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ESTIMATES FOR APPROXIMATION NUMBERS OF SOME CLASSES OF COMPOSITION OPERATORS ON THE HARDY SPACE

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Abstract. We give estimates for the approximation numbers of composition operators on H^2, in terms of some modulus of continuity. For symbols whose image is contained in a polygon, we get that these approximation numbers are dominated by $\mathrm{e}^{-c\sqrt{n}}$. When the symbol is continuous on the closed unit disk and has a domain touching the boundary non-tangentially at a finite number of points, with a good behavior at the boundary around those points, we can improve this upper estimate. A lower estimate is given when this symbol has a good radial behavior at some point. As an application we get that, for the cusp map, the approximation numbers are equivalent, up to constants, to $\mathrm{e}^{-cn/\log n}$, very near to the minimal value e^{-cn}. We also see the limitations of our methods. To finish, we improve a result of El-Fallah, Kellay, Shabankhah and Youssfi, in showing that for every compact set K of the unit circle $\mathbf{T}$ with Lebesgue measure 0, there exists a compact composition operator $C_\varphi\colon H^2 \to H^2$, which is in all Schatten classes, and such that $\varphi = 1$ on K and $|\varphi| < 1$ outside K.

1. Introduction and notation

If the approximation numbers of some classes of operators on Hilbert spaces are well understood (for example, those of Hankel operators: see [17]), it is not the case of those of composition operators. Though their behavior remains mysterious, some recent results are obtained in [15] and [13] for approximation numbers of composition operators on the Hardy space H^2. In [15], it is proved that one always has $a_n(C_\varphi) \gtrsim \mathrm{e}^{-cn}$ for some $c > 0$ [15, Theorem 3.1] and that this speed of decay can only be attained when the symbol φ maps the unit disk $\mathbf{D}$ into a disk centered at 0 of radius strictly less than 1, i.e. $\|\varphi\|_\infty < 1$ [15, Theorem 3.4].

In this paper, we give estimates which are somewhat general, in terms of some modulus of continuity. In Section 2, we obtain an upper estimate when the symbol φ is continuous on the closed unit disk and has an image touching non-tangentially the unit circle at a finite number of points, with a good behavior on the boundary around

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this point. As an application, we show that for symbols φ whose image is contained in a polygon $a_n(C_\varphi) \le a\mathrm{e}^{-b\sqrt{n}}$, for some constants $a, b > 0$; this has to be compared with [13], Proposition 2.7, where it is shown that if φ is a univalent symbol such that $\varphi(\mathbf{D})$ contains an angular sector centered on the unit circle and with opening $\theta\pi$, $0 < \theta < 1$, then $a_n(C_\varphi) \ge a\mathrm{e}^{-b\sqrt{n}}$, for some (other) positive constants a and b, depending only on θ. In Section 3, we obtain a lower bound when φ has a good radial behavior at the contact point. Both proofs use Blaschke products. This allows to recover the estimation $a_n(C_{\lambda_\theta}) \approx \mathrm{e}^{-c\sqrt{n}}$ obtained in [15], Proposition 6.3, and [13], Theorem 2.1 for the lens map λ_θ. In Section 4.1, we give another example, the cusp map, for which $a_n(C_\varphi) \approx \mathrm{e}^{-cn/\log n}$, very near the minimum value e^{-cn}. We end that section by considering a one-parameter class of symbols, first studied by Shapiro and Taylor [23] and seeing the limitations of our methods. In Section 5, we improve a result of Gallardo-Gutiérrez and González (later on generalized by El-Fallah, Kellay, Shabankhah and Youssfi [5, Theorem 3.1]). It is known that for every compact composition operator $C_\varphi\colon H^2 \to H^2$, the set $E_\varphi = \{\mathrm{e}^{i\theta}\,;\ |\varphi^*(\mathrm{e}^{i\theta})| = 1\}$ has Lebesgue measure 0. These authors showed ([6]), with a rather difficult construction, that there exists a compact composition operator $C_\varphi\colon H^2 \to H^2$ such that the Hausdorff dimension of E_φ is equal to 1 (and in [5], it is shown that for any negligible compact set K, there is a Hilbert–Schmidt operator C_φ such that $E_\varphi = K$). We improve this result in showing that for every compact set K of the unit circle $\mathbf{T}$ with Lebesgue measure 0, there exists a compact composition operator $C_\varphi\colon H^2 \to H^2$, which is even in all Schatten classes, and such that $E_\varphi = K$.

Notation. We denote by $\mathbf{D}$ the open unit disk and by $\mathbf{T} = \partial\mathbf{D}$ the unit circle; m is the normalized Lebesgue measure on $\mathbf{T}$: $dm(t) = dt/2\pi$. The disk algebra $A(\mathbf{D})$ is the space of functions which are continuous on the closed unit disk $\overline{\mathbf{D}}$ and analytic in the open unit disk. If H^2 is the usual Hardy space on $\mathbf{D}$, every analytic self-map $\varphi\colon \mathbf{D} \to \mathbf{D}$ (also called *Schur function*) defines, by Littlewood's subordination principle, a bounded operator $C_\varphi\colon H^2 \to H^2$ by $C_\varphi(f) = f\circ\varphi$, called the *composition operator of symbol φ*.

Recall that if $T\colon E \to F$ is a bounded operator between two Banach spaces, the *approximation numbers* $a_n(T)$ of T are defined by:

$$a_n(T) = \inf\{\|T - R\|\,;\ \mathrm{rank}(R) < n\}, \quad n = 1, 2, \dots.$$

The sequence $\big(a_n(T)\big)_n$ is non-increasing and, when F has the Approximation Property, T is compact if and only if $a_n(T)$ tends to 0.

The *Gelfand numbers* $c_n(T)$ are defined by $c_n(T) = \inf\{\|T_{|G}\|\,;\ \mathrm{codim}\, G < n\}$. For compact operators T on Hilbert spaces, one has $c_n(T) = a_n(T)$ (see [9]).

Definition 1.1. A *modulus of continuity* ω is a continuous function

$$\omega\colon [0, A] \to \mathbf{R}^+,$$

which is increasing, sub-additive, and vanishes at zero.

Some examples are:

$$\omega(h) = h^\alpha, \quad 0 < \alpha \le 1; \quad \omega(h) = h\log\frac{1}{h}; \quad \omega(h) = \frac{1}{\log\frac{1}{h}}.$$

For any modulus of continuity ω, there is a concave modulus of continuity ω' such that $\omega \le \omega' \le 2\omega$ (see [18] for example); therefore we may and shall assume that ω

is concave on $[0, A]$. In that case, ω^{-1} is convex, and

$$r_\omega(x) := \frac{\omega^{-1}(x)}{x} \tag{1.1}$$

is non-decreasing.

The notation $u(t) \lesssim v(t)$ means that $u(t) \leq Av(t)$ for some constant $A > 0$ and $u(t) \approx v(t)$ means that both $u(t) \lesssim v(t)$ and $v(t) \lesssim u(t)$.

2. Upper bound and boundary behavior

Definition 2.1. Let ω be a modulus of continuity and φ a symbol in the disk algebra $A(\mathbf{D})$. Let $\xi_0 \in \partial\mathbf{D} \cap \varphi(\overline{\mathbf{D}})$. We say that the symbol φ has an *ω-regular behavior* at ξ_0 if, setting

$$\gamma(t) = \varphi(\mathrm{e}^{it}), \tag{2.1}$$

and $E_{\xi_0} = \{t\,;\ \gamma(t) = \xi_0\}$, there exists $r_0 > 0$ such that

1) for some positive constant $C > 0$, one has, for every $t_0 \in E_{\xi_0}$ and $|t - t_0| \leq r_0$:

$$|\gamma(t) - \gamma(t_0)| \leq C\big(1 - |\gamma(t)|\big). \tag{2.2}$$

2) for some positive constant $c > 0$, one has, for for every $t_0 \in E_{\xi_0}$ and $|t - t_0| \leq r_0$:

$$c\,\omega(|t - t_0|) \leq |\gamma(t) - \gamma(t_0)|. \tag{2.3}$$

The first condition implies that the image of φ touches $\partial\mathbf{D}$ at the point ξ_0, and non-tangentially. The second one implies that φ does not stay long near $\xi_0 = \gamma(t_0)$.

Note that, due to (2.3), the intervals $[t - r_0/2, t + r_0/2]$, for $t \in E_{\xi_0}$ are pairwise disjoint and therefore the set E_{ξ_0} must be finite.

We shall make the following assumption (to avoid the Lipschitz class):

$$\lim_{h \to 0^+} \frac{\omega(h)}{h} = \infty; \quad \text{equivalently} \quad \lim_{h \to 0^+} \frac{\omega^{-1}(h)}{h} = 0. \tag{2.4}$$

Indeed, assume that γ is K-Lipschitz at some point $t_0 \in [0, 2\pi]$, namely $|\varphi(\mathrm{e}^{it}) - \varphi(\mathrm{e}^{it_0})| \leq K\,|t - t_0|$, with $|\varphi(\mathrm{e}^{it_0})| = 1$; then

$$m(\{t \in [0, 2\pi]\,;\ |\varphi(\mathrm{e}^{it}) - \varphi(\mathrm{e}^{it_0})| \leq h\}) \geq m(\{t \in [0, 2\pi]\,;\ |t - t_0| \leq h/K\}) = h/2\pi K;$$

hence this measure in not $o\,(h)$ and the composition operator C_φ is not compact ([16], or [3, Theorem 3.12]).

In order to treat the case where the image of φ is a polygon, we need to generalize the above definition. We ask not only that φ is ω-regular at the points $\xi_1, \ldots, \xi_p$ of contact of $\varphi(\overline{\mathbf{D}})$ with $\partial\mathbf{D}$, but a little bit more.

Definition 2.2. Assume that $\varphi(\overline{\mathbf{D}}) \cap \partial\mathbf{D} = \{\xi_1, \ldots, \xi_p\}$. We say that φ is *globally-regular* if there exists a modulus of continuity ω such that, writing $E_{\xi_j} = \{t\,;\ \gamma(t) = \xi_j\}$, one has, for some $r_1, \ldots, r_p > 0$

$$\mathbf{T} = \bigcup_{j=1}^{p} \big(E_{\xi_j} + [-r_j, r_j]\big)$$

and for some positive constants $C, c > 0$, 1') one has, for $j = 1, \dots, p$, every $t_j \in E_{\xi_j}$ and $|t - t_j| \leq r_j$:

$$|\gamma(t) - \gamma(t_j)| \leq C\big(1 - |\gamma(t)|\big). \tag{2.5}$$

2') one has, for $j = 1, \dots, p$, every $t_j \in E_{\xi_j}$ and $|t - t_j| \leq r_j$:

$$c\,\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)|. \tag{2.6}$$

Let us note that condition 1') is equivalent to say that $\varphi(\overline{\mathbf{D}})$ is contained in a polygon inside $\overline{\mathbf{D}}$ whose vertices contain $\xi_1, \dots, \xi_p$, and these are the only vertices in the boundary $\partial\mathbf{D}$. Of course, we may assume that (2.5) and (2.6) hold only when t is in a neighborhood of t_j, since they will then hold for $|t - t_j| \leq r_j$, provided we change the constants C, c.

Before stating our theorem, let us introduce a notation. If φ is as in Definition 2.2 and $\sigma, \kappa > 0$ are some constants, we set

$$d_N = \left[\sigma \log \frac{\kappa\, 2^{-N}}{\omega^{-1}(\kappa\, 2^{-N})}\right] + 1, \tag{2.7}$$

where [ ] stands for the integer part. For every integer $q \geq 1$, we denote by

$$N = N_q \quad \text{the largest integer such that } p\,N d_N < q \tag{2.8}$$

($N_q = 1$ if no such N exists).

We then have the following result.

Theorem 2.3. *Let φ be a symbol in $A(\mathbf{D})$ whose image touches $\partial\mathbf{D}$ at the points $\xi_1, \dots, \xi_p$, and nowhere else. Assume that φ is globally-regular. Then, there are constants κ, K, $L > 0$, depending only on φ, such that, using the notation* (2.7) *and* (2.8)*, one has, for every $q \geq 1$:*

$$a_q(C_\varphi) \leq K \sqrt{\frac{\omega^{-1}(\kappa\, 2^{-N_q})}{\kappa\, 2^{-N_q}}}. \tag{2.9}$$

Before proving this theorem, let us indicate two applications. In these examples, we can give an upper estimate for all approximation numbers $a_n(C_\varphi)$, $n \geq 1$, because we can interpolate between the integers $N d_N$ and $(N+1)\, d_{N+1}$, which is not the case in general.

1) $\omega(h) = h^\theta$, $0 < \theta < 1$. This is the case for inscribed polygons (see the proof of the foregoing Theorem 2.4; here $\theta = \max\{\theta_1, \dots, \theta_p\}$, where $\theta_1\pi, \dots, \theta_p\pi$ are the values of the angles of the polygon). This is also the case, with $p = 2$, of lens maps λ_θ (see [22], page 27, for the definition; see also [13]). We have here $\omega^{-1}(h) = h^{1/\theta}$. Hence $d_N \approx N$, $N_q \approx \sqrt{q}$, and we then get from (2.9) that $a_q(C_\varphi) \leq \alpha\, 2^{-\delta N}$ for $q \gtrsim N^2$, with $\delta > 0$. Equivalently, for suitable constants $\alpha, \beta > 0$,

$$a_n(C_\varphi) \leq \alpha\, \mathrm{e}^{-\beta\sqrt{n}}, \tag{2.10}$$

which is the result obtained in [13, Theorem 2.1].

2) $\omega(h) = \frac{1}{(\log 1/h)^\alpha}$, $0 < \alpha \leq 1$, as this is the case, when $\alpha = 1$, for the cusp map, defined below in Section 4.1 (with $p = 1$). Then, we have $\omega^{-1}(h) = \mathrm{e}^{-h^{-1/\alpha}}$ and $d_N \approx 2^{N/\alpha}$, so that $N_q \approx \log q$ and $2^{N_q/\alpha} \approx q/\log q$. Now, a simple computation gives

$$a_n(C_\varphi) \leq \alpha\, \mathrm{e}^{-\beta n/\log n}. \tag{2.11}$$

Without assuming some regularity, one has the following general upper estimate.

Theorem 2.4. *Let $\varphi\colon \mathbf{D} \to \mathbf{D}$ be an analytic self-map whose image is contained in a polygon $\mathbf{P}$ with vertices on the unit circle. Then, there exist constants $\alpha, \beta > 0$, β depending only on $\mathbf{P}$, such that*

$$a_n(C_\varphi) \leq \alpha\, \mathrm{e}^{-\beta\sqrt{n}}. \tag{2.12}$$

In [13, Proposition 2.7], it is shown that if φ is a univalent symbol such that $\varphi(\mathbf{D})$ contains an angular sector centered on the unit circle and with opening $\theta\pi$, $0 < \theta < 1$, then $a_n(C_\varphi) \geq \alpha\, \mathrm{e}^{-\beta\sqrt{n}}$, for some (other) positive constants α and β, depending only on θ. Note that the injectivity of the symbol is there necessary, since there exists (see the proof of Corollary 5.4 in [15]), for every sequence (ε_n) of positive numbers tending to 0, a symbol φ whose image is $\mathbf{D} \setminus \{0\}$, and hence contains polygons, which is 2-valent, and for which $a_n(C_\varphi) \lesssim \mathrm{e}^{-\varepsilon_n n}$. This bound may be much smaller than $\mathrm{e}^{-\beta\sqrt{n}}$.

Proof of Theorem 2.3. Recall ([13], Lemma 2.4) that for every Blaschke product B with less than N zeros (each of them being counted with its multiplicity), one has

$$\big[a_N(C_\varphi)\big]^2 \lesssim \sup_{0<h<1, |\xi|=1} \frac{1}{h} \int_{S(\xi,h)} |B(z)|^2 \, dm_\varphi(z), \tag{2.13}$$

where $S(\xi, h) = \{z \in \overline{\mathbf{D}}\,;\ |z - \xi| \leq h\}$ and m_φ is the pull-back measure by φ of the normalized Lebesgue measure m on $\mathbf{T}$.

The proof will come from an adequate choice of a Blaschke product. Fix a positive integer N. Set, for $j = 1, \ldots, p$ and $k = 1, 2, \ldots$,

$$p_{j,k} = (1 - 2^{-k})\xi_j \tag{2.14}$$

and consider the Blaschke product of length pNd (d being a positive integer, to be specified later) given by

$$B(z) = \prod_{j=1}^{p} \prod_{k=1}^{N} \bigg[\frac{z - p_{j,k}}{1 - \overline{p_{j,k}}\, z}\bigg]^d. \tag{2.15}$$

Recall that we have set

$$\gamma(t) = \varphi(\mathrm{e}^{it}). \tag{2.16}$$

To use (2.13), note that if $|\gamma(t) - \xi| \leq h$, then, for some $j = 1, \ldots, p$ and some $t_j \in E_{\xi_j}$, one has $|t - t_j| \leq r_j$ and, by (2.5), $|\gamma(t) - \xi_j| \leq C(1 - |\gamma(t)|) \leq C\,|\gamma(t) - \xi| \leq Ch$. Therefore, denoting by L_j the number of elements of E_{ξ_j} (which is finite by the remark following Definition 2.1),

$$\big[a_N(C_\varphi)\big]^2 \lesssim \sup_{0<h<1} \frac{1}{h} \sum_{j=1}^{p} L_j \int_{\{|\gamma(t)-\xi_j| \leq Ch\} \cap \{|t-t_j| \leq r_j\}} |B[\gamma(t)]|^2 \, \frac{dt}{2\pi},$$

and we only need to majorize the integrals

$$I_j(h) = \int_{\{|\gamma(t)-\xi_j| \leq Ch\} \cap \{|t-t_j| \leq r_j\}} |B\big(\gamma(t)\big)|^2 \, \frac{dt}{2\pi} \cdot$$

Moreover, it suffices, by interpolation, to do that with $h = h_n$, where $h_n = 2^{-n}$.

By (2.6), for $|t - t_j| \leq r_j$ and $|\gamma(t) - \xi_j| \leq Ch_n$, one has $c\,\omega(|t - t_j|) \leq |\gamma(t) - \xi_j| \leq Ch_n = C\,2^{-n}$, which implies that

$$|t - t_j| \leq \omega^{-1}(c^{-1} C\, 2^{-n}). \tag{2.17}$$

Let

$$s_n = \omega^{-1}(c^{-1}C\,2^{-n}). \tag{2.18}$$

One has

$$I_j(h_n) \leq \int_{\{|t-t_j|\leq s_n\}\cap\{|t-t_j|\leq r_j\}} |B\big(\gamma(t)\big)|^2 \,\frac{dt}{2\pi}.$$

For $n \geq N$, we simply majorize $|B\big(\gamma(t)\big)|$ by 1 and we get

$$\frac{1}{h_n}\, I_j(h_n) \leq \frac{1}{h_n}\,\frac{2s_n}{2\pi} = \frac{c^{-1}C}{\pi}\,\frac{1}{c^{-1}C\,2^{-n}}\,\omega^{-1}(c^{-1}C\,2^{-n}) \leq \frac{c^{-1}C}{\pi}\,\frac{\omega^{-1}(c^{-1}C\,2^{-N})}{c^{-1}C\,2^{-N}},$$

since the function $\omega^{-1}(x)/x$ is non-decreasing.

When $n \leq N-1$, we write

$$I_j(h_n) \leq \int_{\{|t-t_j|\leq s_N\}\cap\{|t-t_j|\leq r_j\}} |B\big(\gamma(t)\big)|^2 \,\frac{dt}{2\pi} + \int_{\{s_N<|t-t_j|\leq s_n\}\cap\{|t-t_j|\leq r_j\}} |B\big(\gamma(t)\big)|^2 \,\frac{dt}{2\pi}.$$

The first integral is estimated as above. For the second one, we claim that

Claim 2.5. *For some constant $\chi < 1$, one has, for $j = 1, \dots, p$ and every $t_j \in E_{\xi_j}$:*

$$|B\big(\gamma(t)\big)| \leq \chi^d \quad \textit{when } |t-t_j| > s_N \textit{ and } |t-t_j| \leq r_j. \tag{2.19}$$

To see that, we shall use [13], Lemma 2.3. Let us recall that this lemma asserts that for $w, w_0 \in \mathbf{D}$ satisfying $|w - w_0| \leq M \min(1-|w|, 1-|w_0|)$ for some positive constant M, one has:

$$\left|\frac{w-w_0}{1-\overline{w_0}\,w}\right| \leq \frac{M}{\sqrt{M^2+1}}. \tag{2.20}$$

Let t such that $|t-t_j| \leq r_j$ and $|t-t_j| > s_N$. We have, on the one hand, $\omega(|t-t_j|) \geq \omega(s_N) = c^{-1}C\,2^{-N}$, and, on the other hand, since $|\gamma(t_j)| = |\xi_j| = 1$

$$c\,\omega(|t-t_j|) \leq |\gamma(t)-\gamma(t_j)| \leq C(1-|\gamma(t)|);$$

hence $1-|\gamma(t)| \geq 2^{-N}$.

Let $1 \leq k \leq N$ such that $2^{-k} \leq 1-|\gamma(t)| < 2^{-k+1}$. Since $|p_{j,k}| = 1-2^{-k}$, we have

$$|\gamma(t)-p_{j,k}| \leq |\gamma(t)-\xi_j| + |\xi_j - p_{j,k}| \leq C(1-|\gamma(t)|) + 2^{-k} \leq (2C+1)2^{-k}.$$

Hence

$$|\gamma(t)-p_{j,k}| \leq M \min\big(1-|\gamma(t)|, 1-|p_{j,k}|\big),$$

with $M = 2C+1$. By (2.20), we get $\left|\frac{\gamma(t)-p_{jk}}{1-\overline{p_{j,k}}\,\gamma(t)}\right| \leq \chi$, where $\chi = M/\sqrt{M^2+1}$ is < 1, and therefore $|B[\gamma(t)]| \leq \chi^d$. □

We can now end the proof of Theorem 2.3. We get

$$\begin{aligned}\frac{1}{h_n}\int_{\{s_N<|t-t_j|\leq s_n\}\cap\{|t-t_j|\leq r_j\}} |B\big(\gamma(t)\big)|^2 \,\frac{dt}{2\pi} &\leq \frac{1}{h_n}\,\frac{2s_n}{2\pi}\,\chi^{2d} = \frac{1}{h_n}\,\frac{\omega^{-1}(c^{-1}C\,2^{-n})}{\pi}\,\chi^{2d} \\ &= \frac{c^{-1}C}{\pi}\,\frac{\omega^{-1}(c^{-1}C\,2^{-n})}{c^{-1}C\,2^{-n}}\,\chi^{2d} \\ &\leq \frac{1}{\pi}\,\omega^{-1}(c^{-1}C)\,\chi^{2d},\end{aligned}$$

since $\omega^{-1}(x)/x$ is non-decreasing.

We therefore get, setting $\kappa = c^{-1}C$ and $L = L_1 + \cdots + L_p$,

$$\frac{1}{h_n}\sum_{j=1}^{p} L_j \int_{\{|\gamma(t)-\xi_j|\leq Ch_n\}\cap\{|t-t_j|\leq r_j\}} |B[\gamma(t)]|^2\,\frac{dt}{2\pi} \leq \frac{\kappa L}{\pi}\,\frac{\omega^{-1}(\kappa\,2^{-N})}{\kappa\,2^{-N}} + \frac{L\,\omega^{-1}(\kappa)}{\pi}\chi^{2d}.$$

Choose now $d = d_N$, where d_N is defined by (2.7), with $\sigma = 1/\log(\chi^{-2})$. Then $\chi^{2d} \leq \omega^{-1}(\kappa\,2^{-N})/(\kappa\,2^{-N})$, and, since the Blaschke product B has now $p\,Nd_N$ zeroes, we get, for some positive constant K

$$a_{pNd_N+1}(C_\varphi) \leq K\,\sqrt{\frac{\omega^{-1}(\kappa\,2^{-N})}{\kappa\,2^{-N}}}\,,$$

and that ends the proof of Theorem 2.3. □

Proof of Theorem 2.4. It suffices to consider the case when φ is a conformal map from $\mathbf{D}$ onto $\mathbf{P}$. Indeed, let ψ be such a conformal map. In the general case, our assumption allows to write $\varphi = \psi\circ u$, where $u = \psi^{-1}\circ\varphi\colon \mathbf{D}\to\mathbf{D}$ is analytic. It follows that $C_\varphi = C_u\circ C_\psi$ and that $a_n(C_\varphi) \leq \|C_u\|\,a_n(C_\psi)$. Therefore, we may and shall assume that φ itself is this conformal map.

Let us denote by $\xi_1,\ldots,\xi_p$ the vertices of $\mathbf{P}$. Let $0 < \pi\mu_j < \pi$ be the exterior angle of $\mathbf{P}$ at ξ_j, namely the complement to π of the interior angle; so that

$$\sum_{j=1}^{p}\mu_j = 2, \quad \text{and} \quad 0 < \mu_j < 1.$$

If one sets $\theta_j = 1-\mu_j$, one has $0 < \theta_j < 1$.

We then use the explicit form of φ given by the Schwarz–Christoffel formula [19, page 193]:

$$\varphi(z) = A\int_0^z \frac{dw}{(a_1-w)^{\mu_1}\cdots(a_p-w)^{\mu_p}} + B, \tag{2.21}$$

for some constants $A\neq 0$ and $B\in\mathbf{C}$ and where $a_1,\ldots,a_p\in\partial\mathbf{D}$ are such that $\xi_j = \varphi(a_j)$, $j = 1,\ldots,p$. If, as before, we write $\gamma(t) = \varphi(\mathrm{e}^{it})$, we have $\xi_j = \gamma(t_j)$, with $a_j = \mathrm{e}^{it_j}$ (note that here $E_{\xi_j} = \{t_j\}$).

As we already said, condition (2.5) is trivially satisfied for a polygon.

To end the proof, we use Theorem 2.3 and its Example 1. For that it suffices to show that, for $|t-t_j|$ small enough, we have

$$|\gamma(t)-\xi_j| \approx |t-t_j|^{\theta_j}. \tag{2.22}$$

If $z\in\mathbf{D}$ is close to a_j, it follows from (2.21) that we can write

$$\varphi(z) = A\int_0^z f_j(w)\frac{dw}{(a_j-w)^{\mu_j}} + B,$$

where f_j is holomorphic near a_j and $f_j(a_j)\neq 0$ since

$$|f_j(a_j)| = \prod_{k\neq j,1\leq k\leq p}|a_j-a_k|^{-\mu_k}.$$

Write $f_j(w) = f_j(a_j) + (a_j - w) g_j(w)$ where g_j is holomorphic near a_j. We get

$$\begin{aligned}\varphi(z) &= A f_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \int_0^z g_j(w)(a_j - w)^{\theta_j}\, dw \\ &:= A f_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \psi_j(z),\end{aligned}$$

which can still be written (since $\theta_j > 0$)

$$\varphi(z) = \lambda_j (a_j - z)^{\theta_j} + c_j + \psi_j(z), \tag{2.23}$$

where $\lambda_j \neq 0$, $c_j \in \mathbf{C}$, ψ_j is Lipschitz near a_j and $\xi_j = \varphi(a_j) = c_j + \psi_j(a_j)$. Now, we easily get (2.22). Indeed, for t near t_j, it follows from (2.23) that (recall that $\gamma(t) = \varphi(\mathrm{e}^{it})$ and $\gamma(t_j) = \xi_j$)

$$|\gamma(t) - \gamma(t_j)| = |\lambda_j|\, |\mathrm{e}^{it} - \mathrm{e}^{it_j}|^{\theta_j} + O\left(|t - t_j|\right),$$

which the claimed estimate (2.22) since $\lambda_j \neq 0$ and $|t - t_j|$ is negligible compared to $|t - t_j|^{\theta_j} \approx |\mathrm{e}^{it} - \mathrm{e}^{it_j}|^{\theta_j}$. □

3. Lower bound and radial behavior

We shall consider symbols φ taking real values in the real axis (i.e. its Taylor series has real coefficients) and such that $\lim_{r \to 1^-} \varphi(r) = 1$, with a given speed.

Definition 3.1. We say that an analytic map $\varphi\colon \mathbf{D} \to \mathbf{D}$ is *real* if it takes real values on $]-1, 1[$, and that φ is an *ω-radial symbol* if it is real and there is a modulus of continuity $\omega\colon [0, 1] \to [0, 2]$ such that

$$1 - \varphi(r) \leq \omega(1 - r), \quad 0 \leq r < 1. \tag{3.1}$$

With those definitions and notations, one has:

Theorem 3.2. *Let φ be a real and ω-radial symbol. Then, for the approximation numbers $a_n(C_\varphi)$ of the composition operator C_φ of symbol φ, one has the following lower bound:*

$$a_n(C_\varphi) \geq c \sup_{0<\sigma<1} \sqrt{\frac{\omega^{-1}(a\, \sigma^n)}{a\, \sigma^n}} \exp\left[-\frac{20}{1-\sigma}\right], \tag{3.2}$$

where $a = 1 - \varphi(0) > 0$ and c is another constant depending only on φ.

Observe that, for the lens map λ_θ (see [13, Lemma 2.5]), we have $\omega^{-1}(h) \approx h^{1/\theta}$, so that adjusting $\sigma = 1 - 1/\sqrt{n}$, we get

$$a_n(C_{\lambda_\theta}) \geq c \exp\left(-C\sqrt{n}\right), \tag{3.3}$$

which is the result of [15, Proposition 6.3].

For the cusp map φ (see Section 4.1), we have $\omega^{-1}(h) \approx \mathrm{e}^{-C'/h}$, so that taking $\sigma = \exp(-\log n / 2n)$, we get

$$a_n(C_\varphi) \geq c \exp(-C\, n / \log n). \tag{3.4}$$

We shall use the same methods as for lens maps (see [15, Proposition 6.3]).

We need a lemma. Recall (see [8, pages 194–195] or [20, pages 302–303]) that if (z_j) is a Blaschke sequence, its Carleson constant δ is defined as $\delta = \inf_{j \geq 1} (1 - |z_j|^2)\, |B'(z_j)|$, where B is the Blaschke product whose zeros are the z_j's. Now (see [7, Chapter VII, Theorem 1.1]), every H^∞-interpolation sequence (z_j) is a Blaschke sequence and its Carleson constant δ is connected to its interpolation constant C by the inequalities

$$1/\delta \leq C \leq \kappa/\delta^2 \tag{3.5}$$

where κ is an absolute constant (actually $C \leq \kappa_1(1/\delta)(1+\log 1/\delta)$). Now, if (z_j) is a H^∞-interpolation sequence with constant C, the sequence of the normalized reproducing kernels $f_j = K_{z_j}/\|K_{z_j}\|$ satisfies

$$C^{-1}\big(\sum |\lambda_j|^2\big)^{1/2} \leq \big\| \sum \lambda_j f_j \|_{H^2} \leq C\,\big(\sum |\lambda_j|^2\big)^{1/2} \tag{3.6}$$

(see [15, Lemma 2.2]).

Lemma 3.3. *Let $\varphi\colon \mathbf{D} \to \mathbf{D}$ be an analytic self-map. Let $u = (u_1, \dots, u_n)$ be a finite sequence in $\mathbf{D}$ and set $v_j = \varphi(u_j)$, $v = (v_1, \dots, v_n)$. Denote by δ_v the Carleson constant of the finite sequence v and set*

$$\mu_n^2 = \inf_{1\leq j\leq n} \frac{1-|u_j|^2}{1-|\varphi(u_j)|^2}.$$

Then, for some constant $c' > 0$, we have the lower bound

$$a_n(C_\varphi) \geq c'\,\delta_v^4\,\mu_n. \tag{3.7}$$

Proof. Recall first that the Carleson constant δ of a Blaschke sequence (z_j) is also equal to

$$\delta = \inf_{k\geq 1} \prod_{j\neq k} \rho(z_k, z_j),$$

where $\rho(z,\zeta) = \left|\frac{z-\zeta}{1-\bar z\,\zeta}\right|$ is the pseudo-hyperbolic distance between z and ζ. Now, the Schwarz–Pick Lemma (see [1, Theorem 3.2]) asserts that every analytic self-map of $\mathbf{D}$ contracts the pseudo-hyperbolic distance. Hence $\rho\big(\varphi(u_j), \varphi(u_k)\big) \leq \rho(u_j, u_k)$ and so, if δ_u and δ_v denote the Carleson constants of u and v:

$$\delta_u \geq \delta_v.$$

Let now R be an operator of rank $< n$. There exists a function $f = \sum_{j=1}^n \lambda_j K_{u_j} \in H^2 \cap \ker R$ with $\|f\| = 1$. We thus have

$$\begin{aligned}
\|C_\varphi^* - R\|^2 &\geq \|C_\varphi^*(f) - R(f)\|_2^2 = \|C_\varphi^*(f)\|_2^2 = \Big\| \sum_{j=1}^n \lambda_j K_{v_j}\Big\|_2^2 \\
&\geq C_v^{-2} \sum_{j=1}^n |\lambda_j|^2 \|K_{v_j}\|_2^2 = C_v^{-2} \sum_{j=1}^n \frac{|\lambda_j|^2}{1-|v_j|^2} \geq C_v^{-2}\,\mu_n^2 \sum_{j=1}^n \frac{|\lambda_j|^2}{1-|u_j|^2} \\
&\geq C_u^{-2} C_v^{-2}\,\mu_n^2 \|f\|_2^2 = C_u^{-2} C_v^{-2}\,\mu_n^2 \geq \kappa^{-4}\,\delta_u^4\,\delta_v^4\,\mu_n^2 \geq \kappa^{-4}\,\delta_v^8\,\mu_n^2,
\end{aligned}$$

and hence $a_n(C_\varphi) \geq \kappa^{-2}\,\delta_v^4\,\mu_n$. □

Remark. This lemma allows to give, in the Hardy case, a simpler proof of Theorem 4.1 in [15], avoiding the use of Lemma 2.3 and Lemma 2.4 (concerning the backward shift) in that paper. Recall that this theorem says that for every non-increasing sequence $(\varepsilon_n)_{n\geq 1}$ of positive real numbers tending to 0, there exists a univalent symbol φ such that $\varphi(0) = 0$ and $C_\varphi\colon H^2 \to H^2$ is compact, but $a_n(C_\varphi) \gtrsim \varepsilon_n$ for every $n \geq 1$. Let us sketch briefly the argument.

First of all, we may assume that $\varepsilon_n \leq 1/2$, for all n, and that the sequence $(\varepsilon_n)_n$ is log-convex and decreasing to 0 (see [15, Lemma 2.6]). The symbol φ is defined as $\varphi(z) = \sigma^{-1}(\mathrm{e}^{-1}\sigma(z))$, where σ is the Riemann map $\sigma\colon \mathbf{D} \to \Omega$, with $\sigma(0) = 0$ and $\sigma'(0) > 0$ from $\mathbf{D}$ onto some domain Ω. This domain Ω is defined as follows. Let A be the piecewise linear function on the intervals $(\mathrm{e}^{n-1}, \mathrm{e}^n)$ with $A(\mathrm{e}^{n-1}) = (1/C_0)\log(1/\varepsilon_n)$ (C_0 is a suitable numerical positive constant). Define $\psi\colon \mathbf{R}_+ \to \mathbf{R}_+$ to be linear on $[0,1]$, with $\psi(0) = (1/2C_0)\log(1/\varepsilon_1)$, and $\psi(t) = t/A(t)$ for $t \geq 1$. Then $\Omega = \{w \in \mathbf{C}\,;\ |\Im\mathrm{m}\, w| < \psi(|\Re\mathrm{e}\, w|)\}$.

If $A_j = (1/C_0)\log(1/\varepsilon_{j+1})$, then the numbers $r_j = \sigma^{-1}(\mathrm{e}^j)$ satisfy $\varphi(r_{j+1}) = r_j$ and (see [15, pages 444–447]):

$$\frac{1 - r_{j+1}}{1 - r_j} \geq \exp(-2C_0 A_j).$$

We apply the above Lemma 3.3 with $u_j = r_j$. Then $v_j = \varphi(u_j) = r_{j-1}$. Hence

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1}{2}\,\frac{1 - |u_j|}{1 - |v_j|} = \frac{1}{2}\,\frac{1 - r_j}{1 - r_{j-1}} \geq \frac{1}{2}\,\exp(-2C_0 A_{j-1}) = \frac{1}{2}\,\varepsilon_j^2 \geq \frac{1}{2}\,\varepsilon_n^2.$$

It follows that $\mu_n \geq \varepsilon_n/\sqrt{2}$.

On the other hand, $(r_j)_{j\geq 1}$ is an interpolating sequence (see [15, Lemma 4.6]); hence there is a constant $\delta > 0$ (which does not depend on $n \geq 1$) such that $\delta_v \geq \delta$. Therefore Lemma 3.3 gives

$$a_n(C_\varphi) \geq c\,\delta^4 \varepsilon_n,$$

which gives Theorem 4.1 of [15]. □

Proof of Theorem 3.2. Fix $0 < \sigma < 1$ and define inductively $u_j \in [0,1)$ by $u_0 = 0$ and the relation

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)] \quad \text{with } 1 > u_{j+1} > u_j$$

(using the intermediate value theorem).

Setting $v_j = \varphi(u_j)$, we have $-1 < v_j < 1$,

$$\frac{1 - v_{j+1}}{1 - v_j} = \sigma, \tag{3.8}$$

and

$$1 - v_n = a\,\sigma^n, \quad \text{with } a = 1 - \varphi(0). \tag{3.9}$$

Now observe that, for $1 \leq j \leq n$, one has, due to the positivity of u_j, to (3.1), and the fact that $r_\omega(x) = \omega^{-1}(x)/x$ is increasing,

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1 - u_j}{2(1 - v_j)} \geq \frac{1}{2}\,\frac{\omega^{-1}(1 - v_j)}{1 - v_j} = \frac{1}{2}\,r_\omega(1 - v_j) \geq \frac{1}{2}\,r_\omega(1 - v_n) = \frac{1}{2}\,r_\omega(a\,\sigma^n),$$

which proves that $\mu_n^2 \geq r_\omega(a\,\sigma^n)/2$. Furthermore, the sequence (v_j) satisfies, by (3.8), a condition very similar to Newman's condition with parameter σ. In fact, for $k > j$, we have

$$\frac{|v_k - v_j|}{|1 - v_k v_j|} = \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + v_j(1 - v_k)} \geq \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + (1 - v_k)} = \frac{1 - \sigma^{k-j}}{1 + \sigma^{k-j}}.$$

Analogously, for $j > k$, we have $\frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \frac{1-\sigma^{j-k}}{1+\sigma^{j-k}}$. Thus, as in the proof of [4, Theorem 9.2], we have, for every k,

$$\prod_{j\neq k} \rho(v_j, v_k) = \prod_{j\neq k} \frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \prod_{l=1}^{\infty} \Big(\frac{1-\sigma^l}{1+\sigma^l}\Big)^2 .$$

Consequently, $\delta_v \geq \prod_{l=1}^{\infty} \big(\frac{1-\sigma^l}{1+\sigma^l}\big)^2 \geq \exp\big(-\frac{5}{1-\sigma}\big)$, by [15, Lemma 6.4]. Finally, use (3.7) to get

$$a_n(C_\varphi) \geq c' \, \delta_v^4 \, \mu_n \geq c \, \exp\Big(-\frac{20}{1-\sigma}\Big) \sqrt{r_\omega(a\,\sigma^n)}.$$

Taking the supremum over σ, that ends the proof of Theorem 3.2. □

Remark. The proof shows that

$$a_n(C_\varphi) \geq \sup_{u_1,\ldots,u_n \in (0,1)} \inf_{\substack{f\in\langle K_{u_1},\ldots,K_{u_n}\rangle \\ \|f\|=1}} \|C_\varphi^* f\|, \tag{3.10}$$

where $\langle K_{u_1},\ldots,K_{u_n}\rangle$ is the linear space generated by n distinct reproducing kernels $K_{u_1},\ldots,K_{u_n}$. But if B is the Blaschke product with zeros $u_1,\ldots,u_n$, then $\langle K_{u_1},\ldots,K_{u_n}\rangle = (BH^2)^\perp$, the *model space* associated to B. Hence

$$a_n(C_\varphi) \geq \sup_{B} \inf_{\substack{f\in(BH^2)^\perp \\ \|f\|=1}} \|C_\varphi^* f\|, \tag{3.11}$$

where the supremum is taken over all Blaschke products with n zeros on the real axis $(0,1)$. This has to be compared with the upper bound (which gives (2.13), see [13, proof of Lemma 2.4])

$$a_n(C_\varphi) \leq \inf_{B} \big\|C_{\varphi|BH^2}\big\| = \inf_{B} \sup_{\substack{f\in BH^2 \\ \|f\|=1}} \|C_\varphi f\|, \tag{3.12}$$

where the infimum is over the Blaschke products with less than n zeros (in the Hilbert space H^2, the approximation number $a_n(C_\varphi)$ is equal to the Gelfand number $c_n(C_\varphi)$, which is, by definition, less or equal to $\big\|C_{\varphi|BH^2}\big\|$, since BH^2 is of codimension $< n$).

4. Examples

4.1. The cusp map.

Definition 4.1. The *cusp map* is the conformal mapping φ sending the unit disk $\mathbf{D}$ onto the domain represented on Figure 1.

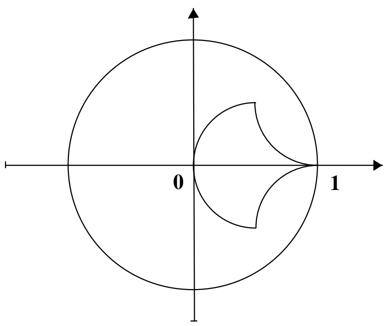

Figure 1. Cusp map domain.

This map was first introduced in [12] (see also [14]). Explicitly, φ is defined as follows.

We first map $\mathbf{D}$ onto the half-disk $\mathbf{D}^+ = \{z \in \mathbf{D}\,;\ \Re\mathrm{e}\, z > 0\}$. To do that, map $\mathbf{D}$ onto itself by $z \mapsto iz$; then map $\mathbf{D}$ onto the upper half-plane $\mathbf{H} = \{z \in \mathbf{C}\,;\ \Im\mathrm{m}\, z > 0\}$ by

$$T(u) = i\,\frac{1+u}{1-u}.$$

Take the square root to map $\mathbf{H}$ in the first quadrant $Q_1 = \{z \in \mathbf{H}\,;\ \Re\mathrm{e}\, z > 0\}$, and go back to the half-disk $\{z \in \mathbf{D}\,;\ \Im\mathrm{m}\, z < 0\}$ by T^{-1}: $T^{-1}(s) = \frac{1+is}{is-1}$; finally, make a rotation by i to go onto $\mathbf{D}^+$. We get

$$\varphi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1}. \tag{4.1}$$

One has $\varphi_0(1) = 0$, $\varphi_0(-1) = 1$, $\varphi_0(i) = -i$ and $\varphi_0(-i) = i$. The half-circle $\{z \in \mathbf{T}\,;\ \Re\mathrm{e}\, z \geq 0\}$ is mapped onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively,

$$\varphi_1(z) = \log \varphi_0(z), \quad \varphi_2(z) = -\frac{2}{\pi}\,\varphi_1(z) + 1, \quad \varphi_3(z) = \frac{1}{\varphi_2(z)}, \tag{4.2}$$

and finally

$$\varphi(z) = 1 - \varphi_3(z). \tag{4.3}$$

Hence

$$1 - \varphi(z) = \frac{1}{1 + \frac{2}{\pi}\,\log\big(1/|\varphi_0(z)|\big) - i\frac{2}{\pi}\arg \varphi_0(z)}. \tag{4.4}$$

φ_2 maps $\mathbf{D}$ onto the semiband $\{z \in \mathbf{C}\,;\ \Re\mathrm{e}\, z > 1 \text{ and } |\Im\mathrm{m}\, z| < 1\}$. One has $\varphi(1) = 1$, $\varphi(-1) = 0$, $\varphi(i) = (1+i)/2$ and $\varphi(-i) = (1-i)/2$.

The domain $\varphi(\mathbf{D})$ is edged by three circular arcs of radii $1/2$ and of respective centers $1/2$, $1 + i/2$ and $1 - i/2$. The real interval $]-1, 1[$ is mapped onto the real interval $]\,0, 1[$ and the half-circle $\{\mathrm{e}^{i\theta}\,;\ |\theta| \leq \pi/2\}$ is sent onto the two circular arcs tangent at 1 to the real axis.

Lemma 4.2.

1) *For* $0 < r < 1$, *let* $\gamma = \frac{\pi}{4} - \arctan r = \arctan[(1-r)/(1+r)]$; *then*

$$\varphi_0(r) = \tan(\gamma/2). \tag{4.5}$$

Hence, when r *tends to* 1_-, *one has*

$$1 - \varphi(r) \sim \frac{\pi}{2}\,\frac{1}{\log(1/\gamma)} \sim \frac{\pi}{2}\,\frac{1}{\log(1/(1-r))}. \tag{4.6}$$

2) *For* $|\theta| < \pi/2$, *one has*

$$\varphi_0(\mathrm{e}^{i\theta}) = -i\,\frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}. \tag{4.7}$$

Hence, when θ tends to 0, one has

$$1 - \varphi(\mathrm{e}^{i\theta}) \sim \frac{\pi}{2} \, \frac{1}{\log(1/|\theta|)}. \tag{4.8}$$

Proof. 1) One has

$$T(ir) = \frac{r - i}{ir - 1} = -\frac{2r}{1 + r^2} + i\, \frac{1 - r^2}{1 + r^2} = -\sin\alpha + i\cos\alpha,$$

with $r = \tan(\alpha/2)$; hence $T(ir) = \cos(\alpha + \pi/2) + i\sin(\alpha + \pi/2) = \mathrm{e}^{i(\alpha+\pi/2)}$. Set $\beta = \frac{\alpha}{2} + \frac{\pi}{4}$; one gets

$$\varphi_0(r) = \frac{\mathrm{e}^{i\beta} - i}{-i\mathrm{e}^{i\beta} + 1} = \frac{\cos\beta}{1 + \sin\beta} = \frac{\sin\gamma}{1 + \cos\gamma} = \tan(\gamma/2)$$

with $\gamma = (\pi/2) - \beta = (\pi/4) - (\alpha/2) = (\pi/4) - \tan^{-1} r$. Then (4.6) follows.

2) Let $\tau = \frac{\pi}{2} - \theta$; one has

$$T(i\mathrm{e}^{i\theta}) = \frac{\mathrm{e}^{i\theta} - i}{i\mathrm{e}^{i\theta} - 1} = \frac{-\cos\theta}{1 + \sin\theta} = \frac{-\sin\tau}{1 + \cos\tau} = -\tan(\tau/2).$$

Note that $0 < \tau/2 < \pi/2$ since $|\theta| < \pi/2$; hence $\tan(\tau/2) > 0$. Therefore

$$\varphi_0(\mathrm{e}^{i\theta}) = \frac{i\sqrt{\tan(\tau/2)} - i}{-i.i\sqrt{\tan(\tau/2)} + 1} = i\, \frac{\sqrt{\tan(\tau/2)} - 1}{\sqrt{\tan(\tau/2)} + 1}.$$

But

$$\tan(\tau/2) = \tan\Big(\frac{\pi}{4} - \frac{\theta}{2}\Big) = \frac{1 - \tan(\theta/2)}{1 + \tan(\theta/2)};$$

it follows that

$$\begin{aligned} \varphi_0(\mathrm{e}^{i\theta}) &= i\, \frac{\sqrt{1 - \tan(\theta/2)} - \sqrt{1 + \tan(\theta/2)}}{\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)}} \\ &= i\, \frac{\big(1 - \tan(\theta/2)\big) - \big(1 + \tan(\theta/2)\big)}{\big(\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)}\big)^2} = -i\, \frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}. \end{aligned}$$

Now, since $\varphi_0(\mathrm{e}^{i\theta}) \sim -i\theta/4$ as θ tends to 0, we get that

$$1 + \frac{2}{\pi}\, \log\big(1/|\varphi_0(\mathrm{e}^{i\theta})|\big) - i\frac{2}{\pi} \arg\varphi_0(\mathrm{e}^{i\theta}) \sim \frac{2}{\pi}\, \log(1/|\theta|)$$

and hence (4.8). □

It follows from this lemma and from Theorem 2.3 and Theorem 3.2 that one has the following estimate.

Theorem 4.3. *For the approximation numbers $a_n(C_\varphi)$ of the composition operator $C_\varphi\colon H^2 \to H^2$ of symbol the cusp map φ, we have*

$$\mathrm{e}^{-c_1 n/\log n} \lesssim a_n(C_\varphi) \lesssim \mathrm{e}^{-c_2 n/\log n}, \quad n = 2, 3, \ldots, \tag{4.9}$$

for some constants $c_1 > c_2 > 0$.

Proof. 1) *Upper estimate.* Note first that, since the domain $\varphi(\mathbf{D})$ is contained in the right half-plane and in the symmetric angular sector of vertex 1 and opening $\pi/2$, there is a constant $C > 0$ such that $|1 - \gamma(t)| \leq C\,(1 - |\gamma(t)|)$ and we have (2.2). Then (4.8) in Lemma 4.2 gives (2.3). The upper estimate is hence given in Theorem 2.3 and (2.11).

2) *Lower estimate.* By Lemma 4.2, (4.6), one has (3.1). Since φ is a real symbol, the upper estimate follows from Theorem 3.2, and (3.4). □

4.2. The Shapiro–Taylor map. This one-parameter map ς_θ, $\theta > 0$, was introduced by Shapiro and Taylor in 1973 [23] and was further studied, with a slightly different definition, in [10, Section 5]. Shapiro and Taylor proved that $C_{\varsigma_\theta}\colon H^2 \to H^2$ is always compact, but is Hilbert–Schmidt if and only if $\theta > 2$. It is proved in [10, Theorem 5.1] that C_{ς_θ} is in the Schatten class S_p if and only if $p > 4/\theta$.

Here, we shall use these maps ς_θ to see the limitations of our previous methods. We first recall their definition.

For $\varepsilon > 0$, we set $V_\varepsilon = \{z \in \mathbf{C}\,;\ \mathfrak{Re}\, z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define

$$f_\theta(z) = z(-\log z)^\theta, \tag{4.10}$$

for $z \in V_\varepsilon$, where $\log z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from $\mathbf{D}$ onto V_ε, which maps $\mathbf{T} = \partial\mathbf{D}$ onto ∂V_ε, defined by $g_\theta(z) = \varepsilon\,\varphi_0(z)$, where φ_0 is given in (4.1).

Then, we define

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta). \tag{4.11}$$

One has $\varsigma_\theta(1) = 1$ and $g_\theta(\mathrm{e}^{it}) \sim -it/4$ as t tends to 0, by Lemma 4.2; hence, when t is near 0,

$$|1 - \varsigma_\theta(\mathrm{e}^{it})| \approx |f_\theta[g_\theta(\mathrm{e}^{it})]| \approx |t|\,[\log(1/|t|)]^\theta.$$

If we were allowed to apply Theorem 2.3, we would get that $a_n(C_{\varsigma_\theta}) \lesssim 1/n^{\theta/4}$, which would be in accordance with the fact that C_{ς_θ} is in the Schatten class S_p if and only if $p > 4/\theta$. However, condition (2.2) is not satisfied: by [10], equations (5.5) and (5.6), one has $1 - |\varsigma_\theta(\mathrm{e}^{it})| \approx |t|(\log 1/|t|)^{\theta-1}$, whereas $|1 - \varsigma_\theta(\mathrm{e}^{it})| \approx |t|(\log 1/|t|)^\theta$.

On the other hand, by the Lemma 4.2 again, $g_\theta(r) \sim \varepsilon(1-r)/4$ as r tends to 1; hence, when r is near to 1,

$$1 - \varsigma_\theta(r) \approx (1-r)\big(\log 1/(1-r)\big)^\theta,$$

so ς_θ is a real ω-radial symbol with $\omega(t) = t(\log 1/t)^\theta$. Hence, we get from Theorem 3.2

$$a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}},$$

taking $\sigma = 1/\mathrm{e}$ in (3.2). However, this lower estimate is not the right one, since C_{ς_θ} is in S_p if and only if $p > 4/\theta$.

5. Contact points

It is well-known (and easy to prove) that for every compact composition operator $C_\varphi\colon H^2 \to H^2$, the set of contact points

$$E_\varphi = \{\mathrm{e}^{i\theta}\,;\ |\varphi^*(\mathrm{e}^{i\theta})| = 1\}$$

has Lebesgue measure 0. A natural question is: to what extent is this negligible set arbitrary? The following partial answer was given by Gallardo-Gutiérrez and González in [6].

Theorem 5.1. (Gallardo-Gutiérrez and González) *There is a compact composition operator C_φ on H^2 such that the Hausdorff dimension of E_φ is one.*

This was generalized by El-Fallah, Kellay, Shabankhah, and Youssfi [5, Theorem 3.1]:

Theorem 5.2. (El-Fallah, Kellay, Shabankhah, and Youssfi) *For every compact set K of measure 0 in $\mathbf{T}$, there exists a Schur function $\varphi \in A(\mathbf{D})$, the disk algebra, such that the associated composition operator C_φ is Hilbert–Schmidt on H^2 and $E_\varphi = K$.*

As an application of our previous results, we shall extend these results, with a very simple proof. Our composition operator will not only be compact, or Hilbert–Schmidt, but in all Schatten classes S_p, and moreover its approximation numbers will be as small as possible.

Theorem 5.3. *Let K be a Lebesgue-negligible compact set of the circle $\mathbf{T}$. Then, there exists a Schur function $\psi \in A(\mathbf{D})$, the disk algebra, such that $E_\psi = K$, $\psi(\mathrm{e}^{i\theta}) = 1$ for all $\mathrm{e}^{i\theta} \in K$, and*

$$a_n(C_\psi) \leq a \exp(-b\, n/\log n). \tag{5.1}$$

In particular, $C_\psi \in \bigcap_{p>0} S_p$.

Proof. According to the Rudin–Carleson theorem [2], we can find $\chi \in A(\mathbf{D})$ such that

$$\chi = 1 \text{ on } K \quad \text{and} \quad |\chi| < 1 \text{ on } \overline{\mathbf{D}} \setminus K.$$

Consider now the cusp map φ, defined in Section 4.1. One has $\varphi \in A(\mathbf{D})$, $\varphi(1) = 1$ and

$$a_n(C_\varphi) \leq a' \exp(-bn/\log n).$$

We now spread the point 1 by composing with the function χ, which is equal to 1 on the whole of K. We check that the composed map $\psi = \varphi \circ \chi$ has the required properties.

That $\psi \in A(\mathbf{D})$ is clear. For $z \in K$, one has $\psi(z) = \varphi(1) = 1$, and for $z \in \overline{\mathbf{D}} \setminus K$, one has $|\chi(z)| < 1$; hence $|\psi(z)| < 1$.

To finish, since $C_\psi = C_\chi \circ C_\varphi$, we have

$$a_n(C_\psi) \leq \|C_\chi\|\, a_n(C_\varphi) \leq a' \,\|C_\chi\|\, \exp(-bn/\log n) := \sigma_n,$$

proving the result (with $a = a' \,\|C_\chi\|$), since clearly $\sum_{n=1}^{\infty} \sigma_n^p < \infty$ for each $p > 0$. □

Actually, we can improve on the previous theorem by proving the following result. This result is optimal because if $\|\psi\|_\infty = 1$, we know (see [15, Theorem 3.4]) that $\liminf_{n\to\infty} [a_n(C_\psi)]^{1/n} = 1$, so we cannot hope to get rid with the forthcoming vanishing sequence $(\varepsilon_n)_n$.

Theorem 5.4. *Let K be a Lebesgue-negligible compact set of the circle $\mathbf{T}$ and $(\varepsilon_n)_n$ a sequence of positive real numbers with limit zero. Then, there exists a Schur function $\varphi \in A(\mathbf{D})$ such that $E_\varphi = K$, $\varphi(\mathrm{e}^{i\theta}) = 1$ for all $\mathrm{e}^{i\theta} \in K$, and*

$$a_n(C_\varphi) \leq C \exp(-n\, \varepsilon_n), \tag{5.2}$$

where C is a positive constant.

This theorem is a straightforward consequence of the following lemma. Recall that the Carleson function of the Schur function $\psi\colon \mathbf{D} \to \mathbf{D}$ is defined by

$$\rho_\psi(h) = \sup_{|\xi|=1} m(\{t \in \mathbf{T}\,;\ |\psi(\mathrm{e}^{it})| \geq 1 - h \text{ and } |\arg(\psi(\mathrm{e}^{it})\,\bar{\xi})| \leq \pi h\}).$$

Lemma 5.5. *Let δ be a nondecreasing positive function on $(0,1]$ tending to 0 as $h \to 0$. Then, there exists a Schur function $\psi \in A(\mathbf{D})$ such that $\psi(1) = 1$, $|\psi(\xi)| < 1$ for $\xi \in \mathbf{T} \setminus \{1\}$, and such that $\rho_\psi(h) \le \delta(h)$, for $h > 0$ small enough.*

Once we have the lemma, in view of the upper bound in [15, Theorem 5.1] for approximation numbers

$$a_n(C_\psi) \lesssim \inf_{0<h<1} \Big[(1-h)^n + \sqrt{\rho_\psi(h)/h}\,\Big], \quad n = 1, 2, \ldots, \tag{5.3}$$

we can adjust the function δ so as to have $a_n(C_\psi) \le K \mathrm{e}^{-n\varepsilon_n}$. Then, we compose ψ with a peaking function χ as in the previous section and the map $\varphi = \psi \circ \chi$ fulfills the requirements of Theorem 5.4, with $C = K\|C_\chi\|$. □

Proof of Lemma 5.5. Consider the domain Ω represented on the Figure 2. This domain is limited by the two hyperbolas $y = 1/x$ and $y = (1/x) + 4\pi$ and to the right-hand side by, say, a semicircle. This limiting semicircle is chosen in order that $\mathfrak{Im}\, w \ge 1$ for $w \in \Omega$. The lower parts of the "saw-teeth" have an imaginary part equal to $4\pi n$. If $a \in \Omega$ is fixed, with $\mathfrak{Im}\, a < 4\pi$, and Ω_n is the part of the domain Ω such that $\mathfrak{Im}\, w < 4\pi n$, the horizontal sizes of the "saw-teeth" are chosen in order that the harmonic measure $\omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n)$ is $\le \delta_n := \delta(1/16\pi(n+1))$. Note that $\partial\Omega \setminus \partial\Omega_n \supseteq \{w \in \partial\Omega\,;\ \mathfrak{Im}\, w > 4\pi n\}$ (see [11, Lemma 4.2]).

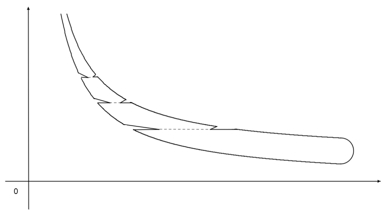

Figure 2. Domain Ω.

By Carathéodory–Osgood's Theorem (see [21], Theorem IX.4.9), there is a unique homeomorphism g from $\overline{\mathbf{D}}$ onto $\overline{\Omega} \cup \{\infty\}$ which maps conformally $\mathbf{D}$ onto Ω and such that $g(0) = a$ and $g(1) = \infty$ (we may choose these two values because if $h\colon \overline{\mathbf{D}} \to \overline{\Omega} \cup \{\infty\}$ is such a map, and u is the automorphism of $\overline{\mathbf{D}}$ such that $u(0) = h^{-1}(a)$ and $u(1) = h^{-1}(\infty)$, then $g = h \circ u$ suits—alternatively, having choosen $h(0) = a$, then, if $h(\mathrm{e}^{i\theta_0}) = \infty$, we take $g(z) = h(\mathrm{e}^{i\theta_0} z)$).

We define $\psi = (g - i)/(g + i)$. Then $\psi\colon \mathbf{D} \to \mathbf{D}$ is a Schur function and $\psi \in A(\mathbf{D})$. Moreover, since the domain Ω is bounded horizontally, we have $\psi(1) = 1$ and $|\psi(\mathrm{e}^{it})| < 1$ for $0 < t < 2\pi$.

Now, $\rho_\psi(h) \le m\big(\{z \in \mathbf{T}\,;\ |\psi(z)| > 1 - h\}\big)$. Writing $g = u + iv$, one has

$$|\psi|^2 = \frac{u^2 + (v-1)^2}{u^2 + (v+1)^2} = 1 - \frac{4v}{u^2 + (v+1)^2}.$$

Since $(1-h)^2 \ge 1 - 2h$, the condition $|\psi(z)| > 1 - h$ implies that $\frac{2v}{u^2+(v+1)^2} \le h$. But $0 < u \le 1 + 2\pi \le 8$ and $(v+1)^2 \le 4v^2$ (since $v \ge 1$); we get hence $\frac{v}{32+2v^2} \le h$, or $\frac{32}{v} + 2v \ge \frac{1}{h}$. Using again the fact that $v \ge 1$, one obtains $2v \ge \frac{1}{h} - 32$, and hence $2v \ge \frac{1}{2h}$ for $0 < h \le 1/64$. Therefore, for $0 < h \le 1/64$,

$$\rho_\psi(h) \le m(\{z \in \mathbf{T}\,;\ \mathfrak{Im}\, g(z) \ge 1/4h\}).$$

Now, for $n \geq 2$ and $1/16\pi(n+1) \leq h < 1/16\pi n$, one gets hence

$$\begin{aligned}\rho_\psi(h) &\leq m(\{z \in \mathbf{T}\,;\ \Im\mathrm{m}\, g(z) > 4\pi n\}) \\ &= \omega_\Omega(a, \{w \in \partial\Omega\,;\ \Im\mathrm{m}\, w > 4\pi n\}) \leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \delta_n \leq \delta(h),\end{aligned}$$

proving Lemma 5.5. □

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