Research Article

Essential Norm of Composition Operators on Banach Spaces of Hölder Functions

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Let (X, d) be a pointed compact metric space, let $0 < \alpha < 1$, and let $\varphi : X \to X$ be a base point preserving Lipschitz map. We prove that the essential norm of the composition operator C_{φ} induced by the symbol φ on the spaces $\lim_{0 \le d} (X, d^{\alpha})$ and $\lim_{0 \le d} (X, d^{\alpha})$ is given by the formula $\|C_{\varphi}\|_{e} = \lim_{t \to 0} \sup_{0 \le d(x, y) \le t} (d(\varphi(x), \varphi(y))^{\alpha} / d(x, y)^{\alpha})$ whenever the dual space $\lim_{0 \le d} (X, d^{\alpha})^{*}$ has the approximation property. This happens in particular when X is an infinite compact subset of a finite-dimensional normed linear space.

1. Introduction

Let (X, d) be a compact metric space with a distinguished point $e \in X$ and $0 < \alpha < 1$. The formula $d^{\alpha}(x, y) = d(x, y)^{\alpha}$ defines a new metric on X, and the metric space (X, d^{α}) is said to be a *Hölder metric space of order* α . As usual, \mathbb{K} denotes the field of real or complex numbers.

The *Lipschitz space* $\text{Lip}_0(X, d^{\alpha})$ is the Banach space of all Lipschitz functions $f : X \to \mathbb{K}$ on the Hölder metric space (X, d^{α}) for which f(e) = 0 under the standard Lipschitz norm

$$L_{\alpha}(f) = \sup\left\{\frac{\left|f(x) - f(y)\right|}{d(x, y)^{\alpha}} : x, y \in X, \ x \neq y\right\}.$$
(1.1)

Notice that the Lipschitz functions on (X, d^{α}) are precisely the *Hölder functions of order* α on (X, d).

The *little Lipschitz space* $\lim_{\alpha \to 0} (X, d^{\alpha})$ is the closed subspace consisting of those functions $f \in \operatorname{Lip}_0(X, d^{\alpha})$ that satisfy the following *local flatness condition*:

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{|f(x) - f(y)|}{d(x,y)^{\alpha}} = 0.$$
(1.2)

The Lipschitz space $\text{Lip}_0(X, d^{\alpha})$ has a canonical predual $\mathcal{F}(X, d^{\alpha})$, the *free Lipschitz space* on (X, d^{α}) , also known as the *Arens-Eells space* in [1], that can be defined as the closed linear span of the point evaluations

$$\delta_x(f) = f(x), \quad (x \in X, \ f \in \operatorname{Lip}_0(X, d^{\alpha})), \tag{1.3}$$

in the dual space $\operatorname{Lip}_0(X, d^{\alpha})^*$. As it turns out, $\mathcal{F}(X, d^{\alpha})$ is itself the dual space of $\operatorname{Lip}_0(X, d^{\alpha})$. The structure of spaces of Lipschitz and Hölder functions and their preduals on general metric spaces was studied by Kalton in [2]. We refer to the book [1] by Weaver for a complete study of the spaces of Lipschitz functions.

We denote by $\mathcal{L}(E)$ the algebra of all bounded linear operators on a Banach space *E*, and by $\mathcal{K}(E)$, the closed ideal of all compact operators on *E*. The *essential norm* $||T||_e$ of an operator $T \in \mathcal{L}(E)$ is just the distance from *T* to $\mathcal{K}(E)$, that is,

$$||T||_e = \inf\{||T - K|| : K \in \mathcal{K}(E)\}.$$
(1.4)

It is clear that an operator $T \in \mathcal{L}(E)$ is compact if and only if $||T||_e = 0$.

Recall that a Banach space *E* is said to have the *approximation property* if the identity operator on *E* can be approximated uniformly on every compact subset of *E* by operators of finite rank.

Let (X, d) be a pointed compact metric space with base point $e \in X$, let $0 < \alpha < 1$, and let $\varphi : X \to X$ be a Lipschitz mapping that preserves the base point, that is, $\varphi(e) = e$ and

$$L(\varphi) := \sup\left\{\frac{d(\varphi(x), \varphi(y))}{d(x, y)} : x, y \in X, \ x \neq y\right\} < \infty.$$
(1.5)

The *composition operator* C_{φ} : $\lim_{\alpha \to \infty} (X, d^{\alpha}) \to \lim_{\alpha \to \infty} (X, d^{\alpha})$ is defined by the expression

$$(C_{\varphi}f)(x) = f(\varphi(x)), \quad (x \in X, \ f \in \operatorname{lip}_0(X, d^{\alpha})).$$
(1.6)

The aim of this paper is to give lower and upper estimates for the essential norm of the composition operator C_{φ} on $\lim_{\alpha} (X, d^{\alpha})$ in terms of φ . Results along these lines were obtained by Montes-Rodríguez [3, 4] and more recently by Galindo and Lindström [5], and also by Galindo et al. [6]. In Section 2, we compute the norm of C_{φ} . We show that

$$\left\|C_{\varphi}\right\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(1.7)

The prototype of a formula as above with $\alpha = 1$ was provided by Weaver in [1, Proposition 1.8.2] for the composition operator C_{φ} on the space $\text{Lip}_0(X, d)$. In Section 3, we give a lower bound for the essential norm of the operator C_{φ} , namely,

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}} \le \left\| C_{\varphi} \right\|_{e}.$$
(1.8)

Section 4 contains our main result. When the dual space $\lim_{\alpha \to 0} (X, d^{\alpha})^*$ has the approximation property, we show that

$$\left\|C_{\varphi}\right\|_{e} \leq \lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(1.9)

The proof of this inequality depends on some results involving shrinking compact approximating sequences on $\lim_{0}(X, d^{\alpha})$. Using the fact that the space $\lim_{0}(X, d^{\alpha})$ is isometrically isomorphic to the second dual of $\lim_{0}(X, d^{\alpha})$, and the relationship between the essential norm of an operator and its adjoint, we derive in Section 5 the same formula for the essential norm of the operator C_{φ} on $\lim_{0}(X, d^{\alpha})$.

It is natural to ask for some examples where the dual space $\lim_{0} (X, d^{\alpha})^*$ has the approximation property. For instance, this happens if X is uniformly discrete, that is, $\inf_{x \neq y} d(x, y) > 0$ (see [2, Proposition 4.4]). Also, $\lim_{0} (X, d^{\alpha})^*$ has the approximation property whenever the space $\lim_{0} (X, d^{\alpha})$ is isomorphic to c_0 and hence $\lim_{0} (X, d^{\alpha})$ is isomorphic to ℓ_{∞} and $\mathcal{F}(X, d^{\alpha})$ is isomorphic to ℓ_1 .

A classical result of Bonic et al. [7] ensures that $\lim_{0}(X, d^{\alpha})$ is isomorphic to c_0 whenever X is an infinite compact subset of a finite-dimensional normed linear space. This result was corrected by Weaver, who asked whether such an isomorphism could be extended to any compact metric space [1, page 98]. Kalton answered this question negatively by proving that a compact convex subset X of a Hilbert space containing the origin has the property that $\lim_{0}(X, d^{\alpha})$ is isomorphic to c_0 if and only if X is finite dimensional [2, Theorem 8.3]. In fact this statement is true for every general Banach space in place of a Hilbert space if $0 < \alpha \leq 1/2$ [2, Theorem 8.5] and for any Banach space that has nontrivial Rademacher type if $0 < \alpha < 1$ [2, Theorem 8.4]. Kalton conjectured that this holds in full generality for all Banach spaces.

Let us recall now that a metric space (X, d) satisfies the *doubling condition* (or has *finite Assouad dimension*) if there is an integer *n* such that for any $\delta > 0$, every closed ball of radius δ can be covered by at most *n* closed balls of radius $\delta/2$. A theorem of Assouad [8] asserts that whenever a metric space (X, d) satisfies the doubling condition, every Hölder metric space (X, d^{α}) Lipschitz embeds in the euclidean space \mathbb{R}^n . Using this result, Kalton observed that if a compact metric space (X, d) satisfies the doubling condition, then the space $\lim_{n \to \infty} (X, d^{\alpha})$ is isomorphic to c_0 [2, Theorem 6.5]. Furthermore, he also showed that the converse is false by means of a counterexample [2, Proposition 6.8].

2. The Norm of C_{φ} **on** $\operatorname{Lip}_0(X, d^{\alpha})$

The aim of this section is to derive a formula for the norm of the composition operator C_{φ} on $\lim_{p \to \infty} (X, d^{\alpha})$ in terms of the Lipschitz constant of φ . A similar expression was already provided by Weaver for the composition operator C_{φ} on the space $\lim_{p \to \infty} (X, d)$, obtaining in [1, Proposition 1.8.2] the following identity:

$$\|C_{\varphi}\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.$$
(2.1)

Theorem 2.1. Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a base point preserving Lipschitz mapping. Then the norm of the composition operator $C_{\varphi} : \lim_{\alpha \to \infty} (X, d^{\alpha}) \to \lim_{\alpha \to \infty} (X, d^{\alpha})$ is given by the expression

$$\|C_{\varphi}\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(2.2)

Proof. We follow the steps of the proof of Weaver's formula. One inequality is formally identical, while the other inequality needs an adjustment of the suitable attaining functions. For any $f \in \lim_{\alpha \to 0} (X, d^{\alpha})$ with $L_{\alpha}(f) \leq 1$, we have

$$L_{\alpha}(C_{\varphi}f) = \sup_{\substack{x \neq y}} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x,y)^{\alpha}}$$

$$\leq \sup_{\substack{\varphi(x) \neq \varphi(y)}} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x),\varphi(y))^{\alpha}} \cdot \sup_{\substack{x \neq y}} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}}$$

$$\leq L_{\alpha}(f) \cdot \sup_{\substack{x \neq y}} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}}$$

$$\leq \sup_{\substack{x \neq y}} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}},$$
(2.3)

and so

$$\|C_{\varphi}\| = \sup_{L_{\alpha}(f) \le 1} L_{\alpha}(C_{\varphi}f) \le \sup_{x \ne y} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(2.4)

For the converse inequality, fix two points $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$ and choose β strictly between α and 1. Define $h : X \to \mathbb{R}$ by

$$h(z) = \frac{d(z,\varphi(y))^{\beta} - d(z,\varphi(x))^{\beta}}{2d(\varphi(x),\varphi(y))^{\beta-\alpha}},$$
(2.5)

and $f: X \to \mathbb{R}$ by

$$f(z) = h(z) - h(e).$$
 (2.6)

It is not hard to show that $f \in \lim_{\alpha \to 0} (X, d^{\alpha})$ with $L_{\alpha}(f) = 1$ (see, for instance, [9]), so

$$\left\|C_{\varphi}\right\| \ge L_{\alpha}(C_{\varphi}f) \ge \frac{\left|f(\varphi(x)) - f(\varphi(y))\right|}{d(x,y)^{\alpha}} = \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}}.$$
(2.7)

Taking supremum over *x* and *y*, we conclude that

$$\|C_{\varphi}\| \ge \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(2.8)

3. The Lower Estimate of the Essential Norm of C_{φ} on $\operatorname{Lip}_0(X, d^{\alpha})$

Next we bound from below the essential norm of C_{φ} on $\lim_{\alpha} (X, d^{\alpha})$ by means of an asymptotic quantity that measures the local flatness of φ .

Theorem 3.1. Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a base point preserving Lipschitz mapping. Then the essential norm of the operator $C_{\varphi} : \lim_{\alpha \to \infty} (X, d^{\alpha}) \to \lim_{\alpha \to \infty} (X, d^{\alpha})$ satisfies the lower estimate

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}} \le \left\| C_{\varphi} \right\|_{e}.$$
(3.1)

We will need the following description of the weak convergence in $\lim_{x \to a} (X, d^{\alpha})$. This result is part of the folklore and it is immediate from the Banach-Steinhaus theorem since $\lim_{x \to a} (X, d^{\alpha})^* = \overline{\text{span}} \{\delta_x : x \in X\}$ (see Weaver [1, Theorem 3.3.3]).

Lemma 3.2. Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\{f_n\}$ a sequence in $\lim_{n \to \infty} (X, d^{\alpha})$. Then $\{f_n\}$ converges to 0 weakly in $\lim_{n \to \infty} (X, d^{\alpha})$ if and only if $\{f_n\}$ is bounded in $\lim_{n \to \infty} (X, d^{\alpha})$ and converges to 0 pointwise on X.

Proof of Theorem 3.1. Since the mapping $\varphi : X \to X$ is Lipschitz, the function

$$t \longmapsto \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}}, \quad (t > 0)$$
(3.2)

is well defined. It is easy to check that

$$\lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}} = \inf_{t > 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(3.3)

Now, for every natural number *n* we can take a real number t_n such that $0 < t_n < [2/n(1 + L(\varphi)^{\alpha})]^{1/\alpha}$ and two points $x_n, y_n \in X$ such that $0 < d(x_n, y_n) < t_n$, satisfying

$$\sup_{0 < d(x,y) < t_n} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}} < \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}} + \frac{1}{n},$$

$$\sup_{0 < d(x,y) < t_n} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}} - \frac{1}{n} < \frac{d(\varphi(x_n),\varphi(y_n))^{\alpha}}{d(x_n,y_n)^{\alpha}}.$$
(3.4)

In this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$0 < d(x_n, y_n) < \left[\frac{2}{n(1+L(\varphi)^{\alpha})}\right]^{1/\alpha},$$

$$\left|\frac{d(\varphi(x_n), \varphi(y_n))^{\alpha}}{d(x_n, y_n)^{\alpha}} - \inf_{t>0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}\right| < \frac{1}{n},$$
(3.5)

for all $n \in \mathbb{N}$, and this last inequality implies that

$$\inf_{t>0} \sup_{0< d(x,y)< t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}} = \lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))^{\alpha}}{d(x_n, y_n)^{\alpha}}.$$
(3.6)

Now take $x_n, y_n \in X$ for which $\varphi(x_n) \neq \varphi(y_n)$ and choose $\beta \in]\alpha, 1[$. Define $h_n, f_n : X \to \mathbb{R}$ by

$$h_n(x) = \frac{d(x,\varphi(y_n))^{\beta} - d(x,\varphi(x_n))^{\beta}}{2d(\varphi(x_n),\varphi(y_n))^{\beta-\alpha}},$$

$$f_n(x) = h_n(x) - h_n(e).$$
(3.7)

Using (3.5), we have $||h_n||_{\infty} \leq 1/n$. Clearly, $f_n \in \lim_{p \to 0} (X, d^{\alpha})$ with $L_{\alpha}(f_n) = L_{\alpha}(h_n) = 1$ and $||f_n||_{\infty} \leq 2/n$ for all $n \in \mathbb{N}$. Moreover, an easy calculation shows that $|f_n(\varphi(x_n)) - f_n(\varphi(y_n))| = d(\varphi(x_n), \varphi(y_n))^{\alpha}$. Since

$$\frac{d(\varphi(x_n),\varphi(y_n))^{\alpha}}{d(x_n,y_n)^{\alpha}} = \frac{\left|f_n(\varphi(x_n)) - f_n(\varphi(y_n))\right|}{d(x_n,y_n)^{\alpha}} \le L_{\alpha}(C_{\varphi}f_n),$$
(3.8)

for all $n \in \mathbb{N}$, we have

$$\lim_{n \to \infty} \frac{d(\varphi(x_n), \varphi(y_n))^{\alpha}}{d(x_n, y_n)^{\alpha}} \le \limsup_{n \to \infty} L_{\alpha}(C_{\varphi}f_n).$$
(3.9)

By Lemma 3.2, $\{f_n\} \to 0$ weakly in $\lim_{n\to\infty} (X, d^{\alpha})$. Thus, if *K* is any compact operator from $\lim_{n\to\infty} (X, d^{\alpha})$ into $\lim_{n\to\infty} (X, d^{\alpha})$, then $\lim_{n\to\infty} L_{\alpha}(Kf_n) = 0$ because compact operators map weakly convergent sequences into norm convergent sequences. It follows that

$$\limsup_{n \to \infty} L_{\alpha}(C_{\varphi}f_{n}) = \limsup_{n \to \infty} \left(L_{\alpha}(C_{\varphi}f_{n}) - L_{\alpha}(Kf_{n}) \right)$$

$$\leq \limsup_{n \to \infty} L_{\alpha}((C_{\varphi} - K)f_{n})$$

$$\leq \|C_{\varphi} - K\|.$$
(3.10)

Combining (3.3), (3.6), (3.9), and (3.10), we conclude that

$$\lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{a}}{d(x, y)^{a}} \le \|C_{\varphi} - K\|.$$
(3.11)

By taking the infimum on both sides of this inequality over all compact operators *K* on $\lim_{k \to 0} (X, d^{\alpha})$, we obtain the lower estimate

$$\lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}} \le \|C_{\varphi}\|_{e}.$$
(3.12)

4. The Upper Estimate of the Essential Norm of C_{φ} on $\operatorname{Lip}_{0}(X, d^{\alpha})$

Now we prove that the lower bound of the essential norm of C_{φ} on $\lim_{0} (X, d^{\alpha})$ obtained in Section 3 is also an upper bound whenever the dual space $\lim_{0} (X, d^{\alpha})^{*}$ has the approximation property.

Theorem 4.1. Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\lim_{p_0} (X, d^{\alpha})^*$ has the approximation property. Let $\varphi : X \to X$ be a base point preserving Lipschitz mapping. Then the essential norm of the composition operator $C_{\varphi} : \lim_{p_0} (X, d^{\alpha}) \to \lim_{p_0} (X, d^{\alpha})$ satisfies

$$\left\|C_{\varphi}\right\|_{e} \leq \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}}.$$
(4.1)

The strategy for the proof of Theorem 4.1 is to work with a sequence $\{K_n\}$ of compact operators on $\lim_{n \to \infty} (X, d^{\alpha})$ that satisfies some prescribed conditions that are stated in Lemma 4.3 below. We borrow this technique from the work of Montes-Rodríguez [3].

First we recall some notions and results. A sequence $\{K_n\}$ is called a *compact* approximating sequence for a separable Banach space *E* if each $K_n : E \to E$ is a compact operator and $\lim_{n\to\infty} ||(I - K_n)f|| = 0$ for every $f \in E$, where *I* denotes the identity operator on *E*. Also, we say that $\{K_n\}$ is *shrinking* if $\lim_{n\to\infty} ||(I - K_n)^*f^*|| = 0$ for every $f^* \in E^*$.

Johnson [10, Theorem 2] showed that both the Banach space $\lim_{\alpha \to 0} (X, d^{\alpha})$ and its dual space $\lim_{\alpha \to 0} (X, d^{\alpha})^*$ are separable.

On the other hand, the *Banach-Mazur distance* between isomorphic Banach spaces *E*, *F* is defined by

$$d(E,F) = \inf \left\{ \|T\| \cdot \left\| T^{-1} \right\| : T \text{ is an isomorphism of } E \text{ onto } F \right\}.$$

$$(4.2)$$

We say that *E* embeds almost isometrically into *F* provided that for every $\varepsilon > 0$, there is a subspace $F_{\varepsilon} \subset F$ such that $d(E, F_{\varepsilon}) < 1 + \varepsilon$.

The next proposition is immediate from a result of Kalton [11].

Proposition 4.2 (see [11, Corollary 3]). Let $\{K_n\}$ be a sequence of compact operators between Banach spaces E and F and let us suppose that $\lim_{n\to\infty} \langle K_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in F^*$ and $f^{**} \in E^{**}$. Then there exists a sequence $\{K_n^c\}$ of compact operators such that $K_n^c \in \operatorname{conv}\{K_m : m \ge n\}$ and $\lim_{n\to\infty} \|K_n^c\| = 0$.

Lemma 4.3. Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\lim_{n \to \infty} (X, d^{\alpha})^*$ has the approximation property. Then there is a shrinking compact approximating sequence $\{K_n\}$ on $\lim_{n \to \infty} \|I - K_n\| \le 1$.

Proof. Since the dual space $\lim_{0} (X, d^{\alpha})^*$ is separable and has the approximation property, it has the metric approximation property and therefore there is a shrinking compact approximating sequence $\{S_n\}$ on $\lim_{0} (X, d^{\alpha})$. We claim that for every $j \in \mathbb{N}$, there exist a natural $n_j \ge j$ and a compact operator K_j on $\lim_{0} (X, d^{\alpha})$ in the convex hull of the set $\{S_m : m \ge n_j\}$ such that $||I - K_j|| < (1 + 1/j)^2$.

Fix $j \in \mathbb{N}$. Now, for the proof of our claim, we use a result that ensures that $\lim_{p \to 0} (X, d^{\alpha})$ embeds almost isometrically into c_0 . We refer to Kalton [2, Theorem 6.6] for a simple proof of this result due to Yoav Benyamini. Thus, there is a closed subspace $F_j \subset c_0$ and an isomorphism T_j : $\lim_{p \to 0} (X, d^{\alpha}) \to F_j$ such that $||T_j|| \cdot ||T_j^{-1}|| < 1 + (1/j)$. Now consider $M_n := T_j S_n T_j^{-1}$, and notice that $\{M_n\}$ is a shrinking compact approximating sequence on F_j . Next, let $\{P_n\}$ be the sequence of projections on c_0 defined by

$$(P_n x)(k) = x(k), \quad (1 \le k \le n), \qquad (P_n x)(k) = 0, \quad (k > n),$$

$$(4.3)$$

for all $x \in c_0$, and let $J : F_j \hookrightarrow c_0$ be the inclusion map. Then consider the sequence of compact operators $D_n := P_n J - J M_n$ defined from F_j into c_0 . Notice that $\lim_{n\to\infty} \langle D_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in c_0^*$ and $f^{**} \in F_j^{**}$. It follows from Proposition 4.2 that there is a sequence of operators $\{D_n^c\}$ such that $D_n^c \in \operatorname{conv}\{D_m : m \ge n\}$ and $\lim_{n\to\infty} \|D_n^c\| = 0$. This gives rise to a pair of shrinking compact approximating sequences $\{P_n^c\}$ and $\{M_n^c\}$ such that, for each $n \in \mathbb{N}, P_n^c \in$ $\operatorname{conv}\{P_m : m \ge n\}, M_n^c \in \operatorname{conv}\{M_m : m \ge n\}$, and $\lim_{n\to\infty} \|P_n^c J - J M_n^c\| = 0$. Now, consider $L_n := T_j^{-1} M_n^c T_j$. We have

$$\|I - L_n\| = \left\|I - T_j^{-1} M_n^c T_j\right\| \le \left\|T_j^{-1}\right\| \cdot \|T_j\| \cdot \|I - M_n^c\|$$
$$\le \left(1 + \frac{1}{j}\right) \cdot \|I - M_n^c\| = \left(1 + \frac{1}{j}\right) \cdot \|J(I - M_n^c)\|$$

$$\leq \left(1 + \frac{1}{j}\right) \cdot \left(\|(I - P_n^c)J\| + \|P_n^cJ - JM_n^c\|\right)$$

$$\leq \left(1 + \frac{1}{j}\right) \cdot \left(1 + \|P_n^cJ - JM_n^c\|\right),$$

(4.4)

for all $n \in \mathbb{N}$. Finally, choose $n_j \ge j$ large enough so that $||P_{n_j}^c J - JM_{n_j}^c|| < 1/j$ and conclude that $||I - L_{n_j}|| < (1 + (1/j))^2$. The claim is proved if we take $K_j := L_{n_j}$.

The proof of the lemma will be finished if we show that $\{K_j\}$ is a shrinking compact approximating sequence on $\lim_{n\to\infty} (X, d^{\alpha})$. Let $f \in \lim_{n\to\infty} (X, d^{\alpha})$ and $\varepsilon > 0$. Since $\lim_{n\to\infty} L_{\alpha}(f - S_n f) = 0$, there exists $m_0 \in \mathbb{N}$ such that $L_{\alpha}(f - S_n f) < \varepsilon$ for $n \ge m_0$. If $j \ge m_0$, using that $K_j \in \operatorname{conv} \{S_m : m \ge n_j\}$ and $n_j \ge j$, we conclude that $L_{\alpha}(f - K_j f) < \varepsilon$. Hence $\lim_{j\to\infty} L_{\alpha}(f - K_j f) = 0$. This shows that $\{K_j\}$ is approximating on $\lim_{n\to\infty} (X, d^{\alpha})$ and similarly it is seen that $\{K_j\}$ is shrinking.

There is another preliminary result that is needed for the proof of Theorem 4.1 and that can be stated as follows.

Lemma 4.4. Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a base point preserving Lipschitz mapping. Let $\{K_n\}$ be a shrinking compact approximating sequence on $\lim_{n \to \infty} |Q(X, d^{\alpha})|$.

Then, for each t > 0*,*

$$\lim_{n \to \infty} \sup_{L_{\alpha}(f) \le 1} \sup_{d(x,y) \ge t} \frac{\left| \left[(I - K_n) f \right] (\varphi(x)) - \left[(I - K_n) f \right] (\varphi(y)) \right|}{d(x,y)^{\alpha}} = 0.$$
(4.5)

Proof. Fix t > 0. Since the inequality $||f||_{\infty} \le \operatorname{diam}(X)^{\alpha} \cdot L_{\alpha}(f)$ is satisfied for all $f \in \operatorname{lip}_{0}(X, d^{\alpha})$, there is a continuous injection $J : (\operatorname{lip}_{0}(X, d^{\alpha}), L_{\alpha}(\cdot)) \hookrightarrow (\operatorname{lip}_{0}(X, d^{\alpha}), || \cdot ||_{\infty})$. Moreover, it follows from the Arzelà-Ascoli Theorem that J is a compact operator, and by Schauder's theorem, its adjoint J^{*} is a compact operator, too.

Let *B* be the unit ball of $(\lim_{0} (X, d^{\alpha}), \|\cdot\|_{\infty})^*$. Since the bounded sequence of operators $\{(I - K_n)^*\}$ converges to zero pointwise on $\lim_{0} (X, d^{\alpha})^*$ and $J^*(B)$ is a relatively compact set in $(\lim_{0} (X, d^{\alpha}), L_{\alpha}(\cdot))^*$, it follows that $\lim_{n\to\infty} ||(I - K_n)^*J^*|| = 0$. Thus, $\lim_{n\to\infty} ||J(I - K_n)|| = 0$. Let $\varepsilon > 0$ be given and choose $m \in \mathbb{N}$ such that if $n \ge m$, then $||(I - K_n)f||_{\infty} < \varepsilon t^{\alpha}/4$ for all $f \in \lim_{0} (X, d^{\alpha})$ with $L_{\alpha}(f) \le 1$. For $n \ge m$, we have

$$\frac{\left|\left[(I-K_n)f\right](\varphi(x)) - \left[(I-K_n)f\right](\varphi(y))\right|}{d(x,y)^{\alpha}} \le \frac{2\left\|(I-K_n)f\right\|_{\infty}}{t^{\alpha}} < \frac{\varepsilon}{2},\tag{4.6}$$

whenever $f \in \lim_{n \to \infty} (X, d^{\alpha})$ with $L_{\alpha}(f) \leq 1$ and $x, y \in X$ such that $d(x, y) \geq t$. Finally, we get

$$\sup_{L_{\alpha}(f)\leq 1} \sup_{d(x,y)\geq t} \frac{\left|\left[(I-K_{n})f\right](\varphi(x)) - \left[(I-K_{n})f\right](\varphi(y))\right|}{d(x,y)^{\alpha}} < \varepsilon,$$
(4.7)

as required.

We now are ready to prove our main result.

Proof of Theorem 4.1. Let $\{K_n\}$ be the sequence of operators on $\lim_0 (X, d^{\alpha})$ provided by Lemma 4.3. Since each K_n is a compact operator, so is the product $C_{\varphi}K_n$: $\lim_0 (X, d^{\alpha}) \rightarrow \lim_0 (X, d^{\alpha})$ and therefore

$$\left\|C_{\varphi}\right\|_{e} \le \left\|C_{\varphi}(I - K_{n})\right\|. \tag{4.8}$$

Next, fix t > 0 and notice that

$$\begin{split} \|C_{\varphi}(I - K_{n})\| &= \sup_{L_{\alpha}(f) \leq 1} L_{\alpha}(C_{\varphi}(I - K_{n})f) \\ &= \sup_{L_{\alpha}(f) \leq 1} \sup_{x \neq y} \frac{\left| \left[(I - K_{n})f \right](\varphi(x)) - \left[(I - K_{n})f \right](\varphi(y)) \right| }{d(x, y)^{\alpha}} \\ &\leq \sup_{L_{\alpha}(f) \leq 1} \sup_{0 < d(x, y) < t} \frac{\left| \left[(I - K_{n})f \right](\varphi(x)) - \left[(I - K_{n})f \right](\varphi(y)) \right| }{d(x, y)^{\alpha}} \\ &+ \sup_{L_{\alpha}(f) \leq 1} \sup_{d(x, y) \geq t} \frac{\left| \left[(I - K_{n})f \right](\varphi(x)) - \left[(I - K_{n})f \right](\varphi(y)) \right| }{d(x, y)^{\alpha}}. \end{split}$$
(4.9)

Then, for every $f \in lip_0(X, d^{\alpha})$, we have

$$\sup_{0 < d(x,y) < t} \frac{\left| \left[(I - K_n) f \right] (\varphi(x) \right) - \left[(I - K_n) f \right] (\varphi(y)) \right|}{d(x,y)^{\alpha}}$$

$$\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{\left| \left[(I - K_n) f \right] (\varphi(x) \right) - \left[(I - K_n) f \right] (\varphi(y)) \right|}{d(\varphi(x), \varphi(y))^{\alpha}} \cdot \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}}$$

$$\leq L_{\alpha} \left[(I - K_n) f \right] \cdot \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}},$$

$$(4.10)$$

so that

$$\sup_{L_{\alpha}(f) \le 1} \sup_{0 < d(x,y) < t} \frac{\left| \left[(I - K_{n})f \right] (\varphi(x)) - \left[(I - K_{n})f \right] (\varphi(y)) \right|}{d(x,y)^{\alpha}} \le \|I - K_{n}\| \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}}.$$
(4.11)

Now, combining the above inequalities, we obtain

$$\|C_{\varphi}\|_{e} \leq \|I - K_{n}\| \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}} + \sup_{L_{a}(f) \leq 1} \sup_{d(x, y) \geq t} \frac{|[(I - K_{n})f](\varphi(x)) - [(I - K_{n})f](\varphi(y))|}{d(x, y)^{\alpha}}.$$
(4.12)

Letting $n \to \infty$ and using Lemmas 4.3 and 4.4, we conclude that

$$\left\|C_{\varphi}\right\|_{e} \leq \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}}.$$
(4.13)

Finally, taking limits as $t \rightarrow 0$ yields the desired inequality.

5. The Essential Norm of C_{φ} **on** $\operatorname{Lip}_{0}(X, d^{\alpha})$

Now we extend the estimates on the essential norm of a composition operator to the spaces $\operatorname{Lip}_0(X, d^{\alpha})$. Recall that $\operatorname{lip}_0(X, d^{\alpha})^{**}$ is isometrically isomorphic to $\operatorname{Lip}_0(X, d^{\alpha})$ whenever (X, d) is a pointed compact metric space and $\alpha \in (0, 1)$ (see [1, Theorem 3.3.3 and Proposition 3.2.2]). As a matter of fact, the mapping $\Delta : \operatorname{lip}_0(X, d^{\alpha})^{**} \to \operatorname{Lip}_0(X, d^{\alpha})$, defined by

$$\Delta(F)(x) = F(\delta_x), \quad (F \in \operatorname{lip}_0(X, d^{\alpha})^{**}, \ x \in X), \tag{5.1}$$

is an isometric isomorphism.

If *T* is a bounded linear operator on a Banach space, then $||T^*|| = ||T||$. However, this identity is no longer true for the essential norm. Since the adjoint of a compact operator is again a compact operator, we always have $||T^*||_e \le ||T||_e$ and therefore $||T^{**}||_e \le ||T||_e$. Axler et al. [12] showed that in fact $||T^{**}||_e = ||T^*||_e$, but they gave a counterexample where $||T^*||_e < ||T||_e$.

Theorem 5.1. Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a point preserving Lipschitz mapping. Then the essential norm of the operator $C_{\varphi} : \operatorname{Lip}_{0}(X, d^{\alpha}) \to \operatorname{Lip}_{0}(X, d^{\alpha})$ satisfies the lower estimate

$$\left\|C_{\varphi}\right\|_{e} \geq \lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(5.2)

If, in addition, the space $\lim_{\alpha \to 0} (X, d^{\alpha})^*$ has the approximation property, then one has the upper estimate

$$\left\|C_{\varphi}\right\|_{e} \leq \lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x, y)^{\alpha}}.$$
(5.3)

Proof. Let us start with the lower estimate. Let $\{f_n\}$ be the weakly null sequence in $\lim_{n \to \infty} (X, d^{\alpha})$ that we constructed for the proof of Theorem 3.1. Then the sequence $\{f_n\}$ is weakly null in $\operatorname{Lip}_0(X, d^{\alpha})$. Thus, if *K* is any compact operator on $\operatorname{Lip}_0(X, d^{\alpha})$, we have $\lim_{n \to \infty} L_{\alpha}(Kf_n) = 0$. Hence, the same computation we performed in Theorem 3.1 yields the lower estimate

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^{\alpha}}{d(x,y)^{\alpha}} \le \left\| C_{\varphi} \right\|_{e}.$$
(5.4)

Now, for the upper estimate, given $F \in \lim_{\alpha \to 0} (X, d^{\alpha})^{**}$ and $x \in X$, notice that

$$\left(\Delta^{-1}C_{\varphi}\Delta\right)(F)(\delta_{x}) = \left(\left(C_{\varphi}\Delta\right)(F)\right)(x) = C_{\varphi}(\Delta(F))(x) = \Delta(F)(\varphi(x))$$

$$= F(\delta_{\varphi(x)}) = F\left(\delta_{x} \circ C_{\varphi} \mid_{\operatorname{lip}_{0}(X, d^{\alpha})}\right) = \left(F \circ \left(C_{\varphi} \mid_{\operatorname{lip}_{0}(X, d^{\alpha})}\right)^{*}\right)(\delta_{x})$$

$$= \left(C_{\varphi} \mid_{\operatorname{lip}_{0}(X, d^{\alpha})}\right)^{**}(F)(\delta_{x}).$$

$$(5.5)$$

Since $\lim_{0} (X, d^{\alpha})^* = \overline{\operatorname{span}} \{ \delta_x : x \in X \}$, we conclude that $\Delta^{-1}C_{\varphi}\Delta = (C_{\varphi}|_{\lim_{0}(X, d^{\alpha})})^{**}$. Finally, using the relationship between the essential norm of an operator and that of its second adjoint, and applying Theorem 4.1, we get

$$\|C_{\varphi}\|_{e} = \|\Delta^{-1}C_{\varphi}\Delta\|_{e} = \|(C_{\varphi}|_{\operatorname{lip}_{0}(X,d^{\alpha})})^{**}\|_{e} \le \|C_{\varphi}|_{\operatorname{lip}_{0}(X,d^{\alpha})}\|_{e} \le \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x),\varphi(y))^{\alpha}}{d(x,y)^{\alpha}},$$
(5.6)

as we wanted.

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