Uniform Distribution Theory **3** (2008), no.1, 35–72

uniform distribution theory

ON SOME OSCILLATING SUMS

J. Arias de Reyna* — J. van de Lune

ABSTRACT. This paper deals with various properties (theoretical as well as computational) of the sums $S_\alpha(n) = \sum_{j=1}^{n}(-1)^{\lfloor j\alpha\rfloor}$ where α is any real number (mostly a positive real quadratic).

Communicated by Michael Drmota

Introduction

This paper deals with the sums

$$S_\alpha(n) = \sum_{j=1}^{n}(-1)^{\lfloor j\alpha\rfloor}$$

where α is any real number. If the value of α is clear from the context we will simply write $S(n)$ instead of $S_\alpha(n)$.

Apparently the interest in these sums was initiated (in 1976) by an unsolved problem proposed by H. D. Ruderman [3]: Prove that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{\lfloor n\sqrt{2}\rfloor}}{n}$$

converges and estimate its value. Indeed, when applying Abel-summation (summation by parts) to this series our sum (with $\alpha = \sqrt{2}$) emerges naturally.

This fact is reflected in the 1978 issue of the Amer. Math. Monthly, where one finds a solution by D. Borwein [7] (with an editorial reference to 11 other solutions).

Many generalizations were presented, one of them being: If α is any real quadratic irrational then the series $\sum_{n=1}^{\infty}(-1)^{\lfloor n\alpha\rfloor}/n^s$ converges for every $s > 0$. Further solutions may be found in Bundschuh [6] and van de Lune [4].

2000 Mathematics Subject Classification: 11K31, 11J70.
Keywords: Exponential sum, continued fraction, quadratic irrational, algorithm.
*Supported by grant MTM2006-05622.

The last related publication seems to be SCHOISSENGEIER [17], where it is proved that if the regular continued fraction expansion of $\beta = \alpha/2$ is $\{b_0; b_1, b_2, \dots\}$ with convergents $\frac{p_k}{q_k}$ then the series $\sum_{n=1}^{\infty}(-1)^{\lfloor n\alpha\rfloor}/n$ converges if and only if the series

$$\sum_{k=0,\ 2\nmid q_k}^{\infty} \frac{(-1)^k \log b_{k+1}}{q_k}$$

converges.

Our sums $S(n)$ are also interesting in themselves as may be seen from the following example: For $\alpha = \sqrt{2}$ we compute $S(n)$ for $n = 1, 2, 3, \dots,$ and keep track of those n for which $S(n)$ assumes a value for the first time (i.e., is larger/smaller than ever before). Here (and in the sequel) we define $S(0) = 0$. The sequence of these n's (the *record-holders*) will be denoted by $t_1, t_2, t_3, \dots$. We found

$t \to$	1	3	8	20	49	119	288	696	1681	4059	9800
$S \to$	-1	1	-2	2	-3	3	-4	4	-5	5	-6

In making our computations we observe that $t_{k+1} = 2t_k + t_{k-1} + 1$ for all $k \geq 1$. Here (and in the sequel) we define $t_0 = 0$. Also, the sequence $\operatorname{sign}(S(t_k))$ appears to be purely periodic.

In this paper we will show that similar results are true for all real quadratic irrationals.

We will see that the behavior of our sums is intimately connected with the regular continued fraction expansions $\alpha = \{a_0; a_1, a_2, a_3, \dots\}$ and $\beta = \alpha/2 = \{b_0; b_1, b_2, b_3, \dots,\}$. In this vein BROUWER and VAN DE LUNE [5] have shown that $S(n) \geq 0$ for all n if and only if the partial quotients a_{2i} are even for all $i \geq 0$. In FOKKINK, FOKKINK and VAN DE LUNE [9] we find a description of a fast algorithm for the computation of $S(n)$ for (very) large n in terms of the regular continued fraction expansion of $\beta = \alpha/2$. For an explicit program we refer to the Appendix. We just mention that by means of this program we easily found that

$$S_{\sqrt{2}}(10^{1000}) = -10, \quad S_{\sqrt{2}}(10^{10000}) = 166, \quad S_{\pi}(10^{10000}) = 11726.$$

A closely related (but somewhat less general) algorithm is given in an interesting paper by O'BRYANT, REZNICK and SERBINOWSKA [15].

It will turn out that the sequence $(-1)^{\lfloor j\alpha\rfloor}$ exhibits various symmetries. We will determine these explicitly in terms of the convergents of α.

We will also show that for any real irrational α the sum $S(n)$ is not bounded, so that the corresponding sequence of record-holders t_k actually is an infinite sequence.

In addition we will prove that for every $j \geq 1$ there is an index k such that $t_j - t_{j-1} = Q_k$, where Q_k is the denominator of a certain regular continued fraction approximant of α, as defined in Section 2. We will translate this into an algorithm which (given sufficiently many initial convergents of α) computes the *entire* sequence of record-holders. We will also give explicit formulae for the numbers t_j and the corresponding Q_k.

Finally we will study the function $H(\alpha) = \Lambda(1 - \alpha)$, where

$$\Lambda(\alpha) = \limsup_{n\to\infty} \frac{S_\alpha(n)}{\log n}.$$

It will turn out that this function is finite for all real α with bounded partial quotients, and, in a certain sense, is a modular function since

$$H(\alpha + 2) = H(\alpha), \qquad H\Big(-\frac{1}{\alpha}\Big) = H(\alpha).$$

We will also present a fast algorithm for the computation of $H(\alpha)$.

In Section 6 we solve the problem of the order of $\sup_{n\leq N} S_\alpha(n)$ for almost all real α (see Theorem 29). This order is similar to that of $\sup_{n\leq N} nD_n^*(j\alpha)$, where D_n^* is the discrepancy of the sequence $\{j\alpha\}$. We think that our proof of the lower bound of (41) might very well be the most straightforward.

In an Appendix we present a fast implementation of the FFL-algorithm (already implicitly described in FFL [9]) to compute $S_\alpha(n)$ for any irrational real α.

1. Preliminary results

We begin with some well known facts.

Lemma 1. *We have*

$$x \in \mathbb{R} \text{ and } n \in \mathbb{Z} \implies \lfloor n + x\rfloor = n + \lfloor x\rfloor \tag{1}$$

$$x \in \mathbb{R} \smallsetminus \mathbb{Z} \implies \lfloor x\rfloor + \lfloor -x\rfloor = -1. \tag{2}$$

$$\alpha \in \mathbb{R} \text{ and } n \geq 0 \implies S_{\alpha+2}(n) = S_\alpha(n) \tag{3}$$

$$\alpha \in \mathbb{R} \smallsetminus \mathbb{Z} \implies S_{-\alpha}(n) = -S_\alpha(n). \tag{4}$$

In the sequel we will always assume that α is real, and that all continued fractions are regular. This also applies to our programs (which have been designed primarily for real quadratic irrationals).

In case $\alpha = \frac{p}{q}$, with $(p, q) = 1$, is rational it is easily seen that the sequence $S(n)$ is (1) periodic (and hence bounded) with period $2q$ if p is odd and (2) unbounded (of true order n) if p is even. Therefore, we will restrict ourselves from now on to irrational real α. In view of Lemma 1 we may, without loss of generality, even restrict ourselves to $0 < \alpha < 1$.

It will turn out that the properties of $S(n)$ heavily depend on the regular continued fraction expansion $\{a_0; a_1, a_2, \dots\}$ of α, in particular on its convergents

$$\frac{P_{-2}}{Q_{-2}} = \frac{0}{1}, \qquad \frac{P_{-1}}{Q_{-1}} = \frac{1}{0}, \qquad \frac{P_j}{Q_j} = \{a_0; a_1, \dots, a_j\}, \quad (j \geq 0) \tag{5}$$

which satisfy the relations

$$P_{j+1} = a_{j+1}P_j + P_{j-1}, \quad Q_{j+1} = a_{j+1}Q_j + Q_{j-1}, \qquad (j \geq -1) \tag{6}$$

and

$$(P_j, Q_j) = 1 \quad (j \geq -2), \quad P_jQ_{j-1} - Q_jP_{j-1} = (-1)^{j-1}, \qquad (j \geq -1). \tag{7}$$

If $a_{n+1} \geq 2$ we will also make use of mediants. These are the irreducible fractions P/Q defined by

$$\frac{P}{Q} = \frac{hP_n + P_{n-1}}{hQ_n + Q_{n-1}}, \qquad (n \geq 0, \quad h = 1, 2, \dots, a_{n+1} - 1).$$

For these fractions we have $PQ_n - P_nQ = (-1)^n$, and we say that P_n/Q_n is the convergent next to the mediant P/Q. Note that α always lies strictly between P/Q and the convergent next to P/Q.

Just for the sake of easy reference we state the following known lemma (see [11, Chapter VII, Exercise 13]).

Lemma 2. *If $\frac{a}{b} < \frac{c}{d}$ with b and $d > 0$ and $cb - ad = 1$, then every fraction $\frac{m}{n}$ with $n > 0$ and*

$$\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$$

satisfies $n \geq b + d$.

Lemma 3. *Let $\frac{P}{Q} = \frac{hP_{n+1}+P_n}{hQ_{n+1}+Q_n}$, with $0 \leq h < a_{n+2}$, be a mediant or a convergent of α.*

If $1 \leq m < Q + Q_{n+1}$, with $Q \nmid m$, then

$$\lfloor m\alpha \rfloor = \Big\lfloor m\frac{P}{Q} \Big\rfloor. \tag{8}$$

In particular this is true for $1 \leq m < Q$.

P r o o f. Suppose the assertion of the Lemma is false. Then we can find an integer k between $m\alpha$ and mP/Q, so that k/m lies strictly between α and P/Q, and is not equal to these extremes. (Indeed $k/m \neq \alpha$ since α is irrational and $k/m \neq P/Q$ since P/Q is in lowest terms and by hypothesis $Q \nmid m$.) Since α

lies between P_{n+1}/Q_{n+1} and P/Q, it follows that k/m is lying strictly between P/Q and P_{n+1}/Q_{n+1},

Now observe that

$$\frac{P}{Q}-\frac{P_{n+1}}{Q_{n+1}}=\frac{P_nQ_{n+1}-Q_nP_{n+1}}{QQ_{n+1}}=-\frac{(-1)^n}{QQ_{n+1}} \tag{9}$$

so that, by Lemma 2, we would have $m > Q + Q_{n+1}$, which contradicts our hypothesis. □

Lemma 4. *Let* $\frac{P}{Q}=\frac{hP_{n+1}+P_n}{hQ_{n+1}+Q_n}$*, with* $0 \le h < a_{n+2}$*, be a mediant or a convergent of* α*. Then*

$$(-1)^{\lfloor Q\alpha\rfloor}=(-1)^{n+P}. \tag{10}$$

Proof. As in the proof of the previous Lemma, α lies between the fractions P/Q and P_{n+1}/Q_{n+1}. By a computation as in (9) we have

$$\Bigl|\frac{P}{Q}-\alpha\Bigr|<\Bigl|\frac{P}{Q}-\frac{P_{n+1}}{Q_{n+1}}\Bigr|=\frac{1}{QQ_{n+1}}.$$

For $n \ge 0$ the two numbers Q and Q_{n+1} are ≥ 1, so that $|Q\alpha - P| < Q_{n+1}^{-1} \le 1$. It follows that $\lfloor Q\alpha\rfloor = P$ if $Q\alpha > P$, and that $\lfloor Q\alpha\rfloor = P-1$ if $Q\alpha < P$. But, by the theory of continued fractions the first case happens when n is even, and the second when n is odd. Hence, (10) is true in both cases. □

The following theorem shows the fundamental connection between the sums $S_\alpha(n)$ and the convergents of α.

Proposition 5. *Let* α *be any irrational real number, and* $\frac{P_n}{Q_n}$ *with* $n \ge 0$ *one of its convergents. Let* $Q_n \le m < Q_n + Q_{n+1}$ *and put* $m = hQ_n + r$ *with* $0 \le r < Q_n$*. Then*

$$\lfloor m\alpha\rfloor=\begin{cases} hP_n+\lfloor r\alpha\rfloor & \text{if } r\ne 0,\\ hP_n & \text{if } r=0 \text{ and } n \text{ is even},\\ hP_n-1 & \text{if } r=0 \text{ and } n \text{ is odd.}\end{cases} \tag{11}$$

Proof. If $r \ne 0$ then $Q_n \nmid m$ and $Q_n \nmid r$. By Lemma 3 we have

$$\lfloor m\alpha\rfloor=\Bigl\lfloor m\frac{P_n}{Q_n}\Bigr\rfloor=\Bigl\lfloor hP_n+r\frac{P_n}{Q_n}\Bigr\rfloor=hP_n+\Bigl\lfloor r\frac{P_n}{Q_n}\Bigr\rfloor=hP_n+\lfloor r\alpha\rfloor.$$

If $r = 0$ then $m = hQ_n$. Also, $m\alpha$ lies between the two numbers mP_n/Q_n and mP_{n+1}/Q_{n+1}, and the distance between these two numbers is

$$\Bigl|m\frac{P_n}{Q_n}-m\frac{P_{n+1}}{Q_{n+1}}\Bigr|=\frac{m}{Q_nQ_{n+1}}\le 1.$$

(For the last inequality observe that always $1 \le Q_n \le Q_{n+1}$. So $Q_nQ_{n+1} \ge Q_n + Q_{n+1} > m$, unless $Q_n = 1$. In this case $Q_n = 1 \le m < 1 + Q_{n+1}$ and the inequality follows.)

When n is even we have

$$m\frac{P_n}{Q_n} = hP_n < m\alpha < m\frac{P_{n+1}}{Q_{n+1}} \le 1 + hP_n$$

and it follows that $\lfloor m\alpha \rfloor = hP_n$.

When n is odd we have

$$m\frac{P_n}{Q_n} = hP_n > m\alpha > m\frac{P_{n+1}}{Q_{n+1}} \ge hP_n - 1.$$

Thus, in this case we get $\lfloor m\alpha \rfloor = hP_n - 1$, and the proof of (11) is complete. □

Corollary 6. *Let α be any irrational real number, and $\frac{P_n}{Q_n}$ with $n \ge 0$ one of its convergents. Let $Q_n \le m < Q_n + Q_{n+1}$ and put $m = hQ_n + r$ with $0 \le r < Q_n$. Then*

$$(a) \qquad P_n \quad \textit{even} \quad \Longrightarrow \quad S(m) = (-1)^n h + S(r). \tag{12}$$

$$(b) \qquad P_n \quad \textit{odd} \quad \Longrightarrow \quad S(m) = \begin{cases} S(Q_n) - S(r) & \textit{if } h \textit{ is odd} \\ S(r) & \textit{if } h \textit{ is even.} \end{cases} \tag{13}$$

P r o o f. We compute the sum

$$S(m) = \sum_{j=1}^{m} (-1)^{\lfloor j\alpha \rfloor} =$$
$$= \sum_{k=0}^{h-1} \sum_{s=1}^{Q_n-1} (-1)^{\lfloor (kQ_n+s)\alpha \rfloor} + \sum_{k=1}^{h} (-1)^{\lfloor kQ_n\alpha \rfloor} + \sum_{s=1}^{r} (-1)^{\lfloor (kQ_n+s)\alpha \rfloor}.$$

Applying (11) we get

$$S(m) = \sum_{k=0}^{h-1} \sum_{s=1}^{Q_n-1} (-1)^{kP_n+\lfloor s\alpha \rfloor} + \sum_{k=1}^{h} (-1)^{kP_n - [\![\, n \text{ is odd} \,]\!]} + \sum_{s=1}^{r} (-1)^{hP_n+\lfloor s\alpha \rfloor}$$

where, following Iverson's notation, $[\![\, X \,]\!] = 1$ if the proposition X is true, and $[\![\, X \,]\!] = 0$ if X is false.

Simplifying we get

$$S(m) = \sum_{k=0}^{h-1} (-1)^{kP_n} S(Q_n - 1) + \sum_{k=1}^{h} (-1)^{kP_n - [\![\, n \text{ is odd} \,]\!]} + (-1)^{hP_n} S(r). \tag{14}$$

By Lemma 3 we have $S_\alpha(Q_n-1) = S_{P_n/Q_n}(Q_n-1)$. Therefore, if P_n is even we have $S_\alpha(Q_n-1)=0$ (see [5, Lemma 5.1]), so that in this case

$$S(m) = (-1)^n h + S(r).$$

If P_n is odd and h even, then the two sums in (14) are equal to 0 and we get

$$S(m) = S(r).$$

Finally, when P_n and h are odd, the first sum in (14) is equal to $S(Q_n-1)$, the second is $(-1)^{n+1}$, and we get

$$S(m) = S(Q_n-1) + (-1)^{n+1} - S(r) = S(Q_n) - S(r)$$

since, by Lemma 4, we have $(-1)^{\lfloor Q_n\alpha\rfloor} = (-1)^{n+P_n} = (-1)^{n+1}$. $\square$

2. Symmetries of the sequence of signs

We consider four types of symmetries of the sequence of signs $(-1)^{\lfloor j\alpha\rfloor}$. Each of them leads to a useful transformation of the sums $S(n)$. In this section we define these symmetries and present some theorems in order to obtain these symmetries from the sequence of convergents of α. It should be noted that our definitions differ slightly from those in [9].

Definition 7. *The integer $n \ge 1$ will be called a **point of repetition** (a REP, for short) if*

$$1 \le k \le n \Longrightarrow (-1)^{\lfloor k\alpha\rfloor} = (-1)^{\lfloor (n+k)\alpha\rfloor}$$

or, equivalently, if $\lfloor k\alpha\rfloor$ and $\lfloor (n+k)\alpha\rfloor$ have the same parity for $1\le k\le n$.

Note. There seem to be α's which do not yield any REP's.

For example, $\alpha = \sqrt{2}$ seems to be such a number.

The first few REP's for $\alpha=\pi$ are: $n = 2$, 7, 14, 21, 28, 35, 42, 49, 56, 226, 452, 678, 904, 1130, 1356, 1582, 1808, 2034, 2260, 2486, 2712, 2938.

Lemma 8. *If n is a REP for α, then*

$$0 \le k \le n \Longrightarrow S_\alpha(n+k) = S_\alpha(n) + S_\alpha(k). \tag{15}$$

Proof. For $k\ge 1$ we have

$$S_\alpha(n+k) = \sum_{j=1}^{n+k}(-1)^{\lfloor j\alpha\rfloor} = \sum_{j=1}^{n}(-1)^{\lfloor j\alpha\rfloor} + \sum_{j=1}^{k}(-1)^{\lfloor (n+j)\alpha\rfloor} = \\ = S_\alpha(n) + S_\alpha(k).$$

As usual we define $S_\alpha(0)=0$, and for $k=0$ the result is trivial. $\square$

Definition 9. *The integer $n \geq 1$ will be called a* ***point of contra-repetition*** *(a CREP, for short) if*

$$1 \leq k \leq n \Longrightarrow (-1)^{\lfloor k\alpha \rfloor} = -(-1)^{\lfloor (n+k)\alpha \rfloor}$$

or, equivalently, if $\lfloor k\alpha \rfloor$ and $\lfloor (n+k)\alpha \rfloor$ have different parities for $1 \leq k \leq n$.

The first few CREP's for $\alpha = \pi$ are: $n = 1$, 3, 106, 113, 339, 565, 791, 1017, 1243, 1469, 1695, 1921, 2147, 2373, 2599, 2825, 3051, 3277, 3503, 3729.

In the same way as in Lemma 8 we prove the following

Lemma 10. *If n is a CREP for α, then*

$$S_\alpha(n+k) = S_\alpha(n) - S_\alpha(k), \qquad 0 \leq k \leq n. \tag{16}$$

The next Theorem provides REP's and CREP's for α. See note at the end of this section regarding the scope of this theorem.

Proposition 11. *Let α be any irrational real number. If $h \in \mathbb{N}$, $n \geq 0$ and $2hQ_n < Q_n + Q_{n+1}$, then*

$$\lfloor (j + hQ_n)\alpha \rfloor = hP_n + \lfloor j\alpha \rfloor, \qquad 1 \leq j \leq hQ_n. \tag{17}$$

Thus hQ_n is a REP for α if hP_n is even, and a CREP if hP_n is odd.

Proof. We may apply (11) with $m = j$ and $m = j + hQ_n$. If $j = kQ_n + r$ with $0 \leq r < Q_n$, then $j + hQ_n = (h+k)Q_n + r$. Thus, by (11),

$$\lfloor j\alpha \rfloor = kP_n + A(n,r), \qquad \lfloor (j + hQ_n)\alpha \rfloor = (h+k)P_n + A(n,r)$$

where $A(n,r)$ depends only on n and r and is the same in both cases.

This proves (17). $\square$

Definition 12. *The integer $n \geq 2$ will be called an* ***end-point of reflection*** *(an EREF, for short) if*

$$1 \leq k \leq n/2 \Longrightarrow (-1)^{\lfloor k\alpha \rfloor} = (-1)^{\lfloor (n+1-k)\alpha \rfloor}$$

or, equivalently, if $\lfloor k\alpha \rfloor$ and $\lfloor (n+1-k)\alpha \rfloor$ have the same parity for $1 \leq k \leq n/2$.

The first few EREF's for $\alpha = \pi$ are given by: $n = 3$, 5, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, 112, 331, 557, 783, 1009, 1235, 1461.

Lemma 13. *If n is an EREF for α, then*

$$S_\alpha(n+1-k) = S_\alpha(n) - S_\alpha(k-1), \qquad (1 \leq k \leq n/2). \tag{18}$$

P r o o f. For $2 \le k \le n/2$ we have

$$S(n) = \sum_{j=1}^{n} (-1)^{\lfloor j\alpha \rfloor} = \sum_{j=1}^{n+1-k} (-1)^{\lfloor j\alpha \rfloor} + \sum_{j=n+1-(k-1)}^{n+1-1} (-1)^{\lfloor j\alpha \rfloor} =$$

$$= S(n+1-k) + \sum_{r=1}^{k-1} (-1)^{\lfloor (n+1-r)\alpha \rfloor} = S(n+1-k) + \sum_{r=1}^{k-1} (-1)^{\lfloor r\alpha \rfloor} =$$

$$= S(n+1-k) + S(k-1).$$

For $k = 1$ the result is clear. $\square$

Definition 14. *The integer $n \ge 2$ will be called an **end-point of contra-reflection** (an ECREF, for short) if*

$$1 \le k \le n/2 \Longrightarrow (-1)^{\lfloor k\alpha \rfloor} = -(-1)^{\lfloor (n+1-k)\alpha \rfloor}$$

or, equivalently, if $\lfloor k\alpha \rfloor$ and $\lfloor (n+1-k)\alpha \rfloor$ have different parities for $1 \le k \le n/2$.

The first few ECREF's for $\alpha = \pi$ are given by: $n = 2$, 4, 6, 13, 211, 218, 225, 444, 670, 896, 1122, 1348, 1574, 1800, 2026, 2252, 2478, 2704, 2930.

Lemma 15. *If n is an ECREF for the real irrational α, then*

$$S_\alpha(n-k) = S_\alpha(n) + S_\alpha(k), \qquad 0 \le k \le n/2. \tag{19}$$

It follows that $S_\alpha(n) = 0$ for n even, and $S_\alpha(n) = (-1)^{\lfloor \frac{n+1}{2}\alpha \rfloor}$ for n odd.

The proof is similar to that of Lemma 13.

Proposition 16. *Let α be any real irrational. If $n \ge -1$ and $1 \le h \le a_{n+2}$ then*

$$\lfloor (Q_n + hQ_{n+1} - j)\alpha \rfloor + \lfloor j\alpha \rfloor = P_n + hP_{n+1} - 1, \qquad 1 \le j < Q_n + hQ_{n+1}. \tag{20}$$

So $Q_n + hQ_{n+1} - 1$ (when ≥ 2) is an EREF for odd $P_n + hP_{n+1}$, and an ECREF when $P_n + hP_{n+1}$ is even.

P r o o f. Let $P = P_n + hP_{n+1}$ and $Q = Q_n + hQ_{n+1}$. The fraction $\frac{P}{Q}$ is a mediant for $1 \le h < a_{n+2}$, whereas $\frac{P}{Q} = \frac{Q_{n+2}}{P_{n+2}}$ if $h = a_{n+2}$. In both cases we may apply Lemma 3 to get

$$\lfloor j\alpha \rfloor = \left\lfloor j\frac{P}{Q} \right\rfloor, \qquad 1 \le j < Q.$$

Observe that (20) is equivalent to

$$\left\lfloor P - j\frac{P}{Q} \right\rfloor + \left\lfloor j\frac{P}{Q} \right\rfloor = P - 1, \qquad 1 \le j < Q.$$

Since $(P, Q) = 1$ and $1 \le j < Q$, the number $r = jP/Q$ is not an integer. So, our assertion is a consequence of (2) in Lemma 1. $\square$

Proposition 17. *Let α be any real irrational. If $n \ge 1$ then*

$$\lfloor (2Q_n - j)\alpha \rfloor + \lfloor j\alpha \rfloor = 2P_n - 1, \qquad 1 \le j < Q_n. \tag{21}$$

Thus $2Q_n - 1$, when ≥ 2, is an ECREF.

P r o o f. By Lemma 3, if $1 \le j < 2Q_n \le Q_n + Q_{n+1}$ and $j \ne Q_n$, then

$$\lfloor j\alpha \rfloor = \Big\lfloor j\frac{P_n}{Q_n} \Big\rfloor. \tag{22}$$

Now our proposition may also be written as

$$\Big\lfloor 2P_n - \frac{jP_n}{Q_n} \Big\rfloor + \Big\lfloor j\frac{P_n}{Q_n} \Big\rfloor = 2P_n - 1.$$

Since jP_n/Q_n is not an integer this follows in the same way as in the previous proof. $\square$

Note. For the cases we have considered, Propositions 11, 16 and 17 give all symmetries associated with α. But we have no proof that no other symmetries exist.

3. The sequence of record-holders

In this section we will study the *record-holders* —those values t of n for which $S_\alpha(t)$ assumes a value for the first time—more thoroughly. The following theorem shows that the record-holders are infinite in number.

Theorem 18. *The sums $S_\alpha(n)$ are not bounded if the real number α is irrational.*

P r o o f. This is a consequence of a more general Theorem of Kesten [2] which says that for subintervals I of $[0, 1)$ the sequence $\big(\sum_{n=1}^{N} c_I(\{n\alpha\}) - N|I|\big)_{N\ge1}$ is bounded if and only if the length of I lies in the group $\mathbb{Z} + \alpha\mathbb{Z}$. Our sums can be written

$$S_\alpha(n) = \sum_{j=1}^{n} (-1)^{\lfloor j\alpha \rfloor} = 2\sum_{j=1}^{n} c_{[0,1/2)}(\{j\alpha/2\} - n$$

and $1/2$ is clearly not in the group $\mathbb{Z} + (\alpha/2)\mathbb{Z}$. $\square$

Definition 19. *Putting $t_0 = 0$ and assuming that we have defined t_k for $k < n$, let t_n be the least integer $t > t_{n-1}$ such that*

$$0 \le j < t \Longrightarrow S(j) \ne S(t). \tag{23}$$

Clearly, if $t > 0$ is a record-holder then $S(t) \ne 0$. We call such a t a *maximum* if $S(t) > 0$, and a *minimum* if $S(t) < 0$.

As we will see, the record-holders are very structured. In particular, this holds true, though not exclusively, for quadratic irrationalities α. For example, we will prove that for every $j \ge 1$ there is a k such that $t_j - t_{j-1} = Q_k$.

Theorem 20. *Let α be any irrational real number, and $k \ge 0$ such that P_k is odd. Define u and v as those record-holders for which*

$$S(u) = \sup_{0\le n<Q_k} S(n), \qquad S(v) = \inf_{0\le n<Q_k} S(n).$$

Then the interval $[Q_k, Q_k + Q_{k+1})$ contains only one record-holder t.

If k is odd then t is a maximum and $t = v + Q_k$. If k is even then t is a minimum and $t = u + Q_k$.

Proof. By Theorem 17 the number $2Q_k - 1$ is an ECREF. By Lemma 15 we thus have

$$S(2Q_k - 1 - j) = S(2Q_k - 1) + S(j), \qquad 0 \le j \le \frac{2Q_k - 1}{2}.$$

Lemma 15 also gives us the value $S(2Q_k - 1) = (-1)^{\lfloor Q_k\alpha\rfloor}$. Thus by Lemma 4 we get $S(2Q_k - 1) = (-1)^{k+1}$, so that

$$S(2Q_k - 1 - j) = (-1)^{k+1} + S(j), \qquad 0 \le j < Q_k. \tag{24}$$

The values of $S(\ell)$ for $Q_k \le \ell < 2Q_k$ differ from those in the interval $0 \le \ell < Q_k$ by $(-1)^{k+1}$. So, for k odd, we get only one record-holder t in the interval $[Q_k, 2Q_k)$ that will be a maximum with $S(t) = S(u)+1$. For k even, we get only one record-holder in $[Q_k, 2Q_k)$ that will be a minimum with $S(t) = S(v) - 1$.

Equation (24) does not yield the exact position of the record-holders. Also, we have not proved that the found record-holder is the only one in the interval $[Q_k, Q_k + Q_{k+1})$. We will clarify these issues by applying Corollary 6: Given ℓ with $Q_k \le \ell < Q_k + Q_{k+1}$ and writting $\ell = hQ_k + r$ with $0 \le r < Q_k$ we will have

$$S(\ell) = \begin{cases} S(Q_k) - S(r) & \text{if } h \text{ is odd} \\ S(r) & \text{if } h \text{ is even.} \end{cases}$$

Thus, the values of $S(\ell)$ for $\ell \in [2Q_k, Q_k + Q_{k+1})$ have already been assumed for $\ell \in [0, 2Q_k)$. This proves that there are no new record-holders in the interval $[2Q_k, Q_k + Q_{k+1})$.

Assuming that k is odd, we know that there is one and only one record-holder $t \in [Q_k, 2Q_k)$ that is a maximum. Also for $\ell \in [Q_k, 2Q_k)$ we have $S(Q_k + r) = S(Q_k) - S(r)$, so that $S(t) = S(Q_k) - S(r)$. For this t to be a maximum, r must be a minimum for $S(r)$ with $r \in [0, Q_k)$. Also r must be the first "time" this minimum is attained. But then $r = v$ and we will have

$$t = v + Q_k, \qquad S(t) = S(Q_k) - S(v).$$

In case k is even we know that the only record-holder $t \in [Q_k, Q_{k+1})$ will be a miminum. So, in this case $t = Q_k + r$ and r must be the maximum of S for the interval $[0, Q_k)$. Hence

$$t = u + Q_k, \qquad S(t) = S(Q_k) - S(u).$$

□

Theorem 21. *Let α be any irrational real number, and $k \geq 0$ such that P_k is even. Define u and v as the record-holders satisfying*

$$S(u) = \sup_{0 \leq n < Q_k} S(n), \qquad S(v) = \inf_{0 \leq n < Q_k} S(n).$$

(a) If k is odd, then the record-holders in the interval $[Q_k, Q_{k+1} + Q_{k+2})$ are minima at the points $v + hQ_k < Q_k + Q_{k+1}$ with $h \geq 1$.

(b) If k is even, then the record-holders contained in $[Q_k, Q_{k+1} + Q_{k+2})$ are maxima at the points $u + hQ_k < Q_k + Q_{k+1}$ with $h \geq 1$.

Proof. Since P_k is even we have by Corollary 6

$$S(\ell) = (-1)^k h + S(r), \qquad Q_k \leq \ell < Q_k + Q_{k+1}$$

where $\ell = hQ_k + r$ with $0 \leq r < Q_k$.

If k is even then $S(\ell) = h + S(r)$. So, in this range of values of ℓ we have $S(\ell) = 1 + S(\ell - Q_k)$. On every interval $[sQ_k, (s+1)Q_k) \subset [Q_k, Q_k + Q_{k+1})$ the values taken by $S(\ell)$ are one unit above the corresponding values taken in the interval $[(s-1)Q_k, sQ_k)$. Therefore, in each of these intervals there is only one record-holder that is a maximum.

The record-holders are at the points $u + sQ_k < Q_k + Q_{k+1}$. In fact, let $\ell < u + sQ_k$ with $\ell = hQ_k + r$. Then either $h = s$ and $r < u$, or $h < s$. In the first case, by the definition of u, we have $S(r) < S(u)$ and $S(\ell) = s + S(r) < s + S(u)$. In the second case $S(\ell) = h + S(r) < s + S(r) \leq s + S(u)$. Therefore, the first time $S(\ell)$ takes the value $s + S(u)$ is at the point $sQ_k + u$.

By a similar argument it may be shown that, when k is odd, the points $v + sQ_k < Q_k + Q_{k+1}$ are the only record-holders in the interval $[Q_k, Q_k + Q_{k+1})$.

Let t be equal to v when k is odd, and equal to u when k is even. Then the last record-holder found satisfies

$$t + sQ_k < Q_k + Q_{k+1} \le t + (s+1)Q_k.$$

It follows that $Q_{k+1} \le t + sQ_k$. So, the last record-holder is larger than or equal to Q_{k+1}.

Since $(P_k, P_{k+1}) = 1$ and P_k is even, the number P_{k+1} is odd. Therefore, we may apply Theorem 20.

Let u' and v' be the record-holders satisfying

$$S(u') = \sup_{0\le n<Q_{k+1}} S(n), \qquad S(v') = \inf_{0\le n<Q_{k+1}} S(n).$$

(When k is odd and $s > 1$ we will have $v' = v + (s-1)Q_k$, and when k is even and $s > 1$ we will have $u' = u + (s-1)Q_k$. But the case $s = 1$ is also possible.)

In case k is odd there is only one record-holder in $[Q_{k+1}, Q_{k+1} + Q_{k+2})$ and it is a minimum at the point $u' + Q_{k+1}$. Since we know that $v + sQ_k$ is a record-holder in this range we will have

$$u' + Q_{k+1} = v + sQ_k, \qquad k \equiv 1 \pmod 2. \tag{25}$$

In case k is even there is only one record-holder in $[Q_{k+1}, Q_{k+1} + Q_{k+2})$ and it is a maximum at the point $v' + Q_{k+1}$. Reasoning as before we obtain

$$v' + Q_{k+1} = u + sQ_k, \qquad k \equiv 0 \pmod 2. \tag{26}$$

This proves that the found record-holders are the only ones in the interval $[Q_k, Q_{k+1} + Q_{k+2})$. □

In what follows the intervals

$$J_k = [Q_k, Q_k + Q_{k+1}) \tag{27}$$

will play a prominent role. For $k \ge 0$ we have $J_k \cap J_{k+1} \ne \emptyset$, but $J_k \cap J_{k+2} = \emptyset$. Theorems 20 and 21 yield all the record-holders contained in J_k when P_k is odd, or in $J_k \cup J_{k+1}$ when P_k is even. Since $Q_0 = 1$, we see that the union of the intervals J_k is all of $[1, +\infty)$. Thus, these two theorems determine all the record-holders. Also, every record-holder t in J_k is given by $t = t' + Q_k$ where t' is a previous record-holder. Namely, t' is either t_m or t_M, or one of these plus $(h-1)Q_k$. But it is not always the largest record-holder less than t. In fact, for some t we have two representations $t = t' + Q_k$ as may be seen from equations (25) and (26).

In the next theorem we will see that there is always a representation of t as $t' + Q_k$ where t' denotes the record-holder immediately preceding t.

Theorem 22. *For every $j \ge 1$ there is a $k \ge 0$ such that $t_j - t_{j-1} = Q_k$.*

Proof. We put $t_0 = 0$. The next record-holder is $t_1 = 1$. If $S(1) = 1$ then t_1 is a maximum, and if $S(1) = -1$ then t_1 is a minimum. Hence, in this case $t_1 - t_0 = 1 = Q_0$.

We proceed by induction. Assuming that $n > 1$ and that, for every $1 \le \ell < n$, we have found a j such that $t_\ell - t_{\ell-1} = Q_j$, we will determine the position of t_n.

Since $Q_0 = 1 \le Q_1 < \cdots$ and $\lim Q_n = \infty$, there is a k such that $t_n \in J_k$. Let $k \ge 0$ be the least integer with this property. We will prove that either $t_n - t_{n-1} = Q_k$ or $t_n - t_{n-1} = Q_{k-1}$.

(a) First consider the case in which P_k is even. Then by Theorem 21 the record-holders in the interval $J_k \cup J_{k+1}$ are of the type $t + hQ_k$ where $t = t_M$ or $t = t_m$ is a record-holder $< Q_k$. Therefore $t_n = t + hQ_k$. If $h > 1$ then $t_{n-1} = t + (h-1)Q_k$ and $t_n - t_{n-1} = Q_k$.

Now consider the subcase $h = 1$. Then $t_n = t + Q_k$ is the first record-holder $\ge Q_k$. Since $t_n > t_1 = 1 = Q_0$, and we know that t_n is the first record-holder in J_k, it follows that $k > 0$. Since $(P_{k-1}, P_k) = 1$ and P_k is even, the number P_{k-1} is odd. By Theorem 20 the interval J_{k-1} contains only one record-holder t'. By our choice of k we have $t_n \notin J_{k-1}$. Since t' is the only record-holder in J_{k-1} and t_n is the first record-holder $\ge Q_k$ it follows that $t' = t_{n-1} < Q_k$. Now we have to distinguish the two cases k odd and k even.

If k is odd, then by Theorem 21 all the record-holders in $J_k \cup J_{k+1}$ (in particular our t_n) are minima at the points $t_m + hQ_k$. On the other hand, by Theorem 20 (since $k-1$ is even) the only record-holder in J_{k-1} is a minimum. Thus $t' = t_m$ since it is a minimum and it is the last record-holder before Q_k. Hence, $t_n = t_m + Q_k = t' + Q_k = t_{n-1} + Q_k$.

If k is even, then by Theorem 21 all the record-holders in $J_k \cup J_{k+1}$ (in particular our t_n) are maxima at the points $t_M + hQ_k$. On the other hand, by Theorem 20 (since $k-1$ is odd) the only record-holder in J_{k-1} is a maximum. Thus $t' = t_M$ since t' is a maximum and it is the last record-holder before Q_k. Hence, $t_n = t_M + Q_k = t' + Q_k = t_{n-1} + Q_k$.

In summary, when P_k is even we always have $t_n - t_{n-1} = Q_k$.

(b) Now we consider the case P_k odd. By Theorem 20, t_n is the only record-holder in J_j and it is either a maximum, for k odd, with $t_n = t_m + Q_k$, or a minimum, for k even, with $t_n = t_M + Q_k$. Since $t_n > t_1 = 1 = Q_0$, and t_n being the only record-holder in J_k, we have $k > 0$.

(b1) If P_{k-1} is odd, again by Theorem 20, there is only one record-holder $t' \in J_{k-1}$. By our choice of k we have $t_n \notin J_{k-1}$, so that $t' < Q_k < Q_{k-1} + Q_k \le t_n$. It follows that $t' = t_{n-1}$.

Now, when k is odd, $k-1$ is even and t' is a minimum. It follows that in this case $t' = t_m$. Similarly, when k is even we have $t' = t_M$. It follows that for k odd $t_n = t_m + Q_k = t' + Q_k = t_{n-1} + Q_k$. Similarly, for k even we have $t_n = t_M + Q_k = t' + Q_k = t_{n-1} + Q_k$.

(b2) If P_{k-1} is even, then, by Theorem 21, the record-holders in $J_{k-1} \cup J_k$ are of the type $t + hQ_{k-1}$, where t is a record-holder with $t < Q_{k-1}$. Since t_n is the only record-holder in J_k, it is the last of the record-holders of the form $t + hQ_{k-1}$, and we can write $t_n = t + h_0 Q_{k-1}$. We have $h_0 > 1$ since

$$t + Q_{k-1} < Q_{k-1} + Q_{k-1} \le a_k Q_{k-1} + Q_{k-1} = Q_k \le t_n.$$

It follows that $t_{n-1} = t + (h_0 - 1)Q_{k-1}$ and $t_n - t_{n-1} = Q_{k-1}$.

So, for P_k odd we have $t_n - t_{n-1} = Q_{k-1}$ if P_{k-1} is even, and $t_n - t_{n-1} = Q_k$ if P_{k-1} is odd. □

Proposition 23. *If P_k is odd then the interval J_k contains one and only one record-holder. If P_k is even then the interval J_k contains a_{k+1} record-holders if $a_k > 1$ and $a_{k+1} + 1$ if $a_k = 1$.*

The only cases in which two of the intervals J_k contain a common record-holder are:

(a) If P_k is even then the only record-holder contained in J_{k+1} is the last record-holder contained in the interval J_k.

(b) If P_k is even and $a_k = 1$ then the only record-holder in J_{k-1} is the first record-holder contained in the interval J_k.

The number of record-holders t satisfying $t > 0$ and $t < Q_n + Q_{n+1}$ is

$$\sup\{m : t_m < Q_n + Q_{n+1}\} = \sum_{\substack{0 \le k \le n \\ P_k \text{ even}}} a_{k+1} + \sum_{\substack{0 \le k \le n \\ P_k \text{ and } P_{k-1} \text{ odd}}} 1. \tag{28}$$

P r o o f. For $k \ge -1$, $J_k \cap J_{k+2} = \emptyset$. So, we only need to consider $J_k \cap J_{k+1}$.

(1) If P_k and P_{k+1} are odd then, by Theorem 20, there is only one record holder $t' \in J_k$ and only one $t \in J_{k+1}$. One of them is a maximum and the other one a minimum. So, for $k \ge 0$ we have $t \ne t'$. (In case $k = -1$ the conclusion can be verified directly.) It follows that $t' \notin J_{k+1}$ and $t \notin J_k$. Therefore, in this case we have $Q_k \le t' < Q_{k+1} < Q_k + Q_{k+1} \le t$, and $J_k \cap J_{k+1} = \emptyset$. Also note that, since t and t' are a maximum and a minimum, by Theorem 20 we will have $t = t' + Q_{k+1}$.

(2) When P_k is even (by Theorem 21) the only record-holders in $J_k \cup J_{k+1} = [Q_k, Q_{k+1} + Q_{k+2})$ are the numbers $t + hQ_k < Q_k + Q_{k+1}$ with $h \ge 1$ and $t < Q_k$ a record-holder. Since P_{k+1} is odd there is only one record-holder in J_{k+1} which (as above) is always contained in J_k. So in this case the only record-holder in J_{k+1} is the last record-holder in J_k and $J_k \cap J_{k+1} \ne \emptyset$. This proves (a).

(3) When $k=0$, we have $J_{-1}=[0,1)$, $J_0=[1,1+Q_1)$ with $P_0=a_0$. Thus, P_0 even implies $a_0\neq 1$ and $J_{-1}\cap J_0=\emptyset$. So, assertion (b) is (vacuously) true for $k=0$. Thus, in what follows we may assume $k\geq 1$.

Assume $k\geq 1$, P_k even and $J_{k-1}\cap J_k\neq\emptyset$. We want to prove that $a_k=1$. Since P_{k-1} is odd there is only one record-holder $t\in J_{k-1}$ and $t=t'+Q_{k-1}$ with $t'<Q_{k-1}$. Since $t\in J_k$ we get $Q_k\leq t'+Q_{k-1}$. Therefore $(a_k-1)Q_{k-1}+Q_{k-2}\leq t'$. Since $k\geq 1$ we have $Q_{k-1}\geq 1$, and $a_k>1$ leads to $t'<Q_{k-1}\leq(a_k-1)Q_{k-1}\leq t'$ which is a contradiction.

Conversely, if $k\geq 1$, P_k is even and $a_k=1$, then since $(P_{k-1},P_k)=1$ the number P_{k-1} is odd and $P_k=P_{k-1}+P_{k-2}$ implies that also P_{k-2} is odd. Then, by case (1) considered above, the only record-holder $t\in J_{k-1}$ and the record-holder $t'\in J_{k-2}$ satisfy $Q_{k-2}<t'\leq Q_{k-1}\leq Q_{k-2}+Q_{k-1}<t$, and $t=t'+Q_{k-1}$. It follows that $Q_{k-2}+Q_{k-1}=Q_k\leq t$ and $t\in J_k$.

We only need to show how many record-holders are contained in J_k when P_k is even. They are all the numbers $t+hQ_k<Q_k+Q_{k+1}=(a_{k+1}+1)Q_k+Q_{k-1}$ with $h\geq 1$ and $t<Q_k$ a particular record-holder. In case $a_k>1$ this record-holder is $t\in J_{k-1}$ and $t\notin J_k$. Thus $Q_{k-1}\leq t<Q_k$. We see that the allowed values of h are $1\leq h\leq a_{k+1}$.

In case $a_k=1$ the only record-holder in J_{k-1} is contained in J_k so that $t<Q_{k-1}$ and it is easily seen that the allowed values of h are given by $1\leq h\leq a_{k+1}+1$.

Formula (28) is an easy consequence of the above results. $\square$

The above theorems justify the following procedure (written in Mathematica Version 5.2) in order to obtain the sequence of "all" record-holders. We assume that we have previously defined the numbers `P[n]` and `Q[n]` for `n < kMax` (`kMax` being an appropriate limit).

Program to compute the record-holders for α

```
T = {0}; (* T will contain the sequence of record-holders *)
t = 0 ;  (* The last obtained record - holder *)
For[n = 0, n <= kMax, n++,
  If[OddQ[P[n]],
  (* then *) If[t < Q[n],  t = t + Q[n]; T = Append[T, t]],
  (* else *) While[t + Q[n] < Q[n] + Q[n + 1], t + = Q[n]; T = Append[T, t]]
       ]
    ]; Print[T]
```

4. The case of a quadratic irrationality

In the case of a real quadratic irrational α the numbers P_k and Q_k can be given explicitly. We did not find the formulas in Proposition 24 in the standard textbooks dealing with continued fractions.

Proposition 24. *Let $\alpha \in \mathbb{Q}(\sqrt{d})$ be a real quadratic irrationality, and let k be the length of the period of the regular continued fraction of α. Then*

$$\begin{aligned} P_{nk+j} &= A_j\omega_1^n + B_j\omega_2^n, \\ Q_{nk+j} &= C_j\omega_1^n + D_j\omega_2^n, \end{aligned} \qquad 0 \le j < k, \quad n \ge n_0 \tag{29}$$

where ω_1 and ω_2 are certain conjugate units in the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$.

P r o o f. Let the continued fraction expansion of α be

$$\alpha = \{a_0; a_1, a_2, \dots, a_h, \overline{b_1, b_2, \dots, b_k}\} \tag{30}$$

and P_n/Q_n the corresponding convergents. We consider the number β with continued fraction $\{0; \overline{b_1, b_2, \dots, b_k}\}$. Let p_n/q_n be the convergents of β. Note that $p_0 = 0$, $p_1 = 1$, $q_0 = 1$ and $q_1 = b_1$.

It is well known that the two quadratic irrationals α and β generate the same field $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d})$.

For $n \ge 0$ and $1 \le j \le k$ we have

$$Q_{h+nk+j} = b_j Q_{h+nk+j-1} + Q_{h+nk+j-2}.$$

In particular

$$Q_{h+nk+1} = b_1 Q_{h+nk} + Q_{h+nk-1} = q_1 Q_{h+nk} + p_1 Q_{h+nk-1}.$$

We claim that in general for $n \ge 0$ and $1 \le j \le k$

$$Q_{h+nk+j} = q_j Q_{h+nk} + p_j Q_{h+nk-1}.$$

We will prove this by induction on the number j. Assuming that we have proved the result for all numbers less than $j+1$ we have

$$\begin{aligned} Q_{h+nk+j+1} &= b_{j+1} Q_{h+nk+j} + Q_{h+nk+j-1} \\ &= b_{j+1}(q_j Q_{h+nk} + p_j Q_{h+nk-1}) + q_{j-1} Q_{h+nk} + p_{j-1} Q_{h+nk-1} \\ &= (b_{j+1} q_j + q_{j-1}) Q_{h+nk} + (b_{j+1} p_j + p_{j-1}) Q_{h+nk-1} \\ &= q_{j+1} Q_{h+nk} + p_{j+1} Q_{h+nk-1}. \end{aligned}$$

We can write this equation in matrix form

$$\mathbf{Q_n} := \begin{pmatrix} Q_{h+kn+k} \\ Q_{h+kn+k-1} \\ Q_{h+kn+k-2} \\ \cdots \\ Q_{h+kn+1} \end{pmatrix} = \mathbf{\Omega} \begin{pmatrix} Q_{h+kn} \\ Q_{h+kn-1} \\ Q_{h+kn-2} \\ \cdots \\ Q_{h+kn-k+1} \end{pmatrix} = \mathbf{\Omega}\ \mathbf{Q_{n-1}}, \qquad n \geq 0$$

where $\mathbf{\Omega}$ is defined by

$$\mathbf{\Omega} = \begin{pmatrix} q_k & p_k & 0 & \dots & 0 \\ q_{k-1} & p_{k-1} & 0 & \dots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ q_1 & p_1 & 0 & \dots & 0 \end{pmatrix}.$$

The linear transformation $\mathbf{\Omega}$ has $k-2$ eigenvalues equal to 0 whereas the other two are the solutions of the equation

$$\begin{vmatrix} q_k - \omega & p_k \\ q_{k-1} & p_{k-1} - \omega \end{vmatrix} = \omega^2 - (q_k + p_{k-1})\omega + (-1)^k = 0.$$

Its two solutions are algebraic integers. We call them ω_1 and ω_2. They are quadratic irrationals in the same field $\mathbb{Q}(\alpha)$. In order to see this we prove that the discriminant of ω is the same as that of β. In fact, β is the solution of the quadratic equation

$$\frac{p_k + \beta p_{k-1}}{q_k + \beta q_{k-1}} = \beta \quad \text{or} \quad q_{k-1}\beta^2 + (q_k - p_{k-1})\beta - p_k = 0.$$

Therefore, the discriminant of β is

$$\begin{aligned} \Delta = (q_k - p_{k-1})^2 + 4q_{k-1}p_k = (q_k - p_{k-1})^2 + 4(q_k p_{k-1} + (-1)^{k-1}) \\ = (q_k + p_{k-1})^2 - 4(-1)^k \end{aligned}$$

which coincides with the discriminant of ω.

Since $\omega_1\omega_2 = (-1)^k$ and ω_1 and ω_2 are not rational, exactly one of them is in absolute value larger than 1. This one we call ω_1.

Let $\mathbf{u_1}$ and $\mathbf{u_2}$ be the corresponding eigenvectors. Every vector in the image of $\mathbf{\Omega}$ is a linear combination of these two vectors. In particular there exist constants A and B such that $\mathbf{Q_0} = A\mathbf{u_1} + B\mathbf{u_2}$.

Then we have

$$\mathbf{Q_n} = \mathbf{\Omega}^n(A\mathbf{u_1} + B\mathbf{u_2}) = A\omega_1^n\mathbf{u_1} + B\omega_2^n\mathbf{u_2}.$$

That is, for $n \geq 0$ and $1 \leq j \leq k$,

$$Q_{h+kn+j} = AU_j\omega_1^n + BV_j\omega_2^n$$

where U_j and V_j are the coordinates of the two eigenvectors. So, U_j and V_j are conjugate numbers in the field $Q(\alpha)$. □

Proposition 25. *Let $\alpha \in \mathbb{Q}(\sqrt{d})$ be a real quadratic irrationality, and k the length of the (pure) period of the continued fraction of α. Then there are natural numbers b, L, K an integer multiple of k, and a function ϕ such that the sequence of record-holders for α satisfies*

$$t_{b+nL+j} - t_{b+nL+j-1} = Q_{nK+\phi(j)}, \qquad 0 \le j < L, \quad n \ge 0. \tag{31}$$

Also, there exists a finite sequence of signs $(\varepsilon_j)_{j=1}^L$ such that

$$\varepsilon_j S(t_{b+nL+j}) > 0, \qquad n \ge 0. \tag{32}$$

Let M be the number of j such that $\varepsilon_j = 1$, and m the number of j with $\varepsilon_j = -1$. Then $m + M = L$, and

$$S(t_{b+nL+j}) = \begin{cases} nM + a_j & \text{if } \varepsilon_j = 1 \\ -nm + a_j & \text{if } \varepsilon_j = -1. \end{cases} \tag{33}$$

P r o o f. Assume that α has the continued fraction (30). For $n \ge 1$ the pair (P_n, P_{n+1}) modulo 2 has only three possible values $(1,0)$, $(0,1)$ and $(1,1)$. Therefore, there exists a multiple K of k (K will be k, $2k$ or $3k$) and $c > h+2$ such that $P_{c-2} \equiv P_{c+K-2}$ and $P_{c-1} \equiv P_{c+K-1}$. Given the periodicity of the partial quotients of α and the recursive formulas (6), we will have

$$P_{c+m} \equiv P_{c+K+m}, \quad (\text{mod } 2), \qquad m \ge -2. \tag{34}$$

Without loss of generality we may assume that K is even (if necessary take $2K$ instead of K).

By Proposition 23 the number of record-holders in the interval J_n depends only on the parity of P_n and the numbers a_n and a_{n+1}. Since K is a multiple of the period k it follows from (34) that for $n \ge c$ the intervals J_n and J_{n+K} contain the same number of record-holders. By Propositions 20 and 21 the characters maximum/minimum of these record-holders will be the same since K has been taken even.

Therefore, the number, characters and relative positions of the record-holders in the union of intervals $\bigcup_{j=0}^K J_{c+nK+j}$ do not depend on n.

Let t_b, t_{b+1}, …, t_{b+L-1} be the record-holders contained in $\bigcup_{j=0}^K J_{c+j}$. Then the record-holders contained in $\bigcup_{j=0}^K J_{c+nK+j}$ will be the numbers t_{b+nL+j} with $0 \le j < L$. The numbers t_{b+nL+j} with a fixed j are either all maxima or all minima. Put $\varepsilon_j = 1$ for a maximum and $\varepsilon_j = -1$ for a minimum. Then, obviously, we will have $\varepsilon_j S(t_{b+nL+j}) > 0$.

If $n \geq b$ and if the record-holder $t_n \in J_v$ then the record-holder $t_{n+L} \in J_{v+K}$. This record-holder will be the only one in the given interval or will have the same position between the record holders in the respective intervals J_v and J_{v+K}. It follows that if $t_n - t_{n-1} = Q_r$ then $t_{n+L} - t_{n+L-1} = Q_{r+K}$. Thus we can define a function ϕ such that

$$t_{b+j+nL} - t_{b+j+nL-1} = Q_{\phi(j)+nK}, \qquad 0 \leq j < L.$$

Finally, let M be equal to the number of maxima in the period of record-holders, and m the number of minima. If t_{b+nL+j} is a maximum, then $t_{b+nL+L+j}$ is also a maximum and $S(t_{b+nL+L+j}) = M + S(t_{b+nL+j})$ since there are M maxima on the period of record-holders. This fact, together with reasoning similar to that for the case of minimum establishes our formula for $S(t_{b+nL+j})$. □

Corollary 26. *With the same notations as in Theorem 24 and Proposition 25*

$$t_{b+nL+j} = E_j + F_j\ \omega_1^{\kappa n} + G_j\ \omega_2^{\kappa n}, \qquad 0 \leq j < L, \quad n > n_1 \tag{35}$$

where $K = \kappa k$, κ being a positive integer.

P r o o f. Summing equations 31 for $0 \leq j < L$ we get

$$t_{b+(n+1)L-1} - t_{b+nL-1} = \sum_{j=0}^{L-1} Q_{nK+\phi(j)}.$$

For every fixed j let $\phi(j) = u_j k + r$ with $0 \leq r < k$. Then $nK + \phi(j) = (n\kappa + u_j)k + r$ and by (29), for $n \geq n_1$, we will have

$$Q_{nK+\phi(j)} = C_j\ \omega_1^{n\kappa+u_j} + D_j\ \omega_2^{n\kappa+u_j}.$$

Thus there exist constants n_1, C'_j and D'_j such that

$$t_{b+(n+1)L-1} - t_{b+nL-1} = C'_j\omega_1^{n\kappa} + D'_j\omega_2^{n\kappa}, \qquad n \geq n_1.$$

Summing this for $n_1 \leq n \leq N-1$ we get

$$t_{b+NL-1} = t_{b+n_1L-1} + \sum_{n=n_1}^{N-1} C'_j\omega_1^{n\kappa} + D'_j\omega_2^{n\kappa}.$$

Summing the geometric series we see that there are constants E_{-1}, F_{-1} and G_{-1} such that

$$t_{b+NL-1} = E_{-1} + F_{-1}\omega_1^{N\kappa} + G_{-1}\omega_2^{N\kappa}, \qquad N > n_1.$$

From this equation, by induction, we obtain (35). For example:

$$\begin{aligned}
&t_{b+NL} = E_{-1} + F_{-1}\omega_1^{N\kappa} + G_{-1}\omega_2^{N\kappa} + Q_{NK+\phi(0)} = \\
&= E_{-1} + F_{-1}\omega_1^{N\kappa} + G_{-1}\omega_2^{N\kappa} + C_0\omega_1^{N\kappa} + D_0\omega_2^{N\kappa} = E_0 + F_0\omega_1^{N\kappa} + G_0\omega_2^{N\kappa}.
\end{aligned}$$

□

5. Connection with modular functions

The functions

$$\Lambda(\alpha) = \limsup_{n\to\infty} \frac{S_\alpha(n)}{\log n}, \quad \lambda(\alpha) = \liminf_{n\to\infty} \frac{S_\alpha(n)}{\log n}$$

are finite at the points α for which $S_\alpha(n) = \mathcal{O}(\log n)$, in particular at real quadratic irrationals.

We may restrict ourselves to one of them since $S_{-\alpha}(n) = -S_\alpha(n)$ (by (2)) so that

$$\lambda(\alpha) = -\Lambda(-\alpha). \tag{36}$$

Defining $H(\alpha) = \Lambda(1-\alpha)$ we have

Theorem 27. *For every real irrational α*

$$H(\alpha+2) = H(\alpha), \qquad H\Big(-\frac{1}{\alpha}\Big) = H(\alpha). \tag{37}$$

P r o o f. By (3) in Lemma 1 we have $\Lambda(\alpha+2) = \Lambda(\alpha)$, so that $H(\alpha+2) = H(\alpha)$.

It is convenient to extend the definition of $S_\alpha(n)$. For any real number x we put

$$S_\alpha(x) = \sum_{1\le n\le x} (-1)^{\lfloor n\alpha\rfloor}.$$

It is easy to show that with this definition we have

$$\Lambda(\alpha) = \limsup_{x\to\infty} \frac{S_\alpha(x)}{\log x}, \quad \lambda(\alpha) = \liminf_{x\to\infty} \frac{S_\alpha(x)}{\log x}.$$

Now we prove that

$$\alpha \in \mathbf{I}, \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad \Rightarrow \quad \Lambda(\alpha) = -\lambda(\beta); \quad \lambda(\alpha) = -\Lambda(\beta). \tag{38}$$

(Here $\mathbf{I}$ stands for the set of all irrational real numbers.) Assume that $\alpha > 1$ is irrational and that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. A well known theorem by Beatty says that $\lfloor n\alpha\rfloor$

and $\lfloor n\beta \rfloor$ then form a partition of the natural numbers. This can be translated into a property of the sums $S_\alpha(n)$. In [15] we found the proof of

$$S_\alpha(x/\alpha) + S_\beta(x/\beta) = \boldsymbol{\mathcal{O}}(1)$$

based on Beatty's theorem.

Dividing by $\log x$ we get

$$\begin{aligned}\Lambda(\alpha) = \limsup_{x\to+\infty} \frac{S_\alpha(x/\alpha)}{\log(x/\alpha)} &= \limsup_{x\to+\infty} \frac{S_\alpha(x/\alpha)}{\log x} = \\ &= \limsup_{x\to+\infty} \frac{\boldsymbol{\mathcal{O}}(1) - S_\beta(x/\beta)}{\log x} = -\liminf_{x\to+\infty} \frac{S_\beta(x/\beta)}{\log(x/\beta)} = -\lambda(\beta).\end{aligned}$$

In the same way we get

$$\lambda(\alpha) = -\Lambda(\beta).$$

We can write the main equation in (38) in the form

$$\Lambda(\alpha) = -\lambda\Big(\frac{\alpha}{\alpha-1}\Big), \qquad \text{for} \quad \alpha \in \mathbf{I}, \quad \alpha > 1. \tag{39}$$

For every irrational $y < 0$ we have

$$\begin{aligned}H(y) = \Lambda(1-y) = -\lambda\Big(\frac{1-y}{-y}\Big) &= \\ &= -\lambda\Big(1 - \frac{1}{y}\Big) = \Lambda\Big(\frac{1}{y} - 1\Big) = \Lambda\Big(1 + \frac{1}{y}\Big) = H\Big(-\frac{1}{y}\Big).\end{aligned}$$

(The first equality is the definition of H, the second an application of (39) with $1 - y > 1$, the third an algebraic identity, the fourth an application of (36), the fifth an application of the first equation in (37), and the last one also an application of the definition of H.)

But then, by the symmetry of this equation, it is true for all $y \in \mathbf{I}$. □

Theorem 28. *For α a quadratic irrational and with the same notations used in Theorem 24 and Proposition 25 we have*

$$\Lambda(\alpha) = \frac{M}{\kappa \log \omega_1} \quad \textit{and} \quad \lambda(\alpha) = -\frac{m}{\kappa \log \omega_1}. \tag{40}$$

Proof. If $M = 0$, then there is at most a finite number of maximum-record-holders and a constant C such that $S(n) \le C$ for all $n \in \mathbb{N}$. So $\Lambda(\alpha) = 0$ and equation (40) is true.

When $M > 0$ there are infinitely many maximum-record-holders. So, given x, there is a record-holder satisfying $t_{b+(n-1)L+j} < x \le t_{b+nL+j}$. By Corollary 35 there is a constant C (of the order of $|\omega_1^\kappa|$) such that $t_{b+nL+j} \le Ct_{b+(n-1)L+j}$. It follows that $\log x \sim \log t_{b+nL+j}$ and we have

$$\Lambda(\alpha) = \limsup_{x\to+\infty} \frac{S_\alpha(x)}{\log x} \le \limsup_{n\to\infty} \frac{S(t_{b+nL+j})}{\log t_{b+nL+j}} =$$
$$= \limsup_{n\to\infty} \frac{nM + a_j}{\log(E_j + F_j\omega_1^{n\kappa} + G_j\omega_2^{n\kappa})} = \frac{M}{\kappa \log \omega_1}.$$

Since this is the limit of $S_\alpha(n)/\log n$ for a particular sequence, it is also less than or equal to the $\limsup = \Lambda(\alpha)$, proving (40). □

6. The order of the sums

O'Bryant, Reznick and Serbinowska wonder in [15] whether $S_\alpha(n) = \boldsymbol{\mathcal{O}}(\log n)$ is the correct type of growth of $S_\alpha(n)$ for a quadratic irrational α . They say that it seems unlikely that $S_\alpha(n) = \boldsymbol{\mathcal{O}}(\log n)$ for almost all α, but a proof of this is elusive. They also ask for necessary and sufficient conditions on α (in terms of its continued fraction expansion) in order to have $S_\alpha(n) = \boldsymbol{\mathcal{O}}(\log n)$.

In this section we will answer the first and second of these questions.

Applying Theorem 28 to a real quadratic irrational α, we have

$$\limsup_n S_\alpha(n)/\log n > \liminf_n S_\alpha(n)/\log n$$

and both are finite real numbers. Hence $S_\alpha(n) = \boldsymbol{\mathcal{O}}(\log n)$ and $S_\alpha(n) = \Omega(\log n)$. Thus, $\boldsymbol{\mathcal{O}}(\log n)$ is the correct rate of growth of $S_\alpha(n)$ for any fixed real quadratic irrational α.

Our Theorem 18 shows that for any irrational α the sum $S_\alpha(n)$ is not bounded. This is about all that can be said in general. In [15] it is proved that for any positive function $\psi(n) \ge 1$ that increases to infinity, we can find an α with $|S_\alpha(n)| \le \psi(n)$ for all n.

There is an important connection between our sums and the concept of discrepancy. Given a sequence of real numbers (x_j) in the interval $[0,1]$, its discrepancy $D_n^*(x_j)$ is defined by

$$D_n^*(x_j) = \sup_{t\in[0,1)} \Bigl|\frac{\#\{j \le n : 0 \le x_j < t\}}{n} - t\Bigr|.$$

It is easy to show that (see, for example, [15])

$$S_\alpha(n) = 2n\Bigl(\frac{\#\{j \le n : \{j\alpha/2\} < \frac12\,\}}{n} - \frac12\Bigr).$$

Here, as usual, $\{x\}$ denotes the fractional part of x. So we have

$$|S_\alpha(n)| \leq 2nD_n^*(\{j\alpha/2\}).$$

There is a substantial literature on the discrepancy of the sequence $\{n\alpha\}$ (a good survey can be found in [10]). For example, the following result is due to Khintchine: *Let $\varphi(n)$ be a positive increasing function. Then*

$$\sup_{n\leq N} nD_n^*(\{j\alpha\}) = \boldsymbol{\mathcal{O}}(\log N \cdot \varphi(\log\log N))$$

for almost all $\alpha \in \mathbb{R}$ if and only if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty.$

It follows that for almost all $\alpha \in (0,1)$ we have

$$\sup_{n\leq N} |S_\alpha(n)| \leq \sup_{n\leq N} nD_n^*(\{j\alpha/2\}) = \boldsymbol{\mathcal{O}}(\log N \cdot \varphi(\log\log N))$$

when $\varphi(n)$ is an increasing positive function with $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$.

We are going to extend this result to the following

Theorem 29. *Let $\psi(x)$ and $\varphi(x)$ be positive increasing functions such that*

$$\int_1^\infty \frac{\mathrm{d}x}{\psi(x)} = +\infty \quad \textit{and} \quad \int_1^\infty \frac{\mathrm{d}x}{\varphi(x)} < +\infty.$$

Then for almost all $\alpha \in (0,1)$ we have

$$\begin{aligned}\Omega(\log N \cdot \psi(\log\log N)) \leq \sup_{n\leq N} |S_\alpha(n)| \leq \\ \leq \sup_{n\leq N} nD_n^*(\{j\alpha/2\}) = \boldsymbol{\mathcal{O}}(\log N \cdot \varphi(\log\log N)).\end{aligned} \tag{41}$$

P r o o f. We only need to prove the first inequality.

By Proposition 23 the number of record-holders $< Q_n + Q_{n+1}$ is larger than or equal to the sum

$$\sum_{\substack{0\leq k\leq n\\ P_k \text{ even}}} a_{k+1}.$$

It follows that the maximum of $|S_\alpha(k)|$ for $k < Q_n + Q_{n+1}$ is, when P_n is even, at least $\frac{1}{2}\, a_{n+1}$. Thus, with $m = n+1$ we have (once more using Iverson's notation)

$$\sup_{n\leq Q_m+Q_{m-1}} |S(n)| \geq \frac{1}{2} \; [\![\; P_{m-1} \text{ even} \;]\!] \; a_m. \tag{42}$$

Now we need some measure theory. Let $E \subset [0,1]$ be the set of those irrational numbers $\alpha \in [0,1]$ for which the partial quotients $a_1 = k_1$, $a_2 = k_2$, …, $a_r = k_r$

take definite values, and let $E(k)$ be the subset of these numbers for which the additional partial quotient a_{r+1} is equal to k. Then we have (see [1, p. 60])

$$\frac{|E|}{3k^2} < |E(k)| \le \frac{2|E|}{k^2}. \tag{43}$$

For every m put $M_\alpha(m) = \sup_{n\le Q_m+Q_{m-1}} |S_\alpha(n)|$. We are going to show that, given the set E (with $k \ge 2$), we have

$$|\{\alpha \in E : M_\alpha(k+3) \ge L\}| \ge \frac{|E|}{200L}.$$

All numbers in E have regular continued fraction expansions starting with

$$\{0; k_1, k_2, \dots, k_r, \dots\}.$$

Thus, they share the same convergents P_0/Q_0, …, P_{r-1}/Q_{r-1}, P_r/Q_r. Since $(P_{r-1}, P_r) = 1$ modulo 2, these two numbers can be: both odd $(1,1)$, the first even and the second odd $(0,1)$, or the first odd and the second even $(1,0)$ (for all numbers in the set E).

Now we decompose E in four disjoint subsets $E_{1,1}$, $E_{1,0}$, $E_{0,1}$, and $E_{0,0}$ defined by:

$$E_{1,1} = \{\alpha \in E : a_{r+1} \equiv 1 \pmod 2, \quad a_{r+2} \equiv 1 \pmod 2\}$$

with similar definitions for $E_{1,0}$, $E_{0,1}$, $E_{0,0}$.

We have to consider three cases.

Assume first that $(P_{r-1}, P_r) \equiv (1,1)$ modulo 2. In this case all the numbers $\alpha \in E_{0,1}$ satisfy

$$P_{r+1} = a_{r+1}P_r + P_{r-1} \equiv P_{r-1} \equiv 1 \pmod 2$$

and

$$P_{r+2} = a_{r+2}P_{r+1} + P_r \equiv P_{r+1} + P_r \equiv 0 \pmod 2.$$

Therefore, by (42), for $\alpha \in E_{0,1}$ we have

$$M_\alpha(r+3) = \sup_{n\le Q_{r+3}+Q_{r+2}} |S_\alpha(n)| \ge \frac{a_{r+3}}{2}$$

so that we have

$$\{\alpha \in E : M_\alpha(r+3) \ge L\} \supset \bigcup_{k\ge 2L} E_{0,1}(k).$$

Here $E_{0,1}(k)$ denotes the set of those $\alpha \in E$ for which a_{r+1} is even, a_{r+2} is odd and $a_{r+3} = k$.

By (43) we get

$$\begin{aligned}|\{\alpha\in E: M_\alpha(r+3)\geq L\}| &\geq \sum_{k\geq 2L}\sum_{u=1}^\infty\sum_{v=1}^\infty \frac{1}{3k^2}\frac{1}{3(2u-1)^2}\frac{1}{3(2v)^2}|E|\\ &= |E|\frac{1}{27}\frac{\pi^2}{8}\frac{\pi^2}{24}\sum_{k\geq 2L}\frac{1}{k^2}\geq |E|\frac{\pi^4}{5184}\frac{1}{2L}\geq\frac{|E|}{200L}.\end{aligned}$$

Similarly, when $(P_{r-1},P_r)\equiv(0,1)$ we get

$$\{\alpha\in E: M_\alpha(r+3)\geq L\}\supset\bigcup_{k\geq 2L}E_{1,1}(k)$$

and an analogous computation gives

$$\begin{aligned}|\{\alpha\in E: M_\alpha(r+3)\geq L\}| &\geq \sum_{k\geq 2L}\sum_{u=1}^\infty\sum_{v=1}^\infty \frac{1}{3k^2}\frac{1}{3(2u-1)^2}\frac{1}{3(2v-1)^2}|E|\\ &= |E|\frac{1}{27}\frac{\pi^2}{8}\frac{\pi^2}{8}\sum_{k\geq 2L}\frac{1}{k^2}\geq |E|\frac{\pi^4}{1728}\frac{1}{2L}\geq\frac{|E|}{200L}.\end{aligned}$$

Finally, in the last case $(P_{r-1},P_r)\equiv(1,0)$ we have

$$\{\alpha\in E: M_\alpha(r+3)\geq L\}\supset\bigcup_{k\geq 2L}E_{0,0}(k)\cup E_{1,0}(k).$$

In this case we also have

$$|\{\alpha\in E: M_\alpha(r+3)\geq L\}|\geq\frac{|E|}{200L}.$$

Therefore, for every set E (defined by conditions on the values of the partial quotients a_j for $j\leq r$) we have

$$|\{\alpha\in E: M_\alpha(r+3)< L\}|\leq\Bigl(1-\frac{1}{200L}\Bigr)|E|. \tag{44}$$

Observe that every set determined by a condition on the numbers $M_\alpha(n)$ for $n\leq m$ is the union of the sets $E=E(a_1,a_2,\dots,a_m)$ determined by the values of the partial quotients a_j with $j\leq m$. In fact, the $M_\alpha(n)$ are determined by the values of $S_\alpha(n)$ for $n<Q_m+Q_{m-1}$ and, by Proposition 5, these values are determined by the a_j with $j\leq m$.

Let $h(x)$ be any positive nondecreasing function such that $\int_1^\infty \mathrm{d}x/h(x)=+\infty$. We claim that $M_\alpha(n)=\Omega(h(n))$ for almost all α, or, more precisely

$$|\{\alpha\in\mathbf{I}: M_\alpha(n)\geq h(n)\text{ for infinitely many values of }n\}|=1. \tag{45}$$

(Here $\mathbf{I}$ denotes the set of all irrational numbers in $[0,1]$).

This is equivalent to showing that for every m_0

$$|\{\alpha \in \mathbf{I} : M_\alpha(n) < h(n) \text{ for } n \geq m_0\}| = 0.$$

For a given value of m_0 let us call this set F. Then we have

$$|F| \leq |\{\alpha \in I : M_\alpha(m_0+3) < h(m_0+3)\}| \leq \Big(1 - \frac{1}{200h(m_0+3)}\Big)$$

because we may apply (44) to every interval $E(a_1, \dots, a_{m_0})$ contained in F. Writing $F_1 = \{\alpha \in I : M_\alpha(m_0+3) < h(m_0+3)\}$ we also have by a similar reasoning

$$\begin{aligned}|F| \leq |\{\alpha \in F_1 : M_\alpha(m_0+6) < h(m_0+6)\}| &\leq \Big(1 - \frac{1}{200h(m_0+6)}\Big)|F_1| \\ &\leq \Big(1 - \frac{1}{200h(m_0+6)}\Big)\Big(1 - \frac{1}{200h(m_0+3)}\Big).\end{aligned}$$

By induction we get

$$|F| \leq \prod_{j=1}^{N}\Big(1 - \frac{1}{200h(m_0+3j)}\Big).$$

Since ψ is increasing, we know that $\sum_{j=1}^{\infty} \frac{1}{h(m_0+3j)} = \infty$. Therefore, the above inequality implies $|F| = 0$. This completes the proof of our claim.

Now, there exists an absolute constant $B > 0$ such that almost everywhere, for sufficiently large n (see [1, p. 65]),

$$Q_n = Q_n(\alpha) < e^{Bn}.$$

Given the positive nondecreasing function $\psi(x)$ with $\int_1^{+\infty} \mathrm{d}x/\psi(x) = +\infty$ we define $h\colon (0,+\infty) \to (0,+\infty)$ in such a way that for all $x > 30$ (note that $h(x)$ is also positive and nondecreasing)

$$\log x \cdot \psi(\log\log x) = h\Big(\frac{\log x - 4}{B}\Big).$$

Then

$$\begin{aligned}\int_1^{+\infty} \frac{\mathrm{d}x}{h(x)} = \int_1^{+\infty} \frac{1}{\log y \cdot \psi(\log\log y)}\frac{\mathrm{d}y}{By} = \\ = \int_1^{+\infty} \frac{\mathrm{d}u}{Bu \cdot \psi(\log u)} = \int^{+\infty} \frac{\mathrm{d}v}{\psi(v)} = +\infty.\end{aligned}$$

Therefore, by (45), for almost all $\alpha \in [0,1]$ we have $h(m) \le M_\alpha(m)$ for infinitely many m. That is

$$h(m) \le \sup_{n<Q_m+Q_{m-1}} |S_\alpha(n)|.$$

For almost all $\alpha \in [0,1]$ we have $Q_n < e^{Bn}$ for $n \ge n_0(\alpha)$. Therefore, for infinitely many m we have

$$h(m) \le \sup_{n\le Q_m+Q_{m-1}} |S_\alpha(n)| \le \sup_{n\le 2e^{Bm}} |S_\alpha(n)|.$$

Taking the integer N such that $N/2 \le 2e^{Bm} < N$ we get

$$\log N \cdot \psi(\log\log N) = h\Big(\frac{\log N - \log 4}{B}\Big) \le \sup_{n\le N} |S_\alpha(n)|$$

which will be true for infinitely many integers N. □

Note 1. In [15] the authors say: "It seems unlikely that $S_\alpha(n)$ is $\mathcal{O}(\log n)$ for almost all α, but a proof of this is elusive". From the above theorem we infer that $\log N \cdot \log\log N \le \sup_{n\le N} |S_\alpha(n)|$ for almost all α. So, indeed, it is not true that $S_\alpha(n) = \mathcal{O}(\log n)$ for almost all α.

Note 2. The conclusion

$$\Omega(\log N \cdot \psi(\log\log N)) \le \sup_{n\le N} nD_n^*(\{j\alpha/2\})$$

is known (see [10]), but our proof by means of $S_\alpha(n)$ seems to be more straightforward.

Example. Theorem 29 is true for almost all α. We can also give concrete examples in which $S_\alpha(n)$ grows quite fast. For example: For $\alpha = (e-1)/(e+1) = \{0, 2, 6, 10, 14, \dots\}$ with $a_k = 4k-2$, all numbers P_{2k} are even. By (28) we easily get that there is a $t < Q_{2n} + Q_{2n+1}$ with $|S(t)| \ge 2n^2$. By induction we get $Q_n < 4^n n!$, so that $\log t \le c\, n \log n$, and consequently $n > c \log t / \log\log t$. It follows that there is an infinite sequence (t_n) tending to infinity, such that

$$|S(t_n)| > c\Big(\frac{\log t_n}{\log\log t_n}\Big)^2.$$

7. Mathematica programs

We now present some Mathematica programs that, given a real quadratic irrational α, will output the regular continued fraction of α, the period of signs of $S(t_n)$, the length of this period, the sign sequence for a period, the numbers ω_1 and ω_2, an initial section of the maxima/minima t_n, a section of the sequence of record-holders, and the corresponding values of $S(t_n)$. As a check it also gives the values of $S(t_n)$ computed with the FFL algorithm described in the Appendix.

We should note here that the function `ContinuedFraction[.]` of Mathematica does *not* always yield the correct value. We have actually detected this malfunctioning when computing `ContinuedFraction[` α `]` for $\alpha = 2 - \sqrt{3} > 0$ by Mathematica Version 5.2.

So, we felt compelled to introduce a correct substitute function:

`RegularContinuedFraction[.]`.

Mathematica Code for the Module RegularContinuedFraction[.].

```
RegularContinuedFraction[α_] := Module[{RCF, x, X, RepeatQ, a, n},
RCF = {}; x = α; X = {x};
RepeatQ = False;  (* = True when a repetition has been detected *)
While[Not[RepeatQ], a = Floor[x];
                    RCF = Append[RCF, a];
                    x = FullSimplify[1/(x - a)];
                    If[MemberQ[X, x], (* then *) RepeatQ = True,
                                      (* else *) X = Append[X, x]]];
n = Position[X, x, {1}, Heads -> False][[1]][[1]] - 1;
(* Output *)   RCF = Append[Take[RCF, n], Drop[RCF, n] ]   ];
```

Mathematica Code for the Module AnalysisOfTheRecordHolders.

```
AnalysisOfTheRecordHolders[α_] := Module[{n, k, x, y, u, v},
(* ============================================================== *)
(* 1. Compute the units : ω1 and ω2. *)
CFα = RegularContinuedFraction[α];
PurePeriodOfCFα = Last[CFα];
LengthPurePeriodα = Length[PurePeriodOfCFα];
β = FromContinuedFraction[{0, PurePeriodOfCFα}];
                                 (* = β of Proposition 24. *)
kLim = LengthPurePeriodα + 2;
CFβ = ContinuedFraction[β, kLim];
                                 (* For β we use lower case p and q *)
p[-2] = 0; p[-1] = 1;
For[k = 0, k < kLim, k++, p[k] = CFβ[[k + 1]]*p[k - 1] + p[k - 2]];
q[-2] = 1; q[-1] = 0;
For[k = 0, k < kLim, k++, q[k] = CFβ[[k + 1]]*q[k - 1] + q[k - 2]];
roots = Solve[
```

```
    x^2 - (q[LengthPurePeriodα] + p[LengthPurePeriodα - 1])x +
                              (-1)^LengthPurePeriodα == 0, x];
ω1 = FullSimplify[x /. roots[[2]]]; ω2 = FullSimplify[x /. roots[[1]]];
                                   (* Interchange if Abs[ω1] < Abs[ω2] *)
If[Abs[ω1] < Abs[ω2], {ω1, ω2} = {ω2, ω1}];        (* Swap *)
(* ================================================================== *)
(*  2. Compute the convergents of α.  *)
(* We need the P[k] for k <= h + 3 * LengthPurePeriodα *)
                                   (* See proof of Theorem (25) *)
h = Length[CFα] - 2;               (* Notation of equation (30) *)
kMax = Max[ h + 3*LengthPurePeriodα, 10];
For[u = 0, u + α < 0, u++];        (* This is not necessary for  α > 0 *)
CF = ContinuedFraction[u + α, kMax]; CF[[1]] = CF[[1]] - u;
                                   (* ! We use CAPITAL P and Q  for  α *)
P[-2] = 0; P[-1] = 1;
For[k = 0, k < kMax, k++, P[k] = CF[[k + 1]]*P[k - 1] + P[k - 2]];
Q[-2] = 1; Q[-1] = 0;
For[k = 0, k < kMax, k++, Q[k] = CF[[k + 1]]*Q[k - 1] + Q[k - 2]];
(* We have computed the necessary P's and Q's  for  α *)
(* ================================================================== *)
(*  3. Compute the sequence of Record - Holders for α. *)
T = {0};              (* T will contain the sequence of record - holders *)
t = 0;                (* t = the last obtained record - holder *)
For[n = 0, n < kMax, n++,
     If[OddQ[P[n]], (* then *) If[t < Q[n], t = t + Q[n]; T = Append[T, t]],
                   (* else *)
                    While[t + Q[n] < Q[n] + Q[n + 1], t = t + Q[n];
                                                  T = Append[T, t]]]];
(* We have computed the list of Record - Holders *)
(* ================================================================== *)
(*  4. Compute K and c satisfying (34) *)
(* h = Length[CF[α] - 2] is the number defined by equation (30) *)
For[n = 0, n < 4, n++,
A[n] = {Mod[P[h + n*LengthPurePeriodα - 1], 2],
                                Mod[P[h + n*LengthPurePeriodα], 2]}];
n = 0; m = 1;
          (* Now we are going to find the first repetition of the A[n] *)
While[A[n] ≠ A[m], If[m < 4, m++, If[n < 3, n++; m = n + 1]]];
m - =  n;
If[OddQ[m*LengthPurePeriodα], m = 2m];
c = h + n*LengthPurePeriodα + 1;
K = m*LengthPurePeriodα;
κ = K/LengthPurePeriodα; (* This is the κ of Corollary 26 *)
    (* We have computed c, K and κ of Corollary 26 *)
(* ================================================================== *)
(* 5. Now compute the Record - Holders < Q[c + K + 2]  *)
If[c + K + 3 >= kMax,
 For[u = 0, u + α < 0, u++];   (* This is not necessary for  α > 0 *)
```

```
 CF = ContinuedFraction[u + α, c + K + 5]; CF[[1]] = CF[[1]] - u;
             (* ! Here we use CAPITAL P and Q *)
 For[k = kMax - 1, k < c+K+5, k++, P[k] = CF[[k + 1]]*P[k - 1] + P[k - 2]];
 For[k = kMax - 1, k < c+K+5, k++, Q[k] = CF[[k + 1]]*Q[k - 1] + Q[k - 2]];
        (* We have used the computed  P's and Q's of 2 *)
 kMax = c + K + 5]; (* End of If[c + K + 3 >= kMax, ... *)
MM = {}; mm = {};
          (* MM and mm will contain the maxima and minima, respectively *)
t = 0;
For[n = 0, n < kMax - 1, n++,    (* We inspect the interval J_n *)
    If[OddQ[P[n]],
        (* Then there is only one record holder *)
        If[t < Q[n], t = t + Q[n];
                     If[OddQ[n], (* then *)  MM = Append[MM, t],
                                 (* else *) mm = Append[mm, t]]],
        (* Else there are several record - holders in J[n] *)
             news = {};
             For[v = 1, t + v*Q[n] < Q[n] + Q[n + 1], v++,
                 news = Append[news, t + v*Q[n]]];
                 If[OddQ[n], (* then *) mm = Union[mm, news],
                             (* else *) MM = Union[MM, news]];
                 t = Last[news]]];
    (* Now MM and mm contain the record - holders < Q[c + K + 2] *)
(* ================================================================== *)
(* 6. Compute the maxima and minima of the period *)
(* The period will be composed of the t_n contained
                                        in [Q[c + 2], Q[c + K + 2]) *)
q0 = Q[c + 2]; q1 = Q[c + K + 2];
Maxima = Select[MM, (# < q1) && (# >= q0) &];
minima = Select[mm, (# < q1) && (# >= q0) &];
M = Length[Maxima]; m = Length[minima];
    (* These are the M and m of Theorem 25 *)
(* ================================================================== *)
(* 7. Compute the period of maxima and minima *)
'period = {}; (* Will contain the period of maxima and minina *)
For[j = 1, j <= M, j++, period = Append[period, {Maxima[[j]], "max"}]];
For[j = 1, j <= m, j++, period = Append[period, {minima[[j]], "min"}]];
period = Sort[period];
Lperiod = Table[period[[j]][[2]], {j, 1, Length[period]}];
      (* Now we get the non - periodic part of the record - holders *)
Maxima = Select[MM, (# > 0) && (# < q0) &];
minima = Select[mm, (# > 0) && (# < q0) &];
M2 = Length[Maxima]; m2 = Length[minima];
(* M2 and m2 are the # of maxima and minima in the Sign - period *)
nonperiodic = {};(* Will contain the non - periodic maxima and minima *)
                 (* These have never been observed, though *)
For[j = 1, j <= M2, j++,
       nonperiodic = Append[nonperiodic, {Maxima[[j]], "max"}]];
```

```
For[j = 1, j <= m2, j++,
       nonperiodic = Append[nonperiodic, {minima[[j]], "min"}]];
nonperiodic = Sort[nonperiodic];
nonperiodic = Table[nonperiodic[[j]][[2]], {j, 1, Length[nonperiodic]}];
period = Append[nonperiodic, Lperiod];
(* ============================================================== *)
(* 8. We simplify the period *)
(* Now we simplify the period {max, {max, min, max}} -> {{max, max, min}} *)
u = Length[Last[period]]; v = Length[period] - 1;
While[(v > 0) && (period[[v]] == Last[Last[period]]),
    period = Insert[period, RotateRight[Last[period], 1], -1];
    period = Delete[period, -2]; period = Delete[period, -2];
    u = Length[Last[period]]; v = Length[period] - 1];
(* We simplify the pure period {max, min, max, min} -> {max, min}  *)
lp = Length[Last[period]]; div = Divisors[lp];
pureperiod = Last[period];
v = 1; CheckValue = False;
While[CheckValue == False, CheckValue = True;
                          d = div[[v]];
                          For[j = 1, j <= d, j++,
                             For[k = 0, k < lp/d, k++,
                               If[pureperiod[[j]] != pureperiod[[d*k + j]],
                                   CheckValue = False]]];
                          v++];
period = Append[Drop[period, -1], Take[Last[period], d]];


(*  9. Print the results of the above analysis *)
Print["* α = ", α];
Print["* Type of S-extremes in period = ", period];
Print["* Length of this period = ", Length[period[[1]]]];
PeriodRecordHolders = Table[S[α, T[[n]]], {n, 2, 1 + 2*Length[period[[1]]]}];
    (* ! We only observed pure periods  ! *)
Print["* Regular CF(α) = ", RegularContinuedFraction[α]];
Print["* Sign sequence of S(t_n) -> ", Sign[PeriodRecordHolders]];
Print["* Units :     ω1 = ", ω1,"       ω2 = ", ω2];
Print["* t_n of Maxima -> ", MM];
Print["* t_n of minima -> ", mm];
T = Delete[T, 1];
Print["* 'All' Record-Holders t_n -> ", T];
(* The next line requires the loading of the FFL routine of the Appendix *)
Print["* S(t_n) -> ", Table[S[α, T[[j]]], {j, 1, Length[period[[1]]]}]];
Print["* κ = ", κ];
Print["* Λ(α) = ", Λ = FullSimplify[  M/(κ*Log[ω1])], " ≈ ", N[Λ]];
Print["* λ(α) = ", λ = FullSimplify[ - m/(κ*Log[ω1])], " ≈ ", N[λ]];
"Done"];
```

One may check the record-holders for α by the following simple program

Mathematica Code for Generating the Record-Holders.

```
α = √2   (* For example *)
  n = 0; s = 0; sMax = 0; sMin = 0;
  While[0 == 0, n += 1;
    If[EvenQ[Floor[n*α]], s += 1, s -= 1];
    If[s > sMax, sMax = s; Print[" n= ", n, "   s= ", s]; Goto[A]];
    If[s < sMin, sMin = s; Print[" n= ", n, "   s= ", s]];
    Label[A]]
```

8. Some remaining open problems

1. In all of our computations, the sequence of the signs of $S_\alpha(t_n)$ always turned out to be *purely* periodic.

   We were unable to prove the consistency of this surprising observation.
2. It seems that there is always a system of recurrence relations for the record-holders t_n. For example, for $\alpha = \sqrt{3}$ we find that

$$\begin{aligned} t_{4n} &= 2\,t_{4n-1} + t_{4n-4} + 1 \\ t_{4n+1} &= t_{4n} + t_{4n-1} + 1 \\ t_{4n+2} &= t_{4n+1} + 2\,t_{4n} + 1 \\ t_{4n+3} &= t_{4n+2} + 2\,t_{4n} + 1. \end{aligned}$$

   We have not pursued this subject any further.
3. It seems that the *inhomogeneous* sums $\sum_{j=1}^{n}(-1)^{\lfloor j\alpha+\beta\rfloor}$ exhibit certain characteristics very similar to those of the *homogeneous* sums dealt with in this paper. For example for the sums $\sum_{j=1}^{n}(-1)^{\lfloor j\sqrt{2}+\frac{1}{2}\rfloor}$ we find the recurrences:

$$\begin{aligned} t_{2n} &= 2\,t_{2n-1} - t_{2n-4} \\ t_{2n+1} &= 3\,t_{2n} + 1. \end{aligned}$$

4. The *probabilistic* distribution of the values of $S_\alpha(n)$ for $n = 1, 2, 3, \ldots$ appears to be very *regular and stable.* Is there a Gaussian distribution lurking in the background?
5. Finally there is the problem of the distribution of $S_\alpha(n)$ over the residue classes mod m (with $m > 2$).
6. Although we did not study general irrational α's, we observed various regularities of $S_\alpha(n)$ for $\alpha =$ a *simple* form composed with the number e.

For example, $\alpha = e$, $e^{1/m}$, $e^{-1/m}$, $\frac{e^{1/m}-1}{e^{1/m}+1}$. It seems that for the last of these the record-holders are given by

$$t_k = \begin{cases} k & \text{if } 1 \le k \le 2m \\ 2(4mr - m + 1)t_{k-1} - t_{k-2} & \text{if } k = 4mr^2 - 2mr + 1,\ (r > 1) \\ 2t_{k-1} - t_{k-2} & \text{if } k \ne 4mr^2 - 2mr + 1,\ (r > 1). \end{cases}$$

9. Appendix. The FFL algorithm

This algorithm is implicitly contained in [9]. It computes the value of $S_\alpha(n)$ for *any* irrational α. Here we present its implementation in Mathematica Version 5.2 and present a proof of it.

In the first part of the algorithm, applying Lemma 1, we determine an irrational $\beta \in (0, 1)$ and a sign σ such that for every natural number n we have $S_\alpha(n) = \sigma S_\beta(n)$.

```
S[α_, M_] := Module[{j, R, m, a, b, k},
(* ========================================================== *)
      (* 1. Compute β and σ such that Sα(n) = σ · Sβ(n) *)
(* ========================================================== *)
σ = 1; β = α;
If[β < 0, β = -β; σ = -σ];                 (* Now β > 0 *)
β - = 2 Floor[β/2];                      (* Now 0 < β < 2 *)
If[β > 1, β = 2 - β; σ = -σ];            (* Now 0 < β < 1 *)
```

In order to compute the value of $S_\beta(n)$ the FFL algorithm uses the denominators q_k of the convergents of the number $\gamma = \beta/2$. We will have to compute these q_k for the indices k satisfying $q_{k-1} \le n < q_k$. By induction we find that $q_k \ge F_{k+1}$, a Fibonacci number. So, we compute q_k for all k such that $F_{k-1} \le \left(\frac{1+\sqrt{5}}{2}\right)^k \le n$.

```
(* ========================================================== *)
      (* 2. Compute the necessary q[k] for M *)
(* ========================================================== *)
β = β/2;
kMax = 1 + Ceiling[Log[2, M + 1]/Log[(1+Sqrt[5])/2]];
CF=ContinuedFraction[ β, kMax ];
```

```
q[-2]=1; q[-1]=0;
For[ k = 0, k < kMax, k++, q[k] = CF[[k+1]]* q[k-1]+ q[k-2] ];
```

Finally, the algorithm depends on two principles:

(A) If $q_{k-1} \leq m < q_k$ and $q_k/2 < m$ then

$$S_\beta(m) = S_\beta(q_k - m - 1) + \begin{cases} (-1)^{k-1} & \text{if } q_k \text{ is even} \\ 0 & \text{if } q_k \text{ is odd.} \end{cases}$$

(B) If $q_{k-1} \leq m < q_k$, $m \leq q_k/2$, and $m = hq_{k-1} + r$ then

$$S_\beta(m) = S_\beta(r) + h \begin{cases} 0 & \text{if } q_{k-1} \text{ is even} \\ (-1)^{k-1} & \text{if } q_{k-1} \text{ is odd.} \end{cases}$$

Now we can supply the code for the function $S_\alpha(n)$.

```
(* ============================================================ *)
      (* 3. The FFL-routine proper *)
(* ============================================================ *)
m=M; R=0;
    (* Throughout the program  we will have  S(M)= S(m)+ R *)
j = kMax - 1;
While[ m > 0, While[q[j] > m, j--];
              (* We have located j with  q[j] <= m < q[j+1] *)
              a = q[j];  b = q[j+1];
              If[ b - m < b/2, (* Then we apply principle A *)
                                  R += If[ EvenQ[b], (-1)^j, 0];
                                  m = b - m - 1,
                                 (* Else we apply principle B *)
                  R += Floor[m/a] * If[EvenQ[a], 0, (-1)^j ];
                                  m = Mod[m,a] ]];
              (* end of  While. *)
(* Output *)  sigma * R ]
(* END of the FFL-routine and S[α,M] *)
```

We prove the two principles of the algorithm.

(A) By Theorem 17 the number $2q_k - 1$ is an ECREF for γ so that

$$\lfloor (2q_k - j)\gamma \rfloor + \lfloor j\gamma \rfloor = 2p_k - 1, \qquad 1 \leq j < q_k. \tag{46}$$

Put $j = 2n$. Since $\gamma = \beta/2$ we have

$$\lfloor (q_k - n)\beta \rfloor + \lfloor n\beta \rfloor = 2p_k - 1, \qquad 1 \leq n < q_k/2.$$

Therefore

$$1 \le n < q_k/2 \quad \Longrightarrow \quad (-1)^{\lfloor n\beta \rfloor} = -(-1)^{\lfloor (q_k - n)\beta \rfloor}.$$

Thus $q_k - 1$ is an ECREF for β and we have

$$S_\beta(q_k - 1 - n) = S_\beta(q_k - 1) + S_\beta(n), \qquad 0 \le n < q_k/2.$$

Moreover, we have

$$S_\beta(q_k - 1) = \sum_{j=1}^{q_k - 1} (-1)^{\lfloor j\beta \rfloor} = \sum_{j=1}^{q_k - 1} (-1)^{\lfloor 2j\gamma \rfloor}.$$

By (46) we have $(-1)^{\lfloor 2j\gamma \rfloor} = -(-1)^{\lfloor 2(q_k - j)\gamma \rfloor}$. If q_k is odd we get $S_\beta(q_k - 1) = 0$, and for q_k even the entire sum is equal to the central term $(-1)^{\lfloor q_k\gamma \rfloor}$. By Lemma 4 this term is equal to $(-1)^{k+p_k}$. Since q_k is even, p_k is odd, and we get $S_\beta(q_k - 1) = (-1)^{k-1}$. Writing $m = q_k - 1 - n$ we get

$$S_\beta(m) = (-1)^{k-1} + S_\beta(q_k - 1 - m), \qquad q_k/2 < m < q_k$$

completing the proof of (A).

(B) Now assume that $q_{k-1} \le m \le q_k/2$. Let h and $0 \le r < q_{k-1}$ be such that $m = hq_{k-1} + r$. Let j be such that $q_{k-1} \le j < j + q_{k-1} \le m < q_k/2$ and let $2j = dq_{k-1} + u$. By applying Proposition 5 twice, first for $2j + 2q_{k-1}$ and then for $2j$, we obtain

$$\begin{aligned}(-1)^{\lfloor (j+q_{k-1})\beta \rfloor} = (-1)^{\lfloor (2j+2q_{k-1})\gamma \rfloor} &= (-1)^{(d+2)p_{k-1} + \lfloor u\gamma \rfloor} = \\ &= (-1)^{dp_{k-1} + \lfloor u\gamma \rfloor} = (-1)^{\lfloor 2j\gamma \rfloor} = (-1)^{\lfloor j\beta \rfloor}.\end{aligned}$$

(This for $u \neq 0$ and by a similar reasoning for $u = 0$.) It follows that

$$S_\beta(m) = \sum_{j=1}^{hq_{k-1}+r} (-1)^{\lfloor j\beta \rfloor} = \sum_{j=1}^{r} (-1)^{\lfloor j\beta \rfloor} + h \sum_{j=1}^{q_{k-1}} (-1)^{\lfloor j\beta \rfloor}.$$

As observed in the proof of (A), the terms of the sum $\sum_{j=1}^{q_{k-1}} (-1)^{\lfloor j\beta \rfloor}$ cancel (the term for $j = 1$ with that for $j = q_{k-1} - 1$, the term with $j = 2$ with that for $j = q_{k-1} - 2, \dots$). If q_{k-1} is odd the sum is equal to $(-1)^{\lfloor 2q_{k-1}\gamma \rfloor}$, and if q_{k-1} is even it is equal to $(-1)^{\lfloor q_{k-1}\gamma \rfloor} + (-1)^{\lfloor 2q_{k-1}\gamma \rfloor}$.

By Proposition 5 we see that $\lfloor 2q_{k-1}\gamma \rfloor$ is equal to $2p_{k-1}$ when $k-1$ is even, and equal to $2p_{k-1} - 1$ when $k-1$ is odd. Thus in each case $(-1)^{\lfloor 2q_{k-1}\gamma \rfloor} = (-1)^{k-1}$. So, when q_{k-1} is odd we have $\sum_{j=1}^{q_{k-1}} (-1)^{\lfloor j\beta \rfloor} = (-1)^{k-1}$.

When q_{k-1} is even we have

$$\sum_{j=1}^{q_{k-1}} (-1)^{\lfloor j\beta \rfloor} = (-1)^{k-1} + (-1)^{\lfloor q_{k-1}\gamma \rfloor}.$$

Also we have $(-1)^{\lfloor q_{k-1}\gamma \rfloor} = (-1)^{p_{k-1}+k-1}$, and q_{k-1} being even, p_{k-1} is odd. Therefore $(-1)^{\lfloor q_{k-1}\gamma \rfloor} = (-1)^k$ and $(-1)^{\lfloor q_{k-1}\gamma \rfloor} + (-1)^{\lfloor 2q_{k-1}\gamma \rfloor} = 0$. This completes the proof of (B).

Acknowledgment. The authors would like to thank Foster Dieckhoff (Kansas City, MO) for his linguistic assistance in preparing this paper, and his interest in our results.

REFERENCES

[1] KINTCHINE, A.YA.: *Continued fractions*, Reprint of the 1964 translation, Dover, Mineola N. Y., 1997.

[2] KRESTEN, H.: *On a conjecture of Erdös and Szüsz related to uniform distribution mod 1*, Acta Arith. **12** (1966), 193–212.

[3] RUDERMAN, H.D.: *Problem 6105**, Amer. Math. Monthly **83** (1976), 573.

[4] VAN DE LUNE, J.: *On the convergence of some "irregularly" oscillating series*, Afdeling Zuivere Wiskunde, Report ZW **86/76**, Mathematical Centre, Amsterdam, 1976.

[5] BROUWER, A.E. – VAN DE LUNE, J.: *A Note on Certain Oscillating Sums*, Afdeling Zuivere Wiskunde, Report ZW **90/76**, Mathematical Centre, Amsterdam, 1976.

[6] BUNDSCHUH, P.: *Konvergenz unendlicher Reihen und Gleichverteilung mod 1*, Arch. Math. **29** (1977), 518–523.

[7] BORWEIN, D.: *Solution to problem no. 6105*, Amer. Math. Monthly **85** (1978), 207–208.

[8] BORWEIN, D. – GAWRONSKI, W.: *On certain sequences of plus and minus ones*, Canad. J. Math. **30** (1978), 170–179.

[9] FOKKINK, R. – FOKKINK, W. – VAN DE LUNE, J.: *Fast Computation of an Alternating Sum*, Nieuw Archief voor Wiskunde **12** (1994), 13–18.

[10] DRMOTA, M.–TICHY, R.F.: *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics **1651**, Springer-Verlag, Berlin, Heidelberg, 1997.

[11] BOURBAKI, N.: *General Topology*, Springer-Verlag, Berlin, 1998, Chapters 5–10.

[12] SCHOISSENGEIER, J.: *The integral mean of discrepancy of the sequence* $(n\alpha)$, Monatsh. Math. **131** (2000), 227–234.

[13] SERBINOWSKA, M.: *A case of an almost alternating series*, Unpublished manuscript (2003), available from the author on request.

[14] SCHOISSENGEIER, J. – TRIČKOVIĆ, S.B.: *On the divergence of a certain series*, J. Math. Anal. Appl. **324** (2006), 238–247.
[15] O'BRYANT, K. – REZNICK, B. – SERBINOWSKA, M.: *Almost alternating sums*, Amer. Math. Monthly **113** (2006), 673–688.
[16] FOSTER, J.H. – SERBINOWSKA, M.: *On the Convergence of a Class of Nearly Alternating Series*, Canad. J. Math. **59** (2007), 85–108.
[17] SCHOISSENGEIER, J.: *On the Convergence of a Series of Bundschuh*, Uniform Distribution Theory **2** (2007), no. 1, 107–113.

Received May 21, 2008
Accepted July 22, 2008

J. Arias de Reyna
Facultad de Matemáticas
Universidad de Sevilla, Apdo. 1160
41080-Sevilla
SPAIN
E-mail: arias@us.es

J. van de Lune
Langebuorren 49
9074 CH Hallum
(formerly at CWI, Amsterdam)
THE NETHERLANDS
E-mail: j.vandelune@hccnet.nl