BISPECTRALITY FOR MATRIX LAGUERRE-SOBOLEV POLYNOMIALS

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Abstract

In this contribution we deal with sequences of polynomials orthogonal with respect to a Sobolev type inner product. A banded symmetric operator is associated with such a sequence of polynomials according to the higher order difference equation they satisfy. Taking into account the Darboux transformation of the corresponding matrix we deduce the connection with a sequence of orthogonal polynomials associated with a Christoffel perturbation of the measure involved in the standard part of the Sobolev inner product. A connection with matrix orthogonal polynomials is stated. The Laguerre-Sobolev type case is studied as an illustrative example. Finally, the bispectrality of such matrix orthogonal polynomials is pointed out.

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1. Introduction

The study of inner products associated with a vector of measures $(d\mu_0, d\mu_1, \cdots, d\mu_N)$ supported on the real line has attracted the interest of many researchers taking into account many properties of standard orthogonal polynomials are lost (see the survey paper [MX15]). In particular, the multiplication operator by x is not symmetric with respect to such inner products and, as a consequence, the corresponding sequences of orthogonal polynomials do not satisfy a three term recurrence relation, that plays a central role in the theory of standard orthogonal polynomials (see [Ch78]). The matrix counterpart of the three term recurrence relation is a tridiagonal matrix that is known in the literature as Jacobi matrix. The spectral theory of such Jacobi matrices is an old topic and yields the so called Favard theorem (see [Ch78]). Assuming you have LU and UL factorization, respectively, of a shifted Jacobi matrix, then the commutation between the matrices in the above factorizations yields new Jacobi matrices whose spectral resolution generates the canonical Christoffel and Geronimus transformations, respectively (see [BM04], [GMM21], [Ga02], [Ga04] [Y02], [Z97], among others). They are the discrete counterpart of the Darboux transformations for second order linear differential operators. When you consider a canonical Christoffel transformation and next a canonical Geronimus transformation of a Jacobi matrix, then the resulting Jacobi matrix has as spectral resolution the so called Uvarov transformation that is a perturbation of the initial spectral measure by adding a Dirac mass point. They appear in the framework of the spectral analysis of fourth order differential operators with polynomial coefficients as analyzed in the pioneering work [HLK40].

The implementation of multiple Christoffel transformations, i. e., an iteration of canonical Christoffel transformations, has been studied in [Ga04]. On the other hand, in [DGM14] the authors focus the attention on multiple Geronimus transformations in a more general framework.

When you deal with a Sobolev inner product associated with a vector of measures as above but $d\mu_k, k = 1, 2, \cdots, N$, are supported on finite subsets of the real line, the so called Sobolev-type inner product appears. The corresponding sequences of orthogonal polynomials are "no so far" of the sequences of standard orthogonal polynomials with respect to the measure $d\mu_0$. This fact was pointed out in [AMRR92] when $N = 1$ and the support of $d\mu_1$ is a point c in the real line that can also be a mass point of the measure $d\mu_0$ and in [MR90] when $d\mu_k = 0, k = 1, 2, \cdots, N - 1$, and support of $d\mu_N$ is a point c in the real line. Algebraic and analytic properties of such orthogonal polynomials have been extensively studied in the literature. In particular, when $d\mu_0$ is the gamma distribution several authors have studied differential operators such that the corresponding eigenfunctions are orthogonal polynomials with respect to Sobolev type inner products assuming the support of the measures $d\mu_k, k = 1, 2, \cdots, N$ is $\{0\}$. The pioneering work [K90] yields an intensive study about the existence and explicit expressions for such differential operators (see [DI15], [KKB98], [KM93], [M19]). When $d\mu_0$ is the beta distribution, a similar analysis was done when the masses are located in one of the end points of the support, i. e., $\{\pm 1\}$ (see [DI18], [M21], [Ma21], [M22]).

Orthogonal polynomials with respect to Sobolev type inner products satisfy higher order recurrence relations associated with a multiplication operator by a polynomial. Such an operator is symmetric with respect to the above inner product. The converse result, an analogue of the Favard's theorem, has been studied in [D93] where a representation of a general symmetric real bilinear form such that there exists a multiplication operator by a polynomial x^{N+1} that is symmetric with respect to such a bilinear form B, i. e., the corresponding sequence of orthogonal polynomials satisfies a symmetric $2N + 3$ recurrence relation, is given. Moreover, the following facts are equivalent (see Corollary 7 in [D93]).

- The multiplication operator by x^{N+1} is a symmetric operator with respect to the bilinear form B, it commutes with the multiplication operator x, i. e., if p, q are polynomials, then $B(x^{N+1}p(x), xq(x)) = B(xp(x), x^{N+1}q(x))$, and $B(x^j, x^k) = B(1, x^{j+k}), 1 \leq j, k \leq N$.
- There exist a function μ_0 and constants $M_k, 1 \leq k \leq N$, such that

$$B(p(x), q(x)) = \int p(x)q(x)d\mu_0(x) + \sum_{k=1}^{N} M_k p^{(k)}(0)q^{(k)}(0).$$

In particular, it was shown in [ELMMR95] that for a Sobolev type inner product

$$\langle f, g\rangle = \int f(x)\, g(x)\, d\mu(x) + \sum_{k=0}^{N} M_k f^{(k)}(c)g^{(k)}(c), \quad M_N > 0,$$

where c is a point in $\mathbb{R}$, the multiplication by $(x - c)^{N+1}$, denoted by E, is a symmetric operator and the sequence of orthogonal polynomials $\{s_n\}_{n\geq 0}$ satisfies a $(2N + 3)$-term recurrence relation of the form

$$(x-c)^{N+1}s_n(x) = \sum_{k=n-N-1}^{n+N+1} a_{n,k}\, s_k(x).$$

In other words, $s_n(x)$ is an eigenfunction of a linear difference operator E in the variable n with eigenvalue $(x-c)^{N+1}$. Notice that, according to [ELMMR95], if you have a Sobolev inner product

$$\langle f, g\rangle = \sum_{k=0}^{N} \int f^{(k)}(x)\, g^{(k)}(x)\, d\mu_k(x)$$

and the multiplication by $(x-c)^{N+1}$ is a symmetric operator with respect to the above inner product, then $d\mu_k(x), k = 1, 2, \cdots, N$, are Dirac deltas supported at $x = c$ and the mass of $d\mu_N(x)$ is a positive real number.

On the other hand, in Theorem 6 [D93] it is proved that the following statements are equivalent.

- The multiplication operator by x^{N+1} is symmetric with respect to the bilinear form B it commutes with the multiplication operator x, i. e., $B(x^{N+1}p(x), xq(x)) = B(xp(x), x^{N+1}q(x))$, where p, q are polynomials.
- There exist a function μ_0 and a positive semi-definite matrix M such that

$$B(p(x), q(x)) = \int p(x)q(x)d\mu_0(x) + (p(0), p'(0), \cdots, p^{(N)}(0))M(q(0), q'(0), \cdots, q^{(N)}(0))^t.$$

This inner product is said to be a nondiagonal Sobolev type inner product. Zeros and asymptotic properties of sequences of orthogonal polynomials with respect to the above inner product have been studied in [AMRR95]. A connection with bispectral problems when $d\mu_0(x)$ is the gamma distribution has been studied in [DI20].

The structure of the manuscript is as follows. In Section 2 we prove that a Darboux transformation of the operator E. i.e., $E = LU$, gives rise to an operator UL which has as eigenfunctions the orthogonal polynomials associated with $(x-c)^{N+1}d\mu(x)$. Furthermore, we prove that UL actually is the $(N+1)$-th power of the *standard* three-term recurrence relation (TTRR in short) that the sequence of polynomials orthogonal with respect to the measure $(x-c)^{N+1}d\mu(x)$ satisfies. Thus, we generalize a result given in [HHLM22] when $N = 1$ concerning the connection between the matrix representation, a five diagonal matrix in terms of the orthonormal basis $s_n(x)$, of the multiplication operator by $(x-c)^2$ and the square of the shifted matrix $J_2 - cI$, where J_2 is the Jacobi matrix associated with the measure $(x-c)^2 d\mu(x)$.

In Section 3 we set a matrix-valued approach by means of [DvA95]. For this regard we consider the specific sequence of Laguerre-Sobolev type orthogonal polynomials to build a monic matrix-valued orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ that satisfy a TTRR with matrix coefficients and we perform a Darboux transformation to find a very interesting connection with results in [DS02]. Namely, we start with a matrix-valued TTRR

$$xP_n(x) = P_{n+1}(x) + (\zeta_{2n+1} + \zeta_{2n})P_n(x) + \zeta_{2n}\zeta_{2n-1}P_{n-1}(x), n \geq 0, P_{-1}(x) = 0,$$

which, after a Darboux transformation, yields a TTRR satisfied by another sequence of monic matrix orthogonal polynomials $\{Q_n\}_{n\geq 0}$

$$xQ_n(x) = Q_{n+1}(x) + (\zeta_{2n+2} + \zeta_{2n+1})Q_n(x) + \zeta_{2n+1}\zeta_{2n}Q_{n-1}(x), n \geq 0, Q_{-1}(x) = 0.$$

Here, the coefficients ζ_n are such that

$$xW_n(x) = W_{n+1}(x) + \zeta_n W_{n-1}(x), n \geq 0, W_{-1}(x) = 0,$$

where $\{W_n\}_{n\geq 0}$ is a sequence of monic matrix orthogonal polynomials given by $W_{2n}(x) = P_n(x^2), n \geq 0$, and $W_{2n+1}(x) = xQ_n(x^2), n \geq 0$.

Finally, in Section 4, we consider the Laguerre-Sobolev type inner product with $\alpha \in \mathbb{N}$, $N = 1$ and $M_1 > 0, M_0 = 0$, to construct a differential operator of order 8 that has every P_n as eigenfunction, showing an underlaying matrix-valued bispectrality. Lastly, we prove that any matrix-valued orthogonal polynomial built from bispectral scalar polynomials with the aid of [DvA95] is bispectral too. Furthermore, we give a general an explicit method to build the corresponding differential operator.

2. Sobolev polynomials under Darboux transformation

Given a probability measure μ supported on an infinite subset of the real line, a point c in the real line and a positive integer N, we consider the following inner products: the one mentioned in the introduction

$$\langle f, g\rangle = \int f(x)\, g(x)\, d\mu(x) + \sum_{j,k=0}^{N} M_{j,k} f^{(j)}(c) g^{(k)}(c), \tag{2.1}$$

where $(M_{j,k})_{j,k=0}^{N}$ is a positive semi-definite matrix of size $(N+1)\times(N+1)$, and other one of the form

$$\langle f, g\rangle_{N+1} = \int f(x)\, g(x)\, (x-c)^{N+1} d\mu(x). \tag{2.2}$$

Now, let us denote by $\{s_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ the sequences of orthonormal polynomials with respect to (2.1) and (2.2), respectively. Immediately, one realizes that, since

$$\langle s_n, p_j\rangle_{N+1} = \langle s_n, (x-c)^{N+1} p_j\rangle_0 = \langle (x-c)^{N+1} s_n, p_j\rangle \tag{2.3}$$

is equal to 0 for $j < n - N - 1$, we have

$$s_n(x) = \sum_{j=n-N-1}^{n} T_{n,j} p_j(x),$$

for some coefficients $T_{n,j}$.

For any two sequences of polynomials $\{\alpha_j\}_{j\geq 0}$ and $\{\beta_j\}_{j\geq 0}$ one can consider the vector notation $\alpha = (\alpha_0, \alpha_1, \dots)^t$ and $\beta = (\beta_0, \beta_1, \dots)^t$. Furthermore, for any inner product $B(\cdot,\cdot)$ we can also consider the bilinear form $B(\alpha, \beta)$ which is nothing more than the semi-infinite matrix whose (j,k)-entry is given by $B(\alpha_j, \beta_k)$. With this notation, if we call $s = (s_0, s_1, \dots)^t$, $p = (p_0, p_1, \dots)^t$, and we define the semi-infinite nonsingular matrix $T = (T_{n,j})_{n,j=0}^{\infty}$, then we have $s = Tp$ and therefore

$$\langle s, s\rangle_{N+1} = \langle Tp, Tp\rangle_{N+1} = TT^*.$$

Recall in this connection that, by definition, the matrix T is not only lower triangular and nonsingular but also has zero entries below the $(N+1)$-th subdiagonal.

On the other hand, we have that the sequence of orthonormal polynomials $\{s_n\}_{n\geq 0}$ satisfies a $(2N+3)$-term recurrence relation of the form

$$(x-c)^{N+1}s_n(x) = \sum_{k=n-N-1}^{n+N+1} h_{n,k}\, s_k(x).$$

This defines a matrix H such that

$$(x-c)^{N+1}s = Hs. \tag{2.4}$$

Since

$$\langle s, s\rangle_{N+1} = \langle Hs, s\rangle = H,$$

we also have the following factorization of H

$$H = TT^{*}.$$

From (2.4) we now have that

$$(x-c)^{N+1}s = TT^{*}s \quad \text{and} \quad (x-c)^{N+1}p = T^{*}Tp. \tag{2.5}$$

This can be summarized as follows.

Theorem 2.1. *For any probability measure μ supported on an infinite subset of the real line, a point c in the real line and a positive integer N, the sequence of Sobolev-type orthonormal polynomials $\{s_n\}_{n\geq 0}$ with respect to*

$$\langle f, g\rangle = \int f(x)\, g(x)\, d\mu(x) + \sum_{j,k=0}^{N} M_{j,k} f^{(j)}(c) g^{(k)}(c),$$

is a Darboux transformation of the sequence of orthonormal polynomials $\{p_n\}_{n\geq 0}$ with respect to

$$\langle f, g\rangle_{N+1} = \int f(x)\, g(x)\, (x-c)^{N+1} d\mu(x),$$

by means of (2.5)*. Namely, if we consider the TTRR satisfied by the sequence of orthonormal polynomials $\{p_n(x)\}_{n\geq 0}$, in vector notation $xp = J_{N+1}p$, the symmetric matrix $(J_{N+1}-c)^{N+1}$ can be factorized as $T^{*}T$, where $s = Tp$. Notice that the matrix $T = (T_{n,j})_{n,j=0}^{\infty}$ can be calculated explicitly*

$$T_{n,j} = \begin{cases} \langle s_n, p_j\rangle_{N+1} & n-N-1 \leq j \leq n, \\ 0 & \textit{elsewhere}. \end{cases}$$

As a straightforward consequence of the above theorem, when in (2.1) $M_{j,k} = 0, j, k = 0, 1, \cdots, N$, we get

Corollary 2.2. *For any probability measure μ supported on an infinite subset of the real line, a point c in the real line and a positive integer N, the sequence of orthonormal polynomials $\{q_n\}_{n\geq 0}$ with respect to*

$$\langle f, g\rangle_0 = \int f(x)\, g(x)\, d\mu(x)$$

is a Darboux transformation of the sequence of orthonormal polynomials $\{p_n\}_{n\geq 0}$ with respect to

$$\langle f, g\rangle_{N+1} = \int f(x)\, g(x)\, (x-c)^{N+1} d\mu(x).$$

Namely, if we consider the TTRR satisfied by $\{p_n\}_{n\geq 0}$ *in vector notation* $xp = J_{N+1}p$*, the symmetric matrix* $(J_{N+1}-c)^{N+1}$ *can be factorized as* C^*C *with* $p = Cq$*. Furthermore,*

$$C_{n,k} = \begin{cases} \langle q_n, p_k\rangle_{N+1} & n-N-1 \leq k \leq n, \\ 0 & \textit{elsewhere.} \end{cases}$$

Finally, let us observe that from the TTRR satisfied by $\{p_n\}_{n\geq 0}$ in vector notation $xp = J_{N+1}p$ we have

$$(x-c)^{N+1}p = (J_{N+1}-c)^{N+1}p.$$

From what we saw above, $(J_{N+1}-c)^{N+1}$ admits an UL-factorization with $U = L^*$, which of course is not unique . We conjecture that all such factorizations give rise to one of the families already considered above.

3. Matrix-valued orthogonal polynomials

In this section we will restrict ourselves to the particular case when $c = 0$. Furthermore, for reasons of space we will simplify the notation by considering $\alpha = 0$, $N = 1$ and the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}dx + f'(0)g'(0). \tag{3.6}$$

The interested reader can verify that the results in the present section hold for more general α and N. Nevertheless we believe that a 2×2 matrix-valued construction with $\alpha = 0$ will suffice to illustrate the situation.

Let us denote by $\{\mathcal{L}_n\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (3.6). Thus we have

$$x^2\mathcal{L}_n(x) = a_n\mathcal{L}_{n+2}(x) + b_n\mathcal{L}_{n+1}(x) + c_n\mathcal{L}_n(x) + b_{n-1}\mathcal{L}_{n-1}(x) + a_{n-2}\mathcal{L}_{n-2}(x), \quad n\geq 2,$$

with

$$a_n = \sqrt{\frac{(2n^2+7n+9)(2n^2-5n+6)(n+4)(n+2)(n+1)^3}{(2n^2+3n+4)(2n^2-n+3)(n+3)}},$$

$$b_n = 4\sqrt{\frac{(4n^7+16n^6+13n^5+10n^4+43n^3+64n^2+84n+36)^2(n+1)}{(2n^2+3n+4)(2n^2-n+3)^2(2n^2-5n+6)(n+3)(n+2)^2}},$$

$$c_n = 2\sqrt{\frac{(12n^8+12n^7-23n^6+57n^5+82n^4-81n^3+37n^2+120n+36)^2}{(2n^2-n+3)^2(2n^2-5n+6)^2(n+2)^2(n+1)^2}}.$$

Let $\{R_{0,n}\}_{n\geq 0}, \{R_{1,n}\}_{n\geq 0}$ be the sequences of polynomials such that for any n

$$\mathcal{L}_n(x) = xR_{1,n}(x^2) + R_{0,n}(x^2).$$

Then, following [DvA95], we build the matrix-valued polynomials

$$R_n(y) = \begin{pmatrix} R_{0,2n}(y) & R_{1,2n}(y) \\ R_{0,2n+1}(y) & R_{1,2n+1}(y) \end{pmatrix}. \tag{3.7}$$

The sequence $\{R_n\}_{n\geq 0}$ satisfies a matrix TTRR

$$xR_n(y) = A^*_{n-1}R_{n-1}(y) + B_nR_n(y) + A_nR_{n+1}(y), n\geq 0, \tag{3.8}$$

with A_n, B_n given, respectively, by

$$
\begin{aligned}
A_{n\,0,0} =& \frac{2\sqrt{8\,n^2+14\,n+9}\sqrt{4\,n^2-5\,n+3}(2\,n+1)^{\frac{3}{2}}\sqrt{n+2}\sqrt{n+1}}{\sqrt{8\,n^2-2\,n+3}\sqrt{4\,n^2+3\,n+2}\sqrt{2\,n+3}},\\
A_{n\,0,1} =& 0,\\
A_{n\,1,0} =& -\frac{4\left(256\,n^7+1408\,n^6+3088\,n^5+3640\,n^4+2692\,n^3+1414\,n^2+570\,n+135\right)\sqrt{n+1}}{\sqrt{8\,n^2+14\,n+9}\sqrt{8\,n^2-2\,n+3}(4\,n^2+3\,n+2)(2\,n+3)\sqrt{n+2}},\\
A_{n\,1,1} =& \frac{2\sqrt{8\,n^2-2\,n+3}\sqrt{4\,n^2+11\,n+9}\sqrt{2\,n+5}\sqrt{2\,n+3}(n+1)^{\frac{3}{2}}}{\sqrt{8\,n^2+14\,n+9}\sqrt{4\,n^2+3\,n+2}\sqrt{n+2}},\\
B_{n\,0,0} =& \frac{2\left(768\,n^8+384\,n^7-368\,n^6+456\,n^5+328\,n^4-162\,n^3+37\,n^2+60\,n+9\right)}{(8\,n^2-2\,n+3)(4\,n^2-5\,n+3)(2\,n+1)(n+1)},\\
B_{n\,0,1} =& -\frac{4\left(128\,n^7+256\,n^6+104\,n^5+40\,n^4+86\,n^3+64\,n^2+42\,n+9\right)\sqrt{2\,n+1}}{(8\,n^2-2\,n+3)\sqrt{4\,n^2+3\,n+2}\sqrt{4\,n^2-5\,n+3}\sqrt{2\,n+3}(n+1)},\\
B_{n\,1,0} =& -\frac{4\left(128\,n^7+256\,n^6+104\,n^5+40\,n^4+86\,n^3+64\,n^2+42\,n+9\right)\sqrt{2\,n+1}}{(8\,n^2-2\,n+3)\sqrt{4\,n^2+3\,n+2}\sqrt{4\,n^2-5\,n+3}\sqrt{2\,n+3}(n+1)},\\
B_{n\,1,1} =& \frac{2\left(768\,n^8+3456\,n^7+6352\,n^6+6744\,n^5+5128\,n^4+2898\,n^3+1099\,n^2+303\,n+63\right)}{(8\,n^2-2\,n+3)(4\,n^2+3\,n+2)(2\,n+3)(n+1)}.
\end{aligned}
$$

Thus $\{R_n\}_{n\geq 0}$ is a sequence of matrix orthonormal polynomials with respect to the positive semi-definite matrix-valued inner product given by

$$
\langle F, G\rangle = \int_0^\infty F(y)\begin{pmatrix} 1 & \sqrt{y} \\ \sqrt{y} & y\end{pmatrix} G^*(y)e^{-y}dy + F(0)\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} G^*(0),
$$

for 2×2 matrix-valued functions F, G.

We now explore the Darboux process for the above matrix TTRR, but applied to the monic matrix orthogonal polynomials. Since the leading coefficient of R_n is given by

$$
\begin{pmatrix}
\frac{\sqrt{4\,n^2-5\,n+3}\sqrt{2\,n+1}}{4\sqrt{8\,n^2-2\,n+3}(2\,n-1)\sqrt{n+1}(n-1)n(2\,n-3)!} & 0 \\
\frac{\left(8\,n^3+6\,n^2-5\,n+3\right)(2\,n+1)}{4\sqrt{8\,n^2-2\,n+3}\sqrt{4\,n^2+3\,n+2}\sqrt{2\,n+3}(2\,n-1)(2\,n-3)\sqrt{n+1}(n-1)n(2\,n-4)!} & \frac{-\sqrt{8\,n^2-2\,n+3}\sqrt{n+1}}{\sqrt{4\,n^2+3\,n+2}\sqrt{2\,n+3}(2n)!}
\end{pmatrix},
$$

we can build explicitly the sequence of monic matrix orthogonal polynomials $\{P_n\}_{n\geq 0}$. They will satisfy a matrix TTRR such that the corresponding Jacobi matrix of (2×2)-blocks can be decomposed in the form LU where L is a lower block triangular matrix and U is a block upper triangular matrix. From Theorem 2.1 the Darboux transformation will give rise to a sequence of monic matrix polynomials $\{Q_n\}_{n\geq 0}$ orthogonal with respect to the weight e^{-y} multiplied by $y = x^2$. More precisely, the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy

$$
\begin{aligned}
xP_n(x) =& P_{n+1}(x) + (\zeta_{2n+1}+\zeta_{2n})P_n(x) + \zeta_{2n}\zeta_{2n-1}P_{n-1}(x), n\geq 0,\\
xQ_n(x) =& Q_{n+1}(x) + (\zeta_{2n+2}+\zeta_{2n+1})Q_n(x) + \zeta_{2n+1}\zeta_{2n}Q_{n-1}(x), n\geq 0,
\end{aligned}
\tag{3.9}
$$

where

$$
\zeta_{2n} = \begin{pmatrix}
-\frac{2\left(16\,n^2-12\,n-9\right)(2\,n-1)^2(n-1)n}{(4\,n^2-5\,n+3)(2\,n+1)} & \frac{4\left(8\,n^3-12\,n^2+4\,n+3\right)n}{(4\,n^2-5\,n+3)(2\,n+1)} \\
-\frac{2\left(16\,n^3-40\,n^2+28\,n-3\right)(2\,n+1)(2\,n-1)^2n}{4\,n^2-5\,n+3} & \frac{2\left(16\,n^3-36\,n^2+29\,n-6\right)(2\,n+1)n}{4\,n^2-5\,n+3}
\end{pmatrix},
$$

$$
\zeta_{2n-1} = \begin{pmatrix}
-\frac{2\left(32\,n^4+8\,n^3-14\,n^2+7\,n+3\right)(2\,n-1)n}{(4\,n^2-5\,n+3)(2\,n+1)} & \frac{4\left(8\,n^3-2\,n+3\right)n}{(4\,n^2-5\,n+3)(2\,n+1)} \\
-\frac{2\left(32\,n^4+16\,n^3-32\,n^2+14\,n+9\right)(2\,n+1)(2\,n-1)n}{4\,n^2-5\,n+3} & \frac{2\left(16\,n^3+4\,n^2-15\,n+12\right)(2\,n+1)n}{4\,n^2-5\,n+3}
\end{pmatrix}.
$$

These TTRR are related through a Darboux transformation. Namely, we have

$$x\begin{pmatrix}P_0\\P_1\\P_2\\P_4\\\vdots\end{pmatrix}=\begin{pmatrix}1&0&0&0&\cdots\\\zeta_2&1&0&0&\cdots\\0&\zeta_4&1&0&\cdots\\0&0&\zeta_6&1&\ddots\\\vdots&\vdots&\ddots&\ddots&\ddots\end{pmatrix}\times\begin{pmatrix}\zeta_1&1&0&0&\cdots\\0&\zeta_3&1&0&\cdots\\0&0&\zeta_5&1&\ddots\\0&0&0&\zeta_7&\ddots\\\vdots&\vdots&\ddots&\ddots&\ddots\end{pmatrix}\begin{pmatrix}P_0\\P_1\\P_2\\P_4\\\vdots\end{pmatrix},$$

$$x\begin{pmatrix}Q_0\\Q_1\\Q_2\\Q_4\\\vdots\end{pmatrix}=\begin{pmatrix}\zeta_1&1&0&0&\cdots\\0&\zeta_3&1&0&\cdots\\0&0&\zeta_5&1&\ddots\\0&0&0&\zeta_7&\ddots\\\vdots&\vdots&\ddots&\ddots&\ddots\end{pmatrix}\times\begin{pmatrix}1&0&0&0&\cdots\\\zeta_2&1&0&0&\cdots\\0&\zeta_4&1&0&\cdots\\0&0&\zeta_6&1&\ddots\\\vdots&\vdots&\ddots&\ddots&\ddots\end{pmatrix}\begin{pmatrix}Q_0\\Q_1\\Q_2\\Q_4\\\vdots\end{pmatrix}.$$

Equation (3.9) may be compared with [DS02, Lemma 3.3].

It is worth to notice that the coefficients for the TTRR of $\{Q_n\}_{n\geq 0}$ are nicer. Indeed,

$$\zeta_{2n+2}+\zeta_{2n+1}=4(n+1)\begin{pmatrix}-(4n+3)(2n+1)&2\\-2\left(4n^2+8n+5\right)(2n+3)(2n+1)&(4n+5)(2n+3)\end{pmatrix},$$

$$\zeta_{2n+1}\zeta_{2n}=4(n+1)n(2n+1)\begin{pmatrix}-(8n+3)(2n-1)&4\\-4\,(2n+3)(2n+1)^2(2n-1)&(8n+5)(2n+3)\end{pmatrix}.$$

This is in concordance with the results of Section 2. Indeed, the sequence of monic matrix polynomials $\{Q_n\}_{n\geq 0}$ is built from a sequence of polynomials satisfying a five term recurrence relation that, as we proved above, is the square iterated of a *standard* TTRR and, as a consequence, we get a very simple expression for their coefficients.

4. Matrix-valued Bispectrality

Once one realizes that the sequence of matrix-valued polynomials $\{R_n\}_{n\geq 0}$ given by (3.7) is related via a Darboux transformation with classical standard orthogonal polynomials it is natural to seek for matrix linear differential equations. As in the previous section we will set $c=0$, $\alpha=0$, $N=1$ and the inner product (3.6), for an initial examplification; at the end we will deal with arbitrary size $N+1$ and general coefficients.

After straightforward computations one can see that for the following 2×2 matrix-valued operator,

$$\mathfrak{D}=\sum_{k=0}^{8}\frac{d^k}{dx^k}D_k(x),$$

acting on the right-hand side of the polynomial $R_n(x)$, yields

$$R_n(x)\mathfrak{D}=\Lambda_n R_n(x),$$

where

$$D_0 = \begin{pmatrix} 0 & 0 \\ -3 & 3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 9\,y-6 & -12 \\ -105\,y+54 & 24\,y \end{pmatrix},$$
$$D_2 = \begin{pmatrix} 27\,y^2+474\,y-72 & -276\,y \\ -6\,(151\,y+459)y & 3\,(19\,y+1100)y \end{pmatrix},$$
$$D_3 = \begin{pmatrix} 24\left(y^2+166\,y+93\right)y & -12\,(53\,y+570)y \\ -4\,(287\,y+6852)y^2 & 8\left(4\,y^2+1278\,y+2205\right)y \end{pmatrix},$$
$$D_4 = \begin{pmatrix} 4\left(y^2+1080\,y+4701\right)y^2 & -8\,(37\,y+2253)y^2 \\ -8\,(47\,y+4908)y^3 & 4\left(y^2+1770\,y+14301\right)y^2 \end{pmatrix},$$
$$D_5 = \begin{pmatrix} 96\,(13\,y+252)y^3 & -32\,(y+348)y^3 \\ -32\,(y+534)y^4 & 384\,(4\,y+123)y^3 \end{pmatrix},$$
$$D_6 = \begin{pmatrix} 96\,(y+101)y^4 & -2208\,y^4 \\ -2656\,y^5 & 32\,(3\,y+443)y^4 \end{pmatrix},$$
$$D_7 = \begin{pmatrix} 1408\,y^5 & -128\,y^5 \\ -128\,y^6 & 1664\,y^5 \end{pmatrix}, \quad D_8 = \begin{pmatrix} 64\,y^6 & 0 \\ 0 & 64\,y^6 \end{pmatrix}$$

and

$$\Lambda_n = \begin{pmatrix} \left(4\,n^3-n+6\right)n & 0 \\ 0 & \left(2\,n^3+3\,n^2+n+3\right)(2\,n+1) \end{pmatrix}.$$

Furthermore, it is easy to check that there is not a linear differential operator of order less than 8 having $\{R_n\}_{n\geq 0}$ as eigenfunctions. On the other hand, the results in [KKB98] prove the existence of a linear differential operator of order $2\alpha+8$ (see [KKB98, Theorem 3.1]). This, of course, is not a coincidence as we will show below.

4.1. **Bispectrality for general size.** Following the construction in [DvA95, page 265], let $N \in \mathbb{N}$ and $\{s_n\}_{n\geq 0}$ be a sequence of orthonormal polynomials, satisfying the $(2N+3)$-term recurrence relation

$$x^{N+1}s_n(x) = \sum_{k=0}^{N+1} \left(c_{n,k}s_{n-k}(x) + \overline{c_{n+k,k}}s_{n+k}(x)\right), n \geq 0,$$

where $c_{n,k}$ are complex numbers, $c_{n,N} = 0$ for any n, the degree of s_n is n, and $s_n = 0$ for $n < 0$.

For any n, let $R_{k,n}$, $k = 0, \dots, n$, be polynomials such that

$$s_n(x) = R_{0,n}(x^{N+1}) + xR_{1,n}(x^{N+1}) + \cdots + x^N R_{N,n}(x^{N+1}).$$

Let

$$R_n(y) = \begin{pmatrix} R_{0,(N+1)n}(y) & R_{1,(N+1)n}(y) & \cdots & R_{N,(N+1)n}(y) \\ R_{0,(N+1)n+1}(y) & R_{1,(N+1)n+1}(y) & \cdots & R_{N,(N+1)n+1}(y) \\ \vdots & \vdots & & \vdots \\ R_{0,(N+1)n+N}(y) & R_{1,(N+1)n+N}(y) & \cdots & R_{N,(N+1)n+N}(y) \end{pmatrix}. \tag{4.10}$$

In [DvA95] it is proved that $\{R_n\}_{n\geq 0}$ is a sequence of matrix polynomials that satisfies a TTRR. We will prove that if there exists a differential operator D having every $s_n(x)$ as eigenfunction, then there is a matrix-valued differential operator that has every $R_n(y)$ as eigenfunction.

Before stating the theorem let us introduce some notation. We denote by w the $(N+1)$-th root of unity $e^{i\frac{2\pi}{N+1}}$. For $y \neq 0$ we denote by $|y|^{\frac{j-1}{N+1}}$ its only positive $(N+1)$-th root. Then all the $(N+1)$-th roots of y are $x = w^j |y|^{\frac{j-1}{N+1}}$ for $j = 0, 1, \dots, N$. Given a differential operator D with coefficients in the variable x we will denote it by $D(x)$ to emphasize the role of variable and by $D\left(w^j|y|^{\frac{j-1}{N+1}}\right)$ the operator obtained after the change of variables $x \to w^j|y|^{\frac{j-1}{N+1}}$.

Theorem 4.1. *Let us assume that there exists a (scalar-valued) linear differential operator $D(x)$ with polynomial coefficients such that*

$$D(x)\, s_n(x) = \lambda_n\, s_n(x), \quad n = 0, 1, \dots .$$

We consider the matrix-valued differential operator $\mathfrak{D}(y)$ acting on the right-hand side, given by

$$\mathfrak{D}(y) = A(y)\, B\, C(y)\, B^{-1}\, A(y)^{-1},$$

where $A(y)$ is a diagonal matrix, B is a constant matrix and $C(y)$ is a (diagonal) matrix-valued operator acting on the right-hand side, all of size $(N+1)\times(N+1)$, such that

$$A(y)_{j,j} = |y|^{\frac{j-1}{N+1}}, \quad B_{j,k} = w^{(j-1)(k-1)}, \quad C(y)_{j,j} = D\left(w^{j-1}|y|^{\frac{1}{N+1}}\right).$$

Then

$$R_n(y)\, \mathfrak{D}(y) = \Lambda_n R_n(y),$$

where R_n are the $(N+1)\times(N+1)$ matrix-valued polinomials given in (4.10) *and Λ_n is the diagonal eigenvalue matrix*

$$\Lambda_n = \begin{pmatrix} \lambda_{(N+1)n} & & & \\ & \lambda_{(N+1)(n+1)} & & \\ & & \ddots & \\ & & & \lambda_{(N+1)N} \end{pmatrix}.$$

Proof. By looking at the entry $(R_n(y)\, A(y)B)_{j,k}$ we have

$$\begin{pmatrix} R_{0,(N+1)n+j-1}(y) & R_{1,(N+1)n+j-1}(y) & \dots & R_{N,(N+1)n+j-1}(y) \end{pmatrix} \times \begin{pmatrix} |y|^{\frac{0}{N+1}} w^{0(k-1)} \\ |y|^{\frac{1}{N+1}} w^{1(k-1)} \\ \cdots \\ |y|^{\frac{N}{N+1}} w^{N(k-1)} \end{pmatrix}$$

which is

$$\begin{aligned} R_{0,(N+1)n+j-1}(y)|y|^{\frac{0}{N+1}} w^{0(k-1)} + \dots + R_{N,(N+1)n+j-1}(y)|y|^{\frac{N}{N+1}} w^{N(k-1)} \\ = s_{(N+1)n+j-1}\left(|y|^{\frac{1}{N+1}} w^{(k-1)}\right). \end{aligned}$$

Since $((R_n(y)\,A(y)B)\ C(y))_{j,k} = D(w^{k-1}|y|^{\frac{1}{N+1}})\,((R_n(y)\,A(y)B)_{j,k})$, we have

$$\begin{aligned}(R_n(y)\,A(y)B\,C(y))_{j,k} =& D(w^{k-1}|y|^{\frac{1}{N+1}})\left(s_{N,(N+1)n+j-1}\left(|y|^{\frac{1}{N+1}}w^{(k-1)}\right)\right)\\ =&\lambda_{(N+1)n+j-1}s_{(N+1)n+j-1}\left(|y|^{\frac{1}{N+1}}w^{(k-1)}\right)\\ =&\lambda_{(N+1)n+j-1}(R_n(y)\,A(y)B)_{j,k}.\end{aligned}$$

This implies that $R_n(y)\,A(y)BC(y) = \Lambda_n R_n(y)\,A(y)B$.

Thus

$$R_n(y)\,A(y)B\,C(y)\,B^{-1}\,A(y)^{-1} = \Lambda_n R_n(y),$$

and we get the desired statement. □

Notice that bispectrality for Krall-Laguerre orthogonal polynomials, an example of standard orthogonal polynomials, when α is a positive integer, has been studied in [DI20] by using a different approach.

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