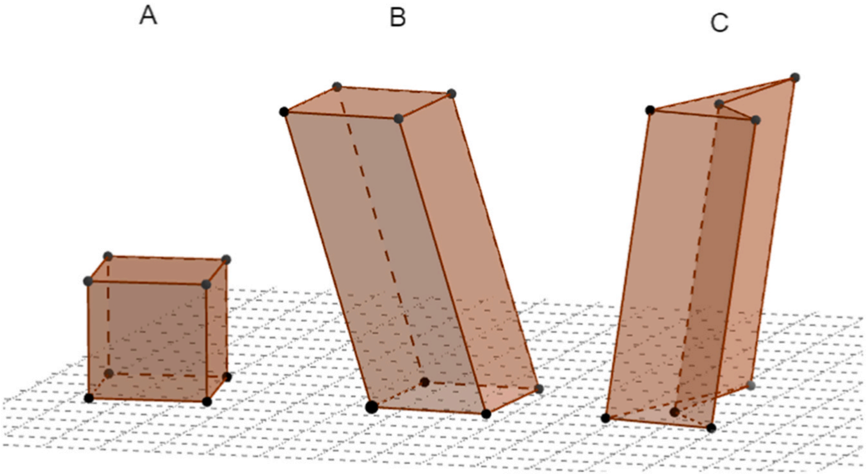

Fig. 1. The three geometric solids constructed in GeoGebra (A and B are convex prisms and C is a concave prism).

Table 1
The questions of the worksheet concerning solid C.

Label	Question
1.C	Give a definition of a geometric concept of which the solid C is an example.
2.C	Are A or B examples of the definition given in question 1. C? Justify your answer.
3.C	What properties or characteristics do you observe in C?
4.C	Could you give another definition of a concept of which C is also an example?
5.C	Considering the following definitions [C1, C2 and C3 below] and the two definitions you gave in questions 1. C and 4. C, would you use any of them to talk to a mathematician? If yes, which one? If not, please propose an alternative definition. And would you use any to teach this geometric content in a Secondary Education classroom? If yes, which one? If not, please propose an alternative definition. In both cases, justify the reason why you have chosen that definition and not the others. Definition C1: oblique prism whose base is a concave quadrilateral. Definition C2: concave polyhedron. Definition C3: solid limited by two concave, parallel and equal polygons that are called bases, and by as many parallelograms as there are sides to each base.

write down on the worksheet.

Then, the data corpus consisted of 9 worksheets with the written answers and the audio recordings (approximately 9 h, 1 h per group).

5.4. Analysis

We first transcribed all the recordings and analyzed the transcripts of each group to identify which parts referred to each of the fifteen questions of the worksheet. In each group, the parts of the transcripts referring to a particular question, together with the written answer to that question, formed an episode, which was labelled with the number and letter of that question. In each episode, we identified the four characteristics of the discourse. Categories of characteristics emerged based on similarities and differences identified between the examples of each characteristic and, specifically, the routines were inferred by determining how the PSTs had interpreted the task situations (both at object and meta level) and what procedures they had implemented. Since the PSTs did not usually explicate their tasks, we inferred them from an analysis of their discourses and actions. In some cases, the tasks became explicit when the PSTs were negotiating their answers to the worksheet with the other members of their group (for instance, "what we have to define now is […]?", group M7, line 7) and, in other cases, when a PST stated that he/she agreed or disagreed with another PST's interpretation of the task. We also compared the PSTs' tasks with the ones that we had expected them to perform. Since this analysis was interpretative, we did a triangulation of investigators to improve its validity (Cohen et al., 2007), i.e., the analysis was first done individually and then discussed between all the researchers until a consensus was reached. The analysis was done using the transcripts and written data, which were in Spanish, and we only translated into English the excerpts that appear in this paper.

Next, we illustrate part of our analysis process by showing part of the episode 1. A of group M7 (keywords are in bold).

1. S1: Question 1. A: Give a definition of a geometric concept of which the solid A is an example.
2. S2: I, the example…, that is, [the] geometric concept that I see the clearest for A is **regular polyhedron,** because since it's a **cube**, it's like the most basic example that occurs to one for a **regular polyhedron**.
3. S1: It has 6 **equal faces**…

4. S3: 6 **equal edges**…
5. S2: No, it has more **edges**…
6. S2: Exactly.
7. S4: What we have to **define** now is the **regular polyhedron**, isn't it?
8. S1: Yes.
9. S2: Yes, a **regular polyhedron** is the one that has **all** its **faces** and **all** its **angles equal**.

From this excerpt and some others of similar nature, the researchers inferred that all the PSTs of group M7 interpreted the task situation as a situation in which they had to formulate a definition of a mathematical object of which solid A was an example. The analysis of their actions permitted us to infer that the PSTs employed a three-step procedure, different from the two-step procedure that the researchers, who were mathematicians, had expected. The steps from the expected procedure were numbered as (1) and (2) in the theoretical framework and the three steps from the PSTs' procedure are called here (1), (2 *) and (3 *) to reflect that the first step was the same, while the other two were different. The three steps of the PSTs' procedure were (1) searching for a signifier of a mathematical object of which the solid A is an example, (2 *) generating a list of mathematical properties of the solid A or of the mathematical object, (3 *) producing a narrative that included both the signifier of the first step and the mathematical properties of the second step.

To answer RQ1, we then compared in each group the discursive characteristics of the PSTs' discourse in the three episodes of each of the five sections of the worksheet. For example, if there were differences (either in procedures or tasks) between the PSTs' routines performed in the three episodes of a section, we analyzed whether those differences appeared between task situations involving prototypical solids and those involving non-prototypical solids.

To answer RQ2, we studied what commognitive conflicts arose between the PSTs' different discourses when defining or selecting non-prototypical solids that did not appear when defining or selecting prototypical solids. We also studied whether those commognitive conflicts arose between the PSTs' discourses and the mathematical discourse of the worksheet, i.e., the mathematical definitions that appeared in Questions 5.A, 5.B and 5.C. We did this by searching for differences in the PSTs' discursive characteristics.

Finally, to answer RQ3, the inferred commognitive conflicts were classified. In a first step, we distinguished between communicational clashes and discourse gaps (Knox & Kontorovich, 2023) depending on whether the PSTs were aware of the interpersonal disagreement (communicational clashes) or whether there were differences in the PSTs' keywords, visual mediators, narratives or routines (discourse gaps). In a second step, we analyzed the conflicts to identify the levels of explicitness of the conflicts, expanding to mathematical discourse the levels that appeared in Nachlieli and Heyd-Metzuyanim (2022). Specifically, we first distinguished if the conflicting narratives were or not accessible to the PSTs, that is, we studied whether there were any PSTs who did not realize that the conflict existed. If there were such PSTs, the conflict was categorized as *CCI*. If there were not and the conflicting narratives were made explicit, we analyzed whether the PSTs had access to the meta-rules that had led to the conflict. If they did not have access, the conflict was categorized as *ECCNR*. If they had access, we further analyzed whether the PSTs positioned themselves in relation to the conflicting narratives when discussing and reflecting on the meta-rules. If they did, the conflict was categorized as *CCRU*. If they did not, the conflict was said to be an *ECC*.

6. Findings

The findings show that, in all the groups, there existed clear differences between the PSTs' discourse in task situations involving the non-prototypical solid C and their discourse in task situations involving solids A and B. Moreover, in two of the nine groups (M7 and M8), some commognitive conflicts appeared in task situations involving solid C, but not in similar task situations involving solids A and B. These commognitive conflicts were also classified according to Knox and Kontorovich's (2023) classification and according to (our adaptation of) the classification in Nachlieli and Heyd-Metzuyanim (2022). We present episodes from groups M7 and M8 to illustrate these findings, since those two groups were the only ones where commognitive conflicts related to non-prototypical solids appeared.

6.1. Differences between defining routines in task situations involving prototypical solids and in task situations involving non-prototypical solids

In this subsection, we present some findings that emerged from the data of group M7. In particular, as answer to RQ1, we will show some differences between the routines that the PSTs employed in task situations with prototypical solids and those they employed in similar task situations with a non-prototypical solid (C). Moreover, as answer to RQ2, we will present a commognitive conflict that arose in a task situation involving a non-prototypical solid and, as answer to RQ3, we will classify it.

When the four PSTs of group M7 were trying to answer Question 1.C of the worksheet, which asked them to define a concept of which the non-prototypical solid C was an example, they followed a routine for defining that was different from the one employed in Questions 1.A and 1.B. In order to contextualize the change of routine of these PSTs, we first need to present the part of the precedent-search-space generated by the two preceding questions (Questions 1.A and 1.B). We begin by describing the routines that the PSTs performed in these two questions, which asked them to define concepts of which the solids A and B were examples, respectively. In those questions, all the PSTs of the group interpreted that they had to formulate definitions of mathematical objects of which the solids A and B were realizations, hence they seemed to interpret the task situations at the object-level as the researchers had expected.

All four PSTs also seemed to agree to perform the same meta-level task in both Questions 1.A and 1.B, which was different from the one expected by the researchers. As mentioned in the theoretical framework, the researchers expected, as mathematicians, the application of a two-step procedure in the three questions of the first section: (1) searching for a signifier of a mathematical object of which the solids A, B or C are realizations; (2) producing a mathematical definition, i.e., a narrative that includes a new signifier and some mathematical properties. That definition should meet as many requirements of a mathematical definition as possible (Tabach & Nachlieli, 2015), which means that it should be precise, include necessary and sufficient properties, be minimal (i.e., it should not include properties that can be deduced from other given properties), be hierarchical (i.e., it should be based on previously defined concepts and be non-circular), etc.

Interpreting PSTs' tasks was not simple in Questions 1.A and 1.B, since they did not fully justify their actions, and therefore some aspects of their meta-level task could remain hidden. However, a careful analysis of the data allowed us to interpret that the PSTs of M7 had performed a three-step procedure while answering Questions 1.A and 1.B. This procedure had the same step (1) as the procedure the researchers expected. On the contrary, their second step, called here (2 *), involved giving a list of mathematical properties of the solid (A or B), and in their third step, called here (3 *), they seemed to produce a narrative that included both the signifier considered in step (1) and the mathematical properties given in the second step (2 *). For instance, when answering Question 1.A ("Give a definition of a geometric concept of which the solid A is an example"), one of the signifiers they proposed in step (1) was "regular hexahedron", while the definition they gave as the product of their third step (3 *) was "regular hexahedron, of which all the faces, edges and angles are equal". It is worth highlighting that this group of PSTs regarded the word given as signifier in step (1) as part of the narrative they gave as definition.

In Question 1.C, the fact that the solid C was non-prototypical for the PSTs (for example, for its non-convexity) seemed to prompt the PSTs of group M7 to interpret the task situation at the meta level in different ways, which led some of them to execute different routines to answer the question. Specifically, their meta-level tasks and the procedures they used turned out to be different.

Data reveal that S1 appeared uneasy with the property of non-convexity of a solid and avoided referring to the convexity and concavity of the solids throughout the session. For example, during a discussion the group had about the use of the terms "convex" and "non-convex" in geometry, S1 did not participate. Furthermore, when S2 later defined the base of C as "a non-convex polygon of 4 sides", S1 immediately stated "which has the shape of a boomerang, to define it somehow." S1 seemed unwilling to accept the inclusion of the word *non-convex* in any definition, whereas S2 was the student that explicated to the rest of the group how to use the words convexity and non-convexity in geometry.

In this episode, the object-level difference between the discourses of S1 and S2 permitted us to observe a difference at meta level in their discourses. This latter difference was based on the different uses S1 and S2 made of the word *definition*, which were interpreted from the meta-rules that governed their actions when answering Question 1.C. On the one hand, S2 performed again the meta-level task of applying the three-step procedure they had used in Questions 1.A and 1.B. In particular, S2 seemed to regard a definition as a narrative that included mathematical properties of the mathematical object to define, without specifying during the discussion whether the signifier given in step (1) was part of the sought definition. On the other hand, S1′s performance of her meta-level task entailed the same step (1) of searching a signifier (as in Questions 1.A and 1.B) and a different second step, here called step (2 **), which consisted of constructing a narrative that includes information that allows readers of the definition to visualize the solid. To accomplish her task at object and meta level, S1 adopted the procedure of generating a 3D geometric solid from a 2D polygon that "replicates" through an axis (i.e., the solid would be generated by the infinite polygons obtained when translating the original polygon in a given direction). S1 regarded her definition as a narrative whose essential requirement was allowing readers to visualize the solid C. We will now present evidence that backs all these previous assertions.

When S2 and S3 suggested that they would include the non-convexity in the requested definition in Question 1.C, S1 asked (in line 86 of the following excerpt) how they could define this polyhedron more precisely. This informed us that S1 needed to add something more to the definition to accomplish the task.

86: S1: Ok, and how could we define more this polyhedron in particular?

87: S2: With the non-convexity, that is regular [solid A], this is parallelepiped [solid B], non-convex [solid C].

88: S1: But if we think of a person who is not seeing it [solid C], how could we define it so that she can visualize the figure? […].

In line 87, student S2 insisted on including the property of being non-convex in the definition, while highlighting that this property played the same role as the properties of being regular or of being a parallelepiped which had been considered when answering Questions 1.A and 1.B, respectively. We have inferred that S1 had been familiar with all the terms proposed to construct the definitions of solids A and B when answering Questions 1.A and 1.B because she had previously endorsed the proposed narratives as definitions without protest. However, in Question 1.C, the proposed mathematical property of non-convexity did not seem to please her. For this reason, S1 made reference to the need of constructing a definition that allowed a person to visualize the solid C when that person was not seeing it ([88]), which showed part of her meta-level task. For S1, the expression "non-convex irregular polyhedron", which had been previously suggested by S2 and S3, was not a possible answer for Question 1.C, since, in her view, it did not sufficient to visualize the solid C. In line 93 of the transcript, S2 upheld that, under S1′s assumption, the parallelepiped B should have also been defined in that way, and it had not been.

93: S2: The parallelepiped B because there are many classes of parallelepipeds. It could have a smaller height, a bigger one… it would be as the parallelepiped B, that we have not defined it so that people can visualize it either, but it is similar to parallelepiped B, as if you took one of the long edges, and caved it in. Then, if you take parallelepiped B and take the edge and cave it in in such a way that it becomes non-convex, it becomes very similar to [the solid] C because C is also parallelepiped. Ok, no, it's not a parallelepiped. No, no, no, no, no, it's not a parallelepiped. I would erase this commentary if I could, it is nonsense what I have said.

This line shows that S2 questioned why the definition they had given in Question 1.B had been endorsed and the one that S2 and S3 had proposed in Question 1.C had not been, despite having applied the same three-step procedure (1), (2 *) and (3 *). In S2′s opinion, the need for visualization had not been taken into account in Question 1.B and, therefore, the definition given in that case was not a definition that allowed the reader to visualize solid B. It seems that S2 did not realize that S1′s main problem was not the employed procedure, but a difference at object level in their discourses which had not surfaced in Question 1.B. This difference at object level made S1′s meta-level task surface, while it had been hidden in the two previous questions. S2 appeared not to realize that S1 had also needed to visualize the solids in Questions 1.A and 1.B, although, in those cases, it looked like the definitions constructed through the three-step procedure employed then had sufficed to allow S1 to visualize the solids. This could be because only well-known words had been used until then (for instance, there had been no mention to the convexity or non-convexity of the solids). In line 93, S2 also asserted that the solid C was similar to the parallelepiped B, although with the difference that one of the long edges was "caved in", which justified the non-convexity of C. S2 assumed that the similarity between the solids B and C implied and, thus, justified the use of the same procedure to accomplish the tasks. We note here that, while S2 was justifying the similarity between the solids B and C, he was unconsciously starting to describe the construction of the solid C as a deformation of the solid B, thus showing a new procedure to answer Question 1.C.

In line 105 (see below), S1 suggested a constructive procedure which consisted of executing the common step (1) and her step (2 **) in order to perform her meta-level task. In lines 106 and 108, S2 tried to defend the use of the constructive procedure he had circumstantially introduced in line 93 by stating that this new procedure, which deformed the solid B to turn it into C, provided a definition that would allow someone who does not see the solid to visualize it. In line 107, S4 expressed, while laughing, her disagreement with S2′s proposal. Later, S2 stated that his proposal was actually an attempt to please S1, which was not simple.

105: S1: I think that it is, if we took this polygon [referring to the base of the prism C], let's go to 2D, if we take this polygon and we reproduce it in one of its axes, imagine that we have it like this….
106: S2: […]. That is, these are 90 degrees. Then, it is as if we took polygon B [referring to the base of solid B], and apart from caving it in, we did one of its angles more acute because they are not 90 degrees, but….
107: S4: [laughing] It is as if we say, we take I don't know what and we squash it, then it turns out to be something else entirely.
108: S2: Well ok, but she has said that how someone that is not seeing it would visualize it…so tell me….
109: S1: But I don't think a person visualizes it that way either. I would define the polygon and this polygon spreads across one of its axes.

Despite the effort S2 had made to convince the others, S1 refused to accept S2′s proposal in line 109, stating that his narrative would not allow someone to visualize the solid. In line 109, she described again her constructive proposal of definition (a *procedural definition*, according to Zaslavsky & Shir, 2005).

Later, S2 agreed with S1 that her proposal was better to visualize the solid and, consequently, S2 and the rest of the group endorsed S1′s narratives given in lines 105 and 109 to construct the definition. Finally, the whole group delved into S1′s proposal, which was based on the use of a polygon and its "replication" from one of "its edges", giving more details on the polygon that would be reproduced and on the way that such a reproduction would be done.

The previous excerpt shows how S2 had shelved his initial three-step procedure and adopted a new one in order to meet the requirements of S1′s meta-level task. This shows the flexible character of S2′s routine when answering Question 1.C, which allowed us to infer a new characteristic of his meta-level task: the flexibility. Although it seemed that S2′ meta-level task was *closed*, in the sense that it was clearly determined by the three-step procedure described above, the flexibility of S2′s routine when defining allowed us to infer that S2′s meta-level task had some other characteristics that had not surfaced before. Specifically, we finally interpreted that S2′s meta-level task was not determined by a unique procedure, but it allowed different procedures (three were observed in this episode) to accomplish the task. Moreover, we inferred that visualization was not a requirement in his meta-level task ([93]), as it actually was for S1. We regard the flexibility of his meta-level task as a sign of a more literate mathematical discourse.

In summary, this episode has permitted us to answer RQ1 by showing differences between the routines that PSTs performed when defining in Question 1.C (which involved a non-prototypical solid) and the routines employed when defining in Questions 1.A and 1.B. In addition, it has allowed us to observe how a task situation may be tackled by different procedures. This episode also shows how difficult the interpretation of performers' meta-level tasks may be, since many aspects of these meta-level tasks may remain hidden if the situation does not demand them. Furthermore, another observed difference in this episode was that, in task situations involving the non-prototypical solid C, a *procedural definition* appeared (like the one proposed by S1), while only *structural definitions* appeared in task situations involving solids A and B (these definitions were the result of the three-step procedure presented at the beginning of this section).

In this episode, as answer to RQ2, we found a commognitive conflict which arose from the different uses S1 and S2 made of the word *definition*. These different uses were observed from the different and misaligned meta-rules that governed their actions when answering Question 1.C, which surfaced when constructing a definition for a non-prototypical solid. While for S1 a definition had to meet the requirement of allowing visualization, this requirement did not exist for S2. Thus, an object-level narrative that was regarded as a definition for S2 (for instance, the expression "non-convex irregular polyhedron") was not regarded as such for S1. Indeed, S2′s argument in line 93, which was an attempt to start a meta-level conversation about the misaligned meta-rules, was futile because the rules governing S2′s and S1′s mathematical discourses were not the same.

We hypothesize that this commognitive conflict was not resolved. The object-level definition constructed from the rules that governed S1′s discourse was endorsed by S2 because of his flexibility in accomplishing the task. S2 regarded as valid the narrative given as definition by S1 when performing her meta-level task because, although visualization was not a requirement for him, it was not a problem either. Indeed, S2 questioned S1′s new requirement of visualization in line 93, but he did not keep discussing this issue, since he seemed to notice that S1 was not *dialogically engaged* in the conversation (Sfard, 2020). This was seen in the conversation when S2 realized that S1 did not want to be a participant of S2′s discourse. For this reason, there was no meta-level discussion about the misaligned meta-rules and, therefore, S1 did not take advantage of this opportunity to learn.

To answer RQ3, the commognitive conflict that appeared in this group was classified. Firstly, we found that the conflict was a *communicational clash* (Knox & Kontorovich, 2023) because (at least some) of the interlocutors were aware of the disagreement, although not of the reason for it. Indeed, the conflict involved conflicting narratives on the requirements of a *good definition*, some of which were aligned with requiring visualization for a definition while others showed flexibility when defining. Those conflicting narratives were accessible to the participants in the discourse (see, for instance, lines 86, 88 and 109 in S1′s discourse and lines 87 and 93 in S2′s discourse). Secondly, this commognitive conflict could also be classified as an *explicit commognitive conflict not reflected upon* (Nachlieli & Heyd-Metzuyanim, 2022), since the PSTs were aware of the commognitive conflict that appeared because of the different uses of the word *definition*, but did not reflect on how to resolve it.

6.2. A commognitive conflict was revealed thanks to the presence of a non-prototypical solid

In this subsection, we would like to show some findings that emerged from the data of group M8. The first one is that the PSTs of this group also had conversations about the signifiers *convex* and *concave*, but only in task situations concerning the non-prototypical solid C. Indeed, the PSTs mentioned that the solid C was convex in Questions 1.C, 2.C, 3.C, 4.C and 5.C (mistakenly, since C is concave) and discussed the definitions of concave and concave in Questions 1.C and 5.C, but those two words were not mentioned at all when discussing Questions 1.A, 1.B, 2.A, 2.B, 3.A, 3.B, 4.A, 4.B, 5.A or 5.B. This shows a clear difference between the keywords and routines employed in task situations with prototypical solids and the keywords and routines employed in task situations with a non-prototypical solid (C). These are some of the differences that were found in answer to RQ1.

The other findings presented in this subsection emerged when studying why the PSTs of M8 wrongly decided that the solid C was convex. After analyzing all the data and the transcripts, we inferred that it was not a simple matter of the PSTs recalling the wrong definition, but that there was an underlying commognitive conflict that was revealed thanks to the presence of the non-prototypical solid C (thus answering RQ2).

In the following, to back the previous assertions, we will show some of the conversations that the PSTs had and we will present and classify the commognitive conflict that was inferred as answer to RQ3.

The first conversation about the keywords concave and convex appeared when the PSTs were tackling the task situation posed in Question 1.C (there was no mention of them in Questions 1.A and 1.B):

54:S2: Well, it's…, it's [solid C is] convex.
55:S1: Convex?
56:S2: Yes, because it has a peak inside.
57:S3: If you join the vertices, that comes out.
58:S1: Yeah, yeah, but I don't know.
59:S2: Sure, I don't know. I don't know why, like here [in the group], the concave and the convex, each person calls it in a different way.
[…].
63:S4: That is how we were taught in secondary school.
64:S1: Convex, with kiss [convex and with kiss are translated in Spanish as convexo and con beso, which are pronounced similarly].
65:S3: Seriously?
66:S2: I learned concave because concave is a hole we dig [in Spanish, concave and I dig are translated as cóncavo and cavo]. Then, it's a hole.
67:S3: I never remembered, I had to look at it up.
68:S4: I had S1′s trick, where the hand is and you kiss it, that's convex.
69:S2: And like this? Muach. Can't you kiss the palm of your hand?
70:S1: No, because usually it is like this.
71:S3: Sure, but that's how it'll stay engraved for you….
72:S2: Okay, okay. Okay then, I think that, that, that…, similar to the previous one, what happens is that it's [solid C is] convex. Because it has an irregularity….

In the previous excerpt, S2 asserted that the solid C was convex ([54]) and both S2 and S3 justified this assertion by using narratives in the discourse of solids: "because it has a peak inside" ([56]) and "if you join the vertices, that comes out" ([57]). However, S1 was hesitant about the definition of that word ([55], [58]), which made S2 confess that he did not really know if it was convex because "here, the concave and the convex, each person calls it in a different way" ([59]). The rest of the conversation shows how the PSTs tried to recall the definitions of the words convex and concave, during which S3 confessed that she never remembered the definitions and had to look them up ([67]), but later endorsed her colleagues' narrative ([71]).

Between Questions 1.C and 5.C, the PSTs of this group incorrectly used the word convex several times in Questions 2.C, 3.C and 4.C, but did not mention it in Questions 2.A, 2.B, 3.A, 3.B, 4.A, 4.B, 5.A or 5.B. In Question 5.C, in which they had to decide which definitions they would choose for the solid C and justify their choices, the words convex and concave appeared again. This shows some differences between the PSTs' keywords and routines when defining and selecting definitions in task situations involving the non-prototypical solid C and those employed in other task situations. This finding complements the answer to RQ1 given in the previous subsection.

To answer the other two research questions, we first need to present an interesting conversation that the PSTs had when answering Question 5.C. In that question, researchers considered that the object-level task was to choose (and justify the choice of) definitions, while the meta-level task was to decide which criteria to employ when making those choices and justifying them. The PSTs of M8 began the task by reading the three definitions proposed in the worksheet in Question 5.C, where they saw that the word concave appeared. This made these PSTs realize that their previous definition of concavity of polygons and solids (discussed in Question 1.C and used in Questions 2.C, 3.C and 4.C) did not coincide with the one used in the definitions of the worksheet and prompted the following conversation.

483:S3: Concave. Isn't it [solid C] convex?
484:S4: That [definition C3 of Table 1] is very confusing, isn't it?
485:S1: But not this one, but this one isn't….
486:S3: Sure, then we have stated that it's [solid C is] convex, and it's concave.
487:S1: Sure, depending on from where you look at it [solid C], right? If you look at it from here [from this perspective].
488:S4: And if you look at it from here [from this other perspective].
[…].
492:S3: Man, it can't be depending on from where you look at it, depending on how you look at it.
493:S1: If you look at it [solid C] this way, yes, it's concave.
494:S3: But you have to look at the base [of solid C], you know? What you're saying is concave or convex is the base.
495:S2: Sure.
496:S1: But you don't look….
497:S2: Sure, because the faces are always going to be rectangles.
498:S3: So this is concave, right? Because if it [the worksheet] says it there, it can't be wrong.
499: S2: Sure, concave is that it's like a hole.
500: S3: But we have made it [solid C] convex.
501: S4: It's true that if you look at it [solid C] from here [from this perspective] … it's concave, but if you look at it from here [from this other perspective] it's convex.
502:S3: Yes, but it can't be depending on from where you look at it [solid C].
503:S2: Let's see, I also tell you that that criterion… everyone has whichever they feel like.
504: S1: Yes, that's it.
505:S3: Sure, everyone has a different criterion.
506: S2: In fact, in the entrance exam to university, the teachers, my teacher even says that to calculate the…, the concave and convex of the second derivative, he says, you write it like this [he makes a gesture with his hand in the shape of a crater] or the face or the sad face and you get out of trouble.
507:S3: Okay.
508:S2: But, come on, if it says there [in the worksheet] concave, it'll be because he has understood that it's [solid C is] concave according to the criterion and the… the definition isn't going to be incorrect.

The conversation of this second excerpt began when the PSTs realized they had previously said that the solid C was convex while the worksheet said that it was concave ([483]-[486]). This prompted S1 and S4 to assert that the definitions of the terms convex and concave depended on the place from where an observer is looking at the figure, a narrative that they repeated more than once (S1 in lines 487, 493; S4 in 488, 501) but is not true in geometry.

To understand those last narratives and other similar ones, we considered as precedents in the PSTs' precedent-search-space all the situations in secondary school where they had studied the graphs of functions. This consideration was made because concavity and convexity of functions is a topic that is studied at secondary school in Spain (and was explicitly mentioned by the PSTs more than once), while the concavity and convexity of solids is not included in the Spanish secondary school curriculum. In secondary school, there is no global agreement about which functions are convex (as S2 stated in line 59 of the first excerpt) and, to avoid problems, students taking the university entrance exams are told that they should clearly state which functions they consider convex and which concave (as S2 said in line 506 of the second excerpt). However, the PSTs of M8 did not seem to be aware that there is only one definition of concavity and convexity in geometry and, thus, that the solid C must always be considered concave.

Bearing in mind that precedent-search-space, we inferred that S1 and S4 seemed to use a meta-rule in the discourse on functions to substantiate an assertion in the discourse on solids: *a mathematical object can be convex or concave depending on which definition we choose*. S3 rejected S1 and S4′s criterion ([492], [502]) and proposed the narrative of line 494, where she stated that a solid is convex or concave if the base is (this is true for prisms, which solid C is, but not in general, a clarification which she omitted). S2 endorsed S3′s narrative ([495]), adding in line 497 that it is true "because the faces are always going to be rectangles" (this narrative, offered to substantiate S3′s, is false because the faces of a prism are always parallelograms, but not always rectangles). S3 added that, although they had previously decided solid C was convex, it had to be concave because all the definitions of it in the worksheet had the word concave in them ([486, 498, 500]). S3 seemed driven by a discursive rule commonly endorsed in the classroom: *the definitions of the* *worksheet must be correct*. Despite this, S4 repeated in line 501 her previous argument (convexity depends on the perspective), which S3 again rejected ([502]). After revisiting their conversation of the first excerpt regarding the precedent of secondary school and the lack of agreement on what a convex function is, S2 concluded the discussion in line 508 by endorsing S3′s narrative in line 498: the definitions of the worksheet cannot be incorrect.

In all these conversations, PSTs did not seem to notice that they sometimes referred to the convexity of functions and sometimes to the convexity of solids, as if the word convex was used the same way in both discourses. They often wished to use the rules of the discourse on functions (the discourse that the PSTs were familiar with from their secondary school precedents) when dealing with the discourse on solids. It seemed that they were recurring to the incorrect *precedent pocket* (Kontorovich, 2021). Since convex and concave functions and convex and concave solids are defined using different meta-rules, those two discourses are incommensurable, which seemed to indicate the existence a commognitive conflict. This conflict was revealed thanks to the presence of the non-prototypical solid C, since the discussions about concavity and convexity only appeared in the questions concerning concave solid C and no mention to those properties was made when discussing convex solids A and B. This completes our answer to RQ2.

To answer RQ3, the commognitive conflict of this episode was classified. Taking into consideration the classification proposed in Knox and Kontorovich (2023), this conflict is a *discursive gap* because the PSTs did not seem to be aware of the conflict. This probably happened because the conflicting narratives in the discourse on figures were not explicated in their conversation. Moreover, this conflict is at the level of *commognitive conflicts inaccessible to one of the parties* (Nachlieli & Heyd-Metzuyanim, 2022), since the PSTs appeared unaware of the fact that the notion of convexity in the discourse of functions is different from the notion of convexity in the discourse of solids.

Lastly, we would like to mention that the PSTs accepted that the solid C was concave consistently with the worksheet definitions, but remained convinced that it is possible to define a convex solid in two different ways, as it happens when defining a convex function in the discourse on functions. In fact, their discussion only stopped when they were influenced by a endorsed classroom rule. In summary, they remained unaware of the conflict and there was no meta-level learning.

7. Discussion and conclusions

In this paper, the commognitive framework was employed to answer three research questions (RQ1, RQ2 and RQ3) related to the differences that may exist between PSTs' discourse when defining and selecting definitions in task situations involving prototypical solids and their discourse in similar task situations involving non-prototypical solids.

Our answer to RQ1, *What differences exist between PSTs' discourse when they define and select definitions in task situations involving prototypical solids* versus *those involving non-prototypical solids?,* was that, in all groups, there existed differences between how they defined and selected definitions for the non-prototypical solid C and for the rest of the solids. We illustrated this fact by showing two episodes in which there were some differences in the keywords and routines that the PSTs employed in task situations involving the defining of prototypical and non-prototypical solids. For instance, in the first episode, the concavity of the solid C was responsible for a difference at meta level surfacing in the discourses of the members of group M7. This supports the idea that the effect of prototypical examples is still very present in the mathematical practice of defining (Clements & Sarama, 2011; Fujita, 2012; Fujita & Jones, 2007), which has consequences in PSTs' training and in their future practice as mathematics teachers. Therefore, in Professional Development programs, there should be more didactical situations in which both prototypical and non-prototypical examples appear (Fischbein & Nachlieli, 1998) so that PSTs' and their future students' conceptualization of a mathematical object includes a more complete variety of realizations.

Additionally, the findings of this paper showed that PSTs had problems when defining or selecting definitions in task situations involving non-convex 3D figures, as Bernabeu and Llinares (2017) had also discovered that primary school students had in task situations with non-convex quadrilaterals. These problems could mean that PSTs see convex 3D figures as prototypical examples and non-convex 3D figures as non-prototypical ones, as Fischbein and Nachlieli (1998) also found happened with students and quadrilaterals.

The second research question of our study (RQ2) was *What commognitive conflicts emerge when defining or selecting definitions in task situations involving non-prototypical solids that do not appear in task situations involving prototypical solids?* We answered RQ2 by showing two commognitive conflicts that had appeared when the PSTs were answering questions related to the solid C, but that were not apparent when the same PSTs were dealing with the other solids. Specifically, in the first episode (concerning group M7), the concavity of the solid C revealed the existence of a commognitive conflict. In the second episode, we showed how a commognitive conflict was revealed when the members of group M8 engaged with conflicting narratives about convexity and concavity coming from two incommensurable discourses. Therefore, the two episodes presented in this study also served to demonstrate connections between the presence of non-prototypical solids and the appearance of commognitive conflicts in the PSTs' discourse, which suggests that these types of examples remain an obstacle for teachers decades after Hershkowitz's (1990) study. Lastly, we claim that these commognitive conflicts could have remained unrevealed without the presence of non-prototypical solids in the worksheet, since we were able to detect incommensurable discourses when the PTS were discussing task situations with non-prototypical solids but not in similar task situations with other solids.

To answer RQ3, *What type of commognitive conflicts emerge as answer to RQ2?*, we first distinguished between communicational clashes and discourse gaps (Knox & Kontorovich, 2023). This allowed us to differentiate between two types of conflicts in a geometric context of constructing and selecting mathematical definitions for solids. Thus, we highlight the potential of Knox and Kontorovich's (2023) study for its applicability to other mathematical contexts and educational levels. To complete our answer to the third question, we also classified those conflicts by using (our adaptation of) the four levels of explicitness of a conflict (Nachlieli & Heyd-Metzuyanim, 2022). Specifically, two of the four levels of explicitness were identified in the two conflicts shown in this paper: *explicit commognitive conflict not reflected upon (ECCNR)* and *commognitive conflict inaccessible to one of the parties (CCI)*. These findings have allowed us to confirm our initial hypothesis, which was that it was possible to extrapolate, from pedagogical discourse to mathematical discourse, the levels of explicitness of commognitive conflicts presented in Nachlieli and Heyd-Metzuyanim (2022). We think that this extrapolation was possible because the levels of explicitness of conflicts do not depend on the concrete content (mathematical or pedagogical) of the discourses.

Moreover, both classifications of commognitive conflicts could give valuable information for the characterization of the process of learning mathematics and, therefore, for the improvement of mathematics teaching. Indeed, the classification of a commognitive conflict as a conflict of a particular type (Knox & Kontorovich, 2023; Nachlieli & Heyd-Metzuyanim, 2022) and the subsequent reflection on the learning or the opportunity to learn that the conflict gave rise to could give us valuable information for teaching PSTs. For instance, knowing that the incommensurable discourses on functions and solids may coexist when PSTs define and that PSTs may be unaware of the incommensurability would permit redesigning teacher training programs that take into account these types of possible commognitive conflicts.

In future works, it would be interesting to study whether knowing the different types of conflicts that appear in the mathematical discourse could be useful, as a teaching tool, to determine how to foster mathematical learning among the students in a class or whether some level of conflict is more advantageous for learning, which is still an open question (Nachlieli & Heyd-Metzuyanim, 2022; Sfard, 2023). Therefore, we believe that the relation between the nature of commognitive conflicts and learning should be further explored (e.g., Nachlieli & Heyd-Metzuyanim, 2022; Sfard, 2023) because the type of commognitive conflict could indicate in which moment of the learning process the PSTs are. Lastly, another future line of research would be classifying the commognitive conflicts found in our data which do not seem to be related to the appearance of non-prototypical examples.

In summary, the commognitive framework has allowed us to detect and characterize in detail the differences between how PSTs define and select definitions in task situations with and without non-prototypical solids, which has implications for teacher training.

Lastly, a limitation of this work is that some aspects of the PSTs' meta-level tasks may have remained hidden, which could be helped in future studies by conducting interviews with the PSTs in order to obtain more data. Those interviews could also look for the causes of some PSTs not being dialogically engaged during some of the discussions. We also acknowledge that focusing on only some pre-service teachers constitutes another limitation of this work. In the future, we could study more pre-service teachers (of Secondary Education and of other educational levels) or focus on a different mathematical content (algebra, calculus, etc.) to obtain a complete picture of how non-prototypical figures influence the way PSTs define and select definitions in mathematics.

CRediT authorship contribution statement

Rocío Toscano: Conceptualization, Funding acquisition, Methodology, Writing – original draft, Writing – review & editing. **Alfonso J. González-Regaña:** Conceptualization, Funding acquisition, Methodology, Writing – original draft, Writing – review & editing. **Verónica Martín-Molina:** Conceptualization, Funding acquisition, Methodology, Writing – original draft, Writing – review & editing. **Aurora Fernández-León:** Conceptualization, Funding acquisition, Methodology, Writing – original draft, Writing – review & editing. **José María Gavilán-Izquierdo:** Conceptualization, Funding acquisition, Methodology, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

None.

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