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A Banzhaf value for games with a proximity relation among the agents

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ABSTRACT

The Banzhaf index is a function determining the power or influence in the decision of a set of agents. The extension of this index to the family of the cooperative games is named Banzhaf value. The relationships of closeness among the agents should modify their power. Games with a priori unions study situations where the closeness relations among the agents are taken into account. In this model the agents are organized in an a priori partition where each element of the partition represents a group of agents with close interests or ideas. The power is determined in two steps, first as a problem among the unions and later, inside each one, the power of each agent is determined. Proximity relations extend this model considering leveled closeness among the agents. In this paper we analyze a version of the Banzhaf value for games with a proximity relation and we show the interest of this value by applying it to the allocation of the power of the political groups in the European Parliament.

1. Introduction

In decision situations for committees or centers of distributed control the quantification of the power of each member is an important element to analyze the final position and the different treatment of each of them. Simple games are a way from the cooperative game theory to represent these situations and to study the power of their elements. A power index for simple games is a function determining the power or influence of the agents in each simple game. One of the most known power indices, the Banzhaf index, was introduced by Penrose [1] in 1946 and later by Banzhaf [2] in 1965. In this context the Banzhaf index was generalized for all cooperative game as the Banzhaf value [3]. The Banzhaf value has been studied in different scenarios incorporating new information over the relationships of the agents (coalition structures [4], communication situations [5], hierarchical relations [6], etc.). Owen [7] proposed a different model with an evident interest for simple games. He considered that agents are organized a priori in groups taking into account the closeness of their interests (ideas). So, besides the game he supposed known a partition of the set of agents in a priori unions based in the relations among the agents. These unions are considered as a starting point for further negotiations. This model allows to determine the power of the agents in a simple game taking into account the a priori unions among them. But closeness is usually a leveled property. For instance, political groups can be organized in a priori ideological unions. The Banzhaf value was studied for this model in [8]. Casajus [9] proposed another version of the Owen model considering also information about the internal estructure of the unions. But the Banzhaf value has not been studied for this version.

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Considering equal every ideological closeness between two political parties is actually a simplification of the situation. Aubin [10] and Butnariu [11] introduced fuzzy sets to describe leveled participation of the players in the coalitions (fuzzy coalitions). Proximity relations are reflexive and transitive fuzzy binary relations. Fernández et al. [12] introduced proximity relations to explain the relations among the players in a cooperative game, extending the Owen model in a natural way for another known value, the Shapley value. Related works are given in Meng [13] and Meng and Zhang[14], but they consider a game with fuzzy coalitions with a crisp system of a priori unions. Hence this model contains a different approach. Others analyze this kind of relations in a probabilistic way, Calvo et al. [15] or Kaniovski and Das [16], but they do not use the Owen model. Now, we propose to use proximity relations to study the power of an agent in a game by the Banzhaf value. Section 2 is dedicated to the preliminaries about cooperative games, a priori unions and fuzzy sets. In section 3 we introduce our Banzhaf value for games with a proximity relation among the agents and particularly the Banzhaf–Owen value is extended to the Casajus version of the Owen model. In section 4 we propose axioms for the proposed value extending known properties of the classical Banzhaf value to our fuzzy situation. Finally, last section shows an example of application of the model as index, analyzing the power of the political groups in the European Parliament in an ad hoc situation.

2. Preliminaries

2.1. Cooperative games

A *cooperative game with transferable utility*, game from now on, is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a mapping satisfying $v(\emptyset) = 0$. The elements of N are named players, the subsets of players are named coalitions and the mapping v is the characteristic function of the game. A *simple game* represents a decision situation by a cooperative game (N, v) where: 1) $v(S) \in \{0, 1\}$ for every $S \subseteq N$, 2) $v(N) = 1$, and 3) $v(S) \leq v(T)$ if $S \subset T \subseteq N$ (monotonicity). A coalition is called winning if $v(S) = 1$ and losing $v(S) = 0$.

A *value* for games is a function ψ which determines for each game (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$ interpreted as a payoff vector. Values for simple games are named power indices.[1] In this case the payoffs mean the power or influence of the agents in the decision. This paper focuses on the Banzhaf value. A swing for a player $i \in N$ in a simple games is a winning coalition S containing player i such that $S \setminus \{i\}$ is losing. The Banzhaf index obtains the probability to get a swing among the coalitions containing a determined agent. Owen [3] extended this index to all the cooperative games. The *Banzhaf value* is a value defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ as

$$\beta_i(N, v) = \sum_{\{S \subset N : i \notin S\}} \frac{1}{2^{|N|-1}} [v(S \cup \{i\}) - v(S)]. \tag{1}$$

2.2. Communication structures

Myerson [5] analyzed the inclusion in a game of information about the communication of the players. Let N be a finite set of players and $L^N = \{\{i, j\} \in N \times N : i \neq j\}$ the set of unordered pairs of different elements in N. We use $ij = \{i, j\}$ from now on. Each undirected graph (N, L) where the set of vertices is N and the set of edges $L \subseteq L^N$ is considered as a communication structure. So, each $L \subseteq L^N$ is called a *communication structure for* N. Myerson defines a *game with communication structure* as a triple (N, v, L) where (N, v) is a game and L is a communication structure for N. A usual cooperative game (N, v) is identified with the game with communication structure (N, v, L^N). Let (N, v, L) be a game with communication structure. A coalition $S \subseteq N$ whose vertices are connected by the links in L is called *connected*. The maximal connected coalitions correspond to the sets of vertices of the connected components of the graph (N, L) and we denote them as N/L. This family N/L is actually a partition of N. If $S \subseteq N$ is a coalition then $L_S = \{ij \in L : i, j \in S\}$ and (S, v, L_S) represents the restriction to S of the characteristic function of the game and the communication structure. We use $S/L = S/L_S$. Given (N, v, L), Myerson introduces a new game $(N, v/L)$ incorporating the information of the communication structure,

$$v/L(S) = \sum_{T \in S/L} v(T) \quad \forall S \subseteq N.$$

This model supposes then that non-connected coalitions do not obtain extra profits with regard to their components and so they are irrelevant. The Banzhaf value was extended for games with communication structure in [18]. The *graph-Banzhaf value* is a function defined by

$$\eta(N, v, L) = \beta(N, v/L). \tag{2}$$

[1] This is a cardinal notion of power rather than other ordinal ones, see [17].

2.3. A priori unions

A *game with a priori unions* [7] is a triple (N, v, P) where $(N, v) \in \mathcal{G}$ and $P = \{N_1, ..., N_m\}$ is a partition of N. Players in N_k for each k have similar interests in the game and they bargain as a block to get a fair payoff. It is supposed that players are interested in the grand coalition N but considering the a priori unions as bargaining elements, so the unions can cooperate among them obtaining profits.[2] The classical situation without a priori unions is identified with the individual partition $P^{ind} = \{\{i\} : i \in N\}$. The *Owen model* is a procedure to define values for games with a priori unions in two steps. Let (N, v, P) with $P = \{N_1, ..., N_m\}$. The *quotient game* is a game (M, v^P) with set of players $M = \{1, ..., m\}$ defined by

$$v^P(Q) = v\left(\bigcup_{q \in Q} N_q\right), \forall Q \subseteq M.$$

Let $k \in M$. For each $S \subset N_k$ the partition P_S of $N \setminus (N_k \setminus S)$ is to replace N_k with S. Given a classical value for games ψ^1, we define a game over each union (N_k, v_k) as $v_k(S) = \psi_k^1\left(M, v^{P_S}\right), \forall S \subseteq N_k$. Finally we solve the game in every union using another value ψ^2. So, for each player $i \in N$ if $k(i)$ is such that $i \in N_{k(i)}$ then the value is $\Psi_i(N, v, P) = \psi_i^2\left(N_{k(i)}, v_{k(i)}\right)$. Owen [8] defined an extension of the Banzhaf value, named *Banzhaf–Owen value* and denoted as β^{ow}, in the sense that $\beta^{ow}(N, v, P^{ind}) = \beta(N, v)$. This extension uses the Owen model with $\psi^1 = \psi^2 = \beta$, namely for every $i \in N$

$$\beta_i^{ow}(N, v, P) = \beta_i(N_{k(i)}, v_{k(i)}) \text{ and } v_k(S) = \beta_k(N_k, v^{P_S}), \ \forall S \subseteq N_k. \tag{3}$$

Casajus [9] proposed a variation of the Owen problem. He considers a partition in a priori unions for the agents, but also information about the bilateral relations which defined the unions. These internal relationships introduce an asymmetry among the players inside a union. In order to represent these situations he used a graph (N, L), as in [5], but now the connected components N/L are the unions and the links inside each component are the bilateral relations determining each union. Hence Casajus uses a value ψ^2 for games with communication structure in the second step in the Owen model taking into account the asymmetry inside the unions. Each triple (N, v, L) is named now *game with a cooperation structure*.[3] But there is no literature about the Banzhaf value for the Casajus version.

2.4. Fuzzy sets and proximity relations

A *fuzzy set* of a finite set N is a function $\tau : N \to [0, 1]$. The support of τ is the set $\text{supp}(\tau) = \{i \in N : \tau(i) \neq 0\}$. The image of τ is the ordered set of the non-null images of the function, $\text{im}(\tau) = \left\{\lambda_1 < \cdots < \lambda_p\right\} = \{\lambda \in (0, 1] : \exists i \in N, \tau(i) = \lambda\}$. For each $t \in (0, 1]$ the *t-cut* of the fuzzy set τ is $[\tau]_t = \{i \in N : \tau(i) \geq t\}$. A (signed) *capacity* over N is a set function $f : 2^N \to \mathbb{R}$ satisfying that $f(\emptyset) = 0$, namely a game. The (signed) *Choquet integral* [20,21] of $\tau \in [0, 1]^N$ with respect to a capacity f is defined by

$$\int \tau \, df = \sum_{k=1}^{p} \left(\lambda_k - \lambda_{k-1}\right) f\left([\tau]_{\lambda_k}\right), \tag{4}$$

where $im(\tau) = \left\{\lambda_1 < \cdots < \lambda_p\right\}$ and $\lambda_0 = 0$. The following properties of the Choquet integral are known:

(C1) $\int e^S \, df = f(S)$, for all $S \subseteq N$, and $e^S(i) = 1$ if $i \in S$ and $e^S(i) = 0$ otherwise.
(C2) $\int t\tau \, df = t \int \tau \, df$, for all $t \in [0, 1]$.
(C3) $\int \tau \, d(a_1 f_1 + a_2 f_2) = a_1 \int \tau \, df_1 + a_2 \int \tau \, df_2$, when $a_1, a_2 \in \mathbb{R}$.
(C4) $\int \tau \, df = a \bigvee_{i \in N} \tau(i)$ if $f([\tau]_t) = a$ for all $t \in (0, 1]$.
(C5) $\int \tau_S \, df' = \int \tau \, df$ if $S \subseteq N$ satisfies $f([\tau]_t) = f'([\tau]_t \cap S)$ for all $t \in (0, 1]$.
(C6) $\int \tau \, df = \sum_{q=1}^{p} (t_k - t_{k-1}) f([\tau]_{t_k})$ for all set $im(\tau) \subseteq \{t_1 < \cdots < t_p\} \subset (0, 1]$ and $t_0 = 0$.

A bilateral *fuzzy relation* over N, see [22], is a function $\rho : N \times N \to [0, 1]$ satisfying the condition $\rho(i, j) \leq \rho(i, i) \wedge \rho(j, j)$. A *proximity relation* over N, is a fuzzy relation ρ satisfying the properties: (Reflexivity) $\rho(i, i) = 1$ for all $i \in N$, and (Symmetry) $\rho(i, j) = \rho(j, i)$ for all $i, j \in N$. If $S \subseteq N$ then the proximity relation ρ restricted to S is ρ_S, a new proximity relation over S with $\rho_S(i, j) = \rho(i, j)$ for all $i, j \in S$. A proximity relation ρ over N can be also seen as fuzzy sets over the set $\overline{L}^N = L^N \cup \{\{ii\} : i \in N\}$ with $1 \in im(\rho)$ and $\{\{ii\} : i \in N\} \subseteq [\rho]_1$. So, we will use $\rho(ij)$ instead $\rho(i, j)$. Every cooperation structure $L \in L^N$ can be seen as a crisp proximity relation (adding the vertices), moreover the cuts of a proximity relation

2 This idea is different in the communication model of Myerson, see the previous section.
3 Actually (N, v, L) is a game communication structure but seen in another way.

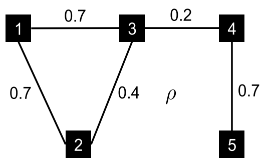


Fig. 1. Fuzzy graph representing a proximity relation.

are cooperation structures. Each set function f over L^N will be identified to a signed capacity with the same letter f over $\overline{L}^N$ defined for all $A \subseteq \overline{L}^N$ by

$$f(A) = \begin{cases} f(L), & \text{if } A = L \cup \{\{ii\} : i \in N\} \\ 0, & \text{otherwise.} \end{cases}$$

3. A Banzhaf value for games with a proximity relation

Fernández et al. [12] studied the Shapley value with proximity relations. The goal of this paper is to define a Banzhaf value for games with a proximity relation among the agents.

Definition 1. A game with a proximity relation is a triple (N, v, ρ) where (N, v) is a game and ρ is a proximity relation over N.

Suppose a game (N, v). Think first about a crisp relation L as in Casajus [9]. If $ij \in L$ then we understand that player j is close to i. If $ij, jk \in L$ and $ik \notin L$ then j is close to i and close to k but in different sense because i is not close to k. We suppose then that j works as a valid intermediary between i and k, moderating the position of all of them and forming a union. Now, we consider a proximity relation ρ. For each pair of agents i, j number $\rho(ij)$ means the level of closeness of the interests or ideas between both of them. This closeness $\rho(ij)$ is also the cohesion level or confidential level of the coalition $\{i, j\}$. For three players $i, j, k \in N$ we take $\rho(ij) \wedge \rho(jk)$ as the maximal confidential level of coalition $\{i, j, k\}$ taking into account the moderation power of j. We can define unions in this context fixing a level of cohesion. So, if we take $t_0 \in (0, 1]$ (we think that this is the minimal reasonable level to get a union) then S is a union if the proximity relation connects S at this level t_0 and S is maximal in this sense. For each $t \in (0, 1]$ the cut $[\rho]_t$ represents a cooperation structure in the Casajus sense and it explains the situation in order of increasing the required level of relation to consider a union.

Suppose for instance a committee formed by five members $N = \{1, 2, 3, 4, 5\}$ such that the a priori bilateral relations among them are well known. Obviously these relationships are not the same and we level them using a proximity relation $\rho(12) = \rho(13) = \rho(45) = 0.7$, $\rho(23) = 0.4$, $\rho(34) = 0.2$ and $\rho(ij) = 0$ otherwise. We represent the proximity relation ρ by a graph with leveled links (see Fig. 1) with ρ. Link ij is not in the graph if $\rho(ij) = 0$. The decision in the committee is taken using a simple game $(N, u_{\{2,3,4\}})$ where $u_{\{2,3,4\}}(S) = 1$ if $S \supseteq \{2, 3, 4\}$ and $u_{\{2,3,4\}}(S) = 0$ otherwise.[4]

In our example the cuts (Fig. 2) show that if $t \in (0, 0.2]$ all the agents form a union but they are asymmetric because of their positions in the graph, if $t \in (0.2, 0.4]$ we get a situation with two a priori unions in the Owen sense because the graphs are complete, if $t \in (0.4, 0.7]$ the cooperation structure follows the Casajus version because there is an asymmetry into one of the unions, and finally if $t \in (0.7, 1]$ then there are not any a priori unions because we have the individual partition P^{ind}. A union is obtained if there is a level such that this coalition is a component in the corresponding cut.

The model in [12] considers a value for games with a priori unions in the Casajus version and the Choquet integral of the proximity relation using this value. But the Banzhaf value in this context has not been studied at the moment. So, first we introduce a Banzhaf value for games with cooperation structure using the Owen model (3) with the graph Banzhaf (2).

Definition 2. Let (N, v, L) be a game with $L \subseteq L^N$ where $N/L = \{N_1, ..., N_m\}$ and $M = \{1, ..., m\}$. If $S \subseteq N_k$ for any $k \in M$ then $v_k(S) = \beta_k \left(M, v^{(N/L)_S}\right)$. The graph Banzhaf–Owen value for each player $i \in N_k$ is $\beta_i^{co}(N, v, L) = \eta_i \left(N_k, v_k, L_{N_k}\right)$.

Remarks. The proposed graph solution is consistent with the other Banzhaf values in this way. Only from the definitions:

- If $L_S = L^S$ for all union $S \in N/L$, namely each component is complete, then $\beta^{co}(N, v, L) = \beta^{ow}(N, v, N/L)$.
- Particularly $\beta^{co}(N, v, \emptyset) = \beta(N, v)$. Observe that $L = \emptyset$ corresponds to P^{ind}.

The graph Banzhaf–Owen value in Definition 2 determines for each player a set function over L^N. So, if $i \in N$ then

$$\beta_i^{co}(N, v)(L) = \beta_i^{co}(N, v, L). \tag{5}$$

[4] Game (N, u_T) with $T \subseteq N$ a non-empty coalition represents the simple game where all the winning coalitions are those containing T. It is known as unanimity game.

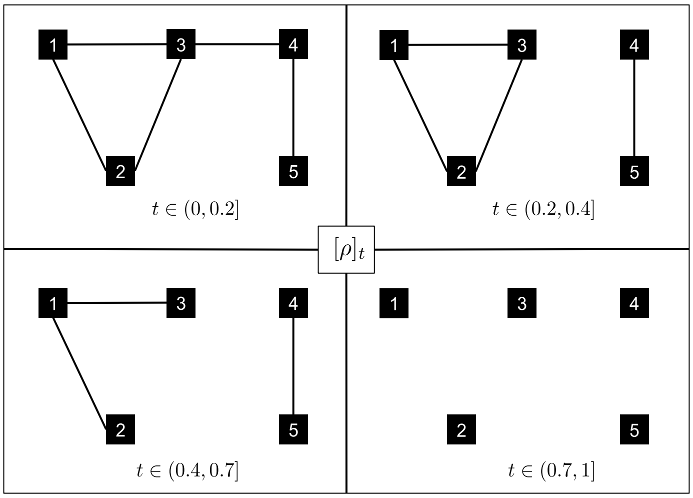


Fig. 2. Cuts of a proximity relation.

Now, we can define our Banzhaf value for games with a proximity relation. We use a Choquet integral (4) of the proximity relation with regard to the above signed capacity.

Definition 3. Let (N, v, ρ) be a game with a proximity relation. The prox-Banzhaf value for each agent $i \in N$ is defined by

$$B_i(N, v, \rho) = \int \rho \, d\beta_i^{co}(N, v).$$

The prox-Banzhaf value meets the purpose of being a Banzhaf value in the sense that it coincides with the classical value when we do not have any a priori relation among the players. We denote as $\rho = 0$ the *trivial proximity relation* satisfying $0(ij) = 0$ if $i \neq j$ and also $0(ii) = 1$. This proximity relation represents the classical situation without a priori unions among the players.

Proposition 1. *The prox-Banzhaf value satisfies* $B(N, v, 0) = \beta(N, v)$.

Proof. We know that β^{co} verifies that $\beta^{co}(N, v, \emptyset) = \beta(N, v)$. Hence as all the cuts of the trivial proximity relation 0 satisfy $[0]_t = \emptyset$ we get that $\beta_i^{co}(N, v)([0]_t) = \beta_i(N, v)$ for all $t \in (0, 1]$. Property (C4) of the Choquet integral and the fact $1 \in im(\rho)$ imply

$$B_i(N, v, 0) = \int 0 \, d\beta_i^{co}(N, v) = \beta_i(N, v). \quad \square$$

Suppose our example in Fig. 1 with the game $u_{\{2,3,4\}}$. When $t \in (0, 0.2]$ the graph is connected and $u_{\{2,3,4\}}/[\rho]_{0.2} = u_{\{2,3,4\}}$, thus

$$\beta^{co}(N, u_{\{2,3,4\}}, [\rho]_{0.2}) = \eta(N, u_{\{2,3,4\}}, [\rho]_{0.2}) = \beta(N, u_{\{2,3,4\}}) = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right).$$

If $t \in (0.2, 0.4]$ then we use the Banzhaf–Owen value with a priori unions. There are two unions $M = \{a = \{1, 2, 3\}, b = \{4, 5\}\}$, and $v_a = \frac{1}{2} u_{\{2,3\}}$, $v_b = \frac{1}{2} u_{\{4\}}$. β^{co} coincides with the Banzhaf value of the above games in each group,

$$\beta^{co}(N, u_{\{2,3,4\}}, [\rho]_{0.4}) = \beta^{ow}(N, u_{\{2,3,4\}}, N/[\rho]_{0.4}) = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right).$$

If $t \in (0.4, 0.7]$ we have the same unions $M = \{a, b\}$ but now player 1 is necessary to get a winning coalition. So, $v_a/L = \frac{1}{2} u_{\{1,2,3\}}$ and

$$\beta^{co}(N, u_{\{2,3,4\}}, [\rho]_{0.7}) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}, 0\right).$$

Finally, if $t \in (0.7, 1]$ we obtain the usual case and also

$$\beta^{co}(N, u_{\{2,3,4\}}, [\rho]_1) = \beta(N, u_{\{2,3,4\}}) = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right).$$

The prox-Banzhaf value in this situation is

$$B(N, u_{\{2,3,4\}}, \rho) = 0.2\left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right) + 0.2\left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right) + 0.3\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}, 0\right) + 0.3\left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right) = \left(\frac{3}{80}, \frac{17}{80}, \frac{17}{80}, \frac{30}{80}, 0\right).$$

4. Axioms for the prox-Banzhaf value

The authors in [12] introduced the following scaling operations over proximity relations. Let ρ be a proximity relation over N. If $a, b \in [0, 1]$ with $a < b$ then ρ_a^b is the *interval scaling* of ρ, a new proximity relation over N defined by

$$\rho_a^b(ij) = \begin{cases} 1, & \text{if } \rho(ij) \geq b \\ \dfrac{\rho(ij) - a}{b - a}, & \text{if } \rho(ij) \in (a, b) \\ 0, & \text{if } \rho(ij) \leq a. \end{cases} \tag{6}$$

Let $a, b \in [0, 1]$ be numbers with $a < b$ and $a \neq 0$ or $b \neq 1$. The *dual interval scaling* of ρ is a new proximity relation over N given by

$$\overline{\rho}_a^b(ij) = \begin{cases} \dfrac{\rho(ij) + a - b}{1 + a - b}, & \text{if } \rho(ij) \geq b \\ \dfrac{a}{1 + a - b}, & \text{if } \rho(ij) \in (a, b) \\ \dfrac{\rho(ij)}{1 + a - b}, & \text{if } \rho(ij) \leq a. \end{cases} \tag{7}$$

Remarks. 1) If $b \in im(\rho)$ with $b < 1$ then $|im(\rho_a^b)| < |im(\rho)|$.
2) If $b \in im(\rho)$ with $b < 1$ and $a \in im(\rho) \cup \{0\}$ then $|im(\overline{\rho}_a^b)| < |im(\rho)|$.
3) If $a = 0$ and $b = 1$ then the dual interval scaling is defined by $\overline{\rho}_0^1(ij) = 1$ if $\rho(ij) = 1$ and $\overline{\rho}_0^1(ij) = 0$ otherwise.

The above scalings allow us to allocate the Choquet integral in particular intervals. This idea is explained in the next lemma. As we said before (see subsection 2.4 in preliminaries) each cut of a proximity relation is identified with a particular element of $\overline{L}^N$.

Lemma 2. *(Fernández et al. [12]) Let ρ be a proximity relation over N. For every pair of numbers $a, b \in [0, 1]$ with $a < b$ and for every set function f over L^N it holds*

$$\int \rho \, df = (b - a) \int \rho_a^b \, df + (1 + a - b) \int \overline{\rho}_a^b \, df.$$

We propose an axiomatization based on classical axioms of the Banzhaf values (for its different versions). The fuzzy information is included into the axioms by the scaling because we can focus on the interval where the axiom is satisfied. Let Ψ be a value determining a payoff vector for each game with a proximity relation.

It is known that the Banzhaf value satisfies the dummy player axiom (see for instance Casajus [23]). Player i is *dummy* in a game (N, v) if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. The dummy player axiom guarantees the payoff $v(\{i\})$ for a dummy player i. This fact changes if there is asymmetry into the component containing the dummy player. An *isolated player* in a proximity relation ρ is a player i satisfying $\rho(ij) = 0$ for all $j \in N \setminus \{i\}$.

Dummy isolated player. *If $i \in N$ is a dummy player in a game (N, v) and i is isolated in a proximity relation ρ then $\Psi_i(N, v, \rho) = v(\{i\})$.*

Next three axioms introduce the delegation or merging situation for proximity relations, classical also in the Banzhaf value (see for instance [23]). Let (N, v) be a game. The amalgamation of players $i, j \in N$ consists of taking the activity of both of them as only one, delegating j to i. The *delegation game* is $(N \setminus \{j\}, v^{ij})$ with $v^{ij}(S) = v(S \cup \{j\})$ if $i \in S$, $v^{ij}(S) = v(S)$

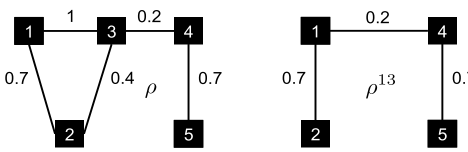


Fig. 3. Amalgamation in a proximity relation.

otherwise. Let ρ be a proximity relation over N and $i, j \in N$ different players with[5] $\rho(ij) = 1$. The proximity relation changes with the amalgamation, ρ^{ij} is a new proximity relation over $N \setminus \{j\}$ given by (see Fig. 3)

$$\rho^{ij}(kl) = \begin{cases} \rho(kl), & \text{if } k, l \neq i \\ \rho(ki) \vee \rho(kj), & \text{otherwise.} \end{cases} \tag{8}$$

Next axiom says that the payoff obtained by both of the players in the game is the same that the payoff obtained by the proxy of them while this is possible (until their relation level) plus the payoff of both of them for the rest of the levels.

Fuzzy amalgamation. *Let (N, v, ρ) be a game with a proximity relation. If $i, j \in N$ are different players with $\rho(ij) > 0$ then*

$$\Psi_i(N, v, \rho) + \Psi_j(N, v, \rho) = \rho(ij)\Psi_i\left(N \setminus \{j\}, v^{ij}, \left(\rho_0^{\rho(i,j)}\right)^{ij}\right) + (1 - \rho(ij))\left[\Psi_i\left(N, v, \overline{\rho}_0^{\rho(ij)}\right) + \Psi_j\left(N, v, \overline{\rho}_0^{\rho(ij)}\right)\right].$$

If $im(\rho) = \{1\}$ then ρ is identified with a cooperation structure L, and the axiom says for every pair of connected players $i, j \in N$ with $ij \in L$,

$$\Psi_i(N, v, L) + \Psi_j(N, v, L) = \Psi_i\left(N \setminus \{j\}, v^{ij}, L^{ij}\right). \tag{9}$$

The amalgamation property is extended also to the isolated players. When two players are isolated one can delegate to the other.

Isolated amalgamation. *Let (N, v, ρ) be a game with a proximity relation. If $i, j \in N$ are different isolated players then*

$$\Psi_i(N, v, \rho) + \Psi_j(N, v, \rho) = \Psi_i\left(N \setminus \{j\}, v^{ij}, \rho^{ij}\right).$$

Players in the group involved in the amalgamation of i, j different to them should not change their payoffs while this situation is working. This fact is the goal of the next axiom, which was introduced for the Banzhaf–Owen value by Amer et al. [24]. Let $i, j, l \in N$ be three different players. Number

$$r_{\{i,j,l\}} = \bigvee_{\{\exists T \in N/[\rho]_t;\ i,j,l \in T\}} t$$

represents the maximal level such that this triple of players are in the same group.

Fuzzy amalgamation neutrality. *If (N, v, ρ) is a game with a proximity relation and i, j different players with $\rho(ij) > 0$ then for all $l \neq i, j$ with $r_{\{i,j,l\}} < \rho(ij)$*

$$\Psi_l(N, v, \rho) = \left(\rho(ij) - r_{\{i,j,l\}}\right)\Psi_l\left(N \setminus \{j\}, v^{ij}, \left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)^{ij}\right) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right)\Psi_l\left(N, v, \overline{\rho}_{r_{\{i,j,l\}}}^{\rho(ij)}\right).$$

If $im(\rho) = \{1\}$ then ρ is identified with a cooperation structure L and the axiom says $\Psi_l(N, v, L) = \Psi_l\left(N \setminus \{j\}, v^{ij}, L^{ij}\right)$.

Modified fairness was introduced in [9] as a modification of the classical axiom of fairness [5]. Suppose a crisp situation, namely a cooperation structure L, and $ij \in L$. Let $S \in N/L$ such that $ij \in L_S$. Fairness is not an option because the number of components can change. If we delete ij then we denote by S^i_{ij}, S^j_{ij} the components in which S is divided containing i and j respectively (they can be the same). Moreover, let $N^i_{ij} = (N \setminus S) \cup S^i_{ij}$ and $N^j_{ij} = (N \setminus S) \cup S^j_{ij}$. Casajus proposed that the

[5] It is possible to define amalgamation when the level of closeness between the players is not 1 but we will not need it.

difference between the payoffs with and without the link of both of the involved players is the same if we do not consider the action of the new union without each player. Namely, we obtain fairness if we delete the new component with the other player if it exists (otherwise it coincides with fairness)

$$\Psi_i(N,v,L)-\Psi_i(N_{ij}^i,v,L_{N_{ij}^i}\setminus\{ij\})=\Psi_j(N,v,L)-\Psi_j(N_{ij}^j,v,L_{N_{ij}^j}\setminus\{ij\}). \tag{10}$$

We extended the modified fairness to a fuzzy situation given by a proximity relation in [12]. In this case, we take into account the mere reduction of the relation between two players. So we have to consider that this reduction of level only concerns to the interval between the reduced level and the original one. Let ρ be a proximity relation over a set of players N with $im(\rho)=\{\lambda_1<\cdots<\lambda_m\}$ and $\lambda_0=0$. Consider $i,j\in N$ two different players with $\rho(ij)=\lambda_k>0$ and let $\rho^*(ij)=\lambda_{k-1}$. If ρ is a proximity relation[6] and $\rho(ij)=1$ then ρ_{-ij} denotes a new proximity relation with $\rho_{-ij}(ij)=0$ and $\rho_{-ij}=\rho$ otherwise. The proximity relation $\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)_{-ij}$ in the following axiom focuses on the relation in the interval where the closeness of ij is reduced. All the cuts of $\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)_{-ij}$ use the same N_{ij}^i and N_{ij}^j if $t\in(0,\rho(ij)-\rho^*(ij)]$ because $\rho^{\rho(ij)}_{\rho(ij)-t}$ is crisp.

Modified fuzzy fairness *Let (N,v,ρ) be a game with a proximity relation and $i,j\in N$ with $\rho(ij)>0$. For each $t\in(0,\rho(ij)-\rho^*(ij)]$ it holds*

$$\begin{aligned}\Psi_i(N,v,\rho)-\Psi_j(N,v,\rho)&=(1-t)\left[\Psi_i\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right)-\Psi_j\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right)\right]\\&+t\left[\Psi_i\left(N_{ij}^i,v,\left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)_{-ij}\right)_{N_{ij}^i}\right)-\Psi_j\left(N_{ij}^j,v,\left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)_{-ij}\right)_{N_{ij}^j}\right)\right].\end{aligned}$$

We prove now that our value satisfies all the axioms described in the above subsection.

Theorem 3. *The prox-Banzhaf value satisfies dummy isolated player, fuzzy amalgamation, isolated amalgamation, fuzzy amalgamation neutrality and modified fuzzy fairness.*

Proof. Suppose always $L\subseteq L^N$ with $N/L=\{N_1,...,N_m\}$ and $M=\{1,...,m\}$. Let also ρ be a proximity relation.
Dummy isolated player. Let $i\in N$ be a dummy player in v and i isolated in L. We take $N_1=\{i\}$. Remember that the Banzhaf value satisfies the dummy player property [23]. Besides set 1 is a dummy player for game $v^{N/L}$, in fact we have for each $Q\subseteq M\setminus\{1\}$

$$v^{N/L}(Q\cup\{1\})-v^{N/L}(Q)=v\left(\bigcup_{q\in Q}N_q\cup\{i\}\right)-v\left(\bigcup_{q\in Q}N_q\right)=v(\{i\})=v^{(N/L)_{\{i\}}}(\{1\}).$$

Therefore $v_1(\{i\})=\beta_1(M,v^{N/L})=v^{N/L}(\{1\})=v(\{i\})$. The graph Banzhaf value satisfies isolated player [19], if i is isolated (dummy or not) in L then $\eta_i(N,v,L)=v(\{i\})$, thus

$$\beta_i^{co}(N,v,L)=\eta_i(N_1,v_1,L_{N_1})=v_1(\{i\})=v(\{i\}).$$

Now suppose ρ a proximity relation and our dummy player i isolated in ρ. We have $\{i\}\in N/[\rho]_t$ for all $t\in(0,1]$, so $\beta_i^{co}(N,v)([\rho]_t)=v(\{i\})$ for each $t\in(0,1]$. Using (C4) we get $B_i(N,v,\rho)=v(\{i\})$.
Fuzzy amalgamation. Let $ij\in L$ with $i,j\in N_1$. We will prove the claim

$$\beta_i^{co}(N,v,L)+\beta_j^{co}(N,v,L)=\beta_i^{co}\left(N\setminus\{j\},v^{ij},L^{ij}\right).$$

The graph Banzhaf value satisfies amalgamation [19] for two players in a link, thus

$$\begin{aligned}\beta_i^{co}(N,v,L)+\beta_j^{co}(N,v,L)&=\eta_i\left(N_1,v_1,L_{N_1}\right)+\eta_j\left(N_1,v_1,L_{N_1}\right)\\&=\eta_i\left(N_1\setminus\{j\},(v_1)^{ij},\left(L_{N_1}\right)^{ij}\right).\end{aligned}$$

On the other hand, $\beta_i^{co}\left(N\setminus\{j\},v^{ij},L^{ij}\right)=\eta_i\left((N\setminus\{j\})_1,\left(v^{ij}\right)_1,\left(L^{ij}\right)_{N_1}\right)$. Since $i,j\in N_1$, then $(N\setminus\{j\})_1=N_1\setminus\{j\}$ and $\left(L_{N_1}\right)^{ij}=\left(L^{ij}\right)_{N_1}$. Observe that the amalgamation of two players i,j connected by a link does not change the number of components of the graph. Namely, if we merge $i,j\in N_1$ we have $(N\setminus\{j\})/L^{ij}=\{N_1\setminus\{j\},N_2,\dots,N_m\}$. We will see that

[6] It is also possible to define this operation when the level is not 1.

$(v_1)^{ij} = \left(v^{ij}\right)_1$. For any $T \subseteq N_1 \setminus \{j\}$ we have that $(v_1)^{ij}(T) = v_1(T \cup \{j\})$ if $i \in T$ and $(v_1)^{ij}(T) = v_1(T) =$ if $i \notin T$. So, Definition 2 implies

$$(v_1)^{ij}(T) = \begin{cases} \beta_1\left(M, v^{(N/L)_{T\cup\{j\}}}\right), & \text{if } i \in T \\ \beta_1\left(M, v^{(N/L)_T}\right), & \text{if } i \notin T. \end{cases}$$

Moreover, for all $T \subseteq N_1 \setminus \{j\}$, $\left(v^{ij}\right)_1(T) = \beta_1\left(M, \left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_T}\right)$. We distinguish two cases:

1) If $i \in T$, we prove the equality $\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_T} = v^{(N/L)_{T\cup\{j\}}}$. Let $Q \subseteq M$. If $1 \notin Q$ it holds $v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, and if $1 \in Q$ then

$$v^{ij}\left(T \cup \bigcup_{q\in Q\setminus\{1\}} N_q\right) = v\left(T \cup \{j\} \cup \bigcup_{q\in Q\setminus\{1\}} N_q\right).$$

Thus

$$\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_T}(Q) = \left\{ \begin{array}{ll} v\left(\bigcup_{q\in Q} N_q\right), & \text{if } 1 \notin Q \\ v\left(T \cup \{j\} \cup \bigcup_{q\in Q\setminus\{1\}} N_q\right), & \text{if } 1 \in Q \end{array} \right\} = v^{(N/L)_{T\cup\{j\}}}(Q).$$

2) If $i \notin T$, we prove the equality $\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_T} = v^{(N/L)_T}$. Let $Q \subseteq M$, then if $1 \notin Q$ it holds $v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, and if $1 \in Q$ we get

$$v^{ij}\left(T \cup \left(\bigcup_{q\in Q\setminus\{1\}} N_q\right)\right) = v\left(T \cup \left(\bigcup_{q\in Q\setminus\{1\}} N_q\right)\right).$$

Thus

$$\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_T}(Q) = \left\{ \begin{array}{ll} v\left(\bigcup_{q\in Q} N_q\right), & \text{if } 1 \notin Q, \\ v\left(T \cup \left(\bigcup_{q\in Q\setminus\{1\}} N_q\right)\right), & \text{if } 1 \in Q \end{array} \right\} = v^{(N/L)_T}(Q).$$

Therefore the claim is true. Now let ρ be a proximity relation and $\rho(ij) = t > 0$. Lemma 2 implies

$$\begin{aligned} B_i(N, v, \rho) + B_j(N, v, \rho) = t \int \rho_0^t \, d(\beta_i^{co}(N, v) + \beta_j^{co}(N, v)) \\ + (1-t)\left[B_i\left(N, v, \overline{\rho}_0^t\right) + B_j\left(N, v, \overline{\rho}_0^t\right)\right]. \end{aligned}$$

We denote $im\left(\rho_0^t\right) = \{\lambda_1 < \cdots < \lambda_m\}$. By the claim we have

$$\begin{aligned} \int \rho_0^t \, d(\beta_i^{co}(N, v) + \beta_j^{co}(N, v)) &= \sum_{k=1}^m \left(\lambda_k - \lambda_{k-1}\right)\left[\beta_i^{co}\left(N, v, \left[\rho_0^t\right]_{\lambda_k}\right) + \beta_j^{co}\left(N, v, \left[\rho_0^t\right]_{\lambda_k}\right)\right] \\ &= \sum_{k=1}^m \left(\lambda_k - \lambda_{k-1}\right) \beta_i^{co}\left(N \setminus \{j\}, v^{ij}, \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right). \end{aligned}$$

Obviously $im(\rho^{ij}) \subseteq im(\rho)$ thus we can write using (C6)

$$B_i(N^{ij}, v^{ij}, (\rho_0^t)^{ij}) = \sum_{k=1}^m \left(\lambda_k - \lambda_{k-1}\right) \beta_i^{co}\left(N^{ij}, v^{ij}, \left[(\rho_0^t)^{ij}\right]_{\lambda_k}\right).$$

If we prove the equality $\left[(\rho_0^t)^{ij}\right]_r = \left(\left[\rho_0^t\right]_r\right)^{ij}$ for all $r \in (0, 1]$ then the proof is finished. The links in both sets without player i are the same because the amalgamation does not affect. So, let $ik \in \left[(\rho_0^t)^{ij}\right]_r$. We have $\rho_0^t(ik) \vee \rho_0^t(jk) \geq r$ and then one of them, for instance ik satisfies $\rho_0^t(ik) \geq r$. But then $ik \in [\rho_0^t]_r$ and $ik \in \left(\left[\rho_0^t\right]_r\right)^{ij}$. The other inclusion follows in the same way.

Isolated amalgamation. Take two players $i, j \in N$ who are isolated in L. Consider $N_1 = \{i\}$ and $N_2 = \{j\}$. Following the proof of dummy isolated player and using that Banzhaf value satisfies amalgamation [25] for any pair of players, we get

$$\begin{aligned}\beta_i^{co}(N,v,L)+\beta_j^{co}(N,v,L) &= \eta_i\left(N_1,v_1,L_{N_1}\right)+\eta_j\left(N_2,v_2,L_{N_2}\right)\\ &= v_1(\{i\})+v_2(\{j\})=\beta_1(M,v^{N/L})+\beta_2(M,v^{N/L})\\ &= \beta_1\left(M\setminus\{2\},\left(v^{N/L}\right)^{12}\right).\end{aligned}$$

We see that for each $Q \subseteq M \setminus \{2\}$

$$\begin{aligned}\left(v^{N/L}\right)^{12}(Q) &= \begin{Bmatrix} v\left(\bigcup_{q\in Q} N_q\right), & \text{if } 1\notin Q, \\ v\left(\bigcup_{q\in Q\setminus\{1\}} N_q \cup \{i,j\}\right), & \text{if } 1\in Q \end{Bmatrix}\\ &= \left(v^{ij}\right)^{N\setminus\{j\}/L^{ij}}(Q).\end{aligned}$$

Hence $\beta_1\left(M\setminus\{2\},\left(v^{N/L}\right)^{12}\right)=\beta_1\left(M\setminus\{2\},\left(v^{ij}\right)^{N\setminus\{j\}/L^{ij}}\right)=\beta_i^{co}\left(N\setminus\{j\},v^{ij},L^{ij}\right)$. Let i,j be isolated in ρ now. As they are isolated we have $im(\rho^{ij})=im(\rho)$ and $[\rho^{ij}]_t=([\rho]_t)^{ij}$ for all $t\in(0,1]$. So, as the signed capacities verifies that $\beta_i(N,v)(L)+\beta_j(N,v)(L)=\beta_i(N\setminus\{j\},v^{ij})(L^{ij})$ then

$$B_i(N,v,\rho)+B_j(N,v,\rho)=B_i(N\setminus\{j\},v^{ij},\rho^{ij}).$$

Fuzzy amalgamation neutrality. We take a cooperation structure L with $N/L=\{N_1,N_2,....,N_m\}$. Suppose the merging of $i,j\in N_1$ and $l\notin N_1$. We see the claim

$$\beta_l^{co}\left(N\setminus\{j\},v^{ij},L^{ij}\right)=\beta_l^{co}(N,v,L).$$

If $l\in N_2$ (or any component different to N_1) then we have $\beta_l^{co}(N,v,L)=\eta_l\left(N_2,v_2,L_{N_2}\right)$ and also

$$\begin{aligned}\beta_l^{co}\left(N\setminus\{j\},v^{ij},L^{ij}\right) &= \eta_l\left((N\setminus\{j\})_2,\left(v^{ij}\right)_2,\left(L^{ij}\right)_{N_2}\right)\\ &= \eta_l\left(N_2,\left(v^{ij}\right)_2,L_{N_2}\right),\end{aligned}$$

where the last equality comes from the fact that $ij\notin N_2$ and then the amalgamation takes place out of the component N_2. So if we see that $\left(v^{ij}\right)_2=v_2$ we have the desired equality. Let $R\subseteq N_2$. By definition, $\left(v^{ij}\right)_2(R)=\beta_2\left(M,\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}\right)$ and $v_2(R)=\beta_2\left(M,v^{(N/L)_R}\right)$. The equality that we need to prove is $\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}=v^{(N/L)_R}$. Let $Q\subseteq M$ with $M=\{1,...,m\}$. We determine the equality in four cases for Q:
1) If $2\notin Q$ and $1\in Q$. As $i,j\in N_1$ we obtain that

$$\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}(Q)=v^{ij}\left(\bigcup_{q\in Q}N_q\right)=v\left(\bigcup_{q\in Q\setminus\{1\}}N_q\cup N_1\right)=v^{(N/L)_R}(Q).$$

2) If $1,2\notin Q$. We get

$$\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}(Q)=v^{ij}\left(\bigcup_{q\in Q}N_q\right)=v\left(\bigcup_{q\in Q}N_q\right)=v^{(N/L)_R}(Q).$$

3) If $2\in Q$ and $1\notin Q$. In this situation,

$$\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}(Q)=v^{ij}\left(\bigcup_{q\in Q\setminus\{2\}}N_q\cup R\right)=v\left(\bigcup_{q\in Q\setminus\{2\}}N_q\cup R\right)=v^{(N/L)_R}(Q).$$

4) If $1,2\in Q$. Both modifications of the games are working and then

$$\begin{aligned}\left(v^{ij}\right)^{(N\setminus\{j\}/L^{ij})_R}(Q) &= v^{ij}\left(\bigcup_{q\in Q\setminus\{1,2\}}N_q\cup(N_k)^{ij}\cup R\right)\\ &= v\left(\bigcup_{q\in Q\setminus\{1,2\}}N_q\cup N_k\cup R\right)=v^{(N/L)_R}(Q).\end{aligned}$$

In all cases the games coincide, therefore the claim is true.

Let $i, j, l \in N$, $\rho(ij) > 0$ and $l \in N \backslash \{i, j\}$, $r_{\{i,j,l\}} < \rho(ij)$. Using Lemma 2 again with numbers $\rho(ij), r_{\{i,j,l\}}$ we have

$$B_l(N, v, \rho) = \left(\rho(ij) - r_{\{i,j,l\}}\right) \int \rho_{r_{\{i,j,l\}}}^{\rho(ij)} \, d\beta_l^{co}(N, v) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right) \int \overline{\rho}_{r_{\{i,j,l\}}}^{\rho(ij)} \, d\beta_l^{co}(N, v)$$
$$= \left(\rho(ij) - r_{\{i,j,l\}}\right) B_l\left(N, v, \rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right) B_l\left(N, v, \overline{\rho}_{r_{\{i,j,l\}}}^{\rho(ij)}\right).$$

We have to prove that $B_l\left(N, v, \rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right) = B_l\left(N^{ij}, v^{ij}, \left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)^{ij}\right)$. Obviously, $im\left((\rho_{r_{\{i,j,l\}}}^{\rho(ij)})^{ij}\right) \subseteq im\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)$, therefore if $im\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_m\}$ we can write by (C6) and the claim

$$B_l\left(N^{ij}, v^{ij}, \left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)^{ij}\right) = \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \beta_l^{co}\left(N^{ij}, v^{ij}, \left[\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)^{ij}\right]_{\lambda_k}\right)$$
$$= \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \beta_l^{co}\left(N, v, \left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_k}\right) = B_l(N, v, \rho_{r_{\{i,j,l\}}}^{\rho(ij)}).$$

Modified fuzzy fairness. Consider L a graph over N with $N/L = \{N_1, ..., N_m\}$. Let $ij \in L$ and suppose $i, j \in N_1$. We denote $N_{ij}^i/(L_{N_{ij}^i} \setminus \{ij\}) = \{(N_1)_i, N_2, ..., N_m\}$. Although the quotient game depends on the graph we get $v^{\left(N_{ij}^i/L_{N_{ij}^i} \setminus \{ij\}\right)_S} = v^{(N/L)_S}$ for each $S \subseteq (N_1)_i$. The graph Banzhaf satisfies decomposability [19], namely the payoff of each player in a game can be calculated by components. Also the graph Banzhaf satisfies fairness [19], for all $ij \in L$ we have $\eta_i(N, v, L) - \eta_i(N, v, L \setminus \{ij\}) = \eta_j(N, v, L) - \eta_j(N, v, L \setminus \{ij\})$. So,

$$\beta_i^{co}(N, v, L) - \beta_i^{co}\left(N_{ij}^i, v, L_{N_{ij}^i} \setminus \{ij\}\right) = \eta_i\left(N_1, v_1, L_{N_1}\right) - \eta_i\left((N_1)_i, v_1, L_{(N_1)_i} \setminus \{ij\}\right)$$
$$= \eta_i\left(N_1, v_1, L_{N_1}\right) - \eta_i\left(N_1, v_1, L_{N_1} \setminus \{ij\}\right)$$
$$= \eta_j\left(N_1, v_1, L_{N_1}\right) - \eta_j\left(N_1, v_1, L_{N_1} \setminus \{ij\}\right)$$
$$= \beta_j^{co}(N, v, L) - \beta_j^{co}\left(N_{ij}^j, v, L_{N_{ij}^j} \setminus \{ij\}\right).$$

We consider ρ a proximity relation with $\rho(ij) > 0$ and $t \in (0, \rho(ij) - \rho^*(ij)]$. Using Lemma 2 for numbers $\rho(ij) - t, \rho(ij)$ and property (C3),

$$B_i(N, v, \rho) - B_j(N, v, \rho) = (1 - t)\left[B_i\left(N, v, \overline{\rho}_{\rho(ij)-t}^{\rho(ij)}\right) - B_j\left(N, v, \overline{\rho}_{\rho(ij)-t}^{\rho(ij)}\right)\right]$$
$$+ t \int \rho_{\rho(ij)-t}^{\rho(ij)} \, d[\beta_i^{co}(N, v) - \beta_j^{co}(N, v)].$$

As $t \in (0, \rho(ij) - \rho^*(ij)]$ then it holds $im\left(\rho_{\rho(ij)-t}^{\rho(ij)}\right) = \{1\}$ and $\left[\rho_{\rho(ij)-t}^{\rho(ij)}\right]_1 = [\rho]_{\rho(ij)}$, thus

$$\beta_i^{co}(N, v)\left(\left[\rho_{\rho(ij)-t}^{\rho(ij)}\right]_1\right) - \beta_j^{co}(N, v)\left(\left[\rho_{\rho(ij)-t}^{\rho(ij)}\right]_1\right) = \beta_i^{co}\left(N_{ij}^i, v\right)\left(\left(\left[\rho_{\rho(ij)-t}^{\rho(ij)}\right]_1\right)_{N_{ij}^i} \setminus \{ij\}\right)$$
$$- \beta_j^{co}\left(N_{ij}^j, v\right)\left(\left(\left[\rho_{\rho(ij)-t}^{\rho(ij)}\right]_1\right)_{N_{ij}^j} \setminus \{ij\}\right).$$

Hence, we obtain by (C3) and (C5),

$$\int \rho_{\rho(ij)-t}^{\rho(ij)} \, d[\beta_i^{co}(N, v) - \beta_j^{co}(N, v)] = \int \left(\left(\rho_{\rho(ij)-t}^{\rho(ij)}\right)_{-ij}\right)_{N_{ij}^i} d\beta_i^{co}\left(N_{ij}^i, v\right)$$
$$- \int \left(\left(\rho_{\rho(ij)-t}^{\rho(ij)}\right)_{-ij}\right)_{N_{ij}^j} d\beta_j^{co}\left(N_{ij}^j, v\right)$$
$$= B_i\left(N_{ij}^i, v, \left(\left(\rho_{\rho(ij)-t}^{\rho(ij)}\right)_{-ij}\right)_{N_{ij}^i}\right)$$
$$- B_j\left(N_{ij}^j, v, \left(\left(\rho_{\rho(ij)-t}^{\rho(ij)}\right)_{-ij}\right)_{N_{ij}^j}\right). \quad \square$$

5. The main result

Now we prove that our value is the only one satisfying the axioms. Our proof uses a characterization of the Banzhaf value obtained by Casajus [23]. He got the uniqueness of the classical value only with two axioms: dummy player and amalgamation. The proof uses the dummy player property and the amalgamation of all the pairs of players. Hence following exactly the same proof of Proposition 1.2 in [23] we propose the next lemma in our context.

Lemma 4. *(Casajus [23]) All the values for games with proximity relation satisfying dummy isolated player and fuzzy isolated amalgamation coincide on the family of games $(N, v, 0)$, namely they coincide on the classical games.*

The characterization theorem is the next one.

Theorem 5. *The prox-Banzhaf value is the only value for games with a proximity relation among the players satisfying dummy isolated player, fuzzy amalgamation, isolated amalgamation, fuzzy amalgamation neutrality and modified fuzzy fairness.*

Proof. Theorem 3 showed that the prox-Banzhaf value satisfies all the proposed axioms. It remains to prove the uniqueness.

Let Ψ^1, Ψ^2 be two different values for games with a proximity relation that satisfy all the axioms. We will prove that for all (N, v, ρ) it holds $\Psi^1(N, v, \rho) = \Psi^2(N, v, \rho)$. If $|N| = 1$ then the equality is always true from the dummy isolated player property. Consider that for all sets of players with cardinality less than n the equality is true and take N with $|N| = n$.

Suppose $|im(\rho)| = 1$. Then $im(\rho) = \{1\}$ and ρ is identified with a cooperation structure L. We will use now an induction on $|L|$. If $L = \emptyset$, namely $\rho = 0$, then the values are equal from Lemma 4. Consider true the equality if $|L| < d$. Let $|L| = d$. There is at least one component $S \in N/L$ with one or more links. If $i, j \in N_k$ with $ij \in L$ then for all $l \notin N_k$ we apply fuzzy amalgamation neutrality (in the crisp version) to get

$$\Psi^1_l(N, v, L) = \Psi^1_l\left(N \setminus \{j\}, v^{ij}, L^{ij}\right) = \Psi^2_l\left(N \setminus \{j\}, v^{ij}, L^{ij}\right) = \Psi^2_l(N, v, L),$$

where the second equality comes because $\left|L^{ij}\right| < d$. Moreover, for every link $ij \in L$ with $i, j \in N_k$, the modified fuzzy fairness axiom (the crisp version (9)) implies

$$\begin{aligned}\Psi^1_i(N, v, L) - \Psi^1_j(N, v, L) &= \Psi^1_i\left(N^i_{ij}, v, L_{N^i_{ij}} \setminus \{ij\}\right) - \Psi^1_j\left(N^j_{ij}, v, L_{N^j_{ij}} \setminus \{ij\}\right)\\ &= \Psi^2_i\left(N^i_{ij}, v, L_{N^i_{ij}} \setminus \{ij\}\right) - \Psi^2_j\left(N^j_{ij}, v, L_{N^j_{ij}} \setminus \{ij\}\right)\\ &= \Psi^2_i(N, v, L) - \Psi^2_j(N, v, L),\end{aligned}$$

where $|L_{N^i_{ij}} \setminus \{ij\}|, |L_{N^j_{ij}} \setminus \{ij\}| < d$. Applying fuzzy amalgamation (the crisp version (10)) we have

$$\begin{aligned}\Psi^1_i(N, v, L) + \Psi^1_j(N, v, L) &= \Psi^1_i\left(N \setminus \{j\}, v^{ij}, L^{ij}\right) = \Psi^2_i\left(N \setminus \{j\}, v^{ij}, L^{ij}\right)\\ &= \Psi^2_i(N, v, L) + \Psi^2_j(N, v, L).\end{aligned}$$

Adding this equality and the previous one we obtain $\Psi^1_i(N, v, L) = \Psi^2_i(N, v, L)$.

Suppose true the equality of the indices when $|im(\rho)| < k$. Let (N, v, ρ) with $|im(\rho)| = k > 1$. Let $i, j \in N$ with $\rho(ij) = t \in (0, 1)$. By fuzzy amalgamation we obtain again

$$\begin{aligned}\Psi^1_i(N, v, \rho) + \Psi^1_j(N, v, \rho) &= t\Psi^1_i\left(N \setminus \{j\}, v^{ij}, \left(\rho^t_0\right)^{ij}\right) + (1-t)\left[\Psi^1_i\left(N, v, \overline{\rho}^t_0\right) + \Psi^1_j\left(N, v, \overline{\rho}^t_0\right)\right]\\ &= t\Psi^2_i\left(N \setminus \{j\}, v^{ij}, \left(\rho^t_0\right)^{ij}\right) + (1-t)\left[\Psi^2_i\left(N, v, \overline{\rho}^t_0\right) + \Psi^2_j\left(N, v, \overline{\rho}^t_0\right)\right]\\ &= \Psi^2_i(N, v, \rho) + \Psi^2_j(N, v, \rho).\end{aligned}$$

because $|N \setminus \{j\}| < n$ and $|im(\overline{\rho}^t_0)| < k$. Also if $r = t - \rho^*(ij)$, where $\rho^*(ij)$ is the level in ρ just before $\rho(ij)$,

$$\begin{aligned}\Psi^1_i(N, v, \rho) - \Psi^1_j(N, v, \rho) &= (1-r)\left[\Psi^1_i\left(N, v, \overline{\rho}^t_{t-r}\right) - \Psi^1_j\left(N, v, \overline{\rho}^t_{t-r}\right)\right]\\ &\quad + r\left[\Psi^1_i\left(N^i_{ij}, v, \left(\left(\rho^t_{t-r}\right)_{-ij}\right)_{N^i_{ij}}\right) - \Psi^1_j\left(N^j_{ij}, v, \left(\left(\rho^t_{t-r}\right)_{-ij}\right)_{N^j_{ij}}\right)\right]\\ &= \Psi^2_i(N, v, \rho) - \Psi^2_j(N, v, \rho),\end{aligned}$$

using modified fuzzy fairness and because $\left|im\left(\left(\left(\rho_{t-r}^t\right)_{-ij}\right)_{N_{ij}^i}\right)\right|, \left|im\left(\overline{\rho}_{t-r}^t\right)\right| < k$. Adding up this equality with the above one we have that

$$\Psi_i^1(N, v, \rho) = \Psi_i^2(N, v, \rho),$$

for all $i \in N$ with another $j \neq i$ such that $\rho(ij) > 0$. Finally, let $l \in N$ such that $\rho(lh) = 0$ for any $h \neq l$. This player l satisfies that $\{l\} \in N/[\rho]_r$ for all $r \in (0, 1]$. Taking i, j with $\rho(ij) = t > 0$ we have $r_{\{i,j,l\}} = 0$. Hence, using fuzzy amalgamation neutrality

$$\begin{aligned}\Psi_l^1(N, v, \rho) &= t\Psi_l^1\left(N \setminus \{j\}, v^{ij}, \left(\rho_0^t\right)^{ij}\right) + (1-t)\Psi_l^1\left(N, v, \overline{\rho}_0^t\right) \\ &= t\Psi_l^2\left(N \setminus \{j\}, v^{ij}, \left(\rho_0^t\right)^{ij}\right) + (1-t)\Psi_l^2\left(N, v, \overline{\rho}_0^t\right) = \Psi_l^2(N, v, \rho),\end{aligned}$$

because $|N \setminus \{j\}| < n$ and $|im(\overline{\rho}_0^t)| < k$. □

Remark. The Banzhaf–Owen value (3) coincides with our prox-Banzhaf one when we use a crisp proximity relation with a graph where all the components are complete. Amer et al. [24] characterized the Banzhaf–Owen value by six axioms. They used dummy player property for all the players (our axiom is weaker), amalgamation for all the players in the same union (equivalent to our fuzzy amalgamation) and amalgamation neutrality (equivalent in this situation to our fuzzy amalgamation neutrality). They used also symmetry within the unions. This axiom is not feasible in our context because of the asymmetry of the players in a proximity relation. Myerson [5] introduced fairness as an axiom substituting symmetry, and later Casajus [9] proposed modified fairness as a modification of fairness for cooperation structures. Hence modified fuzzy fairness is the natural extension of symmetry to these situations. Amer et al. added two more axioms: additivity and many null players. We have reduced the number of axioms to five using amalgamation for isolated players. So, it is not strange to use five axioms, moreover our axiomatization reduced in one the axiomatization of the Banzhaf–Owen value.

6. Application: the power of the political groups in the European Parliament

In this section we illustrate the calculation of the value studied in this work as index and its application. The Treaties of Maastricht (1992) and Lisbon (2009) regulate the functions of the European Parliament in a context of the co-decision procedure with the Council of the European Union. The European Parliament pretends to be the ideologic representation of the European citizens, but currently the channel of voting is the set of national political parties in each member state. Hence, the relations among these groups are partial because of the national and the ideologic interests. Our interest is only to show the application of the model, so we take any legislature. In the seventh legislature there were seven political groups in the European Parliament plus the non-attached seats. So, we consider in our example the following groups corresponding to 2012: 1. European People's Party (Christian Democrats) 265 members, 2. Progressive Alliance of Socialists and Democrats, 183 members. 3. Alliance of Liberals and Democrats for Europe, 84 members. 4. European Conservatives and Reformists, 55 members. 5. Greens/European Free Alliance, 55 members. 6. European United Left – Nordic Green Left, 35 members. 7. Europe of Freedom and Democracy, 29 members. 8. Non-attached Members, 29 members.

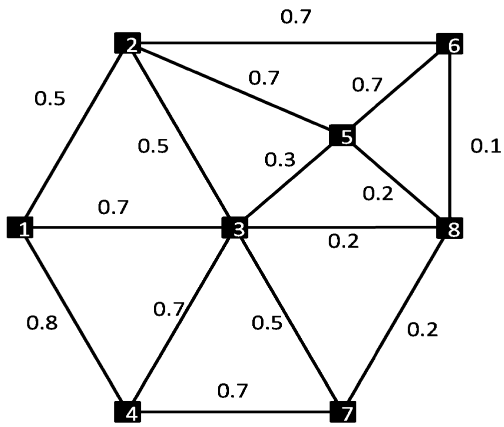
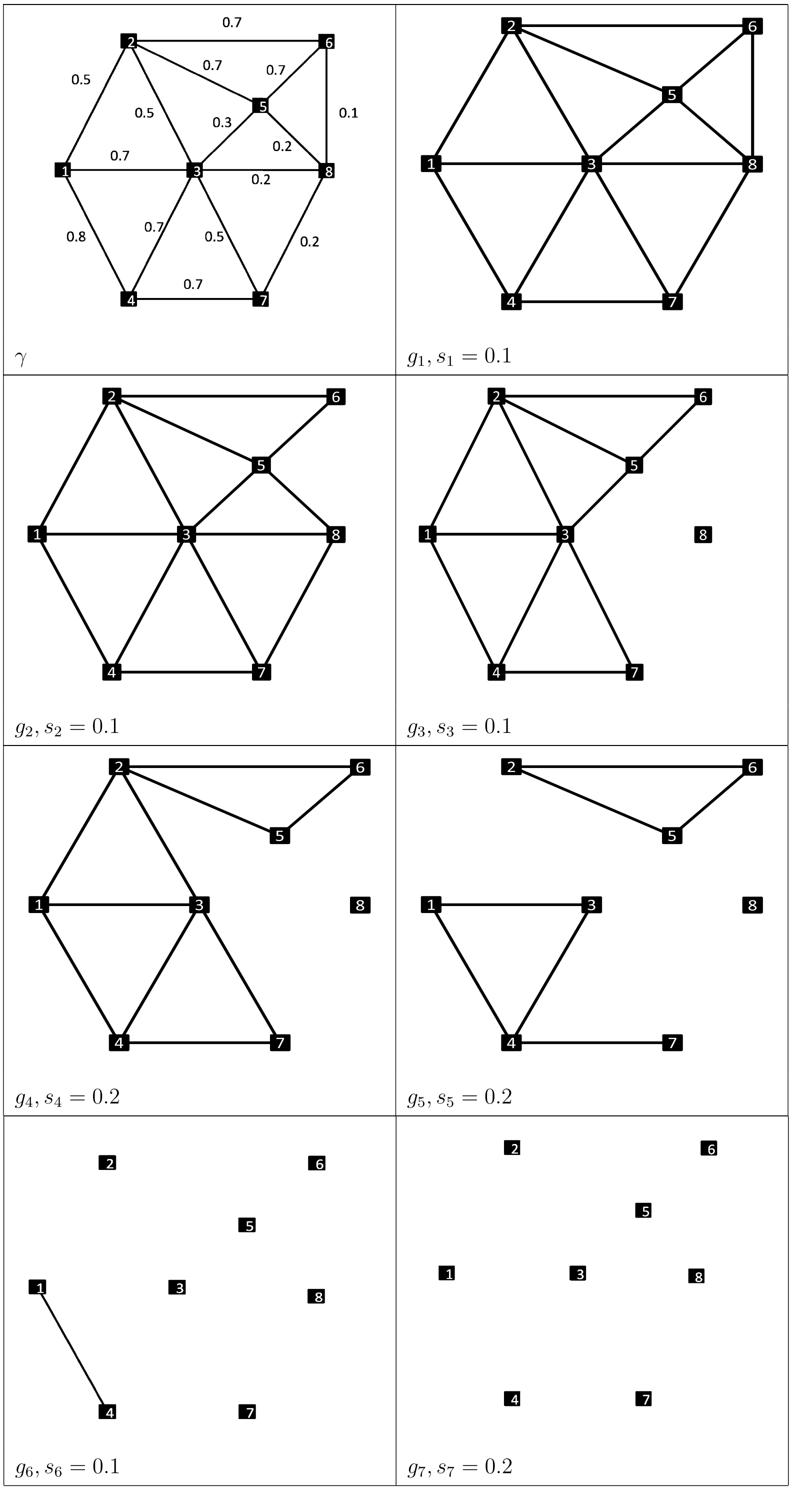


Fig. 4. The EP proximity relation.



We consider the game of the political representation of the groups in the European Parliament in 2012 with 735 seats and a quota of 368. We can represent the voting situation in the committee by a simple game over $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, called the *EP-game*, which is represented by $v(S) = 1$ if the sum of the number of seats of the groups in S is greater or equal to 368, and $v(S) = 0$ otherwise. The fuzzy graph in Fig. 4 is a proximity relation ρ over N, where $\rho(ij)$ is interpreted as the level of coincidence between groups i and j. It can be measured, for instance, by assigning a value in $[0, 1]$ to each

Table 1
Graph Banzhaf–Owen values of the graphs in the cuts of the EP proximity relation.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	0.632813	0.632813	0.640625	0.625	0.5	0.734375	0.734375
2	0.367188	0.367188	0.359375	0.375	0.	0.125	0.265625
3	0.320313	0.320313	0.328125	0.3125	0.3125	0.125	0.234375
4	0.117188	0.117188	0.109375	0.125	0.1875	0.140625	0.140625
5	0.0859375	0.859375	0.078125	0.0625	0.	0.125	0.140625
6	0.0390625	0.0390625	0.03125	0.03125	0.	0.0625	0.078125
7	0.0859375	0.0859375	0.078125	0.09375	0.0625	0.0625	0.078125
8	0.0859375	0.0859375	0.	0.	0.	0.0625	0.078125

Table 2
Comparative of Banzhaf indices (I).

Players	Groups	Votes	$\beta(N, v)$	$\beta^{co}(N, v, g^{\gamma})$	$B(N, v, \gamma)$
1	PPE	265	0.734375	0.632813	0.635938
2	S&D	183	0.265625	0.367188	0.25
3	ADLE	84	0.234375	0.320313	0.28125
4	CRE	55	0.140625	0.117188	0.139063
5	Greens-ALE	55	0.140625	0.085938	0.078125
6	GUE/NGL	35	0.078125	0.039063	0.0390625
7	EDF	29	0.078125	0.085938	0.078125
8	NI	29	0.078125	0.085938	0.0390625

aspect of the ideology, for example, economy, immigration policies, etc., with the condition that the sum of the values of all issues considered is 1. Then, $\rho(ij) = 1$ if both groups have the same ideology in all issues.

The prox-Banzhaf index (our value for simple games) is defined in terms of the Choquet integral and the graph Banzhaf–Owen value. The procedure to compute the index is the following: 1) we get the cuts of the proximity relation, 2) we get the graph Banzhaf–Owen value for the corresponding graph in each level, and 3) we calculate the index using the Choquet integral. Fig. 4 also shows the cuts of the EP proximity relation. Now we have to obtain for each cut β^{co}. We can see in Table 1 the graph Banzhaf–Owen values (Definition 2) of the cuts of the EP proximity relation.

Using the Choquet integral we obtain our index. In Table 2 we compare the index with the classical Banzhaf index and the graph Banzhaf–Owen index. Observe that if we do not consider levels, namely we take the crisp version of the graph, it is connected and the graph Banzhaf–Owen value given in Definition 2 coincides with the graph Banzhaf value. We denote as g^{γ} the crisp version of the EP proximity relation.

We can see how the aggregation of information changes the power of the groups (see Figs. 5 and 6). For instance group 2 has greater power than group 3 with the crisp indices but they exchange their position with the fuzzy index.

7. Conclusions

The Banzhaf value is one of the best methods to determine the power of the different elements in a decision system. The a priori bilateral relations among the agents modify their payoffs but usually these relations are not equal. Proximity relations allow to level the closeness among the agents. We have defined a Banzhaf value using the information of a proximity relation and we have provided the value with an axiomatization. The construction is consistent with the related crisp definitions. Consider $im(\rho) = \{1\}$. In that case ρ is identified with a cooperation structure L and $B(N, v, \rho) = \beta^{co}(N, v, L)$. If L is connected then $B(N, v, \rho) = \eta(N, v, L)$. If for each $S \in N/L$ it holds $L_S = L^S$ (it is complete)

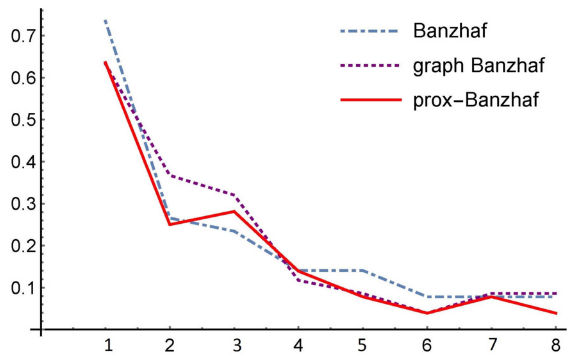


Fig. 5. Comparative of Banzhaf indices (II).

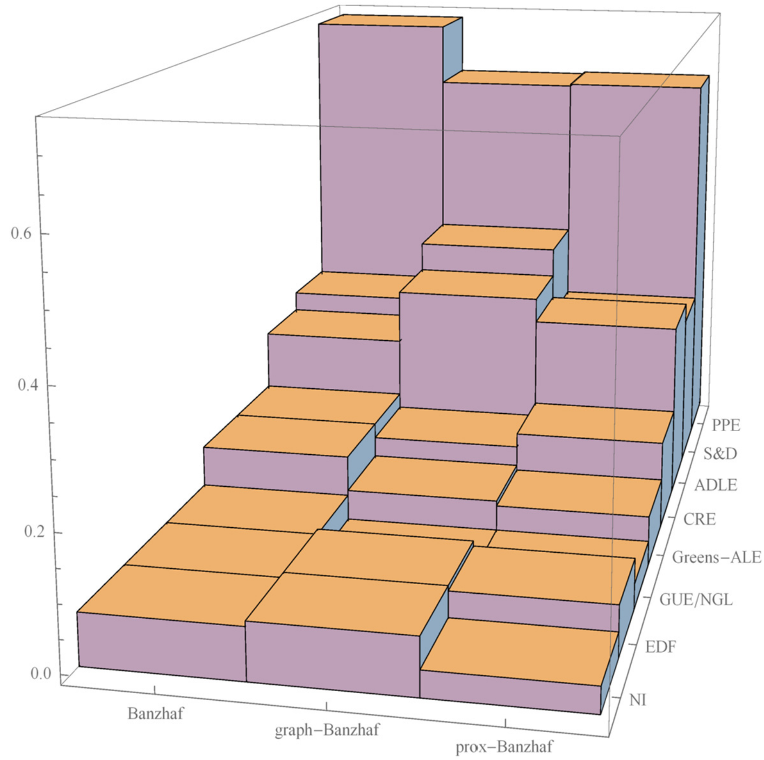


Fig. 6. Comparative of Banzhaf indices (III).

then $B(N, v, \rho) = \beta^{ow}(N, v, N/L)$. If $L = L^N$ then $B(N, v, \rho) = \beta(N, v)$. Finally we have showed the interest of the index through an example.

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