THE DARBOUX PROCESS AND TIME-AND-BAND LIMITING FOR MATRIX ORTHOGONAL POLYNOMIALS

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Abstract. We extend to a situation involving matrix valued orthogonal polynomials a scalar result that originates in work of Claude Shannon in laying the mathematical foundations of information theory and a remarkable series of papers by D. Slepian, H. Landau and H. Pollak at Bell Labs in the 1960's. We show that in this case an **algebraic** miracle that plays a very important role in the classical case survives an application of the so called Darboux process in the matrix valued context.

1. Introduction

This paper can be seen as a combination of three related but independent developments.

The first strand comes from signal processing in its more basic form. C. Shannon, [32], posed the question:

suppose you consider an unknown signal $f(t)$ of finite duration, i.e. the signal is non-zero only in the interval $[-T, T]$. The data you have are the values of the Fourier transform $\mathcal{F}f(k)$ of f for values of k in the interval $[-W, W]$. What is the best use you can make of this data? Needless to say he was thinking of noisy measurements of the Fourier transform, and analytic continuation is out of the question.

Answering this question required finding and exploiting some remarkable mathematical miracles and was accomplished in a series of papers by three workers at Bell labs in the 1960's: David Slepian, Henry Landau and Henry Pollak, see [39, 22, 23, 35, 37]. In a paper by David Slepian, see [38], on the occasion of the John von Neumann lecture given at the SIAM 30th anniversary meeting in 1982, he writes: "There was a lot of serendipity here, clearly. And then our solution, too, seemed to hinge on a lucky accident-namely that we found a second-order differential operator that commuted with an integral operator that was at the heart of the problem".

The second strand has to do with quantum mechanics. E. Schroedinger, [31], observed around 1940 that starting with the free particle for which its quantum mechanical operator is $-D^2$, it was possible to apply to it a certain mathematical process (which unbeknown to Schroedinger had been exploited earlier for other purposes) to obtain a new differential operator of the form $-D^2 + V(x)$ which corresponds to physical systems much more interesting than the original free

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particle. This process can be applied succesively producing more and more interesting systems, and matrix versions of it can be applied to Dirac's equation, see [19]. It is now known in the physics literature as the Infeld-Hull factorization method, although it can be found already in Darboux's book on surfaces from 1899, see [6]. Powerful tools are rediscovered in different fields all the time.

Now for the third strand: starting with the work of people like H. Weyl and others one finds the idea of doing harmonic analysis based on expansions of a function in terms of eigenfunctions of an operator of the form $-D^2+V(x)$. The basic case, Fourier analysis, corresponds to $V(x)=0$. There are both continuous and discrete versions of this developments with the classical moment problem lurking in the background of the discrete version. Two very good sources for this rich material are [1, 42].

With these three important driving forces in mind, we can give a simplified account of what we do in this paper: we start with a (matrix valued) version of the operator $-D^2+V(x)$. We are really dealing with a discrete version of this so that we have a block tridiagonal matrix. With Weyl we set out to do the corresponding harmonic analysis, and we pose the analog of the question of Shannon in this case. More details are given in Section 2.

We then build the analog of the "time-and-band limiting" integral operator which in this case will be a full large matrix which we will denote by M. We then show that the same "lucky accident" that the workers at Bell labs had found is present here too: we can exhibit a (block) tridiagonal matrix, denoted by L such that

$$ML = LM.$$

This algebraic phenomenon has, as in the original case of Shannon, very important numerical consequences: it gives a reliable way to compute the eigenvectors of M, something that cannot be done otherwise.

With Schroedinger or Darboux we then inquire what happens when we apply their process to our analog of $-D^2+V(x)$, then consider with Weyl the corresponding harmonic analysis, build with Shannon the appropriate "time-and-band limiting" operator which we denote by $\widetilde{M}$ and wonder if the previous "lucky accident" will still hold. We find that there is now a (block) heptadiagonal matrix denoted by $\widetilde{L}$, such that

$$\widetilde{M}\widetilde{L} = \widetilde{L}\widetilde{M}.$$

Not only does the algebraic miracle survive the Darboux process, but once again it is crucial in allowing for an accurate computation of the eigenvectors of $\widetilde{M}$.

A few words about the importance of computing accurately the eigenvectors of the matrices $M, \widetilde{M}$. This is best explained in the original set-up of the workers at Bell labs in trying to answer Shannon's question:

what they found was that instead of looking for the unknown $f(t)$ itself one should consider a certain integral operator (alias our M) which has discrete spectrum in the open interval $(0,1)$ and a remarkable "spectral gap": about $[4WT]$ (integer part of $2W\times 2T$) eigenvalues are positive and well separated, and all the remaining ones are essentially zero. They argue that in the presence of noisy data one should try to compute the projection of $f(t)$ on the linear span of the eigenfunctions with "large" eigenvalues. The effective computation of these eigenfunctions is made possible by replacing the integral operator by the commuting differential one (alias our L) alluded to by D.

Slepian (both have simple spectrum). From a theoretical point of view the eigenfunctions of M and L are the same, but using the differential operator instead of the integral one, we have a manageable numerical problem: while the integral operator has a spectrum with eigenvalues that are extremely close together, the differential one has a very spread out spectrum, resulting in a stable numerical computation.

The results in the present paper are a natural extension of the work in [10, 14], where the classical orthogonal polynomials played a central role, to a matrix-valued case involving matrix orthogonal polynomials. In a case such as in [10, 14] or in the present paper where physical and frequency space are of a different nature (one is continuous and the other one is discrete) one can deal (as explained in [10, 14]) with either an integral operator or with a full matrix. Each one of these global objects will depend on two parameters that play the role of (T, W) in Shannon's case, and one will be looking for a commuting local object, i.e., a second order differential operator or a tridiagonal matrix. In this paper we will be dealing with a full matrix and a (block) tri or hepta diagonal one.

The only previous exploration of the commutativity property above in the matrix valued case is in [15]. In the present paper we explore for the first time in a matrix valued context the extend to which the "miracle" survives after an application of the Darboux process. Since the Darboux process preserves bispectrality in this situation, see [18], this new result gives more evidence along the results in [17].

For a very recent account of several computational issues see [2, 21, 25]. For new areas of applications involving (sometimes) vector-valued quantities on the sphere, see [20, 30, 33, 34].

2. Preliminaries

Let $W = W(x)$ be a weight matrix of size R in the open interval (a, b). By this we mean a complex $R \times R$-matrix valued integrable function W on the interval (a, b) such that $W(x)$ is positive definitive almost everywhere and with finite moments of all orders. Let $Q_w(x), w = 0, 1, 2, \ldots$, be a sequence of real valued matrix orthonormal polynomials with respect to the weight $W(x)$. Consider the following two Hilbert spaces: the space $L^2((a, b), W(t)dt)$, denoted here by $L^2(W)$, of all matrix valued measurable matrix valued functions $f(x)$, $x \in (a, b)$, satisfying $\int_a^b \operatorname{tr}(f(x)W(x)f^*(x))\, dx < \infty$ and the space $\ell^2(M_R, \mathbb{N}_0)$ of all real valued $R \times R$ matrix sequences $(C_w)_{w\in\mathbb{N}_0}$ such that $\sum_{w=0}^{\infty} \operatorname{tr}(C_w C_w^*) < \infty$.

The map $\mathcal{F} : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$ given by

$$(A_w)_{w=0}^{\infty} \longmapsto \sum_{w=0}^{\infty} A_w Q_w(x)$$

is an isometry. If the polynomials are dense in $L^2(W)$, this map is unitary with the inverse $\mathcal{F}^{-1} : L^2(W) \longrightarrow \ell^2(M_R, \mathbb{N}_0)$ given by

$$f \longmapsto A_w = \int_a^b f(x)\, W(x)\, Q_w^*(x) dx.$$

We denote our map by $\mathcal{F}$ to remind ourselves of the usual Fourier transform. Here $\mathbb{N}_0$ takes up the role of "physical space" and the interval (a, b) the role of "frequency space". This is, clearly, a noncommutative extension of the problem raised by C. Shannon since he was concerned with scalar valued functions and we are dealing with matrix valued ones.

The *time limiting operator*, at level N, acts on $\ell^2(M_R, \mathbb{N}_0)$ by simply setting equal to zero all the components with index larger than N. We denote it by χ_N. The *band limiting operator*, at level α, acts on $L^2(W)$ by multiplication by the characteristic function of the interval (a, α), $\alpha \leq b$. This operator will be denoted by χ_α. One could consider restricting the band to an arbitrary subinterval (a_1, b_1). However, the algebraic properties exhibited here, see Section 4 and beyond, hold only with this restriction. A similar situation arises in the classical case going all the way back to Shannon.

Consider the problem of determining a function f, from the following data: f has support on the finite set $\{0, \dots, N\}$ and its Fourier transform $\mathcal{F}f$ is known on the compact set $[a, \alpha]$. This can be formalized as follows

$$\chi_\alpha \mathcal{F} f = g = \text{ known }, \qquad \chi_N f = f.$$

We can combine the two equations into

$$Ef = \chi_\alpha \mathcal{F} \chi_N f = g.$$

To analyze this problem we need to compute the singular vectors (and values) of the operator $E : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$. These are given by the eigenvectors of the operators

$$E^*E = \chi_N \mathcal{F}^{-1} \chi_\alpha \mathcal{F} \chi_N \qquad \text{and} \qquad S_2 = EE^* = \chi_\alpha \mathcal{F} \chi_N \mathcal{F}^{-1} \chi_\alpha.$$

The operator E^*E, acting in $\ell^2(M_R, \mathbb{N}_0)$ is just a finite dimensional block-matrix M, and each block is given by

$$(M)_{m,n} = (E^*E)_{m,n} = \int_a^\alpha Q_m(x) W(x) {Q^*}_n w(x) dx, \qquad 0 \leq m, n \leq N.$$

The second operator $S_2 = EE^*$ acts in $L^2((a, \alpha), W(t)dt)$ by means of the integral kernel

$$k(x, y) = \sum_{w=0}^{N} Q_w(x) {Q^*}_w(y).$$

Consider now the problem of finding the eigenfunctions of E^*E and EE^*. For arbitrary N and α there is no hope of doing this analytically, and one has to resort to numerical methods and this is not an easy problem. Of all the strategies one can dream of for solving this problem, none sounds so appealing as that of finding an operator with simple spectrum which would have the same eigenfunctions as the original operators. This is exactly what Slepian, Landau and Pollak did in the scalar case, when dealing with the real line and the actual Fourier transform. They discovered (the analog of) the following properties:

- For each N, α there exists a symmetric tridiagonal matrix L, with simple spectrum, commuting with M.
- For each N, α there exists a selfadjoint differential operator D, with simple spectrum, commuting with the integral operator $S_2 = EE^*$.

To this day nobody has a simple explanation for these miracles, and this paper displays more instances where this holds. Indeed, there has been a systematic effort to see if the "bispectral property" first considered in [9], guarantees the commutativity of these two operators, a global and a local one. A few papers where this question has been taken up, include [10, 11, 12, 13, 14, 17, 28, 29].

3. An example of matrix valued orthogonal polynomials

Given the Jacobi type weight matrix

$$W(x) = [x(2-x)]^{\lambda-3/2}\begin{pmatrix} 1 & x-1 \\ x-1 & 1 \end{pmatrix}, \quad x \in [0,2], \tag{1}$$

appearing in [18, Section 6], one considers the sequences of monic orthogonal polynomials $(P_n(x))_n$. Assume $\lambda > 1/2$. They satisfy the recurrence relation

$$A_n P_{n-1}(x) + B_n P_n(x) + P_{n+1}(x) = xP_n(x),$$

where P_{-1} is the zero matrix and P_0 is the identity matrix. The coefficients A_n and B_n are given by the expressions

$$\begin{aligned} B_n &= \frac{\lambda-1}{2(n+\lambda)(n+\lambda-1)}S + I, \\ A_n &= \frac{n(n+2\lambda-2)}{4(n+\lambda-1)^2}I. \end{aligned}$$

Here I represents the identity matrix of dimension two and

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any value of $\lambda > 1/2$ these polynomials satisfy the first order differential equation

$$P_n'(x)\begin{pmatrix} 1-x & 1 \\ -1 & -1+x \end{pmatrix} + P_n(x)\begin{pmatrix} 1-2\lambda & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-2\lambda-n & 0 \\ 0 & n \end{pmatrix} P_n(x),$$

and the zero order differential equation

$$P_n S = S P_n, \quad n \geq 0.$$

These facts were already pointed out for $\lambda = 1$ in [3, 4].

The norms of the polynomials $(P_n)_n$ are scalar multiples of the identity

$$\|P_n\| = \gamma_n I, \; n \geq 0,$$

where

$$\begin{aligned} \gamma_{2k-1}^2 &= \frac{\sqrt{\pi}(k-1)!(2k-1)!!\Gamma(\frac{2k-1}{2}+\lambda)}{2^k(\lambda+2k-2)(\lambda+2k-3)\dots(\lambda+k-1)\Gamma(\lambda+2k-1)}, \\ \gamma_{2k}^2 &= \frac{\sqrt{\pi}k!(2k-1)!!\Gamma(\frac{2k-1}{2}+\lambda)}{2^k(\lambda+2k-1)(\lambda+2k-2)\dots(\lambda+k)\Gamma(\lambda+2k)}, \quad k \geq 1, \end{aligned}$$

with $\gamma_0^2 = \dfrac{\sqrt{\pi}\Gamma(\lambda-\frac{1}{2})}{\Gamma(\lambda)}$.

We just pointed out that we have a 'bispectral" situation with a differential operator of order two and in the next section we will show that this leads to a (block) tridiagonal L. Later in the paper we will investigate further the relation between the order of the appropriate differential operator and the width of the band of the corresponding operator L, but we are getting ahead of our story.

4. Time and band limiting for matrix polynomials

4.1. **The matrix M.** Now we consider the sequence of orthonormal polynomials $Q_n = \|P_n\|^{-1}P_n = \gamma_n^{-1}P_n$. We fix a natural **even** number N and $\alpha \in (0,2]$ and consider the matrix M, of total size $N\times N$,

$$M = \begin{pmatrix} M^{0,0} & M^{0,1} & \dots & M^{0,\frac{N}{2}-1} \\ M^{1,0} & M^{1,1} & \dots & M^{1,\frac{N}{2}-1} \\ \dots & \dots & \dots & \dots \\ M^{\frac{N}{2}-1,0} & M^{\frac{N}{2}-1,1} & \dots & M^{\frac{N}{2}-1,\frac{N}{2}-1} \end{pmatrix},$$

whose (i,j) block is the 2×2 matrix obtained by taking the inner product of the $i-th$ and $j-th$ normalized matrix valued orthogonal polynomials in the interval $[0,\alpha]$ with $\alpha \le 2$

$$M^{i,j} = \int_0^\alpha Q_i(x)W(x)Q_j(x)^* dx, \quad \text{for } 0 \le i,j \le \frac{N}{2}-1.$$

It should be clear that the restriction to the interval $[0,\alpha]$ implements "band-limiting" while the restriction to the range $0,1,\dots,\frac{N}{2}-1$ takes care of "time-limiting". By definition, the matrix M is a symmetric matrix and the entries of the matrix $M=(M_{rs})_{1\le r,s\le N}$ are related with the entries of the block matrices $M^{i,j} = \begin{pmatrix} M^{i,j}_{11} & M^{i,j}_{12} \\ M^{i,j}_{21} & M^{i,j}_{22}\end{pmatrix}$ by

$$\begin{aligned} M_{2i+1,2j+1} &= M^{i,j}_{11}, \quad M_{2i+1,2j} = M^{i,j-1}_{12}, \\ M_{2i,2j+1} &= M^{i-1,j}_{21}, \quad M_{2i,2j} = M^{i-1,j-1}_{22}. \end{aligned}$$

4.2. **The block tridiagonal matrix L.** The aim of this section is to find all block tridiagonal symmetric matrices L such that

$$ML = LM. \tag{2}$$

Notice that in principle there is not guarantee that we will find any such L except for a scalar multiple of the identity. For the problem at hand we need to exhibit matrices L that have a simple spectrum, so that the eigenvectors of L will automatically be eigenvectors of M.

We found that the vector space of such matrices is of dimension 2. We have found a unique symmetric matrix L satisfying (2) of the form

$$L = \begin{pmatrix} L^{1,1} & L^{1,2} & 0 & \dots & & 0 \\ L^{2,1} & L^{2,2} & L^{2,3} & 0 & \dots & \\ \dots & \ddots & \ddots & \ddots & & \\ & \dots & & L^{\frac{N}{2}-1,\frac{N}{2}-2} & L^{\frac{N}{2}-1,\frac{N}{2}-1} & I \\ 0 & \dots & & & I & 0 \end{pmatrix}.$$

Here 0 and I denote de 2×2 zero and identity matrices respectively. Of course, to this matrix L we can always add a multiple of the identity matrix (of dimension $N\times N$). This is the result of extensive symbolic computation done, independently, in Mathematica and Maxima.

The matrices $L^{k,k}$, $1\le k\le \frac{N}{2}$, and $L^{k,k+1} = L^{k+1,k}$, $1 \le k \le \frac{N}{2}-1$, are given by the expressions

$$L^{k,k} = L^{k,k}_{11} I + L^{k,k}_{12} S, \quad L^{k,k+1} = L^{k,k+1}_{11} I, \tag{3}$$

where

$$L_{11}^{k,k} = (1-\alpha)\frac{(N+2\lambda-4)(\frac{N}{2}+2(\lambda-1)+k-1)(\frac{N}{2}-k)}{(N+2\lambda-3)\sqrt{\frac{N}{2}+2\lambda-3}\sqrt{\frac{N}{2}-1}}, \tag{4}$$

$$L_{12}^{k,k} = (\lambda-1)\frac{(N-2k)(\frac{N}{2}+\lambda-1)(\frac{N}{2}+2\lambda+k-3)}{(\lambda-2+k)(\lambda+k-1)(N+2\lambda-3)\sqrt{N-2}\sqrt{N+4\lambda-6}}, \tag{5}$$

and finally

$$L_{11}^{k,k+1} = \frac{\sqrt{k(2\lambda+k-2)}(\frac{N}{2}+2\lambda+k-2)(N+2\lambda-4)(\frac{N}{2}-k)}{(2\lambda+2(k-1))(N+2\lambda-3)\sqrt{\frac{N}{2}-1}\sqrt{\frac{N}{2}+2\lambda-3}}. \tag{6}$$

Note that (as we should) we obtain zero in (4) and (5) for $k = \dfrac{N}{2}$ and that the value of the expression in (6) is one for $k = \frac{N}{2} - 1$.

Remark 4.1. We are convinced that the spectrum of L is simple. We do not have yet an analytic proof of this fact that covers arbitrary values of even N. Extensive symbolic and numerical experimentation done independently in Maxima and Mathematica leads us to make this assertion.

For instance, from its pentadiagonal structure one has immediately that any eigenvalue of the matrix L has at most geometrical dimension two. The matrix L is similar to the matrix

$$\begin{pmatrix}
la_1 & 0 & ld_1 & \cdots & & & & & & 0 \\
0 & la_2 & 0 & ld_1 & \cdots & & & & & \\
ld_1 & 0 & la_3 & 0 & ld_2 & & & & & \\
0 & ld_1 & 0 & la_4 & \ddots & \ddots & & & & \\
0 & 0 & ld_2 & 0 & la_5 & & & & & \\
\vdots & & & ld_2 & \ddots & \ddots & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & & la_{N-3} & 0 & 1 & 0 \\
 & & & & & & & la_{N-2} & 0 & 1 \\
 & & & & & & 1 & 0 & 0 & 0 \\
0 & \cdots & & & & & 0 & 1 & 0 & 0
\end{pmatrix},$$

with only two nonzero diagonals. Here $la_{2k-1} = L_{11}^{k,k} + L_{12}^{k,k}$ and $la_{2k} = L_{11}^{k,k} - L_{12}^{k,k}$, $k = 1, \ldots, \dfrac{N}{2} - 1$, are the eigenvalues of the diagonal blocks $L^{k,k}$ of L, and $ld_k = L_{11}^{k,k+1}$, $k = 1, \ldots, \dfrac{N}{2} - 2$. All the elements la_k in the main diagonal are different and have the same sign and the entries ld_k are also different and positive. However, these facts do not guarantee, in general, that the spectrum of the matrix is simple, though in this case one has in particular, $la_{2k-1} > la_{2k}$, which might be helpful to reduce the problem.

Alberto, lee por favor con atencion este ultimo párrafo que lo modifique ligeramente para que se vea claro que en general una matriz de esas caractiersiticas con las diagonales con elementos distintos y del mismo signo, etc.. NO tiene por que tener espectro

simple. La última frase *"though in this case one has in particular, $la_{2k-1} > la_{2k}$, which might be helpful to reduce the problem"* **si prefieres la quitamos y la dejamos de reserva en nuestra cocina por si se queja el referee...**

Esta frase *"Getting a proof that the spectrum of L is simple would not finish the job"*, **prefiero tal vez sustituirla por esta:** *"The fact that the spectrum of L is simple is not the only remarkable thing"* **(o similar...pues la frase inicial me suena como una apology de no poder probarlo...cosa que ya entonamos el mea culpa en el Remark 4.1)** What makes L really useful in this context is the observation that its spectrum is nicely "spread out". In fact we are convinced that the spectrum of M itself is also simple but with eigenvalues that are just too close together. The same applies after one step of the Darboux process which is introduced in the next section and then illustrated in the last section.

All of these facts are clearly seen in the section on numerical results at the end of the paper. The same situation (i.e. M has simple spectrum with most eigenvalues too close together) is already well known in the classical case.

5. Time and Band limiting and the Darboux Process

Applying the Darboux process (consisting of factorizing and turning the factors around) to the block tridiagonal Jacobi matrix associated to the sequence $(P_n)_n$, orthogonal with respect to the weight matrix (1), one obtains the weight matrix already given in [18, Section 6]

$$\begin{aligned}\widetilde{W}(x) =&(2-x)^{\lambda-3/2}x^{\lambda-5/2}\begin{pmatrix} 1 & x-1 \\ x-1 & 1\end{pmatrix}\\ &-\frac{\sqrt{\pi}\Gamma(\lambda-3/2)}{2\Gamma(\lambda)}\left(\begin{pmatrix} 2\lambda-2 & -1 \\ -1 & 2\lambda-2\end{pmatrix} - (2\lambda-3)a_0^{-1}\right)\delta_0(x).\end{aligned} \tag{7}$$

Here a_0 is a matrix parameter, which is assumed to be positive definite. As shown in [18, Section 6], for $\lambda = 5/2$ these polynomials satisfy a differential equation of order four, i.e., we see that the Darboux process preserves bispectrality.

Consider, for $\lambda = 5/2$, the sequence of monic polynomials $(\widetilde{P}_n)_n$ orthogonal with respect to the weight matrix $\widetilde{W}$ in (7). Consider the sequence of orthonormal polynomials $(\widetilde{Q}_n)_n$, where $\widetilde{Q}_n = l_n^{-1}\widetilde{P}_n$ where l_n is a lower triangular matrix such that $\|\widetilde{P}_n\|^2 = l_n l_n^*$ is the Cholesky decomposition of the positive definite matrix $\|\widetilde{P}_n\|^2$.

Following the procedure as in section 4.1, one defines the matrix $\widetilde{M}$ of truncated matrix products, of size $N \times N$ for an even natural number N, where the 2×2 matrix blocks $\widetilde{M}^{i,j}$ are defined by:

$$\widetilde{M}^{i,j} = \int_0^{\alpha} \widetilde{Q}_i(x)\widetilde{W}(x)\widetilde{Q}_j(x)^* dx, \quad \text{for } 0 \le i, j \le \frac{N}{2} - 1.$$

We wonder now if the Darboux process will also preserve the commutativity found in the previous section, i.e., we search for a banded symmetric square matrix $\widetilde{L}$ of size N such that

$$\widetilde{M}\widetilde{L} = \widetilde{L}\widetilde{M}. \tag{8}$$

As an example, we fix a diagonal matrix $a_0 = \begin{pmatrix} f & 0 \\ 0 & g\end{pmatrix}$, with $f = 1/3$, $g = 1/4$. We have found a (block) hepta-diagonal matrix $\widetilde{L}$ (**15 scalar diagonals**) with simple spectrum such that (8) holds true. In the outer-most diagonals the first, third, fifth... elements are nonzero, while all the other ones vanish. The dimension of the space of such matrices $\widetilde{L}$ is two (counting scalar multiples of the identity). This is the *first example* of this kind obtained after applying the Darboux process.

If we choose $\lambda = 7/2$ (instead of $\lambda = 5/2$) we had a six order differential operator for the corresponding matrix valued orthogonal polynomial in [18]. Going through the process just indicated above we get a (block) matrix $\widetilde{L}$ with a total of 11 (block) diagonals.

Finally we have found that for $\lambda = 9/2$, when the differential operator in [18] was of order eight, we get a (block) matrix $\widetilde{L}$ with a total of 15 (block) diagonals.

For values of λ that are not half an odd integer, neither the existence of a differential operator nor the commutativity property hold.

It is important to point out that, in all these cases, there does not exist any banded matrix $\widetilde{L}$ satisfying property (8), for an even N large enough, with a narrower band.

6. Some numerical results

In this last section we display the results of some numerical computations. This should make clear the importance of having found, as above, a matrix such as L for a given M.

Our point becomes very clear even if we use a small value of N and a value of α pretty close to 2. If we had chosen a larger value of N the phenomenon in question would be present for an even larger range of values of α.

If our task is to compute the eigenvectors of M we can use the QR algorithm as implemented in LAPACK. The results are recorded below.

Using the choices $\lambda = 5/2$, $N = 8$, $\alpha = 3/2$ we get that the eigenvalues of M are given by

$$0.999999, 0.999999, 0.999836, 0.997203,$$
$$0.929126, 0.677533, 0.138490, 0.0206928.$$

The eigenvalues of L are given by

$$-5.12121, -3.90302, -2.24422, 1.67735,$$
$$-1.38639, 1.13472, 0.288662, -0.203965.$$

Denoting by X_L the matrix of eigenvectors of L and by Y_M the matrix of eigenvectors of M (the eigenvectors of both matrices are theoretically the same up to order) one has for the moduli of the entries of $Y_M^t X_L$ the matrix

$$\begin{pmatrix}
0.80784 & 0.58940 & 0.00003 & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} \\
0.26994 & 0.96288 & 0.00001 & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10{-}6 & 0. \times 10^{-6} & 0. \times 10^{-6} \\
0.00448 & 0.00620 & 0.99997 & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} \\
0.00003 & 0.00001 & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} \\
0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 1.0000 \\
0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 1.0000 & 0. \times 10^{-6} \\
0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 1.0000 & 0. \times 10^{-6} & 0. \times 10^{-6} \\
0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 1.0000 & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6} & 0. \times 10^{-6}
\end{pmatrix}.$$

In theory this should be a permutation matrix.

Now we record the effect of having done one step of the Darboux process, again for $\lambda = 5/2$. Using the choices $N = 12$, $\alpha = 19/10$, $a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$, we get that the eigenvalues of $\widetilde{M}$ are given by

$$1., 1., 1., 1., 1., 1., 1., 0.999998, 0.999735, 0.994943, 0.841168, 0.486855.$$

The eigenvalues of $\widetilde{L}$ are given by

$$\begin{gathered} 1.21377, -1.13634, 1.07435, 0.871046, -0.80146, 0.657959 \\ 0.490068, 0.403528, 0.39213, -0.370528, 0.295294, 0.0281296. \end{gathered}$$

We see that denoting by $X_{\widetilde{L}}$ the matrix of eigenvectors of $\widetilde{L}$ and by $Y_{\widetilde{M}}$ the matrix of eigenvectors of $\widetilde{M}$ (the eigenvectors of both matrices are theoretically the same up to order) one has for the moduli of the entries of $Y^t_{\widetilde{M}} X_{\widetilde{L}}$ the matrix

$$\begin{pmatrix}
1.82056 \times 10^{-17} & 3.04717 \times 10^{-17} \ldots & 0.0420759 & 0.583874 & 0.810689 \ldots \\
8.78204 \times 10^{-17} & 3.1225 \times 10^{-17} \ldots & 0.0253894 & 0.804025 & 0.579406 \ldots \\
1.4962 \times 10^{-16} & 4.22405 \times 10^{-16} \ldots & 0.0431757 & 0.112046 & 0.0704807 \ldots \\
3.40873 \times 10^{-16} & 5.89806 \times 10^{-17} \ldots & 0.997858 & 0.00901028 & 0.0458765 \ldots \\
2.63678 \times 10^{-16} & 1.06165 \times 10^{-15} \ldots & 0.000132559 & 0.0000240362 & 8.22533 \times 10^{-6} \ldots \\
6.80012 \times 10^{-16} & 3.33067 \times 10^{-16} \ldots & 1.3994 \times 10^{-6} & 2.4296 \times 10^{-7} & 5.50382 \times 10^{-7} \ldots \\
7.21645 \times 10^{-16} & 2.13718 \times 10^{-15} \ldots & 9.13815 \times 10^{-10} & 2.13949 \times 10^{-9} & 4.66822 \times 10^{-10} \ldots \\
7.21645 \times 10^{-16} & 1.11022 \times 10^{-15} \ldots & 7.22596 \times 10^{-11} & 1.66854 \times 10^{-11} & 2.2451 \times 10^{-11} \\
7.77156 \times 10^{-16} & 4.80171 \times 10^{-15} \ldots & 1.46445 \times 10^{-14} & 1.80558 \times 10^{-13} & 1.22631 \times 10^{-15} \ldots \\
7.77156 \times 10^{-16} & 1.22125 \times 10^{-15} \ldots & 2.54206 \times 10^{-14} & 1.49012 \times 10^{-14} & 2.24714 \times 10^{-16} \ldots \\
1.11022 \times 10^{-16} & 1. \quad \ldots & 8.1532 \times 10^{-17} & 6.20164 \times 10^{-17} & 9.56808 \times 10^{-18} \ldots \\
1. & 1.94289 \times 10^{-16} \ldots & 2.6975 \times 10^{-16} & 5.38306 \times 10^{-17} & 4.63496 \times 10^{-18} \ldots
\end{pmatrix}.$$

Once again this should be a permutation matrix.

Observe that some of the entries of these matrices are indeed very close to the theoretically correct values, while others are terribly off. The reason is that there are a few eigenvalues of M that are just too close together. This produces numerical instability in the computation of the corresponding eigenvectors. On the other hand all the eigenvalues of L are nicely separated, and the corresponding eigenvectors can be trusted.

In summary, a good way to obtain good numerical values for the eigenvectors of M is to forget about M altogether and to compute numerically the eigenvectors of L. Not only we will then be dealing with a very sparse matrix for which the QR algorithm works very fast (most of the work is avoided) but the problem is numerically very well conditioned. For a discussion of the sensitivity of eigenvectors and their dependence on the separation of the corresponding eigenvalues one can consult [7, 40] as well as [27, pages 15, 222].

These numerical problems are not new in our situation involving matrix valued functions, but already appear in the classical scalar case, and this phenomenon is well documented. The important point is that even in the matrix valued case we can exhibit commuting tridiagonal matrices that play the same role of the "prolate spheroidal differential operator" in the scalar case.

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