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Generalized eigenvalue problem for an interface elliptic equation

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Abstract

In this paper we deal with an eigenvalue problem in an interface elliptic equation. We characterize the set of principal eigenvalues as a level set of a concave and regular function. As application, we study a problem arising in population dynamics. In these problems each species lives in a subdomain, and they interact in a common border, which acts as a geographical barrier; but unlike previous results, we consider the case of different growth rates in each subdomain.

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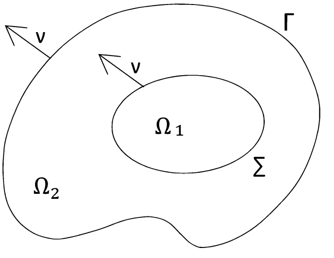

Fig. 1. A possible configuration of the domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$.

1. Introduction

Recently, the following semilinear interface problems have been analyzed

$$\begin{cases} -\Delta u_i = \lambda f_i(x, u_i) & \text{in } \Omega_i, \ i = 1, 2, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbf{R}^N$ with

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma,$$

with Ω_i subdomains, with internal interface $\Sigma = \partial\Omega_1$, and $\Gamma = \partial\Omega_2 \setminus \Sigma$, ν_i is the outward normal to Ω_i, and we call $\nu := \nu_1 = -\nu_2$ (see Fig. 1 where we have illustrated an example of Ω).

In (1), u_i represents the density of a species inhabiting in Ω_i, and they interact on Σ under the so called Kedem-Katchalsky conditions (see [7]), and it means that the flux is proportional to the jump of the function through Σ (see [3], [4], [5], [6] and references therein). Here, $f_i : \Omega_i \times \mathbf{R} \mapsto \mathbf{R}$ are regular functions, $\gamma_i > 0$ stands for the proportional coefficient of the jump and $1/\lambda$, $\lambda > 0$, is a real parameter representing the diffusion coefficient of the species, the same in both subdomains. It seems natural to consider two different diffusion coefficients, one for each species, that is, a problem as

$$\begin{cases} -\Delta u_i = \lambda_i f_i(x, u_i) & \text{in } \Omega_i, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{2}$$

with $\lambda_i \in \mathbf{R}$. Although mathematically it makes sense to consider λ_i as a real parameter, its usual meaning is that $1/\lambda_i$ is the diffusion coefficient in Ω_i, λ_i being a positive parameter in such a case.

As a first step towards the study of (1), it is necessary to analyze the eigenvalue problem

$$\begin{cases} -\Delta u_i = \lambda m_i(x) u_i & \text{in } \Omega_i, \\ u_i > 0 & \text{in } \Omega_i, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{3}$$

where $m_i \in L^\infty(\Omega_i)$, $m_i \not\equiv 0$ in Ω_i. (3) has been analyzed in [9] in the self-adjoint case $\gamma_1 = \gamma_2$. For that, the authors used variational arguments to prove the existence of principal eigenvalue as well as its main properties. The general case $\gamma_1 \neq \gamma_2$ was studied in [8] using a different argument. In [8], to study (3), the authors first analyze the problem

$$\begin{cases} -\Delta u_i + c_i(x)u_i = \lambda u_i & \text{in } \Omega_i, \\ \partial_\nu u_i = \gamma_i(u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{4}$$

where $c_i \in L^\infty(\Omega_i)$. They prove the existence of a unique principal eigenvalue of (4), denoted by $\Lambda_1(c_1, c_2)$. Hence, the study of (3) is equivalent to find the zeros of the map

$$\lambda \in \mathbb{R} \mapsto f(\lambda) := \Lambda_1(-\lambda m_1, -\lambda m_2).$$

The main goal of this paper is to study the following generalized eigenvalue problem:

$$\begin{cases} -\Delta u_i = \lambda_i m_i(x) u_i & \text{in } \Omega_i, \\ u_i > 0 & \text{in } \Omega_i, \\ \partial_\nu u_i = \gamma_i(u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma. \end{cases} \tag{5}$$

Motivated by [8], to study (5) we analyze the zeros of the map

$$(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mapsto F(\lambda_1, \lambda_2) := \Lambda_1(-\lambda_1 m_1, -\lambda_2 m_2),$$

that is, we analyze the set

$$\mathcal{C} := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : F(\lambda_1, \lambda_2) = 0\}.$$

We show that F is a regular function, concave and $F(0,0) = 0$. Hence, for instance, fixed λ_1, there exist at most two values of λ_2 such that $F(\lambda_1, \lambda_2) = 0$. Moreover, due to the concavity of F, it is well known that the set $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : F(\lambda_1, \lambda_2) \geq 0\}$ is convex. In any case, the study of the set $\mathcal{C}$ depends strongly on the signs of m_i. It is obvious that the case where the functions m_i have a definite sign, for example they are positive, is a simpler case than the case where one or both of them change sign.

We summarize the main results.

Our first result deals with the case both m_i non-negative and non-trivial functions (see Fig. 2).

Theorem 1.1. *Assume that $m_i \gneq 0$ in Ω_i, $i = 1, 2$ and define*

$$M_i^0 := \Omega_i \setminus \overline{\{x \in \Omega_i : m_i(x) > 0\}}$$

and assume that ∂M_i^0 is regular and

$$\overline{M_i^0} \subseteq \Omega_i. \tag{6}$$

Then, there exist positive values Λ_i^+, $i = 1, 2$ such that:

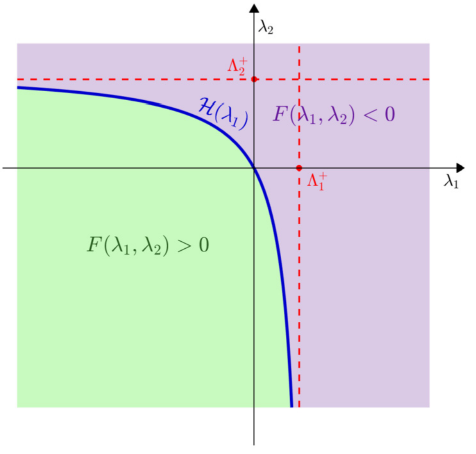

Fig. 2. Case m_1 and m_2 both non-negative and non-trivial verifying (6): we have represented in the plane $\lambda_1 - \lambda_2$ the curve $F(\lambda_1, \lambda_2) = 0$, as well as, the regions where F is negative and positive. In this case, $\mathcal{C} = \{(\lambda_1, \lambda_2) : \lambda_2 = \mathcal{H}(\lambda_1)\}$.

1. *Assume that* $\lambda_1 \geq \Lambda_1^+$. *Then,* $F(\lambda_1, \lambda_2) < 0$ *for all* $\lambda_2 \in \mathbb{R}$.
2. *Assume that* $\lambda_1 < \Lambda_1^+$. *There exists a unique* $\lambda_2 := \mathcal{H}(\lambda_1)$ *such that* $F(\lambda_1, \lambda_2) = 0$ *and*

$$F(\lambda_1, \lambda_2) < 0 \quad \text{for } \lambda_2 > \mathcal{H}(\lambda_1), \qquad F(\lambda_1, \lambda_2) > 0 \quad \text{for } \lambda_2 < \mathcal{H}(\lambda_1).$$

Moreover, the map $\lambda_1 \mapsto \mathcal{H}(\lambda_1)$ *is continuous, decreasing,* $\mathcal{H}(0) = 0$ *and*

$$\lim_{\lambda_1 \to -\infty} \mathcal{H}(\lambda_1) = \Lambda_2^+, \qquad \lim_{\lambda_1 \to \Lambda_1^+} \mathcal{H}(\lambda_1) = -\infty.$$

The values Λ_1^+ and Λ_2^+ will be defined in Section 2.

In the next result we analyze the case m_1 non-negative and non-trivial and m_2 changing sign (see Fig. 3).

Theorem 1.2. *Assume that* $m_1 \gneq 0$ *in* Ω_1 *and verifies (6) and* m_2 *changes sign in* Ω_2. *There exists* $\lambda_1^{\max} \geq 0$ *such that:*

1. *If* $\lambda_1 > \lambda_1^{\max}$, *then* $F(\lambda_1, \lambda_2) < 0$ *for all* $\lambda_2 \in \mathbb{R}$.
2. *If* $\lambda_1 = \lambda_1^{\max}$, *then there exists a unique* $\overline{\lambda}_2$ *such that* $F(\lambda_1^{\max}, \overline{\lambda}_2) = 0$ *and* $F(\lambda_1^{\max}, \lambda_2) < 0$ *for all* $\lambda_2 \in \mathbb{R} \setminus \{\overline{\lambda}_2\}$.
3. *For all* $\lambda_1 < \lambda_1^{\max}$, *there exist* $\lambda_2^- = \mathcal{H}^-(\lambda_1) < \lambda_2^+ = \mathcal{H}^+(\lambda_1)$ *such that*

$$F(\lambda_1, \lambda_2^-) = F(\lambda_1, \lambda_2^+) = 0,$$

and

$$F(\lambda_1, \lambda_2) \begin{cases} < 0 & \text{for } \lambda_2 > \mathcal{H}^+(\lambda_1) \text{ or } \lambda_2 < \mathcal{H}^-(\lambda_1), \\ > 0 & \text{for } \lambda_2 \in (\mathcal{H}^-(\lambda_1), \mathcal{H}^+(\lambda_1)). \end{cases}$$

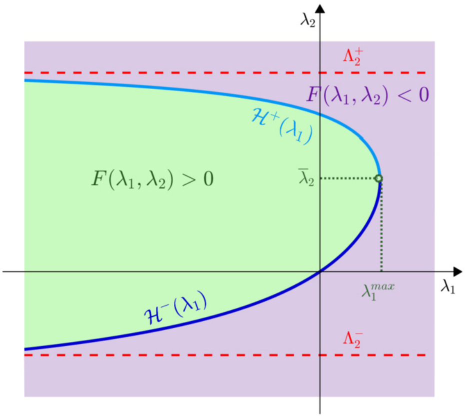

Fig. 3. Case $m_1 \gneq 0$ and verifying (6), m_2 changing sign and $\int_{\Omega_2} m_2 < 0$.

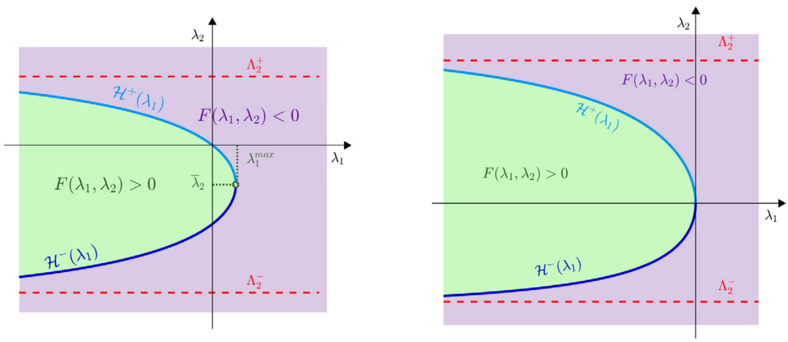

Fig. 4. Case $m_1 \gneq 0$ and verifying (6), m_2 changing sign and $\int_{\Omega_2} m_2 > 0$ (left) and $\int_{\Omega_2} m_2 = 0$ (right).

Moreover, the map $\lambda_1 \mapsto \mathcal{H}^+(\lambda_1)$ (resp. $\mathcal{H}^-(\lambda_1)$) is continuous, decreasing (resp. increasing) and

$$\lim_{\lambda_1 \to -\infty} \mathcal{H}^\pm(\lambda_1) = \Lambda_2^\pm \quad \textit{and} \quad \lim_{\lambda_1 \to \lambda_1^{\max}} \mathcal{H}^\pm(\lambda_1) = \overline{\lambda}_2.$$

4. *Finally,*
   (a) *If $\int_{\Omega_2} m_2 < 0$, then $\lambda_1^{\max} > 0$ and $\overline{\lambda}_2 > 0$.*
   (b) *If $\int_{\Omega_2} m_2 > 0$, then $\lambda_1^{\max} > 0$ and $\overline{\lambda}_2 < 0$.*
   (c) *If $\int_{\Omega_2} m_2 = 0$, then $\lambda_1^{\max} = \overline{\lambda}_2 = 0$.*

Remark 1.3.

1. In Fig. 4 we have represented the cases $m_1 \gneq 0$ in Ω_1 verifying (6), m_2 changes sign in Ω_2, $\int_{\Omega_2} m_2 > 0$ and $\int_{\Omega_2} m_2 = 0$.

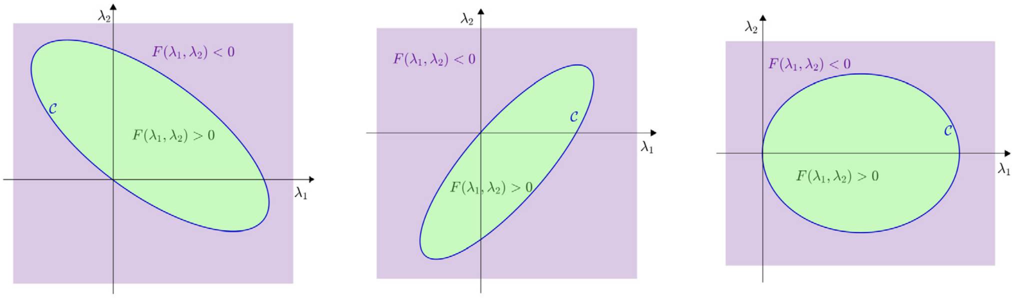

Fig. 5. Cases m_i changing sign. $\int_{\Omega_1} m_1 < 0$ and: $\int_{\Omega_2} m_2 < 0$ (left), $\int_{\Omega_2} m_2 > 0$ (center) and $\int_{\Omega_2} m_2 = 0$ (right).

2. Of course, by symmetry, a similar result holds for m_1 changing sign in Ω_1 and m_2 non-negative, non-trivial and verifying (6).

Finally, we deal with the case of both m_i changing sing.

Theorem 1.4. *Assume that m_i changes sign in Ω_i. Then, there exists a closed curve $\mathcal{C} \subset \mathbf{R}^2$, such that $F(\lambda_1, \lambda_2) = 0$ if and only if $(\lambda_1, \lambda_2) \in \mathcal{C}$. Moreover,*

$$F(\lambda_1, \lambda_2) > 0 \textit{ if and only if } (\lambda_1, \lambda_2) \in int(\mathcal{C}),$$

and

$$F(\lambda_1, \lambda_2) < 0 \textit{ if and only if } (\lambda_1, \lambda_2) \in Ext(\mathcal{C}).$$

Remark 1.5. The form and structure of $\mathcal{C}$ depends strongly on the sign of the integrals of m_i. In all the cases, $(0, 0) \in \mathcal{C}$.

In the following result, we complete the above Theorem, see Fig. 5.

Theorem 1.6. *Assume that m_i changes sign for $i = 1, 2$. There exist $\lambda_1^{\min} \le 0 \le \lambda_1^{\max}$ such that*

1. *If $\lambda_1 < \lambda_1^{\min}$ or $\lambda_1 > \lambda_1^{\max}$, then $F(\lambda_1, \lambda_2) < 0$ for all $\lambda_2 \in \mathbf{R}$.*
2. *If $\lambda_1 = \lambda_1^{\max}$ (resp. $\lambda_1 = \lambda_1^{\min}$) then there exists a unique $\overline{\lambda}_2$ (resp. $\underline{\lambda}_2$) such that $F(\lambda_1^{\max}, \overline{\lambda}_2) = 0$ (resp. $F(\lambda_1^{\min}, \underline{\lambda}_2) = 0$) and $F(\lambda_1^{\max}, \lambda_2) < 0$ (resp. $F(\lambda_1^{\min}, \lambda_2) < 0$) for all $\lambda_2 \in \mathbf{R} \setminus \{\overline{\lambda}_2\}$ (resp. $\lambda_2 \in \mathbf{R} \setminus \{\underline{\lambda}_2\}$).*
3. *If $\lambda_1 \in (\lambda_1^{\min}, \lambda_1^{\max})$ there exist unique $\lambda_2^- = \mathcal{H}^-(\lambda_1) < \lambda_2^+ = \mathcal{H}^+(\lambda_1)$ such that $F(\lambda_1, \lambda_2^{\pm}) = 0$. Moreover,*

$$F(\lambda_1, \lambda_2) \begin{cases} < 0 & \textit{for } \lambda_2 > \mathcal{H}^+(\lambda_1) \textit{ or } \lambda_2 < \mathcal{H}^-(\lambda_1), \\ > 0 & \textit{for } \lambda_2 \in (\mathcal{H}^-(\lambda_1), \mathcal{H}^+(\lambda_1)). \end{cases}$$

Finally,

$$\lim_{\lambda_1 \to \lambda_1^{\min}} \mathcal{H}^{\pm}(\lambda_1) = \underline{\lambda}_2, \quad \textit{and} \quad \lim_{\lambda_1 \to \lambda_1^{\max}} \mathcal{H}^{\pm}(\lambda_1) = \overline{\lambda}_2.$$

We apply these results to the nonlinear problem

$$\begin{cases} -\Delta u_i = \lambda_i m_i(x) u_i - u_i^{p_i} & \text{in } \Omega_i, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{7}$$

with $\lambda_i \in \mathbf{R}$, $m_i \in L^\infty(\Omega_i)$ and $p_i > 1$. Again, we consider the general case $\lambda_i \in \mathbf{R}$ and $m_i \in L^\infty(\Omega_i)$ but of course from a biological point of view only some cases are interesting. For example, when $m_1 \equiv 1$ in Ω_1, λ_1 represents the growth rate of the species u_1, which could be positive or negative. When m_1 changes sign, only the case $\lambda_1 > 0$ should be considered.

We prove (Theorem 5.1) that (7) possesses a positive solution if and only if

$$F(\lambda_1, \lambda_2) < 0.$$

Moreover, in such a case, the solution is the unique positive solution. Hence, we can give the following consequences:

1. Assume that m_1 and m_2 are non-negative and non-trivial functions.
   (a) For λ_1 large ($\lambda_1 > \Lambda_1^+$), there exists a positive solution for all $\lambda_2 \in \mathbf{R}$.
   (b) For $\lambda_1 < \Lambda_1^+$, there exists a value $\lambda_2 = \mathcal{H}(\lambda_1)$ such that (7) possesses a positive solution for $\lambda_2 > \mathcal{H}(\lambda_1)$.
   In both cases, for $\lambda_1 > 0$ we have that there exists a positive solution for negative growth rate (λ_2) of u_2. In the case without interface, this is not possible, that is, even if the population has negative growth in one part of the domain, the interface effect makes it possible for the species to persist throughout the domain.
2. Assume that m_1 is non-negative and non-trivial and m_2 changes sign. Then, if λ_1 is large, then there exists positive solution for all $\lambda_2 \in \mathbf{R}$. However, for $\lambda_1 < \lambda_1^{\max}$, then there exists positive solution for $\lambda_2 < \mathcal{H}^-(\lambda_1)$ or $\lambda_2 > \mathcal{H}^+(\lambda_1)$.
3. Assume that m_1 and m_2 change sign. There exist $\lambda_1^{\min} < \lambda_1^{\max}$ such that for $\lambda_1 > \lambda_1^{\max}$ or $\lambda_1 < \lambda_1^{\min}$, (7) possesses a positive solution for all $\lambda_2 \in \mathbf{R}$. However, for $\lambda_1 \in (\lambda_1^{\min}, \lambda_1^{\max})$, there exist $\mathcal{H}^-(\lambda_1) < \mathcal{H}^+(\lambda_1)$ such that (7) possesses a positive solution only for $\lambda_2 < \mathcal{H}^-(\lambda_1)$ or $\lambda_2 > \mathcal{H}^+(\lambda_1)$.

An outline of the paper is: in Section 2 we include some preliminary results related to scalar eigenvalue problems. Section 3 is devoted to show some general properties of $F(\lambda_1, \lambda_2)$. The main results concerning to the eigenvalue problem (5) are proved in Section 4. Finally, in Section 5, we analyze (7).

2. Preliminary results

2.1. Scalar eigenvalue problem

In this section we recall some results concerning to scalar eigenvalue problems, see [2] for example.

Here G is a $C^{2,\alpha}$, $\alpha \in (0, 1)$, domain of $\mathbf{R}^N$, $\partial G = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint open and closed subsets of ∂G and ν is the outward unit normal vector field. For $c \in L^\infty(G)$,

$h \in C(\Gamma_1)$, $g \in C(\Gamma_2)$, we denote by $\sigma_1^G(-\Delta + c; N+h, N+g)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta\phi + c(x)\phi = \lambda\phi & \text{in} \quad G, \\ \frac{\partial\phi}{\partial\nu} + h\phi = 0 & \text{on} \quad \Gamma_1, \\ \frac{\partial\phi}{\partial\nu} + g\phi = 0 & \text{on} \quad \Gamma_2, \end{cases}$$

and by $\sigma_1^G(-\Delta + c; N+h, D)$ that of the problem

$$\begin{cases} -\Delta\phi + c(x)\phi = \lambda\phi & \text{in} \quad G, \\ \frac{\partial\phi}{\partial\nu} + h\phi = 0 & \text{on} \quad \Gamma_1, \\ \phi = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

We will quote some important properties of $\sigma_1^G(-\Delta + c; N+h, N+g)$ and $\sigma_1^G(-\Delta + c; N+h, D)$. We denote the boundary operator

$$\mathcal{B}(\phi) = \begin{cases} \partial_\nu\phi + h\phi = 0 & \text{on } \Gamma_1, \\ \partial_\nu\phi + g\phi = 0 & \text{on } \Gamma_2, \end{cases} \quad \text{or} \quad \mathcal{B}(\phi) = \begin{cases} \partial_\nu\phi + h\phi = 0 & \text{on } \Gamma_1, \\ \phi = 0 & \text{on } \Gamma_2. \end{cases}$$

Proposition 2.1.

1. *The map $c \in L^\infty(G) \mapsto \sigma_1^G(-\Delta + c; \mathcal{B})$ is continuous and increasing.*
2. *It holds that*

$$\sigma_1^G(-\Delta + c; N+h, N+g) < \sigma_1^G(-\Delta + c; N+h, D).$$

3. *Assume that there exists a positive supersolution, that is, a positive function $\overline{u} \in W^{2,p}(G)$, $p > N$, such that*

$$-\Delta\overline{u} + c(x)\overline{u} \geq 0 \quad \text{in } G, \quad \mathcal{B}(\overline{u}) \geq 0 \quad \text{on } \partial G,$$

and some strict inequalities, then

$$\sigma_1^G(-\Delta + c; \mathcal{B}) > 0.$$

Now, we define

$$\mu(\lambda) := \sigma_1^G(-\Delta - \lambda c; \mathcal{B}), \quad \lambda \in \mathbf{R}.$$

The main properties of $\mu(\lambda)$ are stated in the next result.

Proposition 2.2.

1. *Assume that $c \not\equiv 0$ in G. Then, $\lambda \in \mathbf{R} \mapsto \mu(\lambda)$ is regular and concave.*

2. *Assume that* $c \gneq 0$ *in* G*, define*

$$C_0 := G \setminus \overline{\{x \in G : c(x) > 0\}}, \tag{8}$$

and assume that

$$\overline{C_0} \subseteq G. \tag{9}$$

The map $\lambda \mapsto \mu(\lambda)$ *is decreasing and*

$$\lim_{\lambda \to +\infty} \mu(\lambda) = -\infty, \qquad \lim_{\lambda \to -\infty} \mu(\lambda) = \sigma_1^{C_0}(-\Delta; D).$$

3. *Assume that* c *changes sign, then*

$$\lim_{\lambda \to \pm\infty} \mu(\lambda) = -\infty.$$

Moreover, there exists $\lambda_0 \in \mathbf{R}$ *such that* $\mu'(\lambda_0) = 0$*,* $\mu'(\lambda) > 0$ *for* $\lambda < \lambda_0$ *and* $\mu'(\lambda) < 0$ *for* $\lambda > \lambda_0$*.*

We can describe exactly the sign of the map $\mu(\lambda)$.

Corollary 2.3.

1. *Assume that* $c \gneq 0$ *and the set* C_0 *satisfies (9). Then, there exists a unique zero of the map* $\mu(\lambda)$*, we denote it by* $\lambda_1^+(G, c; \mathcal{B})$*, and as consequence,*

$$\mu(\lambda) \begin{cases} > 0 & \text{if } \lambda < \lambda_1^+(G, c; \mathcal{B}), \\ = 0 & \text{if } \lambda = \lambda_1^+(G, c; \mathcal{B}), \\ < 0 & \text{if } \lambda > \lambda_1^+(G, c; \mathcal{B}). \end{cases}$$

2. *Assume that* c *changes sign.*
   (a) *If* $\mu(\lambda_0) < 0$*, then* $\mu(\lambda) < 0$ *for all* $\lambda \in \mathbf{R}$*.*
   (b) *If* $\mu(\lambda_0) = 0$*, then* λ_0 *is the unique zero of the map* $\mu(\lambda)$*.*
   (c) *If* $\mu(\lambda_0) > 0$*, then there exist two zeros of the map* $\mu(\lambda)$*, we call them* $\lambda_1^-(G, c; \mathcal{B}) < \lambda_0 < \lambda_1^+(G, c; \mathcal{B})$*. As a consequence,*

$$\mu(\lambda) \begin{cases} > 0 & \text{if } \lambda \in (\lambda_1^-(G, c; \mathcal{B}), \lambda_1^+(G, c; \mathcal{B})), \\ = 0 & \text{if } \lambda = \lambda_1^-(G, c; \mathcal{B}) \text{ or } \lambda = \lambda_1^+(G, c; \mathcal{B}), \\ < 0 & \text{if } \lambda < \lambda_1^-(G, c; \mathcal{B}) \text{ or } \lambda > \lambda_1^+(G, c; \mathcal{B}). \end{cases}$$

2.2. *Interface eigenvalue problem*

First, we fix some notations that will be used throughout the paper. For convenience, we write $\mathbf{u} = (u_1, u_2)$ with u_i defined in Ω_i and similarly $\mathbf{c} = (c_1, c_2)$. In order to simplify the notation we write the boundary conditions as

$$\left.\begin{cases} \partial_\nu u_i = \gamma_i(u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 = 0 & \text{on } \Gamma, \end{cases}\right\} \Longleftrightarrow \mathcal{I}(\mathbf{u}) = 0 \quad \text{on } \Sigma \cup \Gamma.$$

We write

$$\mathcal{I}(\mathbf{u}) \succeq 0 \quad \text{on } (\Sigma, \Gamma) \Longleftrightarrow \begin{cases} \partial_\nu u_1 \geq \gamma_1(u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 \leq \gamma_2(u_2 - u_1) & \text{on } \Sigma, \\ \partial_\nu u_2 \geq 0 & \text{on } \Gamma. \end{cases}$$

We consider the Banach spaces

$$\begin{aligned} &L^p := \{\mathbf{u} : u_i \in L^p(\Omega_i)\}, \quad p \geq 1, \\ &H^1 := \left\{\mathbf{u} : u_i \in H^1(\Omega_i)\right\}, \\ &W^{2,p} := \left\{\mathbf{u} : u_i \in W^{2,p}(\Omega_i)\right\}, \quad p \geq 1. \end{aligned}$$

The norm of a function $\mathbf{u}$ is defined as the sum of the norms of u_i in the respective spaces.

On the other hand, given $\mathbf{u} = (u_1, u_2)$ we write $\mathbf{u} \geq 0$ in Ω if $u_i \geq 0$ in Ω_i for $i = 1, 2$ and $\mathbf{u} > 0$ in Ω if both $u_i > 0$ in Ω_i for $i = 1, 2$, and finally $\mathbf{u} \neq 0$ in Ω if $u_i \neq 0$ in a subset of positive measure of Ω_i for some $i = 1, 2$.

Given $c_i \in L^\infty(\Omega_i)$, we denote by $\Lambda_1(\mathbf{c}) = \Lambda_1(c_1, c_2)$ the principal eigenvalue of (see [9])

$$\begin{cases} -\Delta u_i + c_i(x) u_i = \lambda u_i & \text{in } \Omega_i, \\ \mathcal{I}(\mathbf{u}) = 0 & \text{on } \Sigma \cup \Gamma. \end{cases} \tag{10}$$

First, we recall some properties of $\Lambda_1(c_1, c_2)$, see [8].

Definition 2.4. Given $\overline{\mathbf{u}} \geq 0$ in Ω, $\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2)$, $\overline{\mathbf{u}} \in W^{2,p}$, $p > N$, is a strict supersolution of $(-\Delta + \mathbf{c}, \mathcal{I})$ if

$$-\Delta \overline{\mathbf{u}} + \mathbf{c}(x)\overline{\mathbf{u}} \geq 0 \quad \text{in } \Omega, \quad \mathcal{I}(\overline{\mathbf{u}}) \succeq 0 \quad \text{on } (\Sigma, \Gamma),$$

and some of these inequalities are strict.

Proposition 2.5.

1. *Assume that* $\mathbf{c} \leq \mathbf{d}$ *in* Ω. *Then,* $\Lambda_1(\mathbf{c}) \leq \Lambda_1(\mathbf{d})$. *Moreover, if* $\mathbf{c} \neq \mathbf{d}$ *in* Ω, $\Lambda_1(\mathbf{c}) < \Lambda_1(\mathbf{d})$.
2. *Assume that* $\mathbf{c}_n \to \mathbf{c}$ *in* L^∞, *then* $\Lambda_1(\mathbf{c}_n) \to \Lambda_1(\mathbf{c})$.
3. *It holds that*

$$\Lambda_1(\mathbf{c}) < \min\{\sigma_1^{\Omega_1}(-\Delta + c_1; N + \gamma_1), \sigma_1^{\Omega_2}(-\Delta + c_2; N + \gamma_2, N)\}.$$

4. *The map* $\mathbf{c} \in L^\infty \mapsto \Lambda_1(\mathbf{c})$ *is concave.*
5. $\Lambda_1(\mathbf{c}) > 0$ *if and only if there exists a strict positive supersolution* $\overline{\mathbf{u}}$ *of* $(-\Delta + \mathbf{c}, \mathcal{I})$.

3. Generalized interface principal eigenvalue: first properties

The main goal in this paper is to analyze the eigenvalue problem

$$\begin{cases} -\Delta u_i = \lambda_i m_i(x) u_i & \text{in } \Omega_i, \\ u_i > 0 & \text{in } \Omega_i, \\ \mathcal{I}(\mathbf{u}) = 0 & \text{on } \Sigma \cup \Gamma, \end{cases} \tag{11}$$

where $m_i \in L^\infty(\Omega_i)$, $m_i \not\equiv 0$ in Ω_i, $i = 1, 2$. It is obvious that

$$(\lambda_1, \lambda_2) \text{ is an eigenvalue of (11) if and only if } \Lambda_1(-\lambda_1 m_1, -\lambda_2 m_2) = 0.$$

Hence, we define $F : \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$F(\lambda_1, \lambda_2) := \Lambda_1(-\lambda_1 m_1, -\lambda_2 m_2).$$

The following result addresses the concavity of $\Lambda_1(c_1, c_2)$ in each component.

Proposition 3.1. *Fix $c_2 \in L^\infty(\Omega_2)$. Then, the map $c_1 \in L^\infty(\Omega_1) \mapsto \Lambda_1(c_1, c_2) \in \mathbb{R}$ is concave.*

Proof. Denote $G(c_1) := \Lambda_1(c_1, c_2)$, take $c_1^i \in L^\infty(\Omega_1)$, $i = 1, 2$ and $t \in [0, 1]$. Then,

$$\begin{aligned} G(tc_1^1 + (1-t)c_1^2) &= \Lambda_1(tc_1^1 + (1-t)c_1^2, c_2) = \Lambda_1(tc_1^1 + (1-t)c_1^2, tc_2 + (1-t)c_2) \\ &= \Lambda_1(t\mathbf{c} + (1-t)\mathbf{d}), \end{aligned}$$

where $\mathbf{c} = (c_1^1, c_2)$ and $\mathbf{d} = (c_1^2, c_2)$. Using now Proposition 2.5 4., we get that

$$\begin{aligned} G(tc_1^1 + (1-t)c_1^2) = &\ \Lambda_1(t\mathbf{c} + (1-t)\mathbf{d}) \\ \geq &\ t\Lambda_1(\mathbf{c}) + (1-t)\Lambda_1(\mathbf{d}) \\ = &\ t\Lambda_1(c_1^1, c_2) + (1-t)\Lambda_1(c_1^2, c_2) \\ = &\ tG(c_1^1) + (1-t)G(c_1^2). \end{aligned}$$

This completes the proof. □

As a consequence, we deduce the concavity of the map $F(\lambda_1, \lambda_2)$.

Corollary 3.2. *Fixed $\lambda_1 \in \mathbb{R}$, $\lambda_2 \mapsto F(\lambda_1, \lambda_2)$ is concave, and then, there exist at most two values of λ_2 such that $F(\lambda_1, \lambda_2) = 0$. A similar result holds when we fix λ_2.*

In order to simplify the notation, we denote (recall Corollary 2.3)

$$\Lambda_1^{\pm} := \lambda_1^{\pm}(\Omega_1, m_1; N + \gamma_1), \qquad \Lambda_2^{\pm} := \lambda_1^{\pm}(\Omega_2, m_2; N + \gamma_2, N). \tag{12}$$

Observe that if we denote by $\mu(\lambda) = \sigma_1^{\Omega_1}(-\Delta - \lambda m_1; N + \gamma_1)$, then $\mu(0) > 0$. Hence, if m_1 changes sign the existence of $\Lambda_1^- < 0 < \Lambda_1^+$ is guaranteed by Corollary 2.3. If $m_1 \gneq 0$ in Ω_1 then $\Lambda_1^- = -\infty$.

The first result provides upper bounds of $F(\lambda_1, \lambda_2)$.

Lemma 3.3. *It holds:*

$$F(\lambda_1,\lambda_2) < \min\{\sigma_1^{\Omega_1}(-\Delta-\lambda_1 m_1; N+\gamma_1), \sigma_1^{\Omega_2}(-\Delta-\lambda_2 m_2; N+\gamma_2, N)\}, \tag{13}$$

and

$$F(\lambda_1,\lambda_2) \le \frac{-\lambda_1 \int_{\Omega_1} m_1 - \lambda_2 \int_{\Omega_2} m_2 + (\gamma_1+\gamma_2)|\Sigma|}{|\Omega_1|+|\Omega_2|}. \tag{14}$$

Proof. (13) follows from Proposition 2.5 3.

Let $\varphi=(\varphi_1,\varphi_2)$ be a positive eigenfunction associated to $F(\lambda_1,\lambda_2)$. Observe that

$$\begin{aligned} &-\Delta\varphi_i - \lambda_i m_i(x)\varphi_i = F(\lambda_1,\lambda_2)\varphi_i \quad \text{in } \Omega_i, \\ &\mathcal{I}(\varphi)=0 \quad \text{on } \Sigma\cup\Gamma. \end{aligned}$$

Multiplying by $1/\varphi_i$, integrating and adding the two resulting equations, we obtain

$$\begin{aligned} F(\lambda_1,\lambda_2)(|\Omega_1|+|\Omega_2|) = -\lambda_1\int_{\Omega_1} m_1 - \lambda_2\int_{\Omega_2} m_2 - \left(\int_{\Omega_1}\frac{|\nabla\varphi_1|^2}{\varphi_1^2} + \int_{\Omega_2}\frac{|\nabla\varphi_2|^2}{\varphi_2^2}\right) \\ + \int_\Sigma (\varphi_2-\varphi_1)\left(\frac{\gamma_2}{\varphi_2}-\frac{\gamma_1}{\varphi_1}\right). \end{aligned}$$

Observe that

$$\int_\Sigma (\varphi_2-\varphi_1)\left(\frac{\gamma_2}{\varphi_2}-\frac{\gamma_1}{\varphi_1}\right) = (\gamma_1+\gamma_2)|\Sigma| - \int_\Sigma \frac{\gamma_1\varphi_2^2+\gamma_2\varphi_1^2}{\varphi_1\varphi_2},$$

whence we conclude (14). □

Corollary 3.4.

1. $F(\lambda_1,\lambda_2)<0$ *for* $\lambda_1\in(-\infty,\Lambda_1^-]\cup[\Lambda_1^+,+\infty)$ *or* $\lambda_2\in(-\infty,\Lambda_2^-]\cup[\Lambda_2^+,+\infty)$.
2. *Assume that* $F(\lambda_1,\lambda_2)=0$. *Then,*

$$\Lambda_1^- < \lambda_1 < \Lambda_1^+ \quad \textit{and} \quad \Lambda_2^- < \lambda_2 < \Lambda_2^+.$$

Proof. 1. Observe that if $\lambda_1\in(-\infty,\Lambda_1^-]\cup[\Lambda_1^+,+\infty)$, then, by Corollary 2.3, we get $\sigma_1^{\Omega_1}(-\Delta-\lambda_1 m_1; N+\gamma_1)\le 0$. Hence, by (13) we obtain that

$$F(\lambda_1,\lambda_2)<0.$$

2. Since $F(\lambda_1, \lambda_2) = 0$, by (13), we have that

$$0 < \min\{\sigma_1^{\Omega_1}(-\Delta - \lambda_1 m_1; N + \gamma_1), \sigma_1^{\Omega_2}(-\Delta - \lambda_2 m_2; N + \gamma_2, N)\},$$

and then $\sigma_1^{\Omega_1}(-\Delta - \lambda_1 m_1; N + \gamma_1) > 0$ and $\sigma_1^{\Omega_2}(-\Delta - \lambda_2 m_2; N + \gamma_2, N) > 0$, and hence the result concludes by Corollary 2.3. □

The following result will be very useful.

Proposition 3.5. *Assume that $m_i \gneq 0$ in Ω_i and that the set M_2^0 verifies (6). Take two sequences $\{a_n\}$ and $\{b_n\}$ such that*

$$a_n \to a_* \in (-\infty, \infty), \quad b_n \to -\infty \quad \text{as } n \to +\infty.$$

Then, at least for a subsequence,

$$\lim_{n\to\infty} F(a_n, b_n) = \min\{\sigma_1^{\Omega_1}(-\Delta - a_* m_1; N + \gamma_1), \sigma_1^{M_2^0}(-\Delta; D)\}.$$

Proof. Observe that

$$F(a_n, b_n) < \min\{\sigma_1^{\Omega_1}(-\Delta - a_n m_1; N + \gamma_1), \sigma_1^{\Omega_2}(-\Delta - b_n m_2; N + \gamma_2, N)\}.$$

By continuity,

$$\sigma_1^{\Omega_1}(-\Delta - a_n m_1; N + \gamma_1) \to \sigma_1^{\Omega_1}(-\Delta - a_* m_1; N + \gamma_1),$$

and using Proposition 2.2 2., we get

$$\sigma_1^{\Omega_2}(-\Delta - b_n m_2; N + \gamma_2, N) \to \sigma_1^{M_2^0}(-\Delta; D).$$

Hence, $F(a_n, b_n)$ is bounded.

Assume that

$$\sigma_0 := \sigma_1^{M_2^0}(-\Delta; D) < \sigma_1^{\Omega_1}(-\Delta - a_* m_1; N + \gamma_1). \tag{15}$$

Consequently, we conclude that, for a subsequence,

$$F(a_n, b_n) \to F_0 \leq \sigma_0 = \sigma_1^{M_2^0}(-\Delta; D) < \sigma_1^{\Omega_1}(-\Delta - a_* m_1; N + \gamma_1) < \infty \quad \text{as } n \to \infty.$$

Without loss of generality, we consider $\varphi_n = (\varphi_{1n}, \varphi_{2n})$ a positive eigenfunction associated to $F(a_n, b_n)$ such that $\|\varphi_n\|_2 = 1$. Then,

$$\begin{aligned}\int_\Omega |\nabla \varphi_n|^2 - a_n \int_{\Omega_1} m_1 \varphi_{1n}^2 - b_n \int_{\Omega_2} m_2 \varphi_{2n}^2 + \int_\Sigma (\gamma_1 \varphi_{1n}^2 + \gamma_2 \varphi_{2n}^2) - (\gamma_1 + \gamma_2) \int_\Sigma \varphi_{1n}\varphi_{2n} \\ = F(a_n, b_n) \leq C,\end{aligned}$$

where we have denoted

$$\int_{\Omega} |\nabla \varphi_n|^2 = \sum_{i=1}^{2} \int_{\Omega_i} |\nabla \varphi_{in}|^2.$$

Since $b_n < 0$, $\mathbf{m} \geq 0$ in Ω and $a_n \to a_* \in (-\infty, \infty)$, we get that

$$\int_{\Omega} |\nabla \varphi_n|^2 - (\gamma_1 + \gamma_2) \int_{\Sigma} \varphi_{1n}\varphi_{2n} \leq C. \tag{16}$$

Using now the inequalities

$$\int_{\Sigma} u_1 u_2 \leq \frac{1}{2} \left(\int_{\Sigma} u_1^2 + \int_{\Sigma} u_2^2 \right), \tag{17}$$

and that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\int_{\Sigma} v^2 \leq \varepsilon \int_{\Omega_i} |\nabla v|^2 + C(\varepsilon) \int_{\Omega_i} v^2 \quad \forall v \in H^1(\Omega_i), \tag{18}$$

(see for instance Lemma 1 in [1]) and $\|\varphi_n\|_2 = 1$, wet get that

$$\int_{\Sigma} \varphi_{1n}\varphi_{2n} \leq \frac{1}{2} \left(\int_{\Sigma} \varphi_{1n}^2 + \int_{\Sigma} \varphi_{2n}^2 \right) \leq \frac{1}{2} \left(\varepsilon \left(\int_{\Omega_1} |\nabla \varphi_{1n}^2| + \int_{\Omega_2} |\nabla \varphi_{2n}^2| \right) + C(\varepsilon) \right),$$

and then from (16) we get

$$\int_{\Omega} |\nabla \varphi_n|^2 \leq C.$$

Hence,

$$\|\varphi_n\|_{H^1} \leq C_0.$$

Thus,

$$\varphi_n \rightharpoonup \varphi_\infty = (\varphi_{1\infty}, \varphi_{2\infty}) \geq 0 \quad \text{in } H^1, \quad \varphi_n \to \varphi_\infty \quad \text{in } L^2 \text{ and } L^2(\Sigma) \text{ with } \|\varphi_\infty\|_2 = 1.$$

By definition of $F(a_n, b_n)$ we have that

$$\sum_{i=1}^{2}\left(\int_{\Omega_i}\nabla\varphi_{in}\cdot\nabla v_i - a_n\int_{\Omega_1} m_1\varphi_{1n}v_1 - b_n\int_{\Omega_2} m_2\varphi_{2n}v_2\right)+$$
$$+\int_{\Sigma}(\varphi_{2n}-\varphi_{1n})(\gamma_2 v_2-\gamma_1 v_1) = F(a_n,b_n)\left(\int_{\Omega_1}\varphi_{1n}v_1+\int_{\Omega_2}\varphi_{2n}v_2\right),\quad \forall v_i\in H^1(\Omega_i). \tag{19}$$

First, we prove that

$$\varphi_{2\infty}\in H_0^1(M_2^0). \tag{20}$$

Since

$$H_0^1(M_2^0)=\{u\in H^1(\Omega_2): u=0 \text{ in } \Omega_2\setminus M_2^0\},$$

we claim that $\varphi_{2\infty}=0$ in $\Omega_2\setminus M_2^0$.

By contradiction, assume that $\varphi_{2\infty}>0$ in D, for some $D\subset\Omega_2\setminus M_2^0$ and take $v_1=0$ in Ω_1 and $v_2\in C_0^\infty(D)$, $v_2>0$ in D. Then, by (19)

$$-\int_D \Delta v_2\varphi_{2n} - b_n\int_D m_2(x)\varphi_{2n}v_2 = F(a_n,b_n)\int_D \varphi_{2n}v_2. \tag{21}$$

If $\varphi_{2\infty}>0$ in D, then $-b_n\int_D m_2(x)\varphi_{2n}v_2\to\infty$ as $b_n\to-\infty$, a contradiction with (21). Hence, we conclude that $\varphi_{2\infty}=0$ in D. This implies (20).

Taking $v_1\in H^1(\Omega_1)$ and $v_2=0$ in (19), taking limit, we get

$$\int_{\Omega_1}\nabla\varphi_{1n}\cdot\nabla v_1 - a_n\int_{\Omega_1} m_1\varphi_{1n}v_1 + \int_{\Sigma}(\varphi_{2n}-\varphi_{1n})(-\gamma_1 v_1) = F(a_n,b_n)\int_{\Omega_1}\varphi_{1n}v_1,$$

then passing to the limit, taking into account (20) in the boundary integral,

$$\int_{\Omega_1}\nabla\varphi_{1\infty}\cdot\nabla v_1 - a_*\int_{\Omega_1} m_1\varphi_{1\infty}v_1 + \gamma_1\int_{\Sigma}\varphi_{1\infty}v_1 = F_0\int_{\Omega_1}\varphi_{1\infty}v_1.$$

Hence, if $\|\varphi_{1\infty}\|_2\neq 0$, then $F_0=\sigma_1^{\Omega_1}(-\Delta-a_*m_1;N+\gamma_1)$, an absurdum due to $F_0\le\sigma_0$ and (15). Then, $\|\varphi_{1\infty}\|_2=0$. Hence,

$$\|\varphi_{2\infty}\|_2=1.$$

Then, take $v_1=0$ and $v_2\in H_0^1(M_2^0)$ in (19), we obtain

$$\int_{M_0^2}\nabla\varphi_{2\infty}\cdot\nabla v_2 = F_0\int_{M_0^2}\varphi_{2\infty}v_2,$$

which yields that $F_0 = \sigma_1^{M_2^0}(-\Delta; D) = \sigma_0$.

A similar reasoning can be carried out when $\sigma_1^{\Omega_1}(-\Delta - a_* m_1; N + \gamma_1) < \sigma_1^{M_2^0}(-\Delta; D)$.

This finishes the proof. □

4. Proofs of the main results

The main idea of the proof can be summarized as follows. Instead of looking for solutions of $F(\lambda_1, \lambda_2) = 0$ in the general form (λ_1, λ_2), we look for solutions in the particular form $\lambda_2 = \mu\lambda_1$, for all $\mu \in \mathbb{R}$.

Hence, the following map plays an essential role in our study. Given $\mu \in \mathbb{R}$, we define

$$f_\mu(\lambda_1) := \Lambda_1(-\lambda_1 m_1, -\lambda_1 \mu m_2) = F(\lambda_1, \lambda_1 \mu).$$

In the following result we state that, for $\lambda_1 \neq 0$, it is equivalent to solve $F(\lambda_1, \lambda_2) = 0$ to $f_\mu(\lambda_1) = 0$. Specifically, we have:

Proposition 4.1. *Assume that $F(\lambda_1^0, \lambda_2^0) = 0$ and $\lambda_1^0 \neq 0$, then $f_{\mu_0}(\lambda_1^0) = 0$ for $\mu_0 = \lambda_2^0/\lambda_1^0$. Conversely, if $f_{\mu_0}(\lambda_1^0) = 0$ then $F(\lambda_1^0, \lambda_2^0) = 0$ for $\lambda_2^0 = \mu_0\lambda_1^0$.*

In what follows, we explore the particular case $\lambda_1 = 0$.

Proposition 4.2. *Assume that $\lambda_1 = 0$ and denote*

$$g(\lambda_2) := F(0, \lambda_2) = \Lambda_1(0, -\lambda_2 m_2).$$

The map $\lambda_2 \mapsto g(\lambda_2)$ is regular, concave, $g(0) = 0$ and

$$sign(g'(0)) = sign\left(-\int_{\Omega_2} m_2\right). \tag{22}$$

Moreover,

1. *If $m_2 \gneq 0$ in Ω_2 and M_2^0 verifies (6), then $\lambda_2 \mapsto g(\lambda_2)$ is decreasing and*
$$\lim_{\lambda_2 \to +\infty} g(\lambda_2) = -\infty \quad \text{and} \quad \lim_{\lambda_2 \to -\infty} g(\lambda_2) = \min\{\sigma_1^{\Omega_1}(-\Delta; N + \gamma_1), \sigma_1^{M_2^0}(-\Delta; D)\}.$$
In this case, $g(\lambda_2) > 0$ for $\lambda_2 < 0$ and $g(\lambda_2) < 0$ if $\lambda_2 > 0$.
2. *If m_2 changes sign in Ω_2, then*
$$\lim_{\lambda_2 \to \pm\infty} g(\lambda_2) = -\infty.$$
Moreover,
(a) *If $\int_{\Omega_2} m_2 = 0$, then $g'(0) = 0$ and $\lambda_2 = 0$ is the unique root of $g(\lambda_2) = 0$. As a consequence, $g(\lambda_2) < 0$ for $\lambda_2 \neq 0$.*

(b) *If* $\int_{\Omega_2} m_2 < 0$*, then* $g'(0) > 0$ *and there exists* $\lambda_2^+ > 0$ *such that* $g(\lambda_2^+) = 0$*. In this case,*

$$g(\lambda_2) \begin{cases} > 0 & \text{if } \lambda_2 \in (0, \lambda_2^+), \\ < 0 & \text{if } \lambda_2 < 0 \text{ or } \lambda > \lambda_2^+. \end{cases}$$

(c) *If* $\int_{\Omega_2} m_2 > 0$*, then* $g'(0) < 0$ *and there exists* $\lambda_2^- < 0$ *such that* $g(\lambda_2^-) = 0$*. Hence,*

$$g(\lambda_2) \begin{cases} > 0 & \text{if } \lambda_2 \in (\lambda_2^-, 0), \\ < 0 & \text{if } \lambda_2 < \lambda_2^- \text{ or } \lambda_2 > 0. \end{cases}$$

Proof. To begin with, the regularity of g follows by the regularity of the function F. On the other hand, by Proposition 3.1 follows that $g(\lambda)$ is concave. It is obvious that $g(0) = F(0,0) = \Lambda_1(0,0) = 0$. On the other hand, taking $\mathbf{m} = (0, m_2)$ in Proposition 3.17 in [8], we conclude (22).

Finally, observe that by (13) we have

$$g(\lambda_2) < \sigma_1^{\Omega_2}(-\Delta - \lambda_2 m_2; N + \gamma_2, N), \tag{23}$$

whence we deduce that $\lim_{\lambda_2 \to +\infty} g(\lambda_2) = -\infty$ from Proposition 2.2 2. and 3.

1. Assume that $m_2 \gneq 0$ in Ω_2. In this case, g is decreasing. Moreover, by Proposition 3.5, taking $a_n = 0$, we conclude that

$$\lim_{\lambda_2 \to -\infty} g(\lambda_2) = \min\{\sigma_1^{\Omega_1}(-\Delta; N + \gamma_1), \sigma_1^{M_2^0}(-\Delta; D)\}.$$

2. Assume that m_2 changes sign. Then, using (23) and Proposition 2.2 3., we deduce that

$$\lim_{\lambda_2 \to -\infty} g(\lambda_2) = -\infty.$$

Now, from the sign of $g'(0)$ in (22), we conclude the result. □

In the next result, we study in detail the map $\lambda_1 \mapsto f_\mu(\lambda_1)$.

Proposition 4.3. *Fix* $\mu \in \mathbb{R}$*. Then,* $\lambda_1 \mapsto f_\mu(\lambda_1)$ *is regular, concave,* $f_\mu(0) = 0$ *and*

$$sign(f_\mu'(0)) = -sign\left(\gamma_2 \int_{\Omega_1} m_1 + \mu\gamma_1 \int_{\Omega_2} m_2\right). \tag{24}$$

1. *If* $m_1 \gneq 0$ *in* Ω_1*, then*

$$\lim_{\lambda_1 \to +\infty} f_\mu(\lambda_1) = -\infty$$

2. *If m_1 or m_2 changes sign, then*

$$\lim_{\lambda_1\to\pm\infty} f_\mu(\lambda_1) = -\infty.$$

Proof. It is clear that $f_\mu(0) = 0$. The regularity follows by the regularity of F, the concavity of $f_\mu(\lambda_1)$ follows by Proposition 2.5 4., and (24) follows taking $\mathbf{m} = (m_1, \mu m_2)$ in Proposition 3.17 in [8].

On the other hand, by (13) we get

$$f_\mu(\lambda_1) < \min\{\sigma_1^{\Omega_1}(-\Delta - \lambda_1 m_1; N + \gamma_1), \sigma_1^{\Omega_2}(-\Delta - \lambda_1 \mu m_2; N + \gamma_2, N)\},$$

and then $\lim_{\lambda_1\to+\infty} f_\mu(\lambda_1) = -\infty$, and if m_1 or m_2 changes sign, $\lim_{\lambda_1\to-\infty} f_\mu(\lambda_1) = -\infty$. □

For $\int_{\Omega_2} m_2 \neq 0$, we define

$$\mu^* := -\frac{\gamma_2 \displaystyle\int_{\Omega_1} m_1}{\gamma_1 \displaystyle\int_{\Omega_2} m_2}, \tag{25}$$

in such a way that $f'_{\mu^*}(0) = 0$.

4.1. *Case $m_i \gneq 0$ in Ω_i, $i = 1, 2$*

Observe that in this case

$$\mu^* < 0.$$

Proposition 4.4. *Assume that $m_i \gneq 0$ in Ω_i and M_i^0 verify (6) for $i = 1, 2$.*

1. *If $\mu \geq 0$, the unique zero of $f_\mu(\lambda_1)$ is $\lambda_1 = 0$.*
2. *If $\mu < 0$, $\mu \neq \mu^*$, there exists an unique $\lambda_1 = h_1(\mu) \neq 0$ such that $f_\mu(\lambda_1) = 0$. Moreover,*

$$h_1(\mu) \begin{cases} < 0 & \text{if } \mu > \mu^*, \\ = 0 & \text{if } \mu = \mu^*, \\ > 0 & \text{if } \mu < \mu^*. \end{cases}$$

3. *The map $\mu \in (-\infty, 0) \mapsto h_1(\mu)$ is continuous and decreasing. Moreover,*

$$\lim_{\mu\uparrow 0} h_1(\mu) = -\infty, \qquad \lim_{\mu\to-\infty} h_1(\mu) = \Lambda_1^+.$$

Proof. 1. Assume that $\mu \geq 0$. Then, since $\lambda_1 \mapsto f_\mu(\lambda_1)$ is decreasing and $f_\mu(0) = 0$, the result follows.

2. Assume that $\mu < 0$. Recall that $\lambda_1 \mapsto f_\mu(\lambda_1)$ is concave and $f_\mu(0) = 0$. If $\mu > \mu^*$ then $f'_\mu(0) < 0$, and hence there exists a unique $h_1(\mu) < 0$ such that $f_\mu(h_1(\mu)) = 0$. Similarly, when $\mu < \mu^*$ there exists a unique $h_1(\mu) > 0$ such that $f_\mu(h_1(\mu)) = 0$.
3. We will show that $\mu \mapsto h_1(\mu)$ is decreasing. Take now $\mu_1 < \mu_2 < 0$. Observe that $-\mu_1\lambda_1 > -\mu_2\lambda_1$ if $\lambda_1 > 0$ and $-\mu_1\lambda_1 < -\mu_2\lambda_1$ if $\lambda_1 < 0$. Hence, we distinguish several cases:
   (a) Assume that $\mu^* \le \mu_1 < \mu_2 < 0$. In this case, $h_1(\mu_2)$ and $h_1(\mu_1)$ are negative, and then we compare the functions f_{μ_2} and f_{μ_1} for negative values. Indeed, observe that $f_{\mu_2}(\lambda_1) > f_{\mu_1}(\lambda_1)$ for $\lambda_1 < 0$, and then $h_1(\mu_2) < h_1(\mu_1)$.
   (b) Assume that $\mu_1 < \mu^* < \mu_2 < 0$: in this case $h_1(\mu_2) < 0 < h_1(\mu_1)$.
   (c) Assume that $\mu_1 < \mu_2 \le \mu^* < 0$, then $f_{\mu_2}(\lambda_1) < f_{\mu_1}(\lambda_1)$ for $\lambda_1 > 0$, and then $h_1(\mu_2) < h_1(\mu_1)$.

   This shows that $\mu \mapsto h_1(\mu)$ is decreasing.
   We prove now the continuity. Take $\mu_n \in (-\infty, 0) \to \mu_0 < 0$ and consider $\lambda_n := h_1(\mu_n)$. Since $0 = f_{\mu_n}(\lambda_n) = F(\lambda_n, \mu_n\lambda_n)$, by Corollary 3.4 we conclude that

$$\lambda_n < \Lambda_1^+ \quad \text{and} \quad \mu_n\lambda_n < \Lambda_2^+. \tag{26}$$

   Hence, there exists $\overline{\lambda}_1 \in (-\infty, +\infty)$ such that $\lambda_n \to \overline{\lambda}_1$. We have to show that

$$\overline{\lambda}_1 = h_1(\mu_0).$$

   Indeed, observe that

$$0 = f_{\mu_n}(\lambda_n) = \Lambda_1(-\lambda_n m_1, -\lambda_n\mu_n m_2) \to \Lambda_1(-\overline{\lambda}_1 m_1, -\overline{\lambda}_1\mu_0 m_2) = f_{\mu_0}(\overline{\lambda}_1),$$

   that is, $f_{\mu_0}(\overline{\lambda}_1) = 0$. We separate now two cases:
   (a) $\mu_0 \ne \mu^*$: In this case, we assert, that $\overline{\lambda}_1 \ne 0$. Indeed, assume that $\overline{\lambda}_1 = 0$, that is, $h_1(\mu_n) \to 0$. If, for instance, $\mu_0 > \mu^*$, then there exists $\rho_1(\mu_n) \in (h_1(\mu_n), 0)$ such that $f'_{\mu_n}(\rho_1(\mu_n)) = 0$. Since $h_1(\mu_n) \to 0$, then $\rho_1(\mu_n) \to 0$, and as consequence, $f'_{\mu_0}(0) = 0$, a contradiction. This shows that $\overline{\lambda}_1 \ne 0$. Then, since $h_1(\mu_0)$ is the unique nonzero root of $f_{\mu_0}(\lambda_1) = 0$, we have that $\overline{\lambda}_1 = h_1(\mu_0)$.
   (b) $\mu_0 = \mu^*$: in this case $f_{\mu^*}(\overline{\lambda}_1) = 0$ implies that $\overline{\lambda}_1 = 0 = h_1(\mu^*)$.

   This concludes that $\overline{\lambda}_1 = h_1(\mu_0)$, and hence the continuity.
   We claim that

$$h_1(\mu_n) \to -\infty \quad \text{as } \mu_n \to 0. \tag{27}$$

   Assume that $|h_1(\mu_n)| \le C$. Then, we can assume that, at least for a subsequence, $h_1(\mu_n) \to h_1^* < 0$ and hence

$$0 = \Lambda_1(-h_1(\mu_n)m_1, -\mu_n h_1(\mu_n)m_2) \to \Lambda_1(-h_1^* m_1, 0) = 0,$$

   a contradiction because $\Lambda_1(-h_1^* m_1, 0) > 0$. This proves (27).
   By (26), if $\mu \to -\infty$ we can assume that $h_1(\mu) \to h^* \le \Lambda_1^+$ and $h^* > 0$. Then, $\mu h_1(\mu) \to -\infty$. Since

$$0 = f_\mu(h_1(\mu)) = F(h_1(\mu), h_1(\mu)\mu)$$

and by Proposition 3.5

$$0 = F(h_1(\mu), h_1(\mu)\mu) \to \min\{\sigma_1^{\Omega_1}(-\Delta - \lambda^* m_1; N + \gamma_1), \sigma_1^{M_2^0}(-\Delta; D)\},$$

it follows that

$$0 = \min\{\sigma_1^{\Omega_1}(-\Delta - \lambda^* m_1, N + \gamma_1), \sigma_1^{M_2^0}(-\Delta, D)\}.$$

Since $\sigma_1^{M_2^0}(-\Delta, D) > 0$, we conclude that $\sigma_1^{\Omega_1}(-\Delta - h^* m_1, N + \gamma_1) = 0$, that means that $h^* = \Lambda_1^+$, that is

$$\lim_{\mu \to -\infty} h_1(\mu) = \Lambda_1^+.$$

This concludes the proof. □

Once we have studied the map $\mu \mapsto h_1(\mu)$, we need to analyze the map

$$\mu \in (-\infty, 0) \mapsto h_2(\mu) := \mu h_1(\mu).$$

Proposition 4.5. *Assume that $m_i \gneq 0$ in Ω_i and M_i^0 verify (6) for $i = 1, 2$. The map $\mu \in (-\infty, 0) \mapsto h_2(\mu) := \mu h_1(\mu)$ is continuous, increasing,*

$$h_2(\mu) \begin{cases} > 0 & \text{if } \mu > \mu^*, \\ = 0 & \text{if } \mu = \mu^*, \\ < 0 & \text{if } \mu < \mu^*, \end{cases}$$

$$\lim_{\mu \to -\infty} h_2(\mu) = -\infty, \tag{28}$$

and

$$\lim_{\mu \to 0} h_2(\mu) = \Lambda_2^+. \tag{29}$$

Proof. To start with, the continuity and the sign of the map $h_2(\mu)$ follow directly from Proposition 4.4. Moreover, it is clear that

$$\lim_{\mu \to -\infty} h_2(\mu) = \lim_{\mu \to -\infty} \mu h_1(\mu) = -\infty.$$

In order to prove (29) we can argue exactly as the proof of Proposition 4.4.

Finally, using that F is increasing, we prove that the map $\mu \mapsto h_2(\mu)$ is increasing. Take $\mu_1 < \mu_2$ and assume that $h_2(\mu_1) \geq h_2(\mu_2)$. Since $h_1(\mu_1) > h_1(\mu_2)$, then

$$0 = F(h_1(\mu_1), h_2(\mu_1)) < F(h_1(\mu_2), h_2(\mu_1)) \leq F(h_1(\mu_2), h_2(\mu_2)) = 0,$$

a contradiction. □

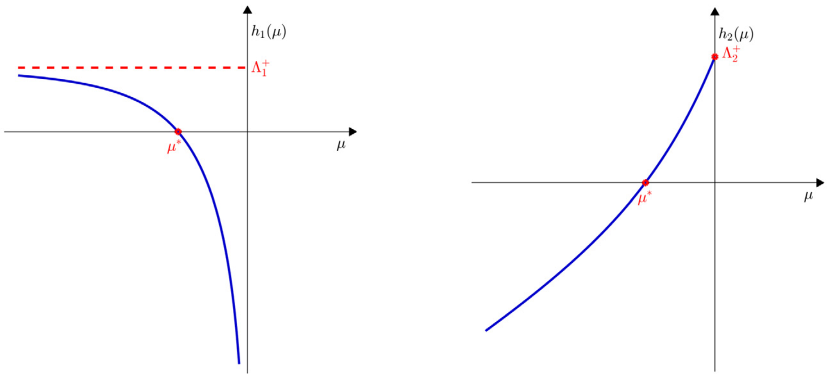

Fig. 6. Case $m_i \gneq 0$ in Ω_i and verifying (6) for $i = 1, 2$. We have represented the functions $\mu \mapsto h_1(\mu)$ (left) and $\mu \mapsto h_2(\mu)$ (right).

In Fig. 6 we have represented the functions $\mu \mapsto h_1(\mu), h_2(\mu)$. Now, we proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1: (see Fig. 2)

1. Observe that by Corollary 3.4, we obtain

$$F(\lambda_1, \lambda_2) < 0 \quad \text{if } \lambda_1 \geq \Lambda_1^+ \text{ or } \lambda_2 \geq \Lambda_2^+.$$

2. Take $\lambda_1 < \Lambda_1^+$. Then, by Proposition 4.4 there exists a unique $\mu = \mu(\lambda_1) < 0$ such that $\lambda_1 = h_1(\mu)$. Take $\lambda_2 = h_2(\mu) = \mu h_1(\mu)$, then

$$F(\lambda_1, \lambda_2) = F(h_1(\mu), h_2(\mu)) = F(h_1(\mu), \mu h_1(\mu)) = f_\mu(h_1(\mu)) = 0.$$

We define the function

$$\mathcal{H}(\lambda_1) := h_2(h_1^{-1}(\lambda_1)), \qquad \lambda_1 < \Lambda_1^+.$$

It is clear that $\mathcal{H}$ is well-defined (observe that h_1^{-1} exists due to that h_1 is a decreasing function), is continuous and

$$F(\lambda_1, \mathcal{H}(\lambda_1)) = 0.$$

Moreover, since fixed λ_1, the map $\lambda_2 \mapsto F(\lambda_1, \lambda_2)$ is concave, it follows that $F(\lambda_1, \lambda_2) > 0$ for $\lambda_2 < \mathcal{H}(\lambda_1)$ and $F(\lambda_1, \lambda_2) < 0$ for $\lambda_2 > \mathcal{H}(\lambda_1)$.
Furthermore, by Propositions 4.4 and 4.5,

$$\lim_{\lambda_1 \uparrow \Lambda_1^+} \mathcal{H}(\lambda_1) = \lim_{\lambda_1 \uparrow \Lambda_1^+} h_2(h_1^{-1}(\lambda_1)) = \lim_{\mu \to -\infty} h_2(\mu) = -\infty,$$

and,

$$\lim_{\lambda_1 \uparrow -\infty} \mathcal{H}(\lambda_1) = \lim_{\lambda_1 \uparrow -\infty} h_2(h_1^{-1}(\lambda_1)) = \lim_{\mu \to 0} h_2(\mu) = \Lambda_2^+.$$

Finally, we prove that $\lambda_1 \mapsto \mathcal{H}(\lambda_1)$ is decreasing. Take $\lambda_1 < \overline{\lambda}_1 < \Lambda_1^+$ and consider $\mathcal{H}(\lambda_1)$ and $\mathcal{H}(\overline{\lambda}_1)$. We are going to show that $\mathcal{H}(\lambda_1) > \mathcal{H}(\overline{\lambda}_1)$. Assume that $\mathcal{H}(\lambda_1) \le \mathcal{H}(\overline{\lambda}_1)$, then

$$0 = F(\lambda_1, \mathcal{H}(\lambda_1)) > F(\overline{\lambda}_1, \mathcal{H}(\lambda_1)) \ge F(\overline{\lambda}_1, \mathcal{H}(\overline{\lambda}_1)) = 0,$$

a contradiction.

This concludes the proof. □

4.2. Case $m_1 \gneq 0$ in Ω_1 and m_2 changes sign in Ω_2

In this case, the results depend on the sign of $\int_{\Omega_2} m_2$. We detail the case

$$\int_{\Omega_2} m_2 < 0,$$

similarly the other cases can be studied (see Remark 4.8). Observe that in this case

$$\mu^* = -\frac{\gamma_2 \int_{\Omega_1} m_1}{\gamma_1 \int_{\Omega_2} m_2} > 0.$$

Proposition 4.6. *Assume that $m_1 \gneq 0$ in Ω_1 and M_1^0 verifies (6), m_2 changes sign in Ω_2 and $\int_{\Omega_2} m_2 < 0$. Then, for each $\mu \neq 0$ there exists a unique $h_1(\mu) \in \mathbf{R}$ such that $f_\mu(h_1(\mu)) = 0$. Moreover,*

$$h_1(\mu) \begin{cases} > 0 & \text{if } \mu > \mu^* , \\ = 0 & \text{if } \mu = \mu^*, \\ < 0 & \text{if } \mu < \mu^*. \end{cases}$$

Furthermore, the map $\mu \in \mathbf{R} \setminus \{0\} \mapsto h_1(\mu) \in \mathbf{R}$ is continuous, and

$$\lim_{\mu \to \pm\infty} h_1(\mu) = 0, \qquad \lim_{\mu \to 0} h_1(\mu) = -\infty.$$

As consequence, there exists $\mu_{\max} > \mu^*$ such that

$$\max_{\mu \neq 0} h_1(\mu) = h_1(\mu_{\max}) := \lambda_1^{max}.$$

Finally, the map $\mu \in \mathbf{R} \setminus \{0\} \mapsto h_1(\mu) \in \mathbf{R}$ is increasing in $(0, \mu_{\max})$ and decreasing in $(-\infty, 0)$ and $(\mu_{\max}, \infty)$.

Proof. The proof of this result is rather similar to the proof of Proposition 4.4, hence we sketch the proof.

Since $f_{\mu_n}(\lambda_1(\mu_n)) = F(h_1(\mu_n), \mu_n h_1(\mu_n)) = 0$, by Corollary 3.4 we get

$$\Lambda_2^- < h_1(\mu_n)\mu_n < \Lambda_2^+,$$

whence we conclude that $h_1(\mu_n) \to 0$ as $\mu_n \to \pm\infty$.

Before proving the monotony, we claim that for any $c \in \mathbb{R}$ there exist at most two values of μ such that

$$h_1(\mu) = c.$$

We argue by contradiction. Assume that for $\mu_1 < \mu_2 < \mu_3$ we get $h_1(\mu_i) = c$ for $i = 1, 2, 3$. Taking $\lambda_2^i = c\mu_i$ we obtain

$$0 = F(c, \lambda_2^i), \quad \lambda_2^1 < \lambda_2^2 < \lambda_2^3,$$

a contradiction because, fixed c, the map $\lambda_2 \mapsto F(c, \lambda_2)$ is concave.

Now, for instance, we show that $h_1(\mu)$ is decreasing in $(-\infty, 0)$. Take $\mu_1 < \mu_2 < 0$ and assume that $h_1(\mu_1) \le h_1(\mu_2)$. Since $h_1(\mu) \to 0$ as $\mu \to -\infty$ and $h_1(\mu) \to -\infty$ as $\mu \to 0$. Then, there exists $c < 0$ such that $h_1(\mu) = c$ possesses at least three solutions. This is a contradiction and proves that $h_1(\mu)$ is decreasing in $(-\infty, 0)$. With a similar argument, it can be proved that $h_1(\mu)$ is decreasing in $(\mu_{\max}, \infty)$ and increasing in $(0, \mu_{\max})$. □

Again, we can deduce properties of the map $h_2(\mu) = \mu h_1(\mu)$.

Proposition 4.7. *Assume that $m_1 \gneq 0$ in Ω_1 and M_1^0 verifies (6), m_2 changes sign in Ω_2 and $\int_{\Omega_2} m_2 < 0$. Then $h_2(\mu) = \mu h_1(\mu)$ is continuous in $\mu \neq 0$, increasing and verifies*

$$h_2(\mu) \begin{cases} > 0 & \text{if } \mu > \mu^* \text{ or } \mu < 0, \\ = 0 & \text{if } \mu = \mu^*, \\ < 0 & \text{if } \mu \in (0, \mu^*), \end{cases}$$

and

$$\lim_{\mu \to 0^\pm} h_2(\mu) = \Lambda_2^\mp, \qquad \lim_{\mu \to \pm\infty} h_2(\mu) = \lambda_2^*,$$

for some $\lambda_2^* \in (0, \Lambda_2^+)$.

Proof. Assume that $\mu_n \to 0^+$, then since $h_2(\mu_n)$ is bounded, at least for a subsequence, $h_2(\mu_n) \to \overline{\lambda_2} < 0$. Observe that since $h_1(\mu_n) \to -\infty$, by Proposition 3.5

$$0 = F(h_1(\mu_n), h_2(\mu_n)) \to \sigma_1^{\Omega_2}(-\Delta - \overline{\lambda_2} m_2; N + \gamma_2, N) = 0,$$

and then $\overline{\lambda_2} = \Lambda_2^-$. Analogously for $\mu_n \to 0^-$.

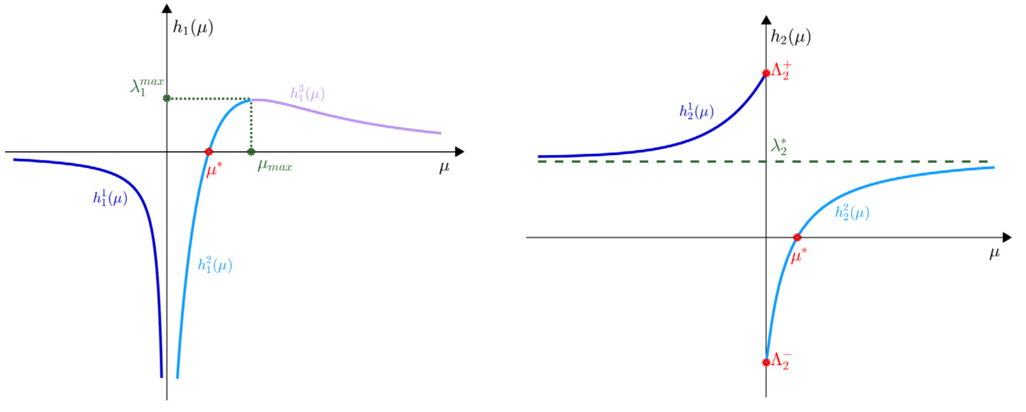

Fig. 7. Representation of the maps $h_1(\mu)$ (left) and $h_2(\mu)$ (right) in the case m_1 non-negative, non-trivial and verifies (6), m_2 changing sign and $\int_{\Omega_2} m_2 < 0$.

On the other hand, assume that $\mu_n \to +\infty$ and $h_2(\mu_n) \to \overline{\lambda_2} < 0$. In this case, $h_1(\mu_n) \to 0$, and then

$$0 = F(-h_1(\mu_n)m_1, -h_2(\mu_n)m_2) \to F(0, -\overline{\lambda_2}m_2),$$

whence $\overline{\lambda_2} = \lambda_2^*$.

Observe that

$$0 = F(0, -\lambda_2^* m_2) < \sigma_1^{\Omega_2}(-\Delta - \lambda_2^* m_2; N + \gamma_2, N)$$

and so $\lambda_2^* < \Lambda_2^+$.

Finally, with an argument similar to the one used in Proposition 4.6 we can conclude that the equation $h_2(\mu) = c$ possesses at most a unique solution. Hence, the monotony of $h_2(\mu)$ follows. This completes the proof. □

In Fig. 7, one may see a representation of the maps $\mu \mapsto h_1(\mu), h_2(\mu)$.

Remark 4.8.

1. In the case

$$\int_{\Omega_2} m_2 > 0$$

we can obtain a similar result switching μ by $-\mu$ (see Fig. 8).

2. When

$$\int_{\Omega_2} m_2 = 0$$

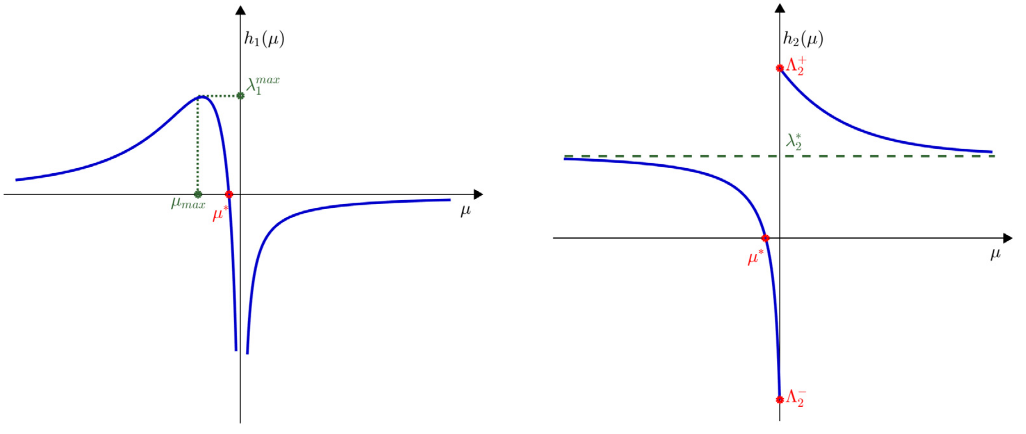

Fig. 8. Representations of the maps $h_1(\mu)$ (left) and $h_2(\mu)$ (right) in the case m_1 non-negative, non-trivial and verifies (6), m_2 changing sign and $\int_{\Omega_2} m_2 > 0$.

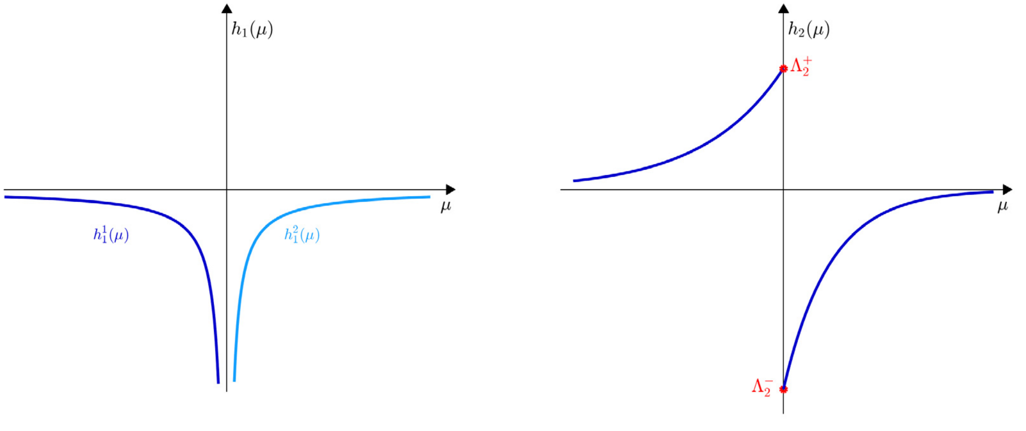

Fig. 9. We have represented the maps $h_1(\mu)$ (left) and $h_2(\mu)$ (right) in the case m_1 non-negative, non-trivial and verifies (6), m_2 changing sign and $\int_{\Omega_2} m_2 = 0$.

observe that $f'_\mu(0) < 0$ for all μ (see (24)), and then $h_1(\mu) < 0$ for all μ. The global behavior of $h_1(\mu)$ at $\mu = 0$ and $\mu \to \pm\infty$ is similar to Proposition 4.6 (see Fig. 9).

Proof of Theorem 1.2: (See Fig. 3.) Assume that $\int_{\Omega_2} m_2 < 0$ (see Fig. 7). We introduce the following notation:

$$h_1(\mu) := \begin{cases} h_1^1(\mu) & \text{if } \mu < 0, \\ h_1^2(\mu) & \text{if } \mu \in (0, \mu_{\max}], \\ h_1^3(\mu) & \text{if } \mu > \mu_{\max}. \end{cases}$$

1. If $\lambda_1 > \lambda_1^{\max}$, then there does not exist $\mu \in \mathbb{R}$ such that $\lambda_1 = h_1(\mu)$. Hence, $F(\lambda_1, \lambda_2) \neq 0$ for all $\lambda_2 \in \mathbb{R}$, in fact, $F(\lambda_1, \lambda_2) < 0$ for all $\lambda_2 \in \mathbb{R}$. Indeed, if for some $\overline{\lambda_2}$ we have

$F(\lambda_1, \overline{\lambda_2}) > 0$, then there exists at least λ_2^0 such that $F(\lambda_1, \lambda_2^0) = 0$. Then, for some μ_0 we have $\lambda_1 = h_1(\mu_0)$, a contradiction.

2. If $\lambda_1 = \lambda_1^{\max}$, there exists a unique $\mu_{max} > \mu^*$ such that $\lambda_1^{\max} = h_1(\mu_{\max})$, and then $\lambda_2^{\max} = h_2(\mu_{\max}) > 0$ and $F(\lambda_1^{\max}, \lambda_2^{\max}) = 0$.
3. We fix $\lambda_1 \in (0, \lambda_1^{\max})$. Then (see Fig. 7), there exist μ_2, μ_3 with $\mu^* < \mu_2 < \mu_{max} < \mu_3$ such that $h_1^i(\mu_i) = \lambda_1$ $i = 2, 3$. To these values correspond two different values of $h_2(\mu_i)$. Moreover, as $\lambda_1 \to 0$, then $\mu_2 \to \mu^*$ and $\mu_3 \to +\infty$, and this case $h_2(\mu_2) = h_2^2(\mu_2) \to h_2^2(\mu^*) = 0$ and $h_2(\mu_3) = h_2^2(\mu_3) \to \lambda_2^*$.
4. On the other hand, when $\lambda_1 \in (-\infty, 0)$. There exist $\mu_1 < 0 < \mu_2 < \mu^*$ such that $\lambda_1 = h_1^i(\mu_i)$ $i = 1, 2$, with $\mu_1 \to -\infty$ and $\mu_2 \to \mu^*$ as $\lambda_1 \to 0$. Then, $h_2(\mu_1) = h_2^1(\mu_1) \to \lambda_2^*$ and $h_2(\mu_2) = h_2^2(\mu_2) \to 0$.
   Observe that when $\lambda_1 \to -\infty$ then $\mu_1 \to 0^-$ and $\mu_2 \to 0^+$, and hence $h_2(\mu_1) \to \Lambda_2^+$ and $h_2(\mu_2) \to \Lambda_2^-$.

With this construction, we can define

$$\mathcal{H}^+(\lambda_1) := \begin{cases} h_2((h_1^3)^{-1}(\lambda_1)) & \text{if } \lambda_1 \in (0, \lambda_1^{\max}], \\ h_2((h_1^1)^{-1}(\lambda_1)) & \text{if } \lambda_1 \leq 0, \end{cases}$$

and

$$\mathcal{H}^-(\lambda_1) := h_2((h_1^2)^{-1}(\lambda_1)) \quad \text{if } \lambda_1 \leq \lambda_1^{\max}.$$

Observe that thanks to the monotony of the maps h_1^i for $i = 1, 2, 3$, $\mathcal{H}^+$ and $\mathcal{H}^-$ are well defined. Moreover,

$$\lim_{\lambda_1 \to 0^+} \mathcal{H}^+(\lambda_1) = \lim_{\lambda_1 \to 0^+} h_2((h_1^3)^{-1}(\lambda_1)) = \lim_{\mu \to +\infty} h_2(\mu) = \lambda_2^*,$$

and

$$\lim_{\lambda_1 \to 0^-} \mathcal{H}^+(\lambda_1) = \lim_{\lambda_1 \to 0^-} h_2((h_1^1)^{-1}(\lambda_1)) = \lim_{\mu \to -\infty} h_2(\mu) = \lambda_2^*.$$

As a consequence, $\mathcal{H}^+$ is continuous.

On the other hand,

$$\lim_{\lambda_1 \to \lambda_1^{\max}} \mathcal{H}^+(\lambda_1) = \lim_{\mu \to \mu_{\max}} h_2(\mu) = \overline{\lambda}_2,$$

$$\lim_{\lambda_1 \to \lambda_1^{\max}} \mathcal{H}^-(\lambda_1) = \lim_{\mu \to \mu_{\max}} h_2(\mu) = \overline{\lambda}_2.$$

Finally,

$$\lim_{\lambda_1 \to -\infty} \mathcal{H}^+(\lambda_1) = \lim_{\mu \to 0^+} h_2(\mu) = \Lambda_2^+,$$

and

$$\lim_{\lambda_1 \to -\infty} \mathcal{H}^-(\lambda_1) = \lim_{\mu \to 0^-} h_2(\mu) = \Lambda_2^-.$$

We show that $\mathcal{H}^+(\lambda_1)$ is decreasing. Take $\lambda_1^1 < \lambda_1^2$.

1. When $\lambda_1^1 < \lambda_1^2 < 0$: then $(h_1^1)^{-1}(\lambda_1^2) < (h_1^1)^{-1}(\lambda_1^1) < 0$ and so $h_2((h_1^1)^{-1}(\lambda_1^2)) < h_2((h_1^1)^{-1}(\lambda_1^1))$. This concludes that

$$\mathcal{H}^+(\lambda_1^1) > \mathcal{H}^+(\lambda_1^2).$$

2. Assume now that $\lambda_1^1 < 0 < \lambda_1^2$: in this case $(h_1^1)^{-1}(\lambda_1^1) < 0 < (h_1^3)^{-1}(\lambda_1^2)$ and then $h_2((h_1^1)^{-1}(\lambda_1^1)) > 0 > h_2((h_1^3)^{-1}(\lambda_1^2))$, that is $\mathcal{H}^+(\lambda_1^1) > \mathcal{H}^+(\lambda_1^2)$.
3. Finally when $0 < \lambda_1^1 < \lambda_1^2$: in this case $0 < (h_1^3)^{-1}(\lambda_1^2) < (h_1^3)^{-1}(\lambda_1^1)$. Again, $\mathcal{H}^+(\lambda_1^1) > \mathcal{H}^+(\lambda_1^2)$.

We can argue in the same manner for $\mathcal{H}^-$. This completes the proof. □

Remark 4.9. Case $\int_{\Omega_2} m_2 > 0$ can be handled in an analogous way, but the case $\int_{\Omega_2} m_2 = 0$ deserves a comment. In this case, $\mathcal{H}^+$ and $\mathcal{H}^-$ should be defined as follows:

$$\mathcal{H}^+(\lambda_1) := \begin{cases} h_2((h_1^1)^{-1}(\lambda_1)) & \text{if } \lambda_1 < 0, \\ 0 & \text{if } \lambda_1 = 0, \end{cases}$$

and

$$\mathcal{H}^-(\lambda_1) := \begin{cases} h_2((h_1^2)^{-1}(\lambda_1)) & \text{if } \lambda_1 < 0, \\ 0 & \text{if } \lambda_1 = 0. \end{cases}$$

4.3. *Case m_i changes sign, $i = 1, 2$*

Consider in this case

$$\int_{\Omega_1} m_1 < 0, \quad \int_{\Omega_2} m_2 < 0,$$

and then

$$\mu^* < 0.$$

Proposition 4.10. *Assume that m_i changes sign for $i = 1, 2$ and $\int_{\Omega_1} m_1 < 0$, $\int_{\Omega_2} m_2 < 0$. Then, for each $\mu \in \mathbf{R}$ there exists a unique value $\lambda_1 = h_1(\mu)$ such that $f_\mu(\lambda_1) = 0$. Moreover, the map $\mu \in \mathbf{R} \mapsto h_1(\mu)$ is continuous,*

$$h_1(\mu) \begin{cases} > 0 & \textit{if } \mu > \mu^*, \\ = 0 & \textit{if } \mu = \mu^*, \\ < 0 & \textit{if } \mu < \mu^*, \end{cases}$$

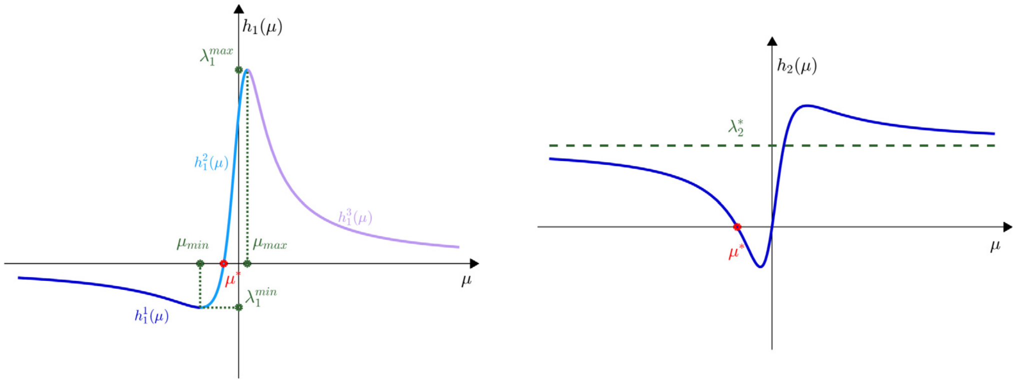

Fig. 10. Functions $h_1(\mu)$ (left) and $h_2(\mu)$ (right) in the case m_1 and m_2 changing sign and $\int_{\Omega_1} m_1 < 0$ and $\int_{\Omega_2} m_2 < 0$.

and

$$\lim_{\mu\to\pm\infty} h_1(\mu) = 0.$$

As consequence, there exist $\mu_{\min} < \mu^* < \mu_{\max}$ *such that*

$$h_1(\mu_{\min}) = \min_{\mu\in\mathbf{R}} h_1(\mu) = \lambda_1^{\min} < 0, \quad h_1(\mu_{max}) = \max_{\mu\in\mathbf{R}} h_1(\mu) = \lambda_1^{\max} > 0.$$

Finally, the map $\mu \mapsto h_1(\mu)$ *is decreasing in* $(-\infty, \mu_{\min})$ *and* $(\mu_{\max}, \infty)$ *and increasing in* $(\mu_{\min}, \mu_{\max})$.

For $h_2(\mu)$, we can deduce the following

Proposition 4.11. *Assume that* m_i *changes sign for* $i = 1, 2$ *and* $\int_{\Omega_1} m_1 < 0$, $\int_{\Omega_2} m_2 < 0$. *Then, the map* $\mu \in \mathbf{R} \mapsto h_2(\mu)$ *is continuous,*

$$h_2(\mu) \begin{cases} > 0 & \textit{if } \mu < \mu^* \textit{ or } \mu > 0, \\ = 0 & \textit{if } \mu = \mu^* \textit{ and } \mu = 0, \\ < 0 & \textit{if } \mu \in (\mu^*, 0). \end{cases}$$

Moreover,

$$\lim_{\mu\to\pm\infty} h_2(\mu) = \lambda_2^*.$$

As a consequence, there exists $\mu_{\min} \in (\mu^*, 0)$ *such that*

$$h_2(\mu_{\min}) = \min_{\mu\in\mathbf{R}} h_2(\mu) = \lambda_2^* < 0.$$

We have represented in Figs. 10 and 11 some examples of the maps $h_1(\mu)$ and $h_2(\mu)$ in the case m_1 and m_2 changing sign and $\int_{\Omega_1} m_1 < 0$ and $\int_{\Omega_2} m_2 < 0$.

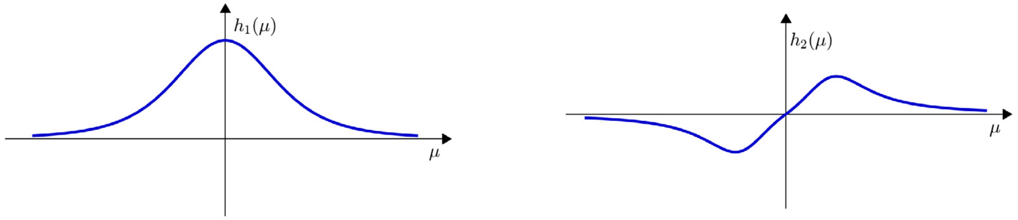

Fig. 11. Functions $h_1(\mu)$ (left) and $h_2(\mu)$ (right) in the case m_1 and m_2 changing sign and $\int_{\Omega_1} m_1 < 0$ and $\int_{\Omega_2} m_2 = 0$.

Proof of Theorem 1.6: 1. By Proposition 4.10, we deduce that

$$F(\lambda_1, \lambda_2) < 0 \quad \text{if } \lambda_1 > \lambda_1^{\max}, \text{ or } \lambda_1 < \lambda_1^{\min}, \text{ or } \lambda_2 \geq \lambda_2^+ \text{ or } \lambda_2 \leq \lambda_2^-$$

2. Now, we introduce some notation:

$$h_1(\mu) := \begin{cases} h_1^1(\mu) & \text{if } \mu < \mu_{\min}, \\ h_1^2(\mu) & \text{if } \mu \in [\mu_{\min}, \mu_{\max}], \\ h_1^3(\mu) & \text{if } \mu > \mu_{\max}. \end{cases}$$

(a) When $\lambda_1 = \lambda_1^{\max}$, there exists a unique value of μ, $\mu = \mu_{\max}$ such that $h_1(\mu_{\max}) = \lambda_1$. The value $h_2(\mu_{\max}) := \overline{\lambda}_2$ verifies that $F(\lambda_1^{\max}, \overline{\lambda}_2) = 0$.
(b) Take now $\lambda_1 \in (0, \lambda_1^{\max})$. Then, there exist $\mu^* < \mu_2 < \mu_3$ such that $\lambda_1 = h_1^i(\mu_i)$ $i = 2, 3$, specifically, $\lambda_1 = h_1^2(\mu_2) = h_1^3(\mu_3)$.
Moreover, $\mu_2 \to \mu^*$ and $\mu_3 \to +\infty$ as $\lambda_1 \to 0$. For these values, $h_2(\mu_2) \to 0$ and $h_2(\mu_3) \to \lambda_2^*$. Observe that $h_2(0) = 0$.
(c) Consider the case $\lambda_1 \in (\lambda_1^{\min}, 0)$. There exists a unique value of $\mu_1 < \mu_2 < \mu^*$ such that $\lambda_1 = h_1^i(\mu_i)$ $i = 1, 2$, in fact, $\lambda_1 = h_1^1(\mu_1) = h_1^2(\mu_2)$.
In this case, as $\lambda_1 \to 0$, then with $\mu_1 \to -\infty$ and $\mu_2 \to \mu^*$. Hence, $h_2(\mu_1) \to \lambda_2^*$ and $h_2(\mu_2) \to 0$.
(d) The case $\lambda_1 = \lambda_1^{\min}$ is analogous to the first case.

Now, we define the maps

$$\mathcal{H}^+(\lambda_1) := \begin{cases} h_2((h_1^3)^{-1}(\lambda_1)) & \text{if } \lambda_1 \in (0, \lambda_1^{\max}], \\ h_2((h_1^1)^{-1}(\lambda_1)) & \text{if } \lambda_1 \in [\lambda_1^{\min}, 0] \end{cases}$$

and

$$\mathcal{H}^-(\lambda_1) := h_2((h_1^2)^{-1}(\lambda_1)) \quad \text{if } \lambda_1 \in [\lambda_1^{\min}, \lambda_1^{\max}].$$

This completes the proof. □

5. Semilinear interface problems

In this section we study the semilinear problem (7).

Theorem 5.1. *Problem (7) possesses a positive solution if and only if* $F(\lambda_1, \lambda_2) < 0$. *In case the existence, the positive solution is unique.*

Proof. Assume that there exists at least a positive solution (u_1, u_2) of (7). Then, using Proposition 2.5 1.,

$$0 = \Lambda_1(-\lambda_1 m_1 + u_1^{p_1-1}, -\lambda_2 m_2 + u_2^{p_2-1}) > \Lambda_1(-\lambda_1 m_1, -\lambda_2 m_2) = F(\lambda_1, \lambda_2),$$

whence we deduce that $F(\lambda_1, \lambda_2) < 0$.

On the other hand, assume that $F(\lambda_1, \lambda_2) < 0$. Let $\varphi = (\varphi_1, \varphi_2)$ be a positive eigenfunction associated to $F(\lambda_1, \lambda_2)$, then

$$\underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2) = \varepsilon(\varphi_1, \varphi_2), \qquad \overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2) = K(1, 1),$$

it is a pair of sub-supersolution for ε small and K large. Indeed, K and ε must verify

$$K^{p_i-1} \geq |\lambda_i| \|m_i\|_{L^\infty(\Omega_i)}, \qquad \varepsilon^{p_i-1}\|\varphi_i\|_{L^\infty(\Omega_i)} \leq -F(\lambda_1, \lambda_2) \quad i = 1, 2.$$

Clearly, we can take ε small and K large verifying both inequalities and such that $\underline{\mathbf{u}} \leq \overline{\mathbf{u}}$ in Ω.

The uniqueness follows by Theorem 4.3 in [8]. □

Data availability

No data was used for the research described in the article.

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