AUTOMORPHISMS, DERIVATIONS AND GRADINGS OF THE SPLIT QUARTIC CAYLEY ALGEBRA.

VICTOR BLASCO AND ALBERTO DAZA-GARCIA

Abstract. The split quartic Cayley algebra is a structurable algebra which has been used to give constructions of Lie algebras of type D4. Here, we calculate it's group of automorphisms, it's algebra of derivations and it's gradings.

1. Introduction

Structurable algebras are a class of algebras with involution introduced by Allison in 1978 [Ali78] as a generalization of Jordan algebras. They are a generalization in the sense that they also have a Tits-Kantor-Koecher (TKK) construction of a Lie algebra. One of these algebras is the split quartic Cayley algebra which is used for example in [Ali91] to give constructions of Lie algebras of type D4. Here, we calculate its group of automorphism, the algebra of derivations and it's gradings up to isomorphism.

The structure is as follows: in section 2 we define the split quartic Cayley algebra and give a multiplication table, in section 3 we calculate it's group of automorphisms and it's algebra of derivations and in section 4 we calculate it's automorphisms.

We are going to work over an algebraically closed field $\mathbb{F}$ of characteristic different from $2, 3$ and 5. Groups are going to be considered abelian and it's neutral element will be denoted by e, unless we work with specific groups with their own notation.

2. The split quartic Cayley algebra

This section is devoted to introduce the split Cayley algebra. In order to do so, we recall a modified Cayley Dickson process introduced in [AF84] starting with the algebra $\mathcal{B} = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$. Take $\mu \in \mathbb{F}^{\times}$. We denote by t the trace of $\mathcal{B}$. Define $b^{\theta} = -b + \frac{1}{2}t(b)1$ for all $b \in \mathcal{B}$. Let $\mathcal{A} = \mathcal{B} \oplus s\mathcal{B} = \{b_1 + sb_2 \mid b_1, b_2 \in \mathcal{B}\}$. We define a product and an involution in $\mathcal{A}$ by:

$$(b_1 + sb_2)(b_3 + sb_4) = (b_1 b_3 + \mu(b_2 b_4^{\theta})^{\theta}) + s(b_1^{\theta} b_4 + (b_2^{\theta} b_3^{\theta})^{\theta})$$
$$\overline{b_1 + sb_2} = b_1 - sb_2^{\theta}$$

We call this algebra $\mathfrak{CD}(\mathcal{B}, \mu)$. Notice, that since we are in an algebraically closed field, the morphism $b_1 + sb_2 \mapsto b_1 + \sqrt{\mu} sb_2$ is an isomorphism from $\mathfrak{CD}(\mathcal{B}, \mu)$ to $\mathfrak{CD}(\mathcal{B}, 1)$. Hence, from now on, we are going to work with the algebra $\mathfrak{CD}(\mathcal{B}, 1)$. We call this algebra the **split quartic Cayley algebra** check with the isomorphism in [AF84, Proposition 6.5] and the definition in [Ali90].

Call $x_1 = (1, 1, -1, -1), x_2 = (1, -1, 1, -1), x_3 = (1, -1, -1, 1)$. Call $\mathcal{K} = \mathbb{F}1 \oplus \mathbb{F}s$ which is a subalgebra of $\mathcal{A}$ isomorphic to $\mathbb{F} \times \mathbb{F}$ via the automorphism given by

2020 *Mathematics Subject Classification.* Primary 17A01; Secondary 17A30; 17A36.

Key words and phrases. Structurable algebras; automorphisms; gradings.

The second author acknowledges support by the F.P.I. grant PRE2018-087018. He also aknowledges support by grant MTM2017-83506-C2-1-P (AEI/FEDER, UE) and by grant PID2021-123461NB-C21, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

$1 \mapsto (1,1)$, $s \mapsto (1,-1)$. Then, the action $\circ\colon \mathcal{K} \times \mathcal{A} \to \mathcal{A}$ given by $g \circ x = xg$ for all $g \in \mathcal{K}, x \in \mathcal{A}$, endows $\mathcal{A}$ with a structure of left $\mathcal{K}$-module, which is a free $\mathcal{K}$-module spanned by $1, x_1, x_2, x_3$. If we identify $\mathcal{K}$ with $\mathbb{F} \times \mathbb{F}$ and call ex the involution given by $\mathrm{ex}(x,y) = (y,x)$,the multiplication and the involution follows from the following rules:

$$\begin{gathered}(f1)(g1) = (fg)1,\ (gx_i)(f1) = (fg)x_i = (\overline{f}1)(fx_i)\\ (fx_i)(gx_i) = (f\overline{g})1,\ (fx_i)(gx_j) = (\overline{fg}x_k)\end{gathered} \tag{2.1}$$

for all $f, g \in \mathcal{K}$ and $\{i,j,k\} = \{1,2,3\}$ and

$$\overline{f_0 1 + f_1 x_1 + f_2 x_2 + f_3 x_3} = \mathrm{ex}(f_0)1 + f_1 x_1 + f_2 x_2 + f_3 x_3 \tag{2.2}$$

for all $f_0, f_1, f_2, f_3 \in \mathcal{K}$.

Remark 2.1. Notice that if we define the subspaces $\mathcal{S} = \{x \in \mathcal{A} \mid \overline{x} = -x\}, \mathcal{H} = \{x \in \mathcal{A} \mid \overline{x} = x\}, \mathcal{M} = \{x \in \mathcal{H} \mid sx + xs = 0\}$, we get that:

$$\mathcal{S} = \mathbb{F}s,\ \mathcal{H} = \mathbb{F}1 \oplus \left(\bigoplus_{i=1}^{3} \mathcal{K}x_i\right),\ \mathcal{M} = \bigoplus_{i=1}^{3} \mathcal{K}x_i,\ \mathrm{Alg}_{\mathbb{F}}(\mathcal{S}) = \mathcal{K}$$

Remark 2.2. There is a $\mathbb{Z}_2^2$ grading of $\mathcal{A}$ given by $\mathcal{A}_{(\bar{0},\bar{0})} = \mathcal{K}$, $\mathcal{A}_{(\bar{0},\bar{1})} = \mathcal{K}x_1$, $\mathcal{A}_{(\bar{1},\bar{0})} = \mathcal{K}x_2$ and $\mathcal{A}_{(\bar{1},\bar{1})} = \mathcal{K}x_3$. We call this grading the **standard quartic grading** and denote it by Γ_{SQ}.

3. Automorphisms and derivations

In this section we calculate the groups of automorphisms and the algebra of derivations of $(\mathcal{A}, -)$ (i.e. those automorphisms and derivations which commute with the involution). We begin with some easy properties:

Lemma 3.1. *Let* $\varphi \in \mathrm{Aut}(\mathcal{A}, -)$ *and* $d \in \mathrm{Der}(\mathcal{A}, -)$

(1) $\varphi(\mathcal{S}) = \mathcal{S}$, $\varphi(\mathcal{H}) = \mathcal{H}$, $d(\mathcal{S}) \subseteq \mathcal{S}$ *and* $d(\mathcal{H}) \subseteq \mathcal{H}$
(2) $\varphi(\mathcal{K}) = \mathcal{K}$, $\varphi(\mathcal{M}) = \mathcal{M}$, $d(\mathcal{K}) = 0$ *and* $d(\mathcal{M}) \subseteq \mathcal{M}$.

Proof. Each conteinment '$\subseteq$' in (1) is due to the fact that the involution commutes with φ and d. The equalities follow from the fact that φ is invertible.

Since $\varphi(\mathcal{S}) = \mathcal{S}$, There is $\lambda \in \mathbb{F}^\times$ such that $\varphi(s)e = \lambda s$. Since $\varphi(1) = 1$, we get $\varphi(\mathcal{K}) = \mathcal{K}$. If $m \in \mathcal{H}$ and $sm + ms = 0$, applying φ we get that $\lambda(s\varphi(m) + \varphi(m)s) = 0$. Hence $\varphi(\mathcal{M}) \subseteq \mathcal{M}$. We get the equality since φ is invertible. Since d is a derivation $d(1) = 0$. Using (1), there is β such that $d(s) = \beta s$. Since $0 = d(1) = d(s^2) = 2\beta 1$, we get that $d(\mathcal{K}) = 0$. Finally, if $m \in \mathcal{M}$ $0 = d(sm + ms) = sd(m) + d(m)s$. Using (1) it follows $d(\mathcal{M}) \subseteq \mathcal{M}$. □

Now, we will start calculating the automorphisms. In order to do so, we let S_3 be the symmetric on 3 elements, and we will need the following lemma.

Lemma 3.2. *Let* $\varphi \in \mathrm{Aut}(\mathcal{A}, -)$. *There is a permutation in* S_3 *which we denote* σ_φ *such that* $\varphi(\mathcal{K}x_i) = \mathcal{K}x_{\sigma_\varphi(i)}$ *for all* $i \in \{1,2,3\}$.

Proof. Due to lemma 3.1 there are $r_1, r_2, r_3 \in \mathcal{K}$ such that $\varphi(x_i) = r_1x_1 + r_2x_2 + r_3x_3$. Let i be such that $r_i \neq 0$. Then since $1 = \varphi(x_i)^2 = r_1\overline{r_1} + r_3\overline{r_3} + r_3\overline{r_3} + \overline{r_2 r_3}x_1 + \overline{r_1 r_3}x_2 + \overline{r_1 r_2}x_3$. That, due to remark 2.2 means that $r_1r_2 = r_2r_3 = r_3r_1 = 0$ since up to scalar, the only zero divisors in $\mathbb{F} \times \mathbb{F}$ are $(1,0)$ and $(0,1)$, this implies that $r_j, r_k = 0$ for $\{i,j,k\} = \{1,2,3\}$. Hence $\varphi(x_1) = r_ix_i$ and $r_i\overline{r_i} = 1$. Since due to lemma 3.1 $\varphi(\mathcal{K}x_i) = \varphi(\mathcal{K})\varphi(x_i) = \mathcal{K}r_ix_i$, then we have proved that there is a map

$\sigma_\varphi\colon \{1,2,3\} \to \{1,2,3\}$ such that $\varphi(\mathcal{K}x_i) = \mathcal{K}x_{\sigma(i)}$ for all $i \in \{1,2,3\}$. Since φ is invertible this map is a permutation. ☐

Remark 3.3. Let σ be a permutation in S_3. We denote by $f_\sigma\colon \mathcal{A} \to \mathcal{A}$ the map defined as $f_\sigma(r_0 1 + r_1x_1 + r_2x_2 + r_3x_3) = r_0 1 + r_1x_{\sigma(1)} + r_2x_{\sigma(2)} + r_3x_{\sigma(3)}$. Using (2.1) is not hard to check that this is an automorphism of $(\mathcal{A}, -)$. Moreover the map $\theta\colon S_3 \to \mathrm{Aut}(\mathcal{A}, -)$ defined by $\sigma \mapsto f_\sigma$ is a monomorphism of groups and we denote it's image by H.

If we have an algebra with involution $(\mathcal{B}, -)$, a group G and a grading $\Gamma\colon \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, we denote $\mathrm{Aut}(\mathcal{B}, \Gamma, -) := \{\varphi \in \mathrm{Aut}(\mathcal{B}, -) \mid \varphi(\mathcal{B}_g) = \mathcal{B}_g\ \forall g \in G\}$,

Lemma 3.4. $\mathrm{Aut}(\mathcal{A}, -) \cong \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -) \rtimes H$

Proof. Let $\varphi \in \mathrm{Aut}(\mathcal{A}, -)$. We are going to show that $\varphi \circ f^{-1}_{\sigma_\varphi} \in \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$. Since θ as defined in Remark 3.3 is an automorphism $f^{-1}_{\sigma_\varphi} = f_{\sigma_\varphi^{-1}}$. By definition $\varphi(\mathcal{K}x_i) = \mathcal{K}x_{\sigma_\varphi(i)}$ for all i. Hence, $\varphi \circ f^{-1}_{\sigma_\varphi} \in \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$. Therefore, $\mathrm{Aut}(\mathcal{A}, -) = \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)H$. Finally, $f_\sigma \in \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$ if and only if $\sigma = \mathrm{id}$. therefore $\mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -) \cap H = \{\mathrm{id}\}$ finally it is not hard to show that $\mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$ is a normal subgroup so the result follows.

☐

We denote by S^1 the subgroup of $\mathcal{K}^\times$ whose underlying set is $\{r \in \mathcal{K}^\times \mid r\overline{r} = 1\}$ and we denote by C_2 the ciclic group of order 2 generated by σ. We can define an action on $\mathcal{K}$ by $\sigma(s) = -s$. Like this we identify C_2 with $\mathrm{Aut}(\mathcal{K})$.

Lemma 3.5. $\mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -) \cong (S^1 \times S^1) \rtimes C_2$ *with product given by* $(r_1, r_2, g) \star (s_1, s_2, h) = (r_1g(s_1), r_2g(s_2), gh)$.

Proof. Consider the morphism $\theta\colon (S^1 \times S^1) \rtimes Aut(\mathcal{K}) \to \mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$ given by $\theta(r_1, r_2, \psi)(s_0 + s_1x_1 + s_2x_2 + s_3x_3) = \psi(s_0) + \psi(s_1)(r_1x_1) + \psi(s_2)(r_2x_2) + \psi(s_3)(r_3x_3)$. Where $r_3 = \overline{r_1r_2}$. Using (2.1) and (2.2) it is clear that $\theta(r_1, r_2, \psi)$ is an automorphism. Since $r_i(\overline{r}_ix_i) = x_i$ we can check that $\theta(r_1, r_2, \psi)^{-1} = \theta(\psi^{-1}(\overline{r}_1), \psi^{-1}(\overline{r}_2), \psi^{-1})$.

Clearly θ is injective. Moreover, if φ is an element of $\mathrm{Aut}(\mathcal{A}, \Gamma_{SQ}, -)$, then, let $\psi = \varphi_{|\mathcal{K}1}$, $\varphi(x_1) = r_1x_1$, $\varphi(x_2) = r_2x_2$ and $\varphi(x_3) = r_3x_3$. Since $x_i^2 = 1$ for $i = 1, 2$, we get that $r_i\overline{r_i} = 1$. Since $x_1x_2 = x_3$ we get that $r_3 = \overline{r_1r_2}$. Hence, it's easy to show that $\theta(r_1, r_2, \psi) = \varphi$. Since $\mathrm{Aut}(\mathcal{K})$ consist on the identity and the involution sending s to $-s$, it's easy to check that it is isomorphic to C_2. ☐

We can finish calculating the automorphisms with the following proposition:

Theorem 3.6. $\mathrm{Aut}(\mathcal{A}, -) \cong ((S^1 \times S^1) \rtimes C_2) \rtimes S_3$

Proof. This is a consequence of Lemma 3.4 and Lemma 3.5 ☐

Finally, we calculate the derivations. In order to do so, for two given numbers $\lambda, \beta \in \mathbb{F}$ we define the map $d_{(\lambda,\beta)}\colon \mathcal{A} \to \mathcal{A}$ by $d(r_0 + r_1x_1 + r_2x_2 + r_3x_3) = \lambda r_1(sx_1) + \beta r_2(sx_2) - (\lambda + \beta)r_3(sx_3)$.

Theorem 3.7. $\mathrm{Der}(\mathcal{A}, -) = \{d_{\lambda,\beta} \mid \lambda, \beta \in \mathbb{F}\}$

Proof. From Lemma 3.1 we get that for any $d \in \mathrm{Der}(\mathcal{A}, -)$, $d(\mathcal{M}) \subseteq \mathcal{M}$. Therefore, for $i, j, k = \{1, 2, 3\}$ we get that $d(x_i) = r_ix_i + r_jx_j + r_kx_k$ for some $r_i, r_j, r_k \in \mathcal{K}$. Since $0 = d(1) = d(x_i^2) = x_id(x_i) + d(x_i)x_i$ we get that $r_i + \overline{r_i} + 2(\overline{r_j}x_k + \overline{r_k}x_j) = 0$. Therefore, there is some $\lambda_i \in \mathbb{F}$ such that $d(x_i) = \lambda_i(sx_i)$. Moreover, since $\lambda_3(sx_3) = d(x_3) = d(x_1x_2) = x_1d(x_2) + d(x_1)x_2$, it follows that $\lambda_3 = -\lambda_1 - \lambda_2$. Finally, since $d(\mathcal{K}) = 0$ and using the properties of the derivations, it follows that $d(r_0 + r_1x_1 + r_2x_2 + r_3x_3) = \lambda_1r_1(sx_1) + \lambda_2r_2(sx_2) - (\lambda_1 + \lambda_2)r_3(sx_3)$. Now, checking that $d_{1,0}$ and $d_{0,1}$ are derivations is easy and since they span $\{d_{\lambda,\beta} \mid \lambda, \beta \in \mathbb{F}\}$ we get the equality.

□

4. Gradings

Given an algebra with involution $(\mathcal{A}, -)$ and a group G, a **G-grading** Γ on $\mathcal{A}$ is a vector space decomposition:

$$\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$$

satisfying that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ and $\overline{\mathcal{A}_g} \subseteq \mathcal{A}_g$ for all $g, h \in G$. If the grading is fixed we refer to $\mathcal{A}$ as a **G-graded algebra with involution**. We say that an element x is **homogeneous** if there is some $g \in G$ such that $x \in \mathcal{A}_g$. In this case we say that x has degree g and we denote it as $\deg(x) = g$. We say that a subspace V of $\mathcal{A}$ is a **graded subspace** if $V = \bigoplus_{g\in G}(V \cap \mathcal{A}_g)$ in this case we will denote $V_g = V \cap \mathcal{A}_g$.

Remark 4.1. For a G grading Γ, $\mathcal{S}$ and $\mathcal{H}$ are graded subspaces (see [AC20, lemma 3.8]). Moreover, since $\mathcal{S} = s\mathbb{F}$ and $s^2 = 1$, we have that $\deg(s)^2 = e$ where e is the neutral element of G.

Given two G-graded algebras with involution $(\mathcal{A}, -)$ and $(\mathcal{B}, -)$ we say that they are **isomorphic** if there exist an isomorphism of algebras with involution $\varphi\colon \mathcal{A} \to \mathcal{B}$ satisfying that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$.

Given a G-grading Γ and a H-grading Γ' of $(\mathcal{A}, -)$ we say that Γ' is a **coarsening** of Γ (or that Γ is a **refinement** of Γ') if for every $h \in H$ there is a $g \in G$ such that $\mathcal{A}_g \subseteq \mathcal{A}_h$.

The basic facts about gradings can be found in [EK13].

Example 4.2. Given the split quartic Cayley algebra $(\mathcal{A}, -)$ and $\{i, j, k\} = \{1, 2, 3\}$ we can define the $\mathbb{Z}_2$-grading $\Gamma^i_S\colon \mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ with $\mathcal{A}_{\bar{0}} = \mathcal{K} \oplus \mathcal{K}x_i$ and $\mathcal{A}_{\bar{1}} = \mathcal{K}x_j \oplus \mathcal{K}x_k$.

These gradings are a coarsening of the standard quartic grading Γ_{SQ}. Moreover, given $i \neq j$ and a permutation σ with $\sigma(i) = j$ we get that Γ^i_s is isomorphic to Γ_j via the automorphism f_σ with the notation of 3.3

In this section we are going find up to isomorphism the gradings on $(\mathcal{A}, -)$. We start with a lemma:

Lemma 4.3. *For a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ on $(\mathcal{A}, -)$, the subspaces $\mathcal{K}$ and $\mathcal{M}$ are graded subspaces.*

Proof. In any algebra with involution $\mathcal{S}$ and $\mathbb{F}1$ are graded subspaces. Hence, $\mathcal{K} = \mathbb{F}1 \oplus \mathcal{S}$ is a graded subspace.

Let $m \in \mathcal{M}$ and let $a_g \in \mathcal{A}_g$ be such that $m = \sum_{g\in G} a_g$. Let g_0 be the degree of s and for every $g \in G$ denote by π_g the projection on $\mathcal{A}_g$ with respect to the decomposition given by the grading. Since $0 = sm + ms$ and $0 = \pi_g(sm + ms) = sa_g + a_g s$, we get that for every $g \in G$, $a_g \in \mathcal{M}$. Therefore, $\mathcal{M}$ is a graded subspace. □

Since $s^2 = 1$, it's easy to deduce using (2.1) that for any $m \in \mathcal{M}$ $s(sm) = m$. Therefore, we can define two subspaces of $\mathcal{M}$:

$$\mathcal{M}_\sigma = \{m \in \mathcal{M} \mid sm = \sigma m\} \quad \text{for } \sigma = \pm$$

And $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$.

Lemma 4.4. *For a G grading Γ, $\deg(s) = e$ if and only if $\mathcal{M}_+$ and $\mathcal{M}_-$ are graded subspaces.*

Proof. Let $\deg(s) = e$. In this case, if $m \in \mathcal{M}_g$, for some $g \in G$ then, there are $m_+ \in \mathcal{M}_+$ and $m_- \in \mathcal{M}_-$ such that $m = m_+ + m_-$. Since $sm = m_+ - m_- \in \mathcal{M}_g$ we get that $m_\sigma \frac{1}{2}(m + (\sigma sm)) \in \mathcal{M}_g$ for $\sigma = \pm$. Hence, $\mathcal{M}_g = (\mathcal{M}_+ \cap \mathcal{M}_g) \oplus (\mathcal{M}_- \cap \mathcal{M}_g)$. From that is easy to check that $\mathcal{M}_+$ and $\mathcal{M}_-$ are graded. If $\mathcal{M}_+$ and $\mathcal{M}_-$ are graded, let $g = \deg(s)$. Then, let $m \in (\mathcal{M}_+)$ for some $h \in G$ it should happen that $h = \deg(m) = \deg(sm) = gh$ and so $g = e$. □

Remark 4.5. Notice that $\mathcal{M}_+ = \frac{1}{2}(1+s)\mathcal{M}$ and $\mathcal{M}_- = \frac{1}{2}(1-s)\mathcal{M}$ so we are going to call $e_+ = \frac{1}{2}(1+s)$ and $e_- = \frac{1}{2}(1-s)$.

We denote as $b\colon \mathcal{M} \times \mathcal{M} \to \mathbb{F}$ the bilinear form which satisfies $xy = b(x,y)1 + \lambda s + m$ for $\lambda \in \mathbb{F}$ and $m \in \mathcal{M}$.

Lemma 4.6. *For any G grading on $\mathcal{A}$, b is a non-degenerate homogeneous bilinear form (i.e. $b(\mathcal{M}_g, \mathcal{M}_h) = 0$ if and only if $gh = e$).*

Proof. In order to show that b is non degenerate, we take $m \in \mathcal{M}$. then, there are $r_1, r_2, r_3 \in \mathcal{K}$ such that $m = r_1x_1 + r_2x_2 + r_3x_3$. Without loss of generality, we suppose that $r_1 \neq 0$. Then, either $r_1 = \beta e_\sigma$ for $\sigma = \pm$ or $r_1\overline{r_1} = \beta x_1$ in both cases with $\beta \neq 0$. In the first case $b(x, e_{-\sigma}x_1) = \beta$ and in the second case $b(x, r_1x_1) = \beta$.

In order to show you that it is homogeneous, we take $x \in \mathcal{M}_g$ and $y \in \mathcal{M}_h$. Then, $xy \in \mathcal{A}_{gh}$. Suppose that $gh \neq e$. If $\deg(s) = gh$, then, $\mathcal{A}_{gh} = \mathbb{F}s \oplus \mathcal{M} \cap \mathcal{A}_{gh}$ and in other case $\mathcal{A}_{gh} = \mathcal{M} \cap \mathcal{A}_{gh}$

□

Example 4.7. Let G be an abelian group, $i \in \mathbb{F}$ such that $i^2 = -1$ and $\zeta \in \mathbb{F}$ a primitive cubic root of unit.

(1) For $g_1, g_2 \in G$ denote by $\Gamma_{SQ}(G, g_1, g_2)$ the grading on $(\mathcal{A}, -)$ given by $\deg(s) = e$, $\deg(e_+x_1) = g_1$, $\deg(e_+x_2) = g_2$, $\deg(e_+x_3) = (g_1g_2)^{-1}$, $\deg(e_-x_1) = g_1^{-1}$, $\deg(e_-x_2) = g_2^{-1}$ and $\deg(e_-x_3) = g_1g_2$.
(2) For $g, g_1, g_2 \in G$ with g an element of order 2, denote by $\Gamma_{SQ}(G, g, g_1, g_2)$ the grading given by $\deg(s) = g$, $deg(x_1) = g_1$, $deg(x_2) = g_2$, $deg(x_3) = g_1g_2$, $deg(sx_1) = gg_1$, $deg(sx_2) = gg_2$ and $deg(sx_3) = gg_1g_2$
(3) For $\lambda \in \mathbb{F}^\times$ and $h, g, f \in G$ such that $g^2 = f^2 = h^{-1}$ and $g \neq f$, we denote by $\Gamma_S(G, \lambda, h, g, f)$ the grading in which $\deg(s) = e$, $\deg(e_+x_1) = h = \deg(e_-x_1)^{-1}$, $\deg(e_+(x_2 + \lambda x_3)) = g$, $\deg(e_+(-\lambda^{-1}x_2 + x_3)) = f$
(4) For $h, g \in G$ with h of order 2, we denote by $\Gamma_S^1(G, h, g)$ the grading induced by $\deg(s) = h$, $\deg(e_+x_2 + e_-x_3) = g$ and $\deg(e_-x_2 + e_+x_3) = g^{-1}$.
(5) For $h, g \in G$ such that h has order 2 and g has order 4 we denote by $\Gamma_S^2(G, h, g)$ the grading for which $\deg(s) = h$, $\deg(x_1) = g^2$ and $\deg(x_2 + ix_3) = g$.
(6) For $h, g, f \in G$ with h, g and f of order 2, we denote by $\Gamma_S^3(G, h, g, f)$ the grading for which $\deg(s) = h$ and $\deg(x_2 + x_3) = g$ and $\deg(x_2 - x_3) = f$.
(6) For $g_1, g_2 \in G$ of order 3 and $g_1 \neq g_2 \neq (g_1g_2)^{-1}$, we denote by $\Gamma(G, g_1, g_2)$ the grading given by $\deg(s) = e$, $\deg(e_+(x_1 + \zeta x_2 + \zeta^2 x_3)) = g_1$, $\deg(e_+(x_1 + \zeta^2 x_2 + \zeta x_3)) = g_2$, $\deg(e_+(x_1 + x_2 + x_3)) = (g_1g_2)^{-1}$
(7) For $h, g_1 \in G$ such that h has order 2 and g has order 3, we denote by $\Gamma(G, h, g)$ the grading given by $\deg(s) = h$ and $\deg(x_1 + \zeta x_2 + \zeta^2 x_3) = g$.

Given a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and a H-grading $\Gamma'\colon \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, we say that the gradings are **compatible** if $\mathcal{A} = \bigoplus_{(g,h) \in G \times H} \mathcal{A}_g \cap \mathcal{A}_h$.

Proposition 4.8. *If Γ is a grading is compatible with Γ_{SQ}, then it is isomorphic to either $\Gamma_{SQ}(G,g_1,g_2)$ for some $g_1,g_2\in G$ as in example 4.7or to $\Gamma_{SQ}(G,g,g_1,g_2)$ for some g,g_1,g_2 as in example 4.7.*

Proof. If $\Gamma\colon \mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ is such a grading with $\deg(s)=e$. Since it is compatible with Γ_{SQ}, for every $i=1,2,3$ there should be a g_i such that $\mathcal{A}_{g_i}\cap\mathcal{K}x_i=\mathcal{K}x_i$ or g_i and g_i' such that $(\mathcal{A}_{g_i}\cap\mathcal{K}x_i)\oplus(\mathcal{A}_{g_i'}\cap\mathcal{K}x_i)$. In the first case, $\deg(e_+x_i)=\deg(e_-x_i)=g_i$. In the second case, since $s(\mathcal{A}_h\cap\mathcal{K}x_i)=(\mathcal{A}_h\cap\mathcal{K}x_i)$ for $h=g_i$ or $h=g_i'$, we can assume that $\mathcal{A}_{g_i}\cap\mathcal{K}x_i=e_+x_i$ and that $\mathcal{A}_{g_i'}\cap\mathcal{K}x_i=e_-x_i$. Since $(e_+x_i)(e_-x_i)=e_+$, we get that in both cases $\deg(e_+x_i)\deg(e_-x_i)=e$. Finally, since $(e_-x_1)(e_-x_2)=e_+x_3$, we get that $g_3=g_1^{-1}g_2^{-1}$. Hence $\Gamma=\Gamma_{SQ}(G,g_1,g_2)$.

If $\Gamma\colon\mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ is such a grading with $\deg(s)=g$ for g an order 2 element. Since each $\mathcal{K}x_i$ are graded, if for $\sigma=\pm$, $e_\sigma x_i$ is homogeneous, $\deg(e_\sigma x_i)=\deg(s(e_\sigma x_i))=g\deg(e_\sigma x_i)$. Hence, for every $i=1,2,3$ there is a group element g_i there is an invertible $r_i\in\mathcal{K}$ such that $\deg(r_ix_i)=g_i$ since the field is algebraically closed, we can assume that $r_i\overline{r_i}=1$. Since $(r_ix_i)^2=1$, $g_i^2=e$. Moreover, since $(r_1x_1)(r_2x_2)=(\overline{r_1r_2})x_3$, we can assupe that $r_3=\overline{r_1r_2}$ and that $g_3=g_1g_2$. Hence, Γ is isomorphic to $\Gamma_{SQ}(G,g,g_1,g_2)$ via the morphism $\theta(r_1,r_2,Id)$ with the notation of 3.5.

□

Proposition 4.9. *If $\Gamma\colon\mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ is a grading compatible with Γ_S^i for some $i=1,2,3$ but not with Γ_{SQ} then either it is isomorphic to $\Gamma_S(G,\lambda,h,g,f)$ with elements as in example 4.7 or it is isomorphic to $\Gamma_S^i(G,h,g)$ for $i=1,2$ with the notation as in example 4.7.*

Proof. Up to isomorphism we can suppose that it is compatible with Γ_S^1. Due to lemma 4.3, $\mathcal{K}x_1$ is a graded subspace.

If $\deg(s)=e$, since $s\mathcal{K}x_1=\mathcal{K}x_1$, then e_+x_1 and e_-x_2 are homogeneous. Let $\deg(e_+x_1)=h$. Since $(e_+x_1)(e_-x_1)$, we get that $\deg(e_-x_1)=h^{-1}$. We are going to prove that e_+x_2 cannot be homogeneous. We prove it by contradiction. If it is homogeneous of degree g, $(e_+x_2)(e_+x_1)=e_-x_3$ is homogeneous of degree gh. Necesarily, there should be a $\lambda\in\mathbb{F}$ such that $e_+(\lambda x_2+x_3)$ is homogeneous of degree f. Necessarily $f\neq g$ otherwise e_+x_3 is homogeneous and multiplying by e_+x_1, we get that e_-x_1 is homogeneous and that means that the grading is compatible with Γ_{SQ}. Now, multiplying by e_+x_1 we get that $e_(x_2+\lambda x_3)$ is homogeneous of degree fh. Since $b(e_-(x_2+\lambda x_3),e_+x_2)=\frac{1}{2}$ and $b(e_-(x_2+\lambda x_3),e_+(\lambda e_2,e_3))=\lambda$ and since $\ker(b(e_-(x_2+\lambda x_3)),\ \cdot\ )_{|(\mathcal{K}x_2\oplus\mathcal{K}_{x_3})\cap\mathcal{M}_+}$ has to be a graded subspace, then $\lambda=0$. Therefore, e_+x_3 is homogeneous and as we saw before, this leads to a contradiction with the fact that Γ is not compatible with Γ_{SQ}.

Due to the previous discussion, we can assume (because we can multiply by scalar) that there are $\lambda,\beta\in\mathbb{F}^\times$ such that $e_+(x_2+\lambda x_3)$ is homogeneous of degree g and $e_+(\beta x_2+x_3))$ is homogeneous of degree f and both are linearly independent. Multiplying by e_+x_1 you get that $e_-(\lambda x_2+x_3)$ is homogeneous of degree gh and that $e_-(x_2+\beta x_3)$ is homogeneous of degree fh. Call $\varphi=b(e_-(\lambda x_2+x_3),\ \cdot\ )_{|(\mathcal{K}x_2\oplus\mathcal{K}_{x_3})\cap\mathcal{M}_+}$. Since $\ker(\varphi$ has to be a graded subspace of $x_2\oplus\mathcal{K}_{x_3})\cap\mathcal{M}_+$ and $\varphi(e_+(x_2+\lambda x_3))=\lambda\neq 0$, necessarily, $0=\varphi(e_+(\beta x_2+x_3))=\frac{1}{2}(\lambda\beta+1)$. In order to see that $g^2=f^2=h^{-1}$, we see that the square of $(e_+(x_2+\lambda x_3))$ and of $e_+(-\lambda^{-1}x_2+x_3)$ are nonzero multiples of e_-x_1. Therefore, $\beta=-\lambda^{-1}$ and therefore, Γ is isomorphic to $\Gamma_S(G,\lambda,h,g,f)$.

If $\deg(s)=h$ for $h\neq e$, clearly $h^2=e$. By lemma 4.4 we know that there is an invertible $r_1\in\mathcal{K}$ such that r_1x_1 is homogeneous of degree f. Using the automorphism $\theta(\frac{1}{\sqrt{r_1\overline{r_1}}}\overline{r_1},1,Id)$ we can assume that $r_1=1$. We are going to prove by contradiction that for no $r_2 \in \mathcal{K}$, r_2x_2 is homogeneous. Suppose it is. If r_2 is multiple of e_σ for some $\sigma = \pm$, then $h\deg(r_2x_2) = \deg(s(r_2x_2)) = \deg(r_2x_2)$ which would be a contradiction. If r_2 is invertible, since $s(r_2x_2) = (sr_2)x_2$, $x_1(r_2x_2) = \overline{r_2}x_3$ and $(s\overline{r_2})x_3$ are homogeneous, Γ would be compatible with Γ_{SQ}. Hence, there are $r_2, r_3 \in \mathcal{K}\setminus 0$ such that $r_2x_2+r_3x_3$ is homogeneous of degree g, then multiplying by x_1 we get that $\overline{r_3}x_2 + \overline{r_2}x_3$ is homogeneous of degree gf.

If there is no $r_2x_2 + r_3x_3$ homogeneous with r_2, r_3 invertible, then for an homogeneous element like this, $(r_2x_2 + r_3x_3)^2 = 2\overline{r_2r_3}x_1 = 0$ since $r_2r_3 \in \mathbb{F}e_\sigma$ for some $\sigma = \pm$. We can suppose then that $r_2 = \lambda e_\sigma$ and $r_3 = \beta e_{-\sigma}$. Moreover, by scaling the element we can suppose that $\lambda = 1$. Call $x = r_2x_2 + r_3x_3$. Since x_1x has is a linear combination of x and sx then, either $x_1x = x$ in which case $\beta = 1$ and $\deg(x_1) = e$ or $x_1x = sx$, in which case $\beta = -1$ and $\deg(x_1) = \deg(s)$. Using $\theta(s, 1, Id)$ if necessary, we can suppose that $\beta = 1$ and $\deg(x_1) = e$. Hence, $x = e_+x_2 + e_-x_3$ is homogeneous of degree g and since there should be an homogeneous element which doesn't belong to $\operatorname{span}\{x, sx, x_1x, (sx_1)x\} = \operatorname{span}\{x, sx\}$, using the same arguments we see that $y = e_-x_2 + e_+x_3$ is homogeneous. Since $xy = 1+x_1$, we get that $b(x, y) \neq 0$ and so y is homogeneous of degree g^{-1}. Hence, Γ is isomorphic to $\Gamma_S^1(G, h, g)$

Finally, assume that $r_2x_2+r_3x_3$ is homogeneous with r_2, r_3 invertible, using the automorphism $\theta(1, \frac{1}{\sqrt{r_2\overline{r_2}}}\overline{r_2})$ and multiplying by scalar we can assume that $r_2 = 1$. Since $(x_2+r_3x_3)^2 = 2\overline{r_3}x_1$ and it is homogeneous, we can assume that $r_3 \in \mathbb{F}1 \cup \mathbb{F}s$. Using if necesary the automorphism $\theta(s, 1, \mathrm{id})$ we can suppose that $r_3 = \lambda 1$ for some $\lambda \in \mathbb{F}^\times$. If $\deg(x_1) \neq e$, since $b(x, x_1x) \neq 0$ we get that $b(x, x_1x) = 0$ and that means that $\lambda^2 = -1$. Since in this case $-\lambda x_1x = x_2 - \lambda x_3$, any choice of λ would be an homogeneous element. Hence the grading is isomorphic to $\Gamma_S^2(G, h, g)$. Finally, if $\deg(x_1) = e$, $\lambda = \pm 1$. Since $\mathcal{K}x + \mathcal{K}(x_1x) = \mathbb{F}x \oplus \mathbb{F}sx$ we need to complete with another homogeneous element. By the same argument it has to be $y = x_2 - x_3$ so the grading is $\Gamma_S^3(G, h, g, f)$ where $\deg(x_2 + x_3) = g$. □

Proposition 4.10. *Let $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ be a grading on $(\mathcal{A}, -)$ which is not compatible with any Γ_S^i. Then it is isomorphic either to $\Gamma(G, g_1, g_2)$ for g_1, g_2 of order 3 or to $\Gamma(G, h, g)$ for h of order 2 and g of order 3.*

Proof. If $\deg(s) = e$ we start by proving that $e_\sigma x_i$ cannot be homogeneous. Since we can use the automorphisms f_τ and $\theta(1, 1, \mathrm{ex})$, we can prove it for $i = 1$ and $\sigma = +$. In this case, $\ker(b(e_+x_1, \cdot)) \cap \mathcal{M}_- = \mathbb{F}e_-x_2 \oplus \mathbb{F}e_-x_3$ is homogeneous. Then, since b is non degenerate, there should be an homogeneous element of degree g, $x = e_-(x_1 + \lambda_2x_2 + \lambda_3x_3)$ with $\lambda_2, \lambda_3 \in \mathbb{F}$. Since $b(e_+x_1, x)$ and $b(x^2, x)$ are not 0, it follows that e_+x_1 and x^2 have the same degree. Since $y = \frac{1}{2}x^2 - \lambda_2\lambda_3e_+x_1 = e_+(\lambda_3x_2 + \lambda_2x_3)$, if $\lambda_2\lambda_3 \neq 0$ then, since y^2 is homogeneous, then e_-x_1 is homogeneous and then $\ker(b(e_+x_1, \cdot)) \cap \ker(b(e_-x_1, \cdot)) = \mathcal{K}x_2 \oplus \mathcal{K}x_3$ is graded and because of that this grading is compatible with Γ_S^1. If $\lambda_2 \neq 0$ but $\lambda_3 = 0$. Since $x^2(e_+x_1)$ is homogeneous, e_-x_2 is homogeneous. Hence $z = x(e_-x_2) = e_+(\lambda_3x_1 + x_3)$ is homogeneous. Since $b(x, z) = \lambda_3 \neq 0$ it follows that z and e_+x_1 have the same degree and so $z - \lambda_3e_+x_1 = e_+x_3$ is homogeneous. Therefore, $(e_+x_2)(e_+x_3) = e_-x_1$ is homogeneous and it follows as before that it is not compatible with Γ_S^1. If $\lambda_2 = \lambda_3 = 0$ we have it because of the same argument.

Let $x = e_+(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)$ be an homogeneous element of degree g. It follows that $\lambda_1\lambda_2\lambda_3 \neq 0$. Hence, by scalar multiplication we can assume that $\lambda_1\lambda_2\lambda_3 = 1$. Take another homogeneous element $y = e_+(\beta_1x_1 + \beta_2x_2 + \beta_3x_3)$ of degree h with $\beta_1\beta_2\beta_3 = 1$ such that $g \neq h$ (which should exist since the grading is not compatible with Γ_S^1). We can check that $(x^2)^2 = 4x$ and $(y^2)^2 = 4y$. That means that $g^3 = e$ and $h^3 = e$. Moreover, its easy to see that $b(x, x^2) = 6\lambda_1\lambda_2\lambda_3 \neq 0$ and

$b(y, y^2) = 6\beta_1\beta_2\beta_3$. And because $h^2g \neq e$ we deduce that $b(x, y^2) = b(y, x^2) = 0$. That implies that

$$\lambda_1\lambda_2\beta_3 + \lambda_2\lambda_3\beta_1 + \lambda_3\lambda_1\beta_2 = \beta_1\beta_2\lambda_1 + \beta_2\beta_3\lambda_1 + \beta_3\beta_1\lambda_2 = 0 \tag{4.1}$$

Moreover, since $hg \neq g^2$ and $gh \neq h^2$, we deduce that $xy \neq x^2$ and $xy \neq y^2$. Since $xy = e_+[(\lambda_2\beta_3 + \lambda_3\beta_2)x_1 + (\lambda_1\beta_3 + \lambda_3\beta_1)x_2 + (\lambda_2\beta_1 + \lambda_1\beta_2)x_3]$ using (4.1) we see that $xy = e_+(-\lambda_2\lambda_3\beta_1\lambda_1^{-1}x_1 - \lambda_1\lambda_3\beta_2\lambda_2^{-1}x_1 - \lambda_1\lambda_2\beta_3\lambda_3^{-1}x_3)$. Therefore, if we call $z' = -xy$ we can see that it's coefficients products equals to 1. Hence, for $z = \frac{1}{2}z'^2$ we get that $z^2 = 2z'$ and $(z^2)^2 = 4z$. And we can check that the map sending $x \mapsto \deg(e_+(x_1 + \zeta x_2 + \zeta^2 x_3)) = g_1$, $y \mapsto \deg(e_+(x_1 + \zeta^2 x_2 + \zeta x_3)) = g_2$ and $z \mapsto \deg(e_+(x_1 + x_2 + x_3)) = (g_1g_2)^{-1}$ is an isomorphism and so the grading is isomorphic to $\Gamma(G, g, h)$.

If $\deg(s) = h$, as before, $\mathcal{K}x_i$ cannot be a graded subspace.

If all the homogeneous elements $x = r_1x_1 + r_2x_2 + r_3x_3$ such that r_1, r_2 and r_3 are non zero, then, the projection of x^2 in $\mathcal{M}$ is $y = \overline{r_2r_3}x_1 + \overline{r_1r_3}x_3 + \overline{r_1r_2}x_3$ which is homogeneous. If r_1 and r_2 are not invertible, then this is in $\mathcal{M}_\sigma$ for $\sigma = \pm$ and it would happen that $\deg(y) = \deg(sy)$ which can't happen unless $r_3 = 0$. If r_1 is not invertible but r_2 and r_3 are invertible, we use y to show a contradiction. Hence r_1, r_2 and r_3 are invertible. Using the map $\theta(\frac{1}{\sqrt{r_1\overline{r_1}}}\overline{r_1}, \frac{1}{\sqrt{r_2\overline{r_2}}}\overline{r_2})$ we can assume that r_1 and r_2 are scalars. If $\deg(x) = g$, since $(x^2)^2 = (r_1r_2r_3)x$, we can assume that either $r_3 \in \mathbb{F}s$ or $r_3 \in \mathbb{F}1$. Since we can scale we can suppose that $r_1r_2r_3 = 1$ or $r_1r_2r_3 = s$. In the first case and in the second case $g^3 = e$ $g^3h = e$ if $g^3h = e$ we can multiply by s and use the automorphism $\theta(s, s, \mathrm{id})$ and we are in the first case. Now, either there is an element like this whose degree has order 3 or there are 3 linearly independent elements whose degree is 3. In the second case, necessarily, since $\mathcal{K}$ has dimension 2, there must be an element of degree 3 such that r_1, r_2 or r_3 is 0 so we don't consider it here. Now, since the projections of $x, x^2, x^2x, sx, s(x^2x)$ on $\mathcal{M}$ span $\mathcal{M}$, necessarily, $b(x, x^2) = 6r_1r_2r_3 \neq 0$ and that implies $r_3 \in \mathbb{F}1$. Now, up to scalar, we can suppose that $r_1r_2r_3 = 1$ and we can check that the map sending $x \to x_1 + \zeta x_2 + \zeta x_3$ induces an isomorphism of algebras. Therefore, Γ is isomorphic to $\Gamma(G, h, g)$.

Finally, we will show that these are all the possibilities. Indeed, if there is an homogeneous element $x = r_1x_1 + r_2x_2$ of degree g for r_1 and r_2 different from 0, since $x^2 = 2\overline{r_1r_2}x_3$ necessarily, we get that $r_1r_2 = 0$. Hence, we can suppose that there is $\lambda_1, \lambda_2 \in \mathbb{F}$ such that $x = \lambda_1e_+x_1 + \lambda_2e_-x_2$. Moreover, $sx = \lambda_1e_+x_1 - \lambda_2e_-x_2$ is also homogeneous of degree gh. We can show that all homogeneous elements $y = t_1x_1 + t_2x_2 + t_3x_3$ with $t_1, t_2, t_3 \in \mathcal{K}$ have t_1, t_2 or t_3 equal to 0. Otherwise, $xy = e_-\overline{t_2} + e_+\overline{t_3} + (e_+\overline{t_3})x_1 + (e_-\overline{t_3})x_2 + (e_-\overline{t_2} + e_+\overline{t_3})x_3$ so either y or xy has coefficients which are not invertible and arguing as before, this is impossible. Hence, all the homogeneous elements in $\mathcal{M}$ should be of the form $\lambda_ie_\sigma x_i \pm \lambda_je_{-\sigma}x_j$ for $i \neq j$ and $\lambda_i, \lambda_j \in \mathbb{F}$. We can finally show, that if $x = \lambda_1e_+x_1 + \lambda_2e_-x_2$ is homogeneous, there should be $\beta_2, \beta_3 \in \mathbb{F}^\times$ such that $y = \beta_2e_+ + \beta_3e_-$ is homogeneous. But since $xy = \lambda_2\beta_2e_- + \lambda_1\beta_2e_-x_3 + \lambda_2\beta_3e_+x_1$ and that would imply that e_- is homogeneous since $\mathcal{K}$ and $\mathcal{M}$ are homogeneous subspaces. But this would be a contradiction with the fact that $\deg(s) \neq \deg(1)$. □

References

[Ali78] B.N. Allison, *A class of nonassociative algebras with involution containing the class of Jordan algebras*, Math. Ann. **237** (1978), 133–156.

[Ali91] B.N. Allison, *Construction of 3x3 matrix Lie algebra of type D4*, Journal of Algebra **143** (1991), 63–92.

[Ali90] B.N. Allison, *Simple structurable algebras of skew-dimension 1*, Comm. Algebra **18** (1990), 1245–1279.

[AF84] B.N. Allison and J.R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, Trans. Amer. Math. Soc. **283** (1984), 185–210.

[AC20] D. Aranda-Orna and A.S. Cordova-martinez *Gradings on tensor products of compositon algebras and on the Smirnov algebra* , Linear algebra and it's applications **548** (2020), 1–36.

[EK13] A. Elduque and M. Kochetov, *Gradings on simple Lie algebras*, Mathematical Surveys and Monographs **189**, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013.

Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain
Email address: victorblasco98@gmail.com

Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain
Email address: albertodg@unizar.es