Researching how professional mathematicians construct new mathematical definitions: A case study

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Abstract

In this work, we study the mathematical practice of defining by mathematics researchers. Since research is an important part of many professional mathematicians, understanding how they do research is a necessary step before thinking about future researchers' undergraduate and postgraduate education. We focus on the defining process associated with the generalization of existing definitions as a way of constructing new ones. Data of this qualitative study come from a case study whose subject is a mathematics researcher in the area of differential geometry. We have interviewed this researcher and collected her research documents. From our analysis of the data, we have identified four phases in the defining process (Finding an opportunity to generalize an existing concept, Proposing a new definition, Justifying that the new definition is valid and Continuing the chain of definitions), which we will describe in detail in the Results section.

Keywords: defining; generalizing; geometry; mathematicians

Introduction

In recent years, research in mathematics education at university level has gained more and more relevance. For example, some authors have studied the 'advanced mathematical thinking', like Tall, who characterized it 'by two important components:

precise mathematical definitions […] and logical deductions of theorems based upon them' [1,p.495].

Other authors prefer the term 'advancing mathematical activity', like Rasmussen, Zandieh, King & Teppo [2]. The word 'advancing' is chosen in order to 'emphasize the progression and evolution of students' reasoning in relation to their previous activity' [2,p. 51] and the word 'activity' reflects their 'characterization of progression in mathematical thinking as acts of participation in a variety of different socially or culturally situated mathematical practices' [2,p. 52]. Among the advancing mathematical activities, the authors include mathematical practices like defining, symbolizing, algorithmatizing, etc. Several authors have analyzed how mathematicians construct proofs, how they review them and how they present them to students in advanced courses, see [3-4]. Furthermore, Alcock [5] has inquired about the mathematicians' perspectives on how students of an introductory course are being taught and how they learn about proofs.

There has also been interest in how mathematicians do research at university. For example, Burton highlighted the importance of 'research practices of mathematicians' [6,p.1] and she proposed 'a model for how mathematicians come to know' [6,p.11]. She also noted that this type of study has relevance for teaching and learning mathematics at all levels. Other authors, like Mejia-Ramos and Weber [7] and Weber and Mejia-Ramos [8] focused on mathematicians' practices, like how and why mathematicians read proofs during their research in mathematics. All of these authors have considered mathematicians as subjects of study.

Since we are interested in future mathematicians' education and research being an important part of the professional life of many of them, understanding how mathematicians do research becomes our focus of interest. According to Weber, Inglis and Mejia-Ramos 'if the goal of instruction is to have students engage in experts' practices, then it is necessary to have an accurate understanding of what those practices are' [9,p.38].

The study of mathematical processes of construction of knowledge is important in itself because mathematics has a dynamical nature and there exists a duality of process/product in the processes of construction of mathematical content: defining/definition, proving/proof, etc. (see Lakatos [10]). Therefore, the focal point is not only the set of assertions that can be done (axioms, propositions, theorems, etc.) but rather the ways in which these assertions have been produced by mathematics as a cultural product (the 'practices of mathematicians').

Those 'practices of mathematicians' have been considered by Mason as enquiry activities that 'explore ideas, develop notations, define terms, and prove theorems' [11,p.190].

Among the practices of mathematicians, the importance of the defining process has been highlighted by several authors. For Freudenthal, 'establishing a definition can be an essential feat, more essential than finding a proposition or a proof' [12,p.134]. De Villiers states that 'the construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching' [13,p.249]. Zandieh and Rasmussen [14] also studied the notion of defining as a mathematical activity. Indeed, they constructed a framework in order to explain students' progress when they move from informal to more formal ways of reasoning.

More recently, Ouvrier-Buffet talks about the need to study the defining process used by mathematicians. She presented a model of the mathematical defining processes, for which she used both classroom activities and interviews with professional mathematicians (see [15-17]). While she focused on the definitions that appear during processes of proof or when solving a problem, we will study the defining process associated with generalization of existing definitions. In our study, we use the term ‘generalization’ in the sense of Harel and Tall, to ‘mean the process of applying a given argument in a broader context’ [18,p. 38].

In the present paper, we wish to contribute to a better understanding of the process of defining of professional mathematicians when they do research. We consider that investigating in depth how to characterize this process will help our understanding of how future mathematicians’ undergraduate and postgraduate education should be. In order to research this process of defining, we will consider mathematicians as subjects of study and strive to identify the characteristics of their process of construction of definitions when they generalize existing ones.

Conceptual Framework

Our conceptual framework combines the concept of ‘constructive definition’ of Freudenthal [12] and the tools ‘operators’ and ‘control structures’ proposed by Ouvrier-Buffet [15].

For Freudenthal [12], there are two types of activities of defining in mathematics: descriptive and constructive. The activity of defining that lead us to a descriptive definition ‘outlines a known object by singling out a few characteristic properties’, while the one that leads us to a constructive definition ‘models new objects out of familiar ones’ [12,p.457]. This last type of defining activity is usually linked to the practice of generalizing.

Furthermore, Ouvrier-Buffet proposed to ‘characterize mathematically (through an epistemological study of the concept) the nature of the definition construction’

[15,p.263]. In order to do this, she used the tools *operators* and *control structures.* In her own words, ‘an operator is a tool for action that may allow the transformation of the problem’ [15,p.263]. For example, ‘the most important Lakatosian operator to be found during a research process is certainly the use and the generation of examples and counter-examples’ [15,p.266].

On the other hand, a control structure

> allows the observer to describe how the subject judges the adequacy and validity of an action…The control structure is constituted by all the means needed in order to make choices, to make decisions, as well as to express judgment,… the control structure ensures the non-contradictory nature of the conception (in Balacheff’s sense): among the control instruments we must find decision tools legitimizing the use of an operator or the state of the ongoing problem (the problem should be declared solved or not) [15, p.263].

According to Ouvrier-Buffet [15,p.264-265], the main operator for Popper ‘is the generation of refutations through counter-examples’ and the main control structure is ‘resistance to refutations’.

In this paper, our aim is to characterize the defining process of professional mathematician researchers when they generalize existing definitions and thus obtain constructive definitions. In order to do this, we will not use the whole model presented by Ouvrier-Buffet [15-17] but rather her tools *operators* and *control structures.* The reason for this choice is that her model describes very well the defining process that appears in situations of proof (or when solving a problem) but we found that it did not fit completely the defining processes associated with generalization of existing definitions. We will give more details about our decision in the section ‘Discussion and Conclusions’.

Methodology

We have conducted a qualitative study. We will describe as follows the participants, context, instrument and analysis.

Participants and context

In our study, the participants are researchers in mathematics who have at least one paper published in a journal included in the Journal Citation Reports (JCR) list of the Web of Knowledge. We use this definition because, in Spain, the success of a mathematician's research career is generally measured by the number of papers that he/she has published in journals that are in the JCR list.

Firstly, we contacted several professional mathematicians and asked them, in informal interviews, about the role of defining in their research. This way, we identified more than one mathematician whose research focus was proposing new definitions by generalizing previous definitions. Since their research is very technical and specialized, we have decided that it would aid readers' comprehension if we conducted a case study with only one of these mathematicians, who we will call Alice. Another reason for using this methodology is that we are not interested in making assertions about how all mathematicians work (which would be both very bold and probably wrong) but rather what an actual one does.

The reason why we have chosen Alice as the subject of our case study is that she is very articulate and she was interested in mathematics education and in participating in a study of the type explained here. Alice is a young researcher who has a Ph.D. degree in mathematics and has been part of the faculty of more than one big public university in Spain for several years. Similar to all the mathematicians who were interviewed, she has published several papers on differential geometry in journals included in the JCR list of the Web of Knowledge.

Instrument

The data of this case study come from different sources:

- A preliminary interview with Alice in which the authors of this paper told her about the goals of this study and asked her about the general aspects of her research and about the importance of defining in it. This interview lasted approximately 60 minutes.
- Written research documents (Ph.D. thesis, peer-reviewed published papers, etc.).
- Three more in-depth interviews that served to delve into Alice's research work and to clarify some aspects that were relevant to it. Each of these interviews lasted around 40 minutes. Since she preferred not to be recorded, the authors of this paper took numerous and detailed notes that were later used in their analysis. Additionally, Alice drew several diagrams to summarize how authors in her field of research usually introduce new definitions. She then did similar diagrams to summarize her own research. We show one of these diagrams (in her own handwriting) in Figure 1. In order to aid readers' comprehension, we made computer versions of her other diagrams (Figures 2, 3, 4 and 5).

Analysis

We used the strategy of moving from the particular to the general [3].

Focusing on constructive definitions in Freudenthal's sense, we analysed Alice's process of defining when she generalizes previously known definitions in order to obtain new ones in the field of differential geometry. We followed three steps in our analysis, which we will explain now more in depth.

First step

We asked Alice to describe how she constructs definitions as part of her research. She was able to explain how she had defined several new spaces and gave us detailed explanations about their construction. We wrote down her description of the process of defining in several cases and realized that four different objectives appeared in all of them. Therefore, we decided to divide her process in four sequential parts, each of them linked to one of these objectives. We called each of these parts of the process 'phases' and found a name that captured their objective: Finding an opportunity to generalize an existing concept, Proposing a new definition, Justifying that the new definition is valid and Continuing the chain of definitions. In the following, we will give an overview of these phases (which will be completed in the Results section) and present our names for them.

In the first phase, Alice studies what other authors have already done to find well-known definitions and how they appeared. She discovered that a popular activity in her field is to propose a new definition by generalizing an existing object, which later inspired her to construct her own definitions. She exemplified this activity of generalizing by drawing diagrams similar to the one in Figure 1 to support her explanations. We made computer versions of these diagrams and have included them as Figures 2, 3 and 4. Since this part of Alice's process consists of a review of the literature and an identification of the way in which the defining process works, we call it 'Finding an opportunity to generalize an existing concept'.

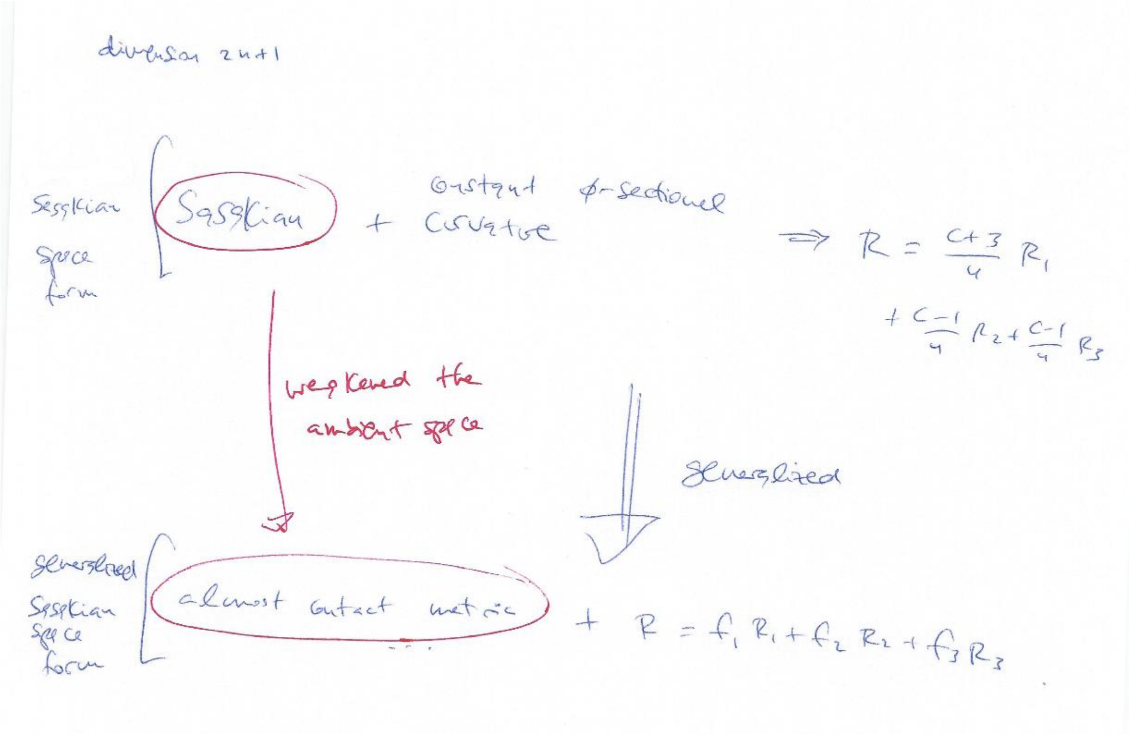

Figure 1. Alice's diagram (in her own handwriting) of the introduction of generalized Sasakian space forms by Alegre, Blair and Carriazo [20]

In the second phase, Alice introduces her own definitions by generalizing previously known objects, which she exemplified for us by using diagrams similar to Figure 1 and by her own publications (both her papers and her Ph.D. thesis). In each of these publications, Alice pointed out the new definitions she had introduced and which existing concepts each of them generalized. We decided to call this phase ‘Proposing a new definition’.

In the third phase, Alice told us that it is not enough to propose a new definition but that one has to justify its validity. This is usually done by checking that there exist new examples that satisfy the new definition but that did not satisfy the previously known one. Indeed, the generalization of an object must not be a mere change of name or description. We have called this phase ‘Justifying that the new definition is valid’.

In the fourth (and last) phase, Alice studies her new definition to see if it can lead to the construction of new definitions, that is, if another 'first phase' of a defining process can appear or if the definition that has been constructed constitutes a dead-end. Our name for this phase is 'Continuing the chain of definitions'.

Second step

In the second step of our case study, we identified which operators and control structures appeared in the multiple examples of construction of definitions that Alice presented to us. We refer to the Results section for a detailed explanation of which operators and control structures appear in each phase.

Third step

Finally, we compared the operators and control structures that appeared in the four phases of construction of all of Alice's examples of definitions. We found that the same operators and control structures appeared in the first phase of all of them, and that the same was true for the second, third and fourth phases. This led us to characterize each phase by the identification and use of operators and control structures in it.

Results

In this section, we present the results of our research on Alice's process of construction of definition, adopting the way of a case study. We describe the four phases that we identified in the first step of our analysis. In each of the phases, we present both Alice's descriptions and our analysis of them, obtaining a characterization of each phase by means of operators and control structures.

Phase 'Finding an opportunity to generalize an existing concept'

During her interview, Alice told us that the first thing that she always does when trying to find new opportunities for research is to do a review of the existing literature in a particular field in order to identify research techniques that other mathematicians have used before. She told us that she had realized that a popular activity in the field of differential geometry consists of defining new spaces by taking a well-known definition and generalizing it. This can be done by weakening or removing some of the conditions of the existing definition, which gives a new class of objects that includes the previous one. Then it is crucial to find examples of the newly defined objects that were not previously known, that is, to check that this new class of objects is not the merely a relabeling the original one. This way, she managed to identify chains of logical steps that take from definition to definition and chains of inclusive classes (since the new class of objects that is defined always includes the previous ones).

When we analysed Alice's descriptions of this phase, we realized that, from our point of view, what she actually does is to identify operators and control structures (although she does not call them by those names). We will show these descriptions and analysis by means of two examples.

Example 1. Tricerri and Vanhecke [19] defined the generalized complex space forms from the complex space forms. It was previously known that the 'complex space forms', that is, the Kähler manifolds with constant holomorphic curvature c, always have a curvature tensor of the form $R=(c/4)(R_1+R_2)$, where R_1 and R_2 are tensors that do not depend on the example. Then Tricerri and Vanhecke [19] decided to define the 'generalized complex space forms' as almost Hermitian manifolds with curvature tensor $R=f_1R_1+f_2R_2$, where f_1 and f_2 are functions (almost Hermitian manifolds include the Kähler ones, it is a weaker condition to ask). Then these and other authors looked for and found examples of ‘generalized complex space forms’ that were not ‘complex space forms’.

Alice summarized this process by pointing out that these authors introduced new spaces by ‘weakening the ambient space’ and ‘generalizing the curvature tensor’. Then they proved that the new class of examples included the previous one but was not the same. Alice illustrated this process with a diagram, which we have adapted into Figure 2.

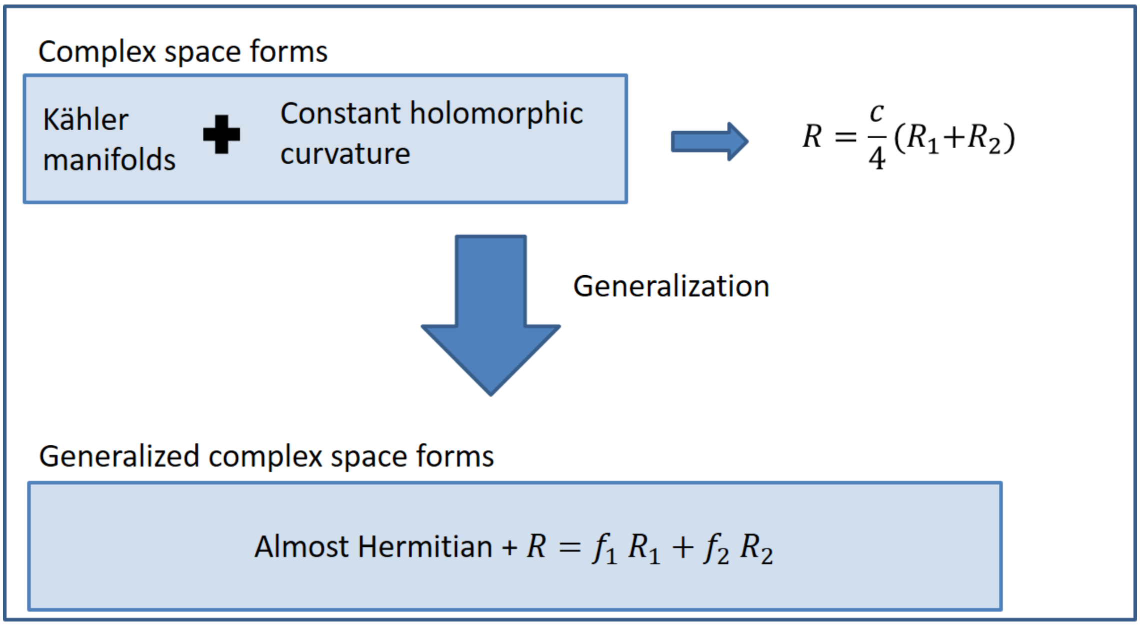

Figure 2. Introduction of generalized complex space forms by Tricerri and Vanhecke [19]

Example 2. Alice told us that Alegre, Blair and Carriazo [20] followed a similar process when they introduced the ‘generalized Sasakian space forms’ by generalizing the ‘Sasakian space forms’. The Sasakian space forms, that is, the odd-dimensional Sasakian spaces with constant ϕ-sectional curvature c, always have a curvature tensor of the form $R=((c+3)/4)R_1+((c-1)/4)(R_2+R_3)$, where R_1, R_2 and R_3 are tensors that do not depend on the example. This led Alegre, Blair and Carriazo [20] to define the ‘generalized Sasakian space forms’ as almost contact manifolds with curvature tensor

$R=f_1R_1+f_2R_2+f_3R_3$ where f_1, f_2 and f_3 are functions (almost contact manifolds include the Sasakian ones, it is a weaker condition to impose). Then they also found examples of 'generalized Sasakian space forms' that were not 'Sasakian space forms'.

Alice pointed out that the 'generalized Sasakian space forms' are a generalization of 'Sasakian space forms' and that this generalization is done again by 'weakening the ambient space' and 'generalizing the curvature tensor'. Finally, the new class of objects must be shown to be bigger than the previous one by showing new examples. Alice also drew a picture to describe this process, which we included in the Methodology section as Figure 1 and we later transformed into Figure 3.

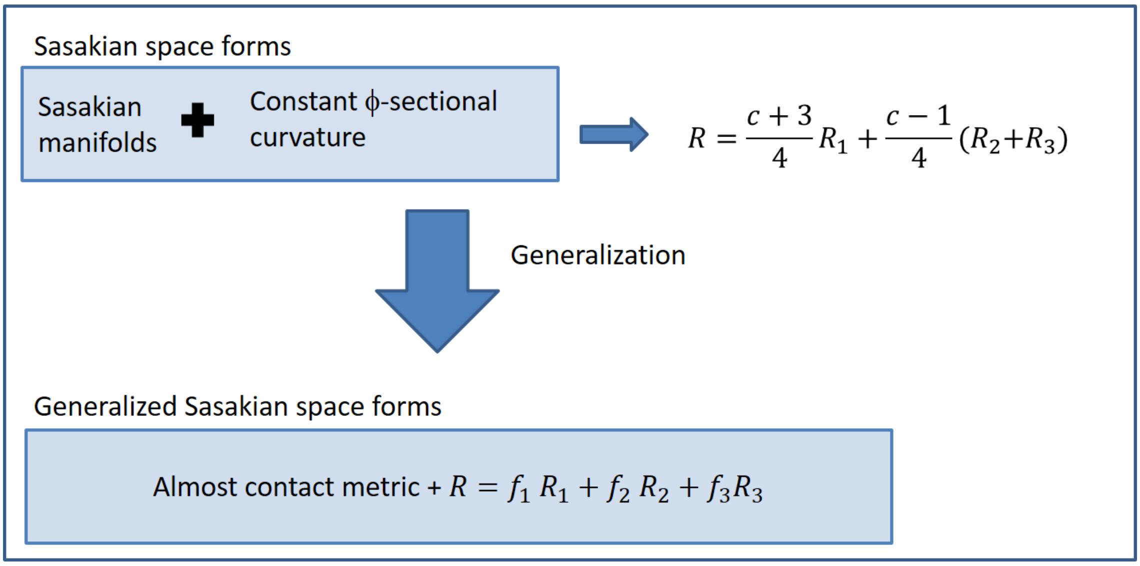

Figure 3. Introduction of 'generalized Sasakian space forms' by Alegre, Blair and Carriazo [20]

Using Figures 2 and 3, Alice noticed the great similarities between the introduction of 'generalized complex space forms' and 'generalized Sasakian space forms'. She described how similarly the new definitions appeared, how they were checked to be valid and how new examples of the newly defined manifolds were found.

Using our conceptual framework, the second step of our analysis led us to notice that she was describing the following 'operators' and 'control structures':

Operator 1: generalizing the ambient space. In the first example, this was done by replacing 'Kähler manifold' by 'almost Hermitian manifold'. In the second one, by replacing 'Sasakian manifold' by 'almost contact metric manifold'.

Operator 2: generalizing the curvature tensor. In the first example, this was done by replacing the constant by functions, that is, they changed $R=(c/4)(R_1+R_2)$ to $R=f_1R_1+f_2R_2$. In the second example, by replacing $R=((c+3)/4)R_1+((c-1)/4)(R_2+R_3)$ by $R=f_1R_1+f_2R_2+f_3R_3$.

Operator 3: transforming the spaces in a particular class in order to discover new examples in that class. It is very common to obtain new examples by deforming or changing other ones. This often gives infinitely many examples from a single one.

Control structure 1: checking that the original class of examples is included strictly in the new class defined by Operators 1 and 2. This is done by finding an example in the new class that is not in the original class.

Control structure 2: checking that the transformed examples obtained by using Operator 3 are indeed examples of the same type. For instance, Alegre, Blair and Carriazo [20] got 'generalized Sasakian space forms' through conformal changes of metrics, D-homothetic deformations and warped products of other 'generalized Sasakian space forms'. It is important to notice that these transformations do not always provide examples of the same type but that this kind of result is also interesting for mathematics researchers.

Phase 'Proposing a new definition'

Alice showed us how she and her co-authors had proposed their own definitions by generalizing existing objects. We will present here two of these definitions. It is important to note that Example 4 is a generalization of Example 3.

Example 3. Alice and her colleagues first focused on the (κ,μ)-spaces, which are a generalization of the Sasakian manifolds. They noticed that the '(κ,μ)-spaces forms', which are (κ,μ)-spaces with constant ϕ-sectional curvature, always have curvature tensor of the form

$$R = ((c+3)/4)R_1 + ((c-1)/4)R_2 + ((c-3)/4-\kappa)\ R_3 + R_4 + (1/2)\ R_5 + (1-\mu)R_6,$$

where $R_1,\ldots R_6$ are tensors that do not depend on the example. Then they decided to define 'generalized (κ,μ)-space forms' as almost contact metric manifolds with curvature tensor $R=f_1R_1+f_2R_2+f_3R_3+f_4R_4+f_5R_5+f_6R_6$, where $f_1, \ldots, f_6$ are functions.

In this way, they created new objects out of previously known ones, they proposed a 'constructive definition' (Freudenthal [12]). Alice also illustrated this process by drawing a picture that we later adapted into Figure 4 to help readability.

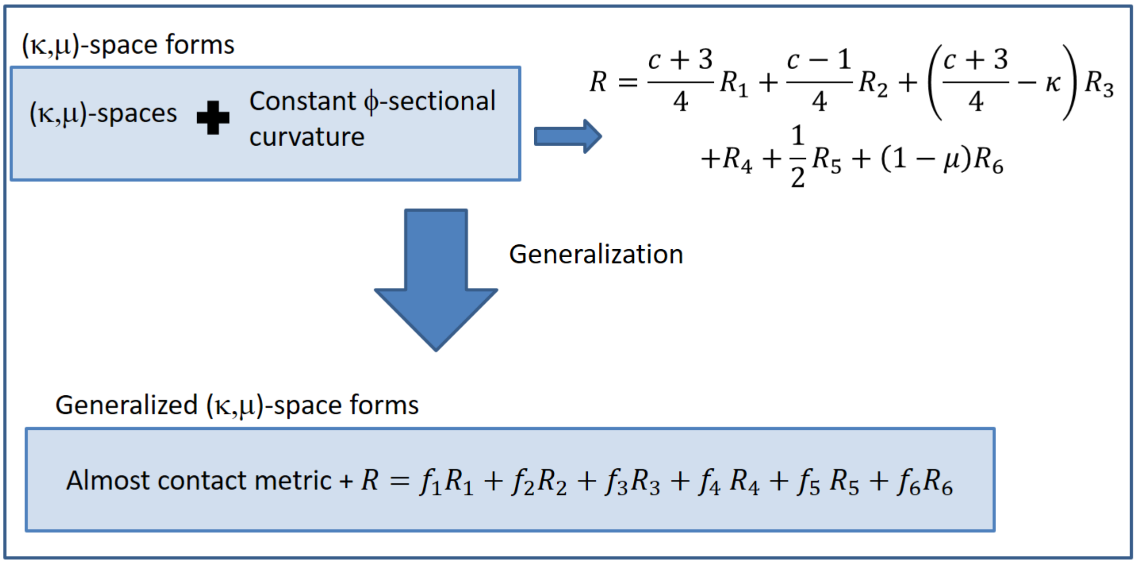

Figure 4. Introduction of 'generalized (κ,μ)-space forms' by Alice and her colleagues

Therefore, from our point of view as researchers in mathematics education, she used Operators 1 and 2. The first operator appeared when she decided to change the ambient space from '(κ,μ)-spaces' to 'almost contact metric manifolds'. The second operator when she replaced the constants by functions in the writing of the curvature tensor.

Example 4. After introducing and studying the 'generalized (κ,μ)-spaces', Alice and her coauthors studied the (κ,μ,ν)-spaces, which include the (κ,μ)-spaces, and in turn include the Sasakian manifolds. Alice summarized these inclusions in a diagram in her own handwriting, which we later converted into Figure 5 to help readability.

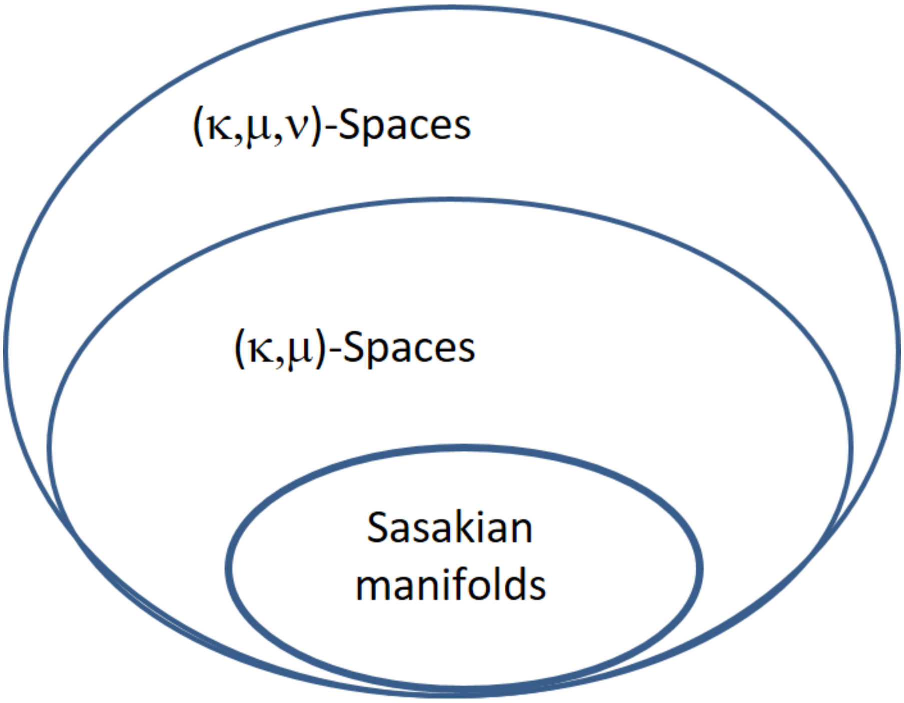

Figure 5. The relation between the different spaces studied

The '(κ,μ,ν)-space forms', which are (κ,μ,ν)-spaces with constant ϕ-sectional curvature, always have a curvature tensor that is a linear combination of constants and tensors $R_1,\ldots, R_8$. This led Alice and some of her colleagues to define 'generalized

(κ,μ,ν)-space forms' as almost contact metric manifolds with curvature tensor $R=f_1R_1+f_2R_2+f_3R_3+f_4R_4+f_5R_5+f_6R_6+f_7R_7+f_8R_8$, where $f_1,\ldots, f_8$ are functions.

Therefore, in our analysis, we realized that Alice and her colleagues had used again Operators 1 and 2. The first operator appeared when they decided to change the ambient space from (κ,μ,ν)-spaces to almost contact metric manifolds. The second operator when they changed the constants for functions in the writing of the curvature tensor.

Phase 'Justifying that the new definition is valid'

Alice said that this step is crucial and that a definition constructed in the previous phase is meaningless without it. She told us that, after proposing a new definition, it is always necessary to prove that the new class of objects contains new examples that were not included in the previously known definition. That is, it must be shown that a new type of spaces has been introduced instead of a mere relabelling of old. In Example 3, this was satisfied by finding 'generalized (κ,μ)-space forms' that were not '(κ,μ)-space forms'. In Example 4, by finding 'generalized (κ,μ,ν)-space forms' that were not '(κ,μ,ν)-space forms'.

From our point of view, Alice used Control structure 1 each time that she and her coauthors checked that a new definition is valid by finding these new examples. This way, they proved that their definitions are indeed constructive rather than descriptive, since they introduced new objects instead of new labels for previously known ones.

Phase 'Continuing the chain of definitions'

In this final phase, the newly-defined objects were studied. Alice and her colleagues analysed what properties (if any) they inherited from the objects that were generalized, what other properties were different and why.

For instance, Alice commented that they applied D-homothetic deformations to ‘generalized (κ,μ)-space forms’ (the spaces introduced in Example 3) in order to look for new examples. In this case, they discovered that these deformed spaces were not, in general, ‘generalized (κ,μ)-space forms’. This result does not invalidate the constructed definition but rather shows an essential characteristic that distinguishes it from the spaces that were generalized and hence makes the new definition more interesting. We identified this part of the defining process as Alice using Operator 3 and Control structure 2.

This phase sometimes leads to the creation of a new definition when a result inspires the author to wonder if removing or changing a particular condition of the examples will introduce another class of objects that includes the previous one, or if that condition is unnecessary. However, before introducing a new definition, it is always necessary to do a new review of the literature to find out if the spaces are already known. This phase, therefore, can mean the beginning of a new cycle of phases for the author.

To conclude this section, we show in Figure 6 a general vision of the four phases that we have identified and described, as well as the operators and control structures that appear in each of them.

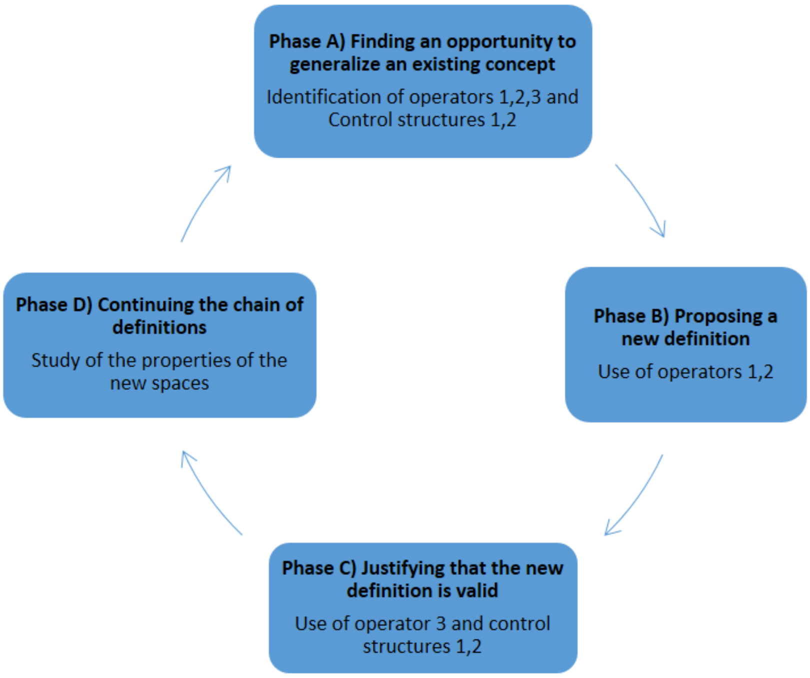

Figure 6. The four phases that we have identified in Alice's research process and the operators and control structures that appear in each of them

Discussion and conclusions

Weber [21] points out the necessity of understanding how mathematicians think and behave. He and his colleagues focused on the mathematical practice of proof and proving. Furthermore, Rasmussen et al. [2] state that the explicit introduction of advanced mathematical activities as pedagogical content in all levels means that a better understanding of how these activities are carried out by mathematicians is necessary.

In this paper, we have dealt with definition and defining, which according to Freudenthal ‘can be an essential feat, more essential than finding a proposition or a proof’ [12,p.134]. In particular, we have studied the defining process of a professional mathematician researcher through a case study. We have focused on the introduction of ‘constructive definitions’ rather than ‘descriptive definitions’ [12]. Moreover, we wish to complement the work that Ouvrier-Buffet did about definitions that appear in the ‘interplay between definitions and proofs’ [17,p.2216] by focusing our paper on definitions that appear during a process of generalizing. This last type of definitions is important because, although at the elementary level most definitions are descriptive, constructive definitions often appear at the undergraduate and postgraduate levels.

Furthermore, most teachers of mathematics subjects at university level in Spain are mathematicians who have a Ph.D. in mathematics, so the undergraduate and postgraduate education of these mathematicians also affects the future education of many other students of multiple disciplines. Therefore, our work can have implications for university education, an emergent field of study for mathematics educators.

In particular, in this paper we study the process of constructing of definitions that appear when generalizing other known ones. We do this by identifying sequential phases that appear during the defining process, and characterizing each of these phases by the use of operators and control structures.

In our work, we have managed to identify four distinctive phases in the process of defining, called: Finding an opportunity to generalize an existing concept, Proposing a new definition, Justifying that the new definition is valid and Continuing the chain of definitions.

We have identified three different operators and two control structures that are linked to these phases. Two of the operators generalize either the space or the curvature tensor and the other one transforms the space. The control structures make sure that the new definitions are valid by finding examples that were not previously known, and by checking if transforming the new objects gives examples of the same type.

Our work is related to that of other researchers like Ouvrier-Buffet [17] and Rasmussen et al. [2]. Indeed, Ouvrier-Buffet also studied the defining process and identified four ‘moments of work’, which ‘do not describe a linear activity, but they are connected: they give a dynamic overall view of the defining activity in the mathematical research' [17,p.2217]. In our paper, we did not use these moments of work for two reasons. The first one is that Ouvrier-Buffet studies defining activities that 'are usually evoked during the study of proofs and of problem solving processes' [17,p.2215], that is, definitions that are usually 'descriptive' in the sense of Freudenthal [12], while we will focus on characterizing the different stages of construction of definitions that appear when mathematician researchers generalize other known definitions. The second reason for which we did not use Ouvrier-Buffet's 'moments of work' is that we wished to emphasize the fact that we are not interested in a particular moment but in the whole stage, until the beginning of the following one. This is the reason we decided to differentiate these stages by calling them 'phases'.

Moreover, Rasmussen et al. [2] consider two dimensions of the defining process: horizontal and vertical mathematizing. They link the first one to the descriptive definitions and the second to the constructive definitions. For them, 'constructive defining creates new objects by building on and extending these known objects' [2,p.67]. Our description of the characteristics of the phases we have identified explains how this 'building on and extending' works. Moreover, Rasmussen et al. point out that 'in constructive defining the majority of the elaboration of a concept lies beyond the initiation of the defining activity, beyond the writing or stating of the definition for the first time' [2,p.68]. This statement is corroborated and detailed in the phases we have identified, which explain how the definition is proposed and what happens before and afterwards. In the description of the phases that we presented before, it can be seen that proposing a new definition is only one phase in the whole defining process.

Finally, we would like to acknowledge some limitations of our work. The first one comes from our methodology (the use of a case study). While focusing on only one mathematics researcher allowed us to better understand how she constructs (and validates) a mathematical definition and also permitted us to present our findings more clearly, this case study provided us with results that cannot be directly extrapolated to all mathematicians. A second limitation of our work is that it is possible that other mathematicians have other criteria for deciding on the adequacy of a new definition. In future works, we are interested in studying these mathematicians' criteria, both in the field of differential geometry and in others. Indeed, since the phases that we have identified (and the operators that appear in each of them) may have been too influenced by the field of research (differential geometry), more research in other fields is needed. Preliminary studies in the field of non-associative algebras hint that our control structures can be extrapolated.

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