FIXED POINT THEOREMS FOR MULTIVALUED NONEXPANSIVE MAPPINGS WITHOUT UNIFORM CONVEXITY

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Abstract. Let X be a Banach space whose characteristic of noncompact convexity is less than 1 and satisfies the non-strict Opial condition. Let C be a bounded closed convex subset of X, $KC(C)$ the family of all compact convex subsets of C and T a nonexpansive mapping from C into $KC(C)$. We prove that T has a fixed point. The non-strict Opial condition can be removed if, in addition, T is an 1-χ-contractive mapping.

1. Introduction

Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued mappings. The first results in this direction were established by J. Markin [12] in a Hilbert space setting and by F. Browder [3] for spaces having a weakly continuous duality mapping. E. Lami Dozo [9] generalized these results to a Banach space satisfying Opial's condition.

By using Edelstein's method of asymptotic centers, T.C. Lim [10] obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space. W. A. Kirk and S. Massa [7] gave an extension of Lim's theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact.

Many questions remain open (see [15] and [16]) about the existence of fixed points for multivalued nonexpansive mappings when the Banach space satisfies geometric properties which assure the existence of a fixed point in the singlevalued case, for instance, if X is a nearly uniformly convex space. In this paper we state some fixed point theorems for multivalued nonexpansive self-mappings, which are more general than the previous results. First, we give a fixed point theorem for a multivalued nonexpansive and 1-χ-contractive mapping in the framework of a Banach space

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whose characteristic of noncompact convexity associated to the separation measure of noncompactness is less than 1. If, in addition, the space satisfies the non-strict Opial condition, we prove, using some properties of χ-minimal sets (see [2, Chapter III] for definitions), that the χ-contractiveness assumption can be removed. In particular, this result gives a partial answer to the above open question.

2.PRELIMINARIES AND NOTATIONS

Let X be a Banach space. We denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $K(X)$ (resp. $KC(X)$) the family of all nonempty compact (resp. compact convex) subsets of X. On $CB(X)$ we have the Hausdorff metric H given by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \quad A,B\in CB(X)$$

where for $x \in X$ and $E \subset X$ $d(x,E) := \inf\{d(x,y) : y \in E\}$ is the distance from the point x to the subset E.

If C is a closed convex subset of X, then a multivalued mapping $T : C \to CB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \leq k\|x-y\|, \quad x,y \in C,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|, \quad x,y \in C.$$

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameter} \leq d\},$$

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radius} \leq d\}.$$

Then a multivalued mapping $T : C \to 2^X$ is called γ-condensing (resp. 1-γ-contractive) where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset B of C with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp.} \quad \gamma(T(B)) \leq \gamma(B)).$$

Here $T(B) = \cup_{x\in B}Tx$. Note that a multivalued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous. There is another different kind of continuity for set-valued operators: $T : X \to CB(X)$ is said to be continuous on X (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see [1] and [5]) that both definitions of continuity are equivalent if Tx is compact for every $x \in X$. We say that $x \in C$ is a fixed point of T if and only if x is contained in Tx.

In the next section we shall use the following result for multivalued mappings (see also [14]).

Theorem 2.1 ([4]). *Let X be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous γ-condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ on D then $Fix(F) \neq \emptyset$. (Here $I_D(x)$ is called the inward set at x defined by $I_D(x) := \{x+\lambda(y-x) : \lambda \geq 0, y \in D\}$).*

Let us recall some definitions of properties satisfied by a Banach space X:

Definition 2.1. *(a) X is said to be nearly uniformly convex (NUC) if it is reflexive and its norm is uniformly Kadec-Klee, that is, for any positive number ϵ there exists a corresponding number $\delta = \delta(\epsilon) > 0$ such that for any sequence $\{x_n\}$*

$$\left.\begin{array}{c} \|x_n\| \leq 1 \quad n = 1, 2, ... \\ w - \lim_n x_n = x \\ sep(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\} \geq \epsilon \end{array}\right\} \Longrightarrow \|x\| \leq 1 - \delta.$$

(b) X is said to satisfy the Opial condition if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|.$$

If the inequality is non strict we say that X satisfies the non-strict Opial condition.

3.ASYMPTOTIC CENTERS AND MODULI OF NONCOMPACT CONVEXITY

In this section we shall consider, apart from α and χ, another measure of noncompactness. The separation measure of noncompactness of a nonempty bounded subset B of X is defined by

$$\beta(B) = \sup\{\epsilon : \text{ there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geq \epsilon\}.$$

Definition 3.1. *Let X be a Banach space and $\phi = \alpha,\ \beta$ or χ. The modulus of noncompact convexity associated to ϕ is defined in the following way*

$$\Delta_{X,\phi}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \textit{ is convex, } \phi(A) \geq \epsilon\}.$$

(B_X is the unit ball of X).

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\epsilon_\phi(X) = \sup\{\epsilon \geq 0 : \Delta_{X,\phi}(\epsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\epsilon) \leq \Delta_{X,\beta}(\epsilon) \leq \Delta_{X,\chi}(\epsilon),$$

and consequently

$$\epsilon_\alpha(X) \geq \epsilon_\beta(X) \geq \epsilon_\chi(X).$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated with β and χ,

$$\Delta_{X,\beta}(\epsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X,\ x = w - \lim_n x_n,\ sep(\{x_n\}) \geq \epsilon\},$$

$$\Delta_{X,\chi}(\epsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X,\ x = w - \lim_n x_n,\ \chi(\{x_n\}) \geq \epsilon\}.$$

It is known that X is NUC if and only if $\epsilon_\phi(X) = 0$, where ϕ is $\alpha,\ \beta$ or χ. The above-mentioned definitions and properties can be found in [2].

Let C be a nonempty bounded closed subset of X and $\{x_n\}$ a bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$r(C, \{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in C\},$$

$$A(C, \{x_n\}) = \{x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\})\}.$$

It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set as C is.

Next, we present a theorem which gives a connection between the asymptotic center of a sequence and the modulus of noncompact convexity and it will play a crucial role in our results. Previously, recall the following notation of regularity and the lemma below.

Definition 3.2. *Let $\{x_n\}$ and C be as above. Then $\{x_n\}$ is called regular with respect to (w.r.t.) C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.*

Lemma 3.1 (Goebel[6], Lim[11]). *Let $\{x_n\}$ and C be as above. Then, there always exists a subsequence of $\{x_n\}$ which is regular w.r.t. C.*

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Theorem 3.1. *Let C be a closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be a bounded sequence in C which is regular w.r.t. C. Then*

$$r_C(A(C, \{x_n\})) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).$$

Moreover, if X satisfies the non-strict Opial condition then

$$r_C(A(C, \{x_n\})) \leq (1 - \Delta_{X,\chi}(1^-))r(C, \{x_n\}).$$

Proof. Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. Since $\overline{co}(\{x_n\}) \subset C$ is a weakly compact set, we can find a subsequence $\{y_n\}$ of $\{x_n\}$ weakly convergent to a point $z \in C$. Without loss of generality we assume that the limit $\lim_{n\neq m} \|y_n - y_m\|$ exists (see [2,Theorem III.1.5]). Since $\{x_n\}$ is regular w.r.t. C , $r = r(C, \{y_n\})$ and then, the weakly lower semicontinuity of the norm implies

$$r \leq \limsup_n \|y_n - z\| \leq \liminf_m \limsup_n \|y_n - y_m\| = \lim_{n\neq m} \|y_n - y_m\|.$$

Hence $\beta(\{y_n\}) \geq r$.

On the other hand, if X satisfies the non-strict Opial condition, it is easy to deduce that $\chi(\{y_n : n \in \mathbb{N}\}) = \limsup_n \|y_n - z\|$. Indeed, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|y_n - y\| < \limsup_n \|y_n - y\| + \epsilon$ for all $n \geq n_0$, and hence $\chi(\{y_n : n \in N\}) \leq \limsup_n \|y_n - y\|$.

Conversely let us suppose that $\{y_n : n \in \mathbb{N}\}$ can be covered by finitely many balls with radius $r < \limsup_n \|y_n - y\|$. Consider a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\lim_n \|z_n - z\| = \limsup_n \|y_n - z\|$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ contained in a ball $B(x, r)$ for some $x \in X$. Therefore we obtain

$$\limsup_k \|z_{n_k} - x\| \leq r < \limsup_n \|y_n - z\| = \lim_k \|z_{n_k} - z\|,$$

contradicting the fact that X satisfies the non-strict Opial condition, because $z_{n_k} \rightharpoonup z$.

Thus, in this case we have $\chi(\{y_n : n \in \mathbb{N}\}) \geq r$.

Assume x lies in A. Since $r = \limsup_n \|y_n - x\|$, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|y_n - x\| < r + \epsilon$ for all positive integer n greater than or equal to n_0. Hence, the sequence

$$\left\{ \frac{y_n - x}{r + \epsilon} \right\}_{n \geq n_0}$$

is contained in the unit ball of X, converges weakly to $\dfrac{z - x}{r + \epsilon}$ and $\beta\left(\left\{ \frac{y_n - x}{r+\epsilon} \right\}\right) \geq \frac{r}{r+\epsilon}$.

If X satisfies the non-strict Opial condition X we also have that $\chi\left(\left\{ \frac{y_n - x}{r+\epsilon} \right\}\right) \geq \frac{r}{r+\epsilon}$. Therefore we deduce

$$\|x - z\| \leq \left(1 - \Delta_{X,\beta,\tau}\left(\frac{r}{r + \epsilon}\right)\right)(r + \epsilon),$$

and in the second assumption

$$\|x - z\| \leq \left(1 - \Delta_{X,\chi,\tau}\left(\frac{r}{r + \epsilon}\right)\right)(r + \epsilon).$$

Since the last inequality is true for every $\epsilon > 0$ and for every $x \in A$, we obtain the inequalities in the statement.

Remark 3.1.-It must be noted that the regularity assumption is necessary in the above theorem. Indeed, consider the product space $X = \ell_\infty^2 \otimes \ell_2$, where $\ell_\infty^2 := (\mathbb{R}^2, \|\cdot\|_\infty)$, with the norm

$$\|(x,y)\| = \left(\|x\|_\infty^2 + \|y\|_2^2 \right)^{\frac{1}{2}}, \; x \in \ell_\infty^2, \, y \in \ell_2.$$

First, we are going to prove that

$$\Delta_{X,\alpha}(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}.$$

Since X contains isometrically ℓ_2, it is easy to deduce that

$$\Delta_{X,\alpha}(\epsilon) \leq \Delta_{\ell_2,\alpha}(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$$

(see [2, Chapter I]).

Let us now study the reverse inequality. Taking in mind that $\alpha(A) \leq 2\chi(A)$ ([2]) for each bounded subset of X, it is clear that

$$\Delta_{X,\alpha}(\epsilon) \geq \Delta_{X,\chi}\left(\frac{\epsilon}{2}\right),$$

for all $\epsilon > 0$. Let us estimate the value of $\Delta_{X,\chi}(\frac{\epsilon}{2})$. Since X is reflexive, we have (see [2, Chapter V])

$$\Delta_{X,\chi}\left(\frac{\epsilon}{2}\right) = \inf\{1 - \|z\| : w - \lim_n z_n = z, \; \|z_n\| \leq 1, \; \chi(\{z_n\}) \geq \frac{\epsilon}{2}\}.$$

Let $\{(x_n, y_n)\}$ be a sequence in the unit ball of X weakly convergent to a vector $(x_o, y_o) \in X$ such that $\chi(\{(x_n, y_n)\}) \geq \frac{\epsilon}{2}$.

It follows that $\lim_n x_n = x_o$ and $\{y_n\}$ is weakly convergent to y_o in ℓ_2. Taking a subsequence if necessary, we can assume that $\lim_n \|y_n - y_o\|_2$ and $\lim_n \|y_n\|_2$ exist, and the supports of $y_n - y_o$ and y_o are nearly disjoint, that is

$$\lim_n \|y_n\|_2^2 = \|y_o\|_2^2 + \lim_n \|y_n - y_o\|_2^2,$$

On the other hand, it is not difficult to see that X satisfies the Opial condition. In fact, it satisfies the uniform Opial condition with the same modulus of Opial associated with ℓ_2. Then

$$\chi(\{(x_n, y_n)\}) = \limsup_n \|(x_n, y_n) - (x_o, y_o)\| = \lim_n \|y_n - y_o\|_2 \geq \frac{\epsilon}{2}.$$

Hence

$$\begin{aligned} 1 \geq \lim_n \|(x_n, y_n)\|^2 &= \lim_n \|x_n\|_\infty^2 + \|y_n\|_2^2 \\ &= \|x_o\|_\infty^2 + \|y_o\|_2^2 + \lim_n \|y_n - y_o\|_2^2 \\ &= \|(x_o, y_o)\|^2 + \lim_n \|y_n - y_o\|_2^2 \\ &\geq \|(x_o, y_o)\|^2 + \frac{\epsilon^2}{4}. \end{aligned}$$

Thus
$$\Delta_{X,\chi}(\frac{\epsilon}{2}) \geq 1-\sqrt{1-\frac{\epsilon^2}{4}},$$
following the required inequality.

Moreover, since X is reflexive and satisfies the uniform Opial condition then $\Delta_{X,\chi}(1^-)=1$ (see Chapter V in [2] for details).

If $x_n \in \mathbb{R}^2$ is the sequence defined by $x_{2n-1}=(-1,0)$ y $x_{2n}=(1,0)$ for each $n \in \mathbb{N}$, we consider the sequence $z_n=(x_n,0)\in X$.

Denote B the unit ball of ℓ^2_∞ and let $C=B\times\{0\}$. Clearly C is a weakly compact convex subset of X which contains $\{z_n\}$. It is not difficult to see that $r(C,\{z_n\})=1$ and $A(C,\{z_n\})=\{((0,y),0) : y\in[-1,1]\}$. Then $r_C(A(C,\{z_n\}))=1$, while $1-\Delta_{X,\alpha}(1^-)=\frac{\sqrt{3}}{2}$ and $1-\Delta_{X,\chi}(1^-)=0$ are less than one.

4. Fixed point theorems

In order to prove our first result, we need the following proposition which is proved along the proof of the Kirk-Massa theorem as it appears in [16].

Proposition 4.1. *Let C be a nonempty weakly compact and separable subset of a Banach space X, $T : C \to K(C)$ a nonexpansive mapping and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n,Tx_n)=0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*
$$Tx\cap A\neq\emptyset, \quad \forall x\in A:=A(C,\{z_n\}).$$

Assume that C is a nonempty weakly compact convex subset of a Banach space X and $T : C\to KC(C)$ is a nonexpansive and 1-χ-contractive self-mapping. Consider a bounded sequence $\{x_n\}$ in C such that T satisfies the condition
$$Tx\cap A\neq\emptyset, \quad \forall x\in A:=A(C,\{x_n\}).$$

For a fixed element $x_0\in A$ and an arbitrary $\mu\in(0,1]$, the contraction $T_\mu : A\to KC(C)$ defined by
$$T_\mu x=\mu x_0+(1-\mu)Tx, \quad x\in A$$
verifies the hypotheses of Theorem 2.1. Indeed, let B be a bounded and nonprecompact subset of C. Since T is 1-χ-contractive and $T_\mu(B)=\mu x_0+(1-\mu)T(B)$ we have
$$\chi(T_\mu(B))=\chi(\mu x_0+(1-\mu)T(B))=\chi((1-\mu)T(B))= \\ (1-\mu)\chi(T(B))\leq(1-\mu)\chi(B)<\chi(B).$$

Thus T_μ is χ-condensing. Moreover, since A is convex, T_μ satisfies the same boundary condition as T does, i.e. we have
$$T_\mu x\cap A\neq\emptyset, \quad \forall x\in A.$$

Hence by Theorem 2.1 T_μ has a fixed point $z_\mu \in A$ and we can find a sequence $\{z_n\}$ in A satisfying $\lim_n d(z_n, Tz_n) = 0$. Notice that this conclusion is true for every A closed bounded convex subset of C satisfying $Tx \cap A \neq \emptyset, \quad \forall x \in A$.

With this observation we are able to prove our main result.

Theorem 4.1. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\beta(X) < 1$, and $T : C \to KC(C)$ be a nonexpansive and 1-χ-contractive nonexpansive mapping. Then T has a fixed point.*

Proof. Let $x_0 \in C$ be fixed and, for each $n \geq 1$, define

$$T_n x := \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx, \quad x \in C.$$

Then T_n is a multivalued contraction and hence has a fixed point x_n by Nadler's theorem ([13]). It is easily seen that $\lim_n d(x_n, Tx_n) = 0$. By Lemma 3.1 we may assume that $\{x_n\}$ is regular w.r.t. C and using Proposition 4.1 we can also assume that

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A(C, \{x_n\}).$$

Since condition $\epsilon_\beta(X) < 1$ implies reflexivity [2], we apply Theorem 3.1 to obtain

$$r_C(A) \leq \lambda r(C, \{x_n\}),$$

where $\lambda := 1 - \Delta_{X,\beta}(1^-) < 1$.

According to the previous observation, we can take a sequence $\{x_n^1\}$ in A satisfying $\lim_n d(x_n^1, Tx_n^1) = 0$ and again reasoning as above we can assume that $\{x_n^1\}$ is regular w.r.t. C and

$$Tx \cap A^1 \neq \emptyset, \quad \forall x \in A^1 := A(C, \{x_n^1\}).$$

Again applying Theorem 3.1 we obtain

$$r_C(A^1) \leq \lambda r(C, \{x_n^1\}).$$

On the other hand, since $\{x_n^1\} \subset A$

$$r(C, \{x_n^1\}) \leq r_C(A)$$

and then

$$r_C(A^1) \leq \lambda r_C(A).$$

By induction, for each $m \geq 1$ we construct A^m and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\})$, $\{x_n^m\}_n \subset A^{m-1}$ such that $\lim_n d(x_n^m, Tx_n^m) = 0$ and

$$r_C(A^m) \leq \lambda^m r_C(A).$$

Choose $x_m \in A^m$. We shall prove that $\{x_m\}_m$ is a Cauchy sequence. For each $m \geq 1$ we have for any positive integer n

$$\|x_{m-1} - x_m\| \leq \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \leq \operatorname{diam} A^{m-1} + \|x_n^m - x_m\|.$$

Taking upper limit as $n \to \infty$

$$\begin{aligned}\|x_{m-1} - x_m\| &\leq \mathrm{diam} A^{m-1} + \limsup_n \|x_n^m - x_m\| = \mathrm{diam} A^{m-1} + r(C, \{x_n^m\}) \\ &\leq \mathrm{diam} A^{m-1} + r_C(A^{m-1}) \\ &\leq 2r_C(A^{m-1}) + r_C(A^{m-1}) = 3r_C(A^{m-1}) \leq 3\lambda^{m-1} r_C(A).\end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $x \in C$ such that x_m converges to x. Let us see that x is a fixed point of T. For each $m \geq 1$,

$$d(x_m, Tx_m) \leq \|x_m - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tx_m) \leq 2\|x_m - x_n^m\| + d(x_n^m, Tx_n^m)$$
.

∎

Taking upper limit as $n \to \infty$

$$d(x_m, Tx_m) \leq 2 \limsup_n \|x_m - x_n^m\| \leq 2\lambda^{m-1} r_C(A).$$

Finally, taking limit in m in both sides we obtain $\lim_m d(x_m, Tx_m) = 0$ and the continuity of T implies that $d(x, Tx) = 0$ i.e. $x \in Tx$.

Remark 4.1.-The inductive construction of the sequence $\{A^m\}_m$ in Theorem 4.1, also appears in [17, Theorem 3.2], though only two steps are done.

Remark 4.2.- Note that Theorem 4.1 does not hold if nonexpansiveness assumption is removed. Indeed, if B_2 is the closed unit ball of l_2 and $T : B_2 \to B_2$ is defined by

$$T(x) = T(x_1, x_2, ...) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...),$$

then T is an 1-χ-contraction without a fixed point.

We do not know if χ-contractiveness condition can be dropped in the above theorem. In fact, it is an open problem if every nonexpansive mapping $T : C \to KC(C)$ is 1-χ-contractive even for single-valued mappings. However, if X is either a separable or a reflexive Banach space and satisfies the non-strict Opial condition this assertion is true, as we prove in the next theorem.

Theorem 4.2. *If X is either a separable or reflexive Banach space satisfying the non-strict Opial condition, C is a nonempty weakly compact subset of X and $T : C \to K(C)$ is a nonexpansive mapping, then T is 1-χ-contractive.*

Proof. Let B be an infinite subset of C. Since $T(B)$ is an infinite and bounded set there exists a sequence $\{y_n\} \subset T(B)$ which is χ-minimal (see [2, Chapter III] for definitions and properties concerning χ-minimality). Since χ is strictly minimalizable we can assume that

$$\chi(\{y_n : n \in \mathbb{N}\}) = \chi(T(B)).$$

Since C is weakly compact, there is a subsequence of $\{y_n\}$ which is weakly convergent to some $y \in C$. Taking a subsequence, if necessary, we can suppose that $y_n \rightharpoonup y$ and $\lim_n \|y_n - y\|$ exists. As in the proof of Theroem 3.1 we have $\chi(\{y_n : n \in \mathbb{N}\}) = \lim_n \|y_n - y\|$.

Choose $x_n \in B$ such that $y_n \in Tx_n$. Taking a subsequence, if necessary, and following the above argument we assume that $x_n \rightharpoonup u \in C$, $\lim_n \|x_n - u\|$ exists and $\chi(\{x_n : n \in \mathbb{N}\}) = \lim_n \|x_n - u\|$.

On the other hand, because T is compact valued, we can take $u_n \in Tu$ verifying

$$\|y_n - u_n\| = d(y_n, Tu) \leq H(Tx_n, Tu) \leq \|x_n - u\|, \quad n \geq 1.$$

By the compactness of Tu, we may assume that $\{u_n\}$ converges (strongly) to a point $v \in Tu$. It follows that

$$\begin{aligned}\chi(T(B)) = \lim_n \|y_n - y\| \leq \limsup_n \|y_n - v\| &= \limsup_n \|y_n - u_n\| \\ &\leq \lim_n \|x_n - u\| = \chi(\{x_n\}) \leq \chi(B),\end{aligned}$$

and T is 1-χ-contractive.

In view of this result, we deduce from Theorem 4.1 the following:

Corollary 4.1. *Let X be a Banach space with $\epsilon_\beta(X) < 1$ which satisfies the non-strict Opial condition. Suppose C is a nonempty weakly compact convex subset of X and $T : C \to KC(C)$ is a nonexpansive mapping. Then T has a fixed point.*

Furthermore, from Theorem 3.1 the method used in the proof of the Theorem 4.1 may be followed to obtain

Theorem 4.3. *Let X be a Banach space with $\epsilon_\chi(X) < 1$ which satisfies the non-strict Opial condition. Suppose C is a nonempty weakly compact convex subset of X and $T : C \to KC(C)$ is a nonexpansive mapping. Then T has a fixed point.*

This theorem extends the Kirk-Massa theorem, in the sense that we do not need the compactness of asymptotic center of a bounded sequence with respect to a bounded closed convex subset of X. Next example, due to Kuczumov and Prus, illustrates this fact.

Example.-([8]) Let X_m be the space ℓ_2 renormed as follows. For $x = \sum_{k=1}^{\infty} x_k e_k$ ($\{e_k\}$ denotes the standard basis in ℓ_2) set

$$\|x\|_m = \sup_n \left(x_n^2 + \frac{1}{m+1} \sum_{k=n+1}^{\infty} x_k^2 \right)^{\frac{1}{2}}, \quad m \geq 1.$$

Clearly $\|\cdot\|_m$ is equivalent to the usual norm in ℓ_2. X_m is NUC for each $m \geq 1$, and it is easy to check that it satisfies the non-strict Opial condition. Thus, the conclusion of Theorem 4.3 holds for these spaces. However, by non-strict Opial condition we have for any $x \in X_m$

$$\limsup \|x - e_n\| \geq 1,$$

while for all k, n with $k < n$

$$\|\frac{1}{\sqrt{m+1}}e_k - e_n\|_m = 1.$$

Thus we conclude

$$A(X_m, \{e_n\}) \supseteq \frac{1}{\sqrt{m+1}}\overline{co}\{e_n\}$$

and, in particular $A(X_m, \{e_n\})$ is not compact.

Note that we cannot apply Lami-Dozo's theorem [9] to obtain a fixed point because X_m does not satisfy strict Opial condition.

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