Dynamics of deterministic and stochastic systems containing colored noise and delay or memory

(Dinámica de sistemas deterministas y estocásticos que contienen ruido coloreado y retardo o memoria)

Memoria escrita por

Fengling Wang

para optar al grado de doctora del
Programa Oficial de Doctorado "Matemáticas"

V º B º El Director del Trabajo *La Autora*

Tomás Caraballo Garrido Fengling Wang

Dpto. Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas
Universidad de Sevilla

Enero 2024

This work has been partially supported by the Spanish Ministerio de Ciencia e Innovación (AEI) and FEDER under project PID2021-122991NB-C21, and China Scholarship Council, No. 202106990035.

Acknowledgements

I would like to thank everyone who has helped me during the completion of this PhD thesis.

To begin with, I want to thank my supervisor, Prof. Tomás Caraballo Garrido. It was his advice during the writing process and careful revision of the thesis after completion that enabled me to complete this thesis. In academics, he is knowledgeable and insightful, and he is a wise, serious, and good adviser. In daily life, he is optimistic about everything, cares about his students, and is a good friend worth learning from in all aspects. I would like to thank him for all the help and care he gave me during the year I stayed in Sevilla.

Then, I would like to thank my supervisor at Southwest University, Prof. Yangrong Li. I appreciate his teaching and helping me during the 7 years from my master's degree to my doctoral degree. Grateful to Prof. Bixiang Wang for helping me to be able to study with Prof. Caraballo. With thanks to Renhai Wang and Wenjie Hu for advising and helping me when I was confused. Thanks to Jiaohui Xu, Lingyu Li, and Dandan Yang for helping me in my studies and life.

Also, thanks to the staff of the IMUS Secretariat for their care during the stay in Sevilla. To the Graduate School and the School of Mathematics and Statistics of Southwest University for providing good policy support for my visit abroad, and to the CSC (China Scholarship Council) for providing financial support for the living expenses in Spain.

My final thanks go to my family and friends for consistently supporting me and allowing me to live a carefree life. Thanks to my dear mommy, daddy, grandparents, and myself, for continuing to make it this far. Wishing everyone all the best for the future!

Contents

Introduction

Infinite dimensional dynamical systems mainly consider the long-time behavior of solutions of nonlinear dissipative evolutionary partial differential equations arising from natural sciences such as physics, fluid dynamics, life sciences, and atmospheric sciences.

For autonomous (time-independent) deterministic infinite-dimensional dynamical systems, the system usually possesses an attractor to which all orbits converge, i.e., the global attractor. Global attractors (which are compact sets in the phase space, attracting the image of particular sets of initial states under the evolution of the dynamical system), in general, are used to describe the long-term dynamical behavior of the system (see, e.g., [75, 82]). During its study, we noticed nonlocal heat/parabolic equations, as well as memory terms.

• *Nonlocal and memory.* Non-local operators appear naturally in phenomena such as elasticity problems, water waves, phase transitions, and flame propagation [19]. Parabolic nonlocal equations have important applications in physical biology and ecology. Evolutionary models with memory terms are widespread. Natural and social phenomena are often influenced not only by their current state but also by their history. Thus problems describing hereditary phenomena in heat conduction and thermodynamics have attracted much attention.
As they are interesting, we have investigated some aspect of them in Chapter 1.

For non-autonomous (time-dependent) stochastic infinite-dimensional dynamical systems, the dynamical behavior of the system is commonly described in terms of a pullback random attractor, which is an extension of random attractors (first introduced in [26, 27]) from autonomous to non-autonomous and was proposed in [10, 12, 86].

A natural extension of the deterministic differential equations model is a system of stochastic differential equations, in which the relevant parameters are modeled as suitable stochastic processes or the stochastic processes are added to the driving system equations. The theory of stochastic differential equations was proposed by K. Itô in 1942. Since then, the theory has been developed in different directions. For example, consider the equation itself as a dynamic system disturbed by noise. The noise term is added to capture the phenomenon that does not exist in the corresponding deterministic model.

We are interested in the following two types of equations and aim to apply them to stochastic situations (see Chapter 2-Chapter 5).

• *Delay differential equations.* In the actual modeling process, the time delay is inevitable and reasonable, which means that the current state depends on the past state. A time-delay system is usually described in the form of differential equations [68]. Based on the boundedness of the time delay variables, the time delay differential equations can be classified into bounded time delay differential equations and unbounded time delay differential equations. For infinite delay equations, choosing a suitable phase space is more difficult than for bounded delay equations, see, e.g., [44, 49].

• *Lattice differential equations.* Lattice systems are widely used in physics, biology, and other fields, to model problems such as nerve impulse propagation, chemical reactions, electric circuits, etc. [24]. Several numerical simulations have revealed that lattice differential equations display a rich variety of dynamical phenomena, including mode formation, traveling waves, and spatial chaos. The solutions and the long-term dynamics of deterministic lattice systems were studied in [41, 85] without delay and [17, 18] with delay, the long-time behavior of stochastic lattice systems has been investigated in [16, 20] without delay and [55, 56, 62] with delay.

There are two theories dealing with the asymptotic qualitative behavior for general stochastic differential equations: the theory of random dynamical systems and the theory of existence and uniqueness of invariant measures for the associated Markov semigroup.

It is stated in [4] that random dynamical systems consist of two basic elements: A noise model and a model of the system perturbed by noise. On this basis, we introduce two kinds of noises that are studied in this thesis as follows:

• *Nonlinear white noise.* An m-dimensional Wiener process $W(t) = \{W_j(t), 1 \leq j \leq m\}$, defined for $t \geq 0$ with state space $\mathbb{R}^m$, is a stochastic process whose components $W_j(t)(j = 1, \dots, m)$, are independent scalar standard Wiener processes. Each W_j is a scalar process that satisfies

(1) $W_j(0) = 0$ a.s.,

(2) $W_j(t) - W_j(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$,

(3) for all time $0 < t_1 < t_2 < \cdots < t_n$, the random variable $W_j(t_1), W_j(t_2) - W_j(t_1), \dots, W_j(t_n) - W_j(t_{n-1})$ are independent ("independent increments").

The process is continuous and a homogenous Markov diffusion process. White noise is the time derivative of the Wiener process. Typically, when the white noise that drives a stochastic partial differential equation is additive or linear multiplicative, the stochastic equation can be converted into a pathwise random equation using the Ornstein-Uhlenbeck transformation, and thus we can deal with such path random equations through the same methods used to deal with deterministic equations. However, currently, there is no way to convert stochastic partial differential equations with nonlinear white noise into pathwise random differential equations. To study the long-time dynamical behavior of stochastic partial differential equations perturbed by nonlinear white noise, B. Wang [90] introduced the concept of weak pullback mean random attractor for mean random dynamical systems and established existence results. Nevertheless, the inspiring results for this method were developed in several earlier papers by T. Caraballo, P. Kloeden, B. Schmalfuß and T. Lorenz (see [11, 53].

• *Nonlinear colored noise.* Colored noise, also known as the Ornstein-Uhlenbeck process, was first proposed and named by G. Uhlenbeck, L. Ornstein, and M. Wang in [83, 97], with the aim of approximating the Wiener process that is nowhere differentiable about the sample paths. By setting $W(s,\omega)=\omega(s)$, we consider a random variable $\zeta_\delta:\Omega\to\mathbb{R}$ defined by

$$\zeta_\delta(\omega)=\frac{1}{\delta}\int_{-\infty}^{0}e^{\frac{s}{\delta}}dW(t,\omega)=-\frac{1}{\delta^2}\int_{-\infty}^{0}e^{\frac{s}{\delta}}d\omega(s),\ \text{for each } \delta>0,\omega\in\Omega.$$

The process $z_\delta(t,\omega)=\zeta_\delta(\theta_t\omega)$ is called an Ornstein-Uhlenbeck process (i.e. the colored noise), which is a stationary Gaussian process with $\mathbb{E}(\zeta_\delta)=0$ and is the unique stationary solution of the stochastic equation:

$$dz+\frac{1}{\delta}zdt=\frac{1}{\delta}dW(t).$$

As mentioned in [74], many physical systems should be simulated using colored noise instead of white noise. Stochastic partial differential equations perturbed by colored noise are pathwise, so the coefficients of the colored noise can be nonlinear functions when we study the random attractors of such equations. Some results are now obtained on the study of stochastic partial differential equations driven by nonlinear colored noise [42, 43].

We usually prove that there exists at least one solution to stochastic differential equations by giving some bounded estimates, with appropriate assumptions for all external forces. If the data of the problem do not satisfy a Lipschitz continuity condition, the solution of stochastic differential equations may not be unique. In this case, we verify the existence of multi-valued dynamical systems generated by all the solutions by proving the continuity and the cocycle property of solutions. By proving the measurability of set-valued maps generated by multiple solutions, we are able to guarantee that these solutions generate a multi-valued random dynamical system. For the theory of multi-valued random dynamical systems, one can refer to [13, 14, 16, 17, 92].

Invariant measures otherwise known as smooth distributions, also characterize the possible long-term behavior of the system [33]. In [89, 91] the author has discussed the existence of invariant probability measures for *autonomous* stochastic systems driven by nonlinear noise. After that, similar studies with different stochastic systems have been largely developed, see, e.g., [23, 55, 56, 98, 99], where [23, 55, 56] also discussed the limiting stability of the invariant probability measure. For *non-autonomous* stochastic systems, as described by Da Prato and Röckner [34, 35], we consider the existence of evolution systems of probability measures of its time inhomogeneous transition operators. In [100, 103], the authors applied their ideas to specific stochastic models.

This PhD thesis is structured in five chapters. In Chapter 1, we study the asymptotic behavior of a nonlocal semilinear degenerate heat equation with degenerate history in a bounded domain, aiming to obtain the *global attractor* of the original equation rather than for the transformed equation obtain by Dafermo's transformation. In Chapter 2, we consider the higher-order continuity of pullback random attractors for nonautonomous random quasilinear equation driven by nonlinear colored noise. The goal is to verify the existence and residual dense continuity of *pullback random bi-spatial attractors* in both square and p-order (where $p > 2$) Lebesgue spaces. In Chapter 3, we mainly consider the long-term behavior of p-Laplace equations with infinite delays driven by nonlinear colored noise, the purpose of which is to obtain the existence of weak solutions (uniqueness cannot be obtained due to the lack of the Lipschitz condition), the regularity of solutions, and the existence of *pullback random attractors*. In Chapter 4, we establish the existence and the limiting behavior of *periodic measures* for periodic stochastic modified Swift-Hohenberg lattice systems with variable delays. In Chapter 5, we are concerned with the asymptotic stability of *evolution systems of probability measures* for non-autonomous stochastic discrete modified Swift-Hohenberg equations driven by locally Lipschitz nonlinear noise.

Now, we present the main results of this thesis.

I Global attractors for deterministic systems with degenerate memory

In Chapter 1, we consider nonlocal semilinear degenerate heat equations.

For semi-linear degenerate parabolic equations, the existence of attractors (global attractors, pullback attractors, random attractors) was studied in [1, 2, 28]. The above papers do not consider memory terms in the formulation. For linear non-degenerate memory, [40] discussed the existence and uniqueness of solutions and the existence of absorbing sets in an appropriate space, [102] derived the existence of global attractors of the original equation, not the Dafermos transformed equation, [67] considered the long-time behavior for semilinear degenerate parabolic equations with non-degenerate memory. Regarding semilinear degenerate memory, [38] studied the asymptotic behavior of the heat equation defined over a bounded domain, and used semigroup theory to obtain the existence of solutions.

In Chapter 1, we consider the asymptotic behavior of nonlocal semilinear degenerate heat equations with degenerate memory on a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 2)$ given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\mathrm{div}[a(x)\nabla u] - \displaystyle\int_{-\infty}^{t} k(t-s)\mathrm{div}[a(x)\nabla u(s)]ds + f(u) = g(x,t), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \\ u(x,t+\tau) = \phi(x,t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{1}$$

where $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, the function $m \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies $0 < h \leq m(r)$ for all $r \in \mathbb{R}$, and $k : \mathbb{R}^+ \to \mathbb{R}$ stands for the memory kernel.

In Section 1.2, we use the Dafermos transformation ((1.7)-(1.9)) to obtain the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\text{div}[a(x)\nabla u] - \displaystyle\int_0^\infty \mu(s)\text{div}[a(x)\nabla \eta^t(s)]ds + f(u) = g(x,t), \\ \dfrac{\partial}{\partial t}\eta^t(s) = u - \dfrac{\partial}{\partial s}\eta^t(s), & \text{in } \Omega \times (\tau,\infty)\times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{2}$$

where

$$\eta^\tau(x,s) = \int_0^s u(x,\tau - r)dr = \int_0^s \phi(x,-r)dr = \int_{-s}^0 \phi(x,r)dr := \eta_0(x,s), \quad \forall s > 0.$$

For any $0 < \epsilon < 1$, we let $a_\epsilon(x) = a(x) + \epsilon$, denote $z_\epsilon(t) = (u_\epsilon(t), \eta_\epsilon^t)^T$ and $z_{0,\epsilon} = (u_0, \eta_0)^T$. To complete the well-posedness of problem (1), we consider the following non-degenerate equation:

$$\begin{cases} \partial_t z_\epsilon = \mathcal{G} z_\epsilon + \mathcal{F}(z_\epsilon), & \text{in } \Omega \times (\tau,\infty), \\ z_\epsilon(x,t) = 0, & \text{on } \partial\Omega \times (\tau,\infty), \\ z_\epsilon(x,\tau) = z_{0,\epsilon} & \text{in } \Omega, \end{cases} \tag{3}$$

where $\mathcal{F}(z_\epsilon) = (-f(u_\epsilon) + g, 0)^T$ and

$$\mathcal{G} z_\epsilon = \begin{pmatrix} m(l(u_\epsilon))\text{div}[a_\epsilon(x)\nabla u_\epsilon] + \int_0^\infty \mu(s)\text{div}[a(x)\nabla \eta_\epsilon^t(s)]ds \\ u_\epsilon - \partial_s \eta_\epsilon \end{pmatrix}.$$

We first prove the existence of the solution of the non-degenerate equation (3), so that when ϵ tends to 0, we can use the conclusion of Theorem 1.6 to obtain the existence, uniqueness, and regularity of the solution to problem (2). And then use results (Lemma 1.4 and Lemma (1.5)) fulfilled by the related operators $\mathcal{F}$ ((1.15)) and $\mathcal{I}$ ((1.17)) respectively, which are derived by the method mentioned in [102], to obtain the existence, uniqueness, and regularity of a solution to problem (1) (see Theorem 1.7). Different from the method of using semigroup theory to obtain the existence of solutions in [38], here we use the Faedo-Galerkin method to prove the existence of solutions of the non-degenerate heat equation (3).

In Section 1.3, we establish an autonomous dynamical system in the Hilbert space $\mathcal{M} = L^2(\Omega) \times L^2_{H_0^a}$, and then show the existence of the global attractor associated with the continuous semigroup $S(t)$ generated by problem (1).

The content of Chapter 1 comes from the paper [93].

II Pullback random attractors for stochastic systems with nonlinear colored noise with or without delay

In Chapter 2 and Chapter 3, we consider random p-Laplace equations.

The p-Laplace partial differential equation often appears in the physical studies about non-Newtonian fluid dynamics. It also occurs in descriptions of phenomena related to nonlinear elasticity, nonlinear filtering, or magnetic field distribution (see [37]). The long-term dynamical behavior (especially the existence of pullback attractors) of the p-Laplace equation has been extensively studied, see e.g., [54, 57] in a general single-valued random dynamical system with Lipschitz continuous conditions. In the absence of Lipschitz continuous condition, the uniqueness of the solution cannot be ensured, thus the long-time dynamics of the p-Laplace equation in multi-valued random dynamical systems are discussed, for instance, in [21, 43, 88]. Note that none of the above papers possesses a delay term.

By a *pullback random attractor* for a non-autonomous random dynamical system (shortly, a cocycle [86]) on X over a probability space Ω, we mean a family $\{\mathcal{A}(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\}$ of compact sets with an invariance property, pullback attraction and measurability in X. In Chapter 3, we prove the existence of pullback random attractors for multi-valued dynamical systems.

A *pullback random bi-spatial attractor* for a regular non-autonomous random dynamical system, which is a pullback random attractor in initial space X such that it is random compact in regular space Y and pullback attracts some subsets of X under the topology of Y, and which generalizes a pullback random attractor to the bi-spatial case and a random bi-spatial attractor in [57] to the non-autonomous case. In Chapter 2, we prove the residual dense continuity (including the existence) of pullback random bi-spatial attractor $\{\mathcal{A}_\lambda(t,\omega)\}$ on both spaces $X := L^2(\mathbb{R}^m)$ and $Y := L^p(\mathbb{R}^m)$ concerning $\lambda \in \Lambda := (0, \infty]$ in single-valued dynamical systems, where $\infty \in \Lambda$ and the infinite noise corresponds to the deterministic equation.

The existence of attractors for models involving hereditary characteristics with infinite delays has been discussed extensively, e.g., in [70, 101, 102] for single-valued dynamical systems and in[13, 14, 15, 76] for multi-valued dynamical systems.

• **Single-valued random differential equations without delay**

In Chapter 2, we consider the higher-order continuity of pullback random attractors for random p-Laplace equations on $\mathbb{R}^m$ driven by nonlinear colored noise

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \alpha u = f(x,u) + g(t,x) + h(x,u)\zeta_\lambda(\theta_t\omega), \ t > \tau, \\ u(\tau, x) = u_\tau(x), \ \tau \in \mathbb{R}, \ x \in \mathbb{R}^m, m \in \mathbb{N}, \end{cases} \tag{4}$$

where $\lambda > 0, p > 2$, $\zeta_\lambda(\theta_t \cdot)$ is the colored noise and $\{\theta_t\}_{t\in\mathbb{R}}$ is a group of self-transformations on the classical Wiener space Ω.

Throughout this chapter, (X, Y) is a limit-identical pair of separable Banach spaces, where the following limit-identical property

$$\{x_n\} \subset X \cap Y, \ \|x_n - x\|_X \to 0, \|x_n - y\|_Y \to 0 \quad \Rightarrow \quad x = y. \tag{5}$$

is equivalent to that $(X \cap Y, \|\cdot\|_X + \|\cdot\|_Y)$ is a Banach space (see [57, Lemma 2.1]). Hausdorff semimetric and Hausdorff metric are respectively defined by

$$d_X(A_1, A_2) = \sup_{a\in A_1} \inf_{b\in A_2} \|a - b\|_X, \quad \rho_X(A_1, A_2) = \max\{d_X(A_1, A_2), d_X(A_2, A_1)\} \tag{6}$$

for $A_1, A_2 \subset X$. It is standard to prove that $d_{X\cap Y} = d_X + d_Y$ and $\rho_{X\cap Y} = \rho_X + \rho_Y$.

In Section 2.2, we establish an abstract existence result (see Theorem 2.8) of a unique pullback random bi-spatial attractor. The main difficulty is to prove the measurability of the attractor in Y by using four verifiable conditions (**B4-B7**). In particular, we assume that the cocycle operator from $X \cap Y$ into Y is strong-weakly continuous (rather than quasi-continuity used in [30]). This condition cannot ensure that the image set of the cocycle operator on a compact subset of $X \cap Y$ is a compact subset of Y. But we can prove that the image set is a closed bounded set in Y. So, we may consider the larger metric space $CB(Y)$ (than $\mathcal{C}(Y)$) of all closed bounded set in Y, equipped with the Hausdorff metric.

In Section 2.3, we establish an abstract result (see Theorem 2.12) on the residual dense continuity of pullback random bi-spatial attractors in $X \cap Y$ by using six verifiable conditions (**C1-C6**), and in its proof, we will use the abstract Baire residual theorem (see [47, theorem 5.1], and the Baire density theorem ([59, Prop. 2]).

It is worth pointing out that the approximation from random (or stochastic) to deterministic equations ($\lambda \to \infty$) is significative (see [87]). In particular, Flandoli et al. [39] pointed out that a deterministic transport equation is unsolved, while the stochastic (or random) version is solved.

In Section 2.4-Section 2.6, we prove the existence and the residual dense continuity of pullback random bi-spatial attractors $\{\mathcal{A}_\lambda(t,\omega)\}$ of the equation (4) on both spaces $X := L^2(\mathbb{R}^m)$ and $Y := L^p(\mathbb{R}^m)$ with respect to $\lambda \in \Lambda := (0, \infty]$.

The content of Chapter 2 comes from the work [61].

• Multi-valued random differential equations with delay

In Chapter 3, we consider the existence of pullback random attractors for non-autonomous p-Laplace equations with infinite delay (representing the history of variables) on a bounded domain $\mathcal{O} \subset \mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u = f(t,x,u) + g(x, u(t-\varrho(t))) \\ \quad + \displaystyle\int_{-\infty}^{0} F(x,l,u(t+l))dl + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega), \ t > \tau, \quad x \in \mathcal{O}, \\ u(t,x) = 0, \ t > \tau, \quad x \in \partial\mathcal{O}, \\ u(\tau+s,x) = \varphi(s,x), \quad s \in (-\infty, 0], \quad x \in \mathcal{O}, \tau \in \mathbb{R}, \end{cases} \tag{7}$$

where $p \geq 2, \lambda > 0$, ζ_δ is the colored noise with correlation time $\delta > 0$, and W is a scalar Wiener process on the classical Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$. The nonlinear drift term f and the nonlinear diffusion term h are continuous functions but not necessarily Lipschitz continuous, and the delay terms $g : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ and $F : \mathcal{O} \times \mathbb{R}_- \times \mathbb{R} \to \mathbb{R}$ are also non-Lipschitz continuous.

Let X be a Hilbert space. To deal with the delay terms g and F in (7), we denote our phase space by

$$C_{\gamma,X} = \{w \in C((-\infty,0];X) : \lim_{\tau\to-\infty} e^{\gamma\tau} w(\tau) \ \text{ exists}\}, \tag{8}$$

where $\gamma > 0$ and we set $\|w\|_{C_{\gamma,X}} := \sup_{\tau\in(-\infty,0]} e^{\gamma\tau}\|w(\tau)\| < \infty$ for all $w \in C_{\gamma,X}$. From [13], we know that $C_{\gamma,X}$ is a separable Banach space.

In Section 3.2, we prove the existence of weak solutions to equation (7). To solve this problem, we use the traditional Galerkin approximations technique, and according to the method in [13], we prove that $u \to \int_{-\infty}^{0} F(x,l,u(t+l))dl$ is continuous from $C_{\gamma,L^2(\mathcal{O})}$ into $L^2(\mathcal{O})$ as shown in Remark 3.3, to obtain the existence of weak solutions (see Theorem 3.11).

In fact, we also try to prove the regularity of pullback random attractors for equation (7). For p-Laplacian equations with bounded delays, Sobolev's compactness theorem and Arzelà-Ascoli's theorem can be applied to prove the regularity of pullback attractors as in [66]. But for the space considered in this paper (shown in (8)), we find that there is no embedding relationship between spaces $C_{\gamma,L^2(\mathcal{O})}$ and $C_{\gamma,W_0^{1,p}(\mathcal{O})}$. Therefore, we can only prove the regularity of the solution (see Theorem 3.14) by applying the method of [21].

In Section 3.3, we show the measurability of multi-valued dynamical systems and pullback attractors and the asymptotic compactness of solutions. To obtain pullback asymptotically compactness of solutions, we will use the same technique as [76, Lemma 5.5]. The measurability of the pullback attractor will be deduced by proving the upper-semicontinuity of multi-valued functions, the closure of a graph on some subspaces of the probability space by using the methods in [13].

The content of Chapter 3 comes from the paper [94].

III Invariant measures for stochastic systems with nonlinear white noise with or without delay

In Chapter 4 and Chapter 5, we consider stochastic modified Swift-Hohenberg lattice systems.

The Swift-Hohenberg equation was first proposed in 1977 by Swift and Hohenberg (see [80]) to study the analogy between the branch of fluid hydrodynamic behavior and the related PDEs as well as continuous phase transitions in thermodynamical systems. About the modified Swift-Hohenberg equation, there have been many papers discussing the existence of attractors (global, pullback, random, uniform) and bifurcations, e.g., [72, 73], yet there are only a few papers on the modified Swift-Hohenberg lattice system, [45] considered the deterministic case and [96] was in the stochastic case.

We consider a Banach space as below:

$$\ell^r = \Big\{ w = (w_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |w_i|^r < +\infty \Big\}, \quad r \geq 1,$$

with the norm $\|w\|_r^r = \sum_{i\in\mathbb{Z}} |w_i|^r$. For $r = 2$, it has the norm $\|w\|^2 = \sum_{i\in\mathbb{Z}} |w_i|^2$ and inner product $(w,v) := \sum_{i\in\mathbb{Z}} w_i v_i$. In addition, we denote the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ by $C(\mathbb{R},\mathbb{R})$.

The existence of *periodic measures* in ℓ^2 for stochastic lattice systems without delay was investigated in [63]. In the delay case, [55, 62] considered the periodic measure of the stochastic lattice system. The above nonlinear terms are all globally Lipschitz continuous, in Chapter 4 we would like to consider variable delays, the nonlinear functions, the modified term, and the cubic terms are all locally Lipschitz continuous.

In contrast to considering the existence of invariant measures in ℓ^2 of time-homogeneous transition operators for **autonomous** stochastic lattice systems driven by nonlinear white noise in [23, 55, 56, 89, 99], in Chapter 5 we use the extended Krylov-Bogolyubov method mentioned by Da Prato and Röckner in [34, 35] to prove the existence of *evolution systems of probability measures* of time-inhomogeneous transition operators for **non-autonomous** stochastic lattice systems (see Theorem 5.14).

• Periodic measures for stochastic lattice systems with delay

In Chapter 4, we take into account the existence and the limiting behavior of periodic measures for the periodic stochastic delay modified Swift-Hohenberg lattice systems on the integer set $\mathbb{Z}$ given by:

$$\begin{cases} du_i(t) + q_1(t)\Big[\big(u_{i+2}(t) - 4u_{i+1}(t) + 6u_i(t) - 4u_{i-1}(t) + u_{i+2}(t)\big) + 2\big(u_{i-1}(t) \\ \quad - 2u_i(t) + u_{i+1}(t)\big)\Big]dt + q_2(t)u_i(t)dt + q_{3,i}(t)|u_{i+1}(t) - u_i(t)|^2 dt + u_i^3(t)dt \\ = f_i(t, u_i(t), u_i(t-\varrho(t)))dt + g_i(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty} \big(h_{i,j}(t) + \sigma_{i,j}(t, u_i(t), u_i(t-\varrho(t)))\big)dW_j(t), \quad t > 0, \\ u_i(s) = \varphi_i(s), \quad s \in [-\rho, 0], \ i \in \mathbb{Z}, \end{cases} \tag{9}$$

which is obtained by a spatial discretization of the periodic continuous modified Swift-Hohenberg equation with a variable time delay on $\mathbb{R}$:

$$\begin{cases} du(t) + q_1(t)\Delta^2 u(t)dt + 2q_1(t)\Delta u(t)dt + q_2(t)u(t)dt + q_3(t)|\nabla u(t)|^2 dt + u^3(t)dt \\ = f(t, u(t), u(t-\varrho(t)))dt + g(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty} (h_j(t,x) + \sigma_j(t, u(t), u(t-\varrho(t)))dW_j(t), \ t > 0, \\ u(s) = \varphi(s), \quad s \in [-\rho, 0]. \end{cases} \tag{10}$$

Here $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ are positive and continuous, $q_3 = (q_{3,i})_{i\in\mathbb{Z}} : \mathbb{R} \to \ell^2$ is continuous, $f_i, \sigma_{j,i} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions for every $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, noise intensity $0 < \epsilon \leq 1$ and delay parameter $\rho > 0$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$ and $h_j : \mathbb{R} \to L^2(\mathbb{R})$ are given, and $(W_j)_{j\in\mathbb{N}}$ is a sequence of standard two-sided real-valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. Furthermore, for a given positive constant $\mathcal{T}$, all time-dependent terms of the system (9) are $\mathcal{T}$-periodic in time (as shown in (4.13)). Equation (10) is a usual Swift-Hohenberg equation when $q_3(t) = 0,\ f(t, u(t), u(t - \varrho(t))) = 0$ and $\epsilon = 0$.

Let us denote by $C([-\rho, 0]; \ell^2)$ the Banach space of all ℓ^2-valued continuous functions on $[-\rho, 0]$ with the norm

$$\|x\|_{C([-\rho,0];\ell^2)} = \sup_{s\in[-\rho,0]} \|x(s)\| = \sup_{s\in[-\rho,0]} \sum_{i\in\mathbb{Z}} |x_i(s)|^2,\ \forall x \in C([-\rho, 0]; \ell^2).$$

In Section 4.3, the aim is to apply Krylov-Bogolyubov's method to prove the existence of periodic measures of the lattice system (9) for all $\epsilon \in [0, 1]$ in the space $C([-\rho, 0], \ell^2)$. We need to prove the tightness of distribution laws of solutions to (9), and the difficulty of proving this tightness is analogous to the fact that the Sobolev embedding is no longer compact when stochastic PDEs are over an unbounded domain. To address this difficulty, we show that the tail of the solution to (9) is uniformly small in $L^2(\Omega, \mathcal{F}, \ell^2)$ (its definition can be found in [90]) using the uniform tail-estimation method proposed in [84]. Furthermore, since the solution $u_t(\cdot)$ of the system (9) depends on the past history and is therefore non-Markovian, we use the method in [71] to ensure that the solution map for stochastic functional differential equations with finite delay possesses Markov property.

In Section 4.4, the objective is to prove, under stronger assumptions, the limiting behavior of periodic measures of the system (9) when the noise intensity $\epsilon \to \epsilon_0 \in [0, 1]$ (see (4.120)). We show that for $\epsilon \in [0, 1]$, the set of all periodic measures of (9) is weakly compact, and any limit point of a tight sequence of periodic measures of the system (9) must be an invariant measure of the associated limiting equation (see Theorem 4.16).

The content of Chapter 4 comes from the paper [95].

• Evolution systems of measures for stochastic lattice systems without delay

In Chapter 5, we consider the stochastic discrete modified Swift-Hohenberg equations with nonlinear noise for $t > \tau$ with $\tau \in \mathbb{R}$ as follows

$$\begin{cases} du_i(t) + (u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i+2})dt \\ \quad + 2(u_{i-1} - 2u_i + u_{i+1})dt + au_i(t)dt + b_i|u_{i+1} - u_i|^2 dt + u_i^3(t)dt \\ = f_i(u_i(t))dt + g_i(t)dt + \epsilon \sum_{j=1}^{\infty} (h_{j,i}(t) + \sigma_{j,i}(t, u_i(t)))dW_j(t), \\ u_i(\tau) = u_{\tau,i}, \quad i \in \mathbb{Z}, \end{cases} \tag{11}$$

where $a > 0$ and $b = (b_i)_{i\in\mathbb{Z}} \in \ell^2$ satisfy assumption **R1** in Section 5.1 (which roughly speaking, means that a should be big and $\|b\|$ quite small), the noise intensity parameter $\epsilon \geq 0,\ f_i, \sigma_{j,i}$ are nonlinear functions and locally Lipschitz continuous, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h(t) = (h_{j,i}(t))_{j\in\mathbb{N},i\in\mathbb{Z}}$ are random sequences, and $(W_j)_{j\in\mathbb{N}}$ is a sequence of standard two-sided real-valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

The system (11) can be viewed as a spatial discretization model of the following non-autonomous continuous modification Swift-Hohenberg equation:

$$\begin{cases} du(t) + \Delta^2 u(t)dt + 2\Delta u(t)dt + au(t)dt + b|\nabla u(t)|^2 dt + u^3(t)dt \\ = f(u(t))dt + g(t,x)dt + \epsilon \sum_{j=1}^{\infty}(h_j(t,x) + \sigma_j(t,u(t)))dW_j(t), \ t > \tau, x \in \mathbb{R}, \\ u(x,\tau) = u_\tau(x), \quad \tau \in \mathbb{R}. \end{cases} \tag{12}$$

The modified nonlinear term, $b|\nabla u(t)|^2$, is derived from some studies of various pattern formation phenomena for certain phase turbulence or phase transitions (see [79]). For this reason, the results obtained in our analysis might be also obtained for other interesting models from the real world containing similar nonlinearities, and we plan to study them in a forthcoming work. The long-term dynamics of the usual Swift-Hohenberg equation (if $b = 0, f(x,t) = 0, \epsilon = 0$ in (12)) have also been reported in [69].

In Section 5.1, we prove the existence and uniqueness of the solution for the system (11) under some appropriate assumptions on the nonlinear drift and diffusion terms. Due to the local Lipschitz continuity of nonlinear terms, we need to use the technique of cut-off functions (see (5.27)) and a stopping time developed by [89] to establish the well-posedness of (11) in the Bochner space $L^2(\Omega, \mathcal{F}, \ell^2)$.

In Section 5.2, we need to establish the tightness of probability measures of solutions. As mentioned previously, we use the method of uniform tail estimates proposed in [84] to overcome the difficulty of lacking the compactness of the usual Sobolev embedding. Based on these uniform tail-estimates, we can show the tightness of distribution laws of solutions and thus the existence of evolution systems of probability measures.

As a further result, we discuss the limiting stability of the evolution system of probability measures for non-autonomous stochastic modified Swift-Hohenberg lattice systems (11). We first prove the Feller property of the transfer operator for the solution of system (11) (see Lemma 5.11), and then we discuss the convergence of probability as noise intensity $\epsilon \to \epsilon_0$ (see Lemma 5.15). Finally, we show the asymptotic stability of evolution systems of probability measures by using Theorem 5.8.

The content of Chapter 5 comes from the work [96].

IV Future work

In investigating the asymptotic properties of solutions of stochastic equations, the averaging principle also plays an important role, which has a wide range of applications, such as in chemistry, fluid dynamics, biology, etc. (see, e.g., [22] and references therein). Bogoliubov and Mitropolsky first studied the deterministic system in [7] and then extended it to stochastic differential equations by Khasminskii in [50]. Therefore, our future work will study the averaging principle for non-autonomous stochastic partial differential equations.

For example, we focus on multi-scale fractional stochastic nonautonomous differential equations on unbounded domains. One needs first to consider the existence and uniqueness of an evolution system of measure for the fast equation, as well as the exponential ergodicity of solutions. Then, based on time discretization and variational methods for stochastic partial differential equations, it is possible to confirm the strong rate of convergence concerning the slow component for a solution of fractional stochastic nonautonomous equations.

Resumen en español–Spanish Summary

Los sistemas dinámicos en dimensión infinita consideran principalmente el comportamiento a largo plazo de soluciones de ecuaciones en derivadas parciales evolutivas disipativas no lineales que surgen en las ciencias aplicadas como la física, la dinámica de fluidos, las ciencias de la vida y las ciencias atmosféricas.

En la teoría de sistemas dinámicos deterministas autónomos (independientes del tiempo) en dimensión infinita, el sistema suele poseer un atractor al que convergen todas las órbitas, es decir, el atractor global. Los atractores globales, en general, se utilizan para describir el comportamiento dinámico a largo plazo del sistema (véase, e.g., [75, 82]), y resulta ser un conjunto compacto en el espacio de fases, que atrae la imagen de determinados conjuntos de estados iniciales bajo la evolución del sistema dinámico. Durante su estudio, observamos ecuaciones no locales de calor/parabólicas, así como conteniendo términos de memoria o retardo.

• *No local y memoria.* Los operadores no locales aparecen de forma natural en fenómenos como problemas de elasticidad, ondas de agua, transiciones de fase y propagación de llamas [19]. Las ecuaciones parabólicas no locales tienen importantes aplicaciones en biología, física y ecología. Los modelos evolutivos con términos de memoria están siendo ampliamente estudiados. Los fenómenos naturales y sociales están a menudo influenciados no sólo por su estado actual sino también por su historia. Así, los problemas que describen fenómenos hereditarios en la conducción del calor y la termodinámica han atraído mucha atención.
Por su interés en el mundo real, hemos investigado este problema en el Capítulo 1.

Para sistemas dinámicos estocásticos infinito-dimensionales no autónomos (dependientes del tiempo), el comportamiento dinámico se describe comúnmente en términos de un atractor pullback aleatorio, que es una extensión de los atractores aleatorios (introducidos por primera vez en [26, 27]) de autónomos a no autónomos y fue propuesto en [10, 12, 86].

Una extensión natural del modelo de ecuaciones diferenciales deterministas es un sistema de ecuaciones diferenciales estocásticas, en el que los parámetros relevantes se modelan como procesos estocásticos adecuados o los procesos estocásticos se añaden a las ecuaciones del sistema. La teoría de las ecuaciones diferenciales estocásticas fue propuesta por K. Itô en 1942. Desde entonces, se ha desarrollado en diferentes direcciones. Por ejemplo, considerando la ecuación en sí misma como un sistema dinámico perturbado por el ruido. El término de ruido se añade para capturar las propiedades de incertidumbre que están presentes en cualquier fenómeno

del mundo real.

Nos interesan los dos tipos de ecuaciones siguientes y pretendemos aplicarlas a situaciones estocásticas (véanse Capítulo 2-Capítulo 5).

• *Ecuaciones diferenciales con retardo.* En el proceso de modelado real, el retardo temporal es inevitable y razonable, ya que la evolución de cualquier fenómeno depende no sólo del estado actual sino de lo que ha ocurrido en el pasado. Un sistema con tiempo de retardo se describe generalmente en forma de ecuaciones diferenciales [68]. A tenor de cómo pueda ser la dependencia del pasado, las ecuaciones diferenciales con retardo se clasifican en ecuaciones diferenciales con retardo de tiempo finito o acotado y de retardo infinito o no acotado. Para las ecuaciones de retardo infinito, la elección de un espacio de fasees adecuado es más difícil que para las ecuaciones de retardo acotado (véanse, e.g., [44, 49]).

• *Ecuaciones diferenciales reticulares.* Los sistemas reticulares se utilizan ampliamente en física, biología y otros campos, para modelizar fenómenos como la propagación del impulso nervioso, las reacciones químicas, los circuitos eléctricos, etc (véase, e.g., [24]). Varias simulaciones numéricas han revelado que este tipo de ecuaciones diferenciales muestran una rica variedad de fenómenos dinámicos, incluyendo la formación de modos, ondas viajeras y caos espacial. En [41, 85] se estudiaron las soluciones, y su dinámica a largo plazo, de sistemas reticulares deterministas sin retardo y en [17, 18] se hicieron con retardo; el comportamiento a largo plazo de los sistemas estocásticos reticulares se ha investigado en [16, 20] sin retardo y en [55, 56, 62] con retardo.

Existen dos teorías que tratan del comportamiento cualitativo asintótico para ecuaciones diferenciales estocásticas generales: la teoría de sistemas dinámicos aleatorios y la teoría de existencia y unicidad de medidas invariantes para el semigrupo de Markov asociado.

Como se afirma en [4], los sistemas dinámicos aleatorios constan de dos elementos básicos: Un modelo de ruido y un modelo del sistema perturbado por el ruido. Sobre esta base, introducimos dos tipos de ruidos que se estudian en esta tesis:

• *Ruido blanco no lineal.* Un proceso Wiener m-dimensional $W(t) = \{W_j(t), 1 \leq j \leq m\}$, definido para $t \geq 0$ con espacio de estados $\mathbb{R}^m$, es un proceso estocástico cuyas componentes $W_j(t)(j = 1, \dots, m)$, son procesos Wiener estándar escalares independientes. Cada W_j es un proceso escalar que satisface

(1) $W_j(0) = 0$ a.s.,

(2) $W_j(t) - W_j(s)$ es $\mathcal{N}(0, t-s)$ para todo $t \geq s \geq 0$,

(3) para todo tiempo $0 < t_1 < t_2 < \cdots < t_n$, las variables aleatorias $W_j(t_1), W_j(t_2) - W_j(t_1), \dots, W_j(t_n) - W_j(t_{n-1})$ son independientes ("incrementos independientes").

El proceso es continuo y un proceso de difusión de Markov homogéneo. El ruido blanco es la derivada temporal del proceso de Wiener en un sentido generalizado. Normalmente, cuando el ruido blanco que impulsa una ecuación en derivadas parciales estocástica es un ruido blanco aditivo o multiplicativo lineal, la ecuación estocástica puede convertirse en una ecuación aleatoria gracias a un cambio de variables que involucra al proceso de Ornstein-Uhlenbeck y, de este modo, podemos tratar dichas ecuaciones aleatorias con los mismos métodos utilizados para analizar ecuaciones deterministas. Sin embargo, actualmente no se conoce cómo se pueden convertir las ecuaciones en derivadas parciales estocásticas con ruido blanco no lineal en ecuaciones en derivadas parciales aleatorias. Para estudiar el comportamiento dinámico a largo plazo de las ecuaciones en derivadas parciales estocásticas perturbadas por ruido blanco no lineal, B. Wang [90] introdujo el concepto de atractores pullback aleatorios débiles en media para sistemas dinámicos aleatorios y estableció resultados de existencia. El origen de esta vía se encuentra en los trabajos anteriores llevados a cabo por T. Caraballo, P. Kloeden, B. Schmalfuß and T. Lorenz (véanse [11, 53].

• *Ruido coloreado no lineal.* El ruido coloreado, también conocido como proceso de Ornstein-Uhlenbeck, fue propuesto y nombrado por primera vez por G. Uhlenbeck, L. Ornstein y M. Wang en [83, 97], con el objetivo de aproximar el proceso de Wiener que no es diferenciable en ninguna parte sobre las trayectorias muestrales. Identificando $W(s,\omega) = \omega(s)$, el ruido coloreado es una variable aleatoria $\zeta_\delta : \Omega \to \mathbb{R}$ definida por

$$\zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^{0} e^{\frac{s}{\delta}} dW(t,\omega) = -\frac{1}{\delta^2}\int_{-\infty}^{0} e^{\frac{s}{\delta}} d\omega(s), \text{ para cadass } \delta > 0, \omega \in \Omega.$$

El proceso $z_\delta(t,\omega) = \zeta_\delta(\theta_t\omega$ se denomina proceso de Ornstein-Uhlenbeck (i.e. el ruido coloreado), que es un proceso gaussiano estacionario con $\mathbb{E}(\zeta_\delta) = 0$ y es la única solución estacionaria de la ecuación estocástica:

$$dz + \frac{1}{\delta} z dt = \frac{1}{\delta} dW.$$

Como se menciona en [74], muchos sistemas físicos deben simularse utilizando ruido coloreado en lugar de ruido blanco. Las ecuaciones diferenciales parciales estocásticas perturbadas por ruido coloreado son aleatorias (y por tanto se pueden estudiar para cada $\omega \in \Omega$ fijo por los métodos deterministas), por lo que los coeficientes del ruido coloreado pueden ser funciones no lineales cuando estudiamos los atractores aleatorios de dichas ecuaciones. Ahora ya se conocen algunos resultados sobre el estudio de ecuaciones en derivadas parciales estocásticas conteniendo ruido coloreado no lineal [42, 43].

En la mayoría de los casos, se suele demostrar que existe al menos una solución de los problemas asociados a las ecuaciones diferenciales estocásticas, proporcionando algunas acotaciones de las mismas, bajo condiciones apropiadas sobre las fuerzas externas. Si no se verifica una condición de continuidad de Lipschitz, la solución de las ecuaciones diferenciales estocásticas puede no ser única. En este caso, se verifica la existencia de sistemas dinámicos multivaluados generados por dichas soluciones demostrando la continuidad y la propiedad del cociclo de las mismas. Se hace imprescindible demostrar también la medibilidad del sistema dinámico aleatorio multivaluado generado por las mismas. Para la teoría de sistemas dinámicos aleatorios multivaluados, se puede consultar [13, 14, 16, 17, 92].

Las medidas invariantes, también conocidas como distribuciones suaves, caracterizan el posible comportamiento a largo plazo del sistema [33]. [89, 91] ha discutido la existencia de medidas invariantes de probabilidad para sistemas estocásticos *autónomos* con ruido no lineal. Después de eso, estudios similares con diferentes sistemas estocásticos se han desarrollado en gran medida, véanse, e.g., [23, 55, 56, 98, 99], donde [23, 55, 56] también discutieron la estabilidad límite de la medida invariante de probabilidad. Para los sistemas estocásticos *no autónomos*, descritos por Da Prato y Röckner [34, 35], se considera la existencia de sistemas de evolución de las medidas de probabilidad de sus operadores de transición inhomogéneos en el tiempo, [100, 103] aplicaron sus ideas a modelos estocásticos específicos.

Esta tesis doctoral se estructura en cinco capítulos. En el Capítulo 1, estudiamos el comportamiento asintótico de una ecuación de calor semilineal degenerada no local con memoria degenerada en un dominio acotado, siendo el objetivo demostrar la existencia del atractor global. En el Capítulo 2, consideramos la continuidad de orden superior de los atractores pullback aleatorios para ecuaciones cuasilineales aleatorias no autónomas conteniendo ruido coloreado no lineal. El objetivo es verificar la existencia y continuidad densa residual de los atractores bi-espaciales pullback aleatorios en espacios de Lebesgue de orden p (donde $p > 2$). En el Capítulo 3, consideramos principalmente el comportamiento a largo plazo de las ecuaciones de tipo p-Laplace con retardos infinitos perturbadas por ruido coloreado no lineal, siendo nuestro propósito demostrar la existencia de soluciones débiles (la unicidad no se puede obtener debido a la falta de condición de Lipschitz), la regularidad de las soluciones, y la existencia de atractores aleatorios. En el Capítulo 4, establecemos la existencia y el comportamiento límite de las medidas periódicas para sistemas estocásticos periódicos modificados de Swift-Hohenberg con retardos variables. En el Capítulo 5, nos ocupamos de la estabilidad asintótica de los sistemas de evolución de las medidas de probabilidad para ecuaciones estocásticas discretas modificadas de Swift-Hohenberg no autónomas impulsadas por ruido local no lineal de Lipschitz.

A continuación, presentamos los principales resultados de esta tesis.

V Atractores globales para sistemas deterministas con memoria degenerada

En el Capítulo 1, consideramos ecuaciones de calor semilineales degeneradas no locales.

Para ecuaciones parabólicas semilineales degeneradas, la existencia de atractores (atractores globales, atractores pullback, atractores aleatorios) fue estudiada en [1, 2, 28]. Los trabajos anteriores no consideran términos de memoria en la formulación. Para memoria lineal no degenerada, [40] discutió la existencia y unicidad de soluciones y la existencia de conjuntos absorbentes en un espacio apropiado. Con esto, en [102] se dedujo la existencia de atractores globales de la ecuación original, no de la ecuación transformada de Dafermos. In [67] se consideró el comportamiento a largo plazo para ecuaciones parabólicas semilineales degeneradas con memoria no degenerada. En cuanto a la memoria degenerada semilineal, [38] estudió el comportamiento asintótico de la ecuación del calor definida sobre un dominio acotado, y utilizó la teoría de semigrupos para obtener la existencia de soluciones.

En el Capítulo 1, consideramos el comportamiento asintótico de las ecuaciones de calor degeneradas semilineales no locales con memoria degenerada en un dominio acotado $\Omega \subset \mathbb{R}^N (N \geq 2)$ dado por:

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\text{div}[a(x)\nabla u] - \displaystyle\int_{-\infty}^{t} k(t-s)\text{div}[a(x)\nabla u(s)]ds + f(u) = g(x,t), \\ u(x,t) = 0, & \text{en } \partial\Omega \times \mathbb{R}, \\ u(x,\tau) = u_0(x), & \text{en } \Omega, \\ u(x,t+\tau) = \phi(x,t), & \text{en } \Omega \times (-\infty, 0], \end{cases} \tag{13}$$

donde $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, la función $m \in C(\mathbb{R}; \mathbb{R}^+)$ satisface $0 < h \leq m(r)$ para todo $r \in \mathbb{R}$, y $k : \mathbb{R}^+ \to \mathbb{R}$ representa el núcleo de memoria.

En la Sección 1.2, utilizamos la transformación de Dafermos ((1.7)-(1.9)) para obtener

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\text{div}[a(x)\nabla u] - \displaystyle\int_{0}^{\infty} \mu(s)\text{div}[a(x)\nabla \eta^t(s)]ds + f(u) = g(x,t), \\ \dfrac{\partial}{\partial t}\eta^t(s) = u - \dfrac{\partial}{\partial s}\eta^t(s), & \text{en } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{en } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x), & \text{en } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s), & \text{en } \Omega \times \mathbb{R}^+, \end{cases} \tag{14}$$

donde

$$\begin{aligned} \eta^\tau(x,s) = \int_0^s u(x,\tau - r)dr &= \int_0^s \phi(x,-r)dr \\ &= \int_{-s}^0 \phi(x,r)dr := \eta_0(x,s), \quad \forall s > 0. \end{aligned}$$

Para cualquier $0 < \epsilon < 1$, definimos $a_\epsilon(x) = a(x) + \epsilon$, denotamos $z_\epsilon(t) = (u_\epsilon(t), \eta_\epsilon^t)^T$ y $z_{0,\epsilon} = (u_0, \eta_0)^T$. Para completar el buen planteamiento del problema (13), consideramos la siguiente ecuación no degenerada:

$$\begin{cases} \partial_t z_\epsilon = \mathcal{G} z_\epsilon + \mathcal{F}(z_\epsilon), & \text{en } \Omega \times (\tau,\infty), \\ z_\epsilon(x,t) = 0, & \text{en } \partial\Omega \times (\tau,\infty), \\ z_\epsilon(x,\tau) = z_{0,\epsilon} & \text{en } \Omega, \end{cases} \tag{15}$$

donde $\mathcal{F}(z_\epsilon) = (-f(u_\epsilon) + g, 0)^T$ y

$$\mathcal{G} z_\epsilon = \begin{pmatrix} m(l(u_\epsilon))\text{div}[a_\epsilon(x)\nabla u_\epsilon] + \int_0^\infty \mu(s)\text{div}[a(x)\nabla \eta_\epsilon^t(s)]ds \\ u_\epsilon - \partial_s \eta_\epsilon \end{pmatrix}.$$

Primero demostramos la existencia de solución de la ecuación no degenerada (15), de modo que cuando ϵ tiende a 0, podemos utilizar la conclusión del Teorema 1.6 para obtener la existencia, unicidad y regularidad de la solución del problema

(14). Y, a continuación, utilizamos los resultados (Lemma 1.4 y Lemma (1.5)) que cumplen los operadores relacionados $\mathcal{F}$ ((1.15)) y $\mathcal{I}$ ((1.17)), respectivamente, que se derivan por el método mencionado en [102], para obtener la existencia, unicidad y regularidad de una solución del problema (13) (ver Teorema 1.7). A diferencia del método de utilizar la teoría de semigrupos para obtener la existencia de soluciones en [38], aquí utilizamos el método de Faedo-Galerkin para demostrar la existencia de soluciones de la ecuación del calor no degenerada (15).

En la Sección 1.3, establecemos un sistema dinámico autónomo en el espacio de Hilbert $\mathcal{M} = L^2(\Omega) \times L^2_{H_0^a}$ y luego mostramos la existencia del atractor global asociado al semigrupo continuo $S(t)$ generado por el problema (13).

El contenido del Capítulo 1 procede fundamentalmente del trabajo [93].

VI Atractores aleatorios pullback para sistemas estocásticos con ruido coloreado no lineal con y sin retardo

En el Capítulo 2 y en el Capítulo 3, consideramos ecuaciones aleatorias de tipo p-Laplace.

La ecuación en derivadas parciales de tipo p-Laplace aparece con frecuencia en los estudios físicos sobre dinámica de fluidos no newtonianos. También aparece en descripciones de fenómenos relacionados con la elasticidad no lineal, el filtraje no lineal o la distribución de campos magnéticos (véase [37]). El comportamiento dinámico a largo plazo (especialmente la existencia de atractores pullback) de la ecuación de tipo p-Laplace se ha estudiado ampliamente, véanse, por ejemplo, [54, 57] en un sistema dinámico aleatorio general univaluado con condiciones continuas de Lipschitz. En ausencia de la condición de Lipschitz, la unicidad de solución no se puede asegurar, y por lo tanto, la dinámica a largo plazo de la ecuación de tipo p-Laplace en sistemas dinámicos aleatorios multivaluados se ha analizado, por ejemplo, en [21, 43, 88]. Nótese que ninguno de los trabajos anteriores posee un término de retardo.

Por *pullback random attractor* para un sistema dinámico aleatorio no autónomo (brevemente, un cocycle [86]) en X sobre un espacio de probabilidad Ω, entendemos una familia $\{\mathcal{A}(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\}$ de conjuntos compactos con las propiedades de invarianza, de atracción pullback y de medibilidad en X. En el Capítulo 3 demostramos la existencia de atractores aleatorios pullback en sistemas dinámicos multivaluados.

Un *atractor aleatorio biespacial pullback* para un sistema dinámico aleatorio regular no autónomo, que es un atractor aleatorio pullback en el espacio inicial X tal que es compacto en el espacio regular Y y pullback atrae algunos subconjuntos de X bajo la topología de Y, generaliza el concepto atractor aleatorio pullback al caso biespacial y el de atractor aleatorio biespacial en [57] al caso no autónomo. En el Capítulo 2, demostramos la continuidad densa residual (incluyendo la existencia) del atractor bi-espacial aleatorio pullback $\{\mathcal{A}_\lambda(t,\omega)\}$ en ambos espacios $X := L^2(\mathbb{R}^m)$ y $Y := L^p(\mathbb{R}^m)$ relativos a $\lambda \in \Lambda := (0,\infty]$ en sistemas dinámicos univaluados, donde $\infty \in \Lambda$ y el ruido infinito se corresponde con la ecuación determinista.

La existencia de atractores para modelos que involucran características hereditarias con retardos infinitos ha sido ampliamente discutida, como [70, 101, 102] en el caso de sistemas dinámicos univaluados y [13, 14, 15, 76] en el de multivaluados.

• **Ecuaciones diferenciales aleatorias univaluadas sin retardo**

En el Capítulo 2, consideramos la continuidad de orden superior de los atractores aleatorios pullback para ecuaciones aleatorias de tipo p-Laplace en $\mathbb{R}^m$ perturbadas por ruido coloreado no lineal

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \alpha u = f(x,u) + g(t,x) + h(x,u)\zeta_\lambda(\theta_t\omega), \ t > \tau, \\ u(\tau,x) = u_\tau(x), \ \tau \in \mathbb{R}, \ x \in \mathbb{R}^m, m \in \mathbb{N}, \end{cases} \tag{16}$$

donde $\lambda > 0, p > 2$, $\zeta_\lambda(\theta_t\cdot)$ es el ruido coloreado y $\{\theta_t\}_{t\in\mathbb{R}}$ es un grupo de autotransformaciones en el espacio clásico de Wiener Ω.

A lo largo de este Capítulo 2, (X,Y) es un par límite-idéntico de espacios de Banach separables, donde la siguiente propiedad

$$\{x_n\} \subset X \cap Y, \ \|x_n - x\|_X \to 0, \|x_n - y\|_Y \to 0 \quad \Rightarrow \quad x = y. \tag{17}$$

es equivalente a que $(X \cap Y, \|| \cdot \|_X + || \cdot \|_Y)$ sea un espacio de Banach (véase [57, Lemma 2.1]). La semimétrica de Hausdorff y la métrica de Hausdorff se definen respectivamente por

$$d_X(A_1,A_2) = \sup_{a\in A_1} \inf_{b\in A_2} \|a-b\|_X, \quad \rho_X(A_1,A_2) = \max\{d_X(A_1,A_2), d_X(A_2,A_1)\} \tag{18}$$

para $A_1, A_2 \subset X$. Es inmediato demostrar que $d_{X\cap Y} = d_X + d_Y$ y $\rho_{X\cap Y} = \rho_X + \rho_Y$.

En la Sección 2.2, establecemos un resultado abstracto de existencia (véase Teorema 2.8) de un único atractor bi-espacial aleatorio pullback. La principal dificultad consiste en demostrar la medibilidad del atractor en Y utilizando cuatro condiciones verificables (**B4-B7**). En particular, suponemos que el operador cociclo de $X \cap Y$ a Y es fuerte-débilmente continuo (en lugar de la cuasi-continuidad utilizada en [30]). Esta condición no puede asegurar que el conjunto imagen del operador cociclo sobre un subconjunto compacto de $X \cap Y$ sea un subconjunto compacto de Y. Pero podemos demostrar que el conjunto imagen es un conjunto acotado cerrado en Y. Por lo tanto, podemos considerar el espacio métrico más grande $CB(Y)$ (que $\mathcal{C}(Y)$) de todos los conjuntos acotados cerrados en Y, equipado con la métrica de Hausdorff.

En la Sección 2.3, establecemos un resultado abstracto (véase Teorema 2.12) sobre la continuidad densa residual de atractores bi-espaciales aleatorios pullback en $X \cap Y$ utilizando seis condiciones verificables (**C1-C6**), y en su demostración, utilizaremos el Teorema residual abstracto de Baire (véase [47, Theorem 5.1], y el Teorema de la densidad de Baire ([59, Prop. 2]).

Cabe señalar que la aproximación de las ecuaciones aleatorias (o estocásticas) a las deterministas ($\lambda \to \infty$) es significativa (véase [87]). En particular, Flandoli et al. [39] señalaron que una ecuación de transporte determinista no se resuelve, mientras que la versión estocástica (o aleatoria) sí se resuelve.

En las secciones 2.4-2.6, demostramos la existencia y la continuidad densa residual de atractores bi-espaciales aleatorios pullback $\{\mathcal{A}_\lambda(t,\omega)\}$ de la ecuación (16) en ambos espacios $X := L^2(\mathbb{R}^m)$ y $Y := L^p(\mathbb{R}^m)$ con respecto a $\lambda \in \Lambda := (0,\infty]$.

El contenido del Capítulo 2 procede del trabajo [61].

• Ecuaciones diferenciales aleatorias multivaluadas con retardo

En el Capítulo 3, consideramos la existencia de atractores aleatorios para ecuaciones de tipo p-Laplace no autónomas con retardo infinito (que representa la historia de las variables) en un dominio acotado $\mathcal{O} \subset \mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u = f(t,x,u) + g(x, u(t-\varrho(t))) \\ \quad + \displaystyle\int_{-\infty}^{0} F(x,l,u(t+l))dl + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega), \ t > \tau, \quad x \in \mathcal{O}, \\ u(t,x) = 0, \ t > \tau, \quad x \in \partial\mathcal{O}, \\ u(\tau+s,x) = \varphi(s,x), \quad s \in (-\infty,0], \quad x \in \mathcal{O}, \tau \in \mathbb{R}, \end{cases} \tag{19}$$

donde $p \geq 2, \lambda > 0$, ζ_δ es el ruido coloreado con tiempo de correlación $\delta > 0$, y W es un proceso escalar de Wiener en el espacio clásico de Wiener $(\Omega, \mathbb{P}, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$. El término no lineal f y el término de difusión no lineal son funciones continuas pero no necesariamente de Lipschitz, y los términos de retardo $g : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ y $F : \mathcal{O} \times \mathbb{R}_- \times \mathbb{R} \to \mathbb{R}$ tampoco se suponen de tipo Lipschitz.

Sea X un espacio de Hilbert. Para tratar con los términos de retardo g y F en (19), denotamos nuestro espacio de fase por

$$C_{\gamma,X} = \{w \in C((-\infty,0];X) : \lim_{\tau\to-\infty} e^{\gamma\tau} w(\tau) \ \text{ existe}\}, \tag{20}$$

donde $\gamma > 0$ y fijamos $\|w\|_{C_{\gamma,X}} := \sup\limits_{\tau\in(-\infty,0]} e^{\gamma\tau}\|w(\tau)\| < \infty$ para todo $w \in C_{\gamma,X}$. Por [13], sabemos que $C_{\gamma,X}$ es un espacio de Banach separable.

En la Sección 3.2, demostramos la existencia de soluciones débiles de la ecuación (19). Para resolver este problema, utilizamos la técnica tradicional de aproximaciones de Galerkin, y de acuerdo con el método en [13], demostramos que $u \to \int_{-\infty}^{0} F(x,l,u(t+l))dl$ es continua de $C_{\gamma,L^2(\mathcal{O})}$ a $L^2(\mathcal{O})$ como se muestra en la Observación 3.3, para obtener la existencia de soluciones débiles (véase Teorema 3.11).

De hecho, también tratamos de demostrar la regularidad de los atractores aleatorios pullback para la ecuación (19). Para ecuaciones de tipo p-Laplace con retardos acotados, se puede aplicar el Teorema de compacidad de Sobolev y el Teorema de Arzelà-Ascoli para demostrar la regularidad de los atractores pullback como en [66]. Pero para el espacio considerado en este trabajo (que se muestra en (20)), nos encontramos con que no hay relación de inclusión entre los espacios $C_{\gamma,L^2(\mathcal{O})}$ y $C_{\gamma,W_0^{1,p}(\mathcal{O})}$. Por tanto, sólo podemos demostrar la regularidad de la solución (véase Teorema 3.14) aplicando el método de [21].

En la Sección 3.3, mostramos la medibilidad de los sistemas dinámicos multivaluados y atractores pullback y la compacidad asintótica de las soluciones. Para obtener la compacidad asintótica pullback de las soluciones, vamos a utilizar la misma técnica que [76, Lemma 5.5]. La medibilidad del atractor pullback se deducirá demostrando la semicontinuidad superior de funciones multivaluadas, el cierre de un grafo en algunos subespacios del espacio de probabilidad utilizando los métodos en [13].

El contenido del Capítulo 3 procede del trabajo [94].

VII Medidas invariantes para sistemas estocásticos con ruido blanco no lineal con y sin retardo

En los capítulos 4 y 5, consideramos los sistemas estocásticos reticulares de Swift-Hohenberg modificados.

La ecuación de Swift-Hohenberg fue propuesta por primera vez en 1977 por Swift y Hohenberg (véase [80]) para estudiar la analogía entre la rama del comportamiento hidrodinámico de fluidos y las ecuaciones en derivadas parciales relacionadas, así como las transiciones de fase continuas en sistemas termodinámicos. Sobre la ecuación de Swift-Hohenberg modificada, han habido muchos trabajos que discuten la existencia de atractores (global, pullback, aleatorio, uniforme) y bifurcaciones, por ejemplo, [72, 73]. Sin embargo, sólo hay varios trabajos sobre el sistema reticular de Swift-Hohenberg modificado: [45] consideró el caso determinista y [96] el caso estocástico.

Consideramos el siguiente espacio de Banach:

$$\ell^r = \left\{ w = (w_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |w_i|^r < +\infty \right\}, \quad r \geq 1,$$

con la norma $\|w\|_r^r = \sum_{i\in\mathbb{Z}} |w_i|^r$. Para $r = 2$, se tiene la norma $\|w\|^2 = \sum_{i\in\mathbb{Z}} |w_i|^2$ y el producto interior $(w, v) := \sum_{i\in\mathbb{Z}} w_i v_i$. Además, denotamos el espacio de todas las funciones continuas de $\mathbb{R}$ en $\mathbb{R}$ por $C(\mathbb{R}, \mathbb{R})$.

En [63] se investigó la existencia de *medidas periódicas* en ℓ^2 para sistemas estocásticos reticulares sin retardo. En el caso de retardo, [55, 62] consideró la medida periódica del sistema reticular estocástico. Los términos no lineales anteriores son todos globalmente de Lipschitz. En el Capítulo 4 queremos considerar retardos variables, las funciones no lineales, el término modificado, y los términos cúbicos son todos localmente Lipschitz continuos.

A diferencia de considerar la existencia de medidas invariantes en ℓ^2 de operadores de transición homogéneos en el tiempo para sistemas estocásticos reticulares **autónomos** conducidos por ruido blanco no lineal en [23, 55, 56, 89, 99], en el Capítulo 5 utilizamos el método de Krylov-Bogolyubov extendido mencionado por Da Prato y Röckner en [34, 35] para demostrar la existencia de *sistemas de evolución de medidas de probabilidad* de operadores de transición homogéneos en el tiempo para sistemas estocásticos reticulares **no autónomos** (véase el Teorema 5.14).

- **Medidas periódicas para sistemas reticulares estocásticos con retardo**

En el Capítulo 4, tenemos en cuenta la existencia y el comportamiento límite de las medidas periódicas para el caso con retardo variable del problema estocástico periódico modificado de Swift-Hohenberg como sistemas reticular en el conjunto de enteros $\mathbb{Z}$ dado por:

$$\begin{cases} du_i(t) + q_1(t)\Big[\big(u_{i+2}(t) - 4u_{i+1}(t) + 6u_i(t) - 4u_{i-1}(t) + u_{i+2}(t)\big) + 2\big(u_{i-1}(t) \\ \quad - 2u_i(t) + u_{i+1}(t)\big)\Big]dt + q_2(t)u_i(t)dt + q_{3,i}(t)|u_{i+1}(t) - u_i(t)|^2 dt + u_i^3(t)dt \\ = f_i(t, u_i(t), u_i(t - \varrho(t)))dt + g_i(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty} \big(h_{i,j}(t) + \sigma_{i,j}(t, u_i(t), u_i(t - \varrho(t)))\big)dW_j(t), \quad t > 0, \\ u_i(s) = \varphi_i(s), \quad s \in [-\rho, 0], \ i \in \mathbb{Z}, \end{cases} \tag{21}$$

que se obtiene mediante una discretización espacial de la ecuación periódica continua modificada de Swift-Hohenberg con un retardo temporal variable en $\mathbb{R}$:

$$\begin{cases} du(t) + q_1(t)\Delta^2 u(t)dt + 2q_1(t)\Delta u(t)dt + q_2(t)u(t)dt + q_3(t)|\nabla u(t)|^2 dt + u^3(t)dt \\ = f(t, u(t), u(t - \varrho(t)))dt + g(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty}(h_j(t, x) + \sigma_j(t, u(t), u(t - \varrho(t)))dW_j(t), \ t > 0, \\ u(s) = \varphi(s), \quad s \in [-\rho, 0]. \end{cases} \tag{22}$$

Aquí $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ son positivas y continuas, $q_3 = (q_{3,i})_{i\in\mathbb{Z}} : \mathbb{R} \to \ell^2$ es continua, $f_i, \sigma_{j,i} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ son funciones localmente Lipschitz continuas para cada $i \in \mathbb{Z}$ y $j \in \mathbb{Z}$, con intensidad de ruido $0 < \epsilon \leq 1$ y parámetro de retardo $\rho > 0$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ tal que $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$ y $h_j : \mathbb{R} \to L^2(\mathbb{R})$, y $(W_j)_{j\in\mathbb{N}}$ es una secuencia de procesos de Wiener estándar de dos caras con valores reales definidos en un espacio de probabilidad filtrado completo $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. Además, para una constante positiva dada $\mathcal{T}$, todos los términos dependientes del tiempo del sistema (21) son $\mathcal{T}$-periódicos en el tiempo (como se muestra en (4.13)). La ecuación (22) es una ecuación usual de Swift-Hohenberg cuando $q_3(t) = 0$, $f(t, u(t), u(t - \varrho(t))) = 0$ y $\epsilon = 0$.

Denotemos por $C([-\rho, 0]; \ell^2)$ el espacio de Banach de todas las funciones continuas con valores en ℓ^2 sobre $[-\rho, 0]$ con la norma

$$\|x\|_{C([-\rho,0];\ell^2)} = \sup_{s\in[-\rho,0]} \|x(s)\| = \sup_{s\in[-\rho,0]} \sum_{i\in\mathbb{Z}} |x_i(s)|^2, \ \forall x \in C([-\rho, 0]; \ell^2).$$

En la Sección 4.3, el objetivo es aplicar el método de Krylov-Bogolyubov para demostrar la existencia de medidas periódicas del sistema reticular (21) para todo $\epsilon \in [0, 1]$ en el espacio $C([-\rho, 0], \ell^2)$. Tenemos que demostrar la estanqueidad de las leyes de distribución de soluciones a (21), y la dificultad de demostrar esta estanqueidad es análoga al hecho de que la incrustación de Sobolev ya no es compacta cuando las PDEs estocásticas son sobre un dominio no limitado. Para hacer frente a esta dificultad, mostramos que la cola de la solución de (21) es uniformemente pequeña en $L^2(\Omega, \mathcal{F}, \ell^2)$ (su definición se puede encontrar en [90]) utilizando el método de estimación de cola uniforme propuesto en [84]. Además, dado que la solución $u_t(\cdot)$ del sistema (21) depende de la historia pasada y por lo tanto no es Markov, utilizamos el método en [71] para encontrar el mapa de solución para ecuaciones diferenciales funcionales estocásticas con retardo finito posee la propiedad de Markov.

En la Sección 4.4, el objetivo es demostrar, bajo supuestos más fuertes, el comportamiento límite de las medidas periódicas del sistema (21) cuando la intensidad del ruido $\epsilon \to \epsilon_0 \in [0,1]$ (véase (4.120)). Demostramos que para $\epsilon \in [0,1]$, el conjunto de todas las medidas periódicas de (21) es débilmente compacto, y cualquier punto límite de una secuencia ajustada de medidas periódicas del sistema (21) debe ser una medida invariante de la ecuación límite asociada (ver Teorema 4.16).

El contenido del Capítulo 4 procede de [95].

• Sistemas de evolución de medidas para sistemas reticulares estocásticos sin retardo

En el Capítulo 5, estudiamos las siguientes ecuaciones estocásticas discretas modificadas de Swift-Hohenberg con ruido no lineal para $t > \tau$ con $\tau \in \mathbb{R}$:

$$\begin{cases} du_i(t) + (u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i+2})dt \\ \quad + 2(u_{i-1} - 2u_i + u_{i+1})dt + au_i(t)dt + b_i|u_{i+1} - u_i|^2 dt + u_i^3(t)dt \\ = f_i(u_i(t))dt + g_i(t)dt + \epsilon \sum_{j=1}^{\infty}(h_{j,i}(t) + \sigma_{j,i}(t, u_i(t)))dW_j(t), \\ u_i(\tau) = u_{\tau,i}, \quad i \in \mathbb{Z}, \end{cases} \tag{23}$$

donde $a > 0$ y $b = (b_i)_{i\in\mathbb{Z}} \in \ell^2$ satisfacen la suposición **R1** de la Sección 5.1 (lo que, a grandes rasgos, significa que a debe ser grande y $\|b\|$ bastante pequeño), el parámetro de intensidad del ruido $\epsilon \geq 0$, $f_i, \sigma_{j,i}$ son funciones no lineales y localmente Lipschitz continuas, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ y $h(t) = (h_{j,i}(t))_{j\in\mathbb{N}, i\in\mathbb{Z}}$ son secuencias aleatorias, y $(W_j)_{j\in\mathbb{N}}$ es una secuencia de procesos de Wiener estándar de dos caras con valores reales definidos en un espacio de probabilidad filtrado completo $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

El sistema (23) puede verse como un modelo de discretización espacial de la siguiente ecuación de Swift-Hohenberg de modificación continua no autónoma:

$$\begin{cases} du(t) + \Delta^2 u(t)dt + 2\Delta u(t)dt + au(t)dt + b|\nabla u(t)|^2 dt + u^3(t)dt \\ = f(u(t))dt + g(t,x)dt + \epsilon \sum_{j=1}^{\infty}(h_j(t,x) + \sigma_j(t, u(t)))dW_j(t), \ t > \tau, x \in \mathbb{R}, \\ u(x,\tau) = u_\tau(x), \quad \tau \in \mathbb{R}. \end{cases} \tag{24}$$

El término no lineal modificado, $b|\nabla u(t)|^2$, se deriva de algunos estudios de diversos fenómenos de formación de patrones para ciertas turbulencias o transiciones de fase (véase [79]). Por esta razón, los resultados obtenidos en nuestro análisis podrían obtenerse también para otros modelos interesantes del mundo real que contengan no linealidades similares, y tenemos previsto estudiarlos en un próximo trabajo. La dinámica a largo plazo de la ecuación usual de Swift-Hohenberg (si $b = 0, f(x,t) = 0, \epsilon = 0$ en (24)) también ha sido reportada en [69].

En la Sección 5.1, probamos la existencia y unicidad de la solución para el sistema (23) bajo algunos supuestos apropiados sobre los términos no lineales de deriva y difusión. Debido a la continuidad local de Lipschitz de los términos no lineales, tenemos que utilizar la técnica de las funciones de corte (véase (5.27)) y un tiempo de parada desarrollado por [89] para establecer la bondad de (23) en el espacio de Bochner $L^2(\Omega, \mathcal{F}, \ell^2)$.

En la Sección 5.2, tenemos que establecer la propiedad "tight" de las medidas de probabilidad de las soluciones. Como se mencionó anteriormente, utilizamos el método de estimaciones de cola uniforme propuesto en [84] para superar la dificultad de la falta de compacidad de la inclusión de Sobolev habitual. Basándonos en estas estimaciones de colas uniformes, podemos demostrar la propiedad "tight" de las leyes de distribución de las soluciones y, por tanto, la existencia de sistemas de evolución de las medidas de probabilidad.

Como resultado adicional, discutimos la estabilidad límite del sistema de evolución de medidas de probabilidad para sistemas estocásticos reticulares no autónomos modificados de Swift-Hohenberg (23). Primero demostramos la propiedad de Feller del operador de transferencia para la solución del sistema (23) (véase Lemma 5.11), y luego discutimos la convergencia de la probabilidad cuando la intensidad de ruido $\epsilon \to \epsilon_0$ (véase Lemma 5.15). Por último, mostramos la estabilidad asintótica de los sistemas de evolución de las medidas de probabilidad utilizando el Teorema 5.8.

El contenido del Capítulo 5 procede del trabajo [96].

VIII Trabajos futuros

En la investigación de las propiedades asintóticas de las soluciones de ecuaciones estocásticas, el principio de promediación también desempeña un papel importante, y tiene una amplia gama de aplicaciones en química, en dinámica de fluidos, en biología, etc. (véase, e.g., [22] y las referencias allí incluidas). Bogoliubov y Mitropolsky estudiaron primero el sistema determinista en [7] y luego Khasminskii lo extendió a ecuaciones diferenciales estocásticas en [50]. Por lo tanto, nuestro trabajo futuro estudiará el principio de promediación para ecuaciones en derivadas parciales estocásticas no autónomas.

Por ejemplo, nos centramos en las ecuaciones diferenciales estocásticas fraccionarias no autónomas multiescala en dominios no acotados. Primero hay que considerar la existencia y unicidad de un sistema de evolución de medidas para la ecuación rápida, así como la ergodicidad exponencial de las soluciones. A continuación, basándose en la discretización temporal y en métodos variacionales para ecuaciones en derivadas parciales estocásticas, es posible confirmar la fuerte tasa de convergencia relativa a la componente lenta para una solución de ecuaciones fraccionarias estocásticas no autónomas.

Part I

Global attractors for deterministic systems with memory

Chapter 1

Nonlocal semilinear degenerate heat equations with degenerate memory

In this chapter, we will study the asymptotic behavior of a nonlocal semilinear degenerate heat equation with degenerate history in a bounded domain. We approximate the degenerate heat equation obtained after the Dafermos transformation with a non-degenerate equation to obtain the existence, uniqueness, and regularity of the solution of the equation. Then, we obtain the existence of the global attractor of the original equation (without Dafermos transformation) by using the method mentioned in [102].

In the next section, we introduce some related spaces and define linear operators. In Sect. 1.2, we use approximation methods to prove the existence, uniqueness, and regularity of the solution. In Sect. 1.3, we establish the existence of the global attractor associated with the semigroup $S(t)$ generated by problem (1).

1.1 Some spaces

Let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$, and $\|\cdot\|$ denote the L^2-norm. Let $W_0^{m,p}(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, where $W^{m,p}(\Omega)$ is the subspace of $L^p(\Omega)$ formed by the functions possessing distributional derivatives up to order $\leq m$ also in $L^p(\Omega)$, where $m \in \mathbb{N}, 1 \leq p \leq \infty$. When $p = 2$, we write $W_0^{m,2}(\Omega) = H_0^m(\Omega)$. Moreover, in space $H_0^1(\Omega)$ we use the inner product

$$(w_1, w_2)_1 = (\nabla w_1, \nabla w_2)$$

with the corresponding norm $\|\cdot\|_1$.

Let $H_0^a(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ with the norm

$$\|w\|_{H_0^a} := \left(\int_\Omega a(x)|\nabla w|^2 dx\right)^{\frac{1}{2}},$$

which is a Hilbert space by means of [8], and with the inner product given by

$$(w_1, w_2)_{H_0^a} = \int_\Omega a(x)\nabla w_1 \cdot \nabla w_2 dx, \quad \text{for every} \quad w_1, w_2 \in H_0^a(\Omega). \tag{1.1}$$

Notice that we only assume that $a \in L^1_{loc}(\Omega)$ and do not require $a \in L^\infty_{loc}(\Omega)$, thus there is no inclusion relationship between $H_0^a(\Omega)$ and $H_0^1(\Omega)$.

Also, some spaces that are useful to us are defined in terms of [65]. Let a positive and self-adjoint linear operator $Q := -\text{div}[a(x)\nabla\cdot]$, which has a domain such that

$$D(Q) = \{w | w \in H_0^a(\Omega) \text{ and } Qw \in L^2(\Omega)\},$$

and it is a Hilbert space with respect to the usual graph scalar product. The next compact and dense embeddings also hold

$$H_0^a(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H_0^a(\Omega))^*, \tag{1.2}$$

where $(H_0^a(\Omega))^*$ is the dual space of $H_0^a(\Omega)$. Then by (1.2), there exists a complete orthonormal system of eigenvectors $\{e_j\}_{j\in\mathbb{N}}$ associated to λ_j such that

$$\begin{cases} Qe_j = \lambda_j e_j, & j = 1, 2, ..., \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, & \lambda_j \to +\infty \text{ as } j \to +\infty. \end{cases}$$

Moreover,

$$\lambda_1 = \inf\left\{ \frac{\|w\|^2_{H_0^a}}{\|w\|^2}; w \in H_0^a(\Omega), w \ne 0 \right\}. \tag{1.3}$$

For every $r > 0$, we define the operator $Q^{\frac{r}{2}}$ via spectrum theory: $Q^{\frac{r}{2}}$ is an unbounded strictly positive and self-adjoint operator in $L^2(\Omega)$ with domain $D(Q^{\frac{r}{2}})$, and $D(Q^{\frac{r}{2}})$ is a dense subset of $L^2(\Omega)$. The space $D(Q^{\frac{r}{2}})$ can be represented by the standard eigenvectors of Q, i.e.,

$$V^r = D(Q^{\frac{r}{2}}) = \left\{ u = \sum_{j=1}^{\infty} (u, e_j)_{L^2} e_j;\ \|u\|^2_{D(Q^{\frac{r}{2}})} = \sum_{j=1}^{\infty} (u, e_j)^2_{L^2} \lambda_j^r < +\infty \right\},$$

which is a Hilbert space with respect to the norm $\|\cdot\|_{V^r}$ and the inner product

$$(w_1, w_2)_{V^r} = (Q^{\frac{r}{2}} w_1, Q^{\frac{r}{2}} w_2).$$

For $r = 0$ we have $V^0 = L^2(\Omega)$, for $r = 1$, we have $V^1 = H_0^a(\Omega)$ and denote by $V^{-1} = (H_0^a(\Omega))^*$ its dual space. In addition, for $r = 2$, we have $V^2 = D(Q)$ with the inner product $(w_1, w_2)_{D(Q)} = (Qw_1, Qw_2)$, and the norm $\|w\|^2_{D(Q)} = \|Qw\|^2$.

For $r \in \mathbb{R}$, let $L^2_\mu(\mathbb{R}^+; V^{r+1})$ be a Hilbert space of functions $w : \mathbb{R}^+ \to V^{r+1}$ endowed with the inner product

$$(w_1, w_2)_{r+1,\mu} = \int_0^\infty \mu(s)(w_1, w_2)_{V^{r+1}} ds.$$

Furthermore, we define the Hilbert spaces

$$\mathcal{H} = L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^a_0(\Omega)) \text{ and } \mathcal{H}^1 = H^a_0(\Omega) \times L^2_\mu(\mathbb{R}^+; D(Q)),$$

endowed with the inner products,

$$(w_1, w_2)_{\mathcal{H}} = (w_1, w_2) + (w_1, w_2)_{1,\mu} \text{ and } (w_1, w_2)_{\mathcal{H}^1} = (w_1, w_2)_{H^a_0} + (w_1, w_2)_{2,\mu},$$

where $w_1, w_2 \in \mathcal{H}$ or $\mathcal{H}^1$, respectively.

Let X be a Banach space, we define L^2_X the space of functions $u(\cdot)$ satisfying

$$\int_{-\infty}^0 e^{\gamma s}\|u(s)\|^2_X ds < \infty, \tag{1.4}$$

where $0 < \gamma < \min\{h\lambda_1, \delta\}$ (λ_1 is defined in (1.3), δ is defined in assumption **(A3)** below). We also define the Hilbert spaces

$$\mathcal{M} = L^2(\Omega) \times L^2_{H^a_0} \text{ and } \mathcal{M}^1 = H^a_0(\Omega) \times L^2_{D(Q)},$$

which are endowed with the inner products

$$(w_1, w_2)_{\mathcal{M}} = (w_1, w_2) + (w_1, w_2)_{L^2_{H^a_0}} \text{ and } (w_1, w_2)_{\mathcal{M}^1} = (w_1, w_2)_{H^a_0} + (w_1, w_2)_{L^2_{D(Q)}},$$

respectively.

1.2 Well-posedness of nonlocal degenerate equations

In this part, we work on the well-posedness of the following nonlocal degenerate equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\text{div}[a(x)\nabla u] - \displaystyle\int_{-\infty}^t k(t-s)\text{div}[a(x)\nabla u(s)]ds & \\ \quad + f(u) = g(x,t), & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x, t+\tau) = \phi(x,t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{1.5}$$

where $m \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies

$$0 < h \le m(r), \quad \forall r \in \mathbb{R}, \tag{1.6}$$

and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, $k : \mathbb{R}^+ \to \mathbb{R}$ represents the memory kernel, the nonlinearity f has a growth order $p - 1$ with

$$2 < p \le \frac{2N}{N - 2 + \alpha} \text{ for } \alpha \in (0, 2),$$

and the measurable, nonnegative diffusion variable $a : \Omega \to [0, +\infty)$ satisfies the assumption:

(A1) $a \in L^1_{loc}(\Omega)$ and $\liminf_{x \to z} \frac{a(x)}{|x-z|^\alpha} > 0$ for some $\alpha \in (0, 2)$ and every $z \in \overline{\Omega}$.

To deal with the delay term concerning u, that appears in the integral form, we use the Dafermos transformation (introduced in [31]), thus considering the new variables

$$u^t(x,s) = u(x,t-s), \quad s \geq 0, \ t \geq \tau, \tag{1.7}$$

and

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^t u(x,r)dr, \quad s \geq 0, \ t \geq \tau, \tag{1.8}$$

using the method of integration by parts, combining (1.5) and (1.8), and the assumption that $k(\infty) = 0$, we obtain

$$\int_{-\infty}^t k(t-s)\text{div}[a(x)\nabla u(s)]ds = -\int_0^\infty k'(s)\text{div}[a(x)\nabla \eta^t(s)]ds. \tag{1.9}$$

Then we set $\mu(s) = -k'(s)$, so that based on the above transformation, we have the following nondelay system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(l(u))\text{div}[a(x)\nabla u] - \displaystyle\int_0^\infty \mu(s)\text{div}[a(x)\nabla \eta^t(s)]ds & \\ \quad + f(u) = g(x,t), & \text{in } \Omega \times (\tau, \infty), \\ \dfrac{\partial}{\partial t}\eta^t(s) = u - \dfrac{\partial}{\partial s}\eta^t(s), & \text{in } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.10}$$

where the initial integrated past history of $u(x,t)$ (does not depend on $u_0(x)$) is derived from equation (1.7)-(1.8):

$$\begin{aligned} \eta^\tau(x,s) = \int_0^s u(x,\tau - r)dr &= \int_0^s \phi(x,-r)dr \\ &= \int_{-s}^0 \phi(x,r)dr := \eta_0(x,s), \quad \forall s > 0. \end{aligned} \tag{1.11}$$

For the final purpose, we need to impose some assumptions, as follows:

(A2) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) \geq 0, \quad \forall s \in \mathbb{R}^+$.

(A3) $\mu'(s) + \delta\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+$, for some $\delta > 0$.

(A4) $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, there exist $\alpha_1, \alpha_2, \beta > 0$ such that for some $p > 2$

$$|f(s)| \leq \alpha_1(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}, \tag{1.12}$$

and

$$f(s)s \geq \alpha_2|s|^p - \beta, \quad \forall s \in \mathbb{R}. \tag{1.13}$$

Moreover, there exists $\alpha_3 > 0$ such that

$$f'(s) > -\alpha_3, \quad \forall s \in \mathbb{R}. \tag{1.14}$$

Remark 1.1. **(A1)** *implies that the set* $\{z \in \overline{\Omega} : a(z) = 0\}$ *is finite and the function* $a(\cdot)$ *could be non-smooth (see [8]). Moreover, the physical motivation for* **(A1)** *is to deal with the media that may be "perfect" insulators or "perfect" conductors (see [8]). When the medium is fully insulated and completely dielectric at some points, it is quite natural that* $a(x)$ *is assumed to vanish at these points (see [36]).*

Remark 1.2. *In order to apply the continuity and compact embedding from Lemma 1.3 in Theorem 1.7, for the nonlinear term* f*, we will consider that the order* p *satisfies* (1.21) *below.*

1.2.1 Some preparation

For each $\alpha \in (0, 2)$ and $N \geq 2$, we introduce a positive number that will be useful later in the Sobolev embedding, like

$$2^*_\alpha := \begin{cases} \dfrac{4}{\alpha} \in (2, \infty), & \text{if} \quad N = 2, \\ \dfrac{2N}{N-2+\alpha} \in (2, \dfrac{2N}{N-2}) & \text{if} \quad N \geq 3. \end{cases}$$

According to [8], we have the following lemma about the continuous and compact embedding of the space $H^a_0(\Omega)$.

Lemma 1.3. *Assume* $a \in L^1_{loc}(\Omega)$ *satisfies* **(A1)**. *Then, in the bounded domain* Ω *of* $\mathbb{R}^N (N \geq 2)$*, the following embeddings hold:*
(i) $H^a_0(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega)$ *continuously;*
(ii) $H^a_0(\Omega) \hookrightarrow W^{1,\theta}_0(\Omega)$ *continuously for every* $\theta \in [1, \frac{2N}{N+\alpha})$*;*
(iii) $H^a_0(\Omega) \hookrightarrow L^q(\Omega)$ *with compact inclusion if* $q \in [1, 2^*_\alpha)$.
Specifically, the embedding $H^a_0(\Omega) \hookrightarrow L^2(\Omega)$ *is compact.*

To use the same technique as in [102], we need to define an operator $\mathcal{F} : L^2_{H^a_0} \to L^2_\mu(\mathbb{R}^+; H^a_0(\Omega))$ by

$$(\mathcal{F}\phi)(s) = \int_{-s}^0 \phi(r)dr, \quad s \in \mathbb{R}^+. \tag{1.15}$$

Then we can derive:

Lemma 1.4. *Assume* **(A1)**-**(A3)** *hold. Then, operator* $\mathcal{F}$ *is a linear and continuous mapping. In addition, for any* $\phi \in L^2_{H^a_0}$*, it holds*

$$\|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))} \leq C_\mu \|\phi\|^2_{L^2_{H^a_0}}, \tag{1.16}$$

where C_μ *is a positive constant independent of* ϕ.

Proof. Conditions **(A2)** and **(A3)** imply that $\mu(\cdot)$ decays exponentially, and allows $\mu(\cdot)$ to have a singularity at $s=0$. Therefore, we truncate $s\in\mathbb{R}^+$ into two parts $s\in[0,1]$ and $s\in[1,\infty)$. It follows from **(A3)** and (1.15) that

$$
\begin{aligned}
&\|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}\\
&=\int_0^\infty \mu(s)\Big\|\int_{-s}^0\phi(r)dr\Big\|^2_{H^a_0}ds\\
&=\int_0^\infty \mu(s)\int_\Omega a(x)\Big|\int_{-s}^0\nabla\phi(r)dr\Big|^2dxds\\
&=\int_0^1 \mu(s)\int_\Omega a(x)\Big|\int_{-s}^0\nabla\phi(r)dr\Big|^2dxds+\int_1^\infty \mu(s)\int_\Omega a(x)\Big|\int_{-s}^0\nabla\phi(r)dr\Big|^2dxds\\
&\le\int_0^1 s\mu(s)\int_\Omega a(x)\Big(\int_{-s}^0|\nabla\phi(r)|^2dr\Big)dxds\\
&\quad+\mu(1)\int_1^\infty e^{-\delta(s-1)}s\int_\Omega a(x)\Big(\int_{-s}^0|\nabla\phi(r)|^2dr\Big)dxds\\
&\le\int_{-r}^1 s\mu(s)ds\int_{-1}^0\Big(\int_\Omega a(x)|\nabla\phi(r)|^2dx\Big)dr\\
&\quad+\mu(1)e^\delta\int_{-r}^\infty se^{-\gamma r}e^{-\delta s}ds\int_{-\infty}^0 e^{\gamma r}\int_\Omega a(x)|\nabla\phi(r)|^2dxdr\\
&\le\int_0^1\mu(s)ds\int_{-1}^0 e^{-\gamma r}e^{\gamma r}\|\phi(r)\|^2_{H^a_0}dr+\mu(1)e^\delta\int_{-r}^\infty se^{\gamma s}e^{-\delta s}ds\|\phi(r)\|^2_{L^2_{H^a_0}}\\
&\le e^\gamma\int_0^1\mu(s)ds\|\phi(r)\|^2_{L^2_{H^a_0}}+\frac{\mu(1)e^\delta}{(\gamma-\delta)^2}\|\phi(r)\|^2_{L^2_{H^a_0}},
\end{aligned}
$$

where we used that $0<\gamma<\delta$. Then we have

$$
\|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}\le\Big(e^\gamma\int_0^1\mu(s)ds+\frac{\mu(1)e^\delta}{(\gamma-\delta)^2}\Big)\|\phi(r)\|^2_{L^2_{H^a_0}},
$$

which means that $\mathcal{F}$ is well-defined and bounded. In addition, the linear property of $\mathcal{F}$ is evident, so we obtain the desired result.□

To obtain the regularity of the solution, we introduce an operator $\mathcal{I}:L^2_{D(Q)}\to L^2_\mu(\mathbb{R}^+;D(Q))$ by

$$
(\mathcal{I}\phi)(s)=\int_{-s}^0\phi(r)dr,\quad s\in\mathbb{R}^+. \tag{1.17}
$$

Using a method similar to Lemma 1.4, we can prove the next result.

Lemma 1.5. *Assume* **(A1)**-**(A3)** *hold. Then the operator $\mathcal{I}$ is a linear and continuous mapping. In addition, for any $\phi\in L^2_{D(Q)}$, it holds*

$$
\|\mathcal{I}\phi\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))}\le C_\mu\|\phi\|^2_{L^2_{D(Q)}}, \tag{1.18}
$$

where C_μ is the same as in (1.16).

1.2.2 Existence and uniqueness of solutions of original equations

For any $0 < \epsilon < 1$, let $a_\epsilon(x) = a(x) + \epsilon$ and denote

$$z_\epsilon(t) = (u_\epsilon(t), \eta_\epsilon^t)^T \quad \text{and} \quad z_{0,\epsilon} = (u_0, \eta_0)^T.$$

To approximate problem (1.10), we consider the following non-degenerate problem:

$$\begin{cases} \partial_t z_\epsilon = \mathcal{G} z_\epsilon + \mathcal{F}(z_\epsilon), & \text{in } \Omega \times (\tau, \infty), \\ z_\epsilon(x,t) = 0, & \text{on } \partial\Omega \times (\tau, \infty), \\ z_\epsilon(x,\tau) = z_{0,\epsilon} & \text{in } \Omega, \end{cases} \tag{1.19}$$

where

$$\mathcal{G} z_\epsilon = \begin{pmatrix} m(l(u_\epsilon))\text{div}[a_\epsilon(x)\nabla u_\epsilon] + \int_0^\infty \mu(s)\text{div}[a(x)\nabla \eta_\epsilon^t(s)]ds \\ u_\epsilon - \partial_s \eta_\epsilon \end{pmatrix},$$

and

$$\mathcal{F}(z_\epsilon) = (-f(u_\epsilon) + g, 0)^T.$$

We first show the existence of solutions to problem (1.19).

Theorem 1.6. *Suppose that* (1.6), **(A1)-(A4)**, *and* $g \in L^2(\Omega)$ *hold. Also assume that* $m(\cdot)$ *is locally Lipschitz, and there exists a constant* $\tilde{h} > 0$ *such that*

$$m(s) \le \tilde{h}, \quad \forall s \in \mathbb{R}. \tag{1.20}$$

Moreover, assume that

$$2 < p \le \frac{2N}{N-2+\alpha}, \quad \alpha \in (0,2). \tag{1.21}$$

Then, for any $z_{0,\epsilon} \in \mathcal{H}$, *there exists a weak solution* $z_\epsilon(\cdot)$ *to problem* (1.19).

Proof. We temper to using the Fadeo-Galerkin method (as [40, 102]) to solve this part.

Let $\{w_j\}_{j=1}^\infty$ be a sequence of smooth functions constituting an orthonormal basis in $L^2(\Omega)$ and orthogonal in $H_0^a(\Omega)$. Also, let $\{\zeta_j\}_{j=1}^\infty$ be an orthonormal basis of $L_\mu^2(\mathbb{R}^+; H_0^a(\Omega))$, whose elements belong to $\mathcal{D}(\mathbb{R}_+; H_0^a(\Omega))$. Here $\mathcal{D}(I; X)$ is the space of infinitely differentiable X-valued functions with compact support in $I \subset \mathbb{R}$, and its dual space is the distribution space over I with values in X^* (dual of X), denoted as $\mathcal{D}'(I; X^*)$.

(1) For any $n \in \mathbb{N}$, let P_n be the orthogonal projection from $L^2(\Omega)$ to a finite-dimensional subspace $W_n = span\{w_1, \dots, w_n\}$, and χ_n be the orthogonal projection from $L_\mu^2(\mathbb{R}^+; H_0^a(\Omega))$ to a finite-dimensional subspace $H_n = span\{\zeta_1, \dots, \zeta_n\}$.

For $T > \tau$ and given $\epsilon > 0, n \in \mathbb{N}$, consider the approximate solution $z_{n,\epsilon}(\cdot) = (u_{n,\epsilon}(\cdot), \eta_{n,\epsilon}^{\cdot})$ in the form

$$u_{n,\epsilon}(t) = \sum_{j=1}^n b_j(t) w_j \quad \text{and} \quad \eta_{n,\epsilon}^t(s) = \sum_{j=1}^n c_j(t)\zeta_j(s),$$

to satisfy

$$\begin{cases}(\partial_t z_{n,\epsilon},(w_k,\zeta_j))_{\mathcal{H}}=(\mathcal{G}z_{n,\epsilon},(w_k,\zeta_j))+(\mathcal{F}(z_{n,\epsilon}),(w_k,\zeta_j)), \quad k,j=0,\dots,n,\\ z_{n,\epsilon}|_{t=\tau}=(P_nu_0,\chi_n\eta_0),\end{cases} \tag{1.22}$$

for a.e. $\tau\le t\le T$, where w_0,ζ_0 are the zero vectors in their respective spaces. Take (w_k,ζ_0) and (w_0,ζ_k) in (1.22), and then use the divergence theorem to obtain the following system:

$$\begin{cases}\dfrac{d}{dt}b_k(t)=m(l(\sum_{j=1}^n b_j(t)w_j))\Big[\sum_{j=1}^n b_j(t)(w_j,w_k)_{H_0^a}+\epsilon\sum_{j=1}^n b_j(t)(w_j,w_k)_1\Big]\\ \quad-\sum_{j=1}^n c_j(\zeta_j,w_k)_{1,\mu}-(f(\sum_{j=1}^n b_j(t)w_j)),w_k)+(g,w_k),\\ \dfrac{d}{dt}c_k(t)=\sum_{j=1}^n b_j(w_j,\zeta_k)_{1,\mu}-\sum_{j=1}^n c_j(\zeta_j',\zeta_k)_{1,\mu},\\ b_k(\tau)=(u_0,w_k),\quad c_k(\tau)=(\eta_0,\zeta_k)_{1,\mu}.\end{cases} \tag{1.23}$$

According to the standard existence theory for ODE, a continuous solution of (1.23) exists on (τ,T_n). Using a priori estimate, it can be deduced that $T_n=+\infty$.

(2) We establish a priori estimate for solutions $z_{n,\epsilon}(t)=(u_{n,\epsilon}(t),\eta^t_{n,\epsilon})$. Multiplying $(1.23)_1$ by b_k and $(1.23)_2$ by c_k, then summing over $k(k=1,2,\dots,n)$, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_{n,\epsilon}(t)\|^2+\|\eta^t_{n,\epsilon}\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\Big)+m(l(u_{n,\epsilon}))\|u_{n,\epsilon}\|^2_{H_0^a}+\epsilon m(l(u_{n,\epsilon}))\|u_{n,\epsilon}\|_1^2 \\ +(\partial_s\eta^t_{n,\epsilon},\eta^t_{n,\epsilon})_{1,\mu}+(f(u_{n,\epsilon}),u_{n,\epsilon})=(g,u_{n,\epsilon}). \tag{1.24}$$

By (1.6), (1.3) and (1.13), we derive

$$\frac{d}{dt}\Big(\|u_{n,\epsilon}(t)\|^2+\|\eta^t_{n,\epsilon}\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\Big)+2h\|u_{n,\epsilon}\|^2_{H_0^a}+2h\epsilon\|u_{n,\epsilon}\|_1^2 \\ +2(\partial_s\eta^t_{n,\epsilon},\eta^t_{n,\epsilon})_{1,\mu}+2\alpha_2\|u_{n,\epsilon}\|_p^p\le 2\beta|\Omega|+\frac{1}{h\lambda_1}\|g\|^2+h\|u_{n,\epsilon}\|^2_{H_0^a}. \tag{1.25}$$

Based on **(A2)**, **(A3)**, and integration by parts, it follows that

$$\begin{aligned}2(\partial_s\eta^t_{n,\epsilon},\eta^t_{n,\epsilon})_{1,\mu}&=2\int_0^\infty\int_\Omega\mu(s)\sqrt{a}\nabla\partial_s\eta^t_{n,\epsilon}(s)\cdot\sqrt{a}\nabla\eta^t_{n,\epsilon}(s)dxds\\ &=-\int_0^\infty\mu'(s)\|\eta^t_{n,\epsilon}\|^2_{H_0^a}ds\ge 0.\end{aligned} \tag{1.26}$$

Then,

$$\frac{d}{dt}\Big(\|u_{n,\epsilon}(t)\|^2+\|\eta^t_{n,\epsilon}\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\Big)+h\|u_{n,\epsilon}\|^2_{H_0^a}+2h\epsilon\|u_{n,\epsilon}\|_1^2+2\alpha_2\|u_{n,\epsilon}\|_p^p \\ \le 2\beta|\Omega|+\frac{1}{h\lambda_1}\|g\|^2. \tag{1.27}$$

By integrating (1.27) between τ and t with $t \in (\tau, T]$, one obtains

$$\|u_{n,\epsilon}(t)\|^2 + \|\eta_{n,\epsilon}^t\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega)} + h\int_\tau^t \|u_{n,\epsilon}(s)\|^2_{H_0^a} ds + 2h\epsilon\int_\tau^t \|u_{n,\epsilon}(s)\|_1^2 ds$$
$$+ 2\alpha_2 \int_\tau^t \|u_{n,\epsilon}(s)\|_p^p ds \le \|u_0\|^2 + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega)} + c_1(T-\tau). \tag{1.28}$$

Therefore, we conclude that

$$\begin{aligned}
&\{u_{n,\epsilon}\}_{n=1}^\infty \text{ is uniformly bounded in} \\
&\qquad L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \cap L^p(\tau, T; L^p(\Omega)), \\
&\{\eta_{n,\epsilon}^t\}_{n=1}^\infty \text{ is uniformly bounded in } L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))),
\end{aligned}$$

and hence there exists a limit function $z_\epsilon(t) = (u_\epsilon(t), \eta_\epsilon^t)$ satisfying

$$\begin{aligned}
&u_\epsilon \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \cap L^p(\tau, T; L^p(\Omega)), \\
&\eta_\epsilon^t \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))),
\end{aligned}$$

such that, as $n \to \infty$, we have

$$\begin{cases}
u_{n,\epsilon} \to u_\epsilon & \text{weak-star in} \quad L^\infty(\tau, T; L^2(\Omega)), \\
u_{n,\epsilon} \to u_\epsilon & \text{weakly in} \quad L^2(\tau, T; H_0^a(\Omega)), \\
u_{n,\epsilon} \to u_\epsilon & \text{weakly in} \quad L^p(\tau, T; L^p(\Omega)), \\
\eta_{n,\epsilon}^t \to \eta_\epsilon^t & \text{weak-star in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))).
\end{cases} \tag{1.29}$$

(3) Give $\pi \in \mathbb{N}$, $v = (\sigma, \varsigma) \in \mathcal{D}(\tau, T; H_0^a(\Omega) \cap L^p(\Omega)) \times \mathcal{D}(\tau, T; \mathcal{D}(\mathbb{R}^+; H_0^a(\Omega)))$ of the form

$$\sigma(t) = \sum_{j=1}^{\pi} \tilde{b}_j(t) w_j \quad \text{and} \quad \varsigma^t(s) = \sum_{j=1}^{\pi} \tilde{c}_j(t)\zeta_j(s),$$

where $\{\tilde{b}_j\}_{j=1}^\pi$ and $\{\tilde{c}_j\}_{j=1}^\pi$ are given functions in $\mathcal{D}(\tau, T)$. Then, by (1.22), let $v = (\sigma, \varsigma) \in \mathcal{D}(\tau, T)$, we have

$$\int_\tau^t (\partial_r z_{n,\epsilon}, v)_{\mathcal{H}} dr = \int_\tau^t \Big[- m(l(u_{n,\epsilon})) \int_\Omega a_\epsilon(x) \nabla u_{n,\epsilon} \cdot \nabla \sigma dx - (\eta_{n,\epsilon}^r, \sigma)_{1,\mu} \tag{1.30}$$
$$- (f(u_{n,\epsilon}), \sigma) + (g, \sigma) + (u_{n,\epsilon}, \varsigma^r)_{1,\mu} - < \partial_s \eta_{n,\epsilon}^r, \varsigma^r > \Big] dr$$

holds in the sense of $\mathcal{D}'(\tau, T)$, and in which $< \cdot, \cdot >$ represents the duality map of $H_\mu^1(\mathbb{R}^+; H_0^a(\Omega))$ and its dual space. It can be inferred from (1.28) that

$$h\int_\tau^t \int_\Omega a_\epsilon(x) |\nabla u_{n,\epsilon}|^2 dx dr = h\int_\tau^t \int_\Omega a(x) |\nabla u_{n,\epsilon}|^2 dx dr + h\epsilon \int_\tau^t \int_\Omega |\nabla u_{n,\epsilon}|^2 dx dr$$
$$= h\int_\tau^t \|u_{n,\epsilon}(r)\|^2_{H_0^a} dr + h\epsilon \int_\tau^t \|u_{n,\epsilon}(r)\|_1^2 dr \le c_2, \tag{1.31}$$

where $c_2 > 0$ and independent of n and ϵ. Therefore, by the result in [101, Theorem 2.7], we find that as $n \to \infty$,

$$\begin{aligned}&\int_\tau^t m(l(u_{n,\epsilon}))\int_\Omega a_\epsilon(x)\nabla u_{n,\epsilon}\cdot\nabla\sigma dxdr\\ &\qquad\to\int_\tau^t m(l(u_\epsilon))\int_\Omega a_\epsilon(x)\nabla u_\epsilon\cdot\nabla\sigma dxdr.\end{aligned} \tag{1.32}$$

Analogously, by $(1.29)_2$ and $(1.29)_4$ yields

$$\int_\tau^t (\eta^r_{n,\epsilon},\sigma)_{1,\mu}dr \to \int_\tau^t (\eta^r_\epsilon,\sigma)_{1,\mu}dr \quad \text{as} \quad n\to\infty, \tag{1.33}$$

and

$$\int_\tau^t (u_{n,\epsilon},\varsigma^r)_{1,\mu}dr \to \int_\tau^t (u_\epsilon,\varsigma^r)_{1,\mu}dr \quad \text{as} \quad n\to\infty. \tag{1.34}$$

In addition, for every $\upsilon \in L^2_\mu(\mathbb{R}^+; H^a_0(\Omega))$, we have

$$<\partial_s\upsilon,\varsigma^t> = -\int_0^\infty \mu'(s)(\upsilon(s),\varsigma^t(s))_{H^a_0}ds - \int_0^\infty \mu(s)(\upsilon(s),(\varsigma^t)'(s))_{H^a_0}ds. \tag{1.35}$$

Using $(1.29)_3$, it can be deduced that

$$\lim_{n\to\infty} <\partial_s\eta^t_{n,\epsilon},\varsigma^t> = <\partial_s\eta^t_\epsilon,\varsigma^t> .$$

If we can verify that

$$\lim_{n\to\infty}\int_\tau^T\int_\Omega |f(u_{n,\epsilon})\sigma|dxdt = \int_\tau^T\int_\Omega |f(u_\epsilon)\sigma|dxdt, \tag{1.36}$$

then, by standard arguments, we have

$$\partial_t z_{n,\epsilon} \to \partial_t z_\epsilon \quad \text{in} \quad \mathcal{D}'(\tau,T;H^a_0(\Omega)\cap L^p(\Omega))\times\mathcal{D}'(\tau,T;\mathcal{D}(\mathbb{R}^+;H^a_0(\Omega))). \tag{1.37}$$

Combining (1.32) to (1.36), and by taking limit as n goes to ∞ in equation (1.30), we can finally obtain that $z_\epsilon(t) = (u_\epsilon(t), \eta^t_\epsilon)$ is a weak solution of (1.19).

Now, our purpose is to prove that (1.36) holds. Notice that

$$\begin{aligned}\|\partial_t u_{n,\epsilon}\|_{L^2(\tau,T;V^{-1}(\Omega))+L^{\frac{p}{p-1}}(\tau,T;L^{\frac{p}{p-1}}(\Omega))} &\le \|m(l(u_{n,\epsilon}))\mathrm{div}[a_\varepsilon(x)\nabla u_{n,\epsilon}]\|_{L^2(\tau,T;V^{-1}(\Omega))}\\ &+\left\|\int_0^\infty \mu(s)\mathrm{div}[a(x)\nabla\eta^t_{n,\epsilon}(s)]ds\right\|_{L^2(\tau,T;V^{-1}(\Omega))} + \|f(u_{n,\epsilon})\|_{L^{\frac{p}{p-1}}(\tau,T;L^{\frac{p}{p-1}}(\Omega))}\\ &+(T-\tau)^{\frac12}\|g\|_{V^{-1}(\Omega)},\end{aligned} \tag{1.38}$$

where $\frac{p}{p-1} \in (1,2]$. By (1.12), we deduce that there exists a number $K > 0$ such that

$$|f(u_{n,\epsilon})|^{\frac{p}{p-1}} \le K(1+|u_{n,\epsilon}|^p), \tag{1.39}$$

which, together with $g \in L^2(\Omega)$, (1.20) and (1.29), we know that $\{\partial_t u_{n,\epsilon}\}$ is bounded in $L^2(\tau, T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(\tau, T; L^{\frac{p}{p-1}}(\Omega))$. Thus, there exists a subsequence such that

$$\partial_t u_{n,\epsilon} \to \partial_t u_\epsilon \quad \text{weakly in} \quad L^2(\tau, T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(\tau, T; L^{\frac{p}{p-1}}(\Omega)). \tag{1.40}$$

By (1.21) and Lemma 1.3, we have the dense embedding

$$H_0^a(\Omega) \hookrightarrow L^p(\Omega),$$

and it follows that $L^{\frac{p}{p-1}}(\Omega) \hookrightarrow V^{-1}(\Omega)$. Moreover, it is true that

$$L^2(\tau, T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(\tau, T; L^{\frac{p}{p-1}}(\Omega)) \subset L^{\frac{p}{p-1}}(\tau, T; V^{-1}(\Omega)). \tag{1.41}$$

By (1.29) and (1.40), we derive

$$u_{n,\epsilon} \rightharpoonup u_\epsilon \text{ in } W^{1,\frac{p}{p-1}}(\tau, T; V^{-1}(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)). \tag{1.42}$$

By the compactness argument [78, P86, Corollary 5] and Lemma 1.3, we have the compact embedding

$$W^{1,\frac{p}{p-1}}(\tau, T; V^{-1}(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \hookrightarrow L^2(\tau, T; L^2(\Omega)).$$

Therefore, there exists a subsequence (relabeled the same) such that

$$u_{n,\epsilon} \to u_\epsilon \quad \text{strongly in} \quad L^2(\tau, T; L^2(\Omega)). \tag{1.43}$$

Since $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, we have

$$f(u_{n,\epsilon}(t,x)) \to f(u_\epsilon(t,x)) \text{ for a.e. } \quad (t,x) \in (\tau, T) \times \Omega,$$

and by (1.39),

$$\|f(u_{n,\epsilon})\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}((\tau,T)\times\Omega)} \leq K|\Omega|(T-\tau) + K\int_\tau^T \|u_{n,\epsilon}\|_p^p dt.$$

From this, using the dominated convergence theorem, we conclude (1.36).

(4) By (1.29), (1.35) and (1.40), we directly obtain that $u_\epsilon \in C([\tau, T]; L^2(\Omega))$ and $\eta_\epsilon^t \in C([\tau, T]; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega)))$ by using [81, Lemma III.1.2] and [40, Theorem, Section 2], respectively. □

Next we obtain the main result as follows:

Theorem 1.7. *Assume that the conditions in Theorem 1.6 hold.*

(i) If $(u_0, \phi) \in \mathcal{M}$, *then there exists a unique solution* $(u(\cdot), \eta^\cdot)$ *to problem* (1.5) *which satisfies*

$$u(\cdot) \in L^\infty([\tau, T]; L^2(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \cap L^p(\tau, T; L^p(\Omega)), \quad \forall \tau < T,$$
$$\eta^\cdot \in L^\infty([\tau, T]; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))), \quad \forall \tau < T.$$

Furthermore, the mapping $(u_0, \mathcal{F}\phi) \to (u(t), \eta^t)$ *is continuous in* $\mathcal{H}$ *for all* $t \geq \tau$.

(ii) If $(u_0, \phi) \in \mathcal{M}^1$, *then there exists a unique solution* $(u(\cdot), \eta^\cdot)$ *to problem* (1.5) *which satisfies*

$$u(\cdot) \in L^\infty([\tau, T]; H_0^a(\Omega)) \cap L^2(\tau, T; D(Q)), \quad \forall \tau < T,$$
$$\eta^\cdot \in L^\infty([\tau, T]; L^2_\mu(\mathbb{R}^+; D(Q))), \quad \forall \tau < T,$$

and the mapping $(u_0, \mathcal{I}\phi) \to (u(t), \eta^t)$ *is continuous in* $\mathcal{H}^1$ *for all* $t \geq \tau$.

Proof. We first prove the existence, uniqueness, and regularity of the solution to problem (1.10) when $\epsilon \to 0$ by the following three steps.

Step 1. From Theorem 1.6, we know that $(u_\epsilon(t), \eta^t_\epsilon)$ is a weak solution of (1.19). By using the same method as for (1.28), we have

$$\|u_\epsilon(t)\|^2 + \|\eta^t_\epsilon\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega)} + h\int_\tau^t \|u_\epsilon(s)\|^2_{H_0^a} ds + 2h\epsilon \int_\tau^t \|u_\epsilon(s)\|_1^2 ds$$
$$+ 2\alpha_2 \int_\tau^t \|u_\epsilon(s)\|_p^p ds \leq \|u_0\|^2 + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega)} + c_1(T - \tau). \tag{1.44}$$

Thus,

$$\begin{aligned} &\{u_\epsilon\} \text{ is uniformly bounded in} \\ &\qquad L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \cap L^p(\tau, T; L^p(\Omega)), \\ &\{\eta^t_\epsilon\} \text{ is uniformly bounded in } L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))), \end{aligned}$$

and, consequently, there exist limit functions

$$u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^a(\Omega)) \cap L^p(\tau, T; L^p(\Omega)),$$
$$\eta^t \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))),$$

such that, there exist subsequences (relabeled the same) of $\{u_\epsilon\}_{\epsilon>0}$ and $\{\eta^t_\epsilon\}_{\epsilon>0}$ which satisfy, as $\epsilon \to 0$,

$$\begin{cases} u_\epsilon \to u & \text{weak-star in} \quad L^\infty(\tau, T; L^2(\Omega)), \\ u_\epsilon \to u & \text{weakly in} \quad L^2(\tau, T; H_0^a(\Omega)), \\ u_\epsilon \to u & \text{weakly in} \quad L^p(\tau, T; L^p(\Omega)), \\ -\epsilon\Delta u_\epsilon \to 0 & \text{weakly in} \quad L^2(\tau, T; H^{-1}(\Omega)), \\ \eta^t_\epsilon \to \eta^t & \text{weak-star in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))). \end{cases} \tag{1.45}$$

Thus, using the same argument as in Theorem 1.6, we can prove that there exists a weak solution $(u(t), \eta^t)$ of (1.10) and it satisfies $u \in C([\tau, T]; L^2(\Omega))$ and $\eta^t \in C([\tau, T]; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega)))$.

Step 2. Next, we will prove the uniqueness and the continuous dependence on the initial data of solutions $z(t) = (u(t), \eta^t)$ of the equation (1.10).

Let z_1, z_2 be two solutions to (1.10) and denote $\bar{z} = z_1 - z_2 = (\bar{u}, \bar{\eta}^t) = (u_1 - u_2, \eta_1^t - \eta_2^t)$. Then, we have

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = m(l(u_1))\text{div}[a(x)\nabla\bar{u}] + (m(l(u_1)) - m(l(u_2)))\text{div}[a(x)\nabla u_2] \\ \quad + \displaystyle\int_0^\infty \mu(s)\text{div}[a(x)\nabla\bar{\eta}^t(s)]ds - (f(u_1) - f(u_2)), \qquad \text{in } \Omega \times (\tau, \infty), \\ \dfrac{\partial}{\partial t}\bar{\eta}^t(s) = \bar{u} - \dfrac{\partial}{\partial s}\bar{\eta}^t(s), \qquad \text{in } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, \qquad \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x) = u_{0,1}(x) - u_{0,2}(x), \qquad \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s) = \eta_{0,1}(x,s) - \eta_{0,2}(x,s), \qquad \text{in } \Omega \times \mathbb{R}^+. \end{cases} \tag{1.46}$$

Since $m(\cdot)$ is locally Lipschitz, for any bounded interval $I \subseteq \mathbb{R}$, there exists Lipschitz constant $L_m = L_m(I) > 0$ such that

$$|m(s_1) - m(s_2)| \le L_m|s_1 - s_2|, \quad \forall s_1, s_2 \in I. \tag{1.47}$$

By (1.44) and $u \in C([\tau, T]; L^2(\Omega))$, we know that there exists a bounded set $\mathcal{O} \subset L^2(\Omega)$ such that $\{u_i(t)\}_{t\in[\tau,T]} \subset \mathcal{O}(i = 1, 2)$. In addition, by the fact that $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, we have $\{l(u_i(t))\}_{t\in[\tau,T]} \subset I(i = 1, 2)$.

Multiplying $(1.46)_1$ by $\bar{u}$ and $(1.46)_2$ by $\bar{\eta}^t$, then by using (1.6), (1.14), the Young inequality and summing the two results we deduce

$$\begin{aligned} \frac{d}{dt}\|\bar{z}\|_{\mathcal{H}}^2 &\le -2m(l(u_1))\|\bar{u}\|_{H_0^a}^2 + 2(m(l(u_1)) - m(l(u_2)))(a(x)\nabla u_2, \nabla\bar{u}) \\ &\quad - 2\int_\Omega (f(u_1) - f(u_2))\bar{u}dx - 2(\partial_s\bar{\eta}^t, \bar{\eta}^t)_{1,\mu} \\ &\le -2m(l(u_1))\|\bar{u}\|_{H_0^a}^2 + 2L_m|l|\|\bar{u}\|\left\|\sqrt{a}\nabla u_2\right\|\left\|\sqrt{a}\nabla\bar{u}\right\| \\ &\quad - 2\int_\Omega (f(u_1) - f(u_2))\bar{u}dx - 2(\partial_s\bar{\eta}^t, \bar{\eta}^t)_{1,\mu} \\ &\le \frac{L_m^2}{2h}|l|^2\|\bar{u}\|^2\|u_2\|_{H_0^a}^2 + \alpha_3\|\bar{u}\|^2 - 2(\partial_s\bar{\eta}^t, \bar{\eta}^t)_{1,\mu}. \end{aligned} \tag{1.48}$$

By [40, P348] and **(A2)**, we find that

$$2(\partial_s\bar{\eta}^t, \bar{\eta}^t)_{1,\mu} = \lim_{s\to\infty}\mu(s)\|\nabla\bar{\eta}^t(s)\|^2 - \int_0^\infty \mu'(s)\|\nabla\bar{\eta}^t(s)\|^2ds \ge 0. \tag{1.49}$$

Integrating (1.48) over τ and t, by (1.49) we have

$$\begin{aligned} \|\bar{z}(t)\|_{\mathcal{H}}^2 &\le \|\bar{z}(\tau)\|_{\mathcal{H}}^2 + \frac{L_m^2}{2h}|l|^2\int_\tau^t \|\bar{u}(s)\|^2\|u_2(s)\|_{H_0^a}^2ds + \alpha_3\int_\tau^t \|\bar{z}(s)\|_{\mathcal{H}}^2ds \\ &\le \|\bar{z}(\tau)\|_{\mathcal{H}}^2 + c_3\int_\tau^t \|\bar{z}(s)\|_{\mathcal{H}}^2ds, \end{aligned} \tag{1.50}$$

where $c_3 = \frac{L_m^2}{2h}|l|^2\|u_2\|_{L^2((\tau,T];H_0^a(\Omega))} + \alpha_3$. Applying the Gronwall inequality to (1.50) yields

$$\|\bar{z}(t)\|_{\mathcal{H}}^2 \leq \|\bar{z}(\tau)\|_{\mathcal{H}}^2 e^{c_3(t-\tau)}, \tag{1.51}$$

and the result is complete.

Step 3. Now, we will show the regularity of solutions to problem (1.10).

Multiplying $(1.10)_1$ by Qu with respect to the inner product of $L^2(\Omega)$, and $(1.10)_2$ by η^t with respect to the inner product of $L^2_\mu(\mathbb{R}^+; D(Q))$, and then adding these two terms together leads to

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{H_0^a}^2 + \|\eta^t\|_{L^2_\mu(\mathbb{R}^+;D(Q))}^2) + 2m(l(u))\|Qu\|^2 + 2\int_0^\infty \mu(s)(\partial_s\eta^t,\eta^t)_{2,\mu}ds\\ &= -2(f(u),Qu) + 2(g,Qu).\end{aligned} \tag{1.52}$$

Notice that $f(s) = s\int_0^1 f'(\iota s)d\iota + f(0)$, from (1.12) we deduce that $|f(0)| \leq \alpha_1$, and along with (1.14) we have

$$\begin{aligned}-2\int_\Omega f(u)Qudx &= -2\int_\Omega Qu\big[u\int_0^1 f'(\iota u)d\iota + f(0)\big]dx\\ &\leq 2\alpha_3\|u\|_{H_0^a}^2 + \frac{h}{2}\|Qu\|^2 + \frac{2\alpha_1^2}{h}|\Omega|.\end{aligned} \tag{1.53}$$

According to Young's inequality,

$$2(g,Qu) \leq \frac{h}{2}\|Qu\|^2 + \frac{2}{h}\|g\|^2. \tag{1.54}$$

By **(A2)**,

$$\begin{aligned}2\int_0^\infty \mu(s)(\partial_s\eta^t,\eta^t)_{2,\mu}ds &= 2\int_0^\infty \mu(s)(Q\partial_s\eta^t,Q\eta^t)ds\\ &= -\int_0^\infty \mu'(s)\|Q\eta^t(s)\|^2ds > 0.\end{aligned} \tag{1.55}$$

Substituting (1.53) to (1.55) into (1.52) yields

$$\frac{d}{dt}(\|u\|_{H_0^a}^2 + \|\eta^t\|_{L^2_\mu(\mathbb{R}^+;D(Q))}^2) + h\|Qu\|^2 \leq 2\alpha_3\|u\|_{H_0^a}^2 + \frac{2\alpha_1^2}{h}|\Omega| + \frac{2}{h}\|g\|^2. \tag{1.56}$$

Using the Gronwall inequality, for all $t \in (\tau, T]$ we have

$$\|u(t)\|_{H_0^a}^2 + \|\eta^t\|_{L^2_\mu(\mathbb{R}^+;D(Q))}^2 + h\int_\tau^t \|u(s)\|_{D(Q)}^2ds \leq \|u_0\|_{H_0^a}^2 + \|\eta_0\|_{L^2_\mu(\mathbb{R}^+;D(Q))}^2 + c_4, \tag{1.57}$$

where $c_4 = 2\alpha_3\|u\|_{L^2((\tau,T];H_0^a(\Omega))}^2 + \left(\frac{2\alpha_1^2}{h}|\Omega| + \frac{2}{h}\|g\|^2\right)(T-\tau)$. Therefore, we can conclude that

$$u \in L^\infty(\tau,T;H_0^a(\Omega)) \cap L^2(\tau,T;D(Q)),$$

$$\eta^t \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; D(Q))).$$

(i) Since $\phi \in L^2_{H^a_0}$, it follows from Lemma 1.4 that $\mathcal{F}\phi \in L^2_\mu(\mathbb{R}^+; H^a_0(\Omega))$. The desired result is obtained by applying **Step 1** and **Step 2** with the initial value $(u_0, \mathcal{F}\phi) \in \mathcal{H}$.

(ii) Since $\phi \in L^2_{D(Q)}$, it follows from Lemma 1.5 that $\mathcal{I}\phi \in L^2_\mu(\mathbb{R}^+; D(Q))$. Due to the initial value $(u_0, \mathcal{I}\phi) \in \mathcal{H}^1$, we can obtain the regularity result by **Step 3**. □

1.3 The existence of a global attractor

In this part, we will show the existence result of global attractors (i.e. a compact, invariant and attracting set) to the problem (1.5) in the sense of the following result:

Theorem 1.8. *[82] A continuous semigroup $\{S(t)\}_{t\geq 0}$ defined in $\mathcal{M}$ has a global attractor, if $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and is asymptotically compact.*

We need to establish an autonomous dynamical system generated by equation (1.5) (under the assumption that g does not depend on t). According to Theorem 1.7, we can define the semigroup $\{S(t)\}_{t\geq 0}$ of solutions on $\mathcal{M}$ by

$$S(t)(u_0, \phi) = (u(t; 0, (u_0, \mathcal{F}\phi)), u_t(\cdot; 0, (u_0, \mathcal{F}\phi))),$$

where $u_t(s) = u(t+s)$ for $s \in (-\infty, 0]$, and $(u(\cdot; 0, (u_0, \mathcal{F}\phi)), \eta^\cdot(\cdot; 0, (u_0, \mathcal{F}\phi)))$ is the unique solution to equation (1.10) with initial values $(u(0), \eta_0) = (u_0, \mathcal{F}\phi)$ and initial time $\tau = 0$.

Given $T > 0$, thanks to $\phi \in L^2_{H^a_0}$ and $u \in L^2(0, T; H^a_0(\Omega))$, we can use a method similar to [102, Lemma 3.6] to immediately obtain that $\{S(t)\}_{t\geq 0}$ is well-defined, that is,

Lemma 1.9. *Assume that the conditions in Theorem 1.6 hold. If $(u_0, \phi) \in \mathcal{M}$, then $S(t)(u_0, \phi) \in \mathcal{M}$.*

1.3.1 Absorption of semigroups

We need to show the estimate of solutions for problem (1.5).

Lemma 1.10. *Assume that the conditions in Theorem 1.6 hold. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that any solution of* (1.5) *satisfies*

$$\|S(t)(u_0, \phi)\|^2_{\mathcal{M}} \leq C_1 e^{-\gamma t} \|(u_0, \phi)\|^2_{\mathcal{M}} + C_2, \quad \forall t \geq 0, (u_0, \phi) \in \mathcal{M}. \tag{1.58}$$

Proof. Denote by $(u(\cdot; 0, (u_0, \mathcal{F}\phi)), \eta^\cdot(\cdot; 0, (u_0, \mathcal{F}\phi)))$ the unique solution of equation (1.10) with initial values $(u_0, \mathcal{F}\phi) \in \mathcal{H}$. Multiplying $(1.10)_1$ by u in $L^2(\Omega)$ and $(1.10)_2$ by η^t in $L^2_\mu(\mathbb{R}^+; H^a_0(\Omega))$, using (1.6) and (1.13), we have

$$\frac{d}{dt}\Big(\|u(t)\|^2 + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+; H^a_0(\Omega))}\Big) + 2h\|u(t)\|^2_{H^a_0} + 2(\partial_s \eta^t, \eta^t)_{1,\mu} + 2\alpha_2 \|u(t)\|^p_p$$

$$\le 2\beta|\Omega| + 2(g,u). \tag{1.59}$$

By **(A3)** and integration by parts, we obtain

$$\begin{aligned} 2(\partial_s \eta^t, \eta^t)_{1,\mu} &= -\int_0^\infty \mu'(s)\|\eta^t(s)\|^2_{H_0^a} ds \\ &\ge \delta \int_0^\infty \mu(s)\|\eta^t(s)\|^2_{H_0^a} ds = \delta \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}. \end{aligned} \tag{1.60}$$

Combining (1.59) and (1.60), from (1.3) and the Young inequality we derive

$$\begin{aligned} &\frac{d}{dt}\Big(\|u(t)\|^2 + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\Big) + \gamma\big(\|u(t)\|^2 + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\big) \\ &+ \frac{h}{2}\|u(t)\|^2_{H_0^a} + 2\alpha_2\|u(t)\|_p^p \le 2\beta|\Omega| + \frac{2}{h\lambda_1}\|g\|^2, \end{aligned} \tag{1.61}$$

where $0 < \gamma < \min\{h\lambda_1, \delta\}$ as in (1.4). On the one hand, by multiplying (1.61) by $e^{\gamma t}$ and integrating it over $(0,t)$, and using (1.16) in Lemma 1.4, we deduce

$$\begin{aligned} &\|u(t)\|^2 + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))} + \frac{h}{2}\int_0^t e^{-\gamma(t-s)}\|u(s)\|^2_{H_0^a} ds \\ &\le e^{-\gamma t}\big(\|u_0\|^2 + \|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))}\big) + \frac{C_0}{\gamma} \\ &\le e^{-\gamma t}\Big(\|u_0\|^2 + C_\mu\|\phi\|^2_{L^2_{H_0^a}}\Big) + \frac{C_0}{\gamma}, \end{aligned} \tag{1.62}$$

where $C_0 = 2\beta|\Omega| + \frac{2}{h\lambda_1}\|g\|^2$. On the other hand, we can infer from (1.62) that

$$\begin{aligned} \|u_t\|^2_{L^2_{H_0^a}} &= \int_{-\infty}^0 e^{\gamma s}\|u(t+s)\|^2_{H_0^a} ds \\ &= e^{-\gamma t}\int_{-\infty}^0 e^{\gamma s}\|\phi(s)\|^2_{H_0^a} ds + \int_0^t e^{-\gamma(t-s)}\|u(s)\|^2_{H_0^a} ds \\ &\le e^{-\gamma t}\|\phi\|^2_{L^2_{H_0^a}} + \frac{2}{h}e^{-\gamma t}\Big(\|u_0\|^2 + C_\mu\|\phi\|^2_{L^2_{H_0^a}}\Big) + \frac{2C_0}{h\gamma}. \end{aligned} \tag{1.63}$$

Therefore, there exist $C_1 = C_1(h, C_\mu) > 0$ and $C_2 = C_2(C_0, \gamma, h) > 0$ such that

$$\begin{aligned} \|S(t)(u_0,\phi)\|^2_{\mathcal{M}} &= \|u(t)\|^2 + \|u_t\|^2_{L^2_{H_0^a}} \\ &\le C_1 e^{-\gamma t}\Big(\|u_0\|^2 + \|\phi\|^2_{L^2_{H_0^a}}\Big) + C_2, \quad \forall t \ge 0, \end{aligned} \tag{1.64}$$

which complete the proof. □

Then, from Lemma 1.10 we can deduce:

Corollary 1.11. *The semigroup $\{S(t)\}_{t\ge 0}$ possesses an absorbing set $\mathcal{B} = \{w \in \mathcal{M} : \|w\|^2_{\mathcal{M}} \le 2C_2\}$.*

1.3.2 Asymptotic compactness of the semigroup

We start with the following result.

Lemma 1.12. *Assume that the conditions in Theorem 1.6 hold. Let* $\{(u_0^n, \phi^n)\}$ *be a sequence such that* $(u_0^n, \phi^n) \to (u_0, \phi)$ *weakly in* $\mathcal{M}$ *as* $n \to \infty$*, and* $S(t)(u_0^n, \phi^n) = (u^n(t), u_t^n)$ *be the solution of* (1.5)*. Then, for every* $T > 0$*, we have*

$$u^n(\cdot) \to u(\cdot) \quad \text{in} \quad C([r,T], L^2(\Omega)), \quad \forall 0 < r < T, \tag{1.65}$$

and

$$u^n(\cdot) \to u(\cdot) \text{ weakly in } L^2(0,T; H_0^a(\Omega)). \tag{1.66}$$

In addition, there exists $c = c(h, C_\mu) > 0$ *such that*

$$\limsup_{n\to\infty} \|u_t^n - u_t\|^2_{L^2_{H_0^a}} \le ce^{-\gamma t} \limsup_{n\to\infty} \left(\|u_0^n - u_0\|^2 + \|\phi^n - \phi\|^2_{L^2_{H_0^a}}\right), \quad \forall t \ge 0. \tag{1.67}$$

Thus, if $(u_0^n, \phi^n) \to (u_0, \phi)$ *in* $\mathcal{M}$ *as* $n \to \infty$*, then*

$$u^n(\cdot) \to u(\cdot) \quad \text{in} \quad L^2(0,T; H_0^a(\Omega)), \quad \forall T > 0, \tag{1.68}$$

$$u_t^n(\cdot) \to u_t(\cdot) \quad \text{in} \quad L^2_{H_0^a}, \quad \forall t \ge 0. \tag{1.69}$$

Proof. (1) Given $T > 0$. From (1.58) and (1.61), we find that $\{\eta_n^t\}$ is bounded in $L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega)))$, $\{u^n\}$ is bounded in $L^\infty(0,T; L^2(\Omega))$ and $L^2(0,T; H_0^a(\Omega)) \cap L^p(0,T; L^p(\Omega))$. Similar to (1.38) in Theorem 1.6, we have $\{\frac{du^n}{dt}\}$ is bounded in $L^2(0,T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(0,T; L^{\frac{p}{p-1}}(\Omega))$, thus there exist a subsequence (still denoted by itself) $\{u^n\}$, a function $u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^a(\Omega)) \cap L^p(0,T; L^p(\Omega))$ with $\frac{du}{dt} \in L^2(0,T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(0,T; L^{\frac{p}{p-1}}(\Omega))$, $\eta^t \in L^\infty(0,T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega)))$, and $\chi \in L^{\frac{p}{p-1}}(0,T; L^{\frac{p}{p-1}}(\Omega))$ such that

$$\begin{cases} u^n \to u & \text{weak-star in} \quad L^\infty(0,T; L^2(\Omega)), \\ u^n \to u & \text{weakly in} \quad L^2(0,T; H_0^a(\Omega)), \\ u^n \to u & \text{weakly in} \quad L^p(0,T; L^p(\Omega)), \\ \eta_n^t \to \eta^t & \text{weak-star in} \quad L^\infty(0,T; L^2_\mu(\mathbb{R}^+; H_0^a(\Omega))), \\ \dfrac{du^n}{dt} \to \dfrac{du}{dt} & \text{weakly in} \quad L^2(0,T; V^{-1}(\Omega)) + L^{\frac{p}{p-1}}(0,T; L^{\frac{p}{p-1}}(\Omega)), \\ f(u^n) \to \chi & \text{weakly in} \quad L^{\frac{p}{p-1}}(0,T; L^{\frac{p}{p-1}}(\Omega)). \end{cases} \tag{1.70}$$

By (1.2), (1.41) and the Aubin-Lions lemma, we can conclude

$$u^n \to u \quad \text{in} \quad L^2(0,T; L^2(\Omega)). \tag{1.71}$$

Thus, we have for a.a. $(t,x) \in (0,T) \times \Omega$,

$$u^n(t,x) \to u(t,x) \quad \text{and} \quad f(u^n(t,x)) \to f(u(t,x)), \tag{1.72}$$

then $\chi = f(u)$ by [64]. Following the proof of Theorem 1.7, we find that $(u(\cdot), \eta^{\cdot})$ is a solution of equation (1.10) with initial $(u_0, \mathcal{F}\phi)$.

Multiplying $(1.10)_1$ by Qu in $L^2(\Omega)$, and $(1.10)_2$ by η^t in $L^2_\mu(\mathbb{R}^+; D(Q))$, we obtain from (1.56)

$$\begin{aligned}\frac{d}{dt}(\|u\|^2_{H^a_0} + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))}) &\le 2\alpha_3\|u\|^2_{H^a_0} + \frac{2\alpha_1^2}{h}|\Omega| + \frac{2}{h}\|g\|^2 \\ &\le C_3(1 + \|u\|^2_{H^a_0}),\end{aligned} \tag{1.73}$$

where $C_3 = \max\left\{2\alpha_3, \frac{2\alpha_1^2}{h}|\Omega| + \frac{2}{h}\|g\|^2\right\}$. Integrating (1.61) over $(t, t+r)$ for $t \ge 0$ and $0 < r < T - t$, and using (1.62) yield

$$\begin{aligned}&\int_t^{t+r} \|u(s)\|^2_{H^a_0} ds \\ &\le \frac{2}{h}\left[\|u(t)\|^2 + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}\right] + \frac{2}{h}\left(2\beta|\Omega| + \frac{2}{h\lambda_1}\|g\|^2\right) r \\ &\le \frac{2}{h}e^{-\gamma t}\left(\|u_0\|^2 + \|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}\right) + \frac{2c_5}{h\gamma} + \frac{2}{h}\left(2\beta|\Omega| + \frac{2}{h\lambda_1}\|g\|^2\right) r \le C_4(1+r),\end{aligned} \tag{1.74}$$

where $C_4 = \max\left\{\frac{2}{h}(2\beta|\Omega| + \frac{2}{h\lambda_1}\|g\|^2), \frac{2}{h}\left[\|(u_0, \mathcal{F}\phi)\|^2_{\mathcal{H}} + \frac{c_5}{\gamma}\right]\right\}$. Integrating (1.73) over $(s, t+r)$ with $s \in (t, t+r)$, and then using (1.74), we obtain

$$\begin{aligned}&\|u(t+r)\|^2_{H^a_0} + \|\eta^{t+r}\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} \\ &\le \|u(s)\|^2_{H^a_0} + \|\eta^s\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3\int_t^{t+r} \|u(s)\|^2_{H^a_0} ds + C_3 r \\ &\le \|u(s)\|^2_{H^a_0} + \|\eta^s\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3C_4(1+r) + C_3 r.\end{aligned} \tag{1.75}$$

Then integrate (1.75) over $(t, t+r)$ again in s, we have

$$\begin{aligned}&\|u(t+r)\|^2_{H^a_0} + \|\eta^{t+r}\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} \\ &\le \frac{1}{r}\int_t^{t+r} \left(\|u(s)\|^2_{H^a_0} + \|\eta^s\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))}\right) ds + C_3C_4(1+r) + C_3 r.\end{aligned} \tag{1.76}$$

From (1.62) we can deduce that

$$\int_0^t \|u(s)\|^2_{H^a_0} ds \le \frac{2}{h}\left(\|u_0\|^2 + \|\mathcal{F}\phi\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}\right) + \frac{2c_5}{h\gamma}e^{\gamma t} \le C_5(1 + e^{\gamma t}), \tag{1.77}$$

where $C_5 = \max\left\{\frac{2}{h}\|(u_0, \mathcal{F}\phi)\|^2_{\mathcal{H}}, \frac{2c_5}{h\gamma}\right\}$. Integrating (1.73) over $(0, t)$, thanks to (1.77), we have

$$\begin{aligned}\|u(t)\|^2_{H^a_0} + \|\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} &\le \|u(0)\|^2_{H^a_0} + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3\int_0^t (1 + \|u(s)\|^2_{H^a_0}) ds \\ &\le \|u(0)\|^2_{H^a_0} + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3 t + C_3C_5(1 + e^{\gamma t}).\end{aligned} \tag{1.78}$$

Since $t < t + r < T$, we can infer from (1.78) that

$$\begin{aligned}
&\frac{1}{r}\int_t^{t+r}\big(\|u(s)\|^2_{H^a_0} + \|\eta^s\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))}\big)ds \\
&\le \|u(0)\|^2_{H^a_0} + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + \frac{C_3}{r}\int_t^{t+r} s ds + \frac{C_3C_5}{r}\int_t^{t+r}(1+e^{\gamma s})ds \\
&\le \|u(0)\|^2_{H^a_0} + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3T + C_3C_5(1+e^{\gamma T}). \qquad (1.79)
\end{aligned}$$

Then, by substituting (1.79) into (1.76), for all $t \ge 0$, we obtain

$$\begin{aligned}
&\|u(t+r)\|^2_{H^a_0} + \|\eta^{t+r}\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} \qquad (1.80)\\
&\le \|u(0)\|^2_{H^a_0} + \|\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;D(Q))} + C_3T + C_3C_5(1+e^{\gamma T}) + C_3C_4(1+r) + C_3 r.
\end{aligned}$$

Thus, $\|u(t)\|_{H^a_0}$ is uniformly bounded in $[r, T]$. By a standard argument in [15, Lemma 17], we find that $u^n(t_n) \to u(t_0)$ in H^a_0 for any sequence $t_n \to t_0$ as $n \to \infty$, $t_n, t_0 \in [r, T]$ with $r \in [0, T]$. Using the compact embedding $H^a_0(\Omega) \subset L^2(\Omega)$, we can derive (1.65).

(2) Let $\hat{z}^n = (\hat{u}^n, \hat{\eta}^t_n) = (u^n - u, \eta^t_n - \eta^t)$ with initial data $\hat{z}^n(0) = (u^n(0) - u_0, \mathcal{F}\phi^n - \mathcal{F}\phi)$, similar to (1.48), we have

$$\begin{aligned}
\frac{d}{dt}\|\hat{z}^n\|^2_{\mathcal{H}} &\le -2m(l(u^n))\|\hat{u}^n\|^2_{H^a_0} + 2L_m|l|\|\hat{u}^n\|\big\|\sqrt{a}\nabla u\big\|\big\|\sqrt{a}\nabla\hat{u}^n\big\| \\
&\quad - 2\int_\Omega (f(u^n) - f(u))\hat{u}^n dx - 2(\partial_s\hat{\eta}^t_n, \hat{\eta}^t_n)_{1,\mu} \\
&\le -\frac{3}{2}h\|\hat{u}^n\|^2_{H^a_0} + \frac{2L_m^2}{h}|l|^2\|\hat{u}^n\|^2\|u\|^2_{H^a_0} \\
&\quad - 2\int_\Omega (f(u^n) - f(u))\hat{u}^n dx - \delta\|\hat{\eta}^t_n\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))}, \qquad (1.81)
\end{aligned}$$

where the last term on the right hand can be obtained from (1.60). By the fact that $0 < \gamma < \min\{h\lambda_1, \delta\}$ and (1.81), we have

$$\begin{aligned}
\frac{d}{dt}\|\hat{z}^n\|^2_{\mathcal{H}} + \gamma\|\hat{z}^n\|^2_{\mathcal{H}} + \frac{h}{2}\|\hat{u}^n\|^2_{H^a_0} &\le \frac{d}{dt}\|\hat{z}^n\|^2_{\mathcal{H}} + \frac{3}{2}h\|\hat{u}^n\|^2_{H^a_0} + \delta\|\hat{\eta}^t_n\|^2_{L^2_\mu(\mathbb{R}^+;H^a_0(\Omega))} \qquad (1.82)\\
&\le \frac{2L_m^2}{h}|l|^2\|\hat{u}^n\|^2\|u\|^2_{H^a_0} - 2\int_\Omega (f(u^n) - f(u))\hat{u}^n dx.
\end{aligned}$$

Multiplying (1.82) by $e^{\gamma t}$, and integrating it over $(0, t)$, we obtain

$$\begin{aligned}
&\|\hat{z}^n(t)\|^2_{\mathcal{H}} + \frac{h}{2}\int_0^t e^{-\gamma(t-s)}\|\hat{u}^n\|^2_{H^a_0}ds \\
&\le e^{-\gamma t}\|\hat{z}^n(0)\|^2_{\mathcal{H}} + \frac{2L_m^2}{h}|l|^2\int_0^t e^{-\gamma(t-s)}\|\hat{u}^n\|^2\|u\|^2_{H^a_0}ds \\
&- 2\int_0^t e^{-\gamma(t-s)}\int_\Omega (f(u^n) - f(u))\hat{u}^n dx ds. \qquad (1.83)
\end{aligned}$$

From (1.65), we deduce that $\|\hat{u}^n(s)\|^2\|u(s)\|^2_{H_0^a} = \|u^n(s) - u(s)\|^2\|u(s)\|^2_{H_0^a} \to 0$ for a.e. $s \in (0,t)$, and $e^{-\gamma(t-s)}\|\hat{u}^n\|^2\|u\|^2_{H_0^a} \leq 2e^{-\gamma(t-s)}(\|u^n(s)\|^2 + \|u(s)\|^2)\|u(s)\|^2_{H_0^a}$ is bounded. Then, by the Lebesgue dominated convergence theorem, it follows that

$$\int_0^t e^{-\gamma(t-s)}\|\hat{u}^n(s)\|^2\|u(s)\|^2_{H_0^a}ds \to 0 \quad \text{as} \quad n \to \infty. \tag{1.84}$$

In addition, by (1.13) and (1.72), then using the Fatou-Lebesgue theorem,

$$\begin{aligned}\limsup_{n\to\infty}\left(-\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u^n)u^n dxds\right) &\leq -\liminf_{n\to\infty}\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u^n)u^n dxds\\ &\leq -\int_0^t e^{-\gamma(t-s)}\int_\Omega \liminf_{n\to\infty} f(u^n)u^n dxds\\ &= -\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u)udxds. \end{aligned}\tag{1.85}$$

Since $u^n \rightharpoonup u$ in $L^p(0,T;L^p(\Omega))$ (see $(1.70)_3$) and $f(u(\cdot)) \in L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(\Omega))$, we have

$$\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u)(u^n - u)dxds \to 0, \tag{1.86}$$

as $n \to \infty$. Since $f(u^n) \to f(u)$ weakly in $L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(\Omega))$ (see $(1.70)_6$), we deduce

$$\int_0^t e^{-\gamma(t-s)}\int_\Omega (f(u^n) - f(u))udxds \to 0, \tag{1.87}$$

as $n \to \infty$. Combining now (1.85)-(1.87),

$$-\int_0^t e^{-\gamma(t-s)}\int_\Omega (f(u^n) - f(u))\hat{u}^n dxds \to 0 \quad \text{as} \quad n \to \infty. \tag{1.88}$$

Therefore, by (1.84), (1.88), and (1.16) in Lemma 1.4, we infer from (1.83) that

$$\begin{aligned}\limsup_{n\to\infty}\int_0^t e^{-\gamma(t-s)}\|\hat{u}^n(s)\|^2_{H_0^a}ds &\leq \frac{2}{h}e^{-\gamma t}\limsup_{n\to\infty}\|\hat{z}^n(0)\|^2_{\mathcal{H}}\\ &= \frac{2}{h}e^{-\gamma t}\limsup_{n\to\infty}(\|u^n(0) - u_0\|^2 + \|\mathcal{F}\phi^n - \mathcal{F}\phi\|_{L^2_\mu(\mathbb{R}^+;H_0^a(\Omega))})\\ &\leq \frac{2}{h}e^{-\gamma t}\limsup_{n\to\infty}\left(\|u^n(0) - u_0\|^2 + C_\mu\|\phi^n - \phi\|_{L^2_{H_0^a}}\right). \end{aligned}\tag{1.89}$$

Eventually, we have

$$\begin{aligned}\limsup_{n\to\infty}\|u^n_t - u_t\|^2_{L^2_{H_0^a}} &= \limsup_{n\to\infty}\int_{-\infty}^{-t} e^{\gamma s}\|u^n(t+s) - u(t+s)\|^2_{H_0^a}ds\\ &+ \limsup_{n\to\infty}\int_{-t}^{0} e^{\gamma s}\|u^n(t+s) - u(t+s)\|^2_{H_0^a}ds\end{aligned}$$

$$= \limsup_{n\to\infty} e^{-\gamma t} \int_{-\infty}^{0} e^{\gamma s} \|\phi^n(s) - \phi(s)\|_{H_0^a}^2 ds$$
$$+ \limsup_{n\to\infty} \int_0^t e^{-\gamma(t-s)} \|u^n(s) - u(s)\|_{H_0^a}^2 ds, \tag{1.90}$$

which together with (1.89) concludes (1.67). Furthermore, if $(u_0^n, \phi^n) \to (u_0, \phi)$ in $\mathcal{M}$, obviously (1.68) and (1.69) can be obtained from (1.67). □

As a consequence of Lemma 1.12, the following corollary about the continuity of $\{S(t)\}_{t\geq 0}$ on $\mathcal{M}$ immediately follows:

Corollary 1.13. *Assume that the conditions in Theorem 1.6 hold, the semigroup $\{S(t)\}_{t\geq 0}$ is continuous from $\mathcal{M}$ to $\mathcal{M}$.*

Then, we will prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geq 0}$.

Lemma 1.14. *Suppose the assumptions in Theorem 1.6 hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in $\mathcal{M}$.*

Proof. For any bounded set B of $\mathcal{M}$ and $t > 0$, let $\{(u^n(\cdot), u_t^n(\cdot))\}$ be a sequence of solutions of the equation (1.5) with initial values $\{(y_n, \phi_n)\}_{n\in\mathbb{N}} \subset B$. Consider the sequence

$$S(t_n)(y_n, \phi_n) = (u(t_n; 0, (y_n, \mathcal{F}\phi_n)), u_{t_n}(\cdot; 0, (y_n, \mathcal{F}\phi_n))) := (u^n(t_n), u_{t_n}^n(\cdot)), \tag{1.91}$$

where $t_n \to \infty$ as $n \to +\infty$. We need to show that the sequence is relatively compact in $\mathcal{M}$.

Given a positive constant T. Let $v^n(\cdot) = u^n(\cdot + t_n - T)$, then

$$v^n(T) = u^n(t_n) \quad \text{and} \quad v_T^n(t) = v^n(T + t) = u^n(t + t_n) = u_{t_n}^n(t).$$

Thus, we denote

$$Y_n = (v^n(T), v_T^n(\cdot)) = (u^n(t_n), u_{t_n}^n(\cdot)) = S(t_n)(y_n, \phi_n),$$

and

$$\xi_n^T = (v^n(0), v_0^n(\cdot)) = (u^n(t_n - T), u_{t_n-T}^n(\cdot)) = S(t_n - T)(y_n, \phi_n).$$

Notice that the sequence (y_n, ϕ_n) is bounded. Since $t_n \to \infty$, there exists n_0 such that if $n \geq n_0$, then $t_n \geq T$. Thus, from Lemma 1.10, we deduce that for all $n \geq n_0$,

$$\|Y_n\|_{\mathcal{M}}^2 = \|S(t_n)(y_n, \phi_n)\|_{\mathcal{M}}^2 \leq 2C_2, \tag{1.92}$$

which means that there exists $Y := (\varpi, o)$ such that passing to a subsequence,

$$Y_n \to Y \quad \text{weakly in} \quad \mathcal{M}. \tag{1.93}$$

Similarly, it follows from Lemma 1.10 that, for all $n \geq n_0$,

$$\|\xi_n^T\|_{\mathcal{M}}^2 = \|S(t_n - T)(y_n, \phi_n)\|_{\mathcal{M}}^2 \leq 2C_2, \tag{1.94}$$

which implies that there exists $\xi^T := (y, \phi)$ such that passing to a subsequence,

$$\xi_n^T \to \xi^T \quad \text{weakly in} \quad \mathcal{M}. \tag{1.95}$$

To prove that the weak convergence in (1.93) is a strong convergence, we only need to prove that for every $\varepsilon > 0$, there exists $n(\varepsilon) > 0$ such that for all $n \geq n(\varepsilon)$,

$$\|Y_n - Y\|_{\mathcal{M}} \leq \varepsilon. \tag{1.96}$$

It follows from the convergence in Lemma 1.12 that $\varpi = v_T(0)$ in $L^2(\Omega)$, $o = v_T$ in $L^2_{H_0^a}$ and $o(s) = v_T(s)$ for a.a. $s \leq 0$. By (1.65), we can deduce that

$$u^n(t_n) = v^n(T) = S(t_n)y_n = S(T)S(t_n - T)y_n \to S(T)y = v(T) = \varpi. \tag{1.97}$$

In what follows, if we can confirm that

$$u^n_{t_n}(\cdot) \to o \quad \text{in} \quad L^2_{H_0^a}, \tag{1.98}$$

then (1.96) holds. By (1.58) and (1.67), we have

$$\begin{aligned} \limsup_{n\to\infty} \|u^n_{t_n}(\cdot) - o\|^2_{L^2_{H_0^a}} &= \limsup_{n\to\infty} \|v^n_T - v_T\|^2_{L^2_{H_0^a}} \\ &\leq e^{-\gamma T} \limsup_{n\to\infty} \left(\|\xi_n^T - \xi^T\|^2_{\mathcal{M}}\right) \leq 4Ce^{-\gamma T}. \end{aligned} \tag{1.99}$$

Finally, by the diagonal method, we can conclude that (1.98) holds for $T \to +\infty$.□

Therefore, we obtain the below existence result of global attractors by using Corollary 1.11, Lemma 1.14, and Corollary 1.13 in the sense of Theorem 1.8.

Theorem 1.15. *Assume that the conditions in Theorem 1.6 hold. Then, the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor in $\mathcal{M}$.*

Part II

Pullback random attractors for stochastic systems with nonlinear colored noise with or without delay

Chapter 2

Single-valued random p-Laplace equations without delay

In this chapter, we introduce the concept of a pullback random bi-spatial attractor for a nonautonomous random dynamical system. We first prove an existence theorem of a pullback random bi-spatial attractor, which includes its measurability, compactness, and attraction in a regular space. Then, the residual dense continuity of a family of pullback random bi-spatial attractors is established from a parameter space into the space of all compact subsets of the regular space equipped with the Hausdorff metric. An application of the abstract results in nonautonomous random quasilinear equations driven by nonlinear colored noise, where the size of noise belongs to $(0, \infty]$ and the infinity size corresponds to the deterministic equation, leads to proving the existence and residual dense continuity of pullback random bi-spatial attractors on $(0, \infty]$ in both square and p-order Lebesgue spaces, where $p > 2$.

In the next section, we will give some definitions and lemmas that will be used. In Sect. 2.2, we show an existence results of pullback random bi-spatial attractors. In Sect. 2.3, we prove the residual dense continuity for pullback random bi-spatial attractors. In Sect. 2.4 and Sect. 2.5, we consider the random p-Laplace equation (4) and use the results of Section 2.2 to prove the existence of pullback random bi-spatial attractors for random equation (4). In Sect. 2.6, we verify the residual dense continuity of the pullback random bi-spatial attractor for random equation (4) by using the results of Section 2.3.

2.1 Some definitions and lemmas

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a group $\{\theta_t : \Omega \to \Omega; t \in \mathbb{R}\}$ of measure-preserving transformations. A measurable mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a **cocycle** on X (see [86]) if, for $t, s \geq 0, \tau \in \mathbb{R}, \omega \in \Omega$,

$$\Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_s\omega)\Phi(s, \tau, \omega),\ \ \Phi(0, \tau, \omega) = I_X.$$

Definition 2.1. *A cocycle Φ on X is called* **regular** *in (X, Y) if*

$$\Phi(t, \tau, \omega)X \subset Y \ \textit{(thus} \subset X \cap Y\textit{)}\ ,\ \forall\ t > 0, \tau \in \mathbb{R}, \omega \in \Omega.$$

A set-valued map $\mathcal{D} : \mathbb{R} \times \Omega \mapsto 2^X \backslash \{\emptyset\}$ is called *a bi-parametric set* in X, while a bi-parametric set $\mathcal{D}$ in X is called **random** in X if $\omega \to d_X(x, F(\omega))$ is $\mathbb{F}$-measurable for $x \in X$, and **compact (closed, bounded)** if $\mathcal{D}(\tau, \omega)$ is compact (closed, bounded) for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Throughout this chapter, $\mathfrak{D}$ is an inclusion-closed universe of some bi-parametric sets in X. (X, Y) is a limit-identical pair of separable Banach spaces whose limit identity property satisfies (5).

Definition 2.2. *Let Φ be a regular cocycle in (X, Y). A bi-parametric set $\mathcal{A}$ is called a* **$\mathfrak{D}$-pullback random** (X, Y)**-attractor** *for Φ if*

(i) $\mathcal{A} \in \mathfrak{D}$;

(ii) $\mathcal{A}$ is compact in $X \cap Y$;

(iii) $\mathcal{A}$ is random in X and in Y respectively;

(iv) $\mathcal{A}$ is invariant under Φ, i.e., for $t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega$,

$$\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{A}(\tau - t, \theta_{-t}\omega) = \mathcal{A}(\tau, \omega); \tag{2.1}$$

(v) $\mathcal{A}$ is $\mathfrak{D}$-pullback attracting in $X \cap Y$, that is, for each $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t \to +\infty} d_{X \cap Y}(\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{D}(\tau - t, \theta_{-t}\omega),\ \mathcal{A}(\tau, \omega)) = 0.$$

Remark 2.3. *(a) The regularity of Φ and invariance in* (2.1) *imply that $\mathcal{A}(\tau, \omega) \subset Y$.*

(b) If $X = Y$, a pullback random bi-spatial attractor reduces to a usual pullback random attractor in [86], while the concept of a pullback random attractor expands the early concept of a random attractor in [26, 27].

(c) Even if $X \neq Y$, a pullback random bi-spatial attractor in (X, Y) must be a pullback random attractor in X.

Definition 2.4. *[86] A bi-parametric set $\mathcal{K}$ in X is called $\mathfrak{D}$-pullback absorbing for a cocycle Φ in X if, for each $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is $T > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{D}(\tau - t, \theta_{-t}\omega) \subset \mathcal{K}(\tau, \omega), \quad \forall\, t \geq T.$$

Definition 2.5. *A regular cocycle Φ in (X, Y) is called $\mathfrak{D}$-pullback asymptotically compact in X (resp. Y, $X \cap Y$) if, for $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t_n \to +\infty$, the sequence*

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n\} \text{ with } x_n \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$$

has a convergent subsequence in X (resp. Y, $X \cap Y$).

We recall two Barie category theorems. A set Q in a topological space Λ is called **nowhere dense** if its closure $\overline{Q}$ does not contain any nonempty open set, and **residual** if the complement Q^c is a union of countably many nowhere dense sets.

Lemma 2.6 (Abstract Baire residual Theorem)**.** *Suppose that Λ is a topological space, Z is a metric space and $F_n : \Lambda \to Z$ is a sequence of continuous mappings such that*

$$\lim_{n\to\infty} F_n(\lambda) = F(\lambda) \text{ in } Z,\ \forall\ \lambda \in \Lambda.$$

Then the **continuity-set** *(i.e. the set of points of continuity) of the mapping $\lambda \to F(\lambda)$ is residual in Λ.*

The above theorem was proved in [59, Prop. 1], while a special case when Λ is a complete metric space was proved in [47, Theorem 5.1].

Recall that Λ is called a **Hausdorff space** if for two points $\lambda_1 \neq \lambda_2$ there are open sets O_1, O_2 such that $\lambda_1 \in O_1$, $\lambda_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Lemma 2.7 (Baire density Theorem)**.** *If Λ is a complete metric space or a compact Hausdorff space, then any residual set in Λ is dense.*

The above lemma can be found in [47] when Λ is a complete metric space and in [59, Prop. 2] when Λ is a compact Hausdorff space.

2.2 Abstract results on the existence of a pullback random bi-spatial attractor

For any bi-parametric set $\mathcal{B}$ in X, the omega-limit set in X is defined by

$$\Upsilon_X(\mathcal{B})(\tau,\omega) := \bigcap_{T>0} \overline{\bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{B}(\tau-t,\theta_{-t}\omega)}$$

for $\tau \in \mathbb{R}$ and $\omega \in \Omega$, where the overline denotes the closure in X. Since Φ is regular in (X,Y), the following set

$$\Upsilon_Y(\mathcal{B})(\tau,\omega) := \bigcap_{T>0} \mathrm{cl}_Y \bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{B}(\tau-t,\theta_{-t}\omega)$$

is well-defined in Y (even if $\mathcal{B}$ does not lie in Y), where cl_Y denotes the closure in Y, and thus $\Upsilon_Y(\mathcal{B})$ is a closed bi-parametric set in Y. It is similar to define the omega-limit set $\Upsilon_{X\cap Y}(\mathcal{B})$.

We now state and prove an abstract result on the unique existence of a pullback random bi-spatial attractor.

Theorem 2.8. *Let Φ be a regular cocycle in (X,Y) and $\mathfrak{D}$ a universe in X.*

(i)If we assume that for $t>0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

(B1) *$\Phi(t,\tau,\omega):\ X \to X$ is continuous;*

(B2) *there is a $\mathfrak{D}$-pullback absorbing set $\mathcal{K} \in \mathfrak{D}$ and $\mathcal{K}$ is random closed in X;*

(B3) *Φ is $\mathfrak{D}$-pullback asymptotically compact in both X and Y, respectively, then Φ has a unique $\mathfrak{D}$-pullback (X,Y)-attractor given by $\mathcal{A} = \Upsilon_X(\mathcal{K})$ which is random in X.*

(ii) If we further assume

(B4) $\Phi(t,\tau,\omega): X\cap Y\to Y$ *is strong-weakly continuous;*

(B5) *there is another* $\mathfrak{D}$*-pullback absorbing set* $\mathcal{M}\in\mathfrak{D}$ *and* $\mathcal{M}$ *is random closed in* $X\cap Y$*;*

(B6) $\omega\to\Phi(t,\tau,\omega)y$ *is* $(\mathbb{F},\mathbb{B}(Y))$*-measurable for each* $y\in X\cap Y$*;*

(B7) $(X\cap Y,\|\cdot\|_X+\|\cdot\|_Y)$ *is separable,*

then $\mathcal{A}$ *is a random set in* Y *and thus it is a* $\mathfrak{D}$*-pullback* **random** (X,Y)*-attractor.*

Proof. (i) By **(B1)**,**(B2)** and the X-part of **(B3)**, the existence theorem in [86] shows that Φ has a unique $\mathfrak{D}$-pullback random attractor in X, given by $\Upsilon_X(\mathcal{K})=:\mathcal{A}\in\mathfrak{D}$. In particular, $\mathcal{A}\in\mathfrak{D}$ and $\mathcal{A}$ is random, invariant, compact, $\mathfrak{D}$-pullback attracting in X.

As the autonomous case in [57], using **(B3)** and the limit-identical property of (X,Y), one can prove that $\Upsilon_X(\mathcal{K})=\Upsilon_Y(\mathcal{K})$, which is compact and $\mathfrak{D}$-pullback attracting in Y.

(ii) The proof of the measurability of $\mathcal{A}$ in Y is similar (but slightly different) to the autonomous case in [30, theorem 20]. We provide here a sketch of the proof.

For the absorbing set $\mathcal{M}$ in **(B5)**, by the similar methods as in [30, theorem 20], one can prove that

$$\mathcal{A}=\Upsilon_X(\mathcal{M})=\Upsilon_{X\cap Y}(\mathcal{M})=\Upsilon_Y(\mathcal{M})=\Upsilon_Y^{\text{dis}}(\mathcal{M}), \tag{2.2}$$

where the last term is the discrete omega-limit set defined by

$$\Upsilon_Y^{\text{dis}}(\mathcal{M})(\tau,\omega):=\bigcap_{m=1}^{\infty}\text{cl}_Y\bigcup_{n=m}^{\infty}\Phi(n,\tau-n,\theta_{-n}\omega)\mathcal{M}(\tau-n,\theta_{-n}\omega).$$

Next, we need to consider the following weak sequential closure (rather than the weak closure in [30, theorem 20]):

$$\text{cl}_{Y_{\text{ws}}}E=\{y\in Y;\ \exists\{y_n\}_{n=1}^{\infty}\subset E \text{ such that } y_n\rightharpoonup y \text{ in } Y\},$$

for $E\subset Y$. We claim that for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\Upsilon_Y^{\text{dis}}(\mathcal{M})(\tau,\omega):=\bigcap_{m=1}^{\infty}\text{cl}_{Y_{\text{ws}}}\bigcup_{n=m}^{\infty}\Phi(n,\tau-n,\theta_{-n}\omega)\mathcal{M}(\tau-n,\theta_{-n}\omega). \tag{2.3}$$

The $\subset$ relationship in (2.3) is obvious. We need to prove the $\supset$ relationship. Suppose y belongs to the right-hand side of (2.3). Then, for each $m\in\mathbb{N}$, there are $\{m_i\}_{i\in\mathbb{N}}$ with $m_i\geq m$ for all $i\in\mathbb{N}$)(where $\{m_i\}$ may not be a subsequence of $\{m\}$) and $x_{m_i}\in\mathcal{M}(\tau-m_i,\theta_{-m_i}\omega)$ such that

$$\Phi(m_i,\tau-m_i,\theta_{-m_i}\omega)x_{m_i}\rightharpoonup y \ \text{ in } Y \text{ as } i\to\infty.$$

Hence the diagonal sequence $\{m_m\}_{m\in\mathbb{N}}$ is a subsequence of $\{m\}$ (because $m_m\geq m\to\infty$) such that

$$\Phi(m_m,\tau-m_m,\theta_{-m_m}\omega)x_{m_m}\rightharpoonup y \ \text{ in } Y \text{ as } m\to\infty.$$

Since $m_m \to \infty$, $\mathcal{M} \in \mathfrak{D}$ and Φ is $\mathfrak{D}$-pullback asymptotically compact in Y (see **(B3)**), it follows that $\{m_m\}$ has a subsequence (not relabeled) such that

$$\Phi(m_m, \tau - m_m, \theta_{-m_m}\omega)x_{m_m} \to z \quad \text{in } Y, \quad \text{as } m \to \infty,$$

and thus $z \in \Upsilon_Y^{\text{dis}}(\mathcal{M})(\tau, \omega)$. By the uniqueness of the limits, we have $y = z \in \Upsilon_Y^{\text{dis}}(\mathcal{M})(\tau, \omega)$. Therefore (2.3) holds true.

By (2.2) and (2.3), it is direct to deduce that

$$\mathcal{A}(\tau, \omega) = \bigcap_{m=1}^{\infty} \text{cl}_{Y_{\text{ws}}} \bigcup_{n=m}^{\infty} \text{cl}_{Y_{\text{ws}}} \Phi(n, \tau - n, \theta_{-n}\omega)\mathcal{M}(\tau - n, \theta_{-n}\omega). \tag{2.4}$$

By **(B5)**, the mapping $\omega \to \mathcal{M}(\tau - n, \omega)$ defines a random closed set in $X \cap Y$, while, by **(B7)**, $X \cap Y$ is a separable Banach space. Hence, [25, Lemma II. 4.1] implies that there is a sequence $\{z_{n_j}\}_{j=1}^{\infty}$ of random elements in $X \cap Y$ such that

$$\mathcal{M}(\tau - n, \omega) = \text{cl}_{X \cap Y}\{z_{n_j}(\omega); j \in \mathbb{N}\}, \ \forall \omega \in \Omega.$$

From which, we deduce from **(B4)** that

$$\begin{aligned} &\text{cl}_{Y_{\text{ws}}} \Phi(n, \tau - n, \theta_{-n}\omega)\mathcal{M}(\tau - n, \theta_{-n}\omega) \\ &= \text{cl}_{Y_{\text{ws}}} \Phi(n, \tau - n, \theta_{-n}\omega)\{z_{n_j}(\theta_{-n}\omega); j \in \mathbb{N}\}. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we obtain that

$$\mathcal{A}(\tau, \omega) = \bigcap_{m=1}^{\infty} \text{cl}_{Y_{\text{ws}}} \bigcup_{n=m}^{\infty} \text{cl}_{Y_{\text{ws}}} \Phi(n, \tau - n, \theta_{-n}\omega)\{z_{n_j}(\theta_{-n}\omega); j \in \mathbb{N}\}.$$

Hence, the similar method as in (2.3) implies that

$$\mathcal{A}(\tau, \omega) = \bigcap_{m=1}^{\infty} \text{cl}_{Y} \bigcup_{n=m}^{\infty} \Phi(n, \tau - n, \theta_{-n}\omega)\{z_{n_j}(\theta_{-n}\omega); j \in \mathbb{N}\}. \tag{2.6}$$

By the same method as in [30, theorem 20], using **(B3)**, **(B4)** and **(B6)**, one can prove that the last set in (2.6) is random in Y, and thus $\mathcal{A}$ is random in Y as desired. □

Remark 2.9. *In Theorem 2.8 (ii), conditions* **(B4)**, **(B5)** *are different from those in [30], while we have to assume* **(B7)**, *which appears to be missed in [30, Theorem 20]. In general, the separability of X and Y cannot imply the separability of $X \cap Y$. But we will prove the separability of $L^2(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ in Lemma 2.15 later.*

2.3 Abstract results for residual dense continuity of pullback random bi-spatial attractors

Recall that $\mathcal{C}(X)$ (resp. $CB(X)$, $B(X)$) denotes the collection of all nonempty compact (resp. closed bounded, bounded) sets in X. It is known that Hausdorff metric ρ_X is a complete metric on $\mathcal{C}(X)$ (see [60, Prop. 2.1]) and a metric on $CB(X)$ (see [47, 48]).

Lemma 2.10. *(i)* $\mathcal{C}(X\cap Y)=\mathcal{C}(X)\cap\mathcal{C}(Y)$ *and* $B(X\cap Y)=B(X)\cap B(Y)$.

(ii) $CB(X\cap Y)\supset B(X)\cap CB(Y)$ *and thus* $CB(X\cap Y)\supset CB(X)\cap CB(Y)$.

(iii) A set-valued mapping $\lambda\mapsto F(\lambda)$ *is continuous at* $\lambda_0\in\Lambda$ *into* $(\mathcal{C}(X\cap Y),\rho_{X\cap Y})$ *iff so it is into* $(\mathcal{C}(X),\rho_X)$ *and* $(\mathcal{C}(Y),\rho_Y)$ *respectively.*

(iv) If A *is a pre-compact set in* $X\cap Y$*, then the closures of* A *in* X*,* Y *and* $X\cap Y$ *are the same, and thus this same closure is compact in* X*,* Y *and* $X\cap Y$ *respectively.*

Proof. (i) By [57, Lemma 2.1], if (X,Y) is limit-identical, then A is compact in $X\cap Y$ iff A is compact in X and in Y respectively. Hence the first equality in (i) holds true, while the second equality follows from $\|\cdot\|_{X\cap Y}=\|\cdot\|_X+\|\cdot\|_Y$.

(ii) Suppose that $D\in B(X)\cap CB(Y)$. By (i), D is bounded in $X\cap Y$. Suppose $x_n\in D$ and $x\in X\cap Y$ such that $\|x_n-x\|_{X\cap Y}\to 0$. Then $\|x_n-x\|_{X\cap Y}\to 0$. Since D is closed in Y, we have $x\in D$, and thus D is closed in $X\cap Y$.

(iii) The assertion (iii) follows from (i) immediately.

(iv) Let A be a pre-compact set in $X\cap Y$ and $y\in \mathrm{cl}_Y A$. Then there is a sequence $y_n\in A$ such that $\|y_n-y\|_Y\to 0$. Since A is pre-compact in $X\cap Y$, passing to a subsequence, we have $\|y_n-z\|_{X\cap Y}\to 0$ for some $z\in X\cap Y$, which implies that $y=z\in \mathrm{cl}_{X\cap Y}A$ and thus $\mathrm{cl}_Y A\subset \mathrm{cl}_{X\cap Y}A$. Obviously, $\mathrm{cl}_{X\cap Y}A\subset \mathrm{cl}_Y A$, and thus they are equal to each other. It is similar to prove that $\overline{A}=\mathrm{cl}_{X\cap Y}A$. □

We need the following result on the strong-weakly continuity of an operator.

Lemma 2.11. *Let* $\Phi:X\to X\cap Y$ *satisfy that* $\Phi:X\to X$ *is continuous.*

(a) if $\Phi:X\cap Y\to Y$ *is strong-weakly continuous, then*

$$\Phi(\mathcal{C}(X\cap Y))\subset CB(Y).$$

(b) On the contrary, if $\Phi(\mathcal{C}(X\cap Y))\subset B(Y)$ *and the embedding* $X^*\hookrightarrow(X\cap Y)^*$ *is dense, then* $\Phi:X\cap Y\to Y$ *is strong-weakly continuous.*

Proof. (a) Given $A\in\mathcal{C}(X\cap Y)$, by the strong-weak continuity of $\Phi:X\cap Y\to Y$, we deduce that $\Phi(A)$ is weakly compact in Y. In particular, $\Phi(A)$ is weakly closed and thus (strongly) closed in Y (because the collection of all weakly closed sets is contained into the collection of all strongly closed sets).

We then prove that $\Phi(A)$ is bounded in Y. Indeed, for each $y^*\in Y$, the mapping $y\to\langle y,y^*\rangle$ is $(Y_{\mathrm{w}},\mathbb{R})$-continuous, where Y_{w} denotes the weak topology. Since $\Phi(A)$ is weakly compact in Y, it follows that the set $\{\langle y,y^*\rangle:\ y\in\Phi(A)\}$ is compact and thus bounded in $\mathbb{R}$ for each $y^*\in Y^*$.

On the other hand, each $y\in\Phi(A)$ can be regarded as a continuous linear functional on Y^*, that is, $y^{**}:Y^*\to\mathbb{R}$ with $y^{**}(y^*)=\langle y,y^*\rangle$ for all $y^*\in Y^*$. By the above proof, the family $\{y^{**}\in Y^{**}:y\in\Phi(A)\}$ is pointwise bounded (acting on each $y^*\in Y^*$). By the resonance theorem, the set $\{y^{**}\in Y^{**}:y\in\Phi(A)\}$ is bounded in Y^{**}. Since $\|y^{**}\|=\|y\|$ for all $y\in Y$, it follows that $\Phi(A)$ is bounded in Y.

(b) Let $x_n\to x_0$ in $X\cap Y$. By the continuity of $\Phi:X\to X$, we have $\Phi x_n\to\Phi x_0$ in X. In particular, $\{\Phi x_n\}$ is bounded in X and

$$\langle\Phi x_n,x^*\rangle\to\langle\Phi x_0,x^*\rangle,\ \forall\, x^*\in X^*. \tag{2.7}$$

Since $\{x_n;\ n \in \mathbb{N}_0\} \cup \{x_0\} \in \mathcal{C}(X \cap Y)$, it follows from the assumption that $\{\Phi x_n\}$ is bounded in Y, and thus $\{\Phi x_n\}$ is bounded in $X \cap Y$.

By the assumption, we know that X^* is dense in $(X \cap Y)^*$. Then the boundedness of $\{\Phi x_n\}$ in $X \cap Y$, together with (2.7), implies that

$$\langle \Phi x_n, y^* \rangle \to \langle \Phi x_0, y^* \rangle,\ \forall\ y^* \in (X \cap Y)^*.$$

Note that $Y^* \hookrightarrow (X \cap Y)^*$, and thus $\Phi x_n \rightharpoonup \Phi x_0$ in Y. $\square$

Let Λ be a topological space (termed as **a parameter space**). For each $\lambda \in \Lambda$, we assign a regular cocycle Φ_λ in (X, Y) and assume that each Φ_λ possesses a $\mathfrak{D}$-pullback random (X, Y)-attractor denoted by $\mathcal{A}_\lambda$. We need some conditions which hold for all $\lambda \in \Lambda$, $t > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

C1. *$\Phi_\lambda(t, \tau, \omega) : X \to X$ is continuous.*

C2. *$\Phi_\lambda(t, \tau, \omega) : X \cap Y \to Y$ is strong-weakly continuous.*

C3. *For each $K \in \mathcal{C}(X \cap Y)$ and $\lambda_0 \in \Lambda$,*

$$\lim_{\lambda \to \lambda_0} \sup_{x \in K} \|\Phi_\lambda(t, \tau, \omega)x - \Phi_{\lambda_0}(t, \tau, \omega)x\|_{X \cap Y} = 0.$$

C4. *$\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$ and each Λ_k is closed in Λ.*

C5. *The union $\mathcal{A}_{\Lambda_k}(s, \omega)$ is pre-compact in $X \cap Y$ for each $s < 0$, where*

$$\mathcal{A}_{\Lambda_k}(\tau, \omega) := \bigcup_{\lambda \in \Lambda_k} \mathcal{A}_\lambda(\tau, \omega),\ \forall\ \tau \in \mathbb{R}, k \in \mathbb{N}, \omega \in \Omega.$$

C6. *For each $k \in \mathbb{N}$, there is $\mathcal{B}_k \in \mathfrak{D}$ such that*

$$\overline{\mathcal{A}_{\Lambda_k}(s, \omega)} \subset \mathcal{B}_k(s, \omega),\ \forall\ s < 0.$$

In order to study the density continuity, we strengthen **C4** by

C4*. *$\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$, each Λ_k is a compact Hausdorff space or a complete metric space, and Λ_k^{in} is residual in Λ_k, where Λ_k^{in} denotes the interior of Λ_k in Λ.*

Theorem 2.12. *Suppose that a family $\{\mathcal{A}_\lambda; \lambda \in \Lambda\}$ of $\mathfrak{D}$-pullback random (X, Y)-attractors satisfies **C1-C6**. Then, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the continuity-set (denoted by $\Lambda^*(\tau, \omega)$) of the mapping*

$$\lambda \mapsto \mathcal{A}_\lambda(\tau, \omega), \quad \Lambda \to (\mathcal{C}(X \cap Y), \rho_{X \cap Y}) \tag{2.8}$$

*is a residual set in Λ. Moreover, $\Lambda^*(\tau, \omega)$ is dense in Λ if **C4*** replaces **C4**.*

Proof. Step 1. *We prove the continuity-set (denoted by $\Lambda_X^*(\tau, \omega)$) of the mapping*

$$\lambda \mapsto \mathcal{A}_\lambda(\tau, \omega), \quad \Lambda \to (\mathcal{C}(X), \rho_X)$$

is residual in Λ. Since $\mathcal{A}_\lambda$ is an attractor in X, by using **C1**, **C4**, **C6** and the X-part of **C3**, **C5**, the assertion follows from [60, Theorem 4], but we use the Baire residual theorem (i.e. Lemma 2.6) when Λ is a topology space, which generalizes the case of a complete metric space used in [47, 48, 60].

Step 2. *We prove under* **C2-C6** *that the continuity-set (denoted by $\Lambda_Y^*(\tau,\omega)$) of the mapping*

$$\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega), \quad \Lambda \to (\mathcal{C}(Y), \rho_Y)$$

is residual in Λ. We fix $k \in \mathbb{N}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and a sequence $\{t_n\}_n$ with $|\tau| < t_n \to +\infty$. By **C5** and $\tau - t_n < 0$, we know that for each $n \in \mathbb{N}$,

$$\mathcal{A}_{\Lambda_k}(\tau - t_n, \theta_{-t_n}\omega) := \bigcup_{\lambda\in\Lambda_k} \mathcal{A}_\lambda(\tau - t_n, \theta_{-t_n}\omega)$$

is a pre-compact set in $X \cap Y$. By Lemma 2.10 (iv),

$$K_n := \overline{\mathcal{A}_{\Lambda_k}(\tau - t_n, \theta_{-t_n}\omega)} = \mathrm{cl}_{X\cap Y}\mathcal{A}_{\Lambda_k}(\tau - t_n, \theta_{-t_n}\omega), \tag{2.9}$$

and thus $K_n \in \mathcal{C}(X \cap Y)$ for each $n \in \mathbb{N}$. By **C2**, the following mapping

$$\Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega) : X \cap Y \to Y$$

is strong-weakly continuous, then it follows from Lemma 2.11 (a) that

$$F_n(\lambda) := \Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)K_n \in CB(Y),\ \forall\, n \in \mathbb{N},\ \lambda \in \Lambda_k.$$

We remark here that by **C1** we have $F_n(\lambda) \in \mathcal{C}(X)$, but it may not belong to $\mathcal{C}(Y)$.

Suppose that $\lambda \to \lambda_0$ in Λ_k, by $K_n \in \mathcal{C}(X \cap Y)$, it follows from **C3** that

$$\begin{aligned} d_Y(F_n(\lambda_0), F_n(\lambda)) &= \sup_{y\in K_n} d_Y(\Phi_{\lambda_0}(t_n, \tau - t_n, \theta_{-t_n}\omega)y, \Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)K_n) \\ &\le \sup_{y\in K_n} \|\Phi_{\lambda_0}(t_n, \tau - t_n, \theta_{-t_n}\omega)y - \Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)y\|_Y \to 0. \end{aligned}$$

Since the last distance is symmetric, we similarly obtain that

$$\lim_{\lambda\to\lambda_0} d_Y(F_n(\lambda), F_n(\lambda_0)) = 0.$$

The above limits imply that for each $n \in \mathbb{N}$,

$$\lambda \mapsto F_n(\lambda),\ \Lambda_k \to (CB(Y), \rho_Y),\ \text{is continuous.} \tag{2.10}$$

We then consider the convergence of $F_n(\cdot)$ as $n \to \infty$. By the invariance and (2.9), we know that, for each $\lambda \in \Lambda_k$,

$$\begin{aligned} d_Y(\mathcal{A}_\lambda(\tau,\omega), F_n(\lambda)) &\le d_Y(\Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)\overline{\mathcal{A}_{\Lambda_k}(\tau - t_n, \theta_{-t_n}\omega)}, F_n(\lambda)) \\ &= d_Y(F_n(\lambda), F_n(\lambda)) = 0. \end{aligned}$$

By $\tau - t_n < 0$, the condition **C6** implies that there is $\mathcal{B}_k \in \mathfrak{D}$ such that

$$K_n \subset \mathcal{B}_k(\tau - t_n, \theta_{-t_n}\omega),\ \forall n \in \mathbb{N}.$$

Then, by the $\mathcal{B}_k$-pullback attraction of $\mathcal{A}_\lambda$ in Y, for each $\lambda \in \Lambda_k$,

$$\begin{aligned} d_Y(F_n(\lambda), \mathcal{A}_\lambda(\tau,\omega)) &= d_Y(\Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)K_n, \mathcal{A}_\lambda(\tau,\omega)) \\ &\leq d_Y(\Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)\mathcal{B}_k(\tau - t_n, \theta_{-t_n}\omega), \mathcal{A}_\lambda(\tau,\omega)) \to 0 \end{aligned}$$

as $n \to \infty$. We conclude that, as $n \to \infty$,

$$F_n(\lambda) \to \mathcal{A}_\lambda(\tau,\omega) \text{ in } (CB(Y), \rho_Y), \ \forall \lambda \in \Lambda_k. \tag{2.11}$$

Both (2.10) and (2.11) are just the assumptions in the Baire residual theorem (see Lemma 2.6) on Λ_k and $Z = CB(Y)$. Hence, the continuity-set (denoted by Λ_Y^k) of the mapping

$$\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega), \ \ \Lambda_k \to (CB(Y), \rho_Y) \tag{2.12}$$

is a residual set in Λ_k. By the compactness of $\mathcal{A}_\lambda$ in Y, the mapping in (2.12) actually maps Λ_k into $(\mathcal{C}(Y), \rho_Y)$.

Now, it is not difficult to prove (see [59, Lemma 3.2]) that the whole mapping

$$\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega), \ \ \Lambda \to (\mathcal{C}(Y), \rho_Y) \tag{2.13}$$

is continuous at each point of $\Lambda_Y^k \cap \Lambda_k^{\text{in}}$ and thus it is continuous at all points of $\cup_{k\in\mathbb{N}}(\Lambda_Y^k \cap \Lambda_k^{\text{in}})$.

It is sufficient to prove that the above union is residual in Λ. Indeed, since Λ_Y^k is residual in Λ_k, there is a sequence $\{Q_{kj}\}_j$ of nowhere dense sets in Λ_k such that

$$\Lambda_k \setminus \Lambda_Y^k = \cup_{j\in\mathbb{N}} Q_{kj}, \ \forall k \in \mathbb{N},$$

where each Q_{kj} is also nowhere dense in the whole space Λ (see [59, Lemma 3.2]). By **C4** and the above equality, we have

$$\begin{aligned} \Lambda \setminus \bigcup_{k\in\mathbb{N}} (\Lambda_Y^k \cap \Lambda_k^{\text{in}}) &= \bigcup_{k\in\mathbb{N}} \Lambda_k \setminus \bigcup_{k\in\mathbb{N}} (\Lambda_Y^k \cap \Lambda_k^{\text{in}}) \\ &\subset \bigcup_{k\in\mathbb{N}} (\Lambda_k \setminus \Lambda_Y^k) \cup (\Lambda_k \setminus \Lambda_k^{\text{in}}) = \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} Q_{kj} \cup (\Lambda_k \setminus \Lambda_k^{\text{in}}). \end{aligned} \tag{2.14}$$

Since Λ_k is closed in Λ as assumed in **C4**, it follows from [59, Lemma 2.1] that $\Lambda_k \setminus \Lambda_k^{\text{in}}$ is nowhere dense in Λ. Hence, we see from (2.14) that $\cup_{k\in\mathbb{N}}(\Lambda_Y^k \cap \Lambda_k^{\text{in}})$ is residual in Λ. Therefore, the larger set $\Lambda_Y^*(\tau,\omega)$ is residual too.

By Lemma 2.10 (iii), the set $\Lambda_{X\cap Y}^*(\tau,\omega) := \Lambda_Y^*(\tau,\omega) \cap \Lambda_X^*(\tau,\omega)$ is just the continuity-set of the mapping in (2.8) (into $\mathcal{C}(X \cap Y)$). Since both $\Lambda_Y^*(\tau,\omega)$ and $\Lambda_X^*(\tau,\omega)$ are residual in Λ, the intersection $\Lambda_{X\cap Y}^*(\tau,\omega)$ is residual in Λ too.

Step 3. *We prove the continuity-set* $\Lambda^*(\tau,\omega)$ *of the mapping in* (2.8) *is dense in* Λ *if* **C4*** *replaces* **C4**. We work on the space $X \cap Y$.

As in **Step 2**, for each $n \in \mathbb{N}$, we put

$$K_n = \overline{\mathcal{A}_{\Lambda_k}(\tau - t_n, \theta_{-t_n}\omega)}^X, F_n(\lambda) := \Phi_\lambda(t_n, \tau - t_n, \theta_{-t_n}\omega)K_n, \ \lambda \in \Lambda_k,$$

then $K_n \in \mathcal{C}(X \cap Y)$ and $F_n(\lambda) \in CB(Y)$. By **C1**, $F_n(\lambda) \in \mathcal{C}(X)$. Hence we see from Lemma 2.10 (ii) that $F_n(\lambda) \in CB(X \cap Y)$.

By the same methods as in (2.10) and (2.11), we can prove that

$$\lambda \mapsto F_n(\lambda),\ \Lambda_k \to (CB(X \cap Y), \rho_{X\cap Y}) \text{ is continuous, } \forall n \in \mathbb{N}$$
$$F_n(\lambda) \to \mathcal{A}_\lambda(\tau, \omega) \text{ in } (CB(X \cap Y), \rho_{X\cap Y}) \text{ as } n \to \infty,\ \forall \lambda \in \Lambda_k.$$

Applying Lemma 2.6 on Λ_k and $Z = CB(X \cap Y)$, we know that the continuity-set $R_k(\tau, \omega)$ of the following mapping

$$\lambda \mapsto \mathcal{A}_\lambda(\tau, \omega), \quad \Lambda_k \to (CB(X \cap Y), \rho_{X\cap Y})$$

is residual in Λ_k. Then the whole mapping

$$\lambda \mapsto \mathcal{A}_\lambda(\tau, \omega), \quad \Lambda \to (\mathcal{C}(X \cap Y), \rho_{X\cap Y})$$

is continuous at all points of $R_k \cap \Lambda_k^{\text{in}}$. Hence the continuity-set satisfies

$$\Lambda^*(\tau, \omega) \supset \bigcup_{k\in\mathbb{N}} (R_k \cap \Lambda_k^{\text{in}}). \tag{2.15}$$

By **C4***, Λ_k^{in} is residual in Λ_k and thus $R_k \cap \Lambda_k^{\text{in}}$ is also residual in Λ_k. By **C4*** again, Λ_k is a compact Hausdorff space or a complete metric space. Then Lemma 2.7 (Baire density theorem) implies that the residual set $R_k \cap \Lambda_k^{\text{in}}$ is dense in Λ_k.

Let now O be a nonempty open set in Λ. Since $\Lambda = \cup_{k\in\mathbb{N}} \Lambda_k$, there is a $k_0 \in \mathbb{N}$ such that $O \cap \Lambda_{k_0}$ is nonempty. Note that the nonempty set $O \cap \Lambda_{k_0}$ is open in Λ_{k_0}, and thus

$$\emptyset \neq (O \cap \Lambda_{k_0}) \cap (R_{k_0} \cap \Lambda_{k_0}^{\text{in}}) \cap \Lambda_{k_0}^{\text{in}}) \subset O \cap \Lambda^*(\tau, \omega),$$

which the inclusion follows from (2.15). Hence, $\Lambda^*(\tau, \omega)$ is dense in Λ.□

In order to prove that the continuity-set $\Lambda^*(\tau, \omega)$ in Theorem 2.12 is independent of $\tau \in \mathbb{R}$, we supply a condition:

C7. *For each $\lambda \in \Lambda$, $t > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, the mapping $y \to \Phi_\lambda(t, \tau, \omega)y$ is uniformly continuous in Y on any compact subset K of $X \cap Y$.*

Note that **C7** is actually equivalent to the (strong) continuity of the cocycle operator $\Phi_\lambda(t, \tau, \omega) : X \cap Y \to Y$. Hence **C7** is stronger than **C2**.

Theorem 2.13. *Suppose that a family $\{\mathcal{A}_\lambda; \lambda \in \Lambda\}$ of $\mathfrak{D}$-pullback random (X, Y)-attractors satisfies **C1-C7**. Then, for each $\omega \in \Omega$, there is a residual set $\Lambda^*(\omega)$ in Λ such that the following mapping*

$$\lambda \mapsto \mathcal{A}_\lambda(\tau, \theta_\tau\omega), \quad \Lambda \to (\mathcal{C}(X \cap Y), \rho_{X\cap Y}) \tag{2.16}$$

*is continuous at all points of $\Lambda^*(\omega)$ for each $\tau \in \mathbb{R}$. Moreover, $\Lambda^*(\omega)$ is dense in Λ if **C4*** replaces **C4**.*

Proof. Under assumptions **C1-C6**, by Theorem 2.12 and its proof (see **Steps 2-3**), for each $\tau \in \mathbb{R}$, $\omega \in \Omega$, the mapping

$$\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega), \quad \Lambda \to (\mathcal{C}(X\cap Y), \rho_{X\cap Y})$$

is continuous at all points of

$$\bigcup_{k\in\mathbb{N}} (R_k(\tau,\omega)\cap \Lambda_k^{\text{in}}) =: R(\tau,\omega), \tag{2.17}$$

where $R_k(\tau,\omega)$ is residual in Λ_k, while $R(\tau,\omega)$ is residual in Λ. Put

$$\Lambda^*(\omega) = \bigcap_{j=-1}^{-\infty} R(j,\theta_j\omega). \tag{2.18}$$

As a countable intersection of residual sets, $\Lambda^*(\omega)$ is residual in Λ.

We prove that each point in $\Lambda^*(\omega)$ *is a (common) point of continuity of the mappings in* (2.16) *for all* $\tau \in \mathbb{R}$ *if we further assume* **C7**.

If $\Lambda^*(\omega) = \emptyset$, the assertion automatically holds. Hence, we assume $\Lambda^*(\omega) \neq \emptyset$.

Given $\lambda_0 \in \Lambda^*(\omega)$ and $\tau \in \mathbb{R}$, we write $\tau = m + t$, where $m \in \{-1,-2,\cdots\}$ and $t > 0$. By (2.18), $\lambda_0 \in \Lambda^*(\omega) \subset R(m,\theta_m\omega)$. By (2.17),

$$\lambda_0 \in R_{k_0}(m,\theta_m\omega)\cap\Lambda_{k_0}^{\text{in}} \subset \Lambda_{k_0}^{\text{in}}, \text{ for some } k_0 \in \mathbb{N},$$

and thus $\Lambda_{k_0}^{\text{in}}$ is an open neighborhood of λ_0 in Λ. Furthermore, for any $\lambda \in \Lambda_{k_0}^{\text{in}}$,

$$\mathcal{A}_\lambda(m,\theta_m\omega) \subset \bigcup_{\lambda\in\Lambda_{k_0}} \mathcal{A}_\lambda(m,\theta_m\omega) \subset \overline{\mathcal{A}_{\Lambda_{k_0}}(m,\theta_m\omega)}^X =: K. \tag{2.19}$$

Since $m < 0$, by **C5**, $\mathcal{A}_{\Lambda_{k_0}}(m,\theta_m\omega)$ is pre-compact in $X\cap Y$ and thus, by Lemma 2.10 (iv), K is compact in $X\cap Y$, in X and in Y respectively.

Given $\varepsilon > 0$, by **C1**, $\Phi_{\lambda_0}(t,m,\theta_m\omega) : X \to X$ is continuous and thus uniformly continuous on $K \in \mathcal{C}(X)$. Consequently, there is $\delta_1 > 0$ such that

$$\|\Phi_{\lambda_0}(t,m,\theta_m\omega)x - \Phi_{\lambda_0}(t,m,\theta_m\omega)y\|_X < \frac{\varepsilon}{4}$$

whenever $\|x-y\|_X < \delta_1$ with $x,y \in K$. Since $K \in \mathcal{C}(X\cap Y)$, by **C7** that there is $\delta_2 \in (0,\delta_1)$ such that

$$\|\Phi_{\lambda_0}(t,m,\theta_m\omega)x - \Phi_{\lambda_0}(t,m,\theta_m\omega)y\|_Y < \frac{\varepsilon}{4}$$

whenever $\|x-y\|_Y < \delta_2$ with $x,y \in K$. Put $\delta = \min\{\delta_1,\delta_2\}$, for all $x,y \in K$ with $\|x-y\|_{X\cap Y} < \delta$

$$\|\Phi_{\lambda_0}(t,m,\theta_m\omega)x - \Phi_{\lambda_0}(t,m,\theta_m\omega)y\|_{X\cap Y} < \frac{\varepsilon}{2}. \tag{2.20}$$

Since $\lambda_0 \in R(m,\theta_m\omega)$, namely, λ_0 is a point of continuity of $\lambda \mapsto \mathcal{A}_\lambda(m,\theta_m\omega)$ in $\mathcal{C}(X\cap Y)$ and thus there is an open neighborhood $N_1(\lambda_0)$ in Λ such that

$$\rho_{X\cap Y}(\mathcal{A}_\lambda(m,\theta_m\omega),\mathcal{A}_{\lambda_0}(m,\theta_m\omega)) < \delta,\ \forall \lambda \in N_1(\lambda_0). \tag{2.21}$$

By **C3**, there is an open neighborhood $N_2(\lambda_0)$ in Λ such that

$$\sup_{x\in K}\|\Phi_\lambda(t,m,\theta_m\omega)x - \Phi_{\lambda_0}(t,m,\theta_m\omega)x\|_{X\cap Y} < \frac{\varepsilon}{2},\ \forall \lambda \in N_2(\lambda_0). \tag{2.22}$$

Let $N(\lambda_0) = \Lambda^{\text{in}}_{k_0} \cap N_1(\lambda_0) \cap N_2(\lambda_0)$, which is open in Λ. Then (2.19) implies that $\mathcal{A}_\lambda(m,\theta_m\omega) \subset K$, for all $\lambda \in N(\lambda_0)$, and thus (2.22) implies that

$$\begin{aligned}&\rho_{X\cap Y}(\Phi_\lambda(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega),\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega))\\ &\le \sup_{x\in K}\|\Phi_\lambda(t,m,\theta_m\omega)x - \Phi_{\lambda_0}(t,m,\theta_m\omega)x\|_{X\cap Y} < \frac{\varepsilon}{2}.\end{aligned} \tag{2.23}$$

For $\lambda \in N(\lambda_0)$ and $x_\lambda \in \mathcal{A}_\lambda(m,\theta_m\omega)$, by (2.21), there is $y_{\lambda_0} \in \mathcal{A}_{\lambda_0}(m,\theta_m\omega)$ such that $\|x_\lambda - y_{\lambda_0}\|_{X\cap Y} < \delta$. Note that $x_\lambda \in K, y_{\lambda_0} \in K$, it follows from (2.20) that

$$\begin{aligned}&d_{X\cap Y}(\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega),\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_{\lambda_0}(m,\theta_m\omega))\\ &\le \sup_{x_\lambda\in\mathcal{A}_\lambda(m,\theta_m\omega)}\|\Phi_{\lambda_0}(t,m,\theta_m\omega)x_\lambda - \Phi_{\lambda_0}(t,m,\theta_m\omega)y_{\lambda_0}\|_{X\cap Y} < \frac{\varepsilon}{2}.\end{aligned}$$

On the contrary, for $y \in \mathcal{A}_{\lambda_0}(m,\theta_m\omega)$ and $\lambda \in N(\lambda_0)$, by (2.21), there is $z_\lambda \in \mathcal{A}_\lambda$ such that $\|y - z_\lambda\|_{X\cap Y} < \delta$ with $y, z_\lambda \in K$. Then (2.20) implies that

$$\begin{aligned}&d_{X\cap Y}(\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_{\lambda_0}(m,\theta_m\omega),\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega))\\ &\le \sup_{y\in\mathcal{A}_{\lambda_0}(m,\theta_m\omega)} d_{X\cap Y}(\Phi_{\lambda_0}(t,m,\theta_m\omega)y,\Phi_{\lambda_0}(t,m,\theta_m\omega)z_\lambda) < \frac{\varepsilon}{2}.\end{aligned}$$

Both inequalities mentioned above imply that

$$\rho_{X\cap Y}(\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega),\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_{\lambda_0}(m,\theta_m\omega)) < \frac{\varepsilon}{2} \tag{2.24}$$

for all $\lambda \in N(\lambda_0)$. By the triangle inequality, both (2.23) and (2.24) imply that

$$\rho_{X\cap Y}(\Phi_\lambda(t,m,\theta_m\omega)\mathcal{A}_\lambda(m,\theta_m\omega),\Phi_{\lambda_0}(t,m,\theta_m\omega)\mathcal{A}_{\lambda_0}(m,\theta_m\omega)) < \varepsilon$$

for all $\lambda \in N(\lambda_0)$. However, by the invariance, we conclude that

$$\lim_{\lambda\to\lambda_0}\rho_{X\cap Y}(\mathcal{A}_\lambda(\tau,\theta_\tau\omega),\mathcal{A}_{\lambda_0}(\tau,\theta_\tau\omega)) = 0.$$

We finally prove that $\Lambda^*(\omega)$ is dense in Λ if **C4*** *replaces* **C4**.

Indeed, by (2.17) and (2.18), we have

$$\Lambda^*(\omega) = \bigcap_{j=-1}^{-\infty} R(j,\theta_j\omega) = \bigcap_{j=-1}^{-\infty}\bigcup_{k\in\mathbb{N}}(R_k(j,\theta_j\omega)\cap\Lambda_k^{\text{in}}) \supset \bigcup_{k\in\mathbb{N}}\Big(\bigcap_{j=-1}^{-\infty} R_k(j,\theta_j\omega)\cap\Lambda_k^{\text{in}}\Big). \tag{2.25}$$

By **C4*** and Lemma 2.7, the set $\bigcap_{j=-1}^{-\infty} R_k(j,\theta_j\omega)\cap\Lambda_k^{\text{in}}$ is residual in Λ_k and thus dense in Λ_k for each $k \in \mathbb{N}$. By the same method as **Step 3** of Theorem 2.12, we see from (2.25) that $\Lambda^*(\omega)$ is dense in Λ. □

2.4 Random quasi-linear equations with nonlinear colored noise

The random quasi-linear p-Laplace equation driven by nonlinear colored noise is read as

$$\begin{cases} u_t + A_p u + \alpha u = f(x,u) + g(t,x) + h(x,u)\zeta_\lambda(\theta_t\omega), \ t > \tau, \\ u(\tau,x) = u_\tau(x), \ \tau \in \mathbb{R}, \ x \in \mathbb{R}^m, m \in \mathbb{N}, \end{cases} \tag{2.26}$$

where the p-Laplace operator is given by $A_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 2$.

We assume that $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable such that for all $x \in \mathbb{R}^m$ and $s \in \mathbb{R}$,

$$f(x,s)s \le -\beta_1|s|^p + \phi_1(x), \quad \phi_1 \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m), \tag{2.27}$$

$$|f(x,s)| \le \beta_2|s|^{p-1} + \phi_2(x), \quad \phi_2 \in L^{\hat{p}}(\mathbb{R}^m), \tag{2.28}$$

$$\frac{\partial f}{\partial s}(x,s) \le -\beta_3|s|^{p-2} + \phi_3(x), \quad \phi_3 \in L^2 \cap L^\infty, \tag{2.29}$$

where $\beta_i > 0$ and $\hat{p} = p/(p-1)$. We also assume $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^m))$ and $h : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable such that

$$|h(x,s)| \le \psi_1(x)|s|^{q-1} + \psi_2(x), \tag{2.30}$$

$$\frac{\partial h}{\partial s}(x,s) \le \psi_3(x)|s|^{q-2} + \psi_4(x), \ \forall\, x \in \mathbb{R}^m, \ s \in \mathbb{R}, \tag{2.31}$$

where $q \in [2,p)$ and $\psi_i \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ (thus $\psi_i \in L^r(\mathbb{R}^m)$ for any $r \ge 1$) for $i = 1,2,3,4$.

As usual [4], we consider a two-sided scalar Wiener process $W(t,\omega)$ on the classical Wiener space $(\Omega, \mathbb{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where

$$\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \ \omega(0) = 0\}, \quad \theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t), \ \forall \omega \in \Omega,$$

$\mathbb{F}$ is the Borel σ-algebra $\mathcal{B}(\Omega)$ with respect to the Fréchet metric and $\mathbb{P}$ is the two-sided Wiener measure on $(\Omega, \mathbb{F})$. Let

$$\Omega_0 = \{\omega \in \Omega : \forall\, \epsilon > 0, \ \exists\, C_\epsilon(\omega) > 0 \text{ s.t. } |\omega(t)| \le C_\epsilon(\omega)e^{\epsilon|t|}\}. \tag{2.32}$$

By [14, Lemma 11], Ω_0 is a θ-invariant full-measure subset of Ω and thus we do not distinguish between Ω_0 and Ω below.

For each $\lambda > 0$, the Ornstein-Uhlenbeck process $\{\zeta_\lambda(\theta_t\omega); t \in \mathbb{R}\}$ is called colored noise.

Lemma 2.14. *([42]) For each $\lambda > 0$ and $\omega \in \Omega$, $t \to \zeta_\lambda(\theta_t\omega)$ is continuous, and tempered, more precisely,*

$$\lim_{t\to\pm\infty} e^{-\epsilon|t|}\zeta_\lambda(\theta_t\omega) = 0, \ \forall\, \epsilon > 0. \tag{2.33}$$

2.4.1 Initial and regular spaces

From now on, we set $X := L^2(\mathbb{R}^m)$ and $Y := L^p(\mathbb{R}^m)$ with the norms $\|\cdot\|$ and $\|\cdot\|_p$ respectively, where $p > 2$. There are not any embedding relationship between X and Y (see the counterexamples in [30]), but we can verify **(B7)** in Theorem 2.8 as follows.

Lemma 2.15. *$(X \cap Y, \|\cdot\| + \|\cdot\|_p)$ is a separable Banach space. Moreover, X^* is dense in $(X \cap Y)^*$.*

Proof. We mainly prove that $X \cap Y$ is separable. Put $O_k = \{x \in \mathbb{R}^m : |x| < k\}$. For each $k \in \mathbb{N}$, it is known that $L^p(O_k)$ is separable and thus there is a countable dense subset $\{u_{kj} : j \in \mathbb{N}\}$ of $L^p(O_k)$. Given any $u \in L^2(O_k) \cap L^p(O_k)$, we can extract a sequence $\{u_k^i\}_i$ from $\{u_{kj} : j \in \mathbb{N}\}$ such that $u_k^i \to u$ in $L^p(O_k)$ as $i \to \infty$. By Hölder inequality, we have

$$\|u_k^i - u\|^2_{L^2(O_k)} = \int_{O_k} |u_k^i(x) - u(x)|^2 dx \le |O_k| \|u_k^i - u\|^2_{L^p(O_k)} \to 0$$

as $i \to \infty$. By the Hölder inequality, $\{u_{kj} : j \in \mathbb{N}\} \in L^2(O_k)$. Hence, foe each $k \in \mathbb{N}$

$$\{u_{kj} : j \in \mathbb{N}\} \text{ is dense in } (L^2(O_k) \cap L^p(O_k), \|\cdot\|_{L^2(O_k) \cap L^p(O_k)}).$$

The null-extension of $v : O_k \to \mathbb{R}$ is denoted by $\widetilde{v}$, namely, $\widetilde{v} = v$ on O_k and $\widetilde{v} = 0$ on O_k^c. We then obtain that

$$\{\widetilde{u}_{kj} : j \in \mathbb{N}\} \text{ is dense in } (L_k^2 \cap L_k^p, \|\cdot\|_{L_k^2 \cap L_k^p}), \quad \forall k \in \mathbb{N}, \tag{2.34}$$

where $L_k^2 := \{\widetilde{v} : v \in L^2(O_k)\} \subset X$ and $L_k^p := \{\widetilde{v} : v \in L^p(O_k)\} \subset Y$.

Given now $w \in L^2(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$, the truncation of w on O_k is defined by

$$w_k(x) = w(x), \ \forall x \in O_k, \quad w_k(x) = 0, \ \forall x \in \mathbb{R}^n \setminus O_k.$$

Obviously, $w_k \in L_k^2 \cap L_k^p$ for each $k \in \mathbb{N}$. Moreover, as $k \to \infty$,

$$\|w_k - w\| \to 0, \ \|w_k - w\|_p \to 0 \text{ and thus } \|w_k - w\|_{X \cap Y} \to 0,$$

which implies that $\cup_{k \in \mathbb{N}}(L_k^2 \cap L_k^p)$ is dense in $X \cap Y$. Therefore, by (2.34)

$$\{\widetilde{u}_{kj} : \ k, j \in \mathbb{N}\} \text{ is dense in } (X \cap Y, \|\cdot\| + \|\cdot\|_p).$$

which means that $X \cap Y$ is separable.

Using the test functions, one can prove (see [57]) the limit-identical property (5) for (X, Y). Then it follows from [57, Lemma 2.1] that $X \cap Y$ is a Banach space. The last assertion on the dense embedding is obvious. □

2.4.2 The cocycle on the initial space X

Consider the Sobolev space $W^{1,p}(\mathbb{R}^m)$ equipped with the norm

$$\|u\|_{W^{1,p}} = (\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}, \quad \forall u \in W^{1,p}(\mathbb{R}^m).$$

We have $W^{1,p}(\mathbb{R}^m) \hookrightarrow Y$, but the embedding is not compact.

Both **(B1)** and **C1** in theorems 2.8, 2.12 are verified as follows.

Lemma 2.16. *For each $\lambda > 0$, $\omega \in \Omega$ and $u_\tau \in X = L^2(\mathbb{R}^m)$, equation (2.26) has a unique weak solution such that*

$$u_\lambda \in C([\tau, \infty); X) \cap L^p_{loc}(\tau, \infty; W^{1,p}(\mathbb{R}^m)). \tag{2.35}$$

Moreover, the solution operator $u_\tau \to u_\lambda(t; \tau, \omega, u_\tau)$ is continuous on X and $\omega \to u_\lambda(t; \tau, \omega, u_\tau)$ is $(\mathbb{F}, \mathbb{B}(X))$ measurable.

Proof. *Existence.* As done in [54], by the Faedo-Galerkin method, one can prove the existence of a solution satisfying (2.35) by using the following a priori estimates:

$$\begin{aligned} &\|u_\lambda(t; \tau, \omega, u_\tau)\|^2 \le c\Upsilon_\lambda(t, \tau, \omega, u_\tau), \\ &\int_\tau^t e^{\alpha(s-t)}(\|\nabla u_\lambda(s)\|_p^p + \|u_\lambda(s)\|_p^p + \|u_\lambda(s)\|^2)ds \le c\Upsilon_\lambda(t, \tau, \omega, u_\tau), \end{aligned} \tag{2.36}$$

where $t > \tau$, c is an intrinsic positive constant and

$$\Upsilon_\lambda(t, \tau, \omega, u_\tau) = e^{-\alpha(t-\tau)}\|u_\tau\|^2 + \int_\tau^t e^{\alpha(s-t)}(\|g(s)\|^2 + |\zeta_\lambda(\theta_s\omega)|^{\frac{2p-2}{p-q}} + 1)ds.$$

The proof of (2.36) is similar to the autonomous case as given in [58].

Uniqueness, continuity and measurability in X. This is standard or deduced from the same method as in Proposition 2.19. □

By Lemma 2.16, for each $\lambda > 0$, one can define a continuous cocycle $\Phi_\lambda : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ by

$$\Phi_\lambda(t, \tau, \omega)u_\tau = u_\lambda(\tau + t; \tau, \theta_{-\tau}\omega, u_\tau) \tag{2.37}$$

for all $t \ge 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in X = L^2(\mathbb{R}^m)$.

2.4.3 Regularity and strong-weak continuity of the cocycle

We prove that the cocycle is regular and strong-weakly continuous in $Y = L^p(\mathbb{R}^m)$, namely, **(B4)** and **C2** hold true.

Proposition 2.17. *Let $\lambda > 0$, $T \ge 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.*

(i) The cocycle Φ_λ is regular in (X, Y), i.e., $\Phi_\lambda(T, \tau, \omega)X \subset Y$.

(ii) The mapping $\Phi_\lambda(T, \tau, \omega) : X \cap Y \to Y$ is strong-weakly continuous.

Proof. (i) Multiplying (2.26) by $|u|^{p-2}u$ and integrating the products over $\mathbb{R}^m$ we obtain that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u\|_p^p + \langle A_p u, |u|^{p-2}u\rangle + \alpha\|u\|_p^p\\ &= \langle f(u), |u|^{p-2}u\rangle + \langle g(t), |u|^{p-2}u\rangle + \zeta_\lambda(\theta_t\omega)\langle h(u), |u|^{p-2}u\rangle.\end{aligned}$$

Note that $\langle A_p u, |u|^{p-2}u\rangle \geq 0$. By (2.27),

$$\langle f(u), |u|^{p-2}u\rangle \leq -\beta_1\|u\|_{2p-2}^{2p-2} + \int_{\mathbb{R}^m} |\phi_1||u|^{p-1}dx \leq -\frac{3}{4}\beta_1\|u\|_{2p-2}^{2p-2} + c\|\phi_1\|^2.$$

By (2.30) and $\frac{2p-2}{p-q} \geq 2$,

$$\begin{aligned}&\zeta_\lambda(\theta_t\omega)\langle h(u), |u|^{p-2}u\rangle \leq |\zeta_\lambda(\theta_t\omega)| \int_{\mathbb{R}^m} (|\psi_1||u|^{q+p-2} + |\psi_2||u|^{p-1})dx\\ &\leq \frac{\beta_1}{8}\|u\|_{2p-2}^{2p-2} + c\|\psi_1\|_{\frac{2p-2}{p-q}}^{\frac{2p-2}{p-q}}|\zeta_\lambda(\theta_t\omega)|^{\frac{2p-2}{p-q}} + c\|\psi_2\|^2|\zeta_\lambda(\theta_t\omega)|^2\\ &\leq \frac{\beta_1}{8}\|u\|_{2p-2}^{2p-2} + c(|\zeta_\lambda(\theta_t\omega)|^{\frac{2p-2}{p-q}} + 1).\end{aligned}$$

We also have $\langle g(t), |u|^{p-2}u\rangle \leq \frac{\beta_1}{8}\|u\|_{2p-2}^{2p-2} + c\|g(t)\|^2$. Therefore,

$$\frac{d}{dt}\|u\|_p^p + p\alpha\|u\|_p^p + \frac{p\beta_1}{2}\|u\|_{2p-2}^{2p-2} \leq c(\|g(t)\|^2 + |\zeta_\lambda(\theta_t\omega)|^{\frac{2p-2}{p-q}} + 1). \tag{2.38}$$

Integrating (2.38) in $t \in [r, s]$, where $\tau + \epsilon \leq r < s \leq \tau + T$ and $\epsilon > 0$, we obtain that

$$\|u(s)\|_p^p \leq \|u(r)\|_p^p + c\int_\tau^{\tau+T} (\|g(t)\|^2 + |\zeta_\lambda(\theta_t\omega)|^{\frac{2p-2}{p-q}} + 1)dt \leq \|u(r)\|_p^p + C_T(\omega),$$

where we use $g \in L^2_{\mathrm{loc}}(\mathbb{R}, X)$ and the continuity of $t \to \zeta_\lambda(\theta_t\omega)$. Integrating the above inequality in $r \in [\tau + \epsilon, s]$ yields

$$\|u(s)\|_p^p \leq \frac{1}{s - \tau - \epsilon}\int_\tau^{\tau+T} \|u(r)\|_p^p dr + C_T(\omega)$$

for all $s \in (\tau+\epsilon, \tau+T]$. By (2.36) (or (2.35)), the above integral is finite. Therefore, for any $\epsilon \in (0, T/2)$,

$$\sup_{t\in[2\epsilon,T]} \|u(\tau + t, \tau, \omega, u_\tau)\|_p^p \leq C_{\epsilon,T}(\omega).$$

(ii) We claim that $\Phi_\lambda(T, \tau, \omega)\mathcal{C}(X \cap Y) \subset B(Y)$. Indeed, for $A \in \mathcal{C}(X \cap Y)$ and $u_\tau \in A$, by (2.38), the solution $u(\cdot; \tau, \theta_{-\tau}, u_\tau)$ satisfies that

$$\frac{d}{dt}\|u\|_p^p \leq c\|g(t)\|^2 + C_T(\theta_{-\tau}\omega), \ \forall t \in [\tau, \tau + T].$$

Integrating it from $t = \tau$ to $\tau + T$ we obtain that

$$\begin{aligned}\|u(\tau+T)\|_p^p &\le \|u_\tau\|_p^p + c\int_\tau^{\tau+T}\|g(t)\|^2 dt + TC_T(\omega)\\ &\le \|A\|_{X\cap Y}^p + c_g + C_T(\omega) \le C_T(\omega) < +\infty.\end{aligned}$$

Hence $\Phi_\lambda(T,\tau,\omega)A \in B(Y)$. Note that X^* is dense in $(X\cap Y)^*$ as in Lemma 2.15. By Lemma 2.11 (b), we know that $\Phi_\lambda(T,\tau,\omega) : X\cap Y \to Y$ is strong-weakly continuous. □

2.4.4 Luzin continuity and measurability of the cocycle in the regular space Y

In order to verify **(B6)**, we set

$$\Omega_N = \{\omega \in \Omega : |\omega(t)| \le Ne^{\frac{\lambda}{2}s},\ \forall s \in \mathbb{R}\},\ \forall N \in \mathbb{N}.$$

By (2.32), $\Omega = \cup_{N=1}^\infty \Omega_N$ and each Ω_N is a closed set in (Ω, d_Ω), where the Fréchet metric is defined by

$$d_\Omega(\omega_1,\omega_2) = \sum_{n=1}^\infty \frac{1}{2^n}\frac{\sup_{s\in[-n,n]}|\omega(s)|}{1+\sup_{s\in[-n,n]}|\omega(s)|},\ \forall\ \omega_1,\omega_2 \in \Omega.$$

We use the convergence of the colored noise in samples ([30, Lemma 22]) to verify **(B6)**.

Lemma 2.18. *Let $\omega_n, \omega_0 \in \Omega_N$ ($N \in \mathbb{N}$) such that $d_\Omega(\omega_n,\omega_0) \to 0$. Then the OU process satisfies*

$$\sup_{s\in[\tau,\tau+T]}|\zeta_\lambda(\theta_s\omega_n) - \zeta_\lambda(\theta_s\omega_0)| \to 0 \text{ as } n \to \infty; \tag{2.39}$$

$$\sup_{n\in\mathbb{N}_0}\sup_{s\in[\tau,\tau+T]}|\zeta_\lambda(\theta_s\omega_n)| \le C,\quad \forall\lambda > 0,\ \tau \in \mathbb{R}, T > 0. \tag{2.40}$$

Proposition 2.19 (Measurability in Y). *For each $y \in X\cap Y$, $T > 0$ and $\tau \in \mathbb{R}$, the mapping $\omega \to \Phi_\lambda(T,\tau,\omega)y$ is $(\mathbb{F},\mathbb{B}(Y))$-measurable and $(\mathbb{F},\mathbb{B}(X\cap Y))$-measurable.*

Proof. Let $\omega_n \to \omega_0$ in Ω_N with fixed $N \in \mathbb{N}$ and $u_n(\cdot) = u(\cdot,\tau,\omega_n,y)$, $n \in \mathbb{N}_0$, the solution of (2.26) with the initial data $u_n(\tau) \equiv y \in X\cap Y$. Then $U_n := u_n - u_0$ satisfies that

$$\frac{dU_n}{dt} + A_p u_n - A_p u_0 + \alpha U_n = f(u_n) - f(u_0) + \zeta_\lambda(\theta_t\omega_n)h(u_n) - \zeta_\lambda(\theta_t\omega_0)h(u_0).$$

Multiplying it by $|U_n|^{p-2}U_n$ and noting the monotonicity of A_p, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|U_n\|_p^p + \alpha\|U_n\|_p^p &= \langle f(u_n) - f(u_0), |U_n|^{p-2}U_n\rangle\\ &\quad + \langle \zeta_\lambda(\theta_t\omega_n)h(u_n) - \zeta_\lambda(\theta_t\omega_0)h(u_0), |U_n|^{p-2}U_n\rangle.\end{aligned}$$

By the integrated mean-value theorem, (2.29) yields

$$I_f := \langle f(u_n) - f(u_0), |U_n|^{p-2}U_n\rangle = \int_{\mathbb{R}^m}\int_0^1 \frac{\partial f}{\partial s}(x, au_n + (1-a)u_0)da|U_n|^p dx$$
$$\leq -\beta_3 \int_{\mathbb{R}^m}\int_0^1 |au_n + (1-a)u_0|^{p-2}da|U_n|^p dx + \|\phi_3\|_\infty \|U_n\|_p^p.$$

Denote by

$$\begin{aligned} I_h &:= \langle \zeta_\lambda(\theta_t\omega_n)h(u_n) - \zeta_\lambda(\theta_t\omega_0)h(u_0), |U_n|^{p-2}U_n\rangle \\ &= \zeta_\lambda(\theta_t\omega_n)\langle h(u_n) - h(u_0), |U_n|^{p-2}U_n\rangle + (\zeta_\lambda(\theta_t\omega_n) - \zeta_\lambda(\theta_t\omega_0))\langle h(u_0), |U_n|^{p-2}U_n\rangle. \end{aligned}$$

By the integrated mean-value theorem again, (2.31) and (2.40) yield that

$$\begin{aligned} I_h^1 &:= \zeta_\lambda(\theta_t\omega_n)\langle h(u_n) - h(u_0), |U_n|^{p-2}U_n\rangle \\ &= \zeta_\lambda(\theta_t\omega_n)\int_{\mathbb{R}^m}\int_0^1 \frac{\partial h}{\partial s}(x, au_n + (1-a)u_0)da|U|^p dx \\ &\leq C\int_{\mathbb{R}^m}\int_0^1 \|\psi_3\|_\infty |au_n + (1-a)u_0|^{q-2}da|U_n|^p dx + C\|\psi_4\|_\infty\|U_n\|_p^p \\ &\leq \beta_3 \int_{\mathbb{R}^m}\int_0^1 |au_n + (1-a)u_0|^{p-2}da|U_n|^p dx + C(\|\psi_3\|_\infty^{\frac{p-2}{p-q}} + \|\psi_4\|_\infty)\|U_n\|_p^p \end{aligned}$$

for all $t \in [\tau, \tau + T]$. Hence $I_f + I_h^1 \leq C\|U_n\|_p^p$. By (2.30),

$$\begin{aligned} I_h^2 &:= (\zeta_\lambda(\theta_t\omega_n) - \zeta_\lambda(\theta_t\omega_0))\langle h(u_0), |U_n|^{p-2}U_n\rangle \\ &\leq |\zeta_\lambda(\theta_t\omega_n) - \zeta_\lambda(\theta_t\omega_0)|C(\|u_0\|_{p+q-2}^{p+q-2} + \|u_n\|_{p+q-2}^{p+q-2}) \\ &\leq C|\zeta_\lambda(\theta_t\omega_n) - \zeta_\lambda(\theta_t\omega_0)|(\|u_0\|^2 + \|u_n\|^2 + \|u_0\|_{2p-2}^{2p-2} + \|u_n\|_{2p-2}^{2p-2}), \end{aligned}$$

where we use $q < p$. Therefore,

$$\frac{d}{dt}\|U_n(t)\|_p^p \leq C\|U_n(t)\|_p^p + C|\zeta_\lambda(\theta_t\omega_n) - \zeta_\lambda(\theta_t\omega_0)|\chi_n(t), \ \forall\, t \in [\tau, \tau + T],$$

where $\chi_n(t) = \|u_0(t)\|^2 + \|u_n(t)\|^2 + \|u_0(t)\|_{2p-2}^{2p-2} + \|u_n(t)\|_{2p-2}^{2p-2}$. By $U_n(\tau) = 0$, the Gronwall lemma implies that

$$\begin{aligned} \|U_n(\tau+T)\|_p^p &\leq C\int_\tau^{\tau+T} e^{-Cs}|\zeta_\lambda(\theta_s\omega_n) - \zeta_\lambda(\theta_s\omega_0)|\chi_n(s)ds \\ &\leq C \sup_{s\in[\tau,\tau+T]} |\zeta_\lambda(\theta_s\omega_n) - \zeta_\lambda(\theta_s\omega_0)| \int_\tau^{\tau+T} \chi_n(s)ds, \end{aligned} \tag{2.41}$$

where $C = C(\tau, T)$. We then claim that the last integrals are bounded. Indeed, by (2.38) and (2.39), we have

$$\frac{d}{dt}\|u_n\|_p^p + p\alpha\|u_n\|_p^p + \frac{p\beta_1}{2}\|u_n\|_{2p-2}^{2p-2}$$

$$\leq c(\|g(t)\|^2 + |\zeta_\lambda(\theta_t\omega_n)|^{\frac{2p-2}{p-q}} + |\zeta_{\lambda_n}(\theta_t\omega)|^2 + 1) \leq C(\|g(t)\|^2 + 1),$$

for all $n \in \mathbb{N}_0$ and $t \in [\tau, \tau + T]$. Then, the Gronwall lemma implies that

$$\frac{p\beta_1}{2}\int_\tau^{\tau+T} e^{p\alpha s}\|u_n(s)\|_{2p-2}^{2p-2} ds \leq e^{p\alpha\tau}\|u_n(\tau)\|_p^p + C\int_\tau^{\tau+T} e^{p\alpha s}(\|g(s)\|^2 + 1)ds.$$

Note that $u_n(\tau) = y \in X \cap Y$ and $g \in L^2_{loc}(\mathbb{R}, X)$. We have

$$\int_\tau^{\tau+T} \|u_n(s)\|_{2p-2}^{2p-2} ds \leq e^{-p\alpha\tau}\int_\tau^{\tau+T} e^{p\alpha s}\|u_n(s)\|_{2p-2}^{2p-2} ds$$
$$\leq C\|y\|_p^p + Ce^{p\alpha T}\int_\tau^{\tau+T}(\|g(s)\|^2 + 1)ds \leq C$$

for all $n \in \mathbb{N}_0$. Hence, by (2.40) and (2.41), we have

$$\|U_n(\tau + T)\|_p^p \to 0 \text{ as } n \to \infty.$$

Hence the mapping $\omega \to \Phi_\lambda(T, \tau, \omega)y$, $\Omega_N \to Y$, is continuous for all $N \in \mathbb{N}$. Note that

$$\lim_{N\to\infty} \mathbb{P}(\Omega_N) = \mathbb{P}\Big(\bigcup_{N=1}^{\infty} \Omega_N\Big) = \mathbb{P}(\Omega) = 1.$$

Therefore, the mapping $\omega \to \Phi_\lambda(T, \tau, \omega)y$ is Luzin continuous from Ω into Y and thus $(\mathbb{F}, \mathbb{B}(Y))$-measurable.

By a similar method as above, it is relatively easy to prove that the mapping $\omega \to \Phi_\lambda(T, \tau, \omega)y$ is continuous from Ω_N into X. Hence the mapping $\omega \to \Phi_\lambda(T, \tau, \omega)y$ is Luzin continuous from Ω into $X \cap Y$ and thus $(\mathbb{F}, \mathbb{B}(X \cap Y))$-measurable. □

2.5 Existence of a pullback random bi-spatial attractor

To obtain a pullback random bi-spatial attractor, we supply a backward tempered assumption: $g \in L^2_{\text{loc}}(\mathbb{R}, X)$ and

$$\sup_{r\leq\tau}\int_{-\infty}^{r} e^{\alpha(s-r)}\|g(s)\|^2 ds < +\infty, \ \forall\ \tau \in \mathbb{R}, \tag{2.42}$$

where α is the positive constant in (2.26). Let $\mathfrak{D}$ be the universe of all tempered bi-parametric sets in X, where a bi-parametric set $\mathcal{D}$ is called tempered if $\mathcal{D}(\tau, \omega)$ is a bounded subset of X such that

$$\lim_{t\to+\infty} e^{-\epsilon t}\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\| = 0, \ \forall\ \epsilon > 0,\ \tau \in \mathbb{R}, \omega \in \Omega. \tag{2.43}$$

2.5.1 Uniform random absorbing sets in $X \cap Y$

We may verify **(B2)** and **(B5)**, namely, the random absorption in X and in $X \cap Y$.

Lemma 2.20. *For $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, there is $T_0 > 2$ such that for all $t \geq T_0$ and $u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$,*

$$\|u_\lambda(\tau; \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \leq c_X(G(\tau) + E_\lambda(\omega)) =: K_\lambda(\tau, \omega), \tag{2.44}$$

$$\sup_{r\in[\tau-1,\tau]} \|u_\lambda(r; \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{X\cap Y} \leq M_\lambda(\tau, \omega), \tag{2.45}$$

$$\sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} \|u_\lambda(r; \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{X\cap Y} \leq \sup_{\lambda\geq\lambda_0} M_\lambda(\tau, \omega) < \infty, \tag{2.46}$$

$$\sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} \|u_\lambda(r; \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{W^{1,p}} \leq c_W \sup_{\lambda\geq\lambda_0} K_\lambda(\tau, \omega) < \infty, \tag{2.47}$$

where $\lambda_0 > 0$, $M_\lambda(\tau, \omega) = c_Y(K_\lambda^{\frac{1}{2}}(\tau, \omega) + K_\lambda^{\frac{1}{p}}(\tau, \omega))$,

$$G(\tau) = 1 + \int_{-\infty}^0 e^{\alpha s}\|g(s + \tau)\|^2 ds, \quad E_\lambda(\omega) = \int_{-\infty}^0 e^{\alpha s}|\zeta_\lambda(\theta_s\omega)|^{\frac{2p-2}{p-q}} ds. \tag{2.48}$$

Proof. By (2.36) and (2.43), one can deduce (2.44) and

$$\int_{\tau-t}^{\tau} e^{\alpha(s-\tau)}(\|\nabla u_\lambda(s)\|_p^p + \|u_\lambda(s)\|_p^p + \|u_\lambda(s)\|^2)ds \leq K_\lambda(\tau, \omega), \tag{2.49}$$

for all $t \geq T_0$. By the energy inequality (2.38), $u_\lambda(\cdot; \tau - t, \theta_{-\tau}\omega, u_{\tau-t})$ satisfies

$$\frac{d}{ds}\|u_\lambda(s)\|_p^p \leq c_2(\|g(s)\|^2 + |\zeta_\lambda(\theta_{s-\tau}\omega)|^{\frac{2p-2}{p-q}} + 1).$$

Integrating it from $s = \hat{r} \in [\tau - 2, \tau - 1]$ to $s = r \in [\tau - 1, \tau]$, we have

$$\begin{aligned}\|u_\lambda(r)\|_p^p &\leq \|u_\lambda(\hat{r})\|_p^p + c_2 \int_{\hat{r}}^{r} (\|g(s)\|^2 + |\zeta_\lambda(\theta_{s-\tau}\omega)|^{\frac{2p-2}{p-q}} + 1)ds \\ &\leq \|u_\lambda(\hat{r})\|_p^p + c_2 \int_{\tau-2}^{\tau} (\|g(s)\|^2 + |\zeta_\lambda(\theta_{s-\tau}\omega)|^{\frac{2p-2}{p-q}} + 1)ds.\end{aligned}$$

Integrating it from $\hat{r} = \tau - 2$ to $\hat{r} = \tau - 1$, we obtain that for all $r \in [\tau - 1, \tau]$,

$$\|u_\lambda(r)\|_p^p \leq \int_{\tau-2}^{\tau-1} \|u_\lambda(s)\|_p^p ds + c_2 \int_{\tau-2}^{\tau} (\|g(s)\|^2 + |\zeta_\lambda(\theta_{s-\tau}\omega)|^{\frac{2p-2}{p-q}} + 1)ds.$$

By (2.36), we know that for all $t \geq T_0 > 2$ and $u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$,

$$\int_{\tau-2}^{\tau-1} \|u_\lambda(s)\|_p^p ds \leq e^{2\alpha} \int_{\tau-2}^{\tau} e^{\alpha(s-\tau)}\|u_\lambda(s)\|_p^p ds \leq e^{2\alpha} K_\lambda(\tau, \omega),$$

and thus

$$\sup_{r\in[\tau-1,\tau]} \|u_\lambda(r; \tau - t)\|_p^p \leq (e^{2\alpha} + c_2)K_\lambda(\tau, \omega).$$

The similar inequality for L^2-norm holds too and thus (2.45) holds true.

Note that T_0 is independent of λ. We obtain that for all $\lambda_0 > 0$ and $t \geq T_0 > 2$,

$$\sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} \|u_\lambda(r;\tau-t)\|_{X\cap Y} \leq \sup_{\lambda\geq\lambda_0} M_\lambda(\tau,\omega).$$

We have

$$\sup_{\lambda\geq\lambda_0} K_\lambda(\tau,\omega) = c_X G(\tau) + c_X \sup_{\lambda\geq\lambda_0} E_\lambda(\omega) < \infty$$

and thus $\sup\{M_\lambda(\tau,\omega) : \lambda \geq \lambda_0\}$ is finite. Hence (2.46) holds true. It is standard to prove (2.47) (see [58] for Wong-Zakai noise).□

As the autonomous case in [58], one can establish the asymptotic tail property and omega-compactness X.

Lemma 2.21. *For $\lambda_0 > 0$, $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have*

$$\lim_{t,R\to\infty} \sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} \int_{|x|\geq R} |u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})(x)|^2 dx = 0,$$

uniformly for $u_{\tau-t} \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$.

Lemma 2.22. *For each $\lambda_0 > 0$, the family $\{\Phi_\lambda : \lambda \geq \lambda_0\}$ of cocycles is* **uniformly $\mathfrak{D}$-pullback omega-compact** *in X, $\kappa_X B(T) \to 0$ as $T \to \infty$, where*

$$B(T) = \Big(\bigcup_{t\geq T} \bigcup_{\lambda\geq\lambda_0} \bigcup_{r\in[\tau-1,\tau]} u_\lambda(r;\tau-t,\theta_{-\tau}\omega,\mathcal{D}(\tau-t,\theta_{-t}\omega)\Big) \tag{2.50}$$

for $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, and the Kuratowski measure of $B \subset X$ defined by

$$\kappa_X(B) = \inf\{r > 0 : B \text{ has a finite-net in } X\}.$$

Proof. It is standard from Lemma 2.21 and Lemma 2.20. □

2.5.2 Large-valued estimates of solutions in Y

Let $m(e)$ be the Lebegue measure of a measurable set $e \subset \mathbb{R}^m$.

Lemma 2.23. *Let $\lambda_0 > 0$, $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. We have*

$$\lim_{\gamma\to\infty} \sup_{t\geq T_0} \sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} m[|u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_0)| \geq \gamma] = 0, \tag{2.51}$$

$$\lim_{\gamma,t\to\infty} \sup_{\lambda\geq\lambda_0} \sup_{r\in[\tau-1,\tau]} \int_{[|u_\lambda|\geq\gamma]} |u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_0)(x)|^2 dx = 0, \tag{2.52}$$

uniformly for $u_0 \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$, where $T_0 \geq 2$ as given in Lemma 2.20 and

$$[|u_\lambda| \geq \gamma] = \mathbb{R}^m[|u_\lambda| \geq \gamma] = \{x \in \mathbb{R}^m, |u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_0)(x)| \geq \gamma\}.$$

Proof. By (2.46) in Lemma 2.20, for all $r \in [\tau-1, \tau]$, $t \geq T_0$ and $u_0 \in \mathcal{D}(\tau-t, \theta_{-t}\omega)$,

$$\begin{aligned} &m(\mathbb{R}^m[|u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_0)| \geq \gamma]) \leq \frac{1}{\gamma^2}\int_{[|u_\lambda|\geq\gamma]} |u_\lambda(r;\tau-t)(x)|^2 dx \\ &\leq \frac{1}{\gamma^2}\|u_\lambda(r;\tau-t)\|^2 \leq \frac{1}{\gamma^2}\sup_{\lambda\geq\lambda_0} M_\lambda^2(\tau,\omega) \to 0 \end{aligned}$$

as $\gamma \to \infty$ in view of $\sup_{\lambda\geq\lambda_0} M_\lambda(\tau,\omega) < \infty$. Hence (2.51) holds true.

Given $\varepsilon > 0$. By the uniform omega compactness in Lemma 2.22, there is $T_\varepsilon \geq T_0$ such that the Kuratowski measure $\kappa_X(B(T_\varepsilon)) < \varepsilon/4$, where $B(T_\varepsilon)$ is given by (2.50), and thus $B(T_\varepsilon)$ has a finite $\varepsilon/2$-net in $L^2(\mathbb{R}^m)$ with centers $\{v_1, v_2, \cdots, v_k\} \subset B(T_\varepsilon)$. By the absolute continuity of the Lebesgue integral, there is $\delta > 0$ such that

$$e \subset \mathbb{R}^m,\ m(e) < \delta \Rightarrow \sup_{1\leq i\leq k}\int_e v_i^2(x)dx < \varepsilon^2/4.$$

Note that $B(T_\varepsilon) \subset B(T_0)$ (as $T_\varepsilon \geq T_0$). By (2.51), there is $\gamma_\delta > 0$ such that

$$m\{x \in \mathbb{R}^m; |v(x)| \geq \gamma_\delta\} < \delta, \quad \text{for all } v \in B(T_\varepsilon).$$

For any $v \in B(T_\varepsilon)$, there is v_{i_0}, where $i_0 \in \{1, 2, \cdots, k\}$, such that $\|v - v_{i_0}\| < \varepsilon/2$, and thus

$$\begin{aligned} &\int_{[|v|\geq\gamma_\delta]} v^2(x)dx \leq 2\int_{[|v|\geq\gamma_\delta]} |v_{i_0}(x)|^2 dx + 2\int_{[|v|\geq\gamma_\delta]} |v(x) - v_{i_0}(x)|^2 dx \\ &< \varepsilon^2/2 + 2\|v - v_{i_0}\|^2 < \varepsilon^2. \end{aligned}$$

Hence (2.52) holds true. □

Lemma 2.24. *Let $\lambda_0 > 0$, $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Then*

$$\lim_{t,\gamma\to\infty}\sup_{\lambda\geq\lambda_0}\int_{[|u_\lambda|\geq\gamma]} |u_\lambda(\tau;\tau-t,\theta_{-\tau}\omega,u_0)(x)|^p dx = 0 \tag{2.53}$$

uniformly for $u_0 \in \mathcal{D}(\tau-t, \theta_{-t}\omega)$.

Proof. The solution $u_\lambda(r;\tau-t,\theta_{-\tau}\omega,u_0)$ satisfies

$$\frac{d}{dr}u_\lambda(r) + A_p u_\lambda + \alpha u_\lambda = f(x,u_\lambda) + g(t,x) + h(x,u_\lambda)\zeta_\lambda(\theta_{r-\tau}\omega). \tag{2.54}$$

For any $\gamma > 0$, multiplying (2.54) by $(u_\lambda - \gamma)_+$, where $a_+ = \max\{a, 0\}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\|(u_\lambda-\gamma)_+\|^2 + \langle A_p u_\lambda, (u_\lambda-\gamma)_+\rangle + \alpha\|(u_\lambda-\gamma)_+\|^2 \\ &= \langle f(x,u_\lambda), (u_\lambda-\gamma)_+\rangle + \langle h(x,u_\lambda), (u_\lambda-\gamma)_+\rangle\zeta_\lambda(\theta_{r-\tau}\omega) + \langle g(r,x), (u_\lambda-\gamma)_+\rangle. \end{aligned} \tag{2.55}$$

On $\mathbb{R}^m[u \geq \gamma]$, we have $u > 0$ and $(u-\gamma)_+/u \leq 1$, then (2.27) implies that

$$\langle f(x,u_\lambda), (u_\lambda-\gamma)_+\rangle = \int_{[u_\lambda\geq\gamma]} f(x,u_\lambda)u_\lambda\frac{(u_\lambda-\gamma)_+}{u_\lambda}dx$$

$$\leq -\beta_1 \int_{[u_\lambda \geq \gamma]} |u_\lambda|^p \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx + \int_{[u_\lambda \geq \gamma]} |\phi_1| dx.$$

It is similar from (2.30) and (2.70) to see that

$$\begin{aligned}
&\langle h(x, u_\lambda), (u_\lambda - \gamma)_+ \rangle \zeta_\lambda(\theta_{r-\tau}\omega) = \zeta_\lambda(\theta_{r-\tau}\omega) \int_{[u_\lambda \geq \gamma]} h(x, u_\lambda) u_\lambda \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx \\
&\leq |\zeta_\lambda(\theta_{r-\tau}\omega)| \int_{[u_\lambda \geq \gamma]} |(|\psi_1| u_\lambda^q + |\psi_2| u_\lambda) \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx \\
&\leq \frac{\beta_1}{2} \int_{[u_\lambda \geq \gamma]} |u_\lambda|^p \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx + c \int_{[u_\lambda \geq \gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{p}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^{\frac{p}{p-1}}) dx.
\end{aligned}$$

We also have $\langle A_p u_\lambda, (u_\lambda - \gamma)_+ \rangle \geq 0$ and

$$\langle g(r, x), (u_\lambda - \gamma)_+ \rangle \leq c \int_{[u_\lambda \geq \gamma]} g^2(r, x) dx + \alpha \|(u_\lambda - \gamma)_+\|^2.$$

Substituting all estimates into (2.55), we obtain that

$$\begin{aligned}
&\frac{d}{dr} \|(u_\lambda - \gamma)_+\|^2 + \beta_1 \int_{[u_\lambda \geq \gamma]} |u_\lambda|^p \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx \\
&\leq 2 \int_{[u_\lambda \geq \gamma]} |\phi_1| dx + c \int_{[u_\lambda \geq \gamma]} g^2(r, x) dx \\
&\quad + c \int_{[u_\lambda \geq \gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{p}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^{\frac{p}{p-1}}) dx. \qquad (2.56)
\end{aligned}$$

Integrating (2.56) from $r = \tau - 1$ to $r = \tau$ we have

$$\begin{aligned}
&\int_{\tau-1}^{\tau} \int_{[u_\lambda \geq \gamma]} |u_\lambda(r)|^p \frac{(u_\lambda - \gamma)_+}{u_\lambda} dx dr \leq \frac{1}{\beta_1} \|(u_\lambda(\tau - 1) - \gamma)_+\|^2 \\
&\quad + \frac{2}{\beta_1} \int_{\tau-1}^{\tau} \int_{[u_\lambda(r) \geq \gamma]} |\phi_1| dx dr + c_6 \int_{\tau-1}^{\tau} \int_{[u_\lambda(r) \geq \gamma]} g^2(r, x) dx dr \\
&\quad + c_7 \int_{\tau-1}^{\tau} \int_{[u_\lambda(r) \geq \gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{p}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^{\frac{p}{p-1}}) dx dr. \qquad (2.57)
\end{aligned}$$

Given $\varepsilon > 0$, by (2.52), there are $\gamma_1 > 0$ and $T_\varepsilon > T_0 > 2$ such that for all $\gamma \geq \gamma_1$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$,

$$\frac{1}{\beta_1} \|(u_\lambda(\tau - 1; \tau - t, \theta_{-\tau}\omega) - \gamma)_+\|^2 \leq \frac{1}{\beta_1} \int_{[u_\lambda \geq \gamma]} u_\lambda^2(\tau - 1)(x) dx < \frac{\varepsilon}{8}. \qquad (2.58)$$

By the absolute continuity of Lebesgue integrals, there is $\delta_1 > 0$ such that

$$e \subset \mathbb{R}^m, \ m(e) < \delta_1 \quad \Rightarrow \quad \int_e |\phi_1| dx < \frac{\beta_1 \varepsilon}{16}.$$

By (2.51), there is $\gamma_2 \geq \gamma_1$ such that for all $\gamma \geq \gamma_2$, $t \geq T_\varepsilon$, $\lambda \geq \lambda_0$,

$$\sup_{r\in[\tau-1,\tau]} m[u_\lambda(r;\tau-t,\theta_{-\tau}\omega) \geq \gamma] < \delta_1$$

and thus for all $\gamma \geq \gamma_2$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$,

$$\frac{2}{\beta_1}\int_{\tau-1}^{\tau}\int_{[u_\lambda(r)\geq\gamma]} |\phi_1| dxdr \leq \frac{2}{\beta_1}\int_{\tau-1}^{\tau} \frac{\beta_1\varepsilon}{16} dr = \frac{\varepsilon}{8}. \tag{2.59}$$

The Lebesgue theorem and absolute continuity imply that

$$\lim_{m(e)\to 0}\int_{\tau-1}^{\tau}\int_e g^2(r,x)dxdr = \int_{\tau-1}^{\tau}\lim_{m(e)\to 0}\int_e g^2(r,x)dxdr = 0,$$

which further implies that there is $\delta_2 > 0$ such that

$$e \subset \mathbb{R}^m,\ m(e) < \delta_2 \quad \Rightarrow \quad \int_{\tau-1}^{\tau}\int_e g^2(r,x)dxdr < \frac{\varepsilon}{8c_6}.$$

By (2.51) again, there is $\gamma_3 \geq \gamma_2$ such that for all $\gamma \geq \gamma_3$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$,

$$\sup_{r\in[\tau-1,\tau]} m[u_\lambda(r;\tau-t,\theta_{-\tau}\omega) \geq \gamma] < \delta_2,$$

and thus for all $\gamma \geq \gamma_3$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$,

$$c_6\int_{\tau-1}^{\tau}\int_{[u_\lambda(r)\geq\gamma]} g^2(r,x)dxdr < \frac{\varepsilon}{8}. \tag{2.60}$$

By (2.70) we know that

$$\sup_{\lambda\geq\lambda_0}\sup_{s\in[-1,0]} (|\zeta_\lambda(\theta_s\omega)|^{\frac{p}{p-q}} + |\zeta_\lambda(\theta_s\omega)|^{\frac{p}{p-1}}) < C(\omega).$$

Hence, by the same methods as in (2.59), there is $\gamma_4 \geq \gamma_3$ such that for all $\gamma \geq \gamma_4$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} &c_7\int_{\tau-1}^{\tau}\int_{[u_\lambda(r;\tau-t)\geq\gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{p}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^{\frac{p}{p-1}})dxdr \\ &\leq C\int_{\tau-1}^{\tau}\int_{[u_\lambda(r;\tau-t)\geq\gamma]} (|\psi_1|^{\frac{p}{p-q}} + |\psi_2|^{\frac{p}{p-1}})dxdr < \frac{\varepsilon}{8}. \end{aligned} \tag{2.61}$$

Substituting (2.58)-(2.61) into (2.57) we obtain that

$$\int_{\tau-1}^{\tau}\int_{[u_\lambda\geq\gamma]} |u_\lambda(r)|^p \frac{(u_\lambda-\gamma)_+}{u_\lambda} dxdr < \frac{\varepsilon}{2},$$

which, together with $u \leq 2(u-\gamma)_+$ on $\mathbb{R}^m[u \geq 2\gamma]$, implies that

$$\int_{\tau-1}^{\tau}\int_{[u_\lambda\geq 2\gamma]} |u_\lambda(r;\tau-t)|^p dxdr < \varepsilon \tag{2.62}$$

for all $\gamma \geq \gamma_4$, $t \geq T_\varepsilon$ and $\lambda \geq \lambda_0$.

Multiplying (2.54) by $(u_\lambda - 2\gamma)_+^{p-1}$ and noting that $\langle A_p u_\lambda, (u_\lambda - \gamma)_+^{p-1}\rangle \geq 0$, we deduce that

$$\frac{1}{p}\frac{d}{dr}\|(u_\lambda - 2\gamma)_+\|_p^p + \alpha\|(u_\lambda - 2\gamma)_+\|_p^p \leq \langle g(r,x), (u_\lambda - 2\gamma)_+^{p-1}\rangle \\ + \langle f(x,u_\lambda), (u_\lambda - 2\gamma)_+^{p-1}\rangle + \zeta_\lambda(\theta_{r-\tau}\omega)\langle h(x,u_\lambda), (u_\lambda - 2\gamma)_+^{p-1}\rangle. \tag{2.63}$$

Since $0 \leq (u-2\gamma)_+ < u$ on $\mathbb{R}^m[u \geq 2\gamma]$, it follows from (2.27) that

$$\langle f(x,u_\lambda), (u_\lambda - 2\gamma)_+^{p-1}\rangle = \int_{[u_\lambda \geq 2\gamma]} f(x,u_\lambda)u_\lambda \frac{(u_\lambda - 2\gamma)_+^{p-1}}{u_\lambda}dx \\ \leq -\beta_1 \int_{[u_\lambda \geq 2\gamma]} u_\lambda^{p-1}(u_\lambda - 2\gamma)_+^{p-1}dx + \int_{[u_\lambda \geq 2\gamma]} |\phi_1|(u_\lambda - 2\gamma)_+^{p-2}dx \\ \leq -\beta_1 \int_{[u_\lambda \geq 2\gamma]} u_\lambda^{p-1}(u_\lambda - 2\gamma)_+^{p-1}dx + \frac{\alpha}{2}\|(u_\lambda - 2\gamma)_+\|_p^p + \int_{[u_\lambda \geq 2\gamma]} |\phi_1|^{\frac{p}{2}}dx.$$

It is similar from (2.30) and (2.70) to see that

$$\langle h(x,u_\lambda), (u_\lambda - 2\gamma)_+^{p-1}\rangle\zeta_\lambda(\theta_{r-\tau}\omega) \\ \leq |\zeta_\lambda(\theta_{r-\tau}\omega)| \int_{[u_\lambda \geq 2\gamma]} (|\psi_1|u_\lambda^{q-1} + |\psi_2|)(u_\lambda - 2\gamma)_+^{p-1}dx \\ \leq \frac{\beta_1}{4}\int_{[u_\lambda \geq 2\gamma]} u_\lambda^{p-1}(u_\lambda - 2\gamma)_+^{p-1}dx \\ + c\int_{[u_\lambda \geq 2\gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{p-1}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|)(u_\lambda - 2\gamma)_+^{p-1}dx \\ \leq \frac{\beta_1}{2}\int_{[u_\lambda \geq 2\gamma]} u_\lambda^{p-1}(u_\lambda - 2\gamma)_+^{p-1}dx \\ + c\int_{[u_\lambda \geq 2\gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{2p-2}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^2)dx,$$

where we use $(u-2\gamma)_+^{2p-2} \leq u^{p-1}(u-2\gamma)_+^{p-1}$ on $\mathbb{R}^m[u \geq 2\gamma]$. Finally,

$$\langle g(r,x), (u_\lambda - 2\gamma)_+^{p-1}\rangle \leq \frac{\beta_1}{2}\int_{[u_\lambda \geq 2\gamma]} (u_\lambda - 2\gamma)_+^{2p-2}dx + \int_{[u_\lambda \geq 2\gamma]} g^2(r,x)dx \\ \leq \frac{\beta_1}{2}\int_{[u_\lambda \geq 2\gamma]} u_\lambda^{p-1}(u_\lambda - 2\gamma)_+^{p-1}dx + c\int_{[u_\lambda \geq 2\gamma]} g^2(r,x)dx$$

Substituting all estimates into (2.63), we obtain that

$$\frac{d}{dr}\|(u_\lambda(r) - 2\gamma)_+\|_p^p \leq c\int_{[u_\lambda \geq 2\gamma]} |\phi_1|^{\frac{p}{2}}dx + c\int_{[u_\lambda \geq 2\gamma]} g^2(r,x)dx \\ + c\int_{[u_\lambda \geq 2\gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{2p-2}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^2)dx.$$

Integrating it from $r \in [\tau-1, \tau)$ to $r = \tau$ we have

$$\begin{aligned}
&\|(u_\lambda(\tau)-2\gamma)_+\|_p^p \le \|(u_\lambda(r)-2\gamma)_+\|_p^p + I_\lambda(\gamma,t), \text{ where}\\
&I_\lambda(\gamma,t) := c\int_{\tau-1}^{\tau}\int_{[u_\lambda\ge 2\gamma]} |\phi_1|^{\frac{p}{2}}dxdr + c\int_{\tau-1}^{\tau}\int_{[u_\lambda\ge 2\gamma]} g^2(r,x)dxdr\\
&+c\int_{\tau-1}^{\tau}\int_{[u_\lambda\ge 2\gamma]} (|\zeta_\lambda(\theta_{r-\tau}\omega)\psi_1|^{\frac{2p-2}{p-q}} + |\zeta_\lambda(\theta_{r-\tau}\omega)\psi_2|^2)dxdr.
\end{aligned}$$

Integrating it from $r = \tau-1$ to $r = \tau$ we obtain that

$$\|(u_\lambda(\tau)-2\gamma)_+\|_p^p \le \int_{\tau-1}^{\tau} \|(u_\lambda(r)-2\gamma)_+\|_p^p dr + I_\lambda(\gamma,t). \tag{2.64}$$

By (2.62), for all $\gamma \ge \gamma_4$, $t \ge T_\varepsilon$ and $\lambda \ge \lambda_0$,

$$\int_{\tau-1}^{\tau} \|(u_\lambda(r;\tau-t)-2\gamma)_+\|_p^p dr \le \int_{\tau-1}^{\tau}\int_{[u_\lambda\ge 2\gamma]} |u_\lambda(r;\tau-t)|^p dxdr < \varepsilon.$$

As done in (2.59)-(2.61), we can take $\gamma_5 \ge \gamma_4$ such that $I_\lambda(\gamma,t) < \varepsilon$ for all $\gamma \ge \gamma_5$, $t \ge T_\varepsilon$ and $\lambda \ge \lambda_0$. Noting that $u \le 2(u-2\gamma)_+$ on $\mathbb{R}^m[u \ge 4\gamma]$, we see from (2.64) that

$$\int_{[u_\lambda \ge 4\gamma]} u_\lambda^p(\tau;\tau-t) \le 2^p\|(u_\lambda(\tau)-2\gamma)_+\|_p^p < 2^{p+1}\varepsilon$$

for all $\gamma \ge \gamma_5$, $t \ge T_\varepsilon$ and $\lambda \ge \lambda_0$. One can prove a similar inequality for the negative part $(u_\lambda+\gamma)_-$ and thus (2.53) holds true. □

Lemma 2.25. *For any $\lambda_0 > 0$, the family $\{\Phi_\lambda : \lambda \ge \lambda_0\}$ of cocycles is uniformly $\mathfrak{D}$-pullback omega compact in Y, namely, $\kappa_Y(\widetilde{B}(T)) \to 0$ as $T \to \infty$, where*

$$\widetilde{B}(T) := \bigcup_{t\ge T}\bigcup_{\lambda\ge\lambda_0} \Phi_\lambda(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega)$$

for $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$.

Proof. Note that $\widetilde{B}(T) \subset Y$ in view of Proposition 2.17. Given $\varepsilon > 0$, by Lemma 2.24, there are two positive numbers γ_ε and T_1 such that

$$\int_{[|v|\ge\gamma_\varepsilon]} |v(x)|^p dx < \varepsilon^p, \ \forall\ v \in \widetilde{B}(T_1). \tag{2.65}$$

By the uniform omega compactness in X, as in Lemma 2.22, there is $T_2 > T_1$ such that $\widetilde{B}(T_2)$ has a finite $(2\gamma_\varepsilon)^{\frac{2-p}{2}}\varepsilon^{\frac{p}{2}}$-net in X with centers $\{v_1, v_2, \cdots, v_k\} \subset \widetilde{B}(T_2)$.

In particular, for any $v \in \widetilde{B}(T_2)$, there is v_i, $i \in \{1, 2, \cdots, k\}$, such that

$$\int_{\mathbb{R}^m} |v(x)-v_i(x)|^2 dx < (2\gamma_\varepsilon)^{2-p}\varepsilon^p. \tag{2.66}$$

We decompose the Euclidean space by $\mathbb{R}^m = O_1 \cup O_2 \cup O_3 \cup O_4$, where

$$O_1 = [|v| \le \gamma_\varepsilon] \cap [|v_i| \le \gamma_\varepsilon], \quad O_2 = [|v| \ge \gamma_\varepsilon] \cap [|v_i| \ge \gamma_\varepsilon],$$
$$O_3 = [|v| \ge \gamma_\varepsilon] \cap [|v_i| \le \gamma_\varepsilon], \quad O_4 = [|v| \le \gamma_\varepsilon] \cap [|v_i| \ge \gamma_\varepsilon].$$

Since $|v(x) - v_i(x)| \le 2\gamma_\varepsilon$ for $x \in O_1$, it follows from (2.66) that

$$\int_{O_1} |v(x) - v_i(x)|^p dx \le 2^{p-2}\gamma_\varepsilon^{p-2} \int_{O_1} |v(x) - v_i(x)|^2 dx < \varepsilon^p.$$

By (2.65), we have $\int_{O_2} |v(x) - v_i(x)|^p dx < 2^p\varepsilon^p$. Note that $|v_i| \le |v|$ on O_3 and $|v| \le |v_i|$ on O_4. By (2.65),

$$\int_{O_3} |v(x) - v_i(x)|^p dx \le 2^p \int_{[|v|\ge\gamma_\varepsilon]} |v(x)|)^p dx < 2^p\varepsilon^p,$$
$$\int_{O_4} |v(x) - v_i(x)|^p dx \le 2^p \int_{[|v_i|\ge\gamma_\varepsilon]} |v_i(x)|)^p dx < 2^p\varepsilon^p.$$

Hence, $\int_{\mathbb{R}^m} |v(x) - v_i(x)|^p dx < (4\varepsilon)^p$ and thus $\widetilde{B}(T_2)$ has a finite 4ε-net in Y. $\square$

2.5.3 Existence of a pullback random bi-spatial attractor for the random p-Laplace equation

We are in a position to prove the existence of a $\mathfrak{D}$-pullback random (X,Y)-attractor, where $X = L^2(\mathbb{R}^m)$ and $Y = L^p(\mathbb{R}^m)$.

Theorem 2.26. *For each $\lambda > 0$, the cocycle Φ_λ has a $\mathfrak{D}$-pullback (X,Y)-attractor given by*

$$\mathcal{A}_\lambda(\tau,\omega) = \Upsilon_X(\mathcal{K}_\lambda)(\tau,\omega) = \Upsilon_Y(\mathcal{M}_\lambda)(\tau,\omega), \ \forall \tau \in \mathbb{R}, \omega \in \Omega, \tag{2.67}$$

such that $\mathcal{A}_\lambda(\tau,\cdot)$ is measurable in X, Y and $X \cap Y$ respectively, where

$$\mathcal{K}_\lambda(\tau,\omega) := \{v \in X; \|v\|^2 \le K_\lambda(\tau,\omega)\},$$
$$\mathcal{M}_\lambda(\tau,\omega) := \{v \in X \cap Y; \|v\|_{X\cap Y} \le M_\lambda(\tau,\omega)\},$$

and the functions K_λ and M_λ are given in Lemma 2.20.

Proof. One can prove (see [29]) that condition (2.42) holds for any exponent, i.e.,

$$\sup_{r\le\tau} \int_{-\infty}^{r} e^{\epsilon(s-r)}\|g(s)\|^2 ds < +\infty, \ \forall \epsilon > 0, \ \tau \in \mathbb{R},$$

which implies that, for any $\epsilon < \alpha$,

$$e^{-\epsilon t}G(\tau - t) = e^{-\epsilon t} + e^{-\epsilon t} \int_{-\infty}^{0} e^{\alpha s}\|g(s + \tau - t)\|^2 ds$$

$$\leq e^{-\epsilon t} + e^{-\epsilon t} \int_{-\infty}^{0} e^{\frac{\epsilon}{2}s} \|g(s+\tau-t)\|^2 ds$$
$$\leq e^{-\epsilon t} + e^{-\frac{\epsilon}{2}t} \int_{-\infty}^{\tau} e^{\frac{\epsilon}{2}(s-\tau)} \|g(s)\|^2 ds \to 0$$

as $t \to \infty$, where $G(\cdot)$ is defined in (2.48). By the tempered property of $\zeta_\lambda(\omega)$ as in Lemma 2.14, one can prove that

$$e^{-\epsilon t} E_\lambda(\theta_{-t}\omega) = e^{-\epsilon t} \int_{-\infty}^{0} e^{\alpha s} |\zeta_\lambda(\theta_{s-t}\omega)|^{\frac{2p-2}{p-q}} ds \to 0$$

as $t \to \infty$. Note that $K_\lambda(\tau,\omega) = c(G(\tau) + E_\lambda(\omega))$, we have $\mathcal{K}_\lambda \in \mathfrak{D}$. Similarly, $\widehat{\mathcal{M}}_\lambda \in \mathfrak{D}$, where

$$\widehat{\mathcal{M}}_\lambda(\tau,\omega) := \{v \in X; \|v\|_X \leq M_\lambda(\tau,\omega)\} \supset \mathcal{M}_\lambda(\tau,\omega)$$

and thus $\mathcal{M}_\lambda \in \mathfrak{D}$. Since $M_\lambda(\tau,\cdot)$ is a random variable, it follows that $\mathcal{M}_\lambda(\tau,\cdot)$ is a random closed set in $X \cap Y$, by Lemma 2.20, $\mathcal{M}_\lambda(\tau,\cdot)$ is a $\mathfrak{D}$-pullback absorbing set in $X \cap Y$. Hence both **(B2)** and **(B5)** in Theorem 2.8 hold true.

By Lemma 2.25, Φ is $\mathfrak{D}$-pullback asymptotically compact in Y, which together with Lemma 2.22 implies that Φ is $\mathfrak{D}$-pullback asymptotically compact in $X \cap Y$, namely, **(B3)** holds true, while **(B1)**, **(B4)**, **(B6)** and **(B7)** have been verified. Applying Theorem 2.8, we know that Φ_λ has a $\mathfrak{D}$-pullback random (X,Y)-attractor $\mathcal{A} \in \mathfrak{D}$, which implies that $\mathcal{A}$ is random in X and in Y.

By Proposition 2.19, we know that $\omega \to \Phi(t,\tau,\omega)$ is $(\mathbb{F}, \mathbb{B}(X \cap Y))$-measurable for each $y \in X \cap Y$. Using this property, the same method as in Theorem 2.8 (ii) shows that $\mathcal{A}$ is random in $X \cap Y$. □

2.6 Residual dense continuity of pullback random bi-spatial attractors

In this section, we will establish the residual dense continuity of $\mathcal{A}_\lambda$ on the parameter space $\Lambda = (0,\infty]$, where $\mathcal{A}_\lambda$ is obtained in Theorem 2.26 for $\lambda \to \infty$. In the case of $\lambda = \infty$, we consider the following deterministic equation

$$\begin{cases} \partial u_{\infty,t} + A_p u_\infty + \alpha u_\infty = f(x,u_\infty) + g(t,x), \ t > \tau, \\ u(\tau,x) = u_\tau(x), \ \tau \in \mathbb{R}, \ x \in \mathbb{R}^m. \end{cases} \tag{2.68}$$

Problem (2.68) is well-posed (see [52]), and thus its solutions generate a non-autonomous dynamical system

$$\Phi_\infty(t,\tau)u_\tau = u_\infty(\tau+t;\tau,u_\tau), \forall\, t \geq 0, \tau \in \mathbb{R}, u_\tau \in X.$$

In order to verify condition **C3** in Theorem 2.12, we need the following convergence of colored noise with respect to the size.

Lemma 2.27. *(i) The OU process converges to zero at infinity, i.e.,*

$$\lim_{\lambda\to+\infty}\sup_{t\in[a,b]}|\zeta_\lambda(\theta_t\omega)|=0,\ \forall[a,b]\subset\mathbb{R},\ \omega\in\Omega. \tag{2.69}$$

(ii) $\lambda\to\zeta_\lambda(\theta_t\omega)$ *is continuous at any* $\lambda\in(0,\infty)$ *uniformly for* $t\in[a,b]$.
(iii) For any $\lambda_0>0$ *and* $[a,b]\subset\mathbb{R}$, *we have*

$$\sup_{\lambda\geq\lambda_0}\sup_{t\in[a,b]}|\zeta_\lambda(\theta_t\omega)|\leq C(\omega),\ \forall\ \omega\in\Omega. \tag{2.70}$$

Proof. By the definition of colored noise and the equality $W(t,\omega)=\omega(t)$, we have

$$\zeta_\lambda(\theta_t\omega)=-\frac{1}{\lambda^2}\int_{-\infty}^0 e^{\frac{s}{\lambda}}\omega(t+s)ds+\frac{1}{\lambda}\omega(t),\quad \forall\lambda>0,t\in\mathbb{R},\omega\in\Omega.$$

It suffices to consider the integral:

$$\widetilde{\zeta}_\lambda(t,\omega):=-\frac{1}{\lambda^2}\int_{-\infty}^0 e^{\frac{s}{\lambda}}\omega(t+s)ds=-\frac{1}{\lambda}\int_{-\infty}^0 e^{s}\omega(t+\lambda s)ds.$$

Given $\varepsilon\in(0,1)$, by $\omega(t)/t\to0$ as $t\to\pm\infty$, there is $T_\varepsilon=T_\varepsilon(\omega)>\max\{0,-b\}$ such that $|\omega(t)|\leq\varepsilon|t|$ for all $|t|\geq T_\varepsilon$.

Let $\lambda_\varepsilon=(T_\varepsilon+b)/\varepsilon>0$. Then, for all $\lambda\geq\lambda_\varepsilon$, $t\in[a,b]$ and $s\in(-\infty,-\varepsilon]$,

$$t+\lambda s\leq -T_\varepsilon,\Rightarrow|\omega(t+\lambda s)|\leq-\varepsilon(t+\lambda s),$$

which further implies that for all $\lambda\geq\lambda_\varepsilon$ and $t\in[a,b]$,

$$\begin{aligned}\frac{1}{\lambda}\int_{-\infty}^{-\varepsilon}e^s|\omega(t+\lambda s)|ds&\leq\frac{\varepsilon}{\lambda}\int_{-\infty}^{-\varepsilon}e^s|t+\lambda s|ds\\ &\leq\varepsilon\frac{|t|}{\lambda}+\varepsilon\int_{-\infty}^{-\varepsilon}e^s|s|ds\leq\varepsilon\frac{|t|}{\lambda}+\varepsilon.\end{aligned}$$

Hence, for all $\lambda\geq\max\{\lambda_\varepsilon,|a|,|b|\}$,

$$\sup_{t\in[a,b]}\frac{1}{\lambda}\int_{-\infty}^{-\varepsilon}e^s|\omega(t+\lambda s)|ds\leq2\varepsilon.$$

On the other hand, by $\omega(t)/t\to0$ as $t\to\pm\infty$, we can find a $C_0=C_0(\omega)>0$ (independent of ε) such that $|\omega(t)|\leq|t|+C_0$ for all $t\in\mathbb{R}$. Hence, for all $\lambda\geq\max\{|a|,|b|,C_0\}$,

$$\sup_{t\in[a,b]}\frac{1}{\lambda}\int_{-\varepsilon}^{0}e^s|\omega(t+\lambda s)|ds\leq\sup_{t\in[a,b]}\int_{-\varepsilon}^0\frac{|t|+\lambda|s|+C_0}{\lambda}ds\leq3\varepsilon.$$

Consequently, for all $\lambda\geq\max\{\lambda_\varepsilon,|a|,|b|,C_0\}$ and $t\in[a,b]$,

$$|\widetilde{\zeta}_\lambda(t,\omega)|\leq\frac{1}{\lambda}\int_{-\infty}^{-\varepsilon}e^s|\omega(t+\lambda s)|ds+\frac{1}{\lambda}\int_{-\varepsilon}^{0}e^s|\omega(t+\lambda s)|ds<5\varepsilon,$$

which implies assertion (i) holds. Assertion (ii) follows from the uniform continuity of $\omega(\cdot)$ on any compact interval, while assertion (iii) follows from (i) and (ii). $\square$

We then prove the convergence of the cocycle Φ_λ as $\lambda \to \infty$ and as $\lambda \to \lambda_0$ respectively, where $\lambda_0 \in (0, \infty)$.

Lemma 2.28. *For each $K \in \mathcal{C}(X \cap Y)$, $T > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{\lambda \to \infty} \sup_{y \in K} \|\Phi_\lambda(T, \tau, \omega)y - \Phi_\infty(T, \tau)y\|_{X \cap Y} = 0, \tag{2.71}$$

$$\lim_{\lambda \to \lambda_0} \sup_{y \in K} \|\Phi_\lambda(T, \tau, \omega)y - \Phi_{\lambda_0}(T, \tau, \omega)y\|_{X \cap Y} = 0, \ \forall \lambda_0 \in (0, \infty). \tag{2.72}$$

Proof. By the difference between two equations (2.26) and (2.68), we know that

$$U_\lambda(\cdot) := u_\lambda(\cdot; \tau, \omega, y) - u_\infty(\cdot; \tau, y), \quad \text{where } \ y \in K,$$

satisfies the following equation

$$\frac{dU_\lambda}{dt} + A_p u_\lambda - A_p u_\infty + \alpha U_\lambda = f(u_\lambda) - f(u_\infty) + \zeta_\lambda(\theta_t \omega) h(u_\lambda). \tag{2.73}$$

Multiplying (2.73) by $|U_\lambda|^{p-2} U_\lambda$ and noting the monotonicity of A_p, we have

$$\frac{1}{p} \frac{d}{dt} \|U_\lambda\|_p^p + \alpha \|U_\lambda\|_p^p = \langle f(u_\lambda) - f(u_\infty), |U_\lambda|^{p-2} U_\lambda \rangle + \zeta_\lambda(\theta_t \omega) \langle h(u_\lambda), |U_\lambda|^{p-2} U_\lambda \rangle.$$

By (2.29), the mean-value theorem implies that for some $\gamma \in (0, 1)$,

$$\langle f(u_\lambda) - f(u_\infty), |U_\lambda|^{p-2} U_\lambda \rangle = \int_{\mathbb{R}^m} \frac{\partial f}{\partial s}(x, \gamma u_\lambda + (1 - \gamma) u_\infty) |U_\lambda|^p dx$$
$$\leq -\beta_3 \int_{\mathbb{R}^m} |\gamma u_\lambda + (1 - \gamma) u_\infty|^{p-2} |U_\lambda|^p dx + \|\phi_3\|_\infty \|U_\lambda\|_p^p \leq c \|U_\lambda\|_p^p.$$

By (2.30) and $q < p$,

$$\zeta_\lambda(\theta_t \omega) \langle h(u_\lambda), |U_\lambda|^{p-2} U_\lambda \rangle \leq |\zeta_\lambda(\theta_t \omega)| \int_{\mathbb{R}^n} |h(u_\lambda)| |U_\lambda|^{p-1} dx$$
$$\leq c |\zeta_\lambda(\theta_t \omega)| \int_{\mathbb{R}^n} (|\psi_1| |u_\lambda|^{q-1}| + |\psi_2|)(|u_\lambda|^{p-1} + |u_\infty|^{p-1}) dx$$
$$\leq c |\zeta_\lambda(\theta_t \omega)| \Big(\|u_\lambda\|_{2p-2}^{2p-2} + \|u_\infty\|_{2p-2}^{2p-2} + 1 \Big).$$

Hence we obtain that

$$\frac{d}{dt} \|U_\lambda(t)\|_p^p \leq c \|U_\lambda(t)\|_p^p + c |\zeta_\lambda(\theta_t \omega)| \Big(\|u_\lambda\|_{2p-2}^{2p-2} + \|u_\infty\|_{2p-2}^{2p-2} + 1 \Big).$$

Applying the Gronwall lemma over $[\tau, \tau + T]$ and noting $U_\lambda(\tau) = 0$, we have

$$\|U_\lambda(\tau + T)\|_p^p \leq c \sup_{t \in [\tau, \tau + T]} |\zeta_\lambda(\theta_t \omega)| \int_\tau^{\tau + T} (\|u_\lambda(t)\|_{2p-2}^{2p-2} + \|u_\infty(t)\|_{2p-2}^{2p-2} + 1) dt. \tag{2.74}$$

The Gronwall lemma on (2.38) implies that

$$\int_{\tau}^{\tau+T} \|u_\lambda(t)\|_{2p-2}^{2p-2} dt$$
$$\leq c\|K\|_Y^p + c\int_{\tau}^{\tau+T} e^{\alpha s}(\|g(s)\|^2 + |\zeta_\lambda(\theta_s\omega)|^{\frac{2p-2}{p-q}} + |\zeta_\lambda(\theta_s\omega)|^2 + 1)ds$$

which is bounded as $\lambda \to \infty$ in view of (2.70). Hence, by (2.69), we see from (2.74) that $\|U_\lambda(\tau+T)\|_p^p \to 0$, i.e.,

$$\sup_{y\in K} \|\Phi_\lambda(T,\tau,\omega)y - \Phi_\infty(T,\tau)y\|_Y \to 0 \text{ as } \lambda \to \infty.$$

The above limit holds in X as well, and thus (2.71) holds true.

By the same method as in the proof of Proposition 2.19, using the convergence of $\zeta_\lambda(\theta_s\omega)$ as $\lambda \to \lambda_0$ (see Lemma 2.27), one can prove (2.72). □

We finally establish the residual dense continuity of $\mathcal{A}_\lambda$ in both $X = L^2(\mathbb{R}^m)$ and $Y = L^p(\mathbb{R}^m)$, where $\mathcal{A}_\lambda$ is the $\mathfrak{D}$-pullback random (X,Y)-attractors and $\mathcal{A}_\infty$ is the $\mathfrak{D}$-pullback attractor for (2.68).

Theorem 2.29. *The continuity-set of the following set-valued mapping*

$$\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega), (0,\infty] \to \mathcal{C}(X\cap Y), \rho_{X\cap Y})$$

is residual and dense in $\Lambda := (0,\infty]$ *for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *where* $X = L^2(\mathbb{R}^m)$, $Y = L^p(\mathbb{R}^m)$ *and* $\rho_{X\cap Y}$ *is the Hausdorff metric. In addition, the mapping* $\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega)$ *is full upper semi-continuous from* Λ *into* $X\cap Y$.

Proof. We need to verify the abstract conditions in Theorem 2.12, **C1** and **C2** (i.e. **(B1)** and **(B4)**) have been verified, while **C3** follows from Lemma 2.28.

The parameter space $\Lambda = (0,\infty]$ can be divided into $\Lambda = \cup_{k=1}^{\infty}\Lambda_k$, where

$$\Lambda_1 = [1,\infty], \quad \Lambda_k = \left[\frac{1}{k}, \frac{1}{k-1}\right], \ k = 2,3,\cdots.$$

Obviously, each Λ_k is closed in Λ and thus **C4** holds. Moreover, Λ_1 is a compact Hausdorff space (it is not a metric space) and $\Lambda_1^{\text{in}} = (0,\infty]$, which is residual in Λ_1, while each Λ_k $(k = 2,3,\cdots)$ is a complete metric space such that Λ_k^{in} is residual in Λ_k. Hence **C4*** holds.

By Lemma 2.20, for each $k \in \mathbb{N}$, we have the following uniform absorbing set

$$\mathcal{B}_k(\tau,\omega) = \{v \in L^2(\mathbb{R}^m), \ \|v\|^2 \leq c_X(G(\tau) + \sup_{\lambda\geq 1/k} E_\lambda(\omega)).$$

Note that $G(\cdot)$ is tempered. For each $\epsilon < \alpha$, as $t \to \infty$,

$$e^{-\epsilon t}\sup_{\lambda\geq 1/k} E_\lambda(\theta_{-t}\omega) = e^{-\epsilon t}\sup_{\lambda\geq 1/k}\int_{-\infty}^{0} e^{\alpha s}|\zeta_\lambda(\theta_{s-t}\omega)|^{\frac{2p-2}{p-q}} ds$$

$$\le e^{-\epsilon t} \sup_{\lambda \ge 1/k} \int_{-\infty}^{-t} e^{\frac{\epsilon}{2}(s+t)} |\zeta_\lambda(\theta_s \omega)|^{\frac{2p-2}{p-q}} ds$$
$$\le e^{-\frac{\epsilon}{2}\epsilon t} C_k(\omega) \int_{-\infty}^{-t} e^{\frac{3\epsilon}{4}s} ds \to 0$$

in view of Lemmas 2.14 and 2.27. Hence $\mathcal{B}_k \in \mathfrak{D}$ for all $k \in \mathbb{N}$. Noting that $\mathcal{A}_\lambda \subset \mathcal{B}_k$ for all $\lambda \ge 1/k$, we know that, for all $k \ge 2$,

$$\mathcal{A}_{\Lambda_k}(\tau,\omega) := \cup_{\lambda \in \Lambda_k} \mathcal{A}_\lambda(\tau,\omega) \subset \mathcal{B}_k(\tau,\omega).$$

For $k = 1$, we have

$$\mathcal{A}_{\Lambda_1}(\tau,\omega) := \cup_{\lambda \in \Lambda_1} \mathcal{A}_\lambda(\tau,\omega) \subset \mathcal{B}_1(\tau,\omega) \cup \mathcal{A}_\infty(\tau,\omega).$$

Since $\mathcal{A}_\infty \in \mathfrak{D}$, we have $\mathcal{B}_1 \cup \mathcal{A}_\infty \in \mathfrak{D}$. Hence **C6** hold true.

It suffices to verify condition **C5** about that $\mathcal{A}_{\Lambda_k}(\tau,\omega)$ is pre-compact in $X \cap Y$. We need to prove that for each $k \in \mathbb{N}$ the set

$$A_k(\tau,\omega) := \cup_{\lambda \ge 1/k} \mathcal{A}_\lambda(\tau,\omega)$$

is pre-compact in $X \cap Y$. Indeed, by the invariance, for any $T > 0$ and $t \ge T$,

$$\begin{aligned} A_k(\tau,\omega) :&= \cup_{\lambda \ge 1/k} \Phi_\lambda(t,\tau-t,\theta_{-t}\omega)\mathcal{A}_\lambda(\tau-t,\theta_{-t}\omega) \\ &\subset \cup_{t \ge T} \Phi_\lambda(t,\tau-t,\theta_{-t}\omega)\mathcal{B}_k(\tau-t,\theta_{-t}\omega). \end{aligned}$$

Since $\mathcal{B}_k \in \mathfrak{D}$ as proved above, it follows from Lemma 2.22 and 2.25 that

$$\kappa_{X\cap Y} A_k(\tau,\omega) \le \kappa_{X\cap Y} \cup_{t \ge T} \Phi_\lambda(t,\tau-t,\theta_{-t}\omega)\mathcal{B}_k(\tau-t,\theta_{-t}\omega) \to 0$$

as $T \to \infty$. Hence $\kappa_{X\cap Y} A_k(\tau,\omega) = 0$ and thus $A_k(\tau,\omega)$ is pre-compact in $X \cap Y$.

Therefore, the residual dense continuity of $\lambda \mapsto \mathcal{A}_\lambda(\tau,\omega)$ in $X \cap Y$ follows from the abstract Theorem 2.12. The full upper semi-continuity in Y follows from the non-autonomous version of an abstract theorem as given by [57, Theorem4.1]. □

Chapter 3

Multi-valued random p-Laplace equations with delay

In this chapter, we mainly consider the long-term behavior of p-Laplace equations with infinite delays driven by nonlinear colored noise on a bounded domain $\mathcal{O} \subset \mathbb{R}^N$. We firstly prove the existence of weak solutions to the equation, but the uniqueness of solutions cannot be guaranteed due to the lack of Lipschitz continuity conditions, and thus generate a multi-valued dynamical system. Moreover, the regularity of solutions is also proved. Then we prove the existence of a pullback attractor. Subsequently, the measurability of the pullback attractor and the multi-valued dynamical system are also proved.

In the next section, we show some spaces and assumptions that will be used later. In Sect. 3.2, we prove the existence of weak solutions to equation (7), and that they generate a multi-valued dynamical system. Moreover, we also prove the regularity of solutions. Sect. 3.3 is dedicated to the existence of the pullback attractor, and proves the measurability of the pullback attractor and the multi-valued dynamical system. Therefore, the existence of pullback random attractors for the equation (7) is obtained.

3.1 Some spaces and assumptions

Let $H = L^2(\mathcal{O})$ equipped with the norm $\|\cdot\|$, and use $\|\cdot\|_s$ to denote the norm in $L^s(\mathcal{O})$ (s can be any positive constant). We denote $W_0^{1,p}(\mathcal{O})$ by V, and the dual space $W^{-1,\hat{p}}(\mathcal{O})$ ($\hat{p}$ is the conjugate of p) of $W_0^{1,p}(\mathcal{O})$ by V^*. We also let $(\cdot,\cdot)$ denote the inner product in $L^2(\mathcal{O})$, and denote the duality product between V and V^* by $\langle\cdot,\cdot\rangle$. In addition, we have the usual chain of dense and compact embedding $V \subset H \subset V^*$.

For $p \geq 2$, we define the p-Laplacian operator $\Delta_p : V \to V^*$ by

$$\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad \langle \Delta_p u, v\rangle = \int_{\mathcal{O}} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \tag{3.1}$$

for all $u, v \in V$. Note that Δ_p is a monotone and hemicontinuous operator as in

[77]. Moreover, by (3.1), for all $u, v \in V$

$$\begin{aligned}\langle \Delta_p u, v\rangle &= \int_{\mathcal{O}} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx \\ &\leq \Big(\int_{\mathcal{O}} |\nabla u|^p dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathcal{O}} |\nabla v|^p dx\Big)^{\frac{1}{p}} = \|u\|_V^{p-1}\|v\|_V, \end{aligned} \tag{3.2}$$

and

$$\|\Delta_p u\|_{V^*} = \sup_{\|v\|_V \leq 1} \langle \Delta_p u, v\rangle \leq \|u\|_V^{p-1}. \tag{3.3}$$

Let X be a Hilbert space, we denote our phase space by

$$C_{\gamma,X} = \{w \in C((-\infty, 0]; X) : \lim_{\tau\to-\infty} e^{\gamma\tau} w(\tau) \ \text{ exists}\}, \tag{3.4}$$

where $\gamma > 0$ and we set $\|w\|_{C_{\gamma,X}} := \sup_{\tau\in(-\infty,0]} e^{\gamma\tau}\|w(\tau)\| < \infty$ for all $w \in C_{\gamma,X}$. From [13], we know that $C_{\gamma,X}$ is a separable Banach space.

By [4, 42], we define a random variable $\zeta_\delta : \Omega \to \mathbb{R}$ by

$$\zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^{0} e^{\frac{s}{\delta}} dW(t, \omega), \text{ for each } \delta > 0.$$

The process $z_\delta(t, \omega) = \zeta_\delta(\theta_t\omega)$ is called an Ornstein-Uhlenbeck process (i.e. the colored noise), which is a stationary Gaussian process with $\mathbb{E}(\zeta_\delta) = 0$ and is the unique stationary solution of the stochastic equation:

$$dz + \frac{1}{\delta} z dt = \frac{1}{\delta} dW.$$

By [43], there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset set (still denoted by) Ω of full measure such that for $\omega \in \Omega$,

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0, \qquad \lim_{t\to\pm\infty} \frac{|\zeta_\delta(\theta_t\omega)|}{t} = 0 \text{ for every } 0 < \delta \leq 1. \tag{3.5}$$

In order to achieve our final result, we need to impose the following assumptions:

H1. The external force fulfills $J(\cdot, x) \in C(\mathbb{R}, H)$, for all $x \in \mathcal{O}$.

H2. $\varrho(\cdot) \in C^1(\mathbb{R}, [0, \rho])$ and $|\varrho'(\cdot)| \leq \rho^* < 1$.

H3. $f : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is continuous and for all $t, r \in \mathbb{R}, x \in \mathcal{O}$,

$$f(t, x, r)r \leq -\beta_1|r|^q + \psi_1(t, x), \tag{3.6}$$

$$|f(t, x, r)| \leq \beta_2|r|^{q-1} + \psi_2(t, x), \tag{3.7}$$

where $q > 2, \beta_1, \beta_2 > 0$, $\psi_1 \in L^\infty_{loc}(\mathbb{R}, L^1(\mathcal{O}) \cap L^\infty(\mathcal{O}))$, $\psi_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$.

H4. $g \in C(\mathbb{R}, \mathbb{R})$ and there is $\beta_3 > 0$ such that

$$|g(x, r)|^2 \leq \beta_3|r|^2 + |\psi_3(x)|^2, \quad \forall r \in \mathbb{R}, \ x \in \mathcal{O}, \tag{3.8}$$

where $\psi_3 \in L^2(\mathcal{O})$.

H5. The nonlinear diffusion term $h : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $t, r \in \mathbb{R}, x \in \mathcal{O}$,

$$|h(t,x,r)| \leq \psi_4(t,x)|r|^{\eta-1} + \psi_5(t,x), \tag{3.9}$$

where $2 \leq \eta < q, \psi_4 \in L_{loc}^{\frac{2q-2}{q-\eta}}(\mathbb{R}, L^{\frac{2q-2}{q-\eta}}(\mathcal{O}))$, $\psi_5 \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$.

H6. $F : \mathcal{O}\times\mathbb{R}_-\times\mathbb{R} \to \mathbb{R}$ is continuous. There exist a scalar function $e^{-\gamma \cdot} m_1(\cdot) \in L^1((-\infty, 0], \mathbb{R})$, and a function $m_0(x, \cdot) \in L^1((-\infty, 0], L^1(\mathcal{O}))$ such that F satisfies

$$|F(x,l,r)| \leq m_1(l)|r| + |m_0(x,l)|, \quad \forall x \in \mathcal{O}, l \in \mathbb{R}_-, r \in \mathbb{R}. \tag{3.10}$$

To simplify the calculation, we will denote

$$m_0 = \int_{-\infty}^0 \|m_0(\cdot, r)\|_1 dr, \tag{3.11}$$

$$m_1 = \int_{-\infty}^0 e^{-\gamma r} m_1(r) dr. \tag{3.12}$$

Remark 3.1. *By* (3.8)*, we can obtain that*

$$\begin{aligned} \|g(\cdot, u(t-\varrho(t)))\|^2 &\leq \beta_3 \|u(t-\varrho(t))\|^2 + \|\psi_3(\cdot)\|^2 \\ &\leq \beta_3 e^{2\gamma\rho} \|u_t\|^2_{C_{\gamma,H}} + \|\psi_3(\cdot)\|^2. \end{aligned}$$

Remark 3.2. *By* (3.10)*, we can deduce that*

$$\begin{aligned} &\Big\| \int_{-\infty}^0 F(\cdot, l, u(t+l)) dl \Big\|^2 \\ &\leq \int_{\mathcal{O}} (\int_{-\infty}^0 [m_1(l)|u(t+l)| + |m_0(x,l)|] dl)^2 dx \\ &\leq 2\int_{\mathcal{O}} (\int_{-\infty}^0 m_1(l)|u(t+l)| dl)^2 dx + 2\int_{\mathcal{O}} (\int_{-\infty}^0 |m_0(x,l)| dl)^2 dx \\ &\leq 2\int_{\mathcal{O}} (\sup_{l\leq 0} e^{\gamma l}|u_t(l)| \int_{-\infty}^0 e^{-\gamma l} m_1(l) dl)^2 dx + 2\int_{\mathcal{O}} (\int_{-\infty}^0 |m_0(x,l)| dl)^2 dx \\ &\leq 2m_1^2 \|u_t\|^2_{C_{\gamma,H}} + 2m_0^2. \end{aligned}$$

Remark 3.3. *Given $n \in \mathbb{N}$. By* **H6***, we know that if $\eta^n \to \eta$ in $C_{\gamma,H}$, then for all $l \leq 0$*

$$F(x, l, \eta^n(l)) \to F(x, l, \eta(l)).$$

Thus, there exists a positive constant $C(M)$ such that, for any $l \in [-M, 0]$,

$$\|F(\cdot, l, \eta^n(l)) - F(\cdot, l, \eta(l))\| \leq C(M).$$

Using Lebesgue's majorant theorem we have for any $M > 0$

$$\int_{-M}^0 \|F(\cdot, l, \eta^n(l)) - F(\cdot, l, \eta(l))\| dl \to 0.$$

For any $\varepsilon > 0$ there exists an $M = M(\varepsilon) > 0$ such that

$$\begin{aligned}
&\int_{-\infty}^{-M} \|F(\cdot, l, \eta^n(l)) - F(\cdot, l, \eta(l))\| dl \\
&\leq \int_{-\infty}^{-M} \int_{\mathcal{O}} \big[m_1(l)(|\eta^n(l)| + |\eta(l)|) + 2|m_0(x,l)|\big] dx dl \\
&\leq \int_{-\infty}^{-M} m_1(l) e^{-\gamma l} \int_{\mathcal{O}} e^{\gamma l}(|\eta^n(l)| + |\eta(l)|) dx dl + 2\int_{-\infty}^{-M} \int_{\mathcal{O}} |m_0(x,l)| dx dl \\
&\leq (\|\eta^n\|_{C_{\gamma,H}} + \|\eta\|_{C_{\gamma,H}}) \int_{-\infty}^{-M} m_1(l) e^{-\gamma l} dl + 2\int_{-\infty}^{-M} \|m_0(x,l)\|_1 dx dl \leq \varepsilon.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that, for $n \geq N$,

$$\Big\| \int_{-\infty}^{0} F(\cdot, l, \eta^n(l)) dl - \int_{-\infty}^{0} F(\cdot, l, \eta(l)) dl \Big\| \leq 2\varepsilon,$$

which implies that $\eta \to \int_{-\infty}^{0} F(x, l, \eta) dl$ is continuous from $C_{\gamma,H}$ into H.

Let us recall some definitions that will be used in our analysis. Denote by $\mathcal{C}(X)$ the collection of all nonempty closed subsets of X.

Definition 3.4. *([13, 88]) A multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ is called a strict multi-valued non-autonomous dynamical system on $C_{\gamma,H}$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta\}_{t\in\mathbb{R}})$ if for all $t, s \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $\varphi \in C_{\gamma,H}$, the following conditions (i)-(ii) are satisfied:*
(i) $\Phi(0, \tau, \omega, \cdot) = I_{C_{\gamma,H}}$;
(ii) $\Phi(t + s, \tau, \omega, \varphi) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \varphi))$.

Definition 3.5. *[88] $\mathcal{A} \in \mathfrak{D}$ is called a $\mathfrak{D}$-pullback attractor for Φ if the following conditions (i)-(iii) are satisfied: for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*
(i) $\mathcal{A}(\tau, \omega)$ is compact in $C_{\lambda,H}$;
(ii) $\mathcal{A}(\tau, \omega)$ is strictly invariant, i.e.

$$\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{A}(\tau - t, \theta_{-t}\omega)) = \mathcal{A}(\tau, \omega), \ \forall t \geq 0.$$

(iii) $\mathcal{A}(\tau, \omega)$ is pullback attracting, that is, for each $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} dist_{C_{\lambda,H}}(\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

Definition 3.6. *([13]) A multi-valued cocycle Φ is said to be* **random** *if*

$$\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H}) \ \text{ is } \ \mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(C_{\gamma,H})\text{-measurable},$$

that is, for any open set O in $C_{\gamma,H}$, the set $\{(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times C_{\gamma,H} : \ \Phi(t, \tau, \omega, x) \cap O \neq \emptyset\}$ belongs to $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(C_{\gamma,H})$.

3.2 Multivalued dynamical systems

Now, we consider the following non-autonomous random p-Laplace equation with infinite delays:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta_p u - \lambda u + f(t,x,u) + g(x,u(t-\varrho(t))) + \displaystyle\int_{-\infty}^{0} F(x,l,u(t+l))dl \\ \quad + J(t,x) + h(t,x,u)\zeta_\delta(\theta_t\omega),\ t>\tau,\quad x\in\mathcal{O}, \\ u(t,x)=0,\ t>\tau,\quad x\in\partial\mathcal{O}, \\ u(\tau+s,x)=\varphi(s,x),\quad s\in(-\infty,0],\quad x\in\mathcal{O},\tau\in\mathbb{R}. \end{cases} \tag{3.13}$$

Consider $T>\tau$ and a function $u:(-\infty,T)\to X$, we can define for any $t\in[\tau,T)$ the mapping $u_t:(-\infty,0]\to X$ by $u_t(s)=u(t+s)$ for all $s\in(-\infty,0]$.

3.2.1 Existence of solutions

In this section, we show the existence of weak solutions for system (3.13). To that end, we assume that

$$8m_1^2 < \lambda^2, \tag{3.14}$$

and

$$2M_1 \le \frac{\lambda}{8} \le \frac{\gamma}{4} \quad \text{and} \quad \vartheta := \beta_1 - qM_1 > 0, \quad \text{where} \quad M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}} > 0. \tag{3.15}$$

Definition 3.7. *Given $T>0$, a function $u(\cdot,\tau,\omega,\varphi)\in C((-\infty,\tau+T);H)\cap L^p(\tau,\tau+T;V)\cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$ is called a weak solution of* (3.13) *on $(\tau,\tau+T)$ with initial function $\varphi\in C_{\gamma,H}$, if for every $\eta\in V\cap L^q(\mathcal{O})$,*

$$\begin{aligned} &\frac{d}{dt}(u,\eta) + \langle\Delta_p u,\eta\rangle + \lambda(u,\eta) \\ &= \int_{\mathcal{O}} f(t,x,u)\eta dx + \int_{\mathcal{O}} g(x,u(t-\varrho(t)))\eta dx + \int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u(t+l))dl\Big)\eta dx \\ &+ \int_{\mathcal{O}} J(t,x)\eta dx + \zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)\eta dx, \end{aligned} \tag{3.16}$$

in the sense of distributions.

It can be inferred from Definition 3.7 and **H3-H6** that $\frac{du}{dt}\in L^{\hat{p}}(\tau,\tau+T;V^*)+L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O}))$. By [82], we know that $u\in C([\tau,\tau+T];H)$. Furthermore, for all $t\in[\tau,\tau+T]$,

$$\begin{aligned} &\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|_p^p + 2\lambda\|u\|^2 = 2\int_{\mathcal{O}} f(t,x,u)udx + 2\int_{\mathcal{O}} g(x,u(t-\varrho(t)))udx \quad (3.17) \\ &+ 2\int_{\mathcal{O}}\Big(\int_{-\infty}^{0} F(x,l,u(t+l))dl\Big)udx + 2\int_{\mathcal{O}} J(t,x)udx + 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx. \end{aligned}$$

In order to show the existence of a weak solution to system (3.13), we first need to establish a priori estimates for weak solutions to equation (3.13).

Lemma 3.8. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, *and* u *be a weak solution of system* (3.13) *with initial condition* $\varphi \in C_{\gamma,H}$. *Then there exists* $c = c(M_1, \lambda, \gamma) > 0$ *such that, for all* $t \in [\tau, \tau + T]$,

$$\|u_t\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}\|J(r,\cdot)\|^2 dr$$
$$+ c\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \tag{3.18}$$

Proof. By (3.17), **H3** and the Young inequality, we have

$$\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|_p^p + \frac{7}{4}\lambda\|u\|^2$$
$$\le -2\beta_1\|u\|_q^q + 2\|\psi_1(t,\cdot)\| + 2\int_\mathcal{O} g(x, u(t-\varrho(t)))udx$$
$$+ 2\int_\mathcal{O}\Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)udx + \frac{4}{\lambda}\|J(t,\cdot)\|^2 + 2\zeta_\delta(\theta_t\omega)\int_\mathcal{O} h(t,x,u)udx.$$

By **H5** and the Young inequality again, we deduce

$$2\zeta_\delta(\theta_t\omega)\int_\mathcal{O} h(t,x,u)udx \le 2\zeta_\delta(\theta_t\omega)\int_\mathcal{O}(\psi_4(t,x)|u|^\eta + \psi_5(t,x)|u|)dx$$
$$\le \beta_1\|u\|_q^q + c\|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + c\|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}.$$

By Remark 3.2 and the Young inequality, for the infinite delay term we derive

$$2\int_\mathcal{O}\Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)udx \le \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4m_0^2}{\lambda} + \frac{\lambda}{2}\|u\|^2. \tag{3.19}$$

Therefore, for all $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|u\|^2 + 2\|u\|_V^p + \beta_1\|u\|_q^q + \frac{5}{4}\lambda\|u\|^2$$
$$\le 2\int_\mathcal{O} g(x,u(t-\varrho(t)))udx + \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\|J(t,\cdot)\|^2$$
$$+ c(1 + \|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + \|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}). \tag{3.20}$$

Multiplying (3.20) by $e^{\lambda t}$ and integrating it over $t \in [\tau, \xi]$, we have for all $\xi \ge \tau$,

$$e^{\lambda\xi}\|u(\xi)\|^2 + 2\int_\tau^\xi e^{\lambda r}\|u(r)\|_V^p dr + \beta_1\int_\tau^\xi e^{\lambda r}\|u(r)\|_q^q dr + \frac{\lambda}{4}\int_\tau^\xi e^{\lambda r}\|u(r)\|^2 dr$$
$$\le e^{\lambda\tau}\|u(\tau)\|^2 + 2\int_\tau^\xi e^{\lambda r}\int_\mathcal{O} g(x,u(r-\varrho(r)))u(r)dr + \frac{4m_1^2}{\lambda}\int_\tau^\xi e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}}dr$$
$$+ \frac{4}{\lambda}\int_\tau^\xi e^{\lambda r}\|J(r,\cdot)\|^2 dr + c\int_\tau^\xi e^{\lambda r}(|\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-1}} + 1)dr, \tag{3.21}$$

where we used that $L^{\frac{2q-2}{q-\eta}}(\mathcal{O}) \subset L^{\frac{q}{q-\eta}}(\mathcal{O})$ and $L^2(\mathcal{O}) \subset L^{\frac{q}{q-1}}(\mathcal{O})$. For the finite delay term we obtain

$$\begin{aligned}
&2\int_\tau^\xi e^{\lambda r}\int_{\mathcal{O}} g(x,u(r-\varrho(r)))u(r)dr \\
&\le M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2dr + \frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|g(\cdot,u(r-\varrho(r)))\|^2dr, \qquad (3.22)
\end{aligned}$$

where $M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}}$ is defined in (3.15). By **H4** and **H2**, we have

$$\begin{aligned}
&\frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|g(\cdot,u(r-\varrho(r)))\|^2dr \qquad (3.23)\\
&\le \frac{\beta_3}{M_1}\int_\tau^\xi e^{\lambda r}\|u(r-\varrho(r))\|^2dr + \frac{1}{M_1}\int_\tau^\xi e^{\lambda r}\|\psi_3(\cdot)\|^2dr \\
&\le \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_{\tau-\rho}^\xi e^{\lambda r}\|u(r)\|^2dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r}dr \\
&\le \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_{-\rho}^0 e^{\lambda(r+\tau)-2\gamma r}e^{2\gamma r}\|u(\tau+r)\|^2dr \\
&+ \frac{\beta_3 e^{\lambda\rho}}{M_1(1-\rho^*)}\int_\tau^\xi e^{\lambda r}\|u(r)\|^2dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r}dr \\
&\le \frac{\beta_3 e^{\lambda\tau}e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}\|\varphi\|^2_{C_{\gamma,H}} + M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2dr + \frac{\|\psi_3(\cdot)\|^2}{M_1}\int_\tau^\xi e^{\lambda r}dr.
\end{aligned}$$

It can be inferred from (3.22) and (3.23) that

$$\begin{aligned}
&2\int_\tau^\xi e^{\lambda r}\int_{\mathcal{O}} g(x,u(r-\varrho(r)))u(r)dr \\
&\le 2M_1\int_\tau^\xi e^{\lambda r}\|u(r)\|^2dr + ce^{\lambda\tau}\|\varphi\|^2_{C_{\gamma,H}} + c\int_\tau^\xi e^{\lambda r}dr. \qquad (3.24)
\end{aligned}$$

By $\frac{q}{q-\eta} < \frac{2q-2}{q-\eta}$, $\frac{q}{q-1} < 2$ and $2M_1 \le \frac{\lambda}{8}$ defined in (3.15), plugging (3.24) into (3.21),

$$\begin{aligned}
&\|u(\xi)\|^2 + \int_\tau^\xi e^{\lambda(r-\xi)}(2\|u(r)\|_V^p + \beta_1\|u(r)\|_q^q + \frac{\lambda}{8}\|u(r)\|^2)dr \\
&\le e^{\lambda(\tau-\xi)}\|u(\tau)\|^2 + ce^{\lambda(\tau-\xi)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^\xi e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \qquad (3.25)\\
&+ \frac{4}{\lambda}\int_\tau^\xi e^{\lambda(r-\xi)}\|J(r,\cdot)\|^2dr + c\int_\tau^\xi e^{\lambda(r-\xi)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.
\end{aligned}$$

Then, multiplying (3.25) by $e^{2\gamma s}$, replacing ξ by $t+s$, and taking the supremum in $s \in [\tau-t, 0]$, we obtain that

$$\sup_{s\in[\tau-t,0]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2$$

$$\begin{aligned}
&\le \sup_{s\in[\tau-t,0]} e^{(2\gamma-\lambda)s}\Big[e^{\lambda(\tau-t)}\|u(\tau)\|^2 + ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} \\
&+ \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr + \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr \\
&+ c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^{\frac{q}{q-1}} + 1)dr\Big] \qquad (3.26) \\
&\le e^{\lambda(\tau-t)}\|u(\tau)\|^2 + ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned}$$

where we have used $\lambda \le 2\gamma$ defined in (3.15). For $s \in (-\infty, \tau - t]$, we consider

$$\begin{aligned}
\sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2 &= \sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u_\tau(t+s-\tau,\tau,\omega,\varphi)\|^2 \\
&= \sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|\varphi(t+s-\tau)\|^2 \\
&= \sup_{s\in(-\infty,\tau-t]} e^{-2\gamma(t-\tau)}e^{2\gamma(t+s-\tau)}\|\varphi(t+s-\tau)\|^2 \\
&= e^{-2\gamma(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} \le e^{-\lambda(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}}. \qquad (3.27)
\end{aligned}$$

Further

$$\begin{aligned}
\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le \max\Big\{ &\sup_{s\in(-\infty,\tau-t]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2, \\
&\sup_{s\in[\tau-t,0]} e^{2\gamma s}\|u(t+s,\tau,\omega,\varphi)\|^2\Big\}. \qquad (3.28)
\end{aligned}$$

Using the fact that $\|u(\tau)\|^2 = \|\varphi(0)\|^2 \le \|\varphi\|^2_{C_{\gamma,H}}$, we deduce from (3.26)-(3.28) that for all $t \ge \tau$,

$$\begin{aligned}
&\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{\lambda(\tau-t)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \qquad (3.29)
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&e^{\lambda t}\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{\lambda\tau}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_\tau^t e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ \frac{4}{\lambda}\int_\tau^t e^{\lambda r}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda r}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \qquad (3.30)
\end{aligned}$$

Hence, by (3.14) and using Gronwall's lemma we have

$$\|u_t(\cdot,\tau,\omega,\varphi)\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(t-\tau)}\|\varphi\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}\|J(r,\cdot)\|^2 dr$$

$$+ c\int_\tau^t e^{(\lambda-\frac{4m_1^2}{\lambda})(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \tag{3.31}$$

which completes the proof. □

Lemma 3.9. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, *and* B *be a bounded set of* $C_{\gamma,H}$. *Then, there exists* $c = c(M_1, \lambda, \gamma, B, T) > 0$ *such that any weak solution* $u(\cdot)$ *of system* (3.13) *with initial condition* $\varphi \in B$ *satisfies*

$$\|u(t,\tau,\omega,\varphi)\|^2 \le e^{-\lambda(t-r)}\|u(r)\|^2 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma$$
$$+ c\int_r^t e^{-\lambda(t-\sigma)}(|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2 + 1)d\sigma + c, \tag{3.32}$$

for all $\tau \le r \le t \le \tau + T$.

Proof. By (3.20) we have

$$\begin{aligned}&\frac{d}{dt}\|u\|^2 + 2\|u\|_V^p + \beta_1\|u\|_q^q + \frac{5}{4}\lambda\|u\|^2\\
&\le 2\int_{\mathcal{O}} g(x, u(t-\varrho(t)))udx + \frac{4m_1^2}{\lambda}\|u_t\|^2_{C_{\gamma,H}} + \frac{4}{\lambda}\|J(t,\cdot)\|^2\\
&\quad + c(1 + \|\psi_4(t,\cdot)\|^{\frac{q}{q-\eta}}_{\frac{q}{q-\eta}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-\eta}} + \|\psi_5(t,\cdot)\|^{\frac{q}{q-1}}_{\frac{q}{q-1}}|\zeta_\delta(\theta_t\omega)|^{\frac{q}{q-1}}).\end{aligned} \tag{3.33}$$

Multiplying (3.33) by $e^{\lambda\sigma}$ and integrating it for $\sigma \in [r,t]$ with $\tau \le r \le t \le \tau + T$, we deduce for all $t \ge r$,

$$\begin{aligned}&\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|_V^p + \beta_1\|u(\sigma)\|_q^q + \frac{1}{4}\lambda\|u(\sigma)\|^2)d\sigma\\
&\le e^{-\lambda(t-r)}\|u(r)\|^2 + 2\int_r^t e^{-\lambda(t-\sigma)}\int_{\mathcal{O}} g(x, u(\sigma-\varrho(\sigma)))u(\sigma)dxd\sigma\\
&\quad + \frac{4m_1^2}{\lambda}\int_r^t e^{-\lambda(t-\sigma)}\|u_\sigma\|^2_{C_{\gamma,H}}d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma\\
&\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1 + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{q}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^{\frac{q}{q-1}})d\sigma.\end{aligned} \tag{3.34}$$

Similar to (3.22)-(3.23), we have

$$\begin{aligned}&2\int_r^t e^{-\lambda(t-\sigma)}\int_{\mathcal{O}} g(x, u(\sigma-\varrho(\sigma)))u(\sigma)dxd\sigma \qquad (3.35)\\
&\le M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{1}{M_1}\int_r^t e^{-\lambda(t-\sigma)}\|g(\cdot, u(\sigma-\varrho(\sigma)))\|^2 d\sigma\\
&\le M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{\beta_3 e^{-\lambda(t-\rho)}}{M_1(1-\rho^*)}\int_{r-\rho}^t e^{\lambda\sigma}\|u(\sigma)\|^2 d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}d\sigma\\
&\le 2M_1\int_r^t e^{-\lambda(t-\sigma)}\|u(\sigma)\|^2 d\sigma + \frac{\beta_3 e^{-\lambda(t-r)}e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}\|u_r\|^2_{C_{\gamma,H}} + c\int_r^t e^{-\lambda(t-\sigma)}d\sigma.\end{aligned}$$

By $2M_1 \leq \frac{\lambda}{8}$ defined in (3.15), we obtain

$$\begin{aligned} &\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|_V^p + \beta_1\|u(\sigma)\|_q^q + \frac{1}{8}\lambda\|u(\sigma)\|^2)d\sigma \\ &\leq e^{-\lambda(t-r)}\|u(r)\|^2 + \frac{\beta_3 e^{2\gamma\rho}}{M_1(1-\rho^*)(2\gamma-\lambda)}e^{-\lambda(t-r)}\|u_r\|^2_{C_{\gamma,H}} \\ &\quad + \frac{4m_1^2}{\lambda}\int_r^t e^{-\lambda(t-\sigma)}\|u_\sigma\|^2_{C_{\gamma,H}}d\sigma + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\ &\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1+|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma. \end{aligned} \tag{3.36}$$

By (3.5), **H1** and $\varphi \in B$, we can infer from (3.18) in Lemma 3.8 that $\|u_t\|^2_{C_{\gamma,H}} \leq C_1 = C_1(B,T)$ for $t \in [\tau, \tau+T]$, and we then have

$$\begin{aligned} &\|u(t)\|^2 + \int_r^t e^{-\lambda(t-\sigma)}(2\|u(\sigma)\|_V^p + \beta_1\|u(\sigma)\|_q^q + \frac{1}{8}\lambda\|u(\sigma)\|^2)d\sigma \\ &\leq e^{-\lambda(t-r)}\|u(r)\|^2 + cC_1 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\ &\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1+|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma. \end{aligned} \tag{3.37}$$

Then we conclude (3.32). □

We can draw the following results immediately from (3.37):

Corollary 3.10. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Let $\tau \in \mathbb{R}, \omega \in \Omega, T > 0$, and let $B \subseteq C_{\gamma,H}$ be a bounded set. Then, there exists $c = c(M_1, \lambda, \gamma, B, T) > 0$ such that any weak solution $u(\cdot)$ of system* (3.13) *with initial condition $\varphi \in B$ satisfies, for all $\tau \leq r \leq t \leq \tau + T$,*

$$\begin{aligned} &\int_r^t (\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \\ &\leq c\|u(r)\|^2 + c\int_r^t \|J(\sigma,\cdot)\|^2 d\sigma + c\int_r^t (1+|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma + c. \end{aligned} \tag{3.38}$$

Proof. Consider $C_2 = \min\{2, \beta_1, \frac{1}{8}\lambda\}$. By (3.37) we have, for all $\tau \leq r \leq t \leq \tau+T$,

$$\begin{aligned} &e^{-\lambda(t-\tau)}C_2\int_r^t (\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \\ &\leq C_2\int_r^t e^{-\lambda(t-\sigma)}(\|u(\sigma)\|_V^p + \|u(\sigma)\|_q^q + \|u(\sigma)\|^2)d\sigma \\ &\leq e^{-\lambda(t-r)}\|u(r)\|^2 + c\int_r^t e^{-\lambda(t-\sigma)}\|J(\sigma,\cdot)\|^2 d\sigma \\ &\quad + c\int_r^t e^{-\lambda(t-\sigma)}(1+|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_\sigma\omega)|^2)d\sigma + c, \end{aligned}$$

which implies (3.38). □

Theorem 3.11. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold and* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. *Then, equation* (3.13) *admits at least one weak solution.*

Proof. (1) First, we consider the Galerkin approximations to equation (3.13). It follows from [64] that $H_0^r(\mathcal{O}) \subset V \cap L^q(\mathcal{O})$ for $r \geq \max\{\frac{N(q-2)}{2q}, \frac{2p+N(p-2)}{2p}\}$. We consider a special basis of H consisting of elements $\{w_j\} \subset H_0^r(\mathcal{O})$, and denote by $W_n = span[w_1, \dots, w_n]$. Let the projector $P_n u = \sum_{j=1}^{n}(u, w_j)w_j$, then $\bigcup_{n\in\mathbb{N}} W_n$ is dense in $V \cap L^q(\mathcal{O})$.

For fixed $n \in \mathbb{N}$, consider $\widetilde{u}^n(t) = \sum_{j=1}^{n} \widetilde{\mu}_j^n(t)w_j$, where $\widetilde{\mu}_j^n$ are required to satisfy the following system:

$$\begin{aligned}
&\frac{d}{dt}(\widetilde{u}^n(t), w_j) + \langle \Delta_p \widetilde{u}^n(t), w_j\rangle + \lambda(\widetilde{u}^n(t), w_j) \\
&= (f(t,x,\widetilde{u}^n(t)), w_j) + (g(x, \widetilde{u}_t^n(-\varrho(t))), w_j) + \Big(\int_{-\infty}^{0} F(x,l,\widetilde{u}_t^n(l))dl, w_j\Big) \\
&\quad + (J(t,\cdot), w_j) + \zeta_\delta(\theta_t\omega)(h(t,x,\widetilde{u}^n(t)), w_j), \quad 1 \leq j \leq n, \qquad (3.39)
\end{aligned}$$

where initial value is $\widetilde{u}^n(\tau + s) = P_n\varphi(s)$ for $s \in (-\infty, 0]$. It follows from [46, Theorem 1.1] the existence of local solutions for (3.39). Now, we show that solutions do exist in $[\tau, \tau + T]$ with $T > 0$.

(2) From (3.32) and setting $r = \tau$ in (3.38), we obtain for all $T > 0$,

$$\{\widetilde{u}^n\} \text{ is bounded in } L^\infty(\tau, \tau+T; H) \cap L^p(\tau,\tau+T;V) \cap L^q(\tau,\tau+T;L^q(\mathcal{O})).$$

By (3.9), the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}
&\int_\tau^t \int_{\mathcal{O}} \zeta_\delta(\theta_r\omega)h(t,x,\widetilde{u}^n(r))w dxdr \\
&\leq \int_\tau^t \Big(\int_{\mathcal{O}} \Big|\zeta_\delta(\theta_r\omega)\psi_4(r,x)|\widetilde{u}^n(r)|^{\eta-1} + \zeta_\delta(\theta_r\omega)\psi_5(r,x)\Big|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}} dr \int_\tau^t \|w\|_q dr \\
&\leq 2^{\frac{1}{q}} \int_\tau^t \Big[\Big(\int_{\mathcal{O}} \big|\zeta_\delta(\theta_r\omega)\psi_4(r,x)\big|^{\frac{q}{q-1}} \big|\widetilde{u}^n(r)\big|^{\frac{q(\eta-1)}{q-1}} dx\Big)^{\frac{q-1}{q}} \\
&\quad + \Big(\int_{\mathcal{O}} \big|\zeta_\delta(\theta_r\omega)\psi_5(r,x)\big|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}\Big]dr \int_\tau^t \|w\|_q dr \\
&\leq c\int_\tau^t \Big[\Big(\Big(\int_{\mathcal{O}}|\zeta_\delta(\theta_r\omega)\psi_4(r,x)|^{\frac{q}{q-\eta}}dx\Big)^{\frac{q-\eta}{q-1}}\Big(\int_{\mathcal{O}}|\widetilde{u}^n(r)|^q dx\Big)^{\frac{\eta-1}{q-1}}\Big)^{\frac{q-1}{q}} \\
&\quad + \Big(\int_{\mathcal{O}}|\zeta_\delta(\theta_r\omega)\psi_5(r,x)|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}\Big]dr \int_\tau^t \|w\|_q dr \\
&\leq c\Big(\int_\tau^t \|\psi_4(r,\cdot)\|_{\frac{2q-2}{q-\eta}}dr + \int_\tau^t \|\widetilde{u}^n\|_q^{\eta-1}dr + \int_\tau^t \|\psi_5(r,\cdot)\|_2 dr\Big)\int_\tau^t \|w\|_q dr,
\end{aligned}$$

which together with (3.7) yields

$$\{f(t,x,\widetilde{u}^n)\} \text{ and } \{\zeta_\delta(\theta_t\omega)h(t,x,\widetilde{u}^n)\} \text{ are bounded in } L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O})).$$

Recall from Lemma 3.9 that

$$\|\widetilde{u}_t^n\|_{C_{\gamma,H}}^2 \leq C_1, \quad \forall t \in [\tau, \tau+T], \quad \varphi \in B \subset C_{\gamma,H}, \quad n \in \mathbb{N}. \tag{3.40}$$

Thus, by (3.40), Remark 3.1 and Remark 3.2, we have $\{g(x, \widetilde{u}_\cdot^n)\}$ is bounded in $L^2(\tau, \tau+T; H)$, and $\{\int_{-\infty}^0 F(x, l, \widetilde{u}_\cdot^n(l))dl\}$ is bounded in $L^2(\tau, \tau+T; H)$. Moreover, we can deduce that $\{\Delta_p \widetilde{u}^n\}$ is bounded in $L^{\hat{p}}(\tau, \tau+T; V^*)$ from (3.5). From the above, we know that $\{\frac{d\widetilde{u}^n}{dt}\}$ is bounded in $L^{\hat{p}}(\tau, \tau+T; H^{-r}(\mathcal{O}))$ by [21].

Hence, there exist a subsequence (relabeled the same) $\{\widetilde{u}^n\}$, an element $\widetilde{u} \in L^\infty(\tau, \tau+T; H) \cap L^p(\tau, \tau+T; V) \cap L^q(\tau, \tau+T; L^q(\mathcal{O}))$ with $\frac{d\widetilde{u}}{dt} \in L^{\hat{p}}(\tau, \tau+T; H^{-r}(\mathcal{O}))$, $\chi_1 \in L^{\hat{p}}(\tau, \tau+T; V^*)$, $\chi_2 \in L^2(\tau, \tau+T; H)$, $\chi_3 \in L^2(\tau, \tau+T; H)$, $\chi_4 \in L^{\frac{q}{q-1}}(\tau, \tau+T; L^{\frac{q}{q-1}}(\mathcal{O}))$, $\chi_5 \in L^{\frac{q}{q-1}}(\tau, \tau+T; L^{\frac{q}{q-1}}(\mathcal{O}))$ such that, up to a subsequence,

$$\begin{cases} \widetilde{u}^n \to \widetilde{u} \ \text{ weakly star in } \ L^\infty(\tau, \tau+T; H), \\ \widetilde{u}^n \to \widetilde{u} \ \text{ weakly in } \ L^p(\tau, \tau+T; V) \ \text{ and } \ L^q(\tau, \tau+T; L^q(\mathcal{O})), \\ \widetilde{u}^n \to \widetilde{u} \ \text{ strongly in } \ L^p(\tau, \tau+T; L^p(\mathcal{O})), \\ \Delta_p \widetilde{u}^n \to \chi_1 \ \text{ weakly in } \ L^{\hat{p}}(\tau, \tau+T; V^*), \\ g(x, \widetilde{u}_\cdot^n) \to \chi_2 \ \text{ weakly in } \ L^2(\tau, \tau+T; H), \\ \displaystyle\int_{-\infty}^0 F(x, l, \widetilde{u}_\cdot^n(l))dl \to \chi_3 \ \text{ weakly in } \ L^2(\tau, \tau+T; H), \\ \dfrac{d\widetilde{u}^n}{dt} \to \dfrac{d\widetilde{u}}{dt} \ \text{ weakly in } \ L^{\hat{p}}(\tau, \tau+T; H^{-r}(\mathcal{O})), \\ f(\cdot, x, \widetilde{u}^n) \to \chi_4 \ \text{ weakly in } \ L^{\frac{q}{q-1}}(\tau, \tau+T; L^{\frac{q}{q-1}}(\mathcal{O})), \\ \zeta_\delta(\theta.\omega) h(\cdot, x, \widetilde{u}^n) \to \chi_5 \ \text{ weakly in } \ L^{\frac{q}{q-1}}(\tau, \tau+T; L^{\frac{q}{q-1}}(\mathcal{O})), \end{cases} \tag{3.41}$$

for all $T > 0$. From [64, Lemma 1.3], we can identify that $\chi_1 = \Delta_p \widetilde{u}$, $\chi_4 = f(\cdot, x, \widetilde{u})$ and $\chi_5 = \zeta_\delta(\theta.\omega) h(\cdot, x, \widetilde{u})$.

By the compact embedding $H \hookrightarrow H^{-r}(\mathcal{O})$ and (3.40), we can infer from the Arzelà-Ascoli theorem that $\widetilde{u}^n \to \widetilde{u}$ in $C([\tau, \tau+T]; H^{-r}(\mathcal{O}))$. Then, by (3.40) again, it is not difficult to prove that for any sequence $t_n \to t_0$ with $t_n, t_0 \in [\tau, \tau+T]$,

$$\widetilde{u}^n(t_n) \to \widetilde{u}(t_0) \ \text{ weakly in } \ H. \tag{3.42}$$

In fact, we want to show that

$$\widetilde{u}^n(\cdot) \to \widetilde{u}(\cdot) \quad \text{in} \quad C([\tau, \tau+T]; H). \tag{3.43}$$

By (3.41), passing to the limit in (3.39), we consider a solution $\widetilde{u} \in C([\tau, \tau+T]; H)$ of a similar problem to (3.13), that is, for all $\eta \in V \cap L^q(\mathcal{O})$

$$\begin{aligned} &\frac{d}{dt}(\widetilde{u}, \eta) + \langle \Delta_p \widetilde{u}, \eta \rangle + \lambda(u, \eta) = (f(t, \cdot, \widetilde{u}), \eta) + (\chi_2, \eta) + (\chi_3, \eta) \\ &\quad + (J(t, \cdot), \eta) + (\zeta_\delta(\theta_t \omega) h(t, \cdot, \widetilde{u}), \eta), \end{aligned} \tag{3.44}$$

with the initial data $\widetilde{u}(\tau+s)=\varphi(s)$ for $s\in(-\infty,0]$. By (3.40), Remark 3.1 and Remark 3.2, for all $\tau\le r\le t\le\tau+T$,

$$\int_r^t \|\chi_2(\sigma)\|^2 d\sigma \le \liminf_{n\to+\infty}\int_r^t \|g(\sigma,\widetilde{u}^n_\sigma)\|^2 d\sigma \le c(t-r),$$

and

$$\int_r^t \|\chi_3(\sigma)\|^2 d\sigma \le \liminf_{n\to+\infty}\int_r^t \left\|\int_{-\infty}^0 F(\sigma,l,\widetilde{u}^n_\sigma(l))dl\right\|^2 d\sigma \le c(t-r).$$

Therefore, by the same method as Lemma 3.9, $\widetilde{u}$ can also satisfies (3.32). We define functions $J_n, J:[\tau,\tau+t]\to\mathbb{R}$ by

$$J_n(t)=\|\widetilde{u}^n(t)\|^2-c\int_\tau^t\|J(\sigma,\cdot)\|^2d\sigma-c\int_\tau^t(|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_\sigma\omega)|^2+1)d\sigma, \tag{3.45}$$

$$J(t)=\|\widetilde{u}(t)\|^2-c\int_\tau^t\|J(\sigma,\cdot)\|^2d\sigma-c\int_\tau^t(|\zeta_\delta(\theta_\sigma\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_\sigma\omega)|^2+1)d\sigma. \tag{3.46}$$

where the c in (3.45) is the same as (3.46). It is clear that J_n and J are non-increasing and continuous functions. By (3.41) and [15, Lemma 11], we can deduce that

$$J_n(t)\to J(t),\quad \text{for a.e. } t\in[\tau,\tau+T]. \tag{3.47}$$

Then, we have

$$\limsup_{n\to+\infty}\|\widetilde{u}^n(t_n)\|\le\|\widetilde{u}(t_0)\|, \tag{3.48}$$

which together with (3.42) implies (3.43). By Remark 3.1 and (3.43), we can deduce that $\chi_2=g(x,\widetilde{u}_\cdot)$.

By [70, Theorem 5], for the initial datum $\varphi\in C_{\gamma,H}$, we know that $P_n\varphi\to\varphi$ in $C_{\gamma,H}$. Indeed,

$$\begin{aligned}
&\sup_{s\le 0} e^{\gamma s}\|\widetilde{u}^n(t+s)-\widetilde{u}(t+s)\|\\
&\le\max\Big\{\sup_{s\in[\tau-t,0]} e^{\gamma s}\|\widetilde{u}^n(t+s)-\widetilde{u}(t+s)\|,\\
&\qquad\sup_{s\in(-\infty,\tau-t]} e^{\gamma(\tau-t)}e^{\gamma(s+t-\tau)}\|\widetilde{u}^n_\tau(t+s-\tau)-\widetilde{u}_\tau(t+s-\tau)\|\Big\}\\
&\le\max\big\{\sup_{s\in[\tau,t]}\|\widetilde{u}^n(s)-\widetilde{u}(s)\|, e^{\gamma(\tau-t)}\|P_n\varphi-\varphi\|_{C_{\gamma,H}}\big\}\to 0,
\end{aligned}$$

which implies that for all $t\in[\tau,\tau+T]$

$$\widetilde{u}^n_t\to\widetilde{u}_t\qquad\text{in } C_{\gamma,H}. \tag{3.49}$$

Therefore, by Remark 3.3, we deduce $\chi_3=\int_{-\infty}^0 F(x,l,\widetilde{u}_\cdot(l))dl$. Finally, we can pass to the limit in (3.39), concluding that $\widetilde{u}$ is a solution of (3.13). □

3.2.2 Regularity of solutions

In order to show a regularity result for the solution of equation (3.13), we need the next lemma. In the proof of the latter, we will use the following Gronwall-type lemma for the estimate of the solutions in the regular space.

Lemma 3.12. *([104]) Let y, g and h be three nonnegative and locally integrable functions on $\mathbb{R}$, thus $\frac{dy}{dt}$ is also locally integrable and*

$$\frac{dy}{dt} + by(t) + g(t) \leq h(t), t \in \mathbb{R}. \tag{3.50}$$

Then, for every $t > \tau$ with $\tau \in \mathbb{R}$, one has

$$y(t) \leq \frac{1}{t-\tau}\int_\tau^t y(r)e^{b(r-t)}dr + \int_\tau^t h(r)e^{b(r-t)}dr. \tag{3.51}$$

In particular, if $b = 0$ then

$$y(t) \leq \frac{1}{t-\tau}\int_\tau^t y(r)dr + \int_\tau^t h(r)dr. \tag{3.52}$$

Lemma 3.13. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$, and B be a bounded set of $C_{\gamma,H}$. Then there exists $c = c(M_1, \lambda, \gamma, B, T) > 0$ such that*

(1) A weak solution $u(\cdot)$ of system (3.13) *with initial condition $\varphi \in B$ satisfies*

$$\int_\tau^t \|u(r)\|_{2q-2}^{2q-2}dr \leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c, \tag{3.53}$$

for all $t \in (\tau, \tau + T]$.

(2) A weak solution $u(\cdot)$ of system (3.13) *with initial condition $\varphi \in B$ and $\varphi(0) \in V \cap L^q(\mathcal{O})$ satisfies*

$$\int_\tau^t \|u(r)\|_{2q-2}^{2q-2}dr \leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c, \tag{3.54}$$

for all $t \in [\tau, \tau + T]$.

Proof. Multiplying (3.13) by $|u|^{q-2}u$, and integrating over $\mathcal{O}$, we have

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{q}\|u\|_q^q) + \int_\mathcal{O} \Delta_p u(|u|^{q-2}u)dx + \lambda\|u\|_q^q \\
&= \int_\mathcal{O} f(t,x,u)(|u|^{q-2}u)dx + \int_\mathcal{O} g(x, u(t-\varrho(t)))(|u|^{q-2}u)dx \\
&+ \int_\mathcal{O}\big(\int_{-\infty}^0 F(x,l,u(t+l))dl\big)(|u|^{q-2}u)dx + \int_\mathcal{O} J(t,x)(|u|^{q-2}u)dx \\
&+ \zeta_\delta(\theta_t\omega)\int_\mathcal{O} h(t,x,u)(|u|^{q-2}u)dx.
\end{aligned} \tag{3.55}$$

It is easy to check that for any $q > 2$,

$$\int_{\mathcal{O}} \Delta_p u(|u|^{q-2}u)dx \geq 0. \tag{3.56}$$

By (3.6) and the Young inequality, we deduce that

$$\begin{aligned}\int_{\mathcal{O}} f(t,x,u)(|u|^{q-2}u)dx &\leq \int_{\mathcal{O}} [-\beta_1(|u|^q + |u|^p) + \psi_1(t,x)]|u|^{q-2}dx \\ &\leq -\beta_1\|u\|_{2q-2}^{2q-2} - \beta_1\|u\|_{p+q-2}^{p+q-2} + \int_{\mathcal{O}} \psi_1(t,x)|u|^{q-2}dx \\ &\leq -\beta_1\|u\|_{2q-2}^{2q-2} + \frac{\lambda}{2}\|u\|_q^q + \frac{1}{2\lambda}\|\psi_1(t,\cdot)\|_{\frac{q}{2}}^{\frac{q}{2}}. \end{aligned}\tag{3.57}$$

By Remark 3.2 and the Young inequality we have

$$\begin{aligned}&\int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u(t+l))dl\Big)(|u|^{q-2}u)dx \\ &\leq \frac{2m_1^2}{\beta_1}\|u_t\|_{C_{\gamma,H}}^2 + \frac{2m_0^2}{\beta_1} + \frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2}. \end{aligned}\tag{3.58}$$

By the Young inequality again,

$$\int_{\mathcal{O}} J(t,x)(|u|^{q-2}u)dx \leq \frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2} + \frac{1}{\beta_1}\|J(t,\cdot)\|^2. \tag{3.59}$$

Jointly with **H5**, we have

$$\begin{aligned}&\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)(|u|^{q-2}u)dx \\ &\leq \zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} (\psi_4(t,x)|u|^{\eta-1} + \psi_5(t,x))(|u|^{q-2}u)dx \\ &\leq \frac{\beta_1}{4}\|u\|_{2q-2}^{2q-2} + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2. \end{aligned}\tag{3.60}$$

For $q > 2$, we substitute (3.56)-(3.60) into (3.55) to yield

$$\begin{aligned}&\frac{d}{dt}\|u\|_q^q + 2\lambda\|u\|_q^q + \frac{\beta_1}{2}\|u\|_{2q-2}^{2q-2} \\ &\leq q\int_{\mathcal{O}} g(x,u(t-\varrho(t)))(|u|^{q-2}u)dx + \frac{q}{2\lambda}\|\psi_1(t,\cdot)\|_{\frac{q}{2}}^{\frac{q}{2}} + \frac{2qm_1^2}{\beta_1}\|u_t\|_{C_{\gamma,H}}^2 + \frac{2qm_0^2}{\beta_1} \\ &+ \frac{q}{\beta_1}\|J(t,\cdot)\|^2 + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2. \end{aligned}\tag{3.61}$$

Then, using (3.51) in Lemma 3.12 over the interval $[\tau, t]$, we have

$$\|u(t)\|_q^q + \frac{\beta_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_q^q dr + q\int_\tau^t e^{\lambda(r-t)}\int_{\mathcal{O}} g(x,u(r-\varrho(r)))(|u(r)|^{q-2}u(r))dxdr\\
&+ c\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr + c\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr\\
&+ c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \qquad (3.62)
\end{aligned}
$$

where $\varepsilon \in (0, t-\tau)$. By the same method as (3.22)-(3.23), for all $\tau < t \leq \tau + T$, we derive

$$
\begin{aligned}
&q\int_\tau^t e^{\lambda(r-t)}\int_{\mathcal{O}} g(x,u(r-\varrho(r)))(|u(r)|^{q-2}u(r))dxdr\\
&\leq \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^{2q-2}_{2q-2}dr + \frac{q}{2M_1}\int_\tau^t e^{\lambda(r-t)}\|g(x,u(r-\varrho(r)))\|^2 dr\\
&\leq \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^{2q-2}_{2q-2}dr + \frac{q\beta_3 e^{\lambda\rho}}{2M_1(1-\rho^*)}\int_{\tau-\rho}^\tau e^{\lambda(r-t)}\|u(r)\|^2 dr\\
&+ \frac{q\beta_3 e^{\lambda\rho}}{2M_1(1-\rho^*)}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}dr\\
&\leq \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^{2q-2}_{2q-2}dr + \frac{qM_1 e^{2\gamma\rho}}{2\gamma-\lambda}e^{-\lambda(t-\tau)}\|u_\tau\|^2_{C_{\gamma,H}}\\
&+ \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}dr, \qquad (3.63)
\end{aligned}
$$

where $M_1 = \frac{\sqrt{\beta_3 e^{\lambda\rho}}}{\sqrt{1-\rho^*}}$. By (3.63) and let $r=\tau$ in (3.36), we have for all $t \in (\tau, \tau+T]$,

$$
\begin{aligned}
&\|u(t)\|_q^q + \frac{\vartheta}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^{2q-2}_{2q-2}dr\\
&\leq \frac{1}{\varepsilon}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_q^q dr + \frac{qM_1}{2}\int_\tau^t e^{\lambda(r-t)}\|u(r)\|^2 dr\\
&+ ce^{-\lambda(t-\tau)}\|u_\tau\|^2_{C_{\gamma,H}} + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr + c\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr\\
&+ c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr\\
&\leq ce^{-\lambda(t-\tau)}\|u_\tau\|^2_{C_{\gamma,H}} + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr\\
&+ c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr, \qquad (3.64)
\end{aligned}
$$

where $\vartheta = \beta_1 - qM_1 > 0$ is defined in (3.15). Since $\varphi \in B$, thus $\|u_r\|^2_{C_{\gamma,H}} \leq C_1 = C_1(B,T)$ for $r \in [\tau, \tau+T]$ as proved in Lemma 3.9. Thus for all $t \in (\tau, \tau+T]$,

$$
\int_\tau^t \|u(r)\|^{2q-2}_{2q-2}dr \leq cC_1 + c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \qquad (3.65)
$$

(2) Applying the general Gronwall inequality to (3.61), we have for all $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\|u(t)\|_q^q + \frac{\beta_1}{2} \int_\tau^t e^{\lambda(r-t)} \|u(r)\|_{2q-2}^{2q-2} dr \\ &\leq e^{-\lambda(t-\tau)} \|u_\tau\|_q^q dr + q \int_\tau^t e^{\lambda(r-t)} \int_{\mathcal{O}} g(x, u(r - \varrho(r)))(|u(r)|^{q-2} u(r)) dx dr \\ &+ c \int_\tau^t e^{\lambda(r-t)} \|u_r\|_{C_{\gamma,H}}^2 dr + c \int_\tau^t e^{\lambda(r-t)} \|J(r, \cdot)\|^2 dr \\ &+ c \int_\tau^t e^{\lambda(r-t)} (|\zeta_\delta(\theta_r \omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r \omega)|^2 + 1) dr, \end{aligned} \tag{3.66}$$

By (3.63) and similar to (3.64), we have

$$\begin{aligned} &\|u(t)\|_q^q + \frac{\vartheta}{2} \int_\tau^t e^{\lambda(r-t)} \|u(r)\|_{2q-2}^{2q-2} dr \\ &\leq e^{-\lambda(t-\tau)} \|u_\tau\|_q^q dr + c e^{-\lambda(t-\tau)} \|u_\tau\|_{C_{\gamma,H}}^2 + c \int_\tau^t e^{\lambda(r-t)} \|u_r\|_{C_{\gamma,H}}^2 dr \\ &+ c \int_\tau^t e^{\lambda(r-t)} (\|J(r, \cdot)\|^2 + |\zeta_\delta(\theta_r \omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r \omega)|^2 + 1) dr. \end{aligned} \tag{3.67}$$

On account of $\varphi \in B$ and $\varphi(0) \in V \cap L^q(\mathcal{O})$, it follows that for all $t \in [\tau, \tau + T]$, (3.54) holds. □

Now, we can show a regularity result for the solution of equation (3.13).

Theorem 3.14. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold and* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varphi \in C_{\gamma,H}$. *Then any weak solutions u to equation* (3.13) *belongs to* $C_w((\tau, \tau + T]; V)$. *In particular, if* $\varphi(0) \in V \cap L^q(\mathcal{O})$, *then* $u \in C_w([\tau, \tau + T]; V)$.

Proof. Given $T > 0$, let $u(\cdot, \tau, \omega, \varphi)$ be a weak solution of equation (3.13), for short denoted by u. Consider that problem

$$(P_u) \begin{cases} \dfrac{dy}{dt} = -\Delta_p y - \lambda y + f(t, x, u) + g(x, u(t - \varrho(t))) \\ \quad + \displaystyle\int_{-\infty}^0 F(x, l, u(t + l)) dl + J(t, x) + h(t, x, u) \zeta_\delta(\theta_t \omega), \quad x \in \mathcal{O}, \\ y(t, x) = 0, \ t > \tau, \quad x \in \partial \mathcal{O}, \\ y(\tau + s, x) = u(\tau + s, x) = \varphi(s, x), \quad s \in (-\infty, 0], \quad x \in \mathcal{O}, \tau \in \mathbb{R}, \end{cases} \tag{3.68}$$

possesses a local solution by [77]. Now, we will show the local solution is a global solution.

Recall from Lemma 3.9 that

$$\|u_t\|_{C_{\gamma,H}}^2 \leq C_1, \quad \forall t \in [\tau, \tau + T], \quad \varphi \in B \subset C_{\gamma,H}. \tag{3.69}$$

For fixed $n \in \mathbb{N}$, consider $\widehat{u}^n(t) = \sum_{j=1}^{n} \widehat{\mu}_j^n(t) w_j$, where $\widehat{\mu}_j^n$ are required to satisfy the following system:

$$\begin{aligned} &\frac{d}{dt}(\widehat{u}^n(t), w_j) + \langle \Delta_p \widehat{u}^n(t), w_j\rangle + \lambda(\widehat{u}^n(t), w_j) = \int_{\mathcal{O}} f(t,x,u(t)) w_j dx \\ &+ \int_{\mathcal{O}} g(x, u_t(-\varrho(t))) w_j dx + \int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u_t(l)) dl\Big) w_j dx \\ &+ \int_{\mathcal{O}} J(t,\cdot) w_j dx + \zeta_\delta(\theta_t\omega) \int_{\mathcal{O}} h(t,x,u(t)) w_j dx, \quad 1 \le j \le n. \end{aligned} \tag{3.70}$$

Multiplying (3.70) by $\widehat{\mu}_j^n(t)$, summing from $j = 1$ until n we have

$$\begin{aligned} &\frac{d}{dt}\|\widehat{u}^n(t)\|^2 + 2\|\nabla \widehat{u}^n(t)\|_p^p + 2\lambda \|\widehat{u}^n(t)\|^2 \\ &= 2\int_{\mathcal{O}} f(t,x,u(t)) \widehat{u}^n(t) dx + 2\int_{\mathcal{O}} g(x, u(t-\varrho(t))) \widehat{u}^n(t) dx \\ &+ 2\int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u_t(l)) dl\Big) \widehat{u}^n(t) dx + 2\int_{\mathcal{O}} J(t,x) \widehat{u}^n(t) dx \\ &+ 2\zeta_\delta(\theta_t\omega) \int_{\mathcal{O}} h(t,x,u(t)) \widehat{u}^n(t) dx. \end{aligned} \tag{3.71}$$

By (3.7) and the Young inequality, we deduce

$$2\int_{\mathcal{O}} f(t,x,u) \widehat{u}^n(t) dx \le \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + \frac{16\beta_2^2}{\lambda}\|u\|_{2q-2}^{2q-2} + \frac{16}{\lambda}\|\psi_2(t,\cdot)\|^2. \tag{3.72}$$

By Remark 3.2, we obtain that

$$2\int_{\mathcal{O}} \Big(\int_{-\infty}^{0} F(x,l,u(t+l)) dl\Big) \widehat{u}^n(t) dx \le \frac{16 m_1^2}{\lambda}\|u_t\|_{C_{\gamma,H}}^2 + \frac{16 m_0^2}{\lambda} + \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2. \tag{3.73}$$

By Remark 3.1, we have

$$2(g(x, u(t-\varrho(t))), \widehat{u}^n(t)) \le \frac{8\beta_3}{\lambda} e^{2\gamma\rho}\|u_t\|_{C_{\gamma,H}}^2 + \frac{8}{\lambda}\|\psi_3(\cdot)\|^2 + \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2. \tag{3.74}$$

By the Young inequality again,

$$2\int_{\mathcal{O}} J(t,x) \widehat{u}^n(t) dx \le \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + \frac{8}{\lambda}\|J(t,\cdot)\|^2. \tag{3.75}$$

Jointly with **H5**, we derive

$$\begin{aligned} &2\zeta_\delta(\theta_t\omega) \int_{\mathcal{O}} h(t,x,u) \widehat{u}^n(t) dx \\ &\le 2\zeta_\delta(\theta_t\omega) \int_{\mathcal{O}} (\psi_4(t,x)|u|^{\eta-1} + \psi_5(t,x)) \widehat{u}^n(t) dx \end{aligned} \tag{3.76}$$

$$\leq \frac{\lambda}{8}\|\widehat{u}^n(t)\|^2 + c\|u\|_{2q-2}^{2q-2} + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{q-1}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.$$

Substituting (3.72)-(3.76) into (3.71),

$$\begin{aligned}&\frac{d}{dt}\|\widehat{u}^n\|^2 + 2\|\nabla\widehat{u}^n(t)\|_p^p + \frac{11}{8}\lambda\|\widehat{u}^n\|^2 \\ &\leq c\|u\|_{2q-2}^{2q-2} + \frac{16}{\lambda}\|\psi_2(t,\cdot)\|^2 + (\frac{16m_1^2}{\lambda} + \frac{8\beta_3}{\lambda}e^{2\gamma\rho})\|u_t\|_{C_{\gamma,H}}^2 + c \\ &+ \frac{8}{\lambda}\|J(t,\cdot)\|^2 + c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{q-1}{q-\eta}}^{\frac{2q-2}{q-\eta}} + c|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.\end{aligned} \tag{3.77}$$

Multiplying (3.77) by $e^{\lambda t}$, and integrating over (τ, t) with $t \in [\tau, \tau+T]$,

$$\begin{aligned}&\|\widehat{u}^n(t)\|^2 + 2\int_\tau^t e^{\lambda(r-t)}\|\nabla\widehat{u}^n(r)\|_p^p dr + \frac{3}{8}\lambda\int_\tau^t e^{\lambda(r-t)}\|\widehat{u}^n(r)\|^2 dr \\ &\leq e^{-\lambda(t-\tau)}\|\varphi\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\ &+ \frac{8}{\lambda}\int_\tau^t e^{\lambda(r-t)}\|J(r,\cdot)\|^2 dr + c\int_\tau^t e^{\lambda(r-t)}(|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr.\end{aligned} \tag{3.78}$$

Using the same method as (3.26)-(3.28), and by (3.64) and (3.69) we have for all $t \in (\tau, \tau+T]$,

$$\begin{aligned}\|\widehat{u}_t^n\|_{C_{\gamma,H}}^2 &\leq ce^{-\lambda(t-\tau)}\|\varphi\|_{C_{\gamma,H}}^2 + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|_{C_{\gamma,H}}^2 dr \\ &\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr \\ &\leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c.\end{aligned} \tag{3.79}$$

Hence, we deduce the existence of global solution to equation (P_u) on $t \in [\tau+\varepsilon, +\infty)$ with $\varepsilon \in (0, t-\tau)$. Analogously, by (3.67), (3.69) and $\varphi(0) \in V \cap L^q(\mathcal{O})$, it follows the existence of the global solution to equation (P_u) on $t \in [\tau, +\infty)$.

Now we prove the uniqueness of the solution for equation (P_u). Let y_1, y_2 be two solutions of (3.68). Taking the inner product of (3.68) with $\widehat{u}^n = y_1 - y_2$, we have

$$\frac{d}{dt}\|\widehat{u}^n\|^2 + 2\langle\Delta_p y_1 - \Delta_p y_2, \widehat{u}^n\rangle + 2\lambda\|\widehat{u}^n\|^2 = 0. \tag{3.80}$$

By [32, Lemma 2.1], we know the following inequality: for every $p \geq 2$, there exists $c > 0$ such that, for all $a_1, a_2 \in \mathbb{R}$,

$$(|a_1|^{p-2}a_1 - |a_2|^{p-2}a_2)(a_1 - a_2) \geq c|a_1 - a_2|^p. \tag{3.81}$$

The above fact yields

$$\langle\Delta_p y_1 - \Delta_p y_2, \widehat{u}^n\rangle = (|\nabla y_1|^{p-2}\nabla y_1 - |\nabla y_2|^{p-2}\nabla y_2, \nabla y_1 - \nabla y_2) \geq c\|\nabla\widehat{u}^n\|_p^p. \tag{3.82}$$

Thus

$$\frac{d}{dt}\|\widehat{u}^n\|^2 + 2c\|\nabla\widehat{u}^n\|_p^p + 2\lambda\|\widehat{u}^n\|^2 \leq 0. \tag{3.83}$$

Multiplying (3.83) by $e^{\lambda t}$, integrating over $t \in [\tau, \xi]$, we have for all $\xi \in [\tau, \tau + T]$,

$$\|\widehat{u}^n(\xi, \tau, \omega, \varphi)\|^2 \leq e^{-\lambda(\xi-\tau)}\|\widehat{u}^n(\tau, \tau, \omega, \varphi)\|^2. \tag{3.84}$$

Let $\xi = t + s$ with $s \leq 0$, then using the same method as (3.26)-(3.28), we have

$$\|\widehat{u}_t^n(\cdot, \tau, \omega, \varphi)\|^2_{C_{\gamma,H}} \leq e^{-\lambda(t-\tau)}\|\widehat{u}_\tau^n\|^2_{C_{\gamma,H}}, \tag{3.85}$$

which, together with $\widehat{u}_\tau^n = 0$, implies the uniqueness of solution to (P_u). Therefore, on account of u is a solution to equation (3.13), it follows that $y_1 = y_2 = u$.

(1) Let $C_3 = \min\{2, \frac{3}{8}\lambda\}$. Then we infer from (3.78) that, for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&C_3 e^{-\lambda(t-\tau)}\int_\tau^t (\|\nabla\widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq C_3\int_\tau^t e^{\lambda(r-t)}(\|\nabla\widehat{u}^n(r)\|_p^p + \|\widehat{u}^n(r)\|^2)dr \\
&\leq e^{-\lambda(t-\tau)}\|\widehat{u}^n(\tau)\|^2 + c\int_\tau^t e^{\lambda(r-t)}\|u(r)\|_{2q-2}^{2q-2}dr + c\int_\tau^t e^{\lambda(r-t)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&\quad + c\int_\tau^t e^{\lambda(r-t)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr,
\end{aligned} \tag{3.86}$$

which means that by (3.64) and (3.69), for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&\int_\tau^t (\|\nabla\widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c.
\end{aligned} \tag{3.87}$$

Similarly, by (3.67), (3.69) and $\varphi(0) \in V \cap L^q(\mathcal{O})$, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\int_\tau^t (\|\nabla\widehat{u}^n(r)\|_p^p dr + \|\widehat{u}^n(r)\|^2)dr \\
&\leq c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr + c.
\end{aligned} \tag{3.88}$$

(2) Multiplying (3.70) by $\frac{d\widehat{\mu}_j^n(t)}{dt}$, summing from $j = 1$ until n we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\nabla\widehat{u}^n(t)\|_p^p + \left\|\frac{d\widehat{u}^n}{dt}\right\|^2 &= -\lambda\int_{\mathcal{O}}\widehat{u}^n\frac{d\widehat{u}^n}{dt}dx + \int_{\mathcal{O}} f(t,x,u)\frac{d\widehat{u}^n}{dt}dx \\
&+ \int_{\mathcal{O}} g(x, u(t-\varrho(t)))\frac{d\widehat{u}^n}{dt}dx + \int_{\mathcal{O}}\Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)\frac{d\widehat{u}^n}{dt}dx
\end{aligned}$$

$$+\int_{\mathcal{O}} J(t,x)\frac{d\widehat{u}^n}{dt}dx+\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)\frac{d\widehat{u}^n}{dt}dx. \tag{3.89}$$

By (3.7) and the Young inequality,

$$\int_{\mathcal{O}} f(t,x,u)\frac{d\widehat{u}^n}{dt}dx\le \frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+2\beta_2^2\|u\|_{2q-2}^{2q-2}+2\|\psi_2(t,\cdot)\|^2. \tag{3.90}$$

The Young inequality once more and (3.9) imply that

$$\begin{aligned}
&-\lambda\int_{\mathcal{O}}\widehat{u}^n\frac{d\widehat{u}^n}{dt}dx+\int_{\mathcal{O}} J(t,x)\frac{d\widehat{u}^n}{dt}dx+\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)\frac{d\widehat{u}^n}{dt}dx\\
&\le \frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+3\lambda^2\|\widehat{u}^n\|^2+3\|J(t,\cdot)\|^2+6\int_{\mathcal{O}}|\zeta_\delta(\theta_t\omega)\psi_4(t,x)|^2|u|^{2\eta-2}dx\\
&\quad+6|\zeta_\delta(\theta_t\omega)|^2\int_{\mathcal{O}}|\psi_5(t,x)|^2dx\\
&\le \frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+3\lambda^2\|\widehat{u}^n\|^2+3\|J(t,\cdot)\|^2+\beta_2^2\|u\|_{2q-2}^{2q-2}\\
&\quad+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+6|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.
\end{aligned} \tag{3.91}$$

By Remark 3.1, we have

$$\int_{\mathcal{O}} g(x,u(t-\varrho(t)))\frac{d\widehat{u}^n}{dt}dx\le \frac{1}{8}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2+2\beta_3e^{2\gamma\rho}\|u_t\|_{C_{\gamma,H}}^2+2\|\psi_3(\cdot)\|^2. \tag{3.92}$$

Similar to (3.73), we obtain

$$\int_{\mathcal{O}}\Big(\int_{-\infty}^0 F(x,l,u(t+l))dl\Big)\frac{d\widehat{u}^n}{dt}dx\le 2m_1^2\|u_t\|_{C_{\gamma,H}}^2+2m_0^2+\frac{1}{4}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2. \tag{3.93}$$

Then, plugging (3.90)-(3.93) into (3.89),

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\widehat{u}^n(t)\|_p^p+\frac{p}{8}\Big\|\frac{d\widehat{u}^n}{dt}\Big\|^2\\
&\le 3\lambda^2p\|\widehat{u}^n\|^2+3p\beta_2^2\|u\|_{2q-2}^{2q-2}+2p(\|\psi_2(t,\cdot)\|^2+\|\psi_3(\cdot)\|^2)\\
&\qquad+(2pm_1^2+2p\beta_3e^{2\gamma\rho})\|u_t\|_{C_{\gamma,H}}^2+2pm_0^2+3p\|J(t,\cdot)\|^2\\
&\qquad+c|\zeta_\delta(\theta_t\omega)|^{\frac{2q-2}{q-\eta}}\|\psi_4(t,\cdot)\|_{\frac{2q-2}{q-\eta}}^{\frac{2q-2}{q-\eta}}+6p|\zeta_\delta(\theta_t\omega)|^2\|\psi_5(t,\cdot)\|^2.
\end{aligned} \tag{3.94}$$

Applying (3.52) in Lemma 3.12 to (3.94) over the interval $(\tau,t]$,

$$\begin{aligned}
&\|\nabla\widehat{u}^n(t)\|_p^p+\frac{p}{8}\int_\tau^t\Big\|\frac{d\widehat{u}^n}{dr}\Big\|^2dr \qquad (3.95)\\
&\le \frac{1}{\varepsilon}\int_\tau^t\|\nabla\widehat{u}^n(r)\|_p^pdr+3\lambda^2p\int_\tau^t\|\widehat{u}^n(r)\|^2dr+2pm_1^2\int_\tau^t\|u_r\|_{C_{\gamma,H}}^2dr\\
&\quad+c\int_\tau^t\|u(r)\|_{2q-2}^{2q-2}dr+c\int_\tau^t(\|J(r,\cdot)\|^2+|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_r\omega)|^2+1)dr,
\end{aligned}$$

where $\varepsilon \in (0, t-\tau)$. By (3.53), (3.69) and (3.87), we have

$$\|\nabla \widehat{u}^n(t)\|_p^p \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr + c, \tag{3.96}$$

for all $\tau + \varepsilon \leq t \leq \tau + T$ with $\varepsilon \in (0, t-\tau)$. Therefore, we can deduce that $\{\widehat{u}^n\}$ is bounded in $L^\infty(\tau+\varepsilon, \tau+T; V)$. Then, by the uniqueness of solution to (P_u) and $u \in C([\tau, \tau+T]; H)$, it follows that $u \in C_w((\tau, \tau+T]; V)$ by [21, Theorem 4].

(3) Integrating (3.94) in $r \in [\tau, t]$, with $\tau \leq t \leq \tau + T$, we have

$$\|\nabla \widehat{u}^n(t)\|_p^p + \frac{p}{8}\int_\tau^t \left\|\frac{d\widehat{u}^n}{dr}\right\|^2 dr \leq \|\widehat{u}^n(\tau)\|_V^p + c\int_\tau^t \|\widehat{u}^n(r)\|^2 dr + c\int_\tau^t (\|u_r\|_{C_{\gamma,H}}^2$$

$$+ \|u(r)\|_{2q-2}^{2q-2}) dr + c\int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr. \tag{3.97}$$

Similar to (2), by (3.54), (3.69), (3.88) and $\widehat{u}^n(\tau) = \varphi(0) \in V \cap L^q(\Omega)$, we obtain that

$$\|\nabla \widehat{u}^n(t)\|_p^p \leq c \int_\tau^t (\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1) dr + c, \tag{3.98}$$

for all $\tau \leq t \leq \tau + T$. As [21, Theorem 4], we have $u \in C_w([\tau, \tau+T]; V)$. □

3.2.3 Generation of a multi-valued cocycle

Now, we define a multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ by

$$\Phi(t, \tau, \omega, \varphi) = \{u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \varphi) : u \text{ is a solution of } (3.13)\} \tag{3.99}$$

for every $(t, \tau, \omega, u_\tau) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma,H}$.

Lemma 3.15. *Suppose that* **H1-H6***,* (3.14)-(3.15) *hold and* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,* $\varphi \in C_{\gamma,H}$*. The mapping* $\Phi(t, \tau, \omega, \varphi)$ *in* (3.99) *is a multi-valued cocycle on* $C_{\gamma,H}$ *over* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$*.*

Proof. By the same argument as in [13, Lemma 5.1], the cocycle property (ii) in Definition 3.4 of Φ can be proved. Lemma 3.9 implies that the set $\Phi(t, \tau, \omega, \varphi)$ is nonempty. Moreover, we are able to verify $\Phi(t, \tau, \omega, \varphi)$ has compact values by using Theorem 3.11. Therefore, we complete the proof in the sense of Definition 3.4. □

Lemma 3.16. *Suppose that* **H1-H6***,* (3.14)-(3.15) *hold and* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,* $\varphi \in C_{\gamma,H}$*. The mapping* $\Phi(t, \tau, \omega, \cdot) : C_{\gamma,H} \to \mathcal{C}(C_{\gamma,H})$ *is upper-semicontinuous.*

Proof. Given $T > 0, n \in \mathbb{N}$, let $\tau \in \mathbb{R}, \omega \in \Omega, \varphi^n, \varphi^0 \in C_{\gamma,H}$ such that $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$. Meanwhile, let η^n such that $\eta^n \in \Phi(t, \tau, \omega, \varphi^n)$, that is,

$$\eta^n = u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \varphi^n).$$

As $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$, without loss of generality, we can assume that

$$\|\varphi^n\|^2_{C_{\gamma,H}} \leq 1 + 2\|\varphi^0\|^2_{C_{\gamma,H}}, \quad \forall n \in \mathbb{N}.$$

Arguing as in the proof of Lemma 3.9 and Corollary 3.10, η^n is bounded in $L^\infty(\tau, \tau + T; H) \cap L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathcal{O}))$. Similar to the proof of Theorem 3.11, we can ensure that there exist $\eta^0 \in \Phi(t, \tau, \omega, \varphi^0)$ and a subsequence of η^n (still denote the same) such that $\eta^n \to \eta^0$ in $C_{\gamma,H}$ for all $t \in [\tau, \tau + T]$. As T is arbitrary, it follows that Φ is upper-semicontinuous. □

3.3 Existence of pullback random attractors

In this part, we need to establish the existence of $\mathfrak{D}$-pullback attractor of Φ, where $\mathfrak{D}$ is the universe of all tempered time-sample sets $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\mathcal{D}(\tau, \omega)$ is a nonempty bounded subset of $C_{\gamma,H}$ and

$$\lim_{t \to -\infty} e^{\frac{\lambda}{2}t} \|\mathcal{D}(\tau + t, \theta_t \omega)\|^2_{C_{\gamma,H}} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}, \tau \in \mathbb{R}, \omega \in \Omega. \tag{3.100}$$

Consider a number α satisfying

$$\alpha \in (0, \lambda - \frac{4m_1^2}{\lambda}). \tag{3.101}$$

We state now an assumption.

H7. The non-autonomous term $J \in L^2_{loc}(\mathbb{R}, H)$ satisfies: For every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^0 e^{\alpha r} \|J(r + \tau, \cdot)\|^2 dr < \infty, \tag{3.102}$$

and for every positive constant c,

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^0 e^{\alpha r} \|J(r + t, \cdot)\|^2 dr = 0. \tag{3.103}$$

3.3.1 Existence of pullback attractors

We will prove the existence of a pullback attractor in $C_{\lambda,H}$.

Lemma 3.17. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold. For each* $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$, *there exists* $T = T(\tau, \omega, \mathcal{D}, \delta) > 0$ *such that the solution of* (3.13) *satisfies*

$$\|u_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{C_{\gamma,H}} \leq c_0 R(\tau, \omega), \tag{3.104}$$

for all $t \geq T$ *and* $u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$, *where* c_0 *is a constant and*

$$R(\tau, \omega) = \int_{-\infty}^0 e^{(\lambda - \frac{4m_1^2}{\lambda})r} (\|J(r + \tau, \cdot)\|^2 + |\zeta_\delta(\theta_r \omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r \omega)|^2 + 1) dr. \tag{3.105}$$

Proof. Using $\tau - t$ instead of τ and $\theta_{-\tau}\omega$ instead of ω in (3.25), we have for all $\xi \in [\tau - t, \tau]$,

$$\begin{aligned}
&\|u(\xi)\|^2 + C_2 \int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|u(r)\|_V^p + \|u(r)\|_q^q + \|u(r)\|^2)dr \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr,
\end{aligned} \tag{3.106}$$

where $\|u(\tau - t)\|^2 \leq \|u_{\tau-t}\|^2_{C_{\gamma,H}}$ and $C_2 = \min\{2, \beta_1, \frac{1}{8}\lambda\}$ is defined in Corollary 3.10. Multiplying (3.106) by $e^{2\gamma s}$, and replacing ξ by $\xi + s$, then taking the supremum in $s \in [\tau - t - \xi, 0]$, we obtain that

$$\begin{aligned}
&\sup_{s\in[\tau-t-\xi,0]} e^{2\gamma s}\|u(\xi + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&\leq \sup_{s\in[\tau-t-\xi,0]} e^{(2\gamma-\lambda)s}\Big[ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr\Big] \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ c\int_{\tau-t}^{\xi} e^{\lambda(r-\xi)}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr,
\end{aligned} \tag{3.107}$$

where we have used $\lambda \leq 2\gamma$ defined in (3.15). For $s \in (-\infty, \tau - t - \xi]$, we consider

$$\begin{aligned}
&\sup_{s\in(-\infty,\tau-t-\xi]} e^{2\gamma s}\|u(\xi + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&= \sup_{s\in(-\infty,\tau-t-\xi]} e^{-2\gamma(\xi-\tau+t)}e^{2\gamma(t+\xi+s-\tau)}\|u_{\tau-t}(t + \xi + s - \tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&= e^{-2\gamma(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} \leq e^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}}.
\end{aligned} \tag{3.108}$$

Therefore, similar to (3.28) we have for all $\xi \in [\tau - t, \tau]$,

$$\begin{aligned}
&\|u_\xi(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{C_{\gamma,H}} \\
&\leq ce^{-\lambda(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + \frac{4m_1^2}{\lambda}e^{-\lambda\xi}\int_{\tau-t}^{\xi} e^{\lambda r}\|u_r\|^2_{C_{\gamma,H}}dr \\
&+ ce^{-\lambda\xi}\int_{\tau-t}^{\xi} e^{\lambda r}(\|J(r,\cdot)\|^2 + |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr.
\end{aligned} \tag{3.109}$$

Then, using the Gronwall lemma we have, for all $\xi \in [\tau - t, \tau]$,

$$\|u_\xi(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{C_{\gamma,H}}$$

$$\le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + ce^{(\frac{4m_1^2}{\lambda}-\lambda)\xi}\int_{\tau-t}^{\xi} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r,\cdot)\|^2$$
$$+ |\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r-\tau}\omega)|^2 + 1)dr$$
$$\le ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + ce^{(\frac{4m_1^2}{\lambda}-\lambda)(\xi-\tau)}\int_{-t}^{\xi-\tau} e^{(\lambda-\frac{4m_1^2}{\lambda})r}$$
$$\times (\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega)|^2 + 1)dr. \tag{3.110}$$

Let $\xi = \tau$, we have

$$\|u_\tau(\cdot, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{C_{\gamma,H}} \le ce^{(\frac{4m_1^2}{\lambda}-\lambda)t}\|u_{\tau-t}\|^2_{C_{\gamma,H}} + cR(\tau,\omega), \tag{3.111}$$

where $R(\tau,\omega)$ is defined in (3.105). Since $u_{\tau-t} \in \mathcal{D}(\tau-t, \theta_{-t}\omega)$ and $\mathcal{D} \in \mathfrak{D}$, we see from (3.14) and (3.100) that

$$ce^{(\frac{4m_1^2}{\lambda}-\lambda)t}\|u_{\tau-t}\|^2_{C_{\gamma,H}} \le e^{-\frac{1}{2}\lambda t}\|\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2_{C_{\gamma,H}} \to 0, \text{ as } t\to\infty. \tag{3.112}$$

Thus, we complete the proof. □

Proposition 3.18. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold. Then, the multi-valued cocycle* Φ *has a closed* $\mathfrak{D}$*-pullback absorbing set* $\mathcal{K} \in \mathfrak{D}$*, given by*

$$\mathcal{K}(\tau,\omega) = \{w \in C_{\gamma,H} : \|w\|^2_{C_{\gamma,H}} \le c_0 R(\tau,\omega)\}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega. \tag{3.113}$$

where $R(\tau,\omega)$ *is defined in* (3.105).

Proof. By (3.104), we know that for $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $T = T(\tau,\omega,\mathcal{D},\delta) > 0$ such that, for all $t \ge T$,

$$\Phi(t, \tau-t, \theta_{-t}\omega, \mathcal{D}(\tau-t,\theta_{-t}\omega)) \subseteq \mathcal{K}(\tau,\omega). \tag{3.114}$$

Now, we show that $\mathcal{K} \in \mathfrak{D}$. Let $\alpha_0 = \min\{\lambda - \frac{4m_1^2}{\lambda} - \alpha, \frac{\lambda}{4}\}$, by (3.105) and (3.113) we have, for $t \le 0$,

$$e^{\frac{\lambda}{2}t}R(\tau+t,\theta_t\omega)$$
$$= e^{\frac{\lambda}{2}t}\int_{-\infty}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau+t,\cdot)\|^2 + |\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1)dr$$
$$= e^{-\frac{\lambda}{2}\tau}e^{\frac{\lambda}{2}(\tau+t)}\int_{-\infty}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}\|J(r+\tau+t,\cdot)\|^2 dr$$
$$+ e^{\frac{\lambda}{4}t}\int_{-\infty}^{0} e^{\alpha r + (\lambda-\frac{4m_1^2}{\lambda}-\alpha)r + \frac{\lambda}{4}t}(|\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1)dr$$
$$\le e^{-\frac{\lambda}{2}\tau}e^{\frac{\lambda}{2}(\tau+t)}\int_{-\infty}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}\|J(r+\tau+t,\cdot)\|^2 dr$$
$$+ e^{\frac{\lambda}{4}t}\int_{-\infty}^{0} e^{\alpha r + \alpha_0(r+t)}(|\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_{r+t}\omega)|^2 + 1)dr. \tag{3.115}$$

By the same method as [105, Lemma 3.4], we can deduce that there exist $b_1 = b_1(\omega,\delta) > 0$ and $b_2 = b_2(\omega,\delta) > 0$ such that, for all $r, t \le 0$

$$0 < e^{\alpha_0(r+t)}|\zeta_\delta(\theta_{r+t}\omega)|^{\frac{2q-2}{q-\eta}} < b_1, \quad 0 < e^{\alpha_0(r+t)}|\zeta_\delta(\theta_{r+t}\omega)|^2 < b_2. \tag{3.116}$$

Hence, by (3.103) and (3.116), it follows from (3.115) that, for $t \le 0$,

$$\lim_{t\to-\infty} e^{\frac{\lambda}{2}t} R(\tau + t, \theta_t\omega) = 0. \tag{3.117}$$

Therefore $\mathcal{K} \in \mathfrak{D}$ as desired. □

Now we establish the $\mathfrak{D}$-pullback asymptotic compactness of Φ.

Lemma 3.19. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold. Then* Φ *is* $\mathfrak{D}$*-pullback asymptotically compact in* $C_{\gamma,H}$.

Proof. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\mathcal{D} \in \mathfrak{D}$, and $x^n \in \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{\tau-t_n})$ with $u_{\tau-t_n} \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$ and $t_n \to \infty$. Then

$$x^n(s) = u^n_\tau(s, \tau - t_n, \theta_{-\tau}\omega, u_{\tau-t_n}), \quad \forall s \in (-\infty, 0], \tag{3.118}$$

where u^n is a solution of (3.13). We need to show $\{u^n_\tau(s, \tau-t_n, \theta_{-\tau}\omega, u_{\tau-t_n})\}_{n=1}^\infty$ has a convergent subsequence in $C_{\gamma,H}$.

(1) We will verify that there exists $\mathcal{W} \in C([-T, 0]; H)$ and a subsequence of $\{x^n\}$ (not relabeled) such that $x^n \to \mathcal{W}$ in $C([-T, 0]; H)$ for every $T > 0$.

Let T be a positive integer, similar to (3.110) and (3.112) in Lemma 3.17, there exist $n_0 = n_0(\tau, \omega, \mathcal{D}) \ge 1$ and $c_T = c_0 e^{(\lambda - \frac{4m_1^2}{\lambda})T} > 0$ such that, for all $n \ge n_0$ and $t_n \ge T$,

$$\|u^n_\xi\|^2_{C_{\gamma,H}} \le c_T R(\tau, \omega), \quad \forall \xi \in [\tau - T, \tau], \tag{3.119}$$

where $R(\tau, \omega)$ is defined in (3.105) and then

$$\|u^n(\xi)\|^2 \le c_T R(\tau, \omega), \quad \forall \xi \in [\tau - T, \tau],\ n \ge n_0. \tag{3.120}$$

Consider

$$Y^n(\xi) = u^n(\xi - T), \quad \forall \xi \in [\tau, \tau + T]. \tag{3.121}$$

By (3.120), we have

$$\|Y^n(\xi)\|^2 \le c_T R(\tau, \omega), \quad \forall \xi \in [\tau, \tau + T],\ n \ge n_0. \tag{3.122}$$

Thus, for fixed T, $\{Y^n\}$ is bounded in $L^\infty(\tau, \tau + T; H)$. Note that Y^n is a solution of the following system:

$$\frac{dY^n}{d\xi} = -\Delta_p Y^n - \lambda Y^n + \widetilde{f}(\xi, x, Y^n) + \widetilde{g}(x, Y^n(\xi - \varrho(\xi))) \tag{3.123}$$

$$+\int_{-\infty}^{0}\widetilde{F}(x,l,Y^n(\xi+l))dl+\widetilde{J}(\xi,x)+\widetilde{h}(\xi,x,Y^n)\zeta_\delta(\theta_\xi\omega),\quad \forall\xi\in[\tau,\tau+T],$$

with initial data $Y^n_\tau = u^n_{\tau-T}$. It can be inferred from (3.119) that

$$\|Y^n_\tau\|^2_{C_{\gamma,H}} \le c_T R(\tau,\omega),\quad \forall n\ge n_0. \tag{3.124}$$

From (3.121) and (3.123), we consider for all $\xi\in[\tau,\tau+T]$,

$$\begin{aligned}&\widetilde{f}(\xi,x,Y^n)=f(\xi-T,x,u^n),\ \widetilde{g}(x,Y^n(\xi-\varrho(\xi)))=g(x,u^n(\xi-T-\varrho(\xi-T)))\\&\widetilde{F}(x,l,Y^n(\xi+l))=F(x,l,u^n(\xi-T+l)),\ \widetilde{h}(\xi,x,Y^n)=h(\xi-T,x,u^n).\end{aligned} \tag{3.125}$$

Therefore, using the same method as (3.106), we have for all $r\in[\tau-T,\tau]$,

$$\begin{aligned}&C_2e^{-\lambda T}\int_{\tau-T}^{\tau}(\|u^n(r)\|^p_V+\|u^n(r)\|^q_q)dr\\&\le C_2\int_{\tau-T}^{\tau}e^{\lambda(r-\tau)}(\|u^n(r)\|^p_V+\|u^n(r)\|^q_q)dr\\&\le ce^{-\lambda T}\|u^n_{\tau-T}\|^2_{C_{\gamma,H}}+\frac{4m_1^2}{\lambda}\int_{\tau-T}^{\tau}e^{\lambda(r-\tau)}\|u^n_r\|^2_{C_{\gamma,H}}dr\\&\quad+c\int_{\tau-T}^{\tau}e^{\lambda(r-\tau)}(\|J(r,\cdot)\|^2+|\zeta_\delta(\theta_{r-\tau}\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_{r-\tau}\omega)|^2+1)dr\\&\le ce^{-\lambda T}\|u^n_{\tau-T}\|^2_{C_{\gamma,H}}+\frac{4m_1^2}{\lambda}\int_{\tau-T}^{\tau}e^{\lambda(r-\tau)}\|u^n_r\|^2_{C_{\gamma,H}}dr\\&\quad+c\int_{-\infty}^{0}e^{\lambda r}(\|J(r+\tau,\cdot)\|^2+|\zeta_\delta(\theta_r\omega)|^{\frac{2q-2}{q-\eta}}+|\zeta_\delta(\theta_r\omega)|^2+1)dr.\end{aligned} \tag{3.126}$$

Plugging (3.119) into (3.126) and thanks to $\lambda>\lambda-\frac{4m_1^2}{\lambda}$, we have

$$\int_{\tau-T}^{\tau}(\|u(r)\|^p_V+\|u(r)\|^q_q)dr\le c_0e^{\lambda T}R(\tau,\omega), \tag{3.127}$$

which means that $\{Y^n\}$ is bounded in $L^p(\tau,\tau+T;V)$ and $L^q(\tau,\tau+T;L^q(\mathcal{O}))$ in view of (3.121). In addition, we are able to show $\{\widetilde{f}(\cdot,x,Y^n)\}$ and $\{\zeta_\delta(\theta_t\omega)\widetilde{h}(\cdot,x,Y^n)\}$ are bounded in $L^{\frac{q}{q-1}}(\tau,\tau+T;L^{\frac{q}{q-1}}(\mathcal{O}))$. Owing to the above estimates, there exists $Y\in L^\infty(\tau,\tau+T;H)\cap L^p(\tau,\tau+T;V)\cap L^q(\tau,\tau+T;L^q(\mathcal{O}))$ such that

$$\begin{cases}Y^n\to Y\ \text{ weakly star in }\ L^\infty(\tau,\tau+T;H),\\ Y^n\to Y\ \text{ weakly in }\ L^p(\tau,\tau+T;V)\ \text{ and }\ L^q(\tau,\tau+T;L^q(\mathcal{O})).\end{cases} \tag{3.128}$$

By Remark 3.1, Remark 3.2 and (3.119), we deduce that $\widetilde{g}(x,Y^n_\cdot)$ is bounded in $L^2(\tau,\tau+T;H)$ and $\left\{\int_{-\infty}^{0}\widetilde{F}(x,l,Y^n_\cdot(l))\right\}$ is bounded in $L^2(\tau,\tau+T;H)$. Similar to the proof of Theorem 3.11, we can come to this conclusion

$$Y^n\to Y\quad\text{in}\quad C([\tau,\tau+T];H). \tag{3.129}$$

Let $\mathcal{W}(s) := Y(s+\tau+T)$ for $s \in [-T, 0]$. By (3.128), then using the diagonal technique we can obtain that there exist a function $\mathcal{W} \in C((-\infty, 0], H)$ and a subsequence of $\{n\}$ (relabeled the same) such that $x^n = u^n_\tau \to \mathcal{W}$ in $C([-T, 0]; H)$ on every interval $[-T, 0]$. Thus, by (3.119) we have for any $T > 0$,

$$\lim_{n\to\infty} \sup_{s\in[-T,0]} e^{2\gamma s}\|u^n_\tau(s)\|^2 = \sup_{s\in[-T,0]} e^{2\gamma s}\|\mathcal{W}(s)\|^2 \leq c_T R(\tau, \omega), \tag{3.130}$$

that is, $\mathcal{W} \in C_{\gamma,H}$ and

$$\|\mathcal{W}(s)\|^2_{C_{\gamma,H}} \leq c_T R(\tau, \omega), \quad \forall s \in [-T, 0], \quad \text{for any} \quad T > 0. \tag{3.131}$$

(2) We will prove that $x^n \to \mathcal{W}$ in $C_{\gamma,H}$. To that end, we consider for every $\varepsilon > 0$ there exists n_ε such that, for all $n \geq n_\varepsilon$

$$\sup_{s\in(-\infty,0]} e^{2\gamma s}\|u^n_\tau(s) - \mathcal{W}(s)\|^2 \leq \varepsilon. \tag{3.132}$$

Due to $\lambda - \frac{4m_1^2}{\lambda} < \lambda < 2\gamma$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$c_0 e^{-[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})]T_\varepsilon} R(\tau, \omega) \leq \frac{\varepsilon}{2}. \tag{3.133}$$

By (3.110) we have

$$\|u^n_\tau(s)\|^2 \leq c_0 e^{-(\lambda-\frac{4m_1^2}{\lambda})s} R(\tau, \omega). \tag{3.134}$$

Then, (3.133) and (3.134) yield

$$\sup_{s\leq -T_\varepsilon} e^{2\gamma s}\|u^n_\tau(s)\|^2 \leq c_0 \sup_{s\leq -T_\varepsilon} e^{[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})]s} R(\tau, \omega) \leq \frac{\varepsilon}{2}. \tag{3.135}$$

Given $k \geq 0$, by (3.131) and (3.133), we have for all $s \in [-(T_\varepsilon + k + 1), -(T_\varepsilon + k)]$

$$\begin{aligned} e^{2\gamma s}\|W(s)\|^2 &\leq c_0 e^{-2\gamma(T_\varepsilon+k)} e^{(\lambda-\frac{4m_1^2}{\lambda})(T_\varepsilon+k+1)} R(\tau, \omega) \\ &\leq c_0 e^{\lambda-\frac{4m_1^2}{\lambda}} e^{-[2\gamma-(\lambda-\frac{4m_1^2}{\lambda})](T_\varepsilon+k)} R(\tau, \omega) \leq \frac{\varepsilon}{2}, \end{aligned} \tag{3.136}$$

which means that

$$\sup_{s\leq -T_\varepsilon} e^{2\gamma s}\|\mathcal{W}(s)\|^2 \leq \frac{\varepsilon}{2}. \tag{3.137}$$

It can be inferred from (3.135) and (3.137) that for all $n \geq n_\varepsilon$

$$\sup_{s\in(-\infty,-T_\varepsilon]} e^{2\gamma s}\|u^n_\tau(s) - \mathcal{W}(s)\|^2 \leq \varepsilon. \tag{3.138}$$

From (1), the convergence of $u^n_\tau(\cdot)$ to $\mathcal{W}$ is true in compact intervals. Therefore, together with (3.138) we conclude (3.132). □

Then, we prove the existence of a pullback attractor in $C_{\lambda,H}$.

Theorem 3.20. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold. Then, the multi-valued cocycle* Φ *generated from the p-Laplace equation* (3.13) *has a unique* $\mathfrak{D}$*-pullback attractor* $\mathcal{A} \in \mathfrak{D}$.

Proof. Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. We can obtain the multi-valued cocycle Φ is upper semi-continuous by the result in Lemma 3.16. Proposition 3.18 yields the existence of a closed $\mathfrak{D}$-pullback absorbing set $\mathcal{K} \in \mathfrak{D}$. Lemma 3.19 gives the asymptotic compactness of the multi-valued cocycle Φ. Thus we come to this conclusion in view of [13, Theorem 3.4]. □

3.3.2 Measurability of the pullback attractor

We recall (see [4]) that Ω, a subspace of $C(\mathbb{R}, \mathbb{R})$, can be equipped with the Fréchet metric

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{t\in[-n,n]}|\omega_1(t) - \omega_2(t)|}{1 + \sup_{t\in[-n,n]}|\omega_1(t) - \omega_2(t)|}, \ \forall \omega_1, \omega_2 \in \Omega,$$

and $\mathcal{F}$ is the Borel σ-algebra $\mathcal{B}(\Omega)$ with respect to the metric.

For $m \in \mathbb{N}$, we introduce the subset of Ω as

$$\Omega_m := \Big\{\omega \in \Omega : |\omega(t)| \leq |t|, \quad \forall |t| \geq m\Big\}. \tag{3.139}$$

Lemma 3.21. *([43]) Let* $\delta \in (0, 1]$ *and* $\Omega_m \subseteq \Omega$ *given by* (3.139) *for* $m \in \mathbb{N}$.

(i) If $\omega^n \to \omega$ *with* $\omega^n, \omega \in \Omega_m$*, then* $\zeta_\delta(\theta_t\omega^n) \to \zeta_\delta(\theta_t\omega)$ *uniformly for* t *in any compact interval of* $\mathbb{R}$ *as* $n \to \infty$.

(ii) Ω_m *is a closed subset of* Ω *and* $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$.

(iii) Given $\omega \in \Omega_m$*, we have for all* $t \leq -m$,

$$|\zeta_\delta(\theta_t\omega)| \leq \frac{2}{\delta}|t| + 1. \tag{3.140}$$

Lemma 3.22. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Let* $T > 0, M > 0,\ \tau \in \mathbb{R}$, $\omega^n \to \omega$ *with* $\omega^n, \omega \in \Omega_m$*. Then there exists* $c = c(\delta, \tau, T, M, \omega) > 0$ *such that the solutions of* (3.13) *satisfy*

$$\|u_t(\cdot, \tau, \theta_{-\tau}\omega^n, \varphi)\|^2_{C_{\gamma,H}} \leq c, \tag{3.141}$$

and

$$\int_\tau^t e^{-\lambda(t-r)}(\|u(r, \tau, \theta_{-\tau}\omega^n, \varphi)\|^p_V + \|u(r, \tau, \theta_{-\tau}\omega^n, \varphi)\|^q_q)dr \leq c, \tag{3.142}$$

for all $n \in \mathbb{N}, t \in [\tau, \tau + T]$ *and the initial condition* $\varphi \in C_{\gamma,H}$ *with* $\|\varphi\|_{C_{\gamma,H}} \leq M$.

Proof. Since $\omega, \omega^n \in \Omega_m$, by Lemma 3.21 (i) there exists $N = N(\delta, T, \tau, \omega) \geq 1$ such that, for all $n \geq N$ and $r \in [\tau, \tau + T]$,

$$|\zeta_\delta(\theta_{r-\tau}\omega^n)| \leq |\zeta_\delta(\theta_{r-\tau}\omega)| + 1,$$

which, with the continuity of $\zeta_\delta(\theta_s\omega)$ in s, imply that there exists $c_1 = c_1(\delta, T, \tau, \omega) > 0$ such that, for all $r \in [\tau, \tau + T]$,

$$|\zeta_\delta(\theta_{r-\tau}\omega)| \le c_1, \quad \text{so} \quad |\zeta_\delta(\theta_{r-\tau}\omega^n)| \le 1 + c_1. \tag{3.143}$$

Replacing ω in (3.18) with $\theta_{-\tau}\omega^n$, and plugging (3.143) into (3.18) imply (3.141). Analogously, replace ω by $\theta_{-\tau}\omega^n$ and let $r = \tau$ in (3.36). Then, by (3.143), we can also figure out (3.142). □

Now, we will show the multi-valued cocycle Φ is random in the sense of Definition 3.6.

Lemma 3.23. *Suppose that* **H1-H6**, (3.14)-(3.15) *hold. Then, for every $\tau \in \mathbb{R}$, the mapping*

$$(t, \omega, \varphi) \to \Phi(t, \tau, \omega, \varphi)$$

is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(C_{\gamma,H})$-measurable.

Proof. (1) Given $n \in \mathbb{N}$, $t^0 > 0$, let $t^n \to t^0$, $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$ and $\varphi^n, \varphi^0 \in C_{\gamma,H}$, $\omega^n \to \omega^0$ with $\omega^n, \omega^0 \in \Omega_m$. By [13, Lemma 2.5], we need to verify the above mapping is upper-semicontinuous, that is, we will show that for any sequence $\chi^n \in \Phi(t^n, \tau, \omega^n, \varphi^n)$, there exists a subsequence χ^{n_k} converging to some $\chi^0 \in \Phi(t^0, \tau, \omega^0, \varphi^0)$ in $C_{\gamma,H}$.

We assume that, for all $n \in \mathbb{N}$,

$$0 \le t^n \le 1 + t^0 \qquad \text{and} \qquad \|\varphi^n\|^2_{C_{\gamma,H}} \le 1 + 2\|\varphi^0\|^2_{C_{\gamma,H}}. \tag{3.144}$$

By the definition of Φ in (3.99), then $\chi^n = u^n_{\tau+t^n}(\cdot, \tau, \theta_{-\tau}\omega^n, \varphi^n)$. On account of (3.141) in Lemma 3.22, we have the sequence $\{u^n(\cdot, \tau, \theta_{-\tau}\omega^n, \varphi^n)\}$ is bounded in $L^\infty(\tau, \tau+1+t^0; H)$. By similar reasons to the ones in (3.142), $\{u^n(\cdot, \tau, \theta_{-\tau}\omega^n, \varphi^n)\}$ is bounded in $L^p(\tau, \tau+1+t^0; V)$ and $L^q(\tau, \tau+1+t^0; L^q(\mathcal{O}))$. Similar to the proof of Theorem 3.11, there exist a subsequence $\{u^{n_k}(\cdot, \tau, \theta_{-\tau}\omega^{n_k}, \varphi^{n_k})\}$ and $u^0(\cdot, \tau, \theta_{-\tau}\omega^0, \varphi^0) \in L^\infty(\tau, \tau + 1 + t^0; H) \cap L^p(\tau, \tau + 1 + t^0; V) \cap L^q(\tau, \tau + 1 + t^0; L^q(\mathcal{O}))$ such that

$$u^{n_k}(\cdot) \to u^0(\cdot) \quad \text{in} \quad C([\tau, \tau + 1 + t^0]; H). \tag{3.145}$$

Thus, for a given $\varepsilon > 0$, we have

$$\begin{aligned}
&\sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^{n_k}(\tau + t^{n_k} + s, \tau, \theta_{-\tau}\omega^{n_k}, \varphi^{n_k}) - u^0(\tau + t^0 + s, \tau, \theta_{-\tau}\omega^0, \varphi^0)\| \\
&\le \sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^{n_k}(\tau + t^{n_k} + s, \tau, \theta_{-\tau}\omega^{n_k}, \varphi^{n_k}) - u^0(\tau + t^{n_k} + s, \tau, \theta_{-\tau}\omega^0, \varphi^0)\| \\
&+ \sup_{s\in[-1-t^0,0]} e^{\gamma s}\|u^0(\tau + t^{n_k} + s, \tau, \theta_{-\tau}\omega^0, \varphi^0) - u^0(\tau + t^0 + s, \tau, \theta_{-\tau}\omega^0, \varphi^0)\| \\
&\le \frac{\varepsilon}{4}.
\end{aligned} \tag{3.146}$$

For $s \in (-\infty, -1-t^0]$, we have

$$\begin{aligned}
&\sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s)-u^0(\tau+t^0+s)\| \\
&\le \sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^{n_k}(\tau+t^{n_k}+s)-u^0(\tau+t^{n_k}+s)\| \\
&\quad+\sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|u^0(\tau+t^{n_k}+s)-u^0(\tau+t^0+s)\| \\
&\le e^{-\gamma t^{n_k}}\|\varphi^{n_k}-\varphi^0\|_{C_{\gamma,H}}+\sup_{s\in(-\infty,-1-t^0]} e^{\gamma s}\|\varphi^0(t^{n_k}+s)-\varphi^0(t^0+s)\|. \qquad (3.147)
\end{aligned}$$

By the fact that $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$, for large k we obtain

$$e^{-\gamma t^{n_k}}\|\varphi^{n_k}-\varphi^0\|_{C_{\gamma,H}} \le \frac{\varepsilon}{4}. \qquad (3.148)$$

Owing to $\varphi^0 \in C_{\gamma,H}$, there exists $\lim\limits_{s\to-\infty} e^{\gamma s}\varphi^0(s) = \varphi \in H$. Thus consider $T > 1+t^0$ we have

$$\begin{aligned}
&\sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi^0(t^{n_k}+s)-\varphi^0(t^0+s)\| \\
&\le \sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi^0(t^{n_k}+s)-\varphi\| + \sup_{s\in(-\infty,-T]} e^{\gamma s}\|\varphi-\varphi^0(t^0+s)\| \le \frac{\varepsilon}{4}. \qquad (3.149)
\end{aligned}$$

For $s \in [-T, -1-t^0]$, we have

$$\sup_{s\in[-T,-1-t^0]} e^{\gamma s}\|\varphi^0(t^{n_k}+s)-\varphi^0(t^0+s)\| \le \frac{\varepsilon}{4}, \qquad (3.150)$$

for large k. It can be inferred from (3.146)-(3.150) that $\chi^{n_k} \to \chi^0$ in $C_{\gamma,H}$ as $k \to \infty$. Then, $\chi^0 = u^0_{\tau+t^0}(\cdot, \tau, \theta_{-\tau}\omega^0, \varphi^0) \in C_{\gamma,H}$.

By the continuity of f, g, h (see **H3-H5**), Remark 3.3 and Lemma 3.21 (i), we know that u^0 is a solution of equation (3.13), then $\chi^0 \in \Phi(t^0, \tau, \omega^0, \varphi^0)$.

(2) Due to (1), we can obtain that the mapping $(t, \omega, \varphi) \to \Phi(t, \tau, \omega, \varphi)$ is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_{\Omega_m} \times \mathcal{B}(C_{\gamma,H})$-measurable, where $\mathcal{F}_{\Omega_m}$ is the trace σ-algebra of $\mathcal{F}$ with respect to Ω_m. Since Ω_m is a closed subset of Ω by Lemma 3.21 (ii), then $\Omega_m \in \mathcal{F}$, which implies that $\mathcal{F}_{\Omega_m} \subset \mathcal{F}$. Then, together with $\Omega = \bigcup\limits_{m=1}^{\infty} \Omega_m$ by Lemma 3.21 (ii), we complete the proof. □

Recall that the graph of a set-valued map $\omega \mapsto F(\omega) : \Omega \to 2^X$ is defined by

$$\mathrm{Gr}(F) = \{(\omega, x) \in \Omega \times X : \ x \in F(\omega)\}.$$

Lemma 3.24. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold and let* $m \in \mathbb{N}$, $\tau \in \mathbb{R}$. *Then, for any* $t \ge 0$, *the map* $\Omega_m \ni \omega \to \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ *is measurable with respect to the* $\mathbb{P}$*-completion of* $\mathcal{F}_{\Omega_m}$. *In addition,* $\Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ *is closed.*

Proof. Based on the above definition of graph, by [5], we have to verify the graph of the map $\omega \to \Phi(t,\tau,\omega,\mathcal{K}(\tau,\omega))$ is closed in $\Omega_m \times C_{\gamma,H}$. Let $\omega^n \to \omega^0$ in Ω_m and $\chi^n \to \chi^0$ in $C_{\gamma,H}$, where $\chi^n \in \Phi(t,\tau,\omega^n,\mathcal{K}(\tau,\omega^n)$. Thus, we only need to prove that $\chi^0 \in \Phi(t,\tau,\omega^0,\mathcal{K}(\tau,\omega^0)$.

On account of $\chi^n \in \Phi(t,\tau,\omega^n,\varphi^n)$ and $\varphi^n \in \mathcal{K}(\tau,\omega^n)$, we have

$$\chi^n(s) = u^n_\tau(t+s,\tau,\theta_{-\tau}\omega^n,\varphi^n),\ \forall s \le -t,$$

where u^n is a solution of (3.13). Then $\varphi^n \to \varphi^0$ in $C_{\gamma,H}$.

By Lemma 3.21 (i), there exists $c_m > 0$ such that, for all $n \in \mathbb{N}$ and $r \in [-m, 0]$,

$$|\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2 \le c_m. \tag{3.151}$$

In view of **H7** and (3.151), applying the Lebesgue theorem, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{-m}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2+1)dr\\ &=\int_{-m}^{0} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^0)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^0)|^2+1)dr.\end{aligned} \tag{3.152}$$

Since $\omega^n \in \Omega_m$, by Lemma 3.21 (i), there exists $c_\delta > 0$ such that, for all $n \in \mathbb{N}$ and $r \in (-\infty, -m]$,

$$|\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} \le c_\delta(|r|^{\frac{2q-2}{q-\eta}}+1) \text{ and } |\zeta_\delta(\theta_r\omega^n)|^2 \le c_\delta(|r|^2+1). \tag{3.153}$$

In view of **H7** and (3.153), applying the Lebesgue theorem, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{-\infty}^{-m} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^n)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^n)|^2+1)dr\\ &=\int_{-\infty}^{-m} e^{(\lambda-\frac{4m_1^2}{\lambda})r}(\|J(r+\tau,\cdot)\|^2 + |\zeta_\delta(\theta_r\omega^0)|^{\frac{2q-2}{q-\eta}} + |\zeta_\delta(\theta_r\omega^0)|^2+1)dr.\end{aligned} \tag{3.154}$$

Thus, $R(\tau,\omega^n) \to R(\tau,\omega^0)$.

Since $\varphi^n \in \mathcal{K}(\tau,\omega^n)$, from Proposition 3.18 it follows that $\|\varphi^n\|^2_{C_{\gamma,H}} \le c_0R(\tau,\omega^n)$. Hence, $\|\varphi^0\|^2_{C_{\gamma,H}} \le c_0R(\tau,\omega^0)$ and $\varphi^0 \in \mathcal{K}(\tau,\omega^0)$. Using a similar argument as the proof of Lemma 3.23, we deduce that $\chi^0 \in \Phi(t,\tau,\omega^0,\varphi^0) \subset \Phi(t,\tau,\omega^0,\mathcal{K}(\tau,\omega^0)$. □

A pullback attractor $\mathcal{A}$ is **random** with respect to the $\mathbb{P}$-completion $\overline{\mathcal{F}}$ of the σ-algebra $\mathcal{F}$, that is,

$$\{\omega \in \Omega : \mathcal{A}(\tau,\omega) \cap O \neq \emptyset\} \in \overline{\mathcal{F}},$$

for any open set $O \subset C_{\gamma,H}$ and $\tau \in \mathbb{R}$.

Theorem 3.25. *Suppose that* **H1-H7**, (3.14)-(3.15) *hold. Then, the multi-valued random non-autonomous dynamical system* Φ *generated from the p-Laplace equation* (3.13) *has a* $\mathfrak{D}$*-pullback random attractor* $\mathcal{A}$ *over* $(\Omega,\overline{\mathcal{F}},\overline{\mathbb{P}})$.

Proof. By Lemma 3.24, the map $\omega \to \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega))$ is measurable w.r.t. the $\mathbb{P}$-completion of $\mathcal{F}_{\Omega_m}$, that is

$$C_m := \{\omega \in \Omega_m : \ \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega)) \cap O \neq \emptyset\} \in \overline{\mathcal{F}}_{\Omega_m}.$$

By Lemma 3.21 (ii), we have

$$\{\omega \in \Omega : \ \Phi(t, \tau, \omega, \mathcal{K}(\tau, \omega)) \cap O \neq \emptyset\} = \bigcup_{m=1}^{\infty} C_m \in \overline{\mathcal{F}},$$

together with the closedness of the graph proved in Lemma 3.24 and Lemma 3.23, we can deduce the conclusion according to [13, Theorem 3.5]. □

Part III

Invariant measures for stochastic systems with nonlinear white noise and with or without delay

Chapter 4

Periodic measures for stochastic lattice systems with delay

In this chapter, the existence and the limiting behavior of periodic measures for the periodic stochastic modified Swift-Hohenberg lattice systems with variable delays are analyzed. We first prove the existence and uniqueness of global solution when the nonlinear $\mathcal{T}$-periodic drift and diffusion terms are locally Lipchitz continuous and linearly growing. Then we show the existence of periodic measures of the system under some assumptions. Finally, by strengthening the assumptions, we prove that the set of all periodic measures is weakly compact, and we also show that every limit point of a sequence of periodic measures of the original system must be a periodic measure of the limiting system when the noise intensity tends to zero.

In the next section, we introduce some assumptions about nonlinear and time-delay terms and prove the existence and uniqueness of solutions to (9). In Sect. 4.2, we establish uniform estimates of solutions in $C([-\rho, 0], \ell^2)$. In Sect. 4.3, we discuss the existence of periodic measures in $C([-\rho, 0], \ell^2)$. In Sect. 4.4, the limit of periodic measures is studied when the noise intensity $\epsilon \to \epsilon_0 \in [0, 1]$.

4.1 Well-posedness of the system

4.1.1 Some spaces and assumptions

Denote by $C_\rho = C([-\rho, 0]; \ell^2)$ the Banach space of all ℓ^2-valued continuous functions on $[-\rho, 0]$ with the norm

$$\|x\|_{C_\rho} = \sup_{s\in[-\rho,0]} \|x(s)\| = \sup_{s\in[-\rho,0]} \sum_{i\in\mathbb{Z}} |x_i(s)|^2, \ \forall x \in C_\rho.$$

For any map $w : [-\rho, \infty) \to \ell^2$, we denote the delay shift (or segment of the map) by

$$w_t(s) = w(t + s), \quad \forall t \geq 0, \ s \in [-\rho, 0].$$

For convenience, define some operators from ℓ^2 to ℓ^2 as below: for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \ \ (B^*u)_i = u_{i-1} - u_i, \qquad (4.1)$$

and

$$(Du)_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}. \tag{4.2}$$

Thus, for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$ and $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, we deduce from (4.1) that

$$A = BB^* = B^*B, \quad (Bu, v) = (u, B^*v). \tag{4.3}$$

and

$$(Au, v) = (Bu, Bv), \quad \|Bu\|^2 = \|B^*u\|^2 \le 4\|u\|^2, \quad \|Au\|^2 \le 16\|u\|^2. \tag{4.4}$$

Analogously, it follows from (4.2) that

$$(Du, v) = (Au, Av), \quad \|Du\|^2 \le 256\|u\|^2. \tag{4.5}$$

In order to achieve our final result, we need to impose the following assumptions:
F0. The delay term $\varrho(\cdot) \in C^1(\mathbb{R}, [0, \rho])$ and satisfies

$$\varrho'(t) \le \rho^* \ \text{ for some } \ \rho^* \le 0, \quad \forall t \ge 0. \tag{4.6}$$

F1. The same delay term as in **F0** satisfies

$$\varrho'(t) \le \rho^* \ \text{ for some } \ \rho^* < 1, \quad \forall t \ge 0. \tag{4.7}$$

F2. $g(\cdot) = (g_i(\cdot))_{i\in\mathbb{Z}} \in C(\mathbb{R}, \ell^2)$ and $h_j(\cdot) = (h_{i,j}(\cdot))_{i\in\mathbb{Z}, j\in\mathbb{N}} \in C(\mathbb{R}, \ell^2)$, which means that for all $t \in \mathbb{R}$

$$\|g(t)\|^2 = \sum_{i\in\mathbb{Z}} |g_i(t)|^2 < \infty \ \text{ and } \ \sum_{j\in\mathbb{N}} \|h_j(t)\|^2 = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{Z}} |h_{i,j}(t)|^2 < \infty. \tag{4.8}$$

F3. The nonlinear drift term $f_i \in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$, $f_i(t, \cdot, \cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$; namely, for any bounded interval $I \subseteq \mathbb{R}$, there exists a constant $L_0 = L_0(I) > 0$, independent of $t \in \mathbb{R}, i \in \mathbb{Z}$ such that

$$\begin{aligned} &|f_i(t, s_1, s_2) - f_i(t, s_3, s_4)| \\ &\le L_0(|s_1 - s_3| + |s_2 - s_4|), \quad \forall t \in \mathbb{R}, s_m \in I (m = 1, 2, 3, 4). \end{aligned} \tag{4.9}$$

F4. For each $i \in \mathbb{Z}$, there exists $\alpha_i > 0$ such that,

$$|f_i(t, s_5, s_6)| \le \beta_0(t)(|s_5| + |s_6|) + \alpha_i, \quad \forall t, s_5, s_6 \in \mathbb{R}, \tag{4.10}$$

where $(\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$ and $\beta_0 : \mathbb{R} \to \mathbb{R}$ is a positive continuous and non-increasing function.

F5. The nonlinear diffusion term $\sigma_{j,i} \in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$, $\sigma_{j,i}(t, \cdot, \cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous; that is, for any bounded interval $I \subseteq \mathbb{R}$, there exists a constant $L_1 = L_1(I) > 0$, independent of $j \in \mathbb{N}, i \in \mathbb{Z}, t \in \mathbb{R}$ such that

$$|\sigma_{i,j}(t, s_1, s_2) - \sigma_{i,j}(t, s_3, s_4)|$$

$$\leq L_1(|s_1 - s_3| + |s_2 - s_4|), \quad \forall s_m \in I(m = 1, 2, 3, 4), t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}. \quad (4.11)$$

F6. For each $t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}$, there exists $\gamma_{i,j} > 0$ such that for all $s_5, s_6 \in \mathbb{R}$

$$|\sigma_{i,j}(t, s_5, s_6)| \leq \beta_j(t)(|s_5| + |s_6|) + \gamma_{i,j}, \quad (4.12)$$

where $(\gamma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}} \in \ell^2$ and $(\beta_j(\cdot))_{j\in\mathbb{N}} : \mathbb{R} \to \ell^2$ is a positive continuous and non-increasing function.

Remarks on F0: Notice that we are assuming $\rho^* \leq 0$ in **F0**, which is more restrictive than the general case $\rho^* < 1$ in **F1**. The reason to consider this assumption is that we will be able to provide in Remark 4.3, an easy way to prove the existence and uniqueness of solutions to our problem under assumption **F0**.

Given $\mathcal{T} > 0$, we assume that all time-dependent functions in (7) are $\mathcal{T}$-periodic in $t \in \mathbb{R}$, which means that, for all $t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}$,

$$\begin{cases} q_1(t+\mathcal{T}) = q_1(t), \quad q_2(t+\mathcal{T}) = q_2(t), \quad q_{3,i}(t+\mathcal{T}) = q_{3,i}(t), \\ \varrho(t+\mathcal{T}) = \varrho(t), \quad g(t+\mathcal{T}) = g(t), \quad h(t+\mathcal{T}) = h(t), \\ \beta_0(t+\mathcal{T}) = \beta_0(t), \quad f_i(t+\mathcal{T}, \cdot, \cdot) = f_i(t, \cdot, \cdot), \\ \sigma_{i,j}(t+\mathcal{T}, \cdot, \cdot) = \sigma_{i,j}(t, \cdot, \cdot), \quad \beta_j(t+\mathcal{T}) = \beta_j(t). \end{cases} \quad (4.13)$$

4.1.2 Existence and uniqueness of solutions

Consider $u = (u_i)_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$, $h_j(t) = (h_{j,i}(t))_{i\in\mathbb{Z}}$, $f(t, u, v) = (f_i(t, u_i, v_i))_{i\in\mathbb{Z}}$ and $\sigma_j(t, u, v) = (\sigma_{j,i}(t, u_i, v_i))_{i\in\mathbb{Z}}$. Based on the above arguments, we study the following stochastic delay modified Swift-Hohenberg lattice system in ℓ^2 for $t > 0$:

$$\begin{cases} du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2dt + u^3(t)dt \\ = f(t, u(t), u(t-\varrho(t)))dt + g(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), u(t-\varrho(t)))\big)dW_j(t), \\ u(s) = \varphi(s), \quad s \in [-\rho, 0], \end{cases} \quad (4.14)$$

where $q_1, q_2 \in C(\mathbb{R}, \mathbb{R}^+)$, $q_3 = (q_{3,i})_{i\in\mathbb{Z}} \in C(\mathbb{R}, \ell^2)$, $\varphi = (\varphi_i)_{i\in\mathbb{Z}} \in C_\rho$.

By **F3** and **F4**, $f(t, \cdot, \cdot) : \ell^2 \times \ell^2 \to \ell^2$ is locally Lipschitz continuous and grows linearly, that is, for every $R > 0$, there exists $L_R^f > 0$ such that, for all $t \in \mathbb{R}$ and $u_1, v_1, u_2, v_2 \in \ell^2$ with $\|u_1\| \leq R$, $\|u_2\| \leq R$, $\|v_1\| \leq R$ and $\|v_2\| \leq R$,

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|^2 \leq L_R^f(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \quad (4.15)$$

and for all $t \in \mathbb{R}$ and $u, v \in \ell^2$,

$$\|f(t, u, v)\|^2 \leq 4\beta_0^2(t)(\|u\|^2 + \|v\|^2) + 2\|\alpha\|^2, \quad (4.16)$$

where $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$. Similarly, it follows from **F5** that $\sigma_j(t, \cdot, \cdot) : \ell^2 \times \ell^2 \to \ell^2$ is locally Lipschitz continuous for $j \in \mathbb{N}$ in the sense that, for every $R > 0$, there exists $L_R^\sigma > 0$ such that for all $t \in \mathbb{R}$ and $u_1, v_1, u_2, v_2 \in \ell^2$ with $\|u_1\| \le R, \|u_2\| \le R, \|v_1\| \le R$ and $\|v_2\| \le R$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t,u_1,v_1) - \sigma_j(t,u_2,v_2)\|^2 \le L_R^\sigma(\|u_1-u_2\|^2 + \|v_1-v_2\|^2). \tag{4.17}$$

For each $j \in \mathbb{N}$, we infer from **F6** that, for all $t \in \mathbb{R}$ and $u, v \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t,u,v)\|^2 \le 4\|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2, \tag{4.18}$$

where $\gamma = (\gamma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}} \in \ell^2$ and $\|\beta(t)\|^2 = \sum_{j\in\mathbb{N}} |\beta_j(t)|^2$.

Definition 4.1. *Suppose $\varphi \in L^2(\Omega, C_\rho)$ is $\mathcal{F}_0$-measurable. Then, a continuous ℓ^2-valued stochastic process u is called a solution of lattice system* (4.14) *if $(u_t)_{t\ge 0}$ is $\mathcal{F}_t$-adapted, $u_0 = \varphi$, for all $T > 0$,*

$$u \in L^2(\Omega, C([-\rho, T], \ell^2)), \tag{4.19}$$

and, for each $t \ge 0$,

$$\begin{aligned} u(t) = \varphi(0) + &\int_0^t \big(-q_1(s)Du(s) + 2q_1(s)Au(s) - q_2(s)u(s) \\ &- q_3(s)|Bu(s)|^2 - u^3(s)\big)ds + \int_0^t (f(s,u(s),u(s-\varrho(s))) + g(s))ds \\ &+ \epsilon \sum_{j=1}^\infty \int_0^t (h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))))dW_j(s), \end{aligned} \tag{4.20}$$

in ℓ^2 for almost all $\omega \in \Omega$.

Now, we will show the existence and uniqueness of solutions of system (4.14). In the particular case of $\rho^* \le 0$ in **F0**, a short and nice proof will be given in Remark 4.3 below.

Theorem 4.2. *Suppose* **F2**-**F6** *hold and $\varphi \in L^2(\Omega, C_\rho)$. Then, system* (4.14) *has a unique solution u in the sense of Definition 4.1. In addition, if* **F1** *also holds, then for any $T \ge 0$,*

$$\mathbb{E}\big(\|u\|^2_{C([-\rho,T],\ell^2)}\big) \le Me^{MT}\Big(\mathbb{E}(\|\varphi\|^2_{C_\rho}) + T + 1\Big), \tag{4.21}$$

where M is a positive constant independent of φ and T.

Proof. (1) The existence and uniqueness of solution follow from a similar argument to the one used by X. Mao [68] in the case of stochastic differential equations with delay in $\mathbb{R}^n$ (see also Caraballo et al. [17] for stochastic lattices and PDE with delay). We omit the details.

(2) Now, we prove the uniform estimates of solutions. By (4.14) and Ito's formula, we have for all $t \in [0, T]$,

$$\begin{aligned}
&\|u(t)\|^2 + 2\int_0^t q_1(s)\|Au(s)\|^2 ds - 4\int_0^t q_1(s)\|Bu(s)\|^2 ds + 2\int_0^t q_2(s)\|u(s)\|^2 ds \\
&+ 2\int_0^t \|u(s)\|_4^4 ds + 2\int_0^t \big(q_3(s)|Bu(s)|^2, u(s)\big) ds \\
&= \|u(0)\|^2 + 2\int_0^t (f(s,u(s),u(s-\varrho(s))), u(s)) ds + 2\int_0^t (g(s), u(s)) ds \\
&+ \epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))\|^2 ds \\
&+ 2\epsilon \sum_{j=1}^{\infty} \int_0^t \big(h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))), u(s)\big) dW_j(s).
\end{aligned} \tag{4.22}$$

By using Young's inequality, we obtain

$$\begin{aligned}
&-2\int_0^t \big(q_3(s)|Bu(s)|^2, u(s)\big) ds \\
&\le 2\int_0^t \sum_{i\in\mathbb{Z}} q_{3,i}(s)\big||u_{i+1}(s) - u_i(s)|^2\big||u_i(s)| ds \\
&\le 8\int_0^t \|q_3(s)\|^2\|u(s)\|^2 ds + \frac{1}{8}\int_0^t \sum_{i\in\mathbb{Z}} \Big|u_{i+1}^2(s) - 2u_{i+1}(s)u_i(s) + u_i^2(s)\Big|^2 ds \\
&\le 8\int_0^t \|q_3(s)\|^2\|u(s)\|^2 ds + 2\int_0^t \|u(s)\|_4^4 ds.
\end{aligned} \tag{4.23}$$

From (4.16) and the fact that $\beta_0 : \mathbb{R} \to \mathbb{R}$ is a positive continuous and non-increasing function as defined in **F4**, for all $t \in [0, T]$,

$$\begin{aligned}
&2\int_0^t (f(s,u(s),u(s-\varrho(s))), u(s)) ds \\
&\le \int_0^t \frac{1}{2\beta_0(s)}\|f(s,u(s),u(s-\varrho(s)))\|^2 ds + 2\int_0^t \beta_0(s)\|u(s)\|^2 ds \\
&\le \int_0^t \frac{4\beta_0^2(s)}{2\beta_0(s)}(\|u(s)\|^2 + \|u(s-\varrho(s))\|^2) ds + \int_0^t \frac{\|\alpha\|^2}{\beta_0(s)} ds + 2\int_0^t \beta_0(s)\|u(s)\|^2 ds \\
&\le 4\int_0^t \beta_0(s)\|u(s)\|^2 ds + \int_0^t \frac{\|\alpha\|^2}{\beta_0(s)} ds + \frac{2}{1-\rho^*}\int_{-\rho}^t \beta_0(s)\|u(s)\|^2 ds \qquad (4.24) \\
&\le (4 + \frac{2}{1-\rho^*})\int_0^t \beta_0(s)\|u(s)\|^2 ds + \min_{t\in[0,\mathcal{T}]} \beta_0(t)\|\alpha\|^2 T + \frac{2\rho}{1-\rho^*}\|\varphi\|_{C_\rho}^2 \max_{t\in[0,\mathcal{T}]} \beta_0(t).
\end{aligned}$$

Using Young's inequality yields

$$2\int_0^t (g(s), u(s)) ds \le \int_0^t \frac{1}{q_2(s)}\|g(s)\|^2 ds + \int_0^t q_2(s)\mathbb{E}(\|u(s)\|^2) ds. \tag{4.25}$$

By (4.18) and the fact that $(\beta_j(\cdot))_{j\in\mathbb{N}} : \mathbb{R} \to \ell^2$ is a positive continuous and non-increasing function as defined in **F6**, for all $t \in [0, T]$,

$$\begin{aligned}
&\epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s))\|^2 ds \\
&\leq 2\epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|h_j(s)\|^2 ds + 2\epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|\sigma_j(s, u(s), u(s - \varrho(s)))\|^2 ds \\
&\leq 2\epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|h_j(s)\|^2 ds + 8\epsilon^2 \int_0^t \|\beta(s)\|^2 (\|u(s)\|^2 + \|u(s - \varrho(s))\|^2) ds \\
&\ + 4\epsilon^2 \int_0^t \|\gamma\|^2 ds \\
&\leq 8\epsilon^2 (1 + \frac{1}{1-\rho^*}) \int_0^t \|\beta(s)\|^2 \|u(s)\|^2 ds + \frac{8\epsilon^2 \rho}{1-\rho^*} \|\varphi\|_{C_\rho}^2 \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2 \\
&\ + 2\epsilon^2 \int_0^t (\sum_{j=1}^{\infty} \|h_j(s)\|^2 + 2\|\gamma\|^2) ds. \qquad (4.26)
\end{aligned}$$

Plugging (4.23)-(4.26) into (4.22),

$$\begin{aligned}
\|u(t)\|^2 \leq \|u(0)\|^2 &+ \int_0^t M_1(s) \|u(s)\|^2 ds + \frac{2\rho}{1-\rho^*} \Big(\max_{t\in[0,\mathcal{T}]} \beta_0(t) \\
&+ 4\epsilon^2 \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2 \Big) \|\varphi\|_{C_\rho}^2 + \Big(\min_{t\in[0,\mathcal{T}]} \beta_0(t) \|\alpha\|^2 + 4\epsilon^2 \|\gamma\|^2 \Big) T \\
&+ \int_0^t \frac{1}{q_2(s)} \|g(s)\|^2 ds + 2\epsilon^2 \int_0^t \sum_{j=1}^{\infty} \|h_j(s)\|^2 ds \\
&+ 2\epsilon \sum_{j=1}^{\infty} \int_0^t \big(h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s))), u(s)\big) dW_j(s). \qquad (4.27)
\end{aligned}$$

where $M_1(s) = 16q_1(s) + 8\|q_3(s)\|^2 + (4 + \frac{2}{1-\rho^*})\beta_0(s) + 8\epsilon^2(1 + \frac{1}{1-\rho^*})\|\beta(s)\|^2$. Then taking the expectation of (4.27), for all $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq r\leq t} \|u(r)\|^2 \Big) \leq \mathbb{E}(\|u(0)\|^2) + \mathbb{E}\Big(\int_0^t M_1(s) \|u(s)\|^2 ds \Big) + \frac{2\rho}{1-\rho^*} \Big(\max_{t\in[0,\mathcal{T}]} \beta_0(t) \\
&+ 4\epsilon^2 \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2 \Big) \mathbb{E}(\|\varphi\|_{C_\rho}^2) + \Big(\min_{t\in[0,\mathcal{T}]} \beta_0(t) \|\alpha\|^2 + 4\epsilon^2 \|\gamma\|^2 \Big) T \\
&+ \mathbb{E}\Big(\int_0^t q_2^{-1}(s) \|g(s)\|^2 ds \Big) + 2\epsilon^2 \mathbb{E}\Big(\int_0^t \sum_{j=1}^{\infty} \|h_j(s)\|^2 ds \Big) \\
&+ 2\epsilon \mathbb{E}\Big(\sup_{0\leq r\leq t} \Big| \int_0^r \sum_{j=1}^{\infty} \big(h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s))), u(s)\big) dW_j(s) \Big| \Big). \qquad (4.28)
\end{aligned}$$

For the last term on the right-hand side of (4.28), by the Burkhölder-Davis-Gundy inequality and (4.26), for all $t \in [0, T]$,

$$\begin{aligned}
&2\epsilon\mathbb{E}\bigg(\sup_{0\le r\le t}\bigg|\int_0^r\sum_{j=1}^{\infty}\big(h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))),u(s)\big)dW_j(s)\bigg|\bigg)\\
&\le C_0\epsilon\mathbb{E}\bigg(\int_0^t\sum_{j=1}^{\infty}\|u(s)\|^2\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2ds\bigg)^{\frac{1}{2}}\\
&\le C_0\epsilon\mathbb{E}\bigg(\sup_{0\le s\le t}\|u(s)\|\bigg(\int_0^t\sum_{j=1}^{\infty}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2ds\bigg)^{\frac{1}{2}}\bigg)\\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le s\le t}\|u(s)\|^2\bigg)+\frac{1}{2}C_0^2\epsilon^2\mathbb{E}\bigg(\int_0^t\sum_{j=1}^{\infty}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2ds\bigg)\\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le s\le t}\|u(s)\|^2\bigg)+4C_0^2\epsilon^2(1+\frac{1}{1-\rho^*})\mathbb{E}\bigg(\int_0^t\|\beta(s)\|^2\|u(s)\|^2ds\bigg)\\
&+\frac{4C_0^2\epsilon^2\rho}{1-\rho^*}\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\mathbb{E}(\|\varphi\|^2_{C_\rho})+C_0^2\epsilon^2\int_0^t(\sum_{j=1}^{\infty}\|h_j(s)\|^2+2\|\gamma\|^2)ds, \qquad (4.29)
\end{aligned}$$

where C_0 is a positive constant. Combining (4.28), (4.29) and $\|u(0)\|^2 \le \|\varphi\|^2_{C_\rho}$, we find

$$\mathbb{E}\bigg(\sup_{-\tau\le r\le t}\|u(r)\|^2\bigg)\le M_2\mathbb{E}(\|\varphi\|^2_{C_\rho})+M_3\mathbb{E}\bigg(\int_0^t\sup_{0\le r\le s}\|u(s)\|^2ds\bigg)+M_4T+M_5, \qquad (4.30)$$

where

$$\begin{aligned}
M_2&=3+\frac{4\rho}{1-\rho^*}\bigg(\max_{t\in[0,\mathcal{T}]}\beta_0(t)+(4\epsilon^2+2C_0^2\epsilon^2)\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\bigg),\\
M_3&=2\max_{t\in[0,\mathcal{T}]}M_1(t)+16C_0^2\epsilon^2(1+\frac{1}{1-\rho^*})\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2,\\
M_4&=2\bigg(\min_{t\in[0,\mathcal{T}]}\beta_0(t)\|\alpha\|^2+(4\epsilon^2+2C_0^2\epsilon^2)\|\gamma\|^2\bigg),\\
M_5&=\max\big\{2\min_{t\in[0,\mathcal{T}]}q_2(t),4\epsilon^2+2C_0^2\epsilon^2\big\}\big(\|g\|^2_{C([0,\mathcal{T}],\ell^2)}+\|h\|^2_{C([0,\mathcal{T}],\ell^2)}\big),
\end{aligned}$$

are independent of φ and T. It yields from (4.30) and the Gronwall inequality that for all $t \in [0, T]$ with $T > 0$,

$$\mathbb{E}\bigg(\sup_{-\tau\le r\le t}\|u(r)\|^2\bigg)\le\big\{M_2\mathbb{E}(\|\varphi\|^2_{C_\rho})+M_4T+M_5\big\}e^{M_3T}. \qquad (4.31)$$

Therefore, the conclusion (4.21) can be obtained. □

Remark 4.3. *We use the method mentioned in [9, Theorem 1] to consider the existence and uniqueness of solutions to the system* (4.14) *in the general case that* $\rho^* \leq 0$ *as assumed in* **F0**.
By **F0** *we find that* $\varrho(t)$ *is non-increasing and non-negative, so there are only the following three possibilities:*

(a) $\lim\limits_{t\to+\infty} \varrho(t) = \gamma$ *for some* $\gamma > 0$.
Since $\varrho(t)$ *is non-increasing, we have* $\inf\limits_{t\in[0,+\infty)} \varrho(t) = \gamma$ *and* $\varrho(t) \geq \gamma$ *for* $0 \leq t \leq \gamma$, *so* $t - \varrho(t) \leq t - \gamma \leq 0$ *for* $0 \leq t \leq \gamma$. *Then, system* (4.14) *on* $[0,\gamma]$ *can be considered as:*

$$\begin{cases} du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2dt + u^3(t)dt \\ \quad = f(t, u(t), \varphi(t-\varrho(t)))dt + g(t)dt \\ \qquad + \epsilon \sum\limits_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), \varphi(t-\varrho(t)))\big)dW_j(t), \qquad \forall t \in [0,\gamma], \\ u(0) = \varphi(0), \end{cases} \tag{4.32}$$

which is a non-delay system. Since $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ *and* $q_3 = (q_{3,i})_{i\in\mathbb{Z}} : \mathbb{R} \to \ell^2$ *are positive, continuous, and* $\mathcal{T}$*-periodic functions as defined in* (4.13), *we can deduce from [96, Theorem 3] that problem* (4.32) *has a unique solution* u *on* $[0,\gamma]$ *such that* $u \in L^2(\Omega, C([0,\gamma],\ell^2))$. *For all* $k \geq 0$, *repeating this procedure, the solution* u *can be extended from the interval* $[k\gamma,(k+1)\gamma]$ *to* $[0,\infty)$, *so that* $u \in L^2(\Omega, C([0,T],\ell^2))$ *for any* $T > 0$.

(b) $\lim\limits_{t\to+\infty} \varrho(t) = 0$, *but* $\varrho(t) > 0$ *for any* $t \geq 0$.
We choose an increasing sequence $\{t_k\}_{k\geq 0}$ *such that* $t_0 = 0, t_k \uparrow \infty$ *and*

$$\varrho(t_{k+1}) = t_{k+1} + \varrho(t_{k+1}) - t_{k+1} > t_k + \varrho(t_{k+1}) - t_{k+1} > 0,$$

which means that $t_{k+1} - \varrho(t_{k+1}) < t_k$. *Similar to* **(a)**, *our system can be solved on* $[t_k, t_{k+1}]$ *for* $k \geq 0$, *and hence the solution* u *can be extended to the entire interval* $[0,\infty)$.

(c) *There exists* $T_\rho > 0$ *such that* $\varrho(t) > 0$ *for* $t < T_\rho$, *but* $\varrho(t) = 0$ *for* $t \geq T_\rho$. *When* $t < T_\rho$, *we can adopt the same method as in* **(b)**. *When* $t \geq T_\rho$, *system* (4.14) *becomes*

$$\begin{aligned} &du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2dt + u^3(t)dt \\ &\quad = f(t, u(t), u(t))dt + g(t)dt \\ &\qquad + \epsilon \sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), u(t))\big)dW_j(t), \forall t \in [T_\rho, \infty), \end{aligned} \tag{4.33}$$

with initial data $u(T_\rho) \in L^2(\Omega, \ell^2)$. *It is easy to find that* (4.33) *is an equation without delay, and similar to* (4.32), *the existence of the solution is obvious.*

4.2 Uniform estimates of solutions

In this part, we will show some estimates of solutions for the stochastic delay lattice system (4.14). For this purpose, we assume that if $q_1(t), \beta_0(t) \in \mathbb{R}^+$ and $q_3(t), \beta(t) \in \ell^2$ are small enough or $q_2(t)$ is large enough, there exists $p \geq 2$ such that

$$\min_{t\in[0,\mathcal{T}]} q_2(t) \geq 16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + 2^{4-\frac{2}{p}} p^{-1}(p-1)^{1-\frac{1}{p}} \max_{t\in[0,\mathcal{T}]} \beta_0(t) + 4(3p-4) \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2. \tag{4.34}$$

For all $t \in \mathbb{R}$, we set

$$\begin{cases} \Theta_1(t) = \frac{p}{2} q_2(t) - 8pq_1(t) - 4p\|q_3(t)\|^2 - 2^{3-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\beta_0(t) \\ \qquad - 2p(3p-4)\|\beta(t)\|^2, \\ \Theta_2(t) = 2^{2-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\beta_0(t) + 8(\frac{p}{2})^{1-\frac{p}{2}}(p-1)^{\frac{p}{2}}\|\beta(t)\|^2, \end{cases} \tag{4.35}$$

and

$$\begin{cases} \overline{\Theta}_1(t) = q_2(t) - 2q_1(t) - 8\|q_3(t)\|^2 - 4\beta_0(t) - 8\|\beta(t)\|^2, \\ \overline{\Theta}_2(t) = 2\beta_0(t) + 8\|\beta(t)\|^2. \end{cases} \tag{4.36}$$

By (4.34) and $p \geq 2$, one can obtain $\Theta_1(t) \geq 0$ directly. Since $p \geq 2$, we can verify that

$$2^{4-\frac{2}{p}}(p-1)^{1-\frac{1}{p}} \geq 4p, \quad 4p(3p-4) \geq 8p,$$

which, together with (4.34), implies $\overline{\Theta}_1(t) \geq 0$.

We also need to assume that

$$\chi = \int_0^{\mathcal{T}} \Big(\Theta_1(s) - \Theta_2(s)e^{\int_{s-\rho}^{s}\Theta_1(r)dr}\Big)ds > 0, \tag{4.37}$$

and

$$\overline{\chi} = \int_0^{\mathcal{T}} \Big(\overline{\Theta}_1(s) - \overline{\Theta}_2(s)e^{\int_{s-\rho}^{s}\overline{\Theta}_1(r)dr}\Big)ds > 0. \tag{4.38}$$

In addition, the following lemma will be very helpful in computing uniform estimates of solutions.

Lemma 4.4. *[55, Lemma 3.1] Suppose $v \in C([t_0-\tau,\infty),\mathbb{R}^+)$ is a solution of the delay inequality*

$$\begin{cases} D^+v(t) \leq -\kappa_1(t)v(t) + \kappa_2(t)v(t-\tau_0(t)) + \kappa_3, \quad t > t_0, \\ v(t_0+s) \leq \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{4.39}$$

where $D^+v(t)$ is the upper right-hand Dini derivative of v at $t, t_0 \in \mathbb{R}$, $\tau > 0$ and $\kappa_3 \geq 0, \phi \in C([-\tau,0],\mathbb{R}^+), \tau_0 \in C([t_0,\infty),(0,\tau])$, $\kappa_1(t)$ and $\kappa_2(t), t \in \mathbb{R}$, are nonnegative, *continuous, $\mathcal{T}$-periodic functions. Assume that the average of the function* $\eta(t) = \kappa_1(t) - \kappa_2(t)e^{\int_{t-\tau}^{t}\kappa_1(r)dr}$ *on* $[0,\mathcal{T}]$ *is positive; that is,*

$$\lambda = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \eta(t)dt > 0. \tag{4.40}$$

Then, there exist positive constants $K = K(\kappa_1,\kappa_2,\tau,\mathcal{T})$ *and* $G = G(\kappa_1,\kappa_2,\tau,\mathcal{T})$ *such that, for* $t \ge t_0$*,*

$$v(t) \le K\phi(0)e^{-\lambda(t-t_0)} + G(\kappa_3 + \|\kappa_2\|_{C([0,\mathcal{T}],\mathbb{R}^+)}\|\phi\|_{C([-\tau,0],\mathbb{R}^+)}). \tag{4.41}$$

We now apply Lemma 4.4 to establish the following uniform estimate.

Lemma 4.5. *Suppose* **F1**-**F6**, (4.34) *and* (4.37) *hold. If* $\varphi \in L^p(\Omega, C_\rho)$ *with* $p \ge 2$*, then for* $\epsilon \in (0,1]$*, the solution of system* (4.14) *satisfies for all* $t \ge -\rho$*,*

$$\mathbb{E}(\|u(t)\|^p) \le C_1\mathbb{E}(\|\varphi\|^p_{C_\rho})(e^{-\lambda t}+1) + C_1, \tag{4.42}$$

where $C_1 > 0$ *may depend on* p*, but not on* t, ϵ, ρ *or* φ*.*

Proof. For every $t \ge 0$ and $R > 0$, we define the stopping time

$$\eta_R = \inf\{s \ge t : \|u(s)\| > R\}, \tag{4.43}$$

where $\eta_R = +\infty$ if $\{s \ge t : \|u(s)\| > R\} = \emptyset$.

Given $\triangle t \ge 0$, by (4.22) and Ito's theorem in [3, P92], we have

$$\begin{aligned}
&\mathbb{E}(\|u((t+\triangle t)\wedge\eta_R)\|^p) + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} q_1(s)\|u(s)\|^{p-2}\|Au(t)\|^2 ds\bigg)\\
&\quad - 2p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} q_1(s)\|u(s)\|^{p-2}\|Bu(s)\|^2 ds\bigg)\\
&\quad + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} q_2(s)\|u(s)\|^p ds\bigg) + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\|u(s)\|_4^4 ds\bigg)\\
&\quad + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(q_3(s)|Bu(s)|^2, u(s)\big) ds\bigg)\\
&= \mathbb{E}(\|u(t)\|^p) + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(f(s,u(s),u(s-\varrho(s))), u(s)\big) ds\bigg)\\
&\quad + p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(g(s), u(s)\big) ds\bigg)\\
&\quad + \frac{p}{2}\epsilon^2\mathbb{E}\bigg(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds\bigg)\\
&\quad + \frac{p(p-2)}{2}\epsilon^2\mathbb{E}\bigg(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-4}\\
&\quad \times \big|\big(h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))), u(s)\big)\big|^2 ds\bigg).
\end{aligned} \tag{4.44}$$

For the last term on the left-hand side of (4.44), we obtain

$$\begin{aligned}
&-p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\big(q_3(s)|Bu(s)|^2,u(s)\big)ds\Big)\\
&\le p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\sum_{i\in\mathbb{Z}}|q_{3,i}(s)|\big|u_{i+1}(s)-u_i(s)|^2\big||u_i(s)|ds\Big)\\
&\le p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\Big(4\sum_{i\in\mathbb{Z}}|q_{3,i}(s)|^2|u_i(s)|^2\\
&\quad+\frac{1}{16}\sum_{i\in\mathbb{Z}}\big|u_{i+1}^2(t)-2u_{i+1}(t)u_i(t)+u_i^2(t)\big|^2\Big)ds\Big) \qquad (4.45)\\
&\le 4p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|q_3(s)\|^2\|u(s)\|^pds\Big)+p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\|u(s)\|_4^4ds\Big).
\end{aligned}$$

For the second term on the right-hand side of (4.44), by (4.16) and Young's inequality

$$ab\le\frac{(p-1)\varepsilon_0}{p}a^{\frac{p}{p-1}}+\frac{1}{p\varepsilon_0^{p-1}}b^p,\quad\forall\varepsilon_0>0,$$

we can deduce

$$\begin{aligned}
&p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\big(f(s,u(s),u(s-\varrho(s))),u(s)\big)ds\Big)\\
&\le p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-1}\|f(s,u(s),u(s-\varrho(s)))\|ds\Big)\\
&\le 2^{2-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0(s)\|u(s)\|^pds\Big)\\
&\quad+2^{4-2p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0^{1-p}(s)\|f(s,u(s),u(s-\varrho(s)))\|^pds\Big)\\
&\le 2^{2-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0(s)\|u(s)\|^pds\Big)+2^{4-2p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\times\\
&\times\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\Big[2^{2p-2}\beta_0(s)\big(\|u(s)\|^p+\|u(s-\varrho(s))\|^p\big)+2^{p-1}\beta_0^{1-p}(s)\|\alpha\|^p\Big]ds\Big)\\
&\le 2^{3-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0(s)\|u(s)\|^pds\Big)\\
&\quad+2^{2-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0(s)\|u(s-\varrho(s))\|^pds\Big)\\
&\quad+2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0^{1-p}(s)\|\alpha\|^pds\Big). \qquad (4.46)
\end{aligned}$$

For the third term on the right-hand side of (4.44), we derive

$$p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\big(g(s),u(s)\big)ds\Big)\le\frac{p}{4}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}q_2(s)\|u(s)\|^pds\Big)$$

$$+ 4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} q_2^{1-p}(s)\|g(s)\|^p ds\Big). \tag{4.47}$$

For the last two terms on the right-hand side of (4.44), by (4.18) we have

$$\begin{aligned}
&\frac{p}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds\Big)\\
&\quad+\frac{p(p-2)}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-4}\\
&\quad\times\big|\big(h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))),u(s)\big)\big|^2 ds\Big) \qquad (4.48)\\
&\le\frac{p(p-1)}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds\Big)\\
&\le\frac{p(p-1)}{2}\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\Big(2\|h(s)\|^2+2\|\sigma_j(s,u(s),u(s-\varrho(s)))\|^2\Big)ds\Big)\\
&\le I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1&=4p(p-1)\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|\beta(s)\|^2\|u(s)\|^p ds\Big),\\
I_2&=4p(p-1)\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|\beta(s)\|^2\|u(s)\|^{p-2}\|u(s-\varrho(s))\|^2 ds\Big),\\
I_3&=p(p-1)\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}(\|h(s)\|^2+2\|\gamma\|^2)ds\Big).
\end{aligned}$$

The Young inequality yields

$$\begin{aligned}
I_2&\le 2p(p-2)\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|\beta(s)\|^2\|u(s)\|^p ds\Big)\\
&\quad+8\big(\frac{p}{2}\big)^{1-\frac{p}{2}}(p-1)^{\frac{p}{2}}\epsilon^2\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}\|\beta(s)\|^2\|u(s-\varrho(s))\|^p ds\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_3&\le\frac{p}{4}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}q_2(s)\|u(s)\|^p ds\Big)\\
&\quad+\frac{2^{p-1}\epsilon^p(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R}q_2^{1-\frac{p}{2}}(s)(\|h(s)\|^2+2\|\gamma\|^2)^{\frac{p}{2}}ds\Big).
\end{aligned}$$

Therefore, by $\epsilon\in(0,1]$ and (4.48),

$$\frac{p}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-2}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds\Big)$$

$$
\begin{aligned}
&+\frac{p(p-2)}{2}\epsilon^2\mathbb{E}\bigg(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R}\|u(s)\|^{p-4}\\
&\quad\times\left|\big(h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))),u(s)\big)\right|^2ds\bigg)\\
&\le\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}\Big[2p(3p-4)\|\beta(s)\|^2+\frac{p}{4}q_2(s)\Big]\|u(s)\|^p\bigg)\\
&+8(\frac{p}{2})^{1-\frac{p}{2}}(p-1)^{\frac{p}{2}}\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}\|\beta(s)\|^2\|u(s-\varrho(s))\|^pds\bigg)\\
&+\frac{2^{p-1}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}q_2^{1-\frac{p}{2}}(s)(\|h(s)\|^2+2\|\gamma\|^2)^{\frac{p}{2}}ds\bigg).
\end{aligned}\tag{4.49}
$$

It follows from (4.45)–(4.49) that for all $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}(\|u((t+\triangle t)\wedge\eta_R)\|^p)\\
&\le\mathbb{E}(\|u(t)\|^p)-\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}\Theta_1(s)\|u(s)\|^pds\bigg)\\
&+\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}\Theta_2(s)\|u(s-\varrho(s))\|^pds\bigg)\\
&+2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}\beta_0^{1-p}(s)\|\alpha\|^pds\bigg)\\
&+4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1}\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}q_2^{1-p}(s)\|g(s)\|^pds\bigg)\\
&+\frac{2^{p-1}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge\eta_R}q_2^{1-\frac{p}{2}}(s)(\|h(s)\|^2+2\|\gamma\|^2)^{\frac{p}{2}}ds\bigg),
\end{aligned}\tag{4.50}
$$

where $\Theta_1(t),\Theta_2(t)$ are defined in (4.35). It can be deduced from (4.50) that for all $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}(\|u(t+\triangle t\|^p1_{\{t+\triangle t<\eta_R\}})\\
&\le\mathbb{E}(\|u(t)\|^p)-\mathbb{E}\bigg(\int_t^{t+\triangle t}\Theta_1(s)\|u(s)\|^pds\bigg)+\mathbb{E}\bigg(\int_t^{t+\triangle t}\Theta_2(s)\|u(s-\varrho(s))\|^pds\bigg)\\
&+\triangle t\bigg[2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\|\alpha\|^p\min_{0\le t\le\mathcal{T}}\beta_0^{p-1}(t)+4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1}\min_{0\le t\le\mathcal{T}}q_2^{p-1}(t)\\
&\times\|g\|^p_{C([0,\mathcal{T}],\ell^2)}+\frac{2^{\frac{3p}{2}-2}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\min_{0\le t\le\mathcal{T}}q_2^{\frac{p}{2}-1}(t)\big(\|h\|^p_{C([0,\mathcal{T}],\ell^2)}+2^{\frac{p}{2}}\|\gamma\|^p\big)\bigg],
\end{aligned}\tag{4.51}
$$

Thanks to $\lim\limits_{n\to\infty}\eta_R=+\infty$ and the continuity of solutions, we find from (4.51) that, for all $t\ge 0$,

$$
D^+\mathbb{E}(\|u(t)\|^p)\le-\Theta_1(t)\mathbb{E}\big(\|u(t)\|^p\big)+\Theta_2(t)\mathbb{E}\big(\|u(t-\varrho(t))\|^p\big)
$$

$$+ 2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\|\alpha\|^p \min_{0\le t\le \mathcal{T}} \beta_0^{p-1}(t) + 4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1} \min_{0\le t\le \mathcal{T}} q_2^{p-1}(t)\|g\|^p_{C([0,\mathcal{T}],\ell^2)}$$
$$+ \frac{2^{\frac{3p}{2}-2}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}} \min_{0\le t\le \mathcal{T}} q_2^{\frac{p}{2}-1}(t)\big(\|h\|^p_{C([0,\mathcal{T}],\ell^2)} + 2^{\frac{p}{2}}\|\gamma\|^p\big). \tag{4.52}$$

Applying Lemma 4.4 to (4.52), then there exists $\lambda = \frac{\chi}{\mathcal{T}} > 0$ such that, for all $t \ge 0$,

$$\mathbb{E}(\|u(t)\|^p) \le c\|\varphi(0)\|^p e^{-\lambda t} + c(1 + \|\varphi\|^p_{C_\rho}). \tag{4.53}$$

Note that for all $t \in [-\rho, 0]$,

$$\mathbb{E}(\|u(t)\|^p) \le \mathbb{E}(\|\varphi\|^p_{C_\rho}), \tag{4.54}$$

which, together with (4.53) and the fact that $\|\varphi(0)\| \le \sup_{s\in[-\rho,0]} \|\varphi(s)\| = \|\varphi\|_{C_\rho}$, can conclude (4.42). □

As a consequence of Lemma 4.5, it can be immediately deduced that:

Lemma 4.6. *Suppose* **F1**-**F6**, (4.34) *and* (4.37) *hold. If* $\varphi \in L^p(\Omega, C_\rho)$ *with* $p \ge 6$, *then for* $\epsilon \in (0, 1]$, *the solution of system* (4.14) *satisfies for all* $t > r \ge 0$,

$$\mathbb{E}(\|u(t) - u(r)\|^{\frac{p}{3}}) \le C_2\big(1 + \mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big)(|t-r|^{\frac{p}{3}} + |t-r|^{\frac{p}{6}}), \tag{4.55}$$

where $C_2 > 0$ *is depending on* p, *but not on* ϵ, ρ, t, r *or* φ.

Proof. From (4.14), we have that, for $t > r \ge 0$,

$$u(t) - u(r) = -\int_r^t q_1(s)[Du(s) - 2Au(s)]ds - \int_r^t q_2(s)u(s)ds$$
$$- \int_r^t q_3(s)|Bu(s)|^2 ds - \int_r^t u^3(s)ds + \int_r^t \big(f(s, u(s), u(s-\varrho(s))) + g(s)\big)ds$$
$$+ \epsilon \sum_{j=1}^{\infty} \int_r^t (h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))))dW_j(s). \tag{4.56}$$

We infer from (4.56) that

$$\mathbb{E}(\|u(t) - u(r)\|^{\frac{p}{3}})$$
$$\le 7^{\frac{p}{3}-1}24^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t q_1(s)\|u(s)\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|ds\Big)^{\frac{p}{3}}$$
$$+ 7^{\frac{p}{3}-1}4^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t \|q_3(s)\|\|u(s)\|^2 ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|^3 ds\Big)^{\frac{p}{3}}$$
$$+ 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t \|f(s, u(s), u(s-\varrho(s)))\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}}\|g\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)}$$
$$+ 7^{\frac{p}{3}-1}\epsilon^{\frac{p}{3}}\mathbb{E}\Big(\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))))dW_j(s)\Big\|^{\frac{p}{3}}\Big). \tag{4.57}$$

For the first and second terms on the right-hand side of (4.57), by using Hölder's inequality and the conclusion of Lemma 4.5, it can be concluded that, for $p \geq 6$ and $t > r \geq 0$,

$$
\begin{aligned}
&7^{\frac{p}{3}-1}24^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t q_1(s)\|u(s)\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|ds\Big)^{\frac{p}{3}}\\
&\leq 7^{\frac{p}{3}-1}24^{\frac{p}{3}}\bigg(\max_{0\leq t\leq \mathcal{T}} q_1^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds\\
&+ 7^{\frac{p}{3}-1}\bigg(\max_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds\\
&\leq c_1\Bigg[\bigg(\max_{0\leq t\leq \mathcal{T}} q_1^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1} + \bigg(\max_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1}\Bigg]\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big)|t-r|^{\frac{p}{3}}. \quad (4.58)
\end{aligned}
$$

Similarly, for $p \geq 6$ and $t > r \geq 0$, the third and fourth terms on the right-hand side of (4.57) satisfy

$$
\begin{aligned}
&7^{\frac{p}{3}-1}4^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t \|q_3(s)\|\|u(s)\|^2 ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|^3 ds\Big)^{\frac{p}{3}}\\
&\leq 7^{\frac{p}{3}-1}4^{\frac{p}{3}}\bigg(\sup_{0\leq t\leq \mathcal{T}} \|q_3(t)\|^{\frac{p}{p-3}}\bigg)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{2p}{3}})ds\\
&\quad + 7^{\frac{p}{3}-1}\bigg(\max_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{p})ds\\
&\leq c_2\Bigg[\bigg(\sup_{0\leq t\leq \mathcal{T}} \|q_3(t)\|^{\frac{p}{p-3}}\bigg)^{\frac{p}{3}-1}\big(1+\mathbb{E}(\|\varphi\|^{\frac{2p}{3}}_{C_\rho})\big)\\
&\quad + \bigg(\max_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\bigg)^{\frac{p}{3}-1}\big(1+\mathbb{E}(\|\varphi\|^{p}_{C_\rho})\big)\Bigg]|t-r|^{\frac{p}{3}}. \quad (4.59)
\end{aligned}
$$

For the fifth term on the right-hand side of (4.57), by (4.16) and (4.42), we have for $t > r \geq 0$,

$$
\begin{aligned}
&7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t \|f(s,u(s),u(s-\varrho(s)))\|ds\Big)^{\frac{p}{3}}\\
&\leq 7^{\frac{p}{3}-1}2^{\frac{p}{6}-1}4^{\frac{p}{3}}|t-r|^{\frac{p}{3}-1}\int_r^t \beta_0^{\frac{p}{3}}(s)\mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds\\
&+ 7^{\frac{p}{3}-1}2^{\frac{p}{6}-1}4^{\frac{p}{3}}|t-r|^{\frac{p}{3}-1}\int_r^t \beta_0^{\frac{p}{3}}(s)\mathbb{E}(\|u(s-\varrho(s))\|^{\frac{p}{3}})ds + 7^{\frac{p}{3}-1}2^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}}\|\alpha\|^{\frac{p}{3}}\\
&\leq c_3\bigg(\max_{0\leq t\leq \mathcal{T}} \beta_0^{\frac{p}{3}}(t)\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big) + \|\alpha\|^{\frac{p}{3}}\bigg)|t-r|^{\frac{p}{3}}. \quad (4.60)
\end{aligned}
$$

By (4.18), (4.42) and the Burkholder-Davis-Gundy inequality, for all $\epsilon \in (0,1]$ and $t > r \geq 0$,

$$
7^{\frac{p}{3}-1}\epsilon^{\frac{p}{3}}\mathbb{E}\Big(\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))))dW_j(s)\Big\|^{\frac{p}{3}}\Big)
$$

$$
\begin{aligned}
&\leq C_0 \mathbb{E}\Big(\int_r^t \sum_{j=1}^\infty \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds \Big)^{\frac{p}{6}} \\
&\leq C_0 \mathbb{E}\bigg(\int_r^t \Big[2\|h(s)\|^2 + 8\|\beta(s)\|^2 (\|u(s)\|^2 + \|u(s-\varrho(s))\|^2) + 4\|\gamma\|^2 \Big] ds \bigg)^{\frac{p}{6}} \\
&\leq 4^{\frac{p}{6}-1} 2^{\frac{p}{6}} C_0 |t-r|^{\frac{p}{6}} \|h\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)} + 4^{\frac{p}{6}-1} 8^{\frac{p}{6}} C_0 |t-r|^{\frac{p}{6}-1} \mathbb{E}\bigg(\int_r^t \|\beta(s)\|^{\frac{p}{3}} \|u(s)\|^{\frac{p}{3}} ds \bigg) \\
&+ 4^{\frac{p}{6}-1} 8^{\frac{p}{6}} C_0 |t-r|^{\frac{p}{6}-1} \mathbb{E}\bigg(\int_r^t \|\beta(s)\|^{\frac{p}{3}} \|u(s-\varrho(s))\|^{\frac{p}{3}} ds \bigg) + 4^{\frac{p}{3}-1} C_0 |t-r|^{\frac{p}{6}} \|\gamma\|^{\frac{p}{3}} \\
&\leq c_4 \bigg(\sup_{0 \leq t \leq \mathcal{T}} \|\beta(t)\|^{\frac{p}{3}} \big(1 + \mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big) + \|h\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)} + \|\gamma\|^{\frac{p}{3}} \bigg) |t-r|^{\frac{p}{6}}. \qquad (4.61)
\end{aligned}
$$

Substituting (4.58) to (4.61) into (4.57), the desired result (4.55) can be obtained. □

Next, we present uniform estimates for the tails of the solution to the system (4.14).

Lemma 4.7. *Suppose* **F1**-**F6**, (4.34) *and* (4.38) *hold. If* $\varphi \in L^2(\Omega, C_\rho)$, *then for* $\epsilon \in (0, 1]$,

$$
\limsup_{k \to \infty} \sup_{t \geq -\rho} \sum_{|i| \geq k} \mathbb{E}\big(|u_i(t)|^2\big) = 0. \qquad (4.62)
$$

Proof. Consider a smooth function $\vartheta : \mathbb{R} \to [0, 1]$ such that

$$
\vartheta(r) = \begin{cases} 0, & \text{for } |r| \leq 1, \\ 1, & \text{for } |r| \geq 2, \end{cases} \qquad (4.63)
$$

and define a constant $c_0 > 0$ such that $|\vartheta'(r)| \leq c_0$ uniformly for $r \in \mathbb{R}$.

Given $k \in \mathbb{N}$, define

$$
\vartheta_k u = (\vartheta_{k,i} u_i)_{i \in \mathbb{Z}} = \Big(\vartheta(\frac{|i|}{k}) u_i \Big)_{i \in \mathbb{Z}} \quad \text{for} \quad u = (u_i)_{i \in \mathbb{Z}}. \qquad (4.64)
$$

By (4.14), we have

$$
\begin{aligned}
& d(\vartheta_k u(t)) + q_1(t)\vartheta_k D u(t) dt - 2q_1(t)\vartheta_k A u(t) dt + q_2(t)\vartheta_k u(t) dt \\
&\quad + q_3(t)\vartheta_k |Bu(t)|^2 dt + \vartheta_k u^3(t) dt \\
&= \vartheta_k f(t, u(t), u(t-\varrho(t))) dt + \vartheta_k g(t) dt \\
&\quad + \epsilon \sum_{j=1}^\infty \big(\vartheta_k h_j(t) + \vartheta_k \sigma_j(t, u(t), u(t-\varrho(t))) \big) dW_j(t).
\end{aligned} \qquad (4.65)
$$

Applying Ito's formula to (4.65), and taking expectation, for all $t \geq 0$, we have

$$
\mathbb{E}(\|\vartheta_k u(t)\|^2) + 2\int_0^t q_1(s) \mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big) ds
$$

$$
\begin{aligned}
&- 4\int_0^t q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds + 2\int_0^t q_2(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 2\int_0^t \mathbb{E}(q_3(s)\vartheta_k|Bu(s)|^2, \vartheta_k u(s))ds + 2\int_0^t \mathbb{E}(\vartheta_k u^3(s), \vartheta_k u(s))ds \\
&= \mathbb{E}(\|\vartheta_k u(0)\|^2) + 2\int_0^t \mathbb{E}(\vartheta_k g(s), \vartheta_k u(s))ds \\
&+ 2\int_0^t \mathbb{E}(\vartheta_k f(s, u(s), u(s-\varrho(s))), \vartheta_k u(s))ds \\
&+ \epsilon^2 \sum_{j=1}^{\infty}\int_0^t \mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2)ds. \qquad (4.66)
\end{aligned}
$$

It follows from (4.66) that, for $\triangle t \geq 0$ and $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2) + 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&- 4\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds + 2\int_t^{t+\triangle t} q_2(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 2\int_t^{t+\triangle t} \mathbb{E}(q_3(s)|Bu(s)|^2, \vartheta_k^2 u(s))ds + 2\int_t^{t+\triangle t} \mathbb{E}(u^3(s), \vartheta_k^2 u(s))ds \qquad (4.67) \\
&= \mathbb{E}(\|\vartheta_k u(t)\|^2) + 2\int_t^{t+\triangle t} \mathbb{E}(\vartheta_k g(s), \vartheta_k u(s))ds \\
&+ 2\int_t^{t+\triangle t} \mathbb{E}(\vartheta_k f(s, u(s), u(s-\varrho(s))), \vartheta_k u(s))ds \\
&+ \epsilon^2 \sum_{j=1}^{\infty}\int_t^{t+\triangle t} \mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2)ds.
\end{aligned}
$$

For the second term on the left-hand side of (4.67), by using the result in [99, Lemma 7], yields

$$
\begin{aligned}
&2\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&\geq 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k Au(s)\|^2)ds - \frac{136c_0}{k}\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds. \qquad (4.68)
\end{aligned}
$$

For the third term on the left-hand side of (4.67), by the Young inequality, we obtain

$$
\begin{aligned}
&4\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds \\
&\leq 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k Au(s)\|^2)ds + 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds. \qquad (4.69)
\end{aligned}
$$

For the penultimate term on the left-hand side of (4.67), by considering the definition of B in (4.1), we derive

$$
- 2\int_t^{t+\triangle t} \mathbb{E}(q_3(s)|Bu(s)|^2, \vartheta_k^2 u(s))ds
$$

$$
\begin{aligned}
&\le 2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big||u_{i+1}(s)-u_i(s)|^2\big||u_i(s)|\Big)ds\\
&\le 4\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big(u_{i+1}^2(s)+u_i^2(s)\big)|u_i(s)|\Big)ds \qquad (4.70)\\
&\le 8\int_t^{t+\triangle t}\sum_{|i|\ge k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|u_i(s)|^4\Big)ds.
\end{aligned}
$$

For the second and third terms on the right-hand side of (4.67), it follows from (4.16) and Young's inequality that

$$
\begin{aligned}
&2\int_t^{t+\triangle t}\mathbb{E}\big(\vartheta_k f(s,u(s),u(s-\varrho(s))),\vartheta_k u(s)\big)ds+2\int_t^{t+\triangle t}\mathbb{E}\big(\vartheta_k g(s),\vartheta_k u(s)\big)ds\\
&\le\frac12\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})\big(4\beta_0^2(s)(|u_i(s)|^2+|u_i(s-\varrho(s))|^2)+2|\alpha_i|^2\big)\Big)ds\\
&\quad+2\int_t^{t+\triangle t}\beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\ge k}g_i^2(s)ds\\
&\quad+\int_t^{t+\triangle t}q_2(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds\\
&\le\int_t^{t+\triangle t}\big(4\beta_0(s)+q_2(s)\big)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+2\int_t^{t+\triangle t}\beta_0(s)\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds\\
&\quad+\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\sum_{|i|\ge k}|\alpha_i|^2ds+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\ge k}g_i^2(s)ds. \qquad (4.71)
\end{aligned}
$$

By (4.18), for $t\ge 0$ and $\epsilon\in(0,1]$, the last term on the right-hand side of (4.67) satisfies

$$
\begin{aligned}
&\epsilon^2\sum_{j=1}^\infty\int_t^{t+\triangle t}\mathbb{E}(\|\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds\\
&\le 2\epsilon^2\int_t^{t+\triangle t}\sum_{|i|\ge k}\sum_{j=1}^\infty\big|h_{j,i}(s)\big|^2ds+2\epsilon^2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})\big(4|\beta_i(s)|^2(|u_i(s)|^2\\
&\quad+|u_i(s-\varrho(s))|^2)+2|\gamma_i|^2\big)\Big)ds\\
&\le 2\int_t^{t+\triangle t}\sum_{|i|\ge k}\sum_{j=1}^\infty\big|h_{j,i}(s)\big|^2ds+8\int_t^{t+\triangle t}\sum_{|i|\ge k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds\\
&\quad+8\int_t^{t+\triangle t}\sum_{|i|\ge k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds+4\int_t^{t+\triangle t}\sum_{|i|\ge k}|\gamma_i|^2ds. \qquad (4.72)
\end{aligned}
$$

Plugging (4.68)-(4.72) into (4.67),

$$
\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2)-\mathbb{E}(\|\vartheta_k u(t)\|^2)
$$

$$\begin{aligned}
&\le -\int_t^{t+\triangle t}\Big[q_2(s)-2q_1(s)-8\sum_{|i|\ge k}|q_{3,i}(s)|^2-4\beta_0(s)-8\sum_{|i|\ge k}|\beta_i(s)|^2\Big]\\
&\quad\times\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+\int_t^{t+\triangle t}\Big[2\beta_0(s)+8\sum_{|i|\ge k}|\beta_i(s)|^2\Big]\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds\\
&\quad+\frac{136c_0}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds+\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\sum_{|i|\ge k}|\alpha_i|^2ds \qquad (4.73)\\
&\quad+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\ge k}g_i^2(s)ds+2\int_t^{t+\triangle t}\sum_{|i|\ge k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds+4\int_t^{t+\triangle t}\sum_{|i|\ge k}|\gamma_i|^2ds.
\end{aligned}$$

Thanks to Lemma 4.5, for all $t>0$,

$$\frac{136c_0}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\le\frac{c_0}{k}\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\le t\le\mathcal{T}}q_1(t)\triangle t, \qquad (4.74)$$

where c_0 is positive and independent of t,ϵ,k,ρ and φ. Given $\varepsilon>0$, there exists $K_1(\varepsilon)\ge 1$ such that, for all $t\ge 0,\triangle t\ge 0$ and $k\ge K_1(\varepsilon)$,

$$\frac{136c}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\le\varepsilon\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\le t\le\mathcal{T}}q_1(t)\triangle t. \qquad (4.75)$$

Combining (4.73) and (4.75),

$$\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2)-\mathbb{E}(\|\vartheta_k u(t)\|^2)\\
&\le-\int_t^{t+\triangle t}\overline{\Theta}_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+\int_t^{t+\triangle t}\overline{\Theta}_2(s)\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds\\
&\quad+\varepsilon\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\le t\le\mathcal{T}}q_1(t)\triangle t+\min_{0\le t\le\mathcal{T}}\beta_0^{-1}(t)\sum_{|i|\ge k}|\alpha_i|^2\triangle t\\
&\quad+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\ge k}g_i^2(s)ds+2\int_t^{t+\triangle t}\sum_{|i|\ge k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds+4\sum_{|i|\ge k}|\gamma_i|^2\triangle t, \qquad (4.76)
\end{aligned}$$

where $\overline{\Theta}_1$ and $\overline{\Theta}_2$ are defined in (4.36). Since $\alpha,\gamma\in\ell^2$, $g(\cdot),h_j(\cdot)\in C(\mathbb{R},\ell^2)$ and are $\mathcal{T}$-periodic in $t\in\mathbb{R}$, for the same $\varepsilon>0$ in (4.75), there exists $K_2(\varepsilon)\ge K_1(\varepsilon)$ such that for all $t\ge 0,\mathcal{T}\ge 0$, and $k\ge K_2(\varepsilon)$,

$$\sum_{|i|\ge k}|\alpha_i|^2+\sum_{|i|\ge k}|\gamma_i|^2<\varepsilon, \qquad (4.77)$$

and, for all $t\in[0,\mathcal{T}]$,

$$\sum_{|i|\ge k}g_i^2(t)+\sum_{|i|\ge k}\sum_{j=1}^{\infty}\big|h_{j,i}(t)\big|^2<\varepsilon. \qquad (4.78)$$

Therefore, for all $t\ge 0$ and $k\ge K_2(\varepsilon)$,

$$D^+\mathbb{E}(\|\vartheta_k u(t)\|^2)\le-\overline{\Theta}_1(t)\mathbb{E}(\|\vartheta_k u(t)\|^2)+\overline{\Theta}_2(t)\mathbb{E}(\|\vartheta_k u(t-\varrho(t))\|^2)$$

$$+\varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho})\max_{0\le t\le\mathcal{T}}q_1(t)+\max_{0\le t\le\mathcal{T}}q_1(t)+\min_{0\le t\le\mathcal{T}}\beta_0^{-1}(t)+\min_{0\le t\le\mathcal{T}}q_2^{-1}(t)+6\bigg). \quad (4.79)$$

By (4.38) and Lemma 4.4, there exists $\overline{\lambda}=\frac{\overline{\chi}}{\mathcal{T}}>0$ such that, for all $t\ge 0$ and $k\ge K_2(\varepsilon)$,

$$\begin{aligned}\mathbb{E}(\|\vartheta_k u(t)\|^2)&\le C_3\mathbb{E}(\|\vartheta_k\varphi(0)\|^2)e^{-\overline{\lambda}t}+C_4\mathbb{E}(\|\vartheta_k\varphi\|^2_{C_\rho}) \quad (4.80)\\ &+C_5\varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho})\max_{0\le t\le\mathcal{T}}q_1(t)+\max_{0\le t\le\mathcal{T}}q_1(t)+\min_{0\le t\le\mathcal{T}}\beta_0^{-1}(t)+\min_{0\le t\le\mathcal{T}}q_2^{-1}(t)+6\bigg),\end{aligned}$$

where $C_3,C_4,C_5>0$ are independent of $\epsilon,\varepsilon,\rho$.

Thanks to $\varphi\in L^2(\Omega,C_\rho)$ and the fact that $[-\rho,0]$ is compact, we infer that the set $\{\varphi(s)\in\ell^2:s\in[-\rho,0]\}$ is compact, that is, for $\varepsilon>0$, there exists $K_3(\varepsilon)\ge K_2(\varepsilon)$ such that for all $s\in[-\rho,0]$ and $k\ge K_3(\varepsilon)$,

$$\sum_{|i|\ge k}|\varphi_i(s)|^2\le\varepsilon. \quad (4.81)$$

Note that for all $t\in[-\rho,0]$,

$$\mathbb{E}(\|\vartheta_k u(t)\|^2)\le\mathbb{E}\bigg(\sum_{|i|\ge k}|\varphi_i(s)|^2\bigg). \quad (4.82)$$

From (4.80) to (4.82), for all $t\ge-\rho$ and $k\ge K_3(\varepsilon)$,

$$\begin{aligned}\sum_{|i|\ge 2k}\mathbb{E}(|u_i(t)|^2)&\le\mathbb{E}(\|\vartheta_k u(t)\|^2)\le C_3\varepsilon e^{-\overline{\lambda}t}+C_4\varepsilon+C_5\varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho})\max_{0\le t\le\mathcal{T}}q_1(t)\\ &+\max_{0\le t\le\mathcal{T}}q_1(t)+\min_{0\le t\le\mathcal{T}}\beta_0^{-1}(t)+\min_{0\le t\le\mathcal{T}}q_2^{-1}(t)+6\bigg),\end{aligned}$$

thus the desired result is obtained. □

We improve the tail estimates established in Lemma 4.7, which are useful for the tightness of a family of probability distributions of solutions in the space C_ρ, as will be shown in Lemma 4.9.

Lemma 4.8. *Suppose* **F1-F6**, (4.34), (4.37) *and* (4.38) *hold. If* $\varphi\in L^2(\Omega,C_\rho)$, *then for* $\epsilon\in(0,1]$,

$$\limsup_{k\to\infty}\sup_{t\ge\rho}\mathbb{E}\bigg(\sup_{t-\rho\le r\le t}\sum_{|i|\ge k}|u_i(r)|^2\bigg)=0. \quad (4.83)$$

Proof. Consider the same smooth function ϑ as defined in Lemma 4.7, thus also satisfying (4.63) and (4.64). It follows from (4.65) that, for all $t\ge\rho$ and $t-\rho\le r\le t$,

$$\|\vartheta_k u(r)\|^2+2\int_{t-\rho}^{r}q_1(s)\big(Au(s),A(\vartheta_k^2u(s))\big)ds$$

$$
\begin{aligned}
&-4\int_{t-\rho}^{r} q_1(s)\big(Au(s),\vartheta_k^2 u(s)\big)ds+2\int_{t-\rho}^{r}(q_3(s)\vartheta_k|Bu(s)|^2,\vartheta_k u(s))ds\\
&+2\int_{t-\rho}^{r} q_2(s)\|\vartheta_k u(s)\|^2ds+2\int_{t-\rho}^{r}(\vartheta_k u^3(s),\vartheta_k u(s))ds\\
&=\|\vartheta_k u(t-\rho)\|^2+2\int_{t-\rho}^{r}(\vartheta_k f(s,u(s),u(s-\varrho(s))),\vartheta_k u(s))ds\\
&+2\int_{t-\rho}^{r}(\vartheta_k g(s),\vartheta_k u(s))ds+\epsilon^2\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\|\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2ds\\
&+2\epsilon\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\big(\vartheta_k^2 u(s),h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s)))\big)dW_j(s). \qquad (4.84)
\end{aligned}
$$

Similar to (4.68), for the second term on the left-hand side of (4.84), we have

$$
\begin{aligned}
&2\int_{t-\rho}^{r} q_1(s)\big(Au(s),A(\vartheta_k^2 u(s))\big)ds\\
&\geq 2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k Au(s)\|^2ds-\frac{136c_0}{k}\int_{t-\rho}^{r} q_1(s)\|u(s)\|^2ds. \qquad (4.85)
\end{aligned}
$$

For the third and fourth terms on the left-hand side of (4.84), the Young inequality yields

$$
\begin{aligned}
&4\int_{t-\rho}^{r} q_1(s)\big(Au(s),\vartheta_k^2 u(s)\big)ds\\
&\leq 2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k Au(s)\|^2ds+2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k u(s)\|^2ds, \qquad (4.86)
\end{aligned}
$$

and

$$
\begin{aligned}
&-2\int_{t-\rho}^{r}(q_3(s)|Bu(s)|^2,\vartheta_k^2 u(s))ds\\
&\leq 4\int_{t-\rho}^{r}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big(u_{i+1}^2(s)+u_i^2(s)\big)|u_i(s)|\Big)ds\\
&\leq 8\int_{t-\rho}^{r}\sum_{|i|\geq k}|q_{3,i}(s)|^2\|\vartheta_k u(s)\|^2ds+2\int_{t-\rho}^{r}\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|u_i(s)|^4ds. \qquad (4.87)
\end{aligned}
$$

Substituting (4.85)-(4.87) into (4.84), taking the supremum in $r\in[t-\rho,t]$ and its expectation, we derive for all $t\geq\rho$ and $t-\rho\leq r\leq t$,

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\Bigg)\leq\mathbb{E}(\|\vartheta_k u(t-\rho)\|^2)+\frac{136c_0}{k}\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds\\
&+2\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+8\int_{t-\rho}^{t}\sum_{|i|\geq k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k f(s,u(s),u(s-\varrho(s)))\|\|\vartheta_k u(s)\|ds\right)\\
&+2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k g(s)\|\|\vartheta_k u(s)\|ds\right) \qquad (4.88)\\
&+\epsilon^2\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds\\
&+2\epsilon\mathbb{E}\left(\sup_{t-\rho\le r\le t}\left|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\big(\vartheta_k u(s),\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\big)dW_j(s)\right|\right).
\end{aligned}
$$

It can be deduced from (4.62) in Lemma 4.7 that for any $\varepsilon>0$, there exists $K_1(\varepsilon)\ge 1$ such that, for all $s\ge -\rho$ and $k\ge K_1(\varepsilon)$,

$$\sum_{|i|\ge k}\mathbb{E}(|u_i(s)|^2)\le\varepsilon, \qquad (4.89)$$

which means that, for all $s\ge-\rho$ and $k\ge K_1(\varepsilon)$,

$$\mathbb{E}(\|\vartheta_k u(s)\|^2)\le\sum_{|i|\ge k}\mathbb{E}(|u_i(s)|^2)\le\varepsilon. \qquad (4.90)$$

Considering the second term on the right-hand side of (4.88), we infer from Lemma 4.5 that there exists $M(\varphi)>0$ such that for all $t\ge\rho$,

$$\frac{136c_0}{k}\int_{t-\rho}^{t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\le\frac{136c_0\rho}{k}\max_{t\in[0,\mathcal{T}]}q_1(t)M(\varphi),$$

which means that, for every $\varepsilon>0$, there exists $K_2(\varphi,\varepsilon)\ge K_1(\varepsilon)$ such that, for all $t\ge\rho$ and $k\ge K_2(\varphi,\varepsilon)$,

$$\frac{136c_0}{k}\int_{t-\rho}^{t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\le\varepsilon. \qquad (4.91)$$

For the third and fourth terms of the right-hand side of (4.88), by (4.90) and $q_3(\cdot)=(q_{3,i}(\cdot))_{i\in\mathbb{Z}}\in\ell^2$, there exists $K_3(\varphi,\varepsilon)>K_2(\varphi,\varepsilon)$ such that, for all $t\ge\rho$ and $k\ge K_3(\varphi,\varepsilon)$,

$$
\begin{aligned}
&2\int_{t-\rho}^{t}q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+8\int_{t-\rho}^{t}\sum_{|i|\ge k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds\\
&\le 2\rho\varepsilon\max_{t\in[0,\mathcal{T}]}q_1(t)+8\rho\varepsilon\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\ge k}|q_{3,i}(t)|^2\le c_5\varepsilon, \qquad (4.92)
\end{aligned}
$$

where $c_5=c_5(\rho,q_1(t),q_3(t))$. For the fifth term of the right-hand side of (4.88), by (4.16), (4.77) and (4.90), there exists $K_4(\varphi,\varepsilon)>K_3(\varphi,\varepsilon)$ such that, for all $t\ge\rho$ and $k\ge K_4(\varphi,\varepsilon)$,

$$2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k f(s,u(s),u(s-\varrho(s)))\|\|\vartheta_k u(s)\|ds\right)$$

$$
\begin{aligned}
&\le \int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds + 4\int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 4\int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds + 2\int_{t-\rho}^{t} \beta_0^{-1}(s)\sum_{|i|\ge k}|\alpha_i|^2 ds \\
&\le 5\max_{t\in[0,\mathcal{T}]}\beta_0(t)\varepsilon + \frac{4}{1-\rho^*}\int_{t-2\rho}^{t}\beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds + 2\rho\min_{t\in[0,\mathcal{T}]}\beta_0^{-1}(t)\varepsilon \\
&\le 5\max_{t\in[0,\mathcal{T}]}\beta_0(t)\varepsilon + \frac{8\rho\varepsilon}{1-\rho^*}\max_{t\in[0,\mathcal{T}]}\beta_0(t) + 2\rho\min_{t\in[0,\mathcal{T}]}\beta_0^{-1}(t)\varepsilon \le c_6\varepsilon, \qquad (4.93)
\end{aligned}
$$

where $c_6 = c_6(\rho, \beta_0(t))$. Meanwhile, by Young's inequality, (4.78) and (4.90), there exists $K_5(\varphi,\varepsilon) > K_4(\varphi,\varepsilon)$ such that, for all $t \ge \rho$ and $k \ge K_5(\varphi,\varepsilon)$,

$$
\begin{aligned}
2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k g(s)\|\|\vartheta_k u(s)\|ds\right) &\le \int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k g(s)\|^2)ds + \int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&\le \rho\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\ge k} g_i^2(t) + \rho\max_{s\ge 0}\mathbb{E}(\|\vartheta_k u(s)\|^2) \le c_7\varepsilon, \qquad (4.94)
\end{aligned}
$$

where $c_7 = c_7(\rho, g)$. Similar to (4.93), for the second-to-last term of the right-hand side of (4.88), by (4.18), (4.77), and (4.78), there exists $K_6(\varphi,\varepsilon) > K_5(\varphi,\varepsilon)$ such that, for all $t \ge \rho, \epsilon \in (0,1]$ and $k \ge K_6(\varphi,\varepsilon)$,

$$
\begin{aligned}
&\epsilon^2\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds \\
&\le 2\rho\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\ge k}\sum_{j=1}^{\infty}\left|h_{j,i}(t)\right|^2 + 8\int_{t-\rho}^{t}\sum_{|i|\ge k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ \frac{8}{1-\rho^*}\int_{t-2\rho}^{t}\sum_{|i|\ge k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds + 4\rho\sum_{|i|\ge k}|\gamma_i|^2 \\
&\le 2\rho\varepsilon + 8\rho\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\ge k}|\beta_i(t)|^2\varepsilon + \frac{16\rho\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\ge k}|\beta_i(t)|^2}{1-\rho^*}\varepsilon + 4\rho\varepsilon \le c_8\varepsilon, \qquad (4.95)
\end{aligned}
$$

where $c_8 = c_8(\rho, \beta(t), h(t), \gamma)$. For the last term of the right-hand side of (4.88), by the Burkhölder-Davis-Gundy inequality and (4.95), for all $t \ge \rho, \epsilon \in (0,1]$ and $k \ge K_6(\varphi,\varepsilon)$,

$$
\begin{aligned}
&2\epsilon\mathbb{E}\left(\sup_{t-\rho\le r\le t}\left|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\big(\vartheta_k u(s), \vartheta_k h_j(s) + \vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\big)dW_j(s)\right|\right) \\
&\le C_0\mathbb{E}\left(\left(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\left|(\vartheta_k u(s), \vartheta_k h_j(s) + \vartheta_k\sigma_j(s,u(s),u(s-\varrho(s))))\right|^2 ds\right)^{\frac{1}{2}}\right) \\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{t-\rho\le r\le t}\|\vartheta_k u(r)\|^2\right)
\end{aligned}
$$

$$+\frac{1}{2}C_0^2\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds$$
$$\leq\frac{1}{2}\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right)+c_9\varepsilon. \tag{4.96}$$

From (4.91) to (4.96) we deduce that, for all $t\geq\rho$ and $k\geq K_6(\varphi,\varepsilon)$,

$$\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\sum_{|i|\geq 2k}|u_i(r)|^2\right)\leq\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right)\leq c\varepsilon, \tag{4.97}$$

where $c=\max\{c_i\}_{i=5,6,7,8,9}$. Therefore, (4.83) is completely proved. □

4.3 Periodic measures for stochastic delay modified Swift-Hohenberg lattice systems

In this section, we will establish the existence of periodic measures of the system (4.14) in C_ρ. Let $B_b(C_\rho)$ be the space of all bounded Borel-measurable functions on C_ρ, and $\mathscr{B}(C_\rho)$ be the Borel σ-algebra on C_ρ.

If $\phi\in B_b(C_\rho)$, then for $0\leq r\leq t$, we define a probability transition operator with delay by

$$(p_{r,t}\phi)(\nu)=\mathbb{E}[\phi(u_t(\cdot;r,\nu))],\quad\text{for all}\quad\nu\in C_\rho. \tag{4.98}$$

In addition, for $\Lambda\in\mathscr{B}(C_\rho)$ and $0\leq r\leq t$, denote a transition probability function by

$$p(r,\nu;t,\Lambda)=(p_{r,t}1_\Lambda)(\nu)=\mathbb{P}\{\omega\in\Omega:u_t(\cdot;r,\nu)(\omega)\in\Lambda\}, \tag{4.99}$$

where $\nu\in C_\rho$ and 1_Λ is the indicator function of Λ. We now recall the definition of periodic measure, namely, a probability measure μ on C_ρ is called a **periodic measure** with period $\mathcal{T}>0$ if

$$\int_{C_\rho}(p_{0,t+\mathcal{T}}\phi)(\nu)d\mu(\nu)=\int_{C_\rho}(p_{0,t}\phi)(\nu)d\mu(\nu),\quad\forall t\geq 0,\nu\in C_\rho. \tag{4.100}$$

4.3.1 Tightness of a family of probability distributions

In this part, we show the tightness of a family of probability distributions of solutions to system (4.14). For this purpose, we need to recall a definition, that is, a family of probability distributions of solutions is called **tight** if for each $\varepsilon\in(0,1]$, there exists a compact set Z_ε of C_ρ such that

$$\mathbb{P}\{\omega\in\Omega:u_t(\cdot;r,\nu)(\omega)\in Z_\varepsilon\}>1-\varepsilon,\quad\text{for all}\quad 0\leq r\leq t\in\mathbb{R},\nu\in C_\rho. \tag{4.101}$$

As verified in Section 4.1, for any $t_0\geq 0$, and any $\mathcal{F}_{t_0}$-measurable $\varphi\in L^2(\Omega,C_\rho)$, the system (4.14) has a unique solution $u(t,t_0,\varphi)$ for $t\in[t_0-\rho,\infty)$. In the following lemma, we consider the initial data $\varphi=0$ at initial time $t_0=0$.

Lemma 4.9. *Suppose* **F1**-**F6**, (4.34), (4.37) *and* (4.38) *hold with* $p \geq 6$. *Then, for* $\epsilon \in (0,1]$, *the distribution laws of the process* $\{u_t(\cdot;0,0)\}_{t\geq 0}$ *to the system* (4.14) *are tight on* C_ρ.

Proof. By Lemma 4.5, Lemma 4.6, Lemma 4.8 and the Arzelà-Ascoli theorem, we can use similar arguments to those in [23, Lemma 4.8] to obtain the conclusion in the sense of (4.101). □

4.3.2 Existence of periodic measures

To obtain the existence of the periodic measures, we need to do some pre-preparation as follows.

We use the same method as in [96] to approximate nonlinear locally Lipschitz continuous functions $f, \sigma_j, |Bu|^2$ and u^3 by using a suitable cut-off function as follows:

$$\zeta_R(s) = \begin{cases} s, & \text{if } |s| \leq R. \\ \dfrac{Rs}{|s|}, & \text{if } |s| > R. \end{cases} \tag{4.102}$$

Then, we find that for all $s, s_1, s_2 \in \mathbb{R}$,

$$|\zeta_R(s_1) - \zeta_R(s_2)| \leq |s_1 - s_2|, \quad |\zeta_R(s)| \leq |s| \wedge R, \quad \zeta_R(0) = 0. \tag{4.103}$$

Given $j \in \mathbb{N}, R > 0$ and $i \in \mathbb{Z}$. For all $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2, v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, let

$$f^R(t,u,v) = \big(f_i(t, \zeta_R(u_i), \zeta_R(v_i))\big)_{i\in\mathbb{Z}}, \quad \sigma_j^R(t,u,v) = \big(\sigma_{j,i}(t, \zeta_R(u_i), \zeta_R(v_i))\big)_{i\in\mathbb{Z}}.$$

It can be deduced from (4.9) that for all $u_1, u_2, v_1, v_2 \in \ell^2$,

$$\begin{aligned} &\|f^R(t,u_1,v_1) - f^R(t,u_2,v_2)\|^2 \\ &\leq 2L_0^2(R)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \quad \forall t \in \mathbb{R}, \end{aligned} \tag{4.104}$$

and by (4.10), for all $u, v \in \ell^2$,

$$\begin{aligned} \|f^R(t,u,v)\|^2 &\leq 2\|f^R(t,u,v) - f^R(t,0,0)\|^2 + 2\|f^R(t,0,0)\|^2 \\ &\leq 2L_0^2(R)(\|u - v\|^2) + 2\|\alpha\|^2, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4.105}$$

Similarly, by (4.11), for all $u_1, u_2, v_1, v_2 \in \ell^2$,

$$\begin{aligned} &\sum_{j\in\mathbb{N}} \|\sigma_j^R(t,u_1,v_1) - \sigma_j^R(t,u_2,v_2)\|^2 \\ &\leq 2L_1^2(R)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \quad \forall t \in \mathbb{R}, \end{aligned} \tag{4.106}$$

and by (4.18), for all $u, v \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j^R(t,u,v)\|^2 \leq 4\|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2$$

$$\leq 4 \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2, \quad \forall t \in \mathbb{R}. \tag{4.107}$$

The approximation of nonlinear terms $|Bu|^2$ and u^3 is the same as (2.25)-(2.28) in [96], that is, for all $u, v \in \ell^2$,

$$\big\||B\zeta_R(u)|^2 - |B\zeta_R(v)|^2\big\|^2 \leq 64R^2\|u-v\|^2, \quad \big\||B\zeta_R(u)|^2\big\|^2 \leq 64R^2\|u\|^2, \tag{4.108}$$

and

$$\|\zeta_R^3(u) - \zeta_R^3(v)\|^2 \leq 9R^4\|u-v\|^2, \quad \|\zeta_R^3(u)\|^2 \leq 9R^4\|u\|^2. \tag{4.109}$$

Replace now $f(t, u(t), u(t-\varrho(t))), \sigma_j(t, u(t), u(t-\varrho(t))), |Bu|^2$ and u^3 with $f^R(t, u^R(t), u^R(t-\varrho(t))), \sigma_j^R(t, u^R(t), u^R(t-\varrho(t)))$, $|B\zeta_R(u^R)|^2$ and $\zeta_R^3(u^R)$, then the existence of a unique solution u_t^R for the approximate stochastic system can be proved by [68]. We define a stopping time by

$$\varsigma_R = \inf\{t \geq 0 : \|u_t^R(\cdot)\| > R\}, \tag{4.110}$$

where $\varsigma_R = +\infty$ if $\{t \geq 0 : \|u_t^R(\cdot)\| > R\} = \emptyset$.

Lemma 4.10. *Suppose* **F1**-**F6** *hold. If* $\varphi \in L^2(\Omega, C_\rho)$*, then for all* $0 \leq s \leq t$,

$$\lim_{R\to\infty} u_t^R(\cdot, s, \varphi) = u_t(\cdot, s, \varphi) \quad \mathbb{P}\text{-}a.s., \tag{4.111}$$

where $u_t(\cdot)$ *is a solution of* (4.14).

Proof. (1) We want to verify that

$$u^{R+1}(t \wedge \varsigma_R) = u^R(t \wedge \varsigma_R) \quad \text{and} \quad \varsigma_{R+1} \geq \varsigma_R, \tag{4.112}$$

a.s. for all $t \geq -\rho$ and $R > 0$. Due to the delay term, it will be different from the treatment of f^R, σ_j^R in [96, Lemma 1]. We only include the proof of f^R here, and as the one for σ_j^R is similar, we omit it.

By (4.104) and the Young inequality, we have for all $t \geq 0$,

$$\begin{aligned}
&2\int_0^{t\wedge\varsigma_R} \Big(f^{R+1}\big(s, u^{R+1}(s), u^{R+1}(s-\varrho(s))\big) \\
&\quad - f^R\big(s, u^R(s), u^R(s-\varrho(s))\big), u^{R+1}(s) - u^R(s)\Big) ds \\
&\leq (1 + 2L_0^2(R)) \int_0^{t\wedge\varsigma_R} \|u^{R+1}(s) - u^R(s)\|^2 ds \\
&\quad + 2L_0^2(R) \int_0^{t\wedge\varsigma_R} \|u^{R+1}(s-\varrho(s)) - u^R(s-\varrho(s))\|^2 ds \\
&\leq (1 + 2L_0^2(R)) \int_0^{t\wedge\varsigma_R} \|u^{R+1}(s) - u^R(s)\|^2 ds + \frac{2L_0^2(R)}{1-\rho^*} \int_{-\rho}^{t\wedge\varsigma_R} \|u^{R+1}(s) - u^R(s)\|^2 ds \\
&\leq \Big(1 + 2L_0^2(R) + \frac{2L_0^2(R)}{1-\rho^*}\Big) \int_0^t \sup_{-\rho\leq r\leq s} \|u^{R+1}(r\wedge\varsigma_R) - u^R(r\wedge\varsigma_R)\|^2 ds,
\end{aligned} \tag{4.113}$$

where we used the fact that $f^{R+1}(s,u^R(s),u^R(s-\varrho(s)))=f^R(s,u^R(s),u^R(s-\varrho(s)))$ due to $\|u_s^R(\cdot)\|\le R$ for all $s\in[0,\varsigma_R)$. We replace (2.41) in [96] with (4.113), then following the arguments in [96, Lemma 1] and noticing that $q_1,q_2:\mathbb{R}\to\mathbb{R}$ and $q_3=(q_{3,i})_{i\in\mathbb{Z}}:\mathbb{R}\to\ell^2$ are positive, continuous and $\mathcal{T}$-periodic functions, we can derive (4.112).

(2) Next we prove that

$$\varsigma:=\lim_{R\to+\infty}\varsigma_R=\infty,\quad \mathbb{P}\text{-almost surely.} \tag{4.114}$$

Similarly to Theorem 4.2 (2) and [96, Lemma 2], we obtain (4.114). Combining (4.112) and (4.114), using the argument of [96, Theorem 3], we can derive (4.111). □

We now prove the existence of periodic measures of stochastic delay modified Swift-Hohenberg lattice system (4.14).

Theorem 4.11. *Suppose* **F1-F6**, (4.34)*,* (4.37) *and* (4.38) *hold with* $p\ge 6$*. Then, system* (4.14) *has a periodic measures on* C_ρ*, for any* $\epsilon\in(0,1]$*.*

Proof. We first need to consider some properties of the transition operator $\{p_{r,t}\}_{r\le t}$ for solutions of system (4.14) as follows:

(1) **Feller property:** Using a similar approach to [96, Lemma 7] and combining Lemma 4.9, we realize that $\{p_{r,t}\}_{0\le r\le t}$ is Feller, that is, for any $0\le r\le t$, if for any $\phi:C_\rho\to\mathbb{R}$ is bounded and continuous, then $p_{r,t}\phi:C_\rho\to\mathbb{R}$ is also bounded and continuous.

(2) **$\mathcal{T}$-periodic:** It follows from [55, Lemma 4.1] that $\{p_{r,t}\}_{0\le r\le t}$ is $\mathcal{T}$-periodic, namely, for any $0\le r\le t$ and $\nu\in C_\rho$, $p(r,\nu;t,\cdot)=p(r+\mathcal{T},\nu;t+\mathcal{T},\cdot)$.

(3) **Markov property:** Given $r\ge 0$ and $\nu\in C_\rho$, we mainly need to prove that the solution $\{u_t(\cdot,r,\nu)\}_{r\le t}$ of system (4.14) is a C_ρ-valued Markov process, that is, for every bounded and continuous function $\phi:C_\rho\to\mathbb{R}$ and for all $0\le r\le s\le t$,

$$\mathbb{E}(\phi(u_t(\cdot,r,\nu))|\mathcal{F}_s)=(p_{s,t}\phi)(z)|_{z=u_s(\cdot,r,\nu)},\quad \mathbb{P}\text{-a.s..} \tag{4.115}$$

Now we briefly provide a standard proof procedure similar to [89, Lemma 4.5]. Since f^R,σ_j^R, $|B\zeta_R(u^R)|^2$ and $\zeta_R^3(u^R)$ satisfy (4.104)-(4.109), we can obtain that for every $\mathcal{F}_0$-measurable random variable $\varphi\in L^2(\Omega,C_\rho)$,

$$\mathbb{E}\big(\phi(u_t^R(\cdot,s,\varphi))|\mathcal{F}_s\big)=\mathbb{E}\big(\phi(u_t^R(\cdot,s,z))\big)|_{z=\varphi},\quad \mathbb{P}\text{-a.s.,} \tag{4.116}$$

which can be derived from [71, P51]. Recall that Lemma 4.10 ensures:

$$\lim_{R\to\infty}u_t^R(\cdot,s,\varphi)=u_t(\cdot,s,\varphi)\ \ \mathbb{P}\text{-a.s..} \tag{4.117}$$

From the uniqueness of the solution, for all $0\le r\le s\le t$, we have

$$u_t(\cdot,r,\varphi)=u_t(\cdot,s,u_s(\cdot,r,\varphi))\ \ \mathbb{P}\text{-a.s.,} \tag{4.118}$$

which together with (4.116)-(4.117) and the Lebesgue dominated convergence theorem, we conclude (4.115).

Then, it can be inferred from (3) that if $\phi \in B_b(C_\rho)$, for any $\nu \in C_\rho$ and $0 \le s \le r \le t$, $\mathbb{P}$-a.s., $(p_{s,t}\phi)(\nu) = (p_{s,r}(p_{r,t}\phi))(\nu)$. Thus the Chapman-Kolmogorov equation can be satisfied as:

$$p(s, \nu; t, \Lambda) = \int_{C_\rho} p(s, \nu; r, dy) p(r, y; t, \Lambda), \tag{4.119}$$

where $\nu \in C_\rho$ and $\Lambda \in \mathscr{B}(C_\rho)$.

Finally, based on the Krylov-Bogolyubov method and the tightness of distributions of solutions shown in Lemma 4.9, we can use the method of Theorem 4.3 in [55] to prove the existence of periodic measures in the sense of (4.100). □

4.4 Limits stability of periodic measures as noise intensity goes to zero

In this section, we consider the limiting behavior of periodic measures of (4.14) as the noise intensity $\epsilon \to 0$. To that end, we need to strengthen assumptions (4.34) as follows:

$$\begin{aligned} \min_{t\in[0,\mathcal{T}]} q_2(t) > 16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + 3 \cdot 2^{3-\frac{2}{p}} p^{-1} (p-1)^{1-\frac{1}{p}} \max_{t\in[0,\mathcal{T}]} \beta_0(t) \\ + 4\big[(3p-4) + 2(2-\frac{2}{p})^{\frac{p}{2}}\big] \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2. \end{aligned} \tag{4.120}$$

Then, we can still apply (4.120) to derive $\Theta_1(t) > 0$ and $\overline{\Theta}_1(t) > 0$ for $t \in \mathbb{R}$. In addition, most importantly, we can also deduce from (4.120) that

$$\Theta_1(t) > \Theta_2(t) \ \text{ and } \ \overline{\Theta}_1(t) > \overline{\Theta}_2(t), \ \forall t \in \mathbb{R}, \tag{4.121}$$

where $\Theta_1(t), \Theta_2(t)$ and $\overline{\Theta}_1(t), \overline{\Theta}_2(t)$ are defined in (4.35) and (4.36), respectively. By (4.98) and the continuity and periodicity of Θ_1, Θ_2 and $\overline{\Theta}_1, \overline{\Theta}_2$, there exists $\varepsilon > 0$ such that $\Theta_1(t) - \Theta_2(t) \ge \varepsilon$ and $\overline{\Theta}_1(t) - \overline{\Theta}_2(t) \ge \varepsilon$. Hence there exists $h_0 > 0$ such that

$$\int_0^{\mathcal{T}} \Big(\Theta_1(s) - \Theta_2(s) e^{\int_{s-h_0}^s \Theta_1(r)dr}\Big) ds > 0, \tag{4.122}$$

and

$$\int_0^{\mathcal{T}} \Big(\overline{\Theta}_1(s) - \overline{\Theta}_2(s) e^{\int_{s-h_0}^s \overline{\Theta}_1(r)dr}\Big) ds > 0. \tag{4.123}$$

Noting that (4.121)-(4.123) exactly satisfy conditions in [56, Lemma 3.1], we can use this lemma so that all the estimates of Section 4.2 are independent of the initial data. Furthermore, to illustrate the dependence of solutions of system (4.14) on the noise intensity ϵ, we denote it as $u_t^\epsilon(\cdot, 0, \varphi)$ in the relevant proofs that follow.

Applying the conclusion of [56, Lemma 3.1] in the proof of Lemma 4.5 immediately leads to the following Lemma.

Lemma 4.12. *Suppose* **F1**-**F6** *and* (4.120) *hold. Then, for every $R > 0$, there exists $T_1 = T_1(R) > 0$ such that the solution of the system* (4.14) *satisfies for all $t \geq T_1$ and $\epsilon \in [0, 1]$,*

$$\mathbb{E}(\|u^\epsilon(t, 0, \varphi)\|^p) \leq C_6,$$

where $\mathbb{E}(\|\varphi\|_{C_\rho}^p) \leq R$ and $C_6 > 0$ is not depending on t, ϵ, ρ, R and φ.

Similar to Lemma 4.6, it can also be deduced from Lemma 4.12 that

Lemma 4.13. *Suppose* **F1**-**F6** *and* (4.120) *hold. Then for every $R > 0$, there exists $T_2 = T_2(R) > 0$ such that the solution of system* (4.14) *satisfies for all $t \geq r \geq T_2$ and $\epsilon \in [0, 1]$,*

$$\mathbb{E}(\|u^\epsilon(t, 0, \varphi) - u^\epsilon(r, 0, \varphi)\|^{\frac{p}{3}}) \leq C_7(|t - r|^{\frac{p}{3}} + |t - r|^{\frac{p}{6}}),$$

where $\mathbb{E}(\|\varphi\|_{C_\rho}^p) \leq R$ and $C_7 > 0$ is independent of ϵ, ρ, R and φ.

By replacing Lemma 4.4 with the conclusion of [56, Lemma 3.1] and applying it to the proof of Lemma 4.8, we can derive the following tail-estimate of solutions:

Lemma 4.14. *Suppose* **F1**-**F6** *and* (4.120) *hold. Then, for every $R > 0$ and $\varepsilon > 0$, there exist $T_3 = T_3(R, \varepsilon) > 0$ and $K_1 = K_1(\varepsilon) \geq K$ such that the solution of system* (4.14) *satisfies for all $t \geq T_3$, $k \geq K_1$ and $\epsilon \in [0, 1]$,*

$$\mathbb{E}\left(\sup_{t-\rho \leq r \leq t} \sum_{|i| \geq k} |u_i^\epsilon(r, 0, \varphi)|^2 \right) \leq \varepsilon, \tag{4.124}$$

where $\varphi \in L^2(\Omega, C_\rho)$ such that $\mathbb{E}(\|\varphi\|_{C_\rho}^2) \leq R$.

We now show the convergence of solutions of system (4.14) w.r.t. noise intensity ϵ as follows:

Lemma 4.15. *Suppose* **F1**-**F6** *and* (4.120) *hold. Then, for every compact set $\mathcal{K}$ of C_ρ, $\delta > 0$, $t \geq 0$ and $\epsilon_0 \geq 0$,*

$$\lim_{\epsilon \to \epsilon_0} \sup_{\varphi \in \mathcal{K}} \mathbb{P}\Big[\|u_t^\epsilon(\cdot, 0, \varphi) - u_t^{\epsilon_0}(\cdot, 0, \varphi)\|_{C_\rho} \geq \delta \Big] = 0. \tag{4.125}$$

Proof. For the sake of simplicity, we let $u_1(t) = u^\epsilon(t, 0, \varphi)$ and $u_2(t) = u^{\epsilon_0}(t, 0, \varphi)$ for $t \geq -\rho$. Given $T > 0$, for every compact set $\mathcal{K}$ of C_ρ, it follows from Theorem 4.2 that there exists $c = c(\mathcal{K}, T) > 0$ such that, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$,

$$\mathbb{E}\Big(\sup_{t \in [-\rho, T]} \|u_1(t)\|^2 \Big) \leq c.$$

Thus, for every $\varepsilon > 0$, there exists $n = n(\varepsilon, \mathcal{K}, T) > 0$ such that, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$,

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[-\rho,T]} \|u_1(t)\| > n\right\}\right) < \frac{1}{2}\varepsilon.$$

Given $\varphi \in \mathcal{K}$ and $\epsilon, \epsilon_0 \in (0, 1]$, we define

$$\Omega_\epsilon = \left\{\omega \in \Omega : \sup_{t\in[-\rho,T]} \|u_1(t)\| \le n \text{ and } \sup_{t\in[-\rho,T]} \|u_2(t)\| \le n\right\}.$$

Then, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$, we have $\mathbb{P}(\Omega \setminus \Omega_\epsilon) < \varepsilon$.

Given $R > 0$, we define a stopping time by

$$T_R = \inf\{t \ge 0 : \|u_1(t)\| > R \;\text{ or }\; \|u_2(t)\| > R\}. \tag{4.126}$$

Generally, $\inf \emptyset = \infty$, and we can see that $T_R \ge T$ for each $\omega \in \Omega_\epsilon$. For any $\varphi \in \mathcal{K}$ and $\delta > 0$,

$$\begin{aligned}
&\sup_{\varphi\in\mathcal{K}} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[-\rho,T]} \|u_1(t) - u_2(t)\| \ge \delta\right\}\right)\\
&= \sup_{\varphi\in\mathcal{K}} \mathbb{P}\left(\left\{\omega \in \Omega_\epsilon : \sup_{t\in[-\rho,T]} \|u_1(t\wedge T_R) - u_2(t\wedge T_R)\| \ge \delta\right\}\right)\\
&+ \sup_{\varphi\in\mathcal{K}} \mathbb{P}\left(\left\{\omega \in \Omega\setminus\Omega_\epsilon : \sup_{t\in[-\rho,T]} \|u_1(t) - u_2(t)\| \ge \delta\right\}\right)\\
&\le \sup_{\varphi\in\mathcal{K}} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[-\rho,T]} \|u_1(t\wedge T_R) - u_2(t\wedge T_R)\| \ge \delta\right\}\right) + \varepsilon.
\end{aligned} \tag{4.127}$$

To complete the proof, we need to show that

$$\lim_{\epsilon\to\epsilon_0} \sup_{\varphi\in\mathcal{K}} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[-\rho,T]} \|u_1(t\wedge T_R) - u_2(t\wedge T_R)\| \ge \delta\right\}\right) = 0. \tag{4.128}$$

Similar to [96, Lemma 11], we can obtain from (4.14) that, for all $t \ge 0$,

$$\begin{aligned}
&\|u_1(t\wedge T_R) - u_2(t\wedge T_R)\|^2 + 2\int_0^{t\wedge T_R} q_1(r)\|A\big(u_1(r) - u_2(r)\big)\|^2 dr\\
&- 4\int_0^{t\wedge T_R} q_1(r)\|B\big(u_1(r) - u_2(r)\big)\|^2 dr + 2\int_0^{t\wedge T_R} q_2(r)\|u_1(r) - u_2(r)\|^2 dr\\
&+ 2\int_0^{t\wedge T_R} \big(u_1^3(r) - u_2^3(r), u_1(r) - u_2(r)\big) dr\\
&+ 2\int_0^{t\wedge T_R} \Big(q_3(r)\big(|Bu_1(r)|^2 - |Bu_2(r)|^2\big), u_1(r) - u_2(r)\Big) dr
\end{aligned}$$

$$
\begin{aligned}
&\le 2\int_0^{t\wedge T_R}\big(f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dr\\
&+3\epsilon_0^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))-\sigma_j(r,u_2(r),u_2(r-\varrho(r)))\|^2dr\\
&+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))\|^2dr\\
&+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|h_j(r)\|^2dr+2|\epsilon-\epsilon_0|\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}(h_j(r),u_1(r)-u_2(r))dW_j(r)\\
&+2|\epsilon-\epsilon_0|\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}(\sigma_j(r,u_1(r),u_1(r-\varrho(r))),u_1(r)-u_2(r))dW_j(r)\\
&+2\epsilon_0\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\big(\sigma_j(r,u_1(r),u_1(r-\varrho(r)))\\
&-\sigma_j(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dW_j(r).
\end{aligned}\tag{4.129}
$$

For the second-to-last term on the left-hand side of (4.129), we obtain

$$
\begin{aligned}
&-2\int_0^{t\wedge T_R}\big(u_1^3(r)-u_2^3(r),u_1(r)-u_2(r)\big)dr\\
&\le 2\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}\big|(u_{1,i}(r)-u_{2,i}(r))(u_{1,i}^2(r)+u_{1,i}(r)u_{2,i}(r)\\
&\quad+u_{2,i}^2(r))\big||u_{1,i}(r)-u_{2,i}(r)|dr\\
&\le\frac{3}{2}\int_0^{t\wedge T_R}(\|u_1(r)\|^2+\|u_2(r)\|^2)\|u_1(r)-u_2(r)\|^2dr\\
&\le 3R^2\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr.
\end{aligned}\tag{4.130}
$$

For the last term on the left-hand side of (4.129), by (4.4) we have

$$
\begin{aligned}
&-2\int_0^{t\wedge T_R}\Big(q_3(r)\big(|Bu_1(r)|^2-|Bu_2(r)|^2\big),u_1(r)-u_2(r)\Big)dr\\
&\le 2\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}|q_{3,i}(r)|\Big||u_{1,i+1}(r)-u_{1,i}(r)|^2\\
&\quad-|u_{2,i+1}(r)-u_{2,i}(r)|^2\Big||u_{1,i}(r)-u_{2,i}(r)|dr\\
&\le 8\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}|q_{3,i}(r)|^2|u_{1,i}(r)-u_{2,i}(r)|^2dr+\frac{1}{8}\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}\big(|u_{1,i+1}(r)-u_{1,i}(r)|\\
&+|u_{2,i+1}(r)-u_{2,i}(r)|\big)^2\Big|u_{1,i+1}(r)-u_{1,i}(r)-(u_{2,i+1}(r)-u_{2,i}(r))\Big|^2dr\\
&\le 8\int_0^{t\wedge T_R}\|q_3(r)\|^2\|u_1(r)-u_2(r)\|^2dr+\frac{1}{4}\int_0^{t\wedge T_R}(\|Bu_1(r)\|^2+\|Bu_2(r)\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&\times \|Bu_1(r)-Bu_2(r)\|^2\\
&\leq 8\int_0^{t\wedge T_R}(\|q_3(r)\|^2+R^2)\|u_1(r)-u_2(r)\|^2dr. \qquad (4.131)
\end{aligned}
$$

Combining (4.126) and (4.15), for the first term on the right-hand side of (4.129), we can deduce

$$
\begin{aligned}
&2\int_0^{t\wedge T_R}\big(f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dr\\
&\leq \int_0^{t\wedge T_R}\|f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r)))\|^2dr\\
&\quad+\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr\\
&\leq L_R^f\int_0^{t\wedge T_R}(\|u_1(r)-u_2(r)\|^2+\|u_1(r-\varrho(r))-u_2(r-\varrho(r))\|^2)dr\\
&\quad+\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr\\
&\leq (L_R^f+1+\frac{L_R^f}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr. \qquad (4.132)
\end{aligned}
$$

Analogously, by (4.126), (4.17) and (4.18), for the second and third terms on the right-hand side of (4.129),

$$
\begin{aligned}
&3\epsilon_0^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))-\sigma_j(r,u_2(r),u_2(r-\varrho(r)))\|^2dr\\
&\quad+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))\|^2dr\\
&\leq \frac{3\epsilon_0^2}{2}(L_R^\sigma+1+\frac{L_R^\sigma}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr+12(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\beta(r)\|^2\\
&\quad\times\|u_1(r)\|^2dr+12(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\beta(r)\|^2\|u_1(r-\varrho(r))\|^2dr\\
&\quad+6(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\gamma\|^2dr\\
&\leq \frac{3\epsilon_0^2}{2}(L_R^\sigma+1+\frac{L_R^\sigma}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr+\frac{12(\epsilon-\epsilon_0)^2}{1-\rho^*}\int_{-\rho}^0\|\beta(r)\|^2\\
&\quad\times\|\varphi(r)\|^2dr+6(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\big[2(1+\frac{1}{1-\rho^*})R^2\|\beta(r)\|^2+\|\gamma\|^2\big]dr. \qquad (4.133)
\end{aligned}
$$

Bringing the combination of (4.131)-(4.133) into (4.129) and taking expectation, we have

$$
\mathbb{E}\Big(\sup_{0\leq r\leq t}\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)
$$

$$
\begin{aligned}
&\leq \int_0^t (16q_1(r) + 8\|q_3(r)\|^2 + c_{10})\mathbb{E}\Big(\sup_{0\leq r\leq s} \|u_1(r\wedge T_R) - u_2(r\wedge T_R)\|^2\Big) ds \\
&+ 3(\epsilon-\epsilon_0)^2 \sum_{j=1}^{\infty} \int_0^{t\wedge T_R} \|h_j(r)\|^2 dr + I_4,
\end{aligned} \tag{4.134}
$$

where $c_{10} = 11R^2 + L_R^f + 1 + \frac{L_R^f}{1-\rho^*} + \frac{3\epsilon_0^2}{2}(L_R^\sigma + 1 + \frac{L_R^\sigma}{1-\rho^*})$ and

$$
\begin{aligned}
I_4 &= 2|\epsilon-\epsilon_0|\mathbb{E}\Big(\sup_{0\leq r\leq t\wedge T_R} \Big|\sum_{j=1}^{\infty}\int_0^r (h_j(s), u_1(s) - u_2(s))dW_j(s)\Big|\Big) \\
&+ 2|\epsilon-\epsilon_0|\mathbb{E}\Big(\sup_{0\leq r\leq t\wedge T_R} \Big|\sum_{j=1}^{\infty}\int_0^r (\sigma_j(s, u_1(s), u_1(s-\varrho(s))), u_1(s) - u_2(s))dW_j(s)\Big|\Big) \\
&+ 2\epsilon_0\mathbb{E}\Big(\sup_{0\leq r\leq t\wedge T_R} \Big|\sum_{j=1}^{\infty}\int_0^r \big(\sigma_j(s, u_1(s), u_1(s-\varrho(s))) \\
&\quad - \sigma_j(s, u_2(s), u_2(s-\varrho(s))), u_1(s) - u_2(s)\big)dW_j(s)\Big|\Big).
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality, we derive from (4.133) that

$$
\begin{aligned}
I_4 &\leq \frac{3}{4}\mathbb{E}\Big(\sup_{0\leq r\leq t} \|u_1(r\wedge T_R) - u_2(r\wedge T_R)\|^2\Big) \\
&+ c(\epsilon-\epsilon_0)^2 C_0^2 \sum_{j=1}^{\infty}\int_0^{t\wedge T_R} \|h_j(r)\|^2 dr + c(\epsilon-\epsilon_0)^2 C_0^2 \int_0^{t\wedge T_R} \big[\|\beta(r)\|^2 \\
&+ \|\gamma\|^2\big] dr + c(\epsilon-\epsilon_0)^2 C_0^2 \int_{-\rho}^0 \|\beta(r)\|^2 \|\varphi(r)\|^2 dr \\
&+ cC_0^2 \int_0^t \mathbb{E}\Big(\sup_{0\leq r\leq s} (\|u_1(r\wedge T_R) - u_2(r\wedge T_R)\|^2\Big) ds.
\end{aligned} \tag{4.135}
$$

Combining (4.134) and (4.135), for every compact set $\mathcal{K}$ of C_ρ, there exists $c_{11} = c_{11}(\mathcal{K}) > 0$ such that, for all $\varphi \in \mathcal{K}$ and $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq r\leq t} \|u_1(r\wedge T_R) - u_2(r\wedge T_R)\|^2\Big) \qquad (4.136)\\
&\leq 4(16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + c_{10}) \int_0^t \mathbb{E}\Big(\sup_{0\leq r\leq s} (\|u_1(r\wedge T_R) \\
&\quad - u_2(r\wedge T_R)\|^2\Big) ds + c_{11}(\epsilon-\epsilon_0)^2 T\Big(\sup_{r\in[0,\mathcal{T}]} \sum_{j=1}^{\infty} \|h_j(r)\|^2 + \sup_{r\in[0,\mathcal{T}]} \|\beta(r)\|^2 + \|\gamma\|^2\Big).
\end{aligned}
$$

From the Gronwall inequality, for all $t \in [0, T]$,

$$
\mathbb{E}\Big(\sup_{0\leq r\leq t} \|u_1(r) - u_2(r)\|^2\Big)
$$

$$\leq c_{11}(\epsilon-\epsilon_0)^2 T\Big(\sup_{r\in[0,\mathcal{T}]}\sum_{j=1}^{\infty}\|h_j(r)\|^2+\sup_{r\in[0,\mathcal{T}]}\|\beta(r)\|^2+\|\gamma\|^2\Big)$$
$$\times e^{4(16\max_{t\in[0,\mathcal{T}]}q_1(t)+8\sup_{t\in[0,\mathcal{T}]}\|q_3(t)\|^2+c_{10})T}, \tag{4.137}$$

which means that, for all $t\in[0,T]$,

$$\mathbb{E}\Big(\sup_{-\rho\leq r\leq t}\|u_1(r)-u_2(r)\|^2\Big)$$
$$\leq c_{11}(\epsilon-\epsilon_0)^2 T\Big(\sup_{r\in[0,\mathcal{T}]}\sum_{j=1}^{\infty}\|h_j(r)\|^2+\sup_{r\in[0,\mathcal{T}]}\|\beta(r)\|^2+\|\gamma\|^2\Big)$$
$$\times e^{4(16\max_{t\in[0,\mathcal{T}]}q_1(t)+8\sup_{t\in[0,\mathcal{T}]}\|q_3(t)\|^2+c_{10})T}, \tag{4.138}$$

which together with Chebyshev's inequality, we can obtain (4.128) as ϵ tends to ϵ_0. The proof can then be completed by (4.127) and (4.128). □

Given $\epsilon\in[0,1]$, let S^ϵ be the collection of all $\mathcal{T}$-periodic measures μ^ϵ of system (4.14). Notice that all estimates in Section 4.2 are valid under (4.120),(4.122) and (4.123), thus for every $\epsilon\in[0,1]$, S^ϵ is nonempty by the argument of Theorem 4.11.

Now, we show the tightness of $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ in the sense of (4.101), then use Theorem 5.1 in [55] to establish the limiting behavior of any sequence of S^ϵ for the system (4.14) as $\epsilon\to 0$.

Theorem 4.16. *Suppose* **F1**-**F6** *and* (4.120) *hold. Then,*

(i) $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ *is tight on* C_ρ.

(ii) If $\mu^{\epsilon_n}\in S^{\epsilon_n}$ *with* $\epsilon_n\to\epsilon_0\in[0,1]$, *then there exist a subsequence* ϵ_{n_k} *and a* $\mathcal{T}$*-periodic measure* $\mu^{\epsilon_0}\in S^{\epsilon_0}$ *such that* $\mu^{\epsilon_n}\rightharpoonup\mu^{\epsilon_0}$.

Proof. (i) Given a compact set $\widetilde{\mathcal{K}}$ of C_ρ. Based on Lemma 4.12-Lemma 4.14, using a similar approach to Lemma 4.9 , it is known that, for every $\varepsilon>0$ and $\varphi\in C_\rho$, there exists $T_\varepsilon>0$ such that, for all $t\geq T_\varepsilon$ and $\epsilon\in[0,1]$,

$$\mathbb{P}\{\omega\in\Omega: u_t^\epsilon(\cdot;0,\varphi)\in\widetilde{\mathcal{K}}\}>1-\varepsilon.$$

Then using the method of [55, Theorem 5.6], we can obtain that for any $\mu^\epsilon\in S^\epsilon$ with $\epsilon\in[0,1]$,

$$\mu^\epsilon(\widetilde{\mathcal{K}})\geq 1-\varepsilon,$$

which implies that $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ is tight in the sense of (4.101).

(ii) It is clear from (i) that $\{\mu^{\epsilon_n}\}_{n=1}^{\infty}$ is tight, which means that there exist a subsequence $\{\mu^{\epsilon_{n_m}}\}_{m=1}^{\infty}$ of $\{\mu^{\epsilon_n}\}_{n=1}^{\infty}$ and a probability measure μ such that $\mu_t^{\epsilon_{n_m}}\rightharpoonup\mu$ as $m\to\infty$. Then, by Lemma 4.15 and [55, Theorem 5.1], we can complete the proof. □

Chapter 5

Evolution systems of measures for stochastic lattice systems without delay

In this chapter, we are concerned with the asymptotic stability of evolution systems of probability measures for non-autonomous stochastic discrete modified Swift-Hohenberg equations driven by local Lipschitz nonlinear noise. We first show the existence of evolution systems of probability measures of the original equation. Then using the theoretical results in [100], it is proved that the evolution system of probability measures of the limit equation is the limit of the evolution system of probability measures when the noise intensity tends to a certain value.

We emphasize that the methods used in this chapter can be applied to other stochastic lattice point systems, stochastic partial differential equations (e.g., containing different kinds of noise sources like Lévy processes), stochastic fractional partial differential equations, etc.

In Sect. 5.1, we show the existence of solutions under suitable assumptions. In Sect. 5.2, we establish the existence of evolution systems of probability measures, and then prove the limiting stability of evolution systems of probability measures in ℓ^2.

5.1 Existence of solutions

5.1.1 Some assumptions

We define a nonlinear function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(s) = s^3, \quad \forall s \in \mathbb{R}, \tag{5.1}$$

which is locally Lipschitz continuous. Indeed, for all $s_1, s_2 \in \mathbb{R}$,

$$|G(s_1) - G(s_2)| = |s_1^3 - s_2^3| = |(s_1 - s_2)(s_1^2 + s_1 s_2 + s_2^2)| \leq \frac{3}{2}(s_1^2 + s_2^2)|s_1 - s_2|. \tag{5.2}$$

Assume that the progressively measurable processes $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h_j(t) = (h_{j,i}(t))_{i\in\mathbb{Z}}$ satisfy:

$$\int_{\tau}^{\tau+T} \mathbb{E}(\|g(t)\|^2 + \sum_{j=1}^{\infty} \|h_j(t)\|^2)dt < \infty, \quad \forall \tau \in \mathbb{R}, T > 0. \tag{5.3}$$

Then suppose that the drift term $f_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, uniformly in $i \in \mathbb{Z}$, namely, for every bounded subset $I \subseteq \mathbb{R}$, there exists a constant $L_I > 0$ such that

$$|f_i(s_1) - f_i(s_2)| \leq L_I|s_1 - s_2|, \quad \forall s_1, s_2 \in I \subseteq \mathbb{R} \ \text{ and } \ i \in \mathbb{Z}. \tag{5.4}$$

Furthermore, for each $i \in \mathbb{Z}$, we assume there exist positive numbers α_i and L_0 such that

$$|f_i(s)| \leq L_0|s| + \alpha_i, \quad \forall s \in \mathbb{R}, \tag{5.5}$$

where $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$.

For the diffusion term, we also assume that $\sigma_{j,i}(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, that is, for any bounded subset $K \subseteq \mathbb{R}$, there exists a constant $L_K > 0$ such that

$$|\sigma_{j,i}(t, s_1) - \sigma_{j,i}(t, s_2)| \leq L_K|s_1 - s_2|, \quad \forall s_1, s_2 \in K, t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}. \tag{5.6}$$

Moreover, we assume that for each $t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$|\sigma_{j,i}(t, s)| \leq L_1|s| + \beta_i(t), \tag{5.7}$$

where L_1 is a positive constant and $\beta = (\beta_i)_{i\in\mathbb{Z}} \in L^2_{loc}(\mathbb{R}, \ell^2)$. Notice that L_0, L_K, and L_1 do not depend on i and j.

Let $B_b(\ell^2)$ (respectively, $C_b(\ell^2)$) be the space of all bounded Borel-measurable (respectively, bounded continuous) functions on ℓ^2, $\mathcal{P}(\ell^2)$ be the set of all probability measures on ℓ^2. Suppose $\{X^\epsilon(t, \tau, x), t \geq \tau \in \mathbb{R}\}$ is a stochastic process with initial value $x \in \ell^2$ and $\epsilon \in (0, \bar{\epsilon}]$ for any $\bar{\epsilon} > 0$.

For $\varphi \in B_b(\ell^2)$ and $\Lambda \in \mathscr{B}(\ell^2)$ (the Borel σ-algebra on ℓ^2), we define a probability transition operator $(P^\epsilon_{\tau,t})_{t\geq\tau}$ of $X^\epsilon(t, \tau, x)$ by

$$(P^\epsilon_{\tau,t}\varphi)(x) = \mathbb{E}[\varphi(X^\epsilon(t, \tau, x))], \quad \text{for all} \ \ x \in \ell^2, \tag{5.8}$$

and a transition probability function

$$P^\epsilon_{\tau,t}(x, \Lambda) = \mathbb{P}\{\omega \in \Omega : X^\epsilon(t, \tau, x) \in \Lambda\}, \quad \text{for all} \ \ x \in \ell^2. \tag{5.9}$$

For $\eta \in \mathcal{P}(\ell^2)$, we define the adjoint operator $(Q^\epsilon_{\tau,t})_{t\geq\tau}$ of $(P^\epsilon_{\tau,t})_{t\geq\tau}$ by

$$Q^\epsilon_{\tau,t}\eta(\Lambda) = \int_{\ell^2} P^\epsilon_{\tau,t}(x, \Lambda)\eta(dx), \quad \text{for all} \ \ x \in \ell^2. \tag{5.10}$$

Moreover, for any $-\infty < \tau \leq r \leq t < +\infty$, the semigroup laws hold:

$$P^\epsilon_{\tau,t} = P^\epsilon_{\tau,r}P^\epsilon_{r,t} \ \text{ and } \ Q^\epsilon_{\tau,t} = Q^\epsilon_{r,t}Q^\epsilon_{\tau,r}. \tag{5.11}$$

Let us recall some definitions that will be used in some of the later proofs.

Definition 5.1. *([6]) A family* $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(\ell^2)$ *of measures on* $(\ell^2, \mathscr{B}(\ell^2))$ *is* **tight** *if for each* $\varepsilon > 0$*, there exists a compact set* Z_ε *of* ℓ^2 *such that*

$$\eta_t(\ell^2 \setminus Z_\varepsilon) < \varepsilon, \quad \textit{for all} \ \ t \in \mathbb{R}.$$

Definition 5.2. *([35])* $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(\ell^2)$ *is an* **evolution system of probability measures** *of* $(P^{\epsilon}_{\tau,t})_{t\geq\tau}$ *if for all* $t \geq \tau \in \mathbb{R}$,

$$Q^{\epsilon}_{\tau,t}\eta_\tau = \eta_t. \tag{5.12}$$

In order to achieve our final result, we need to impose the following assumptions:
R1. There are $a_0 \in (0, a)$ and $b \in \ell^2$ such that $\vartheta > 0$.
R2. For all $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} e^{ar}\mathbb{E}\Big(\|g(r)\|^2 + \|\beta(r)\|^2 + \sum_{j=1}^{\infty}\|h_j(r)\|^2\Big)dr < \infty.$$

where $g, h_j \in C(\mathbb{R}, \ell^2)$ and $\beta \in L^2_{loc}(\mathbb{R}, \ell^2)$ is defined in (5.7).

5.1.2 Estimates of solutions

We denote the following operators from ℓ^2 to ℓ^2 as follows: for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i,$$

and

$$(Du)_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}.$$

Then, for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$ and $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, we infer that

$$A = BB^* = B^*B \text{ and } (Bu, v) = (u, B^*v), \tag{5.13}$$
$$(Au, v) = (Bu, Bv) \text{ and } (Du, v) = (Au, Av), \tag{5.14}$$
$$\|Bu\|^2 = \|B^*u\|^2 \leq 4\|u\|^2, \quad \|Au\|^2 \leq 16\|u\|^2, \quad \|Du\|^2 \leq 256\|u\|^2. \tag{5.15}$$

For $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, we define an operator $F : \ell^2 \to \ell^2$ by

$$(F(u))_i = \big((|(Bu)_i|^2)_{i\in\mathbb{Z}}\big)_i = |u_{i+1} - u_i|^2. \tag{5.16}$$

Hence, it follows from (5.15) and (5.16) that for each $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2, v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} |(F(u))_i - (F(v))_i|^2 = \sum_{i\in\mathbb{Z}} \Big||u_{i+1} - u_i|^2 - |v_{i+1} - v_i|^2\Big|^2 \\
&\leq \sum_{i\in\mathbb{Z}} (|u_{i+1} - u_i| + |v_{i+1} - v_i|)^2 |u_{i+1} - u_i - (v_{i+1} - v_i)|^2 \\
&\leq 2(\|Bu\|^2 + \|Bv\|^2)\|Bu - Bv\|^2 \leq 32(\|u\|^2 + \|v\|^2)\|u - v\|^2.
\end{aligned} \tag{5.17}$$

Now, we consider $u = (u_i)_{i\in\mathbb{Z}}$, $G(u) = (G(u_i))_{i\in\mathbb{Z}}$, $F(u) = (F_i(u_i))_{i\in\mathbb{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ and $\sigma(u) = (\sigma_{j,i}(u_i))_{i\in\mathbb{Z}}$ for $j \in \mathbb{N}$.

By (5.2), we have

$$\|G(u) - G(v)\| \le \frac{3}{2}(\|u\|^2 + \|v\|^2)\|u - v\|, \quad \forall u, v \in \ell^2, \tag{5.18}$$

which implies that $G : \ell^2 \to \ell^2$ is locally Lipschitz continuous.

It follows from (5.17) that $F : \ell^2 \to \ell^2$ is locally Lipschitz continuous, that is, for every bounded set E of ℓ^2, there exists positive constant L_F (which may depend on E) such that for all $u, v \in E$,

$$\|F(u) - F(v)\| \le L_F\|u - v\|. \tag{5.19}$$

It can be inferred from (5.4) that $f : \ell^2 \to \ell^2$ is locally Lipschitz continuous, namely, for every $n \in \mathbb{N}$, there exists a constant $L_2^f(n) > 0$ such that, for all $u, v \in \ell^2$ with $\|u\| \le n$ and $\|v\| \le n$,

$$\|f(u) - f(v)\|^2 \le L_2^f(n)\|u - v\|^2. \tag{5.20}$$

By (5.5), we deduce that, for all $u \in \ell^2$

$$\|f(u)\|^2 \le 2L_0^2\|u\|^2 + 2\|\alpha\|^2. \tag{5.21}$$

Similarly, by (5.6) we find that $\sigma_j(t, \cdot) : \ell^2 \to \ell^2$ is locally Lipschitz continuous. More precisely, for every $n \in \mathbb{N}$, there exists a constant $L_2^\sigma(n) > 0$ such that, for all $u, v \in \ell^2$ with $\|u\| \le n$ and $\|v\| \le n$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t, u) - \sigma_j(t, v)\|^2 \le L_2^\sigma(n)\|u - v\|^2, \quad \forall t \in \mathbb{R}. \tag{5.22}$$

In addition, by (5.7), we have for all $t \in \mathbb{R}, u \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t, u)\|^2 \le 2L_1^2\|u\|^2 + 2\|\beta(t)\|^2. \tag{5.23}$$

Now, we rewrite (11) as the following stochastic modified S-H lattice systems in ℓ^2 for $t > \tau$ with $\tau \in \mathbb{R}$:

$$\begin{cases} du(t) + \big(Du(t) - 2Au(t) + au(t) + bF(u(t)) + G(u(t))\big)dt \\ \quad = f(u(t))dt + g(t)dt + \epsilon \displaystyle\sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t))\big)dW_j(t), \\ u(\tau) = u_\tau = (u_{\tau,i})_{i\in\mathbb{Z}}, \end{cases} \tag{5.24}$$

where $a > 0$ and $b = (b_i)_{i\in\mathbb{Z}} \in \ell^2$ satisfy **R1**.

Definition 5.3. *Suppose $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$. Then a continuous ℓ^2-valued stochastic process u is called a solution of system* (5.24) *if, for all $T > 0$*

$$u \in L^2(\Omega, C([\tau, \tau + T], \ell^2)), \tag{5.25}$$

and, for all $\tau \le t \le \tau + T$ and almost all $\omega \in \Omega$,

$$\begin{aligned} u(t) = u_\tau + \int_\tau^t \big(- Du(s) + 2Au(s) - au(s) - bF(u(s)) - G(u(s))\big) ds \\ + \int_\tau^t (f(u(s)) + g(s))ds + \epsilon \sum_{j=1}^{\infty} \int_\tau^t (h_j(s) + \sigma_j(s, u(s)))dW_j(s) \quad \text{in } \ell^2. \end{aligned} \tag{5.26}$$

To obtain the global existence of solutions of (5.24), we use a classical method proposed in [51]: approximate the nonlinear term using a suitable cut-off function. Since G, F, f and σ_j in (5.24) are nonlinear functions and locally Lipschitz continuous from ℓ^2 to ℓ^2, then, to approximate G, F, f, and σ_j, we choose a globally Lipschitz continuous function as the cut-off function. Given $n \in \mathbb{N}$, define $\zeta_n : \mathbb{R} \to \mathbb{R}$ by

$$\zeta_n(s) = \begin{cases} s, & \text{if } |s| \le n. \\ \dfrac{ns}{|s|}, & \text{if } |s| > n. \end{cases} \tag{5.27}$$

Then, we find that $\zeta_n : \mathbb{R} \to \mathbb{R}$ is increasing and satisfies for all $s, s_1, s_2 \in \mathbb{R}$,

$$|\zeta_n(s_1) - \zeta_n(s_2)| \le |s_1 - s_2|, \quad |\zeta_n(s)| \le |s| \wedge n, \quad \zeta_n(0) = 0. \tag{5.28}$$

For all $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, we set $\zeta_n(u) = (\zeta_n(u_i))_{i\in\mathbb{Z}}$ and define

$$F^n(u) = F(\zeta_n(u)) \quad \text{and} \quad G^n(u) = G(\zeta_n(u)).$$

By (5.17) and (5.28), we obtain that for all $u, v \in \ell^2$,

$$\begin{aligned} \|F^n(u) - F^n(v)\|^2 &= \sum_{i\in\mathbb{Z}} |(F(\zeta_n(u)))_i - (F(\zeta_n(v)))_i|^2 \\ &\le 32(\|\zeta_n(u)\|^2 + \|\zeta_n(v)\|^2)\|\zeta_n(u) - \zeta_n(v)\|^2 \le L_3^F(n)\|u - v\|^2, \end{aligned} \tag{5.29}$$

where $L_3^F(n) = 64n^2$. From the definition of F and (5.29), we can deduce that for all $u \in \ell^2$,

$$\|F^n(u)\|^2 \le L_3^F(n)\|u\|^2. \tag{5.30}$$

It follows from (5.18) and (5.28) that for all $u, v \in \ell^2$,

$$\begin{aligned} \|G^n(u) - G^n(v)\|^2 &= \sum_{i\in\mathbb{Z}} |(G(\zeta_n(u)))_i - (G(\zeta_n(v)))_i|^2 \\ &\le \frac{9}{4}(\|\zeta_n(u)\|^2 + \|\zeta_n(v)\|^2)^2\|\zeta_n(u) - \zeta_n(v)\|^2 \\ &\le L_3^G(n)\|u - v\|^2, \end{aligned} \tag{5.31}$$

where $L_3^G(n) = 9n^4$. By $G^n(0) = G(\zeta_n(0)) = 0$ and (5.31), we have for all $u \in \ell^2$,

$$\|G^n(u)\|^2 \le L_3^G(n)\|u\|^2. \tag{5.32}$$

Given $j, n \in \mathbb{N}$ and $i \in \mathbb{Z}$. For all $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, let

$$f^n(u) = \big(f_i(\zeta_n(u_i))\big)_{i\in\mathbb{Z}}, \quad \sigma_j^n(t,u) = \big(\sigma_{j,i}(t,\zeta_n(u_i))\big)_{i\in\mathbb{Z}}.$$

It can be inferred from (5.4) and (5.28) that $f^n : \ell^2 \to \ell^2$ is globally Lipschitz continuous, thus there exists a constant $L_3^f(n) > 0$ such that for all $u, v \in \ell^2$,

$$\|f^n(u) - f^n(v)\|^2 \le L_3^f(n)\|u - v\|^2, \tag{5.33}$$

and by (5.5) and (5.33), we have for all $u \in \ell^2$,

$$\|f^n(u)\|^2 \le 2\|f^n(u) - f^n(0)\|^2 + 2\|f^n(0)\|^2 \le 2L_3^f(n)\|u\|^2 + 2\|\alpha\|^2. \tag{5.34}$$

Similarly, by (5.6) and (5.7), there exists a constant $L_3^\sigma(n) > 0$ such that for all $u, v \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j^n(t,u) - \sigma_j^n(t,v)\|^2 \le L_3^\sigma(n)\|u - v\|^2, \quad \forall t \in \mathbb{R}. \tag{5.35}$$

and

$$\sum_{j\in\mathbb{N}} \|\sigma_j^n(t,u)\|^2 \le 2L_1^2\|u\|^2 + 2\|\beta(t)\|^2, \quad \forall t \in \mathbb{R}, u \in \ell^2. \tag{5.36}$$

For each $n \in \mathbb{N}$, we now consider the approximate stochastic system for $t > \tau$ with $\tau \in \mathbb{R}$:

$$\begin{cases} du_n(t) + \big(Du_n(t) - 2Au_n(t) + au_n(t) + bF^n(u_n(t)) + G^n(u_n(t))\big)dt \\ \quad = f^n(u_n(t))dt + g(t)dt + \epsilon \sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j^n(t, u_n(t))\big)dW_j(t), \\ u_n(\tau) = u_\tau = (u_{\tau,i})_{i\in\mathbb{Z}}, \end{cases} \tag{5.37}$$

where $a > 0$, $b = (b_i)_{i\in\mathbb{Z}} \in \ell^2$.

Similar to the argument of [4] for stochastic differential equations in $\mathbb{R}^n$, by (5.29)-(5.36), we can prove that for every $n \in \mathbb{N}$, $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$, system (5.37) possesses a unique solution u_n in the sense of Definition 5.3 as the drift term f and the diffusion term σ_j are replaced by f^n and σ_j^n, respectively.

We shall establish the existence and uniqueness of solutions for (5.24) by examining the limit behavior of $\{u_n\}_{n=1}^\infty$ as $n \to \infty$. For this purpose, we define a stopping time

$$\varsigma_n = \inf\{t \ge \tau : \|u_n(t)\| > n\}, \tag{5.38}$$

where $\varsigma_n = +\infty$ if $\{t \ge \tau : \|u_n(t)\| > n\} = \emptyset$.

We first prove the consistency of solutions to (5.37) as follows.

Lemma 5.4. *Suppose* (5.2)-(5.7) *hold. Let* u_n *be the solution of* (5.37). *Then,*

$$u_{n+1}(t \wedge \varsigma_n) = u_n(t \wedge \varsigma_n) \quad \text{and} \quad \varsigma_{n+1} \geq \varsigma_n \quad \text{a.s. for all} \quad t \geq \tau, n \in \mathbb{N}. \tag{5.39}$$

Proof. By (5.37), for all $n \in \mathbb{N}$ and $t \geq \tau$ we have

$$\begin{aligned} & u_{n+1}(t \wedge \varsigma_n) - u_n(t \wedge \varsigma_n) + \int_\tau^{t\wedge\varsigma_n} D(u_{n+1}(s) - u_n(s))ds \\ & \quad - 2\int_\tau^{t\wedge\varsigma_n} A(u_{n+1}(s) - u_n(s))ds + a\int_\tau^{t\wedge\varsigma_n} (u_{n+1}(s) - u_n(s))ds \\ & \quad + \int_\tau^{t\wedge\varsigma_n} b\big(F^{n+1}(u_{n+1}(s)) - F^n(u_n(s))\big)ds \\ & \quad + \int_\tau^{t\wedge\varsigma_n} \big(G^{n+1}(u_{n+1}(s)) - G^n(u_n(s))\big)ds \\ & = \int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^n(u_n(s))\big)ds \\ & \quad + \epsilon\sum_{j=1}^\infty \int_\tau^{t\wedge\varsigma_n} \big(\sigma_j^{n+1}(s, u_{n+1}(s)) - \sigma_j^n(s, u_n(s))\big)dW_j(s). \end{aligned} \tag{5.40}$$

By Ito's formula in [3, P90], we infer from (5.40) that, for all $t \geq \tau$,

$$\begin{aligned} & \|u_{n+1}(t \wedge \varsigma_n) - u_n(t \wedge \varsigma_n)\|^2 + 2\int_\tau^{t\wedge\varsigma_n} \|A(u_{n+1}(s) - u_n(s))\|^2 ds \\ & - 4\int_\tau^{t\wedge\varsigma_n} \|B(u_{n+1}(s) - u_n(s))\|^2 ds + 2a\int_\tau^{t\wedge\varsigma_n} \|u_{n+1}(s) - u_n(s)\|^2 ds \\ & + 2\int_\tau^{t\wedge\varsigma_n} \big(b(F^{n+1}(u_{n+1}(s)) - F^n(u_n(s))), u_{n+1}(s) - u_n(s)\big)ds \\ & + 2\int_\tau^{t\wedge\varsigma_n} \big(G^{n+1}(u_{n+1}(s)) - G^n(u_n(s)), u_{n+1}(s) - u_n(s)\big)ds \\ = & \; 2\int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^n(u_n(s)), u_{n+1}(s) - u_n(s)\big)ds \qquad (5.41) \\ & + \epsilon^2\sum_{j=1}^\infty \int_\tau^{t\wedge\varsigma_n} \|\sigma_j^{n+1}(s, u_{n+1}(s)) - \sigma_j^n(s, u_n(s))\|^2 ds \\ & + 2\epsilon\sum_{j=1}^\infty \int_\tau^{t\wedge\varsigma_n} \big(\sigma_j^{n+1}(s, u_{n+1}(s)) - \sigma_j^n(s, u_n(s)), u_{n+1}(s) - u_n(s)\big)dW_j(s). \end{aligned}$$

Notice that $\|u_n(s)\| \leq n$ for all $s \in [\tau, \varsigma_n)$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} & G^n(u_n(s)) = G(u_n(s)) = G^{n+1}(u_n(s)), \ F^n(u_n(s)) = F(u_n(s)) = F^{n+1}(u_n(s)) \\ & \text{and } f^{n+1}(u_n(s)) = f^n(u_n(s)), \quad \sigma_j^{n+1}(s, u_n(s)) = \sigma_j^n(s, u_n(s)). \end{aligned} \tag{5.42}$$

By (5.29) and (5.42), for the fifth term on the left-hand side of (5.41), we have

$$2\left|\int_\tau^{t\wedge\varsigma_n} \big(b(F^{n+1}(u_{n+1}(s)) - F^n(u_n(s))), u_{n+1}(s) - u_n(s)\big)ds\right|$$

$$\leq \frac{a}{2}\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds + \frac{2\|b\|^2}{a}\int_{\tau}^{t\wedge\varsigma_n} \|F^{n+1}(u_{n+1}(s)) - F^{n+1}(u_n(s))\|^2 ds$$
$$\leq \Big(\frac{a}{2} + \frac{2\|b\|^2 L_3^F(n+1)}{a}\Big)\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds. \tag{5.43}$$

By (5.31) and (5.42), the last term on the left-hand side of (5.41) satisfies

$$2\Big|\int_{\tau}^{t\wedge\varsigma_n} \big(G^{n+1}(u_{n+1}(s)) - G^n(u_n(s)), u_{n+1}(s)-u_n(s)\big)ds\Big|$$
$$\leq \frac{a}{2}\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds + \frac{2}{a}\int_{\tau}^{t\wedge\varsigma_n} \|G^{n+1}(u_{n+1}(s)) - G^{n+1}(u_n(s))\|^2 ds$$
$$\leq \Big(\frac{a}{2} + \frac{2L_3^G(n+1)}{a}\Big)\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds. \tag{5.44}$$

By (5.33) and (5.42), for the first term on the right-hand side of (5.41), we can obtain

$$2\int_{\tau}^{t\wedge\varsigma_n} (f^{n+1}(u_{n+1}(s)) - f^n(u_n(s)), u_{n+1}(s)-u_n(s))ds$$
$$\leq \frac{a}{2}\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds + \frac{2}{a}\int_{\tau}^{t\wedge\varsigma_n} \|f^{n+1}(u_{n+1}(s)) - f^{n+1}(u_n(s))\|^2 ds$$
$$\leq \Big(\frac{a}{2} + \frac{2L_3^f(n+1)}{a}\Big)\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds. \tag{5.45}$$

Similarly, by (5.35) and (5.42), we have

$$\epsilon^2\sum_{j=1}^{\infty}\int_{\tau}^{t\wedge\varsigma_n} \|\sigma_j^{n+1}(s,u_{n+1}(s)) - \sigma_j^n(s,u_n(s))\|^2 ds$$
$$\leq \epsilon^2 L_3^\sigma(n+1)\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s)-u_n(s)\|^2 ds. \tag{5.46}$$

Letting $C_0 = 16 + \frac{2\|b\|^2 L_3^F(n+1)}{a} + \frac{2L_3^G(n+1)}{a} + \frac{2L_3^f(n+1)}{a} + \epsilon^2 L_3^\sigma(n+1)$, substituting (5.43)-(5.46) into (5.41), taking the expectation of it, for all $t \geq \tau$,

$$\mathbb{E}\Big(\sup_{\tau\leq s\leq t}(\|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2)\Big)$$
$$\leq C_0\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau\leq s\leq r}(\|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big)dr + I_0, \tag{5.47}$$

where

$$I_0 = 2\epsilon\mathbb{E}\Big(\sup_{\tau\leq s\leq t\wedge\varsigma_n}\Big|\sum_{j=1}^{\infty}\int_{\tau}^{s}\big(\sigma_j^{n+1}(r,u_{n+1}(r)) - \sigma_j^n(r,u_n(r)),$$
$$u_{n+1}(r) - u_n(r)\big)dW_j(r)\Big|\Big).$$

By the Burkholder-Davis-Gundy inequality, we can infer from (5.35) and (5.42) that there exists a positive constant C_1 such that

$$\begin{aligned}
I_0 &\leq 2\epsilon C_1 \mathbb{E}\Big(\Big(\int_\tau^{t\wedge\varsigma_n}\Big(\|u_{n+1}(r)-u_n(r)\|^2 \\
&\quad\times\sum_{j=1}^{\infty}\|\sigma_j^{n+1}(r,u_{n+1}(r))-\sigma_j^n(r,u_n(r))\|^2\Big)dr\Big)^{\frac{1}{2}}\Big) \\
&\leq 2\epsilon C_1 \mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_{n+1}(r\wedge\varsigma_n)-u_n(r\wedge\varsigma_n)\| \\
&\quad\times\Big(\int_\tau^{t\wedge\varsigma_n}\sum_{j=1}^{\infty}\|\sigma_j^{n+1}(r,u_{n+1}(r))-\sigma_j^{n+1}(r,u_n(r))\|^2dr\Big)^{\frac{1}{2}}\Big) \\
&\leq 2\epsilon C_1\sqrt{L_3^g(n+1)}\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_{n+1}(r\wedge\varsigma_n)-u_n(r\wedge\varsigma_n)\| \\
&\quad\times\Big(\int_\tau^{t\wedge\varsigma_n}(\|u_{n+1}(r)-u_n(r)\|^2)dr\Big)^{\frac{1}{2}}\Big) \\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq s\leq t}\|u_{n+1}(s\wedge\varsigma_n)-u_n(s\wedge\varsigma_n)\|^2\Big) \\
&\quad+C_2\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq s\leq r}\|u_{n+1}(s\wedge\varsigma_n)-u_n(s\wedge\varsigma_n)\|^2\Big)dr, \qquad (5.48)
\end{aligned}$$

where $C_2 = 2C_1^2\epsilon^2L_3^g(n+1)$. Combining (5.47) and (5.48), we find

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\leq s\leq t}(\|u_{n+1}(s\wedge\varsigma_n)-u_n(s\wedge\varsigma_n)\|^2)\Big) \\
&\leq C_3\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq s\leq r}(\|u_{n+1}(s\wedge\varsigma_n)-u_n(s\wedge\varsigma_n)\|^2\Big)dr, \qquad (5.49)
\end{aligned}$$

where $C_3 = 2(C_{0,n}+C_{2,n})$. Then, using the Gronwall inequality to (5.49), we deduce that (5.39) holds. □

Then we show that $\varsigma_n \to \infty$ as $n \to \infty$ almost surely.

Lemma 5.5. *Suppose* (5.2)-(5.7) *hold. Then the sequence of stopping times* $\{\varsigma_n\}_{n=1}^\infty$ *satisfies that*

$$\varsigma := \lim_{n\to\infty}\varsigma_n = \sup_{n\in\mathbb{N}}\varsigma_n = \infty, \qquad \mathbb{P}\text{-almost surely.} \qquad (5.50)$$

Proof. (1) We first need to derive the following uniform estimates for the solution u_n of (5.37):

$$\mathbb{E}\Big(\sup_{\tau\leq r\leq\tau+T}\|u_n(r\wedge\varsigma_n)\|^2\Big) \leq M, \text{ for every } T>0, \qquad (5.51)$$

where

$$M = M_0e^{M_0T}\Big(1+T+\mathbb{E}(\|u_\tau\|^2)\Big)$$

and M_0 is a finite number independent of u_τ, τ, n, T.

By Ito's formula and (5.37), we derive

$$\begin{aligned}
&\|u_n(t\wedge\varsigma_n)\|^2 + 2\int_\tau^{t\wedge\varsigma_n}\|Au_n(s)\|^2 ds - 4\int_\tau^{t\wedge\varsigma_n}\|Bu_n(s)\|^2 ds + 2a\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|^2 ds\\
&+ 2\int_\tau^{t\wedge\varsigma_n}(bF^n(u_n(s)), u_n(s))ds + 2\int_\tau^{t\wedge\varsigma_n}(G^n(u_n(s)), u_n(s))ds\\
&= \|u_\tau\|^2 + 2\int_\tau^{t\wedge\varsigma_n}(f^n(u_n(s)), u_n(s))ds + 2\int_\tau^{t\wedge\varsigma_n}(g(s), u_n(s))ds\\
&+ \epsilon^2\sum_{j=1}^\infty\int_\tau^{t\wedge\varsigma_n}\|h_j(s) + \sigma_j^n(s, u_n(s))\|^2 ds\\
&+ 2\epsilon\sum_{j=1}^\infty\int_\tau^{t\wedge\varsigma_n}u_n(s)(h_j(s) + \sigma_j^n(s, u_n(s)))dW_j(s). \qquad (5.52)
\end{aligned}$$

For the fifth term on the left-hand side of (5.52), by (5.16) and (5.42), we have

$$\begin{aligned}
&-2\int_\tau^{t\wedge\varsigma_n}(bF^n(u_n(s)), u_n(s))ds \le 2\int_\tau^{t\wedge\varsigma_n}\sum_{i\in\mathbb{Z}} b_i\big||u_{n,i+1}(s) - u_{n,i}(s)|^2\big||u_{n,i}(s)|ds\\
&\le 8\int_\tau^{t\wedge\varsigma_n}\|b\|^2\|u_n(s)\|^2 ds\\
&\quad + \frac{1}{8}\int_\tau^{t\wedge\varsigma_n}\sum_{i\in\mathbb{Z}}\big|(u_{n,i+1}(s))^2 - 2u_{n,i+1}(s)u_{n,i}(s) + (u_{n,i}(s))^2\big|^2 ds\\
&\le 8\int_\tau^{t\wedge\varsigma_n}\|b\|^2\|u_n(s)\|^2 ds + 2\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|_4^4 ds. \qquad (5.53)
\end{aligned}$$

For the second term on the right-hand side of (5.52), by (5.34) and the Young inequality, we have

$$\begin{aligned}
&2\int_\tau^{t\wedge\varsigma_n}(f^n(u_n(s)), u_n(s))ds\\
&\le \frac{\sqrt{L_3^f(n)}}{2}\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|^2 ds + \frac{2}{\sqrt{L_3^f(n)}}\int_\tau^{t\wedge\varsigma_n}\|f^n(u_n(s))\|^2 ds\\
&\le \frac{4}{\sqrt{L_3^f(n)}}\int_\tau^{t\wedge\varsigma_n}\|\alpha\|^2 ds + \frac{9}{2}\sqrt{L_3^f(n)}\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|^2 ds\\
&\le \frac{4}{\sqrt{L_3^f(n)}}T\|\alpha\|^2 + \frac{9}{2}\sqrt{L_3^f(n)}\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|^2 ds. \qquad (5.54)
\end{aligned}$$

Thanks to the Young inequality again, the third term on the right-hand side of (5.52) satisfies

$$2\int_\tau^{t\wedge\varsigma_n}(g(s), u_n(s))ds \le \frac{2}{a}\int_\tau^{t\wedge\varsigma_n}\|g(s)\|^2 ds + \frac{a}{2}\int_\tau^{t\wedge\varsigma_n}\|u_n(s)\|^2 ds. \qquad (5.55)$$

By (5.36), the fourth term on the right-hand side of (5.52) fulfills

$$
\begin{aligned}
&\epsilon^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|h_j(s)+\sigma_j^n(s,u_n(s))\|^2 ds \\
&\le 2\epsilon^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|h_j(s)\|^2 ds + 2\epsilon^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|\sigma_j^n(s,u_n(s))\|^2 ds \qquad (5.56)\\
&\le 2\epsilon^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|h_j(s)\|^2 ds + 4\epsilon^2 L_1^2 \int_{\tau}^{t\wedge\varsigma_n} \|u_n(s)\|^2 ds + 4\epsilon^2 \|\beta\|^2_{L^2(\tau,\tau+T;\ell^2)}.
\end{aligned}
$$

Let $C_4 = \frac{4}{\sqrt{L_3^f(n)}} T\|\alpha\|^2 + 4\epsilon^2\|\beta\|^2_{L^2(\tau,\tau+T;\ell^2)}$ and $C_5 = 16 + 8\|b\|^2 + \frac{9}{2}\sqrt{L_3^f(n)} + 4\epsilon^2 L_1^2$. Along with (5.53)-(5.56) and

$$
2\int_{\tau}^{t\wedge\varsigma_n} (G^n(u_n(s)), u_n(s)) ds = 2\int_{\tau}^{t\wedge\varsigma_n} \|u_n(s)\|_4^4 ds,
$$

it holds that, for all $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&\|u_n(t\wedge\varsigma_n)\|^2 + \frac{3}{2}a \int_{\tau}^{t\wedge\varsigma_n} \|u_n(s)\|^2 ds \\
&\le \|u_\tau\|^2 + C_5 \int_{\tau}^{t\wedge\varsigma_n} \|u_n(s)\|^2 ds + \frac{2}{a}\int_{\tau}^{t\wedge\varsigma_n} \|g(s)\|^2 ds + C_4 \qquad (5.57)\\
&+ 2\epsilon^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|h_j(s)\|^2 ds + 2\epsilon \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} u_n(s)(h_j(s)+\sigma_j^n(s,u_n(s)))dW_j(s).
\end{aligned}
$$

We further derive

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u_n(r\wedge\varsigma_n\|^2)\Big) &\le \mathbb{E}(\|u_\tau\|^2) + C_5 \int_{\tau}^{t} \mathbb{E}(\sup_{\tau\le r\le s} \|u_n(r\wedge\varsigma_n)\|^2) ds \\
&+ 2\epsilon\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge\varsigma_n} \Big|\sum_{j=1}^{\infty}\int_{\tau}^{r} u_n(s)\big(h_j(s)+\sigma_j^n(s,u_n(s))\big)dW_j(s)\Big|\Big) \\
&+ C_6 \int_{\tau}^{\tau+T} \mathbb{E}(\|g(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2) ds + C_4, \qquad (5.58)
\end{aligned}
$$

where $C_6 = \max\{\frac{2}{a}, 2\epsilon^2\}$. According to the Burkholder-Davis-Gundy inequality and (5.36), for all $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&2\epsilon\mathbb{E}\Big(\sup_{\tau\le r\le t\wedge\varsigma_n} \Big|\sum_{j=1}^{\infty}\int_{\tau}^{r} u_n(s)\big(h_j(s)+\sigma_j^n(s,u_n(s))\big)dW_j(s)\Big|\Big) \\
&\le 2C_1\epsilon\mathbb{E}\Big(\int_{\tau}^{t\wedge\varsigma_n} \sum_{j=1}^{\infty} \|u_n(s)\|^2 \|h_j(s)+\sigma_j^n(s,u_n(s))\|^2 ds\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq 2C_1\epsilon\mathbb{E}\Big(\sup_{\tau\leq s\leq t}\|u_n(s\wedge\varsigma_n)\|\Big(\sum_{j=1}^{\infty}\int_\tau^{t\wedge\varsigma_n}\|h_j(s)+\sigma_j^n(s,u_n(s))\|^2ds\Big)^{\frac{1}{2}}\Big)$$
$$\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq s\leq t}\|u_n(s\wedge\varsigma_n)\|^2\Big)+2C_1^2\epsilon^2\sum_{j=1}^{\infty}\int_\tau^{t\wedge\varsigma_n}\mathbb{E}\Big(\|h_j(s)+\sigma_j^n(s,u_n(s))\|^2\Big)ds$$
$$\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq s\leq t}\|u_n(s\wedge\varsigma_n)\|^2\Big)+4C_1^2\epsilon^2\sum_{j=1}^{\infty}\int_\tau^{\tau+T}\mathbb{E}(\|h_j(s)\|^2)ds$$
$$+8\epsilon^2C_1^2\|\beta\|^2_{L^2(\tau,\tau+T;\ell^2)}+8\epsilon^2C_1^2L_1^2\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u_n(r\wedge\varsigma_n)\|^2\Big)ds, \tag{5.59}$$

where C_1 is defined in (5.48). It can be inferred from (5.58)-(5.59) that, for all $t\in[\tau,\tau+T]$,

$$\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u_n(r\wedge\varsigma_n)\|^2\Big)$$
$$\leq 2\mathbb{E}(\|u_\tau\|^2)+C_7\int_\tau^t\mathbb{E}(\sup_{\tau\leq r\leq s}\|u_n(r\wedge\varsigma_n)\|^2)ds+C_8, \tag{5.60}$$

where $C_7=2(C_5+8\epsilon^2C_1^2L_1^2)$, and

$$C_8=2C_4+16\epsilon^2C_1^2\|\beta\|^2_{L^2(\tau,\tau+T;\ell^2)}$$
$$+2(C_6+4C_1^2\epsilon^2)\int_\tau^{\tau+T}\mathbb{E}(\|g(s)\|^2+\sum_{j=1}^{\infty}\|h_j(s)\|^2)ds.$$

Applying the Gronwall inequality to (5.60),

$$\mathbb{E}\Big(\sup_{\tau\leq r\leq t}(\|u_n(r\wedge\varsigma_n)\|^2)\Big)\leq(2\mathbb{E}(\|u_\tau\|^2)+C_8)e^{C_7T},\quad\text{for all } t\in[\tau,\tau+T], \tag{5.61}$$

which implies the desired result (5.51).

(2) By (5.38), we have

$$\{\varsigma_n<\tau+T\}\subseteq\Big\{\sup_{\tau\leq r\leq\tau+T}\|u_n(r\wedge\varsigma_n)\|\geq n\Big\}.$$

Then it follows from (5.51) and Chebyshev's inequality that

$$\mathbb{P}\{\varsigma_n<\tau+T\}\leq\mathbb{P}\{\sup_{\tau\leq r\leq\tau+T}\|u_n(r\wedge\varsigma_n)\|\geq n\}$$
$$\leq\frac{1}{n^2}\mathbb{E}\Big(\sup_{\tau\leq r\leq\tau+T}\|u_n(r\wedge\varsigma_n)\|^2\Big)\leq\frac{M}{n^2}.$$

Consequently,

$$\sum_{n=1}^{\infty}\mathbb{P}\{\varsigma_n<\tau+T\}\leq M\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty. \tag{5.62}$$

Thanks to the Borel-Cantelli lemma and (5.62), we can derive

$$\mathbb{P}\Big(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\varsigma_n<\tau+T\}\Big)=0. \tag{5.63}$$

Denoting $\Omega_T=\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{n=m}^{\infty}\{\varsigma_n<\tau+T\}$, for every $\omega\in\Omega\setminus\Omega_T$, we have $\varsigma_n(\omega)\geq\tau+T$ for n sufficiently large. Then, we deduce from the increasing property of ς_n in n that $\varsigma(\omega)\geq\tau+T$ for all $\omega\in\Omega\setminus\Omega_T$. Let $\Omega_0=\bigcup\limits_{T=1}^{\infty}\Omega_T$, then $\mathbb{P}(\Omega_0)=0$ and $\varsigma(\omega)\geq\tau+T$ for all $\omega\in\Omega\setminus\Omega_0$ and $T\in\mathbb{N}$. Thus, we obtain the desired result (5.50). □

Now, we establish the existence and uniqueness of solutions for (5.24).

Theorem 5.6. *Suppose* (5.2)-(5.7) *hold. For any* $u_\tau\in L^2(\Omega,\mathcal{F}_\tau;\ell^2)$, *problem* (5.24) *has a unique solution* u *in the sense of Definition 5.3 and, for every* $T>0$, *it holds*

$$\mathbb{E}\Big(\|u\|^2_{C([\tau,\tau+T],\ell^2)}\Big)\leq M_0e^{M_0T}\Big(1+T+\mathbb{E}(\|u_\tau\|^2)\Big), \tag{5.64}$$

where M_0 *is defined in* (5.51).

Proof. By Lemma 5.4 and Lemma 5.5, we consider a subset Ω_1 of Ω satisfying $\mathbb{P}(\Omega\setminus\Omega_1)=0$ such that, for all $n\in\mathbb{N},\omega\in\Omega_1,t\geq\tau$,

$$u_{n+1}(t\wedge\varsigma_n,\omega)=u_n(t\wedge\varsigma_n,\omega),\ \text{ and } \varsigma(\omega)=\lim_{n\to\infty}\varsigma_n(\omega)=\infty. \tag{5.65}$$

Define a mapping $u:[\tau,\infty)\times\Omega\to\ell^2$ by

$$u(t,\omega)=\begin{cases}u_n(t,\omega), & \text{if } \omega\in\Omega_1 \text{ and } t\in[\tau,\varsigma_n(\omega))\\ u_\tau(\omega), & \text{if } \Omega\setminus\Omega_1 \text{ and } t\in[\tau,\infty).\end{cases} \tag{5.66}$$

By (5.65), for every $\omega\in\Omega_1$ and $t\geq\tau$, there exists $n_0=n_0(t,\omega)\geq1$ such that $\varsigma_n(\omega)>t$ and $u_n(t,\omega)=u_{n_0}(t,\omega)$ for all $n\geq n_0$. Meanwhile, the mapping in (5.66) is well-defined. Combining the fact that u_n is a continuous ℓ^2-valued process with (5.66), we can deduce that u is continuous concerning t in ℓ^2 almost surely. By (5.66), we have

$$\lim_{n\to\infty}u_n(t,\omega)=u(t,\omega),\quad \text{for all }\ \omega\in\Omega_1,t\geq\tau. \tag{5.67}$$

By (5.67), we know that u is $\mathcal{F}_\tau$-adapted since u_n is $\mathcal{F}_\tau$-adapted. Conditions (5.51), (5.67), and Fatou's lemma imply that (5.64) holds true. Moreover, thanks to (5.37), for all $t\geq\tau$, we obtain

$$\begin{aligned}u_n(t\wedge\varsigma_n)=u_\tau+&\int_\tau^{t\wedge\varsigma_n}\big(-Du_n(s)+2Au_n(s)-au_n(s)-bF^n(u_n(s))\\ -G^n(u_n(s))\big)ds+&\int_\tau^{t\wedge\varsigma_n}(f^n(u_n(s))+g(s))ds\end{aligned}$$

$$+\epsilon\sum_{j=1}^{\infty}\int_{\tau}^{t\wedge\varsigma_n}\big(h_j(s)+\sigma_j^n(s,u_n(s))\big)dW_j(s). \tag{5.68}$$

By (5.66), we know that $u_n(t\wedge\varsigma_n)=u(t\wedge\varsigma_n)$ almost surely and for all $s\in[\tau,\varsigma_n)$,

$$F^n(u_n(s))=F(u(s)),\ G^n(u_n(s))=G(u(s))\ \text{ and} \tag{5.69}$$
$$f^n(u_n(s))=f(u(s)),\ \sigma_j^n(s,u_n(s))=\sigma_j(s,u(s))). \tag{5.70}$$

Therefore, it follows from (5.68) that a.s.,

$$\begin{aligned}u(t\wedge\varsigma_n)=u_\tau+&\int_{\tau}^{t\wedge\varsigma_n}\big(-Du(s)+2Au(s)-au(s)-bF(u(s))-G(u(s))\big)ds\\&+\int_{\tau}^{t\wedge\varsigma_n}(f(u(s))+g(s))ds+\epsilon\sum_{j=1}^{\infty}\int_{\tau}^{t\wedge\varsigma_n}(h_j(s)+\sigma_j(s,u(s)))dW_j(s).\end{aligned} \tag{5.71}$$

By (5.50) and (5.71), we find that a.s.,

$$\begin{aligned}u(t)=u_\tau+&\int_{\tau}^{t}\big(-Du(s)+2Au(s)-au(s)-bF(u(s))-G(u(s))\big)ds\\&+\int_{\tau}^{t}(f(u(s))+g(s))ds+\epsilon\sum_{j=1}^{\infty}\int_{\tau}^{t}(h_j(s)+\sigma_j(s,u(s)))dW_j(s),\end{aligned} \tag{5.72}$$

which implies that u is a solution of (5.24).

Now, we prove the uniqueness of solutions for the system (5.24). Let u_1 and u_2 be two solutions of (5.24), for every $n\in\mathbb{N}$ and $T>0$, we define a stopping time by

$$T_n=(\tau+T)\wedge\inf\{t\geq\tau:\|u_1(t)\|>n\ \text{ or }\ \|u_2(t)\|>n\}. \tag{5.73}$$

By (5.24), we have

$$\begin{aligned}&u_1(t\wedge T_n)-u_2(t\wedge T_n)+\int_{\tau}^{t\wedge T_n}D(u_1(s)-u_2(s))ds\\&-2\int_{\tau}^{t\wedge T_n}A(u_1(s)-u_2(s))ds+a\int_{\tau}^{t\wedge T_n}(u_1(s)-u_2(s))ds\\&+\int_{\tau}^{t\wedge T_n}b\big(F(u_1(s))-F(u_2(s))\big)ds+\int_{\tau}^{t\wedge T_n}\big(G(u_1(s))-G(u_2(s))\big)ds\\&=u_1(\tau)-u_2(\tau)+\int_{\tau}^{t\wedge T_n}(f(u_1(s))-f(u_2(s)))ds\\&+\epsilon\sum_{j=1}^{\infty}\int_{\tau}^{t\wedge T_n}(\sigma_j(s,u_1(s))-\sigma_j(s,u_2(s)))dW_j(s).\end{aligned} \tag{5.74}$$

Then, Ito's formula yields

$$\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|^2+2\int_{\tau}^{t\wedge T_n}\|A(u_1(s)-u_2(s))\|^2ds$$

$$
\begin{aligned}
&-4\int_\tau^{t\wedge T_n}\|B(u_1(s)-u_2(s))\|^2ds+2a\int_\tau^{t\wedge T_n}\|(u_1(s)-u_2(s))\|^2ds\\
&+2\int_\tau^{t\wedge T_n}\big(b(F(u_1(s))-F(u_2(s))),u_1(s)-u_2(s)\big)ds\\
&+2\int_\tau^{t\wedge T_n}(G(u_1(s))-G(u_2(s)),u_1(s)-u_2(s))ds\\
=&\ \|u_1(\tau)-u_2(\tau)\|^2+2\int_\tau^{t\wedge T_n}(f(u_1(s))-f(u_2(s)),u_1(s)-u_2(s))ds\\
&+\epsilon^2\sum_{j=1}^\infty\int_\tau^{t\wedge T_n}\|\sigma_j(s,u_1(s))-\sigma_j(s,u_2(s))\|^2ds\\
&+2\epsilon\sum_{j=1}^\infty\int_\tau^{t\wedge T_n}(\sigma_j(s,u_1(s))-\sigma_j(s,u_2(s)),u_1(s)-u_2(s))dW_j(s).
\end{aligned}
\tag{5.75}
$$

Using a similar argument as in Lemma 5.4 to (5.75), there exists $C_9=C_9(n)>0$ such that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le s\le t}(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2)\Big)\\
&\le C_9\mathbb{E}(\|u_1(\tau)-u_2(\tau)\|^2)+C_9\int_\tau^t\sup_{\tau\le s\le r}\mathbb{E}(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2)dr.
\end{aligned}
\tag{5.76}
$$

Applying now the Gronwall lemma to (5.76), for $u_1(\tau)=u_2(\tau)$ in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, we have

$$
\mathbb{E}\Big(\sup_{\tau\le s\le t}(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2)\Big)=0,
\tag{5.77}
$$

which indicates that $\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|=0$ for all $t\in[\tau,\tau+T]$ almost surely. As u_1 and u_2 are continuous in t, then $T_n=\tau+T$ as n is large enough. Thus, for every $T>0$,

$$
\begin{aligned}
&\mathbb{P}(\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|=0,\ \text{for all } t\in[\tau,\tau+T])\\
&=\mathbb{P}(\|u_1(t)-u_2(t)\|=0,\ \ \text{for all}\ \ t\in[\tau,\tau+T])=1.
\end{aligned}
\tag{5.78}
$$

Since T is arbitrary, the uniqueness of the solution follows. □

Remark 5.7. *By Theorem 5.6, for every $\tau\in\mathbb{R}$ and $u_\tau\in L^2(\Omega,\mathcal{F}_\tau;\ell^2)$, the problem* (5.24) *admits an $\mathcal{F}_t$-adapted unique solution $u(\cdot,\tau,u_\tau)\in L^2(\Omega,C([\tau,\tau+T]),\ell^2))$ for all $T>0$. Since $u\in C([\tau,\infty),\ell^2)$ $\mathbb{P}$-a.s., we can show that $u\in C([\tau,\infty),L^2(\Omega,\ell^2))$ by using* (5.64) *and the Lebesgue dominated convergence theorem.*

5.2 Limiting stability of evolution systems of measures

In this section, we will prove the limiting stability of evolution systems of probability measures for the system (5.24) on ℓ^2.

Now, we list the theoretical results that we will use. As explained in Wang et al. [100, Theorem 1.3], the following assumption about probability convergence is required:

R. For each compact set $K \subseteq \ell^2$, $t \geq \tau \in \mathbb{R}, \delta > 0$, $\epsilon_0 \in (0, \bar{\epsilon}]$,

$$\lim_{\epsilon \to \epsilon_0} \sup_{x \in K} \mathbb{P}(\{\omega \in \Omega : \|X^{\epsilon}(t, \tau, x) - X^{\epsilon_0}(t, \tau, x)\| \geq \delta\}) = 0. \tag{5.79}$$

Theorem 5.8. *[100] Assume* **R** *hold and* $(P^{\epsilon_0}_{\tau,t})_{t \geq \tau}$ *is Feller. Let* $\{\eta^{\epsilon_0}_t\}_{t \in \mathbb{R}}$ *be a family of probability measures on* ℓ^2 *for* $\epsilon_0 \in [0, \bar{\epsilon}]$, *and* $\{\eta^{\epsilon_n}_t\}_{t \in \mathbb{R}}$ *be an evolution system of probability measures of* $(P^{\epsilon_n}_{\tau,t})_{t \geq \tau}$ *on* ℓ^2 *with* $\epsilon_n \to \epsilon_0$. *For each* $t \in \mathbb{R}$, *if there exists a subsequence* $\{\epsilon_{n_k}(t)\}_{k=1}^{\infty}$ *of* $\{\epsilon_n\}_{n=1}^{\infty}$ *such that* $\eta_t^{\epsilon_{n_k}(t)} \to \eta^{\epsilon_0}_t$ *weakly, then* $\{\eta^{\epsilon_0}_t\}_{t \in \mathbb{R}}$ *must be an evolution system of probability measures of* $(P^{\epsilon_0}_{\tau,t})_{t \geq \tau}$.

To indicate the dependence of solutions on noise intensity ϵ, we write the solution of system (5.24) as $u^{\epsilon}(t, \tau, u_0)$ in the following.

5.2.1 Uniform estimates the solutions

In this part, we need to define a parameter ϑ by

$$\vartheta := a_0 - 32 - 16\|b\|^2 - 5\sqrt{2}L_0, \tag{5.80}$$

where $a_0 \in (0, a)$, $b = (b_i)_{i \in \mathbb{Z}} \in \ell^2$ and $L_0 > 0$ is defined in (5.5).

Now, we show the following uniform estimates of solutions for stochastic lattice systems (5.24) in $L^2(\Omega, \ell^2)$.

Lemma 5.9. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. Then, for all* $t \geq \tau$ *with* $\tau \in \mathbb{R}$ *and* $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, *the solution* $u^{\epsilon}(\cdot, \tau, u_0)$ *of problem* (5.24) *satisfies,*

$$\begin{aligned} &\mathbb{E}(\|u^{\epsilon}(t, \tau, u_0)\|^2) + \frac{a}{4}\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}(\|u^{\epsilon}(s, \tau, u_0)\|^2)ds \\ &\leq e^{a(\tau - t)}\mathbb{E}(\|u_0\|^2) + ce^{-at}\int_{-\infty}^{t} e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds + c, \end{aligned} \tag{5.81}$$

where c *is a positive constant independent of* ϵ, t *and* u_0.

Proof. By (5.24), Remark 5.7 and Ito's formula, we have

$$\begin{aligned} &d(\|u^{\epsilon}(t)\|^2) + 2(\|Au^{\epsilon}(t)\|^2 - 2\|Bu^{\epsilon}(t)\|^2 + a\|u^{\epsilon}(t)\|^2 + \|u^{\epsilon}(t)\|_4^4)dt \\ &\quad + 2\big(bF(u^{\epsilon}(t)), u^{\epsilon}(t)\big)dt \\ &= 2(f(u^{\epsilon}(t)), u^{\epsilon}(t))dt + 2(g(t), u^{\epsilon}(t))dt + \epsilon^2\sum_{j=1}^{\infty}\|h_j(t) + \sigma_j(t, u^{\epsilon}(t))\|^2dt \\ &\quad + 2\epsilon\sum_{j=1}^{\infty}(h_j(t) + \sigma_j(t, u^{\epsilon}(t)), u^{\epsilon}(t))dW_j(t), \end{aligned} \tag{5.82}$$

which implies

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}(\|u^\epsilon(t)\|^2) + 2\mathbb{E}(\|Au^\epsilon(t)\|^2) - 4\mathbb{E}(\|Bu^\epsilon(t)\|^2) + 2a\mathbb{E}(\|u^\epsilon(t)\|^2)\\
&\quad + 2\mathbb{E}(\|u^\epsilon(t)\|_4^4) + 2\mathbb{E}\big(bF(u^\epsilon(t)), u^\epsilon(t)\big)\\
&= 2\mathbb{E}(f(u^\epsilon(t)), u^\epsilon(t)) + 2\mathbb{E}(g(t), u^\epsilon(t)) + \epsilon^2\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(t) + \sigma_j(t, u^\epsilon(t))\|^2). \quad (5.83)
\end{aligned}$$

The Young inequality yields

$$\begin{aligned}
&2\mathbb{E}\big(bF(u^\epsilon(t)), u^\epsilon(t)\big)\\
&= 2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} b_i\big|u^\epsilon_{i+1}(t) - u^\epsilon_i(t)|^2\big|u^\epsilon_i(t)|\Big)\\
&\le 8\|b\|^2\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{1}{8}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\big|(u^\epsilon_{i+1}(t))^2 - 2u^\epsilon_{i+1}(t)u^\epsilon_i(t) + (u^\epsilon_i(t))^2\big|^2\Big)\\
&\le 8\|b\|^2\mathbb{E}(\|u^\epsilon(t)\|^2) + 2\mathbb{E}(\|u^\epsilon(t)\|_4^4). \quad (5.84)
\end{aligned}$$

By the Young inequality again and (5.21), we derive

$$\begin{aligned}
&2\mathbb{E}(f(u^\epsilon(t)), u^\epsilon(t)) + 2\mathbb{E}(g(t), u^\epsilon(t))\\
&\le 2\sqrt{2}L_0\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{1}{2\sqrt{2}L_0}\mathbb{E}(\|f(u^\epsilon(t))\|^2) + \frac{4}{a}\mathbb{E}(\|g(t)\|^2) + \frac{a}{4}\mathbb{E}(\|u^\epsilon(t)\|^2)\\
&\le \frac{1}{\sqrt{2}L_0}\|\alpha\|^2 + \frac{5\sqrt{2}}{2}L_0\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{4}{a}\mathbb{E}(\|g(t)\|^2) + \frac{a}{4}\mathbb{E}(\|u^\epsilon(t)\|^2). \quad (5.85)
\end{aligned}$$

By (5.23), we have

$$\begin{aligned}
&\epsilon^2\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(t) + \sigma_j(t, u^\epsilon(t))\|^2)\\
&\le 2\epsilon^2\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(t)\|^2) + 4\epsilon^2\|\beta(t)\|^2 + 4\epsilon^2L_1^2\mathbb{E}(\|u^\epsilon(t)\|^2). \quad (5.86)
\end{aligned}$$

Substituting (5.84)-(5.86) into (5.83), there is

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{7}{4}a\mathbb{E}(\|u^\epsilon(t)\|^2)\\
&\le (16 + 8\|b\|^2 + \frac{5\sqrt{2}}{2}L_0 + 4\epsilon^2L_1^2)\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{4}{a}\mathbb{E}(\|g(t)\|^2) + 2\epsilon^2\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(t)\|^2)\\
&\quad + \frac{1}{\sqrt{2}L_0}\|\alpha\|^2 + 4\epsilon^2\|\beta(t)\|^2. \quad (5.87)
\end{aligned}$$

By **R1** and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, we can deduce

$$16 + 8\|b\|^2 + \frac{5\sqrt{2}}{2}L_0 + 4\epsilon^2L_1^2 = 16 + 8\|b\|^2 + \frac{5\sqrt{2}}{2}L_0 + \frac{\vartheta}{2} = \frac{a_0}{2} < \frac{a}{2}.$$

Therefore, by (5.87) and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$,

$$\begin{aligned} &\frac{d}{dt}\mathbb{E}(\|u^\epsilon(t)\|^2) + \frac{5}{4}a\mathbb{E}(\|u^\epsilon(t)\|^2) \\ &\leq c_1\mathbb{E}\big(\|g(t)\|^2 + \|\beta(t)\|^2 + \sum_{j=1}^{\infty}\|h_j(t)\|^2\big) + c_2, \end{aligned} \tag{5.88}$$

where

$$c_1 = \max\{\frac{4}{a}, \frac{\vartheta^2}{4L_1^2}\} = \max\{\frac{4}{a}, \frac{a - 32 - 16\|b\|^2 - 6\sqrt{2}L_0}{4L_1^2}\}, \quad c_2 = \frac{\|\alpha\|^2}{\sqrt{2}L_0}.$$

Multiplying (5.88) by e^{at} and integrating on (τ, t) implies

$$\begin{aligned} &\mathbb{E}(\|u^\epsilon(t,\tau,u_0)\|^2) + \frac{a}{4}\int_\tau^t e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)ds \qquad (5.89) \\ &\leq e^{a(\tau-t)}\mathbb{E}(\|u_0\|^2) + c_1e^{-at}\int_{-\infty}^t e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds + \frac{c_2}{a}. \end{aligned}$$

Thus the proof is complete. □

Then, we show the uniform estimates on the tails of solutions for stochastic lattice systems (5.24) in $L^2(\Omega, \ell^2)$, which helps to establish the tightness of probability measures of solutions.

Lemma 5.10. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. Then, for each compact set* $\mathcal{K}$ *of* ℓ^2 *and* $\varepsilon > 0$*, there exists* $N = N(\mathcal{K}, \varepsilon) \in \mathbb{N}$ *such that, for all* $n \geq N$, $t \geq \tau$ *with* $\tau \in \mathbb{R}$*, the solution* $u^\epsilon(\cdot, \tau, u_0)$ *of problem* (5.24) *with initial data* $u_0 \in \mathcal{K}$ *satisfies*

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]}\sum_{|i|\geq n}\mathbb{E}(|u_i^\epsilon(t,\tau,u_0)|^2) \leq \varepsilon. \tag{5.90}$$

Proof. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function such that $\rho(s) = 0$ for $|s| \leq 1$ and $\rho(s) = 1$ for $|s| \geq 2$. We define a positive constant c_0 such that $|\rho'(s)| \leq c_0$ uniformly for $s \in \mathbb{R}$. Moreover, given $n \in \mathbb{N}$, we set

$$\rho_n = (\rho_{n,i})_{i\in\mathbb{Z}} = \Big(\rho(\frac{|i|}{n})\Big)_{i\in\mathbb{Z}} \text{ and } \rho_n u^\epsilon = (\rho_{n,i}u_i^\epsilon)_{i\in\mathbb{Z}} \text{ for } u^\epsilon = (u_i^\epsilon)_{i\in\mathbb{Z}}.$$

By (5.24),

$$\begin{aligned} d(\rho_n u^\epsilon) &= \big(-\rho_n Du^\epsilon + 2\rho_n Au^\epsilon - a\rho_n u^\epsilon - b\rho_n F(u^\epsilon) - \rho_n G(u^\epsilon)\big)dt \\ &\quad + \rho_n f(u^\epsilon)dt + \rho_n g(t)dt + \epsilon\sum_{j=1}^{\infty}(\rho_n h_j(t) + \rho_n\sigma_j(t,u^\epsilon))dW_j, \end{aligned} \tag{5.91}$$

which, together with Ito's formula, implies

$$d(\|\rho_n u^\epsilon(t)\|^2) + \Big(2(Au^\epsilon(t), A(\rho_n^2u^\epsilon(t))) - 4(Au^\epsilon(t), \rho_n^2u^\epsilon(t)) + 2a\|\rho_n u^\epsilon(t)\|^2$$

$$
\begin{aligned}
&+ 2(bF(u^\epsilon), \rho_n^2 u^\epsilon(t)) + 2(G(u^\epsilon(t)), \rho_n^2 u^\epsilon(t))\Big) dt \\
&= 2(\rho_n f(u^\epsilon), \rho_n u^\epsilon(t))dt + 2(\rho_n g(t), \rho_n u^\epsilon(t))dt \qquad (5.92)\\
&+ \epsilon^2 \sum_{j=1}^{\infty} \|\rho_n h_j(t) + \rho_n \sigma_j(t, u^\epsilon(t))\|^2 dt + 2\epsilon \sum_{j=1}^{\infty} (h_j(t) + \sigma_j(t, u^\epsilon(t)), \rho_n^2 u^\epsilon(t))) dW_j.
\end{aligned}
$$

Taking the expectation of (5.92), by Remark 5.7 we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + 2\mathbb{E}\big(Au^\epsilon(t), A(\rho_n^2 u^\epsilon(t))\big) - 4\mathbb{E}\big(Au^\epsilon(t), \rho_n^2 u^\epsilon(t)\big) \\
&\quad + 2a\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + 2\mathbb{E}(bF(u^\epsilon), \rho_n^2 u^\epsilon(t)) + 2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})|u_i^\epsilon(t)|^4\Big) \\
&= 2\mathbb{E}(\rho_n f(u^\epsilon(t)), \rho_n u^\epsilon(t)) + 2\mathbb{E}(\rho_n g(t), \rho_n u^\epsilon(t)) \\
&\quad + \epsilon^2 \sum_{j=1}^{\infty} \mathbb{E}(\|\rho_n h_j(t) + \rho_n \sigma_j(t, u^\epsilon(t))\|^2). \qquad (5.93)
\end{aligned}
$$

By [99, Lemma 7], the second term on the left-hand side of (5.93) satisfies

$$
2\mathbb{E}\big(Au^\epsilon(t), A(\rho_n^2 u^\epsilon(t))\big) \geq 2\mathbb{E}(\|\rho_n Au^\epsilon(t)\|^2) - \frac{136c_0}{n}\mathbb{E}(\|u^\epsilon(t)\|^2). \qquad (5.94)
$$

For the third term on the left-hand side of (5.93), we have

$$
4\mathbb{E}\big(Au^\epsilon(t), \rho_n^2 u^\epsilon(t)\big) \leq 2\mathbb{E}(\|\rho_n Au^\epsilon(t)\|^2) + 2\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2). \qquad (5.95)
$$

For the penultimate term on the left-hand side of (5.93), by the definition of operator F in (5.16), we obtain

$$
\begin{aligned}
&- 2\mathbb{E}(bF(u^\epsilon), \rho_n^2 u^\epsilon(t)) \\
&\leq 2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})|b_i||u_{i+1}^\epsilon - u_i^\epsilon|^2|u_i^\epsilon|\Big) \\
&\leq 4\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})|b_i|((u_{i+1}^\epsilon)^2 + (u_i^\epsilon)^2)|u_i^\epsilon|\Big) \\
&\leq 64\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})|b_i|^2|u_i^\epsilon|^2\Big) + \frac{1}{16}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})\big((u_{i+1}^\epsilon)^2 + (u_i^\epsilon)^2\big)^2\Big) \\
&\leq c_3 \sum_{|i|\geq n} |b_i|^4 + 2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} \rho^2(\frac{|i|}{n})|u_i^\epsilon|^4\Big). \qquad (5.96)
\end{aligned}
$$

For the first term on the right-hand side of (5.93), by (5.21) we have

$$
\begin{aligned}
&2\mathbb{E}(\rho_n f(u^\epsilon(t)), \rho_n u^\epsilon(t)) \\
&\leq 2\sqrt{2}L_0\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + \frac{1}{2\sqrt{2}L_0}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}} |\rho(\frac{|i|}{n})f_i(u_i^\epsilon(t))|^2\Big)
\end{aligned}
$$

$$\leq 2\sqrt{2}L_0\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + \frac{1}{2\sqrt{2}L_0}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\rho^2(\frac{|i|}{n})(2|\alpha_i|^2 + 2L_0^2|u_i^\epsilon(t)|^2)\Big)$$
$$\leq \frac{1}{\sqrt{2}L_0}\sum_{|i|\geq n}|\alpha_i|^2 + \frac{5\sqrt{2}L_0}{2}\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2). \tag{5.97}$$

For the second term on the right-hand side of (5.93), we have

$$2\mathbb{E}(\rho_n g(t), \rho_n u^\epsilon(t)) \leq \frac{a}{2}\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + \frac{2}{a}\mathbb{E}\Big(\sum_{|i|\geq n} g_i^2(t)\Big). \tag{5.98}$$

By (5.23), we find that the third term on the right-hand side of (5.93) is satisfied by

$$\epsilon^2\sum_{j=1}^{\infty}\mathbb{E}(\|\rho_n h_j(t) + \rho_n\sigma_j(t, u^\epsilon(t))\|^2)$$
$$\leq 2\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\|\rho_n h_j(t)\|^2\Big) + 2\epsilon^2\sum_{j=1}^{\infty}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}|\rho(\frac{|i|}{n})\sigma_{j,i}(t, u_i^\epsilon(t))|^2\Big)$$
$$\leq 2\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\|\rho_n h_j(t)\|^2\Big) + 2\epsilon^2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\rho^2(\frac{|i|}{n})(2|\beta_i(t)|^2 + 2L_1^2|u_i^\epsilon(t)|^2)\Big)$$
$$\leq 2\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\sum_{|i|\geq n} h_{j,i}^2(t)\Big) + 4\epsilon^2\sum_{|i|\geq n}\beta_i^2(t) + 4\epsilon^2 L_1^2\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2). \tag{5.99}$$

Substituting (5.94)-(5.99) into (5.93), we deduce that

$$\frac{d}{dt}\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + \frac{3}{2}a\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2)$$
$$\leq (2 + \frac{5\sqrt{2}L_0}{2} + 4\epsilon^2 L_1^2)\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + \frac{136c_0}{n}\mathbb{E}(\|u^\epsilon(t)\|^2) + c_3\sum_{|i|\geq n}|b_i|^4 \tag{5.100}$$
$$+ \frac{1}{\sqrt{2}L_0}\sum_{|i|\geq n}|\alpha_i|^2 + 4\epsilon^2\sum_{|i|\geq n}\beta_i^2(t) + \frac{2}{a}\mathbb{E}\Big(\sum_{|i|\geq n} g_i^2(t)\Big) + 2\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\sum_{|i|\geq n} h_{j,i}^2(t)\Big).$$

By **R1** and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, we have

$$\frac{d}{dt}\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2) + a\mathbb{E}(\|\rho_n u^\epsilon(t)\|^2)$$
$$\leq \frac{136c_0}{n}\mathbb{E}(\|u^\epsilon(t)\|^2) + c_4\mathbb{E}\Big(\sum_{|i|\geq n} g_i^2(t) + \sum_{|i|\geq n}\beta_i^2(t) + \sum_{j=1}^{\infty}\sum_{|i|\geq n} h_{j,i}^2(t)\Big)$$
$$+ c_5\Big(\sum_{|i|\geq n}|b_i|^4 + \sum_{|i|\geq n}|\alpha_i|^2\Big). \tag{5.101}$$

Multiplying (5.101) by e^{at} and then integrating on (τ, t),

$$\mathbb{E}(\|\rho_n u^\epsilon(t,\tau,u_0)\|^2) \leq e^{-a(t-\tau)}\mathbb{E}(\|\rho_n u_0\|^2) + \frac{136c_0}{n}\int_\tau^t e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)ds$$
$$+ c_4\int_\tau^t e^{a(s-t)}\mathbb{E}\big(\sum_{|i|\geq n} g_i^2(s) + \sum_{|i|\geq n}\beta_i^2(s) + \sum_{j=1}^{\infty}\sum_{|i|\geq n} h_{j,i}^2(s)\big)ds$$
$$+ \frac{c_5}{a}\Big(\sum_{|i|\geq n}|b_i|^4 + \sum_{|i|\geq n}|\alpha_i|^2\Big). \tag{5.102}$$

By **R2** and (5.81), there exists $c_6 > 0$ depending only on $\mathcal{K}$ such that, as $n \to \infty$,

$$\frac{136c_0}{n}\int_\tau^t e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)ds$$
$$\leq \frac{136c_0}{n}c_6\Big(1 + e^{-at}\int_{-\infty}^t e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds\Big) \to 0, \tag{5.103}$$

uniformly for $u_0 \in \mathcal{K}$ and $t \geq \tau$. By **R2** once more, the second term on the right-hand side of (5.102) tends to zero as $n \to \infty$ and $\tau \to -\infty$. In addition, from $\alpha, \beta, b \in \ell^2$ and $\ell^2 \subseteq \ell^4$ we deduce that, as $n \to \infty$,

$$\frac{c_5}{a}\Big(\sum_{|i|\geq n}|b_i|^4 + \sum_{|i|\geq n}|\alpha_i|^2\Big) \leq \frac{c_5}{a}\Big((\sum_{|i|\geq n}|b_i|^2)^2 + \sum_{|i|\geq n}|\alpha_i|^2\Big) \to 0. \tag{5.104}$$

Therefore, for every $\varepsilon > 0$, there exists $N_1 = N_1(\varepsilon) > 0$ such that, for all $n \geq N_1$ and $t \geq \tau$,

$$\mathbb{E}(\|\rho_n u^\epsilon(t,\tau,u_0)\|^2) \leq e^{-a(t-\tau)}\mathbb{E}(\|\rho_n u_0\|^2) + \varepsilon \leq \sum_{|i|\geq n}|u_{0,i}|^2 + \frac{1}{2}\varepsilon. \tag{5.105}$$

Since $u_0 \in \mathcal{K}$, there exists $N_2 = N_2(\varepsilon) > N_1$ such that, for all $n \geq N_2$,

$$\sum_{|i|\geq n}|u_{0,i}|^2 \leq \frac{1}{2}\varepsilon. \tag{5.106}$$

In conclusion, there exists $N = \max\{N_1, N_2\} > 0$ such that, for all $n \geq N$ and $t \geq \tau$,

$$\mathbb{E}\Big(\sum_{|i|\geq 2n}|u_i^\epsilon(t,\tau,u_0)|^2\Big) \leq \mathbb{E}(\|\rho_n u^\epsilon(t,\tau,u_0)\|^2) \leq \varepsilon, \tag{5.107}$$

which completes the proof. □

Now, we show the **Feller property** of the transition operator $(P^\epsilon_{\tau,t})_{t\geq\tau}$ for the solution $u^\epsilon(\cdot,\tau,u_0)$ to satisfy the conditions required by Theorem 5.8.

The transition operator $(P^\epsilon_{s,t})_{t\geq s\in\mathbb{R}}$ is called **Feller**, that is, for any $s \leq t$, if for any $\varphi : \ell^2 \to \mathbb{R}$ bounded and continuous, $P^\epsilon_{s,t}\varphi : \ell^2 \to \mathbb{R}$ is also bounded and continuous.

Lemma 5.11. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. If* $\varphi \in C_b(\ell^2)$*, then* $P^\epsilon_{s,t}\varphi \in C_b(\ell^2)$ *for all* $t \geq s \in \mathbb{R}$.

Proof. Since $\varphi \in C_b(\ell^2)$, there exists $c_7 = c_7(\varphi) > 0$ such that, for all $x \in \ell^2$,

$$|\varphi(x)| \leq c_7, \tag{5.108}$$

which implies that, for all $s_0 \leq t_0$ and $y \in \ell^2$,

$$|(P^\epsilon_{s_0,t_0}\varphi)(y)| = |\mathbb{E}(\varphi(u^\epsilon(t_0, s_0, y)))| \leq c_7. \tag{5.109}$$

Thus, $P^\epsilon_{s_0,t_0}\varphi : \ell^2 \to \mathbb{R}$ is bounded. Consider a sequence $\{v_n\}_{n=1}^\infty$ in ℓ^2 and $v \in \ell^2$ such that $\|v_n - v\| \to 0$, then we will show that

$$\lim_{n\to\infty} \mathbb{E}(\varphi(u^\epsilon(t_0, s_0, v_n))) = \mathbb{E}(\varphi(u^\epsilon(t_0, s_0, v))). \tag{5.110}$$

(1) Note that the set $\{v, v_n\}_{n=1}^\infty$ is compact in ℓ^2. By Lemma 5.9, for every compact set $\mathcal{K} \subseteq \ell^2$, there exists $c_k = c_k(\mathcal{K}) > 0$ such that

$$\mathbb{E}(\|u^\epsilon(t_0, s_0, v)\|^2) \leq c_k^2, \quad \text{for all } \ t_0 \geq s_0, v \in \mathcal{K}. \tag{5.111}$$

Given $n \in \mathbb{N}$, we denote by $\tilde{u}^\epsilon_n(t_0, s_0, v) = (\tilde{u}^\epsilon_{n,i}(t_0, s_0, v))_{i\in\mathbb{Z}}$, where

$$\tilde{u}^\epsilon_{n,i}(t_0, s_0, v) = \begin{cases} u^\epsilon_i(t_0, s_0, v), & \text{if } \ |i| \leq n, \\ 0, & \text{if } \ |i| > n, \end{cases} \tag{5.112}$$

and $u^\epsilon(t_0, s_0, v)$ is a solution to the problem (5.24). For each $m \in \mathbb{N}$ and $\varepsilon > 0$, we infer from Lemma 5.10 that there exists $n_m = n(\varepsilon, \mathcal{K}, m) \in \mathbb{N}$ such that

$$\mathbb{E}(\|u^\epsilon(t_0, s_0, v) - \tilde{u}^\epsilon_{n_m}(t_0, s_0, v)\|^2) \leq \frac{\varepsilon}{2^{4m}}, \quad \text{for all } \ t_0 \geq s_0, v \in \mathcal{K}. \tag{5.113}$$

For every $m \in \mathbb{N}$, we define

$$X^\varepsilon_m := \{u = (u_i)_{i\in\mathbb{Z}} \in \ell^2 : u_i = 0 \ \text{ for } |i| > n_m \ \text{ and } \|u\| \leq \frac{2^{m+1}c_k}{\sqrt{\varepsilon}}\}, \tag{5.114}$$

$$Z^\varepsilon_m := \{u \in \ell^2 : \|u - w\| \leq \frac{1}{2^{m-1}} \ \text{ for some } w \in X^\varepsilon_m\} \ \text{ and } \ Z^\varepsilon = \bigcap_{m=1}^\infty Z^\varepsilon_m. \tag{5.115}$$

Therefore, we deduce from (5.111), (5.113), and Chebyshev's inequality that for all $v \in \mathcal{K}$ and $t_0 \geq s_0$,

$$\begin{aligned} &\mathbb{P}(\{\omega \in \Omega : u^\epsilon(t_0, s_0, v) \notin Z^\varepsilon_m\}) \\ &\leq \mathbb{P}(\{\omega \in \Omega : \tilde{u}^\epsilon_{n_m}(t_0, s_0, v) \notin X^\varepsilon_m\}) \\ &\quad + \mathbb{P}(\{\omega \in \Omega : u^\epsilon(t_0, s_0, v) \notin Z^\varepsilon_m \text{ and } \tilde{u}^\epsilon_{n_m}(t_0, s_0, v) \in X^\varepsilon_m\}) \\ &\leq \mathbb{P}(\{\omega \in \Omega : \|\tilde{u}^\epsilon_{n_m}(t_0, s_0, v)\| > \frac{2^{m+1}c_k}{\sqrt{\varepsilon}}\}) \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}(\{\omega\in\Omega:\|u^\epsilon(t_0,s_0,v)-\tilde{u}^\epsilon_{n_m}(t_0,s_0,v)\|>\frac{1}{2^{m-1}}\})\\
&\leq\frac{\varepsilon}{2^{2m+2}c_k^2}\mathbb{E}(\|u^\epsilon(t_0,s_0,v)\|^2)+2^{2m-2}\mathbb{E}(\|u^\epsilon(t_0,s_0,v)-\tilde{u}^\epsilon_{n_m}(t_0,s_0,v)\|^2)\\
&\leq\frac{\varepsilon}{2^{2m+1}}.
\end{aligned}
\tag{5.116}
$$

Note that Z^ε is closed and totally bounded in ℓ^2, thus Z^ε is compact in ℓ^2. Then for all $v\in\mathcal{K}$ and $t_0\geq s_0$,

$$
\mathbb{P}(\{\omega\in\Omega:u^\epsilon(t_0,s_0,v)\notin Z^\varepsilon\})\leq\sum_{m=1}^{\infty}\frac{\varepsilon}{2^{2m+1}}<\frac{1}{4}\varepsilon,
\tag{5.117}
$$

or equivalently,

$$
\mathbb{P}(\{\omega\in\Omega:u^\epsilon(t_0,s_0,v)\in Z^\varepsilon\})>1-\frac{1}{4}\varepsilon.
\tag{5.118}
$$

Analogously, for all $v_n\in\mathcal{K}$ and $t_0\geq s_0$,

$$
\mathbb{P}(\{\omega\in\Omega:u^\epsilon(t_0,s_0,v_n)\in Z^\varepsilon\})>1-\frac{1}{4}\varepsilon,\quad\text{for all } n\in\mathbb{N}.
\tag{5.119}
$$

(2) Since $v_n\to v$ in ℓ^2, by (5.64) there exists $c_8>0$ such that for all $n\in\mathbb{N}$

$$
\mathbb{E}\Big(\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v_n)\|^2\Big)+\mathbb{E}\Big(\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v)\|^2\Big)\leq c_8,
\tag{5.120}
$$

which, together with Chebyshev's inequality, implies, as $0<r\to+\infty$,

$$
\begin{aligned}
&\mathbb{P}\{\omega\in\Omega:\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v_n)\|>r\}\\
&\leq\frac{1}{r^2}\mathbb{E}\Big(\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v_n)\|^2\Big)\leq\frac{c_8}{r^2}\to0,
\end{aligned}
\tag{5.121}
$$

and

$$
\begin{aligned}
&\mathbb{P}\{\omega\in\Omega:\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v)\|>r\}\\
&\leq\frac{1}{r^2}\mathbb{E}\Big(\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v)\|^2\Big)\leq\frac{c_8}{r^2}\to0.
\end{aligned}
\tag{5.122}
$$

By (5.121) and (5.122), there exists $r_\varepsilon>0$ such that, for all $n\in\mathbb{N}$,

$$
\begin{cases}
\mathbb{P}\{\omega\in\Omega:\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v_n)\|>r_\varepsilon\}<\dfrac{\varepsilon}{4},\\
\mathbb{P}\{\omega\in\Omega:\sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v)\|>r_\varepsilon\}<\dfrac{\varepsilon}{4}.
\end{cases}
\tag{5.123}
$$

Given $n\in\mathbb{N}$, let $\Omega^\varepsilon_n=\tilde{\Omega}^\varepsilon_n\cap\tilde{\Omega}^\varepsilon$, where

$$
\tilde{\Omega}^\varepsilon_n=\Big\{\omega\in\Omega:u^\epsilon(t_0,s_0,v_n)\in Z^\varepsilon\ \text{ and }\ \sup_{s_0\leq t\leq t_0}\|u^\epsilon(t,s_0,v_n)\|\leq r_\varepsilon\Big\},
$$

$$\tilde{\Omega}^\varepsilon = \Big\{\omega \in \Omega : u^\epsilon(t_0, s_0, v) \in Z^\varepsilon \ \text{ and } \sup_{s_0 \le t \le t_0} \|u^\epsilon(t, s_0, v)\| \le r_\varepsilon\Big\}.$$

By (5.118)-(5.119) and (5.123), we obtain

$$\mathbb{P}(\Omega \setminus \Omega_n^\varepsilon) \le \mathbb{P}(\Omega \setminus \tilde{\Omega}_n^\varepsilon) + \mathbb{P}(\Omega \setminus \tilde{\Omega}^\varepsilon) < \varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{5.124}$$

Since φ is uniformly continuous in Z^ε, there exists $\delta > 0$ such that, for all $x_1, x_2 \in Z^\varepsilon$ with $\|x_1 - x_2\| < \delta$,

$$|\varphi(x_1) - \varphi(x_2)| < \varepsilon. \tag{5.125}$$

Now, we define the following stopping times by

$$\begin{aligned} \varsigma_n &= \inf\{t \ge s_0 : \|u^\epsilon(t, s_0, v_n)\| > r_\varepsilon\}, \ n \in \mathbb{N}, \\ \varsigma &= \inf\{t \ge s_0 : \|u^\epsilon(t, s_0, v)\| > r_\varepsilon\}. \end{aligned} \tag{5.126}$$

Similarly, as in Theorem 5.6, we find that there exists $c_9 > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\Big(\sup_{s_0 \le t \le t_0} \|u^\epsilon(t \wedge \varsigma_n \wedge \varsigma, s_0, v_n) - u^\epsilon(t \wedge \varsigma_n \wedge \varsigma, s_0, v)\|^2\Big) \le c_9 \|v_n - v\|^2. \tag{5.127}$$

It is easily deduced that $\varsigma_n(\omega) \ge t_0$ and $\varsigma(\omega) \ge t_0$ for all $\omega \in \Omega_n^\varepsilon$, thus $t_0 \wedge \varsigma_n \wedge \varsigma = t_0$. By (5.127) and Chebyshev's inequality, we obtain

$$\begin{aligned} &\mathbb{P}(\{\omega \in \Omega_n^\varepsilon : \|u^\epsilon(t_0, s_0, v_n) - u^\epsilon(t_0, s_0, v)\| \ge \delta\}) \\ &= \mathbb{P}(\{\omega \in \Omega_n^\varepsilon : \|u^\epsilon(t_0 \wedge \varsigma_n \wedge \varsigma, s_0, v_n) - u^\epsilon(t_0 \wedge \varsigma_n \wedge \varsigma, s_0, v)\| \ge \delta\}) \\ &\le \mathbb{P}(\{\omega \in \Omega : \|u^\epsilon(t_0 \wedge \varsigma_n \wedge \varsigma, s_0, v_n) - u^\epsilon(t_0 \wedge \varsigma_n \wedge \varsigma, s_0, v)\| \ge \delta\}) \\ &\le \frac{c_9}{\delta^2}\|v_n - v\|^2, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.128}$$

By (5.108), (5.124)-(5.125) and (5.128) we derive, as $v_n \to v$ in ℓ^2,

$$\begin{aligned} &\limsup_{n\to\infty} \int_\Omega |\varphi(u^\epsilon(t_0, s_0, v_n)) - \varphi(u^\epsilon(t_0, s_0, v))| d\mathbb{P} \\ &= \limsup_{n\to\infty} \int_{\Omega_n^\varepsilon} |\varphi(u^\epsilon(t_0, s_0, v_n)) - \varphi(u^\epsilon(t_0, s_0, v))| d\mathbb{P} \\ &\quad + \limsup_{n\to\infty} \int_{\Omega \setminus \Omega_n^\varepsilon} |\varphi(u^\epsilon(t_0, s_0, v_n)) - \varphi(u^\epsilon(t_0, s_0, v))| d\mathbb{P} \\ &\le \limsup_{n\to\infty} \int_{\{\omega \in \Omega_n^\varepsilon : \|u^\epsilon(t_0,s_0,v_n) - u^\epsilon(t_0,s_0,v)\| \ge \delta\}} |\varphi(u^\epsilon(t_0, s_0, v_n)) - \varphi(u^\epsilon(t_0, s_0, v))| d\mathbb{P} \\ &\quad + \limsup_{n\to\infty} \int_{\{\omega \in \Omega_n^\varepsilon : \|u^\epsilon(t_0,s_0,v_n) - u^\epsilon(t_0,s_0,v)\| < \delta\}} |\varphi(u^\epsilon(t_0, s_0, v_n)) - \varphi(u^\epsilon(t_0, s_0, v))| d\mathbb{P} \\ &\quad + 2c_7 \mathbb{P}(\{\Omega \setminus \Omega_n^\varepsilon\}) \\ &\le \frac{2c_7 c_9}{\delta^2} \limsup_{n\to\infty} \|v_n - v\|^2 + \varepsilon + 2c_7\varepsilon \le \varepsilon + 2c_7\varepsilon. \end{aligned} \tag{5.129}$$

According to the arbitrariness of ε, (5.110) follows from (5.129) immediately. □

5.2.2 Existence of evolution systems of probability measures

As a consequence of Lemma 5.9, the following Corollary can be immediately proved.

Corollary 5.12. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. Then, for all* $t \in \mathbb{R}$, $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$ *and* $-t < k \in \mathbb{N}$, *the solution* $u^\epsilon(\cdot, \tau, u_0)$ *of problem* (5.24) *satisfies,*

$$\begin{aligned}&\frac{1}{k+t}\int_{-k}^{t}\mathbb{E}(\|u^\epsilon(t,\tau,u_0)\|^2)d\tau + \frac{a}{4(k+t)}\int_{-k}^{t}\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)dsd\tau\\&\le \frac{c}{k+t}\mathbb{E}(\|u_0\|^2) + ce^{-at}\int_{-\infty}^{t} e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds + c,\end{aligned} \tag{5.130}$$

where c *is a positive constant independent of* ϵ, t, k *and* u_0.

Proof. Integrating (5.81) in Lemma 5.9 over $(-k, t)$ in τ, we obtain (5.130). □

Next, we can deduce from Lemma 5.10 that:

Corollary 5.13. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. Then, for each compact set* $\overline{\mathcal{K}}$ *of* ℓ^2, $\varepsilon > 0$ *and* $-t < k \in \mathbb{N}$, *there exists* $\overline{N} = \overline{N}(\overline{\mathcal{K}}, \varepsilon) \in \mathbb{N}$ *such that, for all* $n \ge \overline{N}$, $t \in \mathbb{R}$, *the solution* $u^\epsilon(\cdot, \tau, u_0)$ *of problem* (5.24) *with initial data* $u_0 \in \overline{\mathcal{K}}$ *satisfies,*

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]}\frac{1}{k+t}\int_{-k}^{t}\sum_{|i|\ge n}\mathbb{E}(|u_i^\epsilon(t,\tau,u_0)|^2)d\tau \le \varepsilon. \tag{5.131}$$

Proof. By (5.102) in Lemma 5.10, we have

$$\begin{aligned}&\mathbb{E}\Big(\sum_{|i|\ge 2n}|u_i^\epsilon(t,\tau,u_0)|^2\Big)\\&\le e^{-a(t-\tau)}\mathbb{E}\Big(\sum_{|i|\ge n}|u_{0,i}|^2\Big) + \frac{136c_0}{n}\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)ds\\&\quad + c_4\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}\Big(\sum_{|i|\ge n} g_i^2(s) + \sum_{|i|\ge n}\beta_i^2(s) + \sum_{j=1}^{\infty}\sum_{|i|\ge n} h_{j,i}^2(s)\Big)ds\\&\quad + \frac{c_5}{a}\Big(\sum_{|i|\ge n}|b_i|^4 + \sum_{|i|\ge n}|\alpha_i|^2\Big).\end{aligned} \tag{5.132}$$

Then, integrating (5.132) for τ over $(-k, t)$, we deduce

$$\begin{aligned}&\frac{1}{k+t}\int_{-k}^{t}\mathbb{E}\Big(\sum_{|i|\ge 2n}|u_i^\epsilon(t,\tau,u_0)|^2\Big)d\tau\\&\le \frac{1}{a(k+t)}\mathbb{E}\Big(\sum_{|i|\ge n}|u_{0,i}|^2\Big) + \frac{136c_0}{n(k+t)}\int_{-k}^{t}\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)dsd\tau\end{aligned}$$

$$+ c_4 e^{-at} \int_{-\infty}^{t} e^{as}\mathbb{E}\Big(\sum_{|i|\geq n} g_i^2(s) + \sum_{|i|\geq n} \beta_i^2(s) + \sum_{j=1}^{\infty}\sum_{|i|\geq n} h_{j,i}^2(s)\Big)ds$$
$$+ \frac{c_5}{a}\Big(\sum_{|i|\geq n} |b_i|^4 + \sum_{|i|\geq n} |\alpha_i|^2\Big). \tag{5.133}$$

By (5.130), for $u_0 \in \overline{\mathcal{K}}$ and $-t < k$, there exists $c_{10} > 0$ depending only on $\overline{\mathcal{K}}$ such that, as $n \to \infty$,

$$\frac{136c_0}{n(k+t)} \int_{-k}^{t}\int_{\tau}^{t} e^{a(s-t)}\mathbb{E}(\|u^\epsilon(s,\tau,u_0)\|^2)dsd\tau \tag{5.134}$$
$$\leq \frac{136c_0}{n(k+t)} c_{10}\Big(1 + e^{-at}\int_{-\infty}^{t} e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds\Big) \to 0,$$

which, together with **R2** and (5.104), completes the proof. □

Now, we show the existence of evolution systems of probability measures for system (5.24) on ℓ^2 in the sense of Definition 5.2. In particular, we shall prove the tightness of the probability measure of the solution in the sense of Definition 5.1.

Theorem 5.14. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. For any* $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, *the transition operator* $(P^\epsilon_{\tau,t})_{t\geq\tau}$ *for* (5.24) *has an evolution system of probability measures* $\{\mu^\epsilon_t\}_{t\in\mathbb{R}}$ *on* ℓ^2.

Proof. For $n \in \mathbb{N}$, we denote by $\tilde{u}^\epsilon_n(t,\tau,u_0) = (\tilde{u}^\epsilon_{n,i}(t,\tau,u_0))_{i\in\mathbb{Z}}$, where

$$\tilde{u}^\epsilon_{n,i}(t,\tau,u_0) = \begin{cases} u^\epsilon_i(t,\tau,u_0), & \text{if } |i| \leq n, \\ 0, & \text{if } |i| > n. \end{cases} \tag{5.135}$$

Given $m \in \mathbb{N}, \varepsilon > 0$ and $k, h \in \mathbb{N}$ with $k \geq h$, by Corollary 5.12 and **R2**, there exists $C_h = C(h) > 0$ (independent on ϵ, k) such that

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} \frac{1}{k-h}\int_{-k}^{-h} \mathbb{E}(\|u^\epsilon(-h,\tau,0)\|^2)d\tau$$
$$\leq ce^{ah}\int_{-\infty}^{0} e^{as}\mathbb{E}\big(\|g(s)\|^2 + \|\beta(s)\|^2 + \sum_{j=1}^{\infty}\|h_j(s)\|^2\big)ds + c \leq C_h^2. \tag{5.136}$$

For each $m \in \mathbb{N}$ and $\varepsilon > 0$, we can indicate from Corollary 5.13 that there exists $n_m = n(\varepsilon, h, m) \in \mathbb{N}$ such that

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} \frac{1}{k-h}\int_{-k}^{-h} \mathbb{E}(\|u^\epsilon(-h,\tau,0) - \tilde{u}^\epsilon_{n_m}(-h,\tau,0)\|^2)d\tau \leq \frac{\varepsilon}{2^{4m}}. \tag{5.137}$$

Then, we define

$$\overline{X}^\varepsilon_m(h) := \{u = (u_i)_{i\in\mathbb{Z}} \in \ell^2 : u_i = 0 \text{ for } |i| > n_m \text{ and } \|u\| \leq \frac{2^m C_h}{\sqrt{\varepsilon}}\}, \tag{5.138}$$

$$\overline{Z}^{\varepsilon}_m(h) := \{u \in \ell^2 : \|u - w\| \leq \frac{1}{2^m} \ \text{ for some } w \in \overline{X}^{\varepsilon}_m(h)\}. \tag{5.139}$$

Using the same method as (5.116), by (5.136)-(5.137), we can verify that

$$\begin{aligned}
&\frac{1}{k-h}\int_{-k}^{-h} \mathbb{P}\Big\{\omega \in \Omega : u^{\epsilon}(-h,\tau,0) \notin \overline{Z}^{\varepsilon}_m(h)\Big\} d\tau \\
&\leq \frac{1}{k-h}\int_{-k}^{-h} \mathbb{P}\Big\{\omega \in \Omega : \|\tilde{u}^{\epsilon}_{n_m}(-h,\tau,0)\| > \frac{2^m C_h}{\sqrt{\varepsilon}}\Big\} d\tau \\
&+ \frac{1}{k-h}\int_{-k}^{-h} \mathbb{P}\Big\{\omega \in \Omega : \|u^{\epsilon}(-h,\tau,0) - \tilde{u}^{\epsilon}_{n_m}(-h,\tau,0)\| > \frac{1}{2^m}\Big\} d\tau \\
&\leq \frac{\varepsilon}{2^{2m}C_h^2}\frac{1}{k-h}\int_{-k}^{-h} \mathbb{E}(\|u^{\epsilon}(-h,\tau,0)\|^2) d\tau \\
&+ \frac{2^{2m}}{k-h}\int_{-k}^{-h} \mathbb{E}(\|u^{\epsilon}(-h,\tau,0) - \tilde{u}^{\epsilon}_{n_m}(-h,\tau,0)\|^2) d\tau \leq \frac{\varepsilon}{2^{2m-1}}.
\end{aligned} \tag{5.140}$$

Define a probability measure $\eta^{\epsilon}_{k,h}(\cdot) = \frac{1}{k-h}\int_{-k}^{-h} \mathbb{P}\big\{\omega \in \Omega : u^{\epsilon}(-h,\tau,0) \in \cdot\big\} d\tau$ on ℓ^2. Let $\overline{Z}^{\varepsilon}(h) = \bigcap\limits_{m=1}^{\infty} \overline{Z}^{\varepsilon}_m(h)$, we know that $\overline{Z}^{\varepsilon}(h)$ is compact in ℓ^2. By (5.140), we have

$$\eta^{\epsilon}_{k,h}(\ell^2 \setminus \overline{Z}^{\varepsilon}(h)) \leq \sum_{m=1}^{\infty} \eta^{\epsilon}_{k,h}(\ell^2 \setminus \overline{Z}^{\varepsilon}_m(h)) < \varepsilon, \tag{5.141}$$

which means that $\{\eta^{\epsilon}_{k,h}\}_{k \geq h}$ is tight on ℓ^2 for each fixed $h \in \mathbb{N}$ in the sense of Definition 5.1. Thus, there exists $\eta^{\epsilon}_h \in \mathcal{P}(\ell^2)$ and an index subsequence (still denoted by $\{\eta^{\epsilon}_{k,h}\}$) such that

$$\eta^{\epsilon}_{k,h} \rightharpoonup \eta^{\epsilon}_h \ \text{ as } \ k \to \infty. \tag{5.142}$$

For any fixed $t \in \mathbb{R}$, consider $h \in \mathbb{N}$ such that $-h \leq t$ and define

$$\mu^{\epsilon}_t := Q^{\epsilon}_{-h,t}\eta^{\epsilon}_h, \tag{5.143}$$

where $Q^{\epsilon}_{-h,t}$ is defined in (5.10). By [35, Theorem 3.1], we know that the definition in (5.143) is independent of h. Then, by (5.11) and (5.143), for any $t \geq \tau \geq -h$, with $h \in \mathbb{N}$, we have

$$\begin{aligned}
Q^{\epsilon}_{\tau,t}\mu^{\epsilon}_{\tau} &= Q^{\epsilon}_{\tau,t}Q^{\epsilon}_{-h,\tau}\eta^{\epsilon}_h \\
&= (P^{\epsilon}_{\tau,t})^*(P^{\epsilon}_{-h,\tau})^*\eta^{\epsilon}_h \\
&= (P^{\epsilon}_{-h,\tau}P^{\epsilon}_{\tau,t})^*\eta^{\epsilon}_h \\
&= (P^{\epsilon}_{-h,t})^*\eta^{\epsilon}_h = Q^{\epsilon}_{-h,t}\eta^{\epsilon}_h = \mu^{\epsilon}_t,
\end{aligned} \tag{5.144}$$

which completes the proof in the sense of Definition 5.2. □

5.2.3 Limits stability of evolution systems of probability measures as noise intensity approaches a certain value

Now, we temper to prove the probability convergence of the solution u of (5.24) to satisfy the assumption **R** required by Theorem 5.8.

Lemma 5.15. *Suppose* (5.3) *and* (5.18)-(5.23) *hold. For every compact set* $\widehat{\mathcal{K}}$ *of* ℓ^2, $\delta > 0$, $t \geq \tau \in \mathbb{R}$ *and* $\epsilon_0 \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$,

$$\lim_{\epsilon \to \epsilon_0} \sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}(\{\omega \in \Omega : \|u^\epsilon(t, \tau, u_0) - u^{\epsilon_0}(t, \tau, u_0)\| \geq \delta\}) = 0. \tag{5.145}$$

Proof. For every $T > 0$ and any bounded set $\widehat{\mathcal{K}} \subseteq \ell^2$ we know from Theorem 5.6 that there exists $c = c(\widehat{\mathcal{K}}, T) > 0$ such that for all $u_0 \in \widehat{\mathcal{K}}$ and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$

$$\mathbb{E}\Big(\sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t, \tau, u_0)\|^2 \Big) \leq c,$$

which, together with Chebyshev's inequality, can imply that for every $\varepsilon > 0$, there exists $n = n(\varepsilon, \widehat{\mathcal{K}}, T) > 0$ such that, for all $u_0 \in \widehat{\mathcal{K}}$ and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$,

$$\mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t, \tau, u_0)\| > n \right\} \right) < \frac{1}{2}\varepsilon.$$

Given $u_0 \in \widehat{\mathcal{K}}$ and $\epsilon, \epsilon_0 \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, we define

$$\Omega_\epsilon = \left\{ \omega \in \Omega : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t, \tau, u_0)\| \leq n \text{ and } \sup_{t \in [\tau, \tau+T]} \|u^{\epsilon_0}(t, \tau, u_0)\| \leq n \right\}.$$

Thus, for all $u_0 \in \widehat{\mathcal{K}}$ and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, we have $\mathbb{P}(\Omega \setminus \Omega_\epsilon) < \varepsilon$. Define a stopping time by

$$\overline{T}_n = \inf\{t \geq \tau : \|u^\epsilon(t)\| > n \;\text{ or }\; \|u^{\epsilon_0}(t)\| > n\}. \tag{5.146}$$

where $\overline{T}_n = +\infty$ if $\{t \geq \tau : \|u^\epsilon(t)\| > n \;\text{ or }\; \|u^{\epsilon_0}(t)\| > n\} = \emptyset$. Then $\overline{T}_n \geq \tau + T$ for each $\omega \in \Omega_\epsilon$.

For any $u_0 \in \widehat{\mathcal{K}}$ and $\delta > 0$, we have

$$\begin{aligned} &\sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t, \tau, u_0) - u^{\epsilon_0}(t, \tau, u_0)\| \geq \delta \right\} \right) \\ &= \sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}\left(\left\{ \omega \in \Omega_\epsilon : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t \wedge \overline{T}_n, \tau, u_0) - u^{\epsilon_0}(t \wedge \overline{T}_n, \tau, u_0)\| \geq \delta \right\} \right) \\ &+ \sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}\left(\left\{ \omega \in \Omega \setminus \Omega_\epsilon : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t, \tau, u_0) - u^{\epsilon_0}(t, \tau, u_0)\| \geq \delta \right\} \right) \end{aligned} \tag{5.147}$$

$$\leq \sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t \wedge \overline{T}_n, \tau, u_0) - u^{\epsilon_0}(t \wedge \overline{T}_n, \tau, u_0)\| \geq \delta\right\}\right) + \varepsilon.$$

Now we just need to confirm that

$$\lim_{\epsilon \to \epsilon_0} \sup_{u_0 \in \widehat{\mathcal{K}}} \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t \in [\tau, \tau+T]} \|u^\epsilon(t \wedge \overline{T}_n, \tau, u_0) - u^{\epsilon_0}(t \wedge \overline{T}_n, \tau, u_0)\| \geq \delta\right\}\right) = 0 \tag{5.148}$$

to complete the proof. By (5.24) and (5.146), for all $t \geq \tau$, we have

$$\begin{aligned}
& u^\epsilon(t \wedge \overline{T}_n) - u^{\epsilon_0}(t \wedge \overline{T}_n) + \int_\tau^{t \wedge \overline{T}_n} D(u^\epsilon(s) - u^{\epsilon_0}(s)) ds \\
& - 2\int_\tau^{t \wedge \overline{T}_n} A(u^\epsilon(s) - u^{\epsilon_0}(s)) ds + a\int_\tau^{t \wedge \overline{T}_n} \big(u^\epsilon(s) - u^{\epsilon_0}(s)\big) ds \\
& + b\int_\tau^{t \wedge \overline{T}_n} \big(F(u^\epsilon(s)) - F(u^{\epsilon_0}(s))\big) ds + \int_\tau^{t \wedge \overline{T}_n} \big(G(u^\epsilon(s)) - G(u^{\epsilon_0}(s))\big) ds \\
& = \int_\tau^{t \wedge \overline{T}_n} \big(f(u^\epsilon(s)) - f(u^{\epsilon_0}(s))\big) ds + \epsilon \int_\tau^{t \wedge \overline{T}_n} \sum_{j=1}^\infty \big(h_j(s) + \sigma_j(t, u^\epsilon(s))\big) dW_j(s) \\
& - \epsilon_0 \int_\tau^{t \wedge \overline{T}_n} \sum_{j=1}^\infty \big(h_j(s) + \sigma_j(t, u^{\epsilon_0}(s))\big) dW_j(s) \\
& = \int_\tau^{t \wedge \overline{T}_n} \big(f(u^\epsilon(s)) - f(u^{\epsilon_0}(s))\big) ds + \sum_{j=1}^\infty \int_\tau^{t \wedge \overline{T}_n} \Big[(\epsilon - \epsilon_0)\big(h_j(s) + \sigma_j(s, u^\epsilon(s))\big) \\
& + \epsilon_0\big(\sigma_j(s, u^\epsilon(s)) - \sigma_j(s, u^{\epsilon_0}(s))\big)\Big] dW_j(s).
\end{aligned} \tag{5.149}$$

Thanks to Ito's formula, for all $t \geq \tau$,

$$\begin{aligned}
& \|u^\epsilon(t \wedge \overline{T}_n) - u^{\epsilon_0}(t \wedge \overline{T}_n)\|^2 + 2\int_\tau^{t \wedge \overline{T}_n} \|A(u^\epsilon(s) - u^{\epsilon_0}(s))\|^2 ds \\
& - 4\int_\tau^{t \wedge \overline{T}_n} \|B(u^\epsilon(s) - u^{\epsilon_0}(s))\|^2 ds + 2a\int_\tau^{t \wedge \overline{T}_n} \|u^\epsilon(s) - u^{\epsilon_0}(s)\|^2 ds \\
& + 2\int_\tau^{t \wedge \overline{T}_n} \big(b(F(u^\epsilon(s)) - F(u^{\epsilon_0}(s))), u^\epsilon(s) - u^{\epsilon_0}(s)\big) ds \\
& + 2\int_\tau^{t \wedge \overline{T}_n} \big(G(u^\epsilon(s)) - G(u^{\epsilon_0}(s)), u^\epsilon(s) - u^{\epsilon_0}(s)\big) ds \\
& \leq 2\int_\tau^{t \wedge \overline{T}_n} \big(f(u^\epsilon(s)) - f(u^{\epsilon_0}(s)), u^\epsilon(s) - u^{\epsilon_0}(s)\big) ds \\
& + 3(\epsilon - \epsilon_0)^2 \sum_{j=1}^\infty \int_\tau^{t \wedge \overline{T}_n} \|h_j(s)\|^2 ds + 3(\epsilon - \epsilon_0)^2 \sum_{j=1}^\infty \int_\tau^{t \wedge \overline{T}_n} \|\sigma_j(s, u^\epsilon(s))\|^2 ds
\end{aligned}$$

$$+ 3\epsilon_0^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} \|\sigma_j(s, u^{\epsilon}(s)) - \sigma_j(s, u^{\epsilon_0}(s))\|^2 ds + I_1, \tag{5.150}$$

where

$$\begin{aligned} I_1 =& 2|\epsilon - \epsilon_0| \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} (h_j(s), u^{\epsilon}(s) - u^{\epsilon_0}(s)) dW_j(s) \\ &+ 2|\epsilon - \epsilon_0| \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} (\sigma_j(s, u^{\epsilon}(s)), u^{\epsilon}(s) - u^{\epsilon_0}(s)) dW_j(s) \\ &+ 2\epsilon_0 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} \big(\sigma_j(s, u^{\epsilon}(s)) - \sigma_j(s, u^{\epsilon_0}(s)), u^{\epsilon}(s) - u^{\epsilon_0}(s)\big) dW_j(s). \end{aligned}$$

For the first term on the right-hand side of (5.150), by (5.20), we have

$$\begin{aligned} &2\int_{\tau}^{t\wedge \overline{T}_n} \big(f(u^{\epsilon}(s)) - f(u^{\epsilon_0}(s)), u^{\epsilon}(s) - u^{\epsilon_0}(s)\big) ds \\ &\le (\frac{a}{2} + \frac{2L_2^f(n)}{a}) \int_{\tau}^{t\wedge \overline{T}_n} \|u^{\epsilon}(s) - u^{\epsilon_0}(s)\|^2 ds, \end{aligned} \tag{5.151}$$

and for the third and fourth terms on the right-hand side of (5.150), by (5.22) and (5.23), we obtain

$$\begin{aligned} &3(\epsilon - \epsilon_0)^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} \|\sigma_j(s, u^{\epsilon}(s))\|^2 ds \\ &+ 3\epsilon_0^2 \sum_{j=1}^{\infty} \int_{\tau}^{t\wedge \overline{T}_n} \|\sigma_j(s, u^{\epsilon}(s)) - \sigma_j(s, u^{\epsilon_0}(s))\|^2 ds \\ &\le 6(\epsilon - \epsilon_0)^2 \int_{\tau}^{t\wedge \overline{T}_n} \|\beta(s)\|^2 ds + 6(\epsilon - \epsilon_0)^2 L_1^2 \int_{\tau}^{t} \|u^{\epsilon}(s\wedge \overline{T}_n)\|^2 ds \\ &\quad + 3\epsilon_0^2 L_2^{\sigma}(n) \int_{\tau}^{t\wedge \overline{T}_n} \|u^{\epsilon}(s) - u^{\epsilon_0}(s)\|^2 ds. \end{aligned} \tag{5.152}$$

By the same method as (5.43)-(5.44), taking expectation, there exists a positive constant $C_{10} = C_{10}(n)$ such that, for all $t \ge \tau$,

$$\begin{aligned} &\mathbb{E}\Big(\sup_{\tau\le s\le t} (\|u^{\epsilon}(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2)\Big) \\ &\le C_{10} \int_{\tau}^{t} \mathbb{E}\Big(\sup_{\tau\le s\le r} (\|u^{\epsilon}(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2\Big) dr + 6(\epsilon - \epsilon_0)^2 \int_{\tau}^{t\wedge \overline{T}_n} \|\beta(s)\|^2 ds \\ &\quad + 6(\epsilon - \epsilon_0)^2 L_1^2 \int_{\tau}^{t\wedge \overline{T}_n} \mathbb{E}(\|u^{\epsilon}(s)\|^2) ds + I_2, \end{aligned} \tag{5.153}$$

where

$$I_2 = 2|\epsilon - \epsilon_0|\mathbb{E}\Big(\sup_{\tau\le s\le t\wedge \overline{T}_n}\Big|\sum_{j=1}^{\infty}\int_\tau^s (h_j(r), u^\epsilon(r) - u^{\epsilon_0}(r))dW_j(r)\Big|\Big)$$
$$+ 2|\epsilon - \epsilon_0|\mathbb{E}\Big(\sup_{\tau\le s\le t\wedge \overline{T}_n}\Big|\sum_{j=1}^{\infty}\int_\tau^s (\sigma_j(r, u^\epsilon(r)), u^\epsilon(r) - u^{\epsilon_0}(r))dW_j(r)\Big|\Big)$$
$$+ 2\epsilon_0\mathbb{E}\Big(\sup_{\tau\le s\le t\wedge \overline{T}_n}\Big|\sum_{j=1}^{\infty}\int_\tau^s \big(\sigma_j(r, u^\epsilon(r)) - \sigma_j(r, u^{\epsilon_0}(r)), u^\epsilon(r) - u^{\epsilon_0}(r)\big)dW_j(r)\Big|\Big).$$

By the Burkholder-Davis-Gundy inequality, we derive from (5.22) and (5.23) that

$$I_2 \le \frac{3}{4}\mathbb{E}\Big(\sup_{\tau\le s\le t}(\|u^\epsilon(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2)\Big)$$
$$+ 8(\epsilon-\epsilon_0)^2 C_1^2 \sum_{j=1}^{\infty}\int_\tau^t \mathbb{E}(\|h_j(s)\|^2)ds$$
$$+ 16(\epsilon-\epsilon_0)^2 C_1^2 \int_\tau^t \|\beta(s)\|^2 ds + 16(\epsilon-\epsilon_0)^2 L_1^2 n^2 T$$
$$+ 8\epsilon_0^2 C_1^2 L_2^\sigma(n)\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le s\le r}(\|u^\epsilon(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2\Big)dr, \tag{5.154}$$

where C_1 is defined in (5.48). Therefore, there exists $C_{11} = C_{11}(n) > 0$ such that, for all $t \in [\tau, \tau + T]$,

$$\mathbb{E}\Big(\sup_{\tau\le s\le t}(\|u^\epsilon(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2)\Big)$$
$$\le c\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le s\le r}(\|u^\epsilon(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2\Big)dr$$
$$+ C_{11}T(\epsilon-\epsilon_0)^2\Big(\sup_{\tau\le s\le t}\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(s)\|^2) + \sup_{\tau\le s\le t}\|\beta(s)\|^2\Big). \tag{5.155}$$

By the Gronwall inequality applied to (5.155), we derive that, for all $t \in [\tau, \tau + T]$,

$$\mathbb{E}\Big(\sup_{\tau\le s\le t}(\|u^\epsilon(s\wedge \overline{T}_n) - u^{\epsilon_0}(s\wedge \overline{T}_n)\|^2)\Big)$$
$$\le C_{11}T(\epsilon-\epsilon_0)^2\Big(\sup_{\tau\le s\le t}\sum_{j=1}^{\infty}\mathbb{E}(\|h_j(s)\|^2) + \sup_{\tau\le s\le t}\|\beta(s)\|^2\Big)e^{cT}. \tag{5.156}$$

Using Chebyshev's inequality, we obtain the result (5.148) when $\epsilon \to \epsilon_0$, and then the proof can be completed by combining (5.147). □

Given $t \in \mathbb{R}$ and $\epsilon \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, let S_t^ϵ be the collection of all evolution systems of probability measures $\{\mu_t^\epsilon\}_{t\in\mathbb{R}}$ of $(P_{\tau,t}^\epsilon)_{t\ge\tau}$ for system (5.24). Now, we show the tightness of $\bigcup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} S_t^\epsilon$ in the sense of Definition 5.1, and the limit stability of any sequence of S_t^ϵ for non-autonomous stochastic modified S-H lattice system (5.24).

Theorem 5.16. *Suppose* (5.18)-(5.23) *and* **R1**-**R2** *hold. Then, for each* $t \in \mathbb{R}$,

(i) $\bigcup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} S_t^\epsilon$ *is tight on* ℓ^2.

(ii) If $\mu_t^{\epsilon_n} \in S_t^{\epsilon_n}$ *with* $\epsilon_n \to \epsilon_0 \in [0, \frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]$, *then there exists a subsequence (still denoted by itself) and* $\mu_t^{\epsilon_0} \in S_t^{\epsilon_0}$ *such that* $\mu_t^{\epsilon_n} \to \mu_t^{\epsilon_0}$ *weakly.*

Proof. (i) Given $t \in \mathbb{R}, m \in \mathbb{N}$ and a compact set $\mathcal{K}$ of ℓ^2, by Lemma 5.9, Lemma 5.10, and (5.135), for all $\varepsilon > 0$, there exist $n_m = n(t, m, \varepsilon) \in \mathbb{N}$, $\mathcal{T}_m = \mathcal{T}(t, \mathcal{K}, m, \varepsilon) \leq t$ and $c_t = c(t) > 0$ such that

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} \sup_{\tau\leq\mathcal{T}_m} \sup_{u_0\in\mathcal{K}} \mathbb{E}(\|u^\epsilon(t,\tau,u_0)\|^2) \leq c_t^2, \tag{5.157}$$

$$\sup_{\epsilon\in[0,\frac{\sqrt{\vartheta}}{2\sqrt{2}L_1}]} \sup_{\tau\leq\mathcal{T}_m} \sup_{u_0\in\mathcal{K}} \mathbb{E}(\|u^\epsilon(t,\tau,u_0) - \tilde{u}_{n_m}^\epsilon(t,\tau,u_0)\|^2) \leq \frac{\varepsilon}{2^{4m}}. \tag{5.158}$$

For every $m \in \mathbb{N}$, we define

$$\widehat{X}_m^\varepsilon(t) := \{u = (u_i)_{i\in\mathbb{Z}} \in \ell^2 : u_i = 0 \ \text{ for } |i| > n_m \ \text{ and } \|u\| \leq \frac{2^m c_t}{\sqrt{\varepsilon}}\}, \tag{5.159}$$

$$\widehat{Z}_m^\varepsilon(t) := \{u \in \ell^2 : \|u - w\| \leq \frac{1}{2^m} \ \text{ for some } w \in \widehat{X}_m^\varepsilon(t)\}, \tag{5.160}$$

and $\widehat{Z}^\varepsilon(t) = \bigcap_{m=1}^\infty \widehat{Z}_m^\varepsilon(t)$. Note that $\widehat{Z}^\varepsilon(t)$ is compact in ℓ^2.

Let us pick a bounded ball $B_M := \{x \in \ell^2 : \|x\| \leq M\}$ of ℓ^2 with $M \in \mathbb{N}$. By (5.9), (5.10), and the definition of $\{\mu_t^\epsilon\}_{t\in\mathbb{R}}$, we have

$$\begin{aligned}
\mu_t^\epsilon(\ell^2 \setminus \widehat{Z}^\varepsilon(t)) &= Q_{\tau,t}^\epsilon \mu_\tau^\epsilon(\ell^2 \setminus \widehat{Z}^\varepsilon(t)) \\
&= \int_{\ell^2} \mathbb{P}\Big(\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}^\varepsilon(t)\}\Big)\mu_\tau^\epsilon(dx) \\
&\leq \int_{B_M} \mathbb{P}\Big(\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}^\varepsilon(t)\}\Big)\mu_\tau^\epsilon(dx) \\
&\quad + \int_{\ell^2\setminus B_M} \mathbb{P}\Big(\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}^\varepsilon(t)\}\Big)\mu_\tau^\epsilon(dx) \qquad (5.161) \\
&\leq \sup_{x\in B_M} \mathbb{P}\Big(\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}^\varepsilon(t)\}\Big)\mu_\tau^\epsilon(B_M) + \mu_\tau^\epsilon(\ell^2 \setminus B_M).
\end{aligned}$$

It follows from (5.157) and (5.158) that for all $x \in B_M$ and $\widetilde{\mathcal{T}}_m = \widetilde{\mathcal{T}}(t, B_M, m, \varepsilon) \geq \tau$,

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}^\varepsilon(t)\} \\
&\leq \sum_{m=1}^\infty \mathbb{P}\{\omega \in \Omega : u^\epsilon(t,\tau,x) \notin \widehat{Z}_m^\varepsilon(t)\} \\
&\leq \sum_{m=1}^\infty \mathbb{P}\{\omega \in \Omega : \|\tilde{u}_{n_m}^\epsilon(t,\tau,x)\| > \frac{2^m c_t}{\sqrt{\varepsilon}}\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{\infty}\mathbb{P}\{\omega\in\Omega:\|u^{\epsilon}(t,\tau,x)-\tilde{u}^{\epsilon}_{n_m}(t,\tau,x)\|>\frac{1}{2^m}\} \qquad (5.162)\\
&\leq\sum_{m=1}^{\infty}\frac{\varepsilon}{2^{2m}c_t^2}\mathbb{E}(\|u^{\epsilon}(t,\tau,x)\|^2)+\sum_{m=1}^{\infty}2^{2m}\mathbb{E}(\|u^{\epsilon}(t,\tau,x)-\tilde{u}^{\epsilon}_{n_m}(t,\tau,x)\|^2)\leq\frac{2\varepsilon}{3}.
\end{aligned}
$$

For all $r\in\mathbb{R}$, we deduce

$$
\mu^{\epsilon}_r(\ell^2)=\mu^{\epsilon}_r\left(\bigcup_{M\in\mathbb{N}}B_M\right)=\mu^{\epsilon}_r\left(B_1\bigcup\Big(\bigcup_{M\in\mathbb{N}}(B_{M+1}\setminus B_M)\Big)\right)=\lim_{M\to\infty}\mu^{\epsilon}_r(B_M),
$$

which together with $\mu^{\epsilon}_r\in\mathcal{P}(\ell^2)$ means that $\lim_{M\to\infty}\mu^{\epsilon}_r(B_M)=1$. Thus we can obtain that $\lim\limits_{M\to\infty}\mu^{\epsilon}_r(\ell^2\setminus B_M)=0$, that is, for every $\varepsilon>0$, there exists $M=M(\epsilon,\varepsilon,t)\in\mathbb{N}$ such that

$$
\mu^{\epsilon}_{\widetilde{\mathcal{T}}(t,B_M,\varepsilon)}(\ell^2\setminus B_M)\leq\frac{\varepsilon}{3}. \qquad (5.163)
$$

Combining (5.162) and (5.163), it follows that, for every $\varepsilon>0$, there exists $M=M(\epsilon,\varepsilon,t)\in\mathbb{N}$ and $\tau=\widetilde{\mathcal{T}}(t,B_M,\varepsilon)\leq t$ such that

$$
\mu^{\epsilon}_t(\ell^2\setminus\widehat{Z}^{\varepsilon}(t))\leq\varepsilon. \qquad (5.164)
$$

(ii) We deduce from (i) that $\{\mu^{\epsilon_n}_t\}$ is tight, which means that there exists a subsequence (not relabeled) and probability measures $\mu^{\epsilon^*}_t$ such that $\mu^{\epsilon_n}_t\rightharpoonup\mu^{\epsilon^*}_t$ for each $t\in\mathbb{R}$. Then, by Lemma 5.11, Lemma 5.15, Theorem 5.14, and Theorem 5.8 we can complete the proof. □

Bibliography

[1] C. T. Anh and P. Q. Hung, Global existence and long-time behavior of solutions to a class of degenerate parabolic equations, *Ann. Polon. Math.*, **93** (2008), 217-230.

[2] C. T. Anh and P. Q. Hung, Global attractors for a class of degenerate parabolic equations, *Acta Math. Vietnam.*, **34** (2009), 213-231.

[3] L. Arnold, *Stochastic differential equations: Theory and applications*, Wiley-Interscience, New York, 1974.

[4] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.

[5] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, 1990.

[6] P. Billingsley, *Convergence of Probability Measures*, Second edition. John Wiley & Sons, Inc., New York, 1999.

[7] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic methods in the theory of non-linear oscillations*, Gordon and Breach Science Publishers, New York, 1961.

[8] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem, *Nonlinear Differ. Equ. Appl.*, **7** (2000), 187-199.

[9] T. Caraballo and J. Real, Partial differential equations with delayed random perturbations: existence uniqueness and stability of solutions, *Stoch. Anal. Appl.*, **11** (1993), 497-511.

[10] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, Pullback attractors of nonautonomous and stochastic multivalued dynamical systems, *Set-Valued Anal.*, **11** (2003), 153-201.

[11] T. Caraballo, P. E. Kloeden and B. Schmalfuß, Exponentially stable stationary solutions for stochastic evolution equations and their perturbation, *Appl. Math. Optim.*, **50** (2004), 183-207.

[12] T. Caraballo and J. A. Langa, On the upper semicontinuity of cocycle attractors for nonautonomous and random dynamical systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **10** (2003), 491-513.

[13] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß and J. Valero, Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete contin. Dyn. Syst.*, **21** (2008), 415-443.

[14] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß and J. Valero, Asymptotic behavior of a stochastic semilinear dissipative functional equation without uniqueness of solutions, *Discrete contin. Dyn. Syst. Ser. B*, **14** (2010), 439-455.

[15] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß and J. Valero, Global attractor for a non-autonomous integrodifferential equation in materials with memory, *Nonlinear Anal.*, **73** (2010), 183-201.

[16] T. Caraballo, F. Morillas and J. Valero, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, *J. Differ. Equ.*, **253** (2012), 667-693.

[17] T. Caraballo, F. Morillas and J. Valero, On differential equations with delay in Banach spaces and attractors for retarded lattice dynamical systems, *Discrete Cont. Dyn. Systems, Series A*, **34** (2014), 51-77.

[18] T. Caraballo, F. Morillas and J. Valero, Attractors for non-autonomous retarded lattice dynamical systems, *Nonauton. Dyn. Syst.*, **2** (2015), 31-51.

[19] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Long-time behaviour of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms, *Nonlinear Anal.*, **121** (2015), 3-18.

[20] T. Caraballo, X. Y. Han, B. Schmalfuß and J. Valero, Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise, *Nonlinear Anal.*, **130** (2016), 255-278.

[21] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Asymptotic behaviour of nonlocal p-Laplacian reaction-diffusion problems, *J. Math. Anal. Appl.*, **459** (2018), 997-1015.

[22] S. Cerrai and M. Freidlin, Averaging principle for a class of stochastic reaction-diffusion equations, *Probab. Theory Related Fields*, **144** (2009), 137-177.

[23] Z. Chen and B. X. Wang, Weak mean attractors and invariant measures for stochastic Schrödinger delay lattice systems, *J. Dyn. Differ. Equ.*, **35** (2023), 3201-3240.

[24] S. N. Chow, *Lattice dynamical systems Dynamical Systems*, Lecture Notes in Math., vol. 1822, Springer-Verlag, Berlin, (2003) 1-102.

[25] I. D. Chueshov, *Monotone random systems theory and applications*, vol. 1779, Springer Science & Business Media, 2002.

[26] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Related Fields*, **100** (1994), 365-393.

[27] H. Crauel, D. Arnaud and F. Flandoli, Random Attractors, *J. Dynam. Differential Equations*, **9** (1997), 307-341.

[28] H. Y. Cui and Y. R. Li, Existence and upper semicontinuity of random attractors for stochastic degenerate parabolic equations with multiplicative noises, *Appl. Math. Comput.*, **271** (2015), 777-789.

[29] H. Y. Cui, J. A. Langa and Y. R. Li, Regularity and structure of pullback attractors for reaction-diffusion type systems without uniqueness, *Nonlinear Anal. TMA*, **140** (2016), 208-235.

[30] H. Y. Cui, J. A. Langa and Y. R. Li, Measurability of random attractors for quasi strong-to-weak continuous random dynamical systems, *J. Dynam. Differ. Equ.*, **30** (2018), 1873-1898.

[31] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.*, **37** (1970), 297-308.

[32] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **15** (1998), 493-516.

[33] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[34] G. Da Prato and M. Röckner, Dissipative stochastic equations in Hilbert space with time dependent coefficients, *Rend. Lincei Mat. Appl.*, **17** (2006), 397-403.

[35] G. Da Prato and M. Röckner, A note on evolution systems of measures for time-dependent stochastic differential equations, *Progress in Probability*, **59** (2009), 115-122.

[36] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. I: Physical origins and classical methods, Springer-Verlag, Berlin, 1985.

[37] J. R. Esteban and J. L. Vázquez, On the equation of turbulent filtration in one-dimensional porous media, *Nonlinear Anal.*, **210** (1986), 1303-1325.

[38] J. C. O. Faria and C. M. Webler, Existence of a global attractor for the heat equation with degenerate memory, *J. Dyn. Diff. Equ.*, **35** (2023), 845-864.

[39] F. Flandoli, M. Gubinelli and E. Priola, Well-posedness of the transport equation by stochastic perturbation, *Invent. Math.*, **180** (2010), 1-53.

[40] C. Giorgi, V. Pata and A. Marzochi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *Nonlinear Differ. Equ. Appl.*, **5** (1998), 333-354.

[41] A. H. Gu and P. E. Kloeden, Asymptotic behavior of a nonautonomous p-Laplacian lattice system, *Int. J. Bifurc. Chaos Appl. Sci. Eng.*, **26** (2016), 1650174.

[42] A. H. Gu and B. X. Wang, Asymptotic behavior of random FitzHugh-Nagumo systems driven by colored noise, *Discrete Contin. Dyn. Syst., Ser. B*, **23** (2018), 1689-1720.

[43] A. H. Gu and B. X. Wang, Random attractors of reaction-diffusion equations without uniqueness driven by nonlinear colored noise, *J. Math. Anal. Appl.*, **486** (2020), 123880.

[44] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Fwzkcial. Ekvac.*, **21** (1978), 11-41.

[45] Y. He, C. Q. Li and J. T. Wang, Invariant measures and statistical solutions for the nonautonomous discrete modified Swift-Hohenberg equation, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 3819-3837.

[46] Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay Lecture Notes in Mathematics*, vol. 1473, Springer-Verlag, Berlin, 1991.

[47] L. T. Hoang, E. J. Olson and J. C. Robinson, On the continuity of global attractors, *Proc. Amer. Math. Soc.*, **143** (2015), 4389-4395.

[48] L. T. Hoang, E. J. Olson and J. C. Robinson, Continuity of pullback and uniform attractors, *J. Differ. Equ.*, **264** (2018), 4067-4093.

[49] F. Kappel and W. Schappacher, Some considerations to the fundamental theory of infinite delay equations, *J. Differ. Equ.*, **37** (1980), 141-183.

[50] R. Z. Khasminskii, On the principle of averaging the Itô stochastic differential equations, *Kibernetica*, **4** (1968), 260-279.

[51] J. Kim, Invariant measures for a stochastic nonlinear Schrödinger equation, *Indiana Univ. Math. J.*, **55** (2006), 687-717.

[52] P. E. Kloeden, J. Simsen and M. S. Simsen, Asymptotically autonomous multivalued cauchy problems with spatially variable exponents, *J. Math. Anal. Appl.*, **445** (2017), 513-531.

[53] P. E. Kloeden and T. Lorenz, Mean-square random dynamical systems, *J. Differential Equations*, **253** (2012), 1422-1438.

[54] A. Krause, M. Lewis and B. X. Wang, Dynamics of the non-autonomous stochastic p-Laplace equation driven by multiplicative noise, *Appl. Math. Comput.*, **246** (2014), 365-376.

[55] D. S. Li, B. X. Wang and X. H. Wang, Periodic measures of stochastic delay lattice systems, *J. Differ. Equ.*, **272** (2021), 74-104.

[56] D. S. Li, B. X. Wang and X. H. Wang, Limiting behavior of invariant measures of stochastic delay lattice systems, *J. Dyn. Differ. Equ.*, **34** (2022), 1453-1487.

[57] Y. R. Li, A. H. Gu and J. Li, Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, *J. Differ. Equ.*, **258** (2015), 504-534.

[58] Y. R. Li, S. Yang and Q. H. Zhang, Continuous Wong-Zakai approximations of random attractors for quasi-linear equations with nonlinear noise, *Qual. Theory Dyn. Syst.*, **19** (2020), 87.

[59] Y. R. Li, S. Yang and G. Q. Long, Continuity of random attractors on a topological space and fractional delayed FitzHugh-Nagumo equations with WZ-noise, *Discrete Contin. Dyn. Syst. B*, **27** (2022), 5977-6008.

[60] Y. R. Li and S. Yang, Hausdorff sub-norm spaces and continuity of random attractors for bi-stochastic g-Navier-Stokes equations with respect to tempered forces, *J. Dyn. Diff. Equ.*, **35** (2023), 543-574.

[61] Y. R. Li, F. L. Wang and T. Caraballo , Higher-order continuity of pullback random attractors for random quasilinear equations with nonlinear colored noise, *J. Dynam. Differ. Equ.*, (2024), Doi:org/10.1007/s10884-023-10333-8.

[62] Y. S. Lin, Y. Y. Li and D. S. Li, Periodic measures of impulsive stochastic neural networks lattice systems with delays, *J. Math. Phys.*, **63** (2022), 122702.

[63] Y. S. Lin and D. S. Li, Periodic measures of impulsive stochastic Hopfield-type lattice systems, *Stoch. Anal. Appl.*, **40** (2022), 914-930.

[64] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Gauthier-Villar, Paris, 1969.

[65] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin, 1972.

[66] L. F. Liu and X. L. Fu, Existence and upper semi-continuity of pullback attractors of a p-Laplacian equation with delay, *J. Math. Phys.*, **58** (2017), 082702.

[67] S. Ma and B. You, Global attractors for a class of degenerate parabolic equations with memory, *Discrete Contin. Dyn. Syst. Ser. B*, **28** (2023), 2044-2055.

[68] X. R. Mao, *Stochastic differential equations and applications (Second Edition)*, Cambbridge: Woodhead Publishing Limited, 2011.

[69] G. Marino and S. Mosconi, Existence and asymptotic behavior of nontrivial solutions to the Swift-Hohenberg equation, *J. Differ. Equ.*, **263** (2017), 8581-8605.

[70] P. Marín-Rubio, J. Real and J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case, *Nonlinear Anal.*, **74** (2011), 2012-2030.

[71] S.-E.A. Mohammed, *Stochastic Functional Differential Equations*, Longman, Harlow/New York, 1986.

[72] J. Y. Park and S. H. Park, Pullback attractor for a non-autonomous modified Swift-Hohenberg equation, *Comput. Math. Appl.*, **67** (2014), 542-548.

[73] M. Polat, Global attractor for a modified Swift-Hohenberg equation, *Comput. Math. Appl.*, **57** (2009), 62-66.

[74] L. Ridolfi, P. D'Odorico and F. Laio, *Noise-Induced Phenomena in the Environmental Sciences*, Cambridge University Press: New York, 2011.

[75] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, vol. 28, Cambridge University Press, 2001.

[76] R. A. Samprogna and T. Caraballo, Pullback attractor for a dynamic boundary non-autonomous problem with infinite delay, *Discrete Contin. Dyn. Syst. Ser. B*, **23** (2018), 509-523.

[77] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Providence, 1997.

[78] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65-96.

[79] G. L. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames-I. derivation of basic equations, *Acta Astron.*, **4** (1977), 1177-1206.

[80] J. B. Swift and E.C. Hohenberg, Hydrodynamic fluctuations at the convective instability, *Phys. Rev. A*, **15** (1977), 319-328.

[81] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.

[82] R. Temam, *Infinite Dimensional Dynamical System in Mathematics and Physics*, Springer, New York, 1997.

[83] G. E. Uhlenbeck and L. S. Ornstein, On the theory of Brownian motion, *Phys. Rev.*, **36** (1930), 823-841.

[84] B. X. Wang, Attractors for reaction-diffusion equations in unbounded domains, *Phys. D*, **128** (1999), 41-52.

[85] B. X. Wang, Dynamics of systems on infinite lattices, *J. Differ. Equ.*, **221** (2006), 224-245.

[86] B. X. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equ.*, **253** (2012), 1544-1583.

[87] B. X. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014), 269-300.

[88] B. X. Wang, Multivalued non-autonomous random dynamical systems for wave equations without uniqueness, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 2011-2051.

[89] B. X. Wang, Dynamics of stochastic reaction-diffision lattice systems driven by nonlinear noise, *J. Math. Anal. Appl.*, **477** (2019), 104-132.

[90] B. X. Wang, Weak pullback attractors for mean random dynamical systems in bochner spaces, *J. Dynam. Differ. Equ.*, **31** (2019), 2177-2204.

[91] B. X. Wang, Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise, *J. Differ. Equ.*, **268** (2019), 1-59.

[92] F. L. Wang and Y. R. Li, Random attractors for multi-valued multi-stochastic delayed p-Laplace lattice equations, *J. Differ. Equ. Appl.*, **27** (2021), 1232-1258.

[93] F. L. Wang, Y. R. Li and T. Caraballo, Asymptotic behavior of nonlocal semilinear degenerate heat equations with degenerate memory, Submitted to *Analysis and Applications*.

[94] F. L. Wang, T. Caraballo and Y. R. Li, Dynamics of multi-valued retarded p-Laplace equations driven by nonlinear colored noise, *J. Math. Phys.*, **64** (2023), 102701.

[95] F. L. Wang, T. Caraballo, Y. R. Li and R. H. Wang, Periodic measures for the stochastic delay modified Swift-Hohenberg lattice systems, *Commun. Nonlinear Sci. Numer. Simulat.*, **125** (2023), 107341.

[96] F. L. Wang, T. Caraballo, Y. R. Li and R. H. Wang, Asymptotic stability of evolution systems of probability measures of stochastic discrete modified Swift-Hohenberg equations, *Stoch PDE: Anal Comp.*, (2023), Doi:org/10.1007/s40072-023-00307-8.

[97] M. C. Wang and G. E. Uhlenbeck, On the theory of Brownian motion, II, *Rev. Modern Phys.*, **17** (1945), 323-342.

[98] R. H. Wang, B. L. Guo and B. X. Wang, Well-posedness and dynamics of fractional FitzHugh-Nagumo systems on $\mathbb{R}^N$ driven by nonlinear noise, *Sci. China Math.*, **64** (2021), 2395-2436.

[99] R. H. Wang, Long-time dynamics of stochastic lattice plate equations with nonlinear noise and damping, *J. Dyn. Differ. Equ.*, **33** (2021), 767-803.

[100] R. H. Wang, T. Caraballo and N. H. Tuan, Asymptotic stability of evolution systems of probability measures for nonautonomous stochastic systems: Theoretical results and applications, *Proc. Amer. Math. Soc.*, **151** (2023), 2449-2458.

[101] J. H. Xu, Z. C. Zhang and T. Caraballo, Non-autonomous nonlocal partial differential equations with delay and memory, *J. Differ. Equ.*, **270** (2021), 505-546.

[102] J. H. Xu, T. Caraballo and J. Valero, Asymptotic behavior of a semilinear problem in heat conduction with long time memory and non-local diffusion, *J. Differ. Equ.*, **327** (2022), 418-447.

[103] D. D. Yang, T. Caraballo and Z. Chen, The periodic and limiting behaviors of invariant measures for 3D globally modified Navier-Stokes equations, *J. Dyn. Diff. Equat.*, (2023), Doi.org/10.1007/s10884-023-10260-8.

[104] W. Q. Zhao, Regularity of random attractors for a degenerate parabolic equations driven by additive noises, *Appl. Math. Comput.*, **239** (2014), 358-374.

[105] W. Q. Zhao and Y. J. Zhang, High-order Wong-Zakai approximations for non-autonomous stochastic p-Laplacian equations on $\mathbb{R}^n$, *Commun. Pur. Appl. Anal.*, **20** (2021), 243-280.