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Biprobabilistic values for bicooperative games

J.M. Bilbao*, J.R. Fernández, N. Jiménez, J.J. López

Escuela Superior de Ingenieros, Matématica Aplicada II, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

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Abstract

The present paper introduces bicooperative games and develops some general values on the vector space of these games. First, we define biprobabilistic values for bicooperative games and observe in detail the axioms that characterize such values. Following the work of Weber [R.J. Weber, Probabilistic values for games, in: A.E. Roth (Ed.), The Shapley Value: Essays in Honor of Lloyd S. Shapley Cambridge University Press, Cambridge, 1988, pp. 101–119], these axioms are sequentially introduced observing the repercussions they have on the value expression. Moreover, compatible-order values are introduced and there is shown the relationship between these values and efficient values such that their components are biprobabilistic values.

Keywords: Bicooperative games; Ternary voting games; Biprobabilistic values; Compatible-order values

1. Introduction

The theory of cooperative games studies situations where a group of people/agents are associated to obtain a profit as a result of their cooperation. Thus, a cooperative game is defined as a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function verifying that $v(\emptyset) = 0$. For each $S \in 2^N$, the worth $v(S)$ can be interpreted as the maximal gain or minimal cost that the players which form coalition S can achieve themselves against the best offensive threat by the complementary coalition $N \setminus S$. Classical market games for economies with private goods are examples of cooperative games. Hence, we can say that a cooperative game has *orthogonal coalitions* (see Myerson [10, Chapter 9]).

Games with non-orthogonal coalitions are games in which the worth of a coalition S depends on the actions of its complementary coalition $N \setminus S$. Clearly, social situations involving externalities and public goods are such cases. For instance, we consider a group of agents with a common good which is causing them expenses or costs. In an external or internal way, a modification (sale, buying, etc.) of this good is proposed to them. This action will suppose a greater profit to them in the case where they all agree with the change proposed about the actual situation of the good. Moreover, even though the patrimonial good can be divisible, we suppose that the greatest value of the selling operation is reached if we consider all the common good.

Situations of this kind may be modeled as follows. We consider ordered pairs (S, T), with $S, T \subseteq N$ and $S \cap T = \emptyset$. Each pair (S, T) yields a partition of N in three groups. Players in S are in favor of the proposal, and players in T object to it. The remaining players in $N \setminus (S \cup T)$ are not convinced of its benefits, but they have no intention of

* Corresponding author. Tel.: +34 954486173.
E-mail address: mbilbao@us.es (J.M. Bilbao).

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objecting to it. We model the above mentioned class of non-orthogonal situations by means of the set of all ordered pairs of disjoint coalitions, that is, the set $3^N = \{(S, T) : S, T \subseteq N, S \cap T = \emptyset\}$ of all *signed coalitions*, and define a function $b : 3^N \to \mathbb{R}$. For each $(S, T) \in 3^N$, the number $b(S, T)$ can be interpreted as the gain (whenever $b(S, T) > 0$) or loss (whenever $b(S, T) < 0$) that S can achieve when T is the opposer coalition and $N \setminus (S \cup T)$ is the neutral coalition. The pair $(\emptyset, N)$ represents the situation if all the players object to the change and $(N, \emptyset)$ represents the situation where all the players wish for the change.

This is the case with multicriteria decision making when underlying scales are bipolar, i.e., a central value exists on each scale and it is considered a neutral value. When building the model, we must then distinguish for a given alternative criterion which have a defender value from those which have a detractor value or a neutral one. This leads us in a natural way to the concept of a bicooperative game introduced in Bilbao [1].

Like for the cooperative case in which each coalition $S \in 2^N$ can be identified with a $\{0, 1\}$-vector, each signed coalition $(S, T) \in 3^N$ can be identified with the $\{-1, 0, 1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, one may imagine that each player can choose one alternative and hence bicooperative games can be seen as a particular case of games with n players and r alternatives (for $r = 3$), introduced by Bolger in [2] and [3]. However, the r possible input alternatives analyzed by Bolger are not ordered and hence the lattice structures of the domains of bicooperative games and games with n players and 3 alternatives are different. For instance, the element $(\emptyset, \emptyset)$ is central in our structure $(3^N, \sqsubseteq)$ and $(0, 0, 0)$ is the least element in $(3^N, \preceq)$, where $\preceq$ is the coordinatewise order. Note that in a bicooperative game, the value 0 is central, and -1, 1 are symmetric extremes. This suggests that bicooperative games are a symmetrization of classical cooperative games and this is the main reason for choosing $b(\emptyset, \emptyset) = 0$. Also it should be noted that bicooperative games *with ordered finite output* are a particular class of the $(3, k)$ hypergraphs introduced by Freixas and Zwicker in [8].

In voting games, each voter has three choices: voting for a proposal, voting against it, and abstaining. Thus, only knowing who is in favor of the proposal is not enough for describing the situation. These games have been studied by Felsenthal and Machover [5] under the name of *ternary voting games*. They generalize the standard voting games by recognizing abstention as an option alongside *yes* and *no* votes. They are formally described by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N, \emptyset) = 1$, $u(\emptyset, N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S, T) \leq u(S', T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as victory, the passing of a bill. In Chua and Huang [4] the Shapley–Shubik index for ternary voting games is considered.

The proposal of Felsenthal and Machover could be refined by introducing a third output for u, which is 0, and represents the 'no decision' situation. More recently, several works by Freixas [6,7] and Freixas and Zwicker [8] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes. These authors have generalized the ternary voting games by the definition of the so-called (j, k) *simple games*. Thus, a bicooperative simple game $b : 3^N \to \{-1, 0, 1\}$ is a $(3, 3)$ simple game such that $b(\emptyset, N) = -1$, $b(\emptyset, \emptyset) = 0$, and $b(N, \emptyset) = 1$.

Let us briefly outline the contents of our work. In the next section, we study some properties and characteristics of the set 3^N, and introduce bicooperative games. The aim of the third section is to analyze the individual valuation of the prospects of the players from their participation in a bicooperative game. Probabilistic values for bicooperative games are defined and a characterization of these values is obtained. Section 4 is devoted to the study of values with the efficiency property. In Section 5 compatible-order values are defined and we prove that they are efficient values such that their components are biprobabilistic values.

2. The formal framework

Let $N = \{1, \ldots, n\}$ be a finite set and let $3^N = \{(A, B) : A, B \subseteq N, A \cap B = \emptyset\}$. Grabisch and Labreuche [9] proposed a relation in 3^N given by

$$(A, B) \sqsubseteq (C, D) \iff A \subseteq C, B \supseteq D.$$

We denote by the symbol $\sqsubset$ the relation defined by means of the strict inclusion, that is, $(A, B) \sqsubset (C, D)$ if and only if $A \subset C$, $B \supset D$. Let us consider the following ordered 3-partitions defined by

$$X = (A, N \setminus (A \cup B), B) \quad \text{and} \quad Y = (C, N \setminus (C \cup D), D).$$

Then $(A, B) \sqsubseteq (C, D) \iff X\,{}^3\!\!\subseteq Y$ and hence the relation $\sqsubseteq$ coincides with the inclusion ${}^3\!\!\subseteq$ for 3-partitions given by Freixas and Zwicker [8, Section 2].

The set $(3^N, \sqsubseteq)$ is a partially ordered set (poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Every pair $\{(A, B), (C, D)\}$ of elements of 3^N has a join

$$(A, B) \vee (C, D) = (A \cup C, B \cap D),$$

and a meet

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

Moreover, $(3^N, \sqsubseteq)$ is a finite distributive lattice. Two pairs (A, B) and (C, D) are *comparable* if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$; otherwise, (A, B) and (C, D) are *incomparable*. A *chain* of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $(3^N, \sqsubseteq)$, all maximal chains have the same number of elements and this number is $2n + 1$. Thus, there can be considered the rank function $\rho : 3^N \to \{0, 1, \ldots, 2n\}$ such that $\rho[(\emptyset, N)] = 0$ and $\rho[(S, T)] = \rho[(A, B)] + 1$ if (S, T) covers (A, B) (i.e., if $(A, B) \sqsubset (S, T)$ and there exists no $(H, J) \in 3^N$ such that $(A, B) \sqsubset (H, J) \sqsubset (S, T)$).

A *join-irreducible* element is an element of a lattice which cannot be represented as a join of elements distinct from itself. For example, the sets

$$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\},$$

ordered by inclusion, form a distributive lattice L. The join-irreducible elements of L are $\emptyset$, $\{1\}$, $\{2\}$ and $\{2, 3\}$. For the distributive lattice 3^N, let P denote the set of all nonzero $\vee$-irreducible elements. Then P is the disjoint union $C_1 + C_2 + \cdots + C_n$ of the chains

$$C_i = \{(\emptyset, N \setminus \{i\}), (i, N \setminus \{i\})\}, \quad 1 \le i \le n.$$

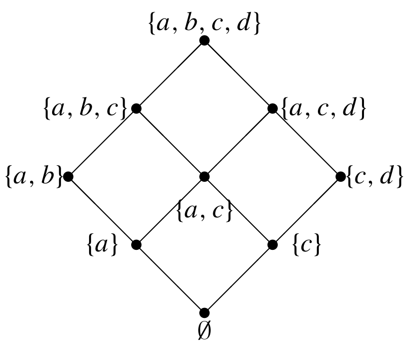
An order ideal of P is a subset I of P such that if $x \in I$ and $y \le x$, then $y \in I$. The set of all order ideals of P, ordered by inclusion, is the distributive lattice $J(P)$, where the lattice operations $\vee$ and $\wedge$ are just the ordinary union and intersection respectively. The fundamental theorem for finite distributive lattices (see [11, Theorem 3.4.1]) states that the map $\varphi : 3^N \to J(P)$ given by

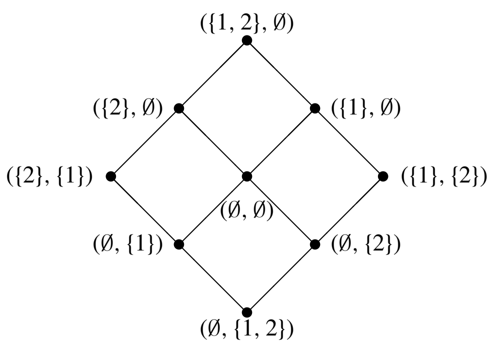
$$(A, B) \overset{\varphi}{\mapsto} \{(X, Y) \in P : (X, Y) \sqsubseteq (A, B)\}$$

is an isomorphism.

Example. Let $N = \{1, 2\}$. Then $P = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\{2\}, \{1\}), (\{1\}, \{2\})\}$ is the disjoint union of the chains $(\emptyset, \{1\}) \sqsubset (\{2\}, \{1\})$ and $(\emptyset, \{2\}) \sqsubset (\{1\}, \{2\})$. We will define $a = (\emptyset, \{1\}), b = (\{2\}, \{1\}), c = (\emptyset, \{2\}), d = (\{1\}, \{2\})$, and hence

$$J(P) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$$





Proposition 1. *The number of maximal chains of the lattice* $(3^N, \sqsubseteq)$ *is* $(2n)!/2^n$.

Proof. The number of maximal chains of 3^N is equal to the number of maximal chains of $J(P)$ and this number is also equal to the number of extensions $e(P)$ of P to a total order (see Stanley [11, Section 3.5]). Since $P = C_1 + \cdots + C_n$, where the chain C_i satisfies $|C_i| = 2$ for $1 \le i \le n$, we can apply the enumeration of lattice paths method from Stanley [11, Example 3.5.4], and obtain

$$e(P) = \binom{2n}{2, \ldots, 2} = \frac{(2n)!}{2^n}. \quad \square$$

Taking into account the above framework, we introduce bicooperative games.

Definition 1. A *bicooperative game* is a pair (N, b), where $N = \{1, \ldots, n\}$ is a finite set of $n \ge 2$ players, $(3^N, \sqsubseteq)$ is a finite distributive lattice and $b : 3^N \to \mathbb{R}$ is a function satisfying $b(\emptyset, \emptyset) = 0$.

The set of all bicooperative games with a fixed set of players N is denoted by $\mathcal{BG}^N$. With respect to addition of games and multiplication of games by real numbers, the set $\mathcal{BG}^N$ is a real vector space. There are three special collections of games in $\mathcal{BG}^N$ taking values in $\{-1, 0, 1\}$: the *identity* games, the *superior unanimity* games and the *inferior unanimity* games which are defined, for any $(S, T) \in 3^N$ such that $(S, T) \neq (\emptyset, \emptyset)$ as follows.

The identity game $\delta_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\delta_{(S,T)}(A, B) = \begin{cases} 1 & \text{if } (A, B) = (S, T), \\ 0 & \text{otherwise.} \end{cases}$$

The superior unanimity game $\overline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\overline{u}_{(S,T)}(A, B) = \begin{cases} 1 & \text{if } (S, T) \sqsubseteq (A, B) \neq (\emptyset, \emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{u}_{(S,T)}(A, B) = \begin{cases} -1 & \text{if } (\emptyset, \emptyset) \neq (A, B) \sqsubseteq (S, T), \\ 0 & \text{otherwise.} \end{cases}$$

The following result is straightforward and therefore the proof will be omitted.

Proposition 2. *The dimension of the vector space* $\mathcal{BG}^N$ *is* $3^n - 1$ *and the sets of games* $\{\delta_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\}$, $\{\overline{u}_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\}$ *and* $\{\underline{u}_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\}$ *are the basis of* $\mathcal{BG}^N$.

3. Biprobabilistic values

A *value on* $\mathcal{BG}^N$ is a mapping $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$ that associates with each game $b \in \mathcal{BG}^N$ a vector $(\Phi_1(b), \ldots, \Phi_n(b)) \in \mathbb{R}^n$, where the real number $\Phi_i(b)$ represents the payoff to player i in the game b. The mapping $\Phi_i : \mathcal{BG}^N \to \mathbb{R}$ is the *value for player* $i \in N$ on $\mathcal{BG}^N$. This value represents an individual assessment for i of his or her expectations from playing bicooperative games. From now on, we will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively.

Definition 2. A value Φ_i for player i on $\mathcal{BG}^N$ is a biprobabilistic value if there exist two collections of real numbers $\{\overline{p}^i_{(S,T)} : (S, T) \in 3^{N\setminus i}\}$ and $\{\underline{p}^i_{(S,T)} : (S, T) \in 3^{N\setminus i}\}$ satisfying $\overline{p}^i_{(S,T)} \ge 0$, $\underline{p}^i_{(S,T)} \ge 0$, $\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} = 1$, $\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} = 1$ such that

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}(b(S \cup i, T) - b(S, T)) + \underline{p}^i_{(S,T)}(b(S, T) - b(S, T \cup i))\right]$$

for every game $b \in \mathcal{BG}^N$.

Observe that in a biprobabilistic value Φ_i, player i estimates his/her participation in the game evaluating his/her marginal contributions $b(S\cup i, T) - b(S, T)$, whenever i joins coalition $S \subseteq N \setminus i$, and his/her marginal contributions $b(S, T) - b(S, T \cup i)$, whenever i leaves coalition $T \cup i$, where $T \subseteq N \setminus i$. If $\overline{p}^i_{(S,T)}$ is the subjective probability that player i joins S and $\underline{p}^i_{(S,T)}$ is the subjective probability that player i leaves $T \cup i$, then $\Phi_i(b)$ is his/her expected payoff for player i in the game b.

We will follow the work of Weber [12] to obtain an axiomatic development of biprobabilistic values for bicooperative games. First, we consider the linearity property.

Linearity axiom: Φ_i satisfies $\Phi_i(\alpha b + \beta w) = \alpha\Phi_i(b) + \beta\Phi_i(w)$, for all $\alpha, \beta \in \mathbb{R}$, and $b, w \in \mathcal{BG}^N$.

Theorem 3. *Let Φ_i be a value for player i on $\mathcal{BG}^N$ which satisfies the linearity axiom. Then there is a unique set of real numbers $\{a^i_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\}$ such that*

$$\Phi_i(b) = \sum_{\{(S,T)\in 3^N:(S,T)\neq(\emptyset,\emptyset)\}} a^i_{(S,T)} b(S, T),$$

for every $b \in \mathcal{BG}^N$.

Proof. The collection of identity games is a basis of $\mathcal{BG}^N$, and each game $b \in \mathcal{BG}^N$ can be written as

$$b = \sum_{\{(S,T)\in 3^N:(S,T)\neq(\emptyset,\emptyset)\}} b(S, T)\delta_{(S,T)}.$$

By the linearity axiom

$$\Phi_i(b) = \sum_{\{(S,T)\in 3^N:(S,T)\neq(\emptyset,\emptyset)\}} \Phi_i(\delta_{(S,T)})b(S, T).$$

Finally, we may write $a^i_{(S,T)} = \Phi_i(\delta_{(S,T)})$ for all $(S, T) \neq (\emptyset, \emptyset)$. □

Next we introduce the concept of *dummy player*, understanding that player i is a dummy player when his/her contributions to signed coalitions $(S\cup i, T)$ formed with his/her incorporation to S and his/her contributions to signed coalitions (S, T) formed with his/her desertion of $T \cup i$ coincide exactly with his/her individual contributions.

Definition 3. A player $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ if, for every $(S, T) \in 3^{N\setminus i}$, it holds that

$$b(S \cup i, T) - b(S, T) = b(\{i\}, \emptyset) \quad \text{and} \quad b(S, T) - b(S, T \cup i) = -b(\emptyset, \{i\}).$$

Note that if i is a dummy player in $b \in \mathcal{BG}^N$, then

$$b(S \cup i, T) - b(S, T \cup i) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}), \quad \text{for all } (S, T) \in 3^{N\setminus i}.$$

Next, a specific game is defined and some properties for dummy players in certain games are given. These properties are a direct consequence of Definition 3 and will be used in the proof of Theorem 5.

Let $i \in N$ and $(A, B) \in 3^{N\setminus i}$. The game $w^i_{(A,B)} : 3^N \to \mathbb{R}$ is defined as follows:

$$w^i_{(A,B)}(S, T) = \begin{cases} w^i_{(A,B)}(S \setminus i, T) & \text{if } i \in S, \\ w^i_{(A,B)}(S, T \setminus i) & \text{if } i \in T, \\ 1 & \text{if } i \notin S \cup T, (\emptyset, \emptyset) \neq (S, T) \sqsubseteq (A, B), \\ 0 & \text{otherwise,} \end{cases}$$

for $(S, T) \in 3^N$.

Proposition 4. *For all $i \in N$, it holds that:*

(1) *Player i is dummy in the superior unanimity game $\overline{u}_{(\{i\},N\setminus i)}$.*
(2) *Player i is dummy in the inferior unanimity game $\underline{u}_{(N\setminus i,\{i\})}$.*
(3) *Player i is dummy in $w^i_{(A,B)}$ for every $(A, B) \in 3^{N\setminus i}$.*

Player i is a dummy in $b \in \mathcal{BG}^N$ if he/she has no meaningful strategic role in the game, since his/her contributions to the coalitions formed with his/her incorporation or desertion coincide. Therefore, the value that this player should expect in the game b must exactly be the sum of his/her marginal contributions. This consideration justifies the introduction of the following axiom.

Dummy axiom: If $i \in N$ is dummy in $b \in \mathcal{BG}^N$, then $\Phi_i(b) = b(\{i\}, \emptyset) - b(\emptyset, \{i\})$.

In the following result, we can observe that if we add the dummy axiom to the linearity axiom, then the value for player i can be expressed as a linear combination of his/her marginal contributions.

Theorem 5. *Let Φ_i be a value for player i on $\mathcal{BG}^N$ which satisfies the linearity and dummy axioms. Then there exist two collections of real numbers $\{\overline{p}^i_{(S,T)} : (S,T) \in 3^{N\setminus i}\}$ and $\{\underline{p}^i_{(S,T)} : (S,T) \in 3^{N\setminus i}\}$ such that for any $b \in \mathcal{BG}^N$,*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}(b(S\cup i, T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S, T\cup i))\right],$$

where $\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} = 1$ and $\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} = 1$.

Proof. Let $i \in N$. Then,

$$\begin{aligned}
\Phi_i(b) &= \sum_{(S,T)\in 3^N} a^i_{(S,T)} b(S,T) \\
&= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N : i\in S\}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N : i\in T\}} a^i_{(S,T)} b(S,T) \\
&= \sum_{\{(S,T)\in 3^{N\setminus i} : (S,T)\neq(\emptyset,\emptyset)\}} a^i_{(S,T)} b(S,T) + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} b(S\cup i, T) + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T\cup i)} b(S, T\cup i) \\
&= \sum_{\{(S,T)\in 3^{N\setminus i} : (S,T)\neq(\emptyset,\emptyset)\}} \left[a^i_{(S,T)} b(S,T) + a^i_{(S\cup i,T)} b(S\cup i, T) + a^i_{(S,T\cup i)} b(S, T\cup i)\right] \\
&\quad + a^i_{(\{i\},\emptyset)} b(\{i\},\emptyset) + a^i_{(\emptyset,\{i\})} b(\emptyset,\{i\}).
\end{aligned}$$

Let us consider the collection of games $w^i_{(A,B)} : 3^N \to \mathbb{R}$, with $(A,B) \in 3^{N\setminus i}$. Since player i is a dummy in $w^i_{(A,B)}$ for each $(A,B) \in 3^{N\setminus i}$ it follows that $\Phi_i(w^i_{(A,B)}) = 0$ by the dummy axiom. If we apply the above equality to $w^i_{(A,B)}$ we get

$$\sum_{\{(S,T)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B)\}} (a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}) = 0.$$

We prove by induction on $\rho[(S,T)]$ (the rank of the signed coalitions) that for all $(S,T) \in 3^{N\setminus i}$ with $(S,T) \neq (\emptyset,\emptyset)$, it holds that $a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0$. First note that the first element in $(3^{N\setminus i}, \sqsubseteq)$ is $(\emptyset, N\setminus i)$ so that $\rho[(\emptyset, N\setminus i)] = 0$. We compute

$$\sum_{\{(S,T)\in 3^{N\setminus i} : (S,T)\sqsubseteq(\emptyset,N\setminus i)\}} (a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}) = a^i_{(\emptyset,N\setminus i)} + a^i_{(\{i\},N\setminus i)} + a^i_{(\emptyset,N)} = 0.$$

Now assume the property for $(H,J) \in 3^{N\setminus i}$ with $\rho[(H,J)] \leq k-1$ and suppose that $(S,T) \in 3^{N\setminus i}$ has $\rho[(S,T)] = k$. Then

$$\begin{aligned}
\Phi_i(w^i_{(S,T)}) &= \sum_{\{(H,J)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(H,J)\sqsubseteq(S,T)\}} (a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} + \sum_{\{(H,J)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(H,J)\sqsubset(S,T)\}} (a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0,
\end{aligned}$$

where the second equality follows from the induction hypothesis, and the third from the dummy axiom. Now, for each $(S,T) \in 3^{N\setminus i}$ define

$$\overline{p}^i_{(\emptyset,\emptyset)} = a^i_{(\{i\},\emptyset)},\ \underline{p}^i_{(\emptyset,\emptyset)} = -a^i_{(\emptyset,\{i\})},\ \overline{p}^i_{(S,T)} = a^i_{(S\cup i,T)},\ \underline{p}^i_{(S,T)} = -a^i_{(S,T\cup i)}.$$

Then we compute

$$\begin{aligned}\Phi_i(b) &= \sum_{(S,T)\in 3^{N\setminus i}} \left[(\underline{p}^i_{(S,T)} - \overline{p}^i_{(S,T)})b(S,T) + \overline{p}^i_{(S,T)}b(S\cup i,T) - \underline{p}^i_{(S,T)}b(S,T\cup i)\right] \\ &= \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}(b(S\cup i,T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S,T\cup i))\right].\end{aligned}$$

By Proposition 4 we have that player i is dummy in game $\overline{u}_{(\{i\},N\setminus i)}$, and hence

$$\begin{aligned}\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} = \sum_{\{(S,T)\in 3^N : i\in S\}} a^i_{(S,T)} \\ &= \sum_{\{(S,T)\in 3^N : i\in S\}} \Phi_i(\delta_{(S,T)}) = \Phi_i\left(\sum_{\{(S,T)\in 3^N : i\in S\}} \delta_{(S,T)}\right) \\ &= \Phi_i(\overline{u}_{(\{i\},N\setminus i)}) = \overline{u}_{(\{i\},N\setminus i)}(\{i\},\emptyset) - \overline{u}_{(\{i\},N\setminus i)}(\emptyset,\{i\}) = 1.\end{aligned}$$

Since player i is dummy in game $\underline{u}_{(N\setminus i,\{i\})}$, we obtain

$$\begin{aligned}\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} -a^i_{(S,T\cup i)} = \sum_{\{(S,T)\in 3^N : i\in T\}} -a^i_{(S,T)} \\ &= \sum_{\{(S,T)\in 3^N : i\in T\}} -\Phi_i(\delta_{(S,T)}) = \Phi_i\left(\sum_{\{(S,T)\in 3^N : i\in T\}} -\delta_{(S,T)}\right) \\ &= \Phi_i(\underline{u}_{(N\setminus i,\{i\})}) = \underline{u}_{(N\setminus i,\{i\})}(\{i\},\emptyset) - \underline{u}_{(N\setminus i,\{i\})}(\emptyset,\{i\}) = 1. \quad \square\end{aligned}$$

Definition 4. The game $b \in \mathcal{BG}^N$ is *monotonic* if

$$(A,B) \sqsubseteq (C,D) \Rightarrow b(A,B) \leq b(C,D).$$

Monotonicity axiom. If $b \in \mathcal{BG}^N$ is monotonic then $\Phi_i(b) \geq 0$.

If we introduce this new axiom in the hypothesis of the above theorem, we can prove that the coefficients $\underline{p}^i_{(S,T)}$ and $\overline{p}^i_{(S,T)}$ are non-negative.

Theorem 6. *Let Φ_i be a value for player i on $\mathcal{BG}^N$ defined by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}(b(S\cup i,T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S,T\cup i))\right],$$

for every game $b \in \mathcal{BG}^N$. If Φ_i satisfies the monotonicity axiom then $\overline{p}^i_{(S,T)} \geq 0$ and $\underline{p}^i_{(S,T)} \geq 0$ for all $(S,T) \in 3^{N\setminus i}$.

Proof. For $(S,T) \in 3^{N\setminus i}$ with $|S| \geq |T|$, consider the game $\overline{\zeta}^i_{(S,T)} : 3^N \to \mathbb{R}$ given by

$$\overline{\zeta}^i_{(S,T)}(A,B) = \begin{cases} \overline{\zeta}^i_{(S,T)}(A, B\setminus i) & \text{if } i \in B, \\ 1 & \text{if } (S,T) \sqsubset (A,B), \\ 0 & \text{otherwise,} \end{cases}$$

and for every $(S,T) \in 3^{N\setminus i}$ with $|S| < |T|$, the game $\overline{\zeta}^i_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\overline{\zeta}^i_{(S,T)}(A,B) = \begin{cases} \overline{\zeta}^i_{(S,T)}(A\setminus i, B) & \text{if } i \in A,\ (A\setminus i, B) \neq (S,T), \\ -1 & \text{if } (S,T) \sqsupset (A,B), \\ 0 & \text{otherwise.} \end{cases}$$

The game $\overline{\zeta}^i_{(S,T)}$ is monotonic, and hence $\Phi_i(\overline{\zeta}^i_{(S,T)}) \geq 0$. We easily find that $\Phi_i(\overline{\zeta}^i_{(S,T)}) = \overline{p}^i_{(S,T)}$, and then $\overline{p}^i_{(S,T)} \geq 0$ for every $(S,T) \in 3^{N\setminus i}$. Similarly, for each $(S,T) \in 3^{N\setminus i}$ with $|S| > |T|$, we consider the game $\underline{\zeta}^i_{(S,T)} : 3^N \to \mathbb{R}$ given by

$$\underline{\zeta}^i_{(S,T)}(A,B) = \begin{cases} \underline{\zeta}^i_{(S,T)}(A, B\setminus i) & \text{if } i \in B, (A, B\setminus i) \neq (S,T), \\ 1 & \text{if } (S,T) \sqsubseteq (A,B), \\ 0 & \text{otherwise,} \end{cases}$$

and for every $(S,T) \in 3^{N\setminus i}$ with $|S| \leq |T|$, the game $\underline{\zeta}^i_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{\zeta}^i_{(S,T)}(A,B) = \begin{cases} \underline{\zeta}^i_{(S,T)}(A\setminus i, B) & \text{if } i \in A, \\ -1 & \text{if } (S,T) \sqsupset (A,B), \\ 0 & \text{otherwise.} \end{cases}$$

The game $\underline{\zeta}^i_{(S,T)}$ is monotonic, and hence $\Phi_i(\underline{\zeta}^i_{(S,T)}) \geq 0$. Since $\Phi_i(\underline{\zeta}^i_{(S,T)}) = \underline{p}^i_{(S,T)}$, we obtain $\underline{p}^i_{(S,T)} \geq 0$ for every $(S,T) \in 3^{N\setminus i}$. $\square$

It is easy to check that every biprobabilistic value satisfies linearity, dummy and monotonicity axioms. Therefore, we obtain the following characterization of biprobabilistic values from the combination of the above results.

Theorem 7. *Let Φ_i be a value for player i on $\mathcal{BG}^N$. The value Φ_i is a biprobabilistic value if and only if Φ_i satisfies the linearity, dummy and monotonicity axioms.*

4. Efficient values

In a cooperative game $v : 2^N \to \mathbb{R}$, it is assumed that all players decide to cooperate among themselves and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them. In this case, a value φ is efficient if

$$\sum_{i\in N} \varphi_i(v) = v(N).$$

In this section, we study the class of values $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$ that provide an equitable distribution of the total saving among the players. Since in a bicooperative game $b : 3^N \to \mathbb{R}$, the amount $b(\emptyset, N)$ is the cost (or expense) incurred when all the players object to a proposal and $b(N,\emptyset)$ is the gain obtained when all players are in its favor, then the net profit is given by $b(N,\emptyset) - b(\emptyset, N)$. Note that for monotonic games the cost $b(\emptyset, N) \leq 0$ and hence $b(N,\emptyset) - b(\emptyset,N) \geq b(N,\emptyset)$. From this perspective, an efficient value must satisfy the following axiom.

Efficiency axiom. Let $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$ be a value. For every $b \in \mathcal{BG}^N$, it holds that

$$\sum_{i\in N} \Phi_i(b) = b(N,\emptyset) - b(\emptyset, N).$$

The following theorem characterizes the values which are efficient.

Theorem 8. *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a value on $\mathcal{BG}^N$, defined by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}(b(S\cup i, T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S, T\cup i))\right],$$

for every game b and for all $i \in N$. Then Φ satisfies the efficiency axiom if and only if it holds that

$$\sum_{i\in N} \overline{p}^i_{(N\setminus i,\emptyset)} = 1,$$

$$\sum_{i\in N} \underline{p}^i_{(\emptyset,N\setminus i)} = 1,$$

$$\sum_{i\in S} \overline{p}^i_{(S\setminus i,T)} - \sum_{i\in T} \underline{p}^i_{(S,T\setminus i)} = \sum_{i\notin S\cup T} \left(\overline{p}^i_{(S,T)} - \underline{p}^i_{(S,T)}\right),$$

for all $(S,T) \in 3^N$, with $(S,T) \notin \{(\emptyset,\emptyset), (\emptyset, N), (N,\emptyset)\}$.

Proof. Let $b \in \mathcal{BG}^N$. Then

$$\begin{aligned}
\sum_{i\in N}\Phi_i(b) &= \sum_{i\in N}\sum_{(S,T)\in 3^{N\setminus i}}\left[\overline{p}^i_{(S,T)}(b(S\cup i,T)-b(S,T))+\underline{p}^i_{(S,T)}(b(S,T)-b(S,T\cup i))\right]\\
&= \sum_{i\in N}\sum_{(S,T)\in 3^{N\setminus i}}\left[\overline{p}^i_{(S,T)}b(S\cup i,T)-\underline{p}^i_{(S,T)}b(S,T\cup i)+\left(\underline{p}^i_{(S,T)}-\overline{p}^i_{(S,T)}\right)b(S,T)\right]\\
&= \sum_{(S,T)\in 3^N} b(S,T)\left[\sum_{i\in S}\overline{p}^i_{(S\setminus i,T)}-\sum_{i\in T}\underline{p}^i_{(S,T\setminus i)}+\sum_{i\notin S\cup T}\left(\underline{p}^i_{(S,T)}-\overline{p}^i_{(S,T)}\right)\right]\\
&= \sum_{(S,T)\notin\{(\emptyset,N),(N,\emptyset)\}} b(S,T)\left[\sum_{i\in S}\overline{p}^i_{(S\setminus i,T)}-\sum_{i\in T}\underline{p}^i_{(S,T\setminus i)}+\sum_{i\notin S\cup T}\left(\underline{p}^i_{(S,T)}-\overline{p}^i_{(S,T)}\right)\right]\\
&\quad + b(N,\emptyset)\sum_{i\in N}\overline{p}^i_{(N\setminus i,\emptyset)} - b(\emptyset,N)\sum_{i\in N}\underline{p}^i_{(\emptyset,N\setminus i)}.
\end{aligned}$$

If the coefficients satisfy the relations of the theorem, then Φ satisfies the efficiency axiom.

Conversely, fix $(S,T)\in 3^N$ such that $(S,T)\neq(\emptyset,\emptyset)$. Applying the above equality to the identity game $\delta_{(S,T)}$, we have

$$\sum_{i\in N}\Phi_i(\delta_{(S,T)}) = \begin{cases}\sum_{i\in N}\overline{p}^i_{(N\setminus i,\emptyset)} & \text{if } (S,T)=(N,\emptyset),\\ -\sum_{i\in N}\underline{p}^i_{(\emptyset,N\setminus i)} & \text{if } (S,T)=(\emptyset,N),\end{cases}$$

and

$$\sum_{i\in N}\Phi_i(\delta_{(S,T)}) = \sum_{i\in S}\overline{p}^i_{(S\setminus i,T)}-\sum_{i\in T}\underline{p}^i_{(S,T\setminus i)}+\sum_{i\notin S\cup T}\left(\underline{p}^i_{(S,T)}-\overline{p}^i_{(S,T)}\right),$$

otherwise. Thus, if Φ satisfies the efficiency axiom, the relations for the coefficients are true. □

A particular case of an efficient value whose i-th component satisfies the linearity, dummy and monotonicity axioms is the value $\Phi_i(b)$ defined, for $b\in\mathcal{BG}^N$, as

$$\Phi_i(b) = \sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}[b(S\cup i, N\setminus(S\cup i)) - b(S, N\setminus S)].$$

Note that, for any bicooperative game $b\in\mathcal{BG}^N$, this value is the Shapley value corresponding to the cooperative game (N,v), where $v: 2^N\to\mathbb{R}$ is defined by $v(A)=b(A,N\setminus A)$ if $A\neq\emptyset$, and $v(\emptyset)=0$. This value is not satisfactory for any bicooperative game in the sense that it only considers the contributions to signed coalitions in which all players take part. Moreover, there are an infinity of different bicooperative games which give rise to the same cooperative game.

5. Compatible-order values

We now consider values which result from a common perception for all players. It is assumed that all of them estimate that $(N,\emptyset)$ is formed as a sequential process where in each step a different player is incorporated into the first coalition or a different player leaves the second one. These sequential processes are obtained considering the different chains from $(\emptyset,N)$ to $(N,\emptyset)$. In each one of these processes, a player can evaluate his/her contribution when he/she is incorporated to a coalition S or his/her contribution when he/she leaves a coalition T. This can be reflected in the vectors of $\mathbb{R}^n$ called *superior marginal worth vectors* and *inferior marginal worth vectors*. With the aim of formalizing this idea, we introduce the following notation.

Given $N=\{1,\dots,n\}$, let $\overline{N}=\{-n,\dots,-1,1,\dots,n\}$. For each $(S,T)\in 3^N$ we define the set $\overline{(S,T)}=S\cup\{-i : i\in N\setminus T\}\subseteq\overline{N}$. Note that this correspondence is *one to one*. For instance, $\overline{(\emptyset,N)}=\emptyset$ and $\overline{(N,\emptyset)}=\overline{N}$. Since $S\cap T=\emptyset \Leftrightarrow S\subseteq N\setminus T$ we see that $i\in\overline{(S,T)}$ and $i>0$ imply $-i\in\overline{(S,T)}$.

In the lattice $(3^N, \sqsubseteq)$, we consider the set of all maximal chains going from $(\emptyset, N)$ to $(N, \emptyset)$ and denote this set by $\Theta(3^N)$. Let $\theta \in \Theta(3^N)$ be the maximal chain

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

and we obtain the associated chain of sets

$$\emptyset \subset \{i_1\} \subset \cdots \subset \{i_1, \ldots, i_j\} \subset \cdots \subset \{i_1, \ldots, i_{2n-1}\} \subset \overline{N}$$

where $\{i_1, \ldots, i_j\} = \overline{(S_j, T_j)}$ for $j = 1, \ldots, 2n$. We define the vector $\theta(i_j) = (i_1, \ldots, i_j)$, where the last component $i_j \in \overline{N}$ satisfies the following property: if $i_j > 0$ then player $i_j \in S_j$ and $i_j \notin S_{j-1}$, that is, i_j is the last player who joins S_j and if $i_j < 0$, then player $-i_j \notin T_j$ and $-i_j \in T_{j-1}$, that is, $-i_j$ is the last player who leaves T_{j-1}. Equivalently, the elements in $\theta(i_j) = (i_1, i_2, \ldots, i_j)$ are written following the order of incorporation or desertion in the chain θ (depending on the sign of each i_k). Therefore, we obtain an equivalence between maximal chains and vectors $\theta = (i_1, \ldots, i_{2n})$. For example, let $N = \{1, 2, 3\}$ and let θ be the maximal chain

$$(\emptyset, N) \sqsubset (\emptyset, \{1, 3\}) \sqsubset (\{2\}, \{1, 3\}) \sqsubset (\{2\}, \{1\}) \sqsubset (\{2\}, \emptyset) \sqsubset (\{2, 3\}, \emptyset) \sqsubset (N, \emptyset).$$

Its associated chain of sets is given by

$$\emptyset \subset \{-2\} \subset \{-2, 2\} \subset \{-2, 2, -3\} \subset \{-2, 2, -3, -1\} \subset \{-2, 2, -3, -1, 3\} \subset \overline{N}.$$

Thus, we can represent the maximal chain by the vector $\theta = (-2, 2, -3, -1, 3, 1)$. If $\theta(i_j) = (i_1, \ldots, i_j)$ we define $\alpha[\theta(i_j)] = (S_j, T_j)$ such that $\overline{(S_j, T_j)} = \{i_1, \ldots, i_j\}$. In particular, $\alpha[\theta(i_{2n})] = (N, \emptyset)$ and $\alpha[\theta(i_1) \setminus i_1] = (\emptyset, N)$.

Definition 5. Let $\theta \in \Theta(3^N)$ and $b \in \mathcal{BG}^N$. The vectors $m^\theta(b), M^\theta(b) \in \mathbb{R}^n$, with

$$m_i^\theta(b) = b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]), \qquad M_i^\theta(b) = b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i]),$$

for $i \in N$, are the inferior and superior marginal worth vectors with respect to θ, respectively. The marginal worth vector with respect to θ is given by $a^\theta(b) \in \mathbb{R}^n$ where

$$a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b), \quad \text{for all } i \in N.$$

Proposition 9. *For any $b \in \mathcal{BG}^N$ and $\theta \in \Theta(3^N)$ it holds that*

$$\sum_{i \in N} a_i^\theta(b) = b(N, \emptyset) - b(\emptyset, N).$$

Proof. Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta(3^N)$. Then

$$\begin{aligned}\sum_{i \in N} a_i^\theta(b) &= \sum_{i \in N} b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]) + b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i]) \\ &= \sum_{j=1}^{2n} [b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j) \setminus i_j])] = b(N, \emptyset) - b(\emptyset, N). \quad \square\end{aligned}$$

Definition 6. A *compatible-order* value on $\mathcal{BG}^N$ is a value $\Psi = (\Psi_1, \ldots, \Psi_n)$ such that there exists a collection $\{p_\theta : \theta \in \Theta(3^N)\}$ satisfying $p_\theta \geq 0$, $\sum_{\theta \in \Theta(3^N)} p_\theta = 1$ and

$$\Psi_i(b) = \sum_{\theta \in \Theta(3^N)} p_\theta a_i^\theta(b),$$

for all $i \in N$ and all $b \in \mathcal{BG}^N$.

A compatible-order value is a value where each player evaluates his/her marginal contributions in the processes of formation of $(N, \emptyset)$ with a common perception of the probability of these processes. The relation between the compatible-order values and the values that satisfy the efficiency axiom is stated in the following theorems.

Theorem 10. *Let $\Psi = (\Psi_1, \ldots, \Psi_n)$ be a compatible-order value on $\mathcal{BG}^N$. Then Ψ satisfies the efficiency axiom and each component of Ψ is a biprobabilistic value.*

Proof. Let $\{p_\theta : \theta \in \Theta(3^N)\}$ be the collection of coefficients such that

$$\begin{aligned}\Psi_i(b) &= \sum_{\theta \in \Theta(3^N)} p_\theta a_i^\theta(b) = \sum_{\theta \in \Theta(3^N)} p_\theta [m_i^\theta(b) + M_i^\theta(b)] \\ &= \sum_{\theta \in \Theta(3^N)} p_\theta [b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i])] + \sum_{\theta \in \Theta(3^N)} p_\theta [b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i])],\end{aligned}$$

for all $i \in N$ and all $b \in \mathcal{BG}^N$. If θ runs over all maximal chains in $\Theta(3^N)$, the sets $\alpha[\theta(i) \setminus i]$ determine all signed coalitions $(S, T) \in 3^{N \setminus i}$ in which i is incorporated in the order and the sets $\alpha[\theta(-i)]$ determine all signed coalitions $(S, T) \in 3^{N \setminus i}$ in which player i has just left the preceding signed coalition in the order. Thus, the above expression can be written as

$$\begin{aligned}\Psi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \Bigg[&\left(\sum_{\{\theta \in \Theta(3^N) : \alpha[\theta(i) \setminus i] = (S,T)\}} p_\theta \right) [b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i])] \\ &+ \left(\sum_{\{\theta \in \Theta(3^N) : \alpha[\theta(-i)] = (S,T)\}} p_\theta \right) [b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i])] \Bigg].\end{aligned}$$

Now for each $(S, T) \in 3^{N \setminus i}$ define

$$\overline{p}^i_{(S,T)} = \sum_{\{\theta \in \Theta(3^N) : \alpha[\theta(i) \setminus i] = (S,T)\}} p_\theta,$$
$$\underline{p}^i_{(S,T)} = \sum_{\{\theta \in \Theta(3^N) : \alpha[\theta(-i)] = (S,T)\}} p_\theta.$$

Then $\overline{p}^i_{(S,T)} \geq 0$ and $\underline{p}^i_{(S,T)} \geq 0$. We claim that

$$\sum_{(S,T) \in 3^{N \setminus i}} \overline{p}^i_{(S,T)} = 1 \quad \text{and} \quad \sum_{(S,T) \in 3^{N \setminus i}} \underline{p}^i_{(S,T)} = 1.$$

To prove the claim, fix i and select all $(S, T) \in 3^{N \setminus i}$ in which player i joins S. Thus we must obtain all chains $\theta \in \Theta(3^N)$ and hence

$$\sum_{(S,T) \in 3^{N \setminus i}} \left(\sum_{\{\theta \in \Theta(3^N) : \alpha[\theta(i) \setminus i] = (S,T)\}} p_\theta \right) = \sum_{\theta \in \Theta(3^N)} p_\theta = 1.$$

Similarly, we show that $\sum_{(S,T) \in 3^{N \setminus i}} \underline{p}^i_{(S,T)} = 1$. It follows that Ψ_i is a biprobabilistic value for all $i \in N$. Next, we show that Ψ is efficient. For every $b \in \mathcal{BG}^N$ we have

$$\begin{aligned}\sum_{i \in N} \Psi_i(b) &= \sum_{i \in N} \sum_{\theta \in \Theta(3^N)} p_\theta a_i^\theta(b) = \sum_{\theta \in \Theta(3^N)} p_\theta \left[\sum_{i \in N} (m_i^\theta(b) + M_i^\theta(b)) \right] \\ &= \sum_{\theta \in \Theta(3^N)} p_\theta [b(N, \emptyset) - b(\emptyset, N)] = b(N, \emptyset) - b(\emptyset, N). \quad \square\end{aligned}$$

Theorem 11. *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a value on $\mathcal{BG}^N$ that satisfies the efficiency axiom and such that each component of Φ is a biprobabilistic value. Then Φ is a compatible-order value.*

Proof. By hypothesis, for all $b \in \mathcal{BG}^N$ and all $i \in N$, we have

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}^i_{(S,T)} (b(S \cup i, T) - b(S, T)) + \underline{p}^i_{(S,T)} (b(S, T) - b(S, T \cup i)) \right].$$

For each $(S, T) \in 3^N$, define

$$A(S, T) = \sum_{j \notin S \cup T} \overline{p}^{j}_{(S,T)} + \sum_{j \in T} \underline{p}^{j}_{(S,T\setminus j)},$$

that is, $A(S, T)$ is the sum of the probabilities of all players that can join S in (S, T) and the probabilities of all players that can leave T in (S, T). For $i \in N$ and $(S, T) \in 3^{N\setminus i}$, define

$$A(i, (S, T)) = \begin{cases} \dfrac{\overline{p}^{i}_{(S,T)}}{A(S,T)} & \text{if } A(S,T) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for all $(S, T) \in 3^N$ such that $i \in T$, define

$$A(-i, (S, T)) = \begin{cases} \dfrac{\underline{p}^{i}_{(S,T\setminus i)}}{A(S,T)} & \text{if } A(S,T) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A(i, (S, T))$ is the quotient between the assigned probability for player i from his/her union to S in (S, T) and the sum of probabilities $A(S, T)$. In $A(-i, (S, T))$ we consider the quotient between the assigned probability for player i from his/her desertion from T in (S, T) and the sum of the probabilities $A(S, T)$.

For $\theta = (i_1, i_2, \ldots, i_{2n}) \in \Theta(3^N)$, define the product

$$p_\theta = A(i_1, \alpha[\theta(i_1) \setminus i_1])A(i_2, \alpha[\theta(i_2) \setminus i_2]) \cdots A(i_{2n}, \alpha[\theta(i_{2n}) \setminus i_{2n}]).$$

Since $i_1 < 0$ the first factor is equal to

$$A(i_1, (\emptyset, N)) = \frac{\underline{p}^{-i_1}_{(\emptyset,N\setminus -i_1)}}{\sum_{j\in N} \underline{p}^{j}_{(\emptyset,N\setminus j)}} = \underline{p}^{-i_1}_{(\emptyset,N\setminus -i_1)},$$

where the last equality follows from the efficiency axiom (see Theorem 8). Since $i_{2n} > 0$ the last factor is

$$A(i_{2n}, (N \setminus i_{2n}, \emptyset)) = \frac{\overline{p}^{i_{2n}}_{(N\setminus i_{2n},\emptyset)}}{\overline{p}^{i_{2n}}_{(N\setminus i_{2n},\emptyset)}} = 1.$$

The collection $\{p_\theta : \theta \in \Theta(3^N)\}$ satisfies that all $p_\theta \geq 0$, and

$$\begin{aligned} \sum_{\theta\in\Theta(3^N)} p_\theta &= \sum_{\{i_1\in\overline{N}:i_1<0\}} \sum_{\{i_2\notin\{i_1\}:\alpha[\theta(i_2)]\in 3^N\}} \cdots \sum_{i_{2n}\notin\{i_1,\ldots,i_{2n-1}\}} p_{(i_1,\ldots,i_{2n})} \\ &= \sum_{i\in N} \underline{p}^{i}_{(\emptyset,N\setminus i)} = 1. \end{aligned}$$

Thus $\{p_\theta : \theta \in \Theta(3^N)\}$ is a finite probability distribution. Let Ψ be the compatible-order value associated with this probability distribution, that is,

$$\Psi_i(b) = \sum_{\theta\in\Theta(3^N)} p_\theta a_i^\theta(b) = \sum_{\theta\in\Theta(3^N)} p_\theta [m_i^\theta(b) + M_i^\theta(b)],$$

for all $i \in N$ and $b \in \mathcal{BG}^N$. Then for all $i \in N$ we have

$$\begin{aligned} \Psi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \Bigg[&\left(\sum_{\{\theta\in\Theta(3^N):\alpha[\theta(i)\setminus i]=(S,T)\}} p_\theta \right) [b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i])] \\ &+ \left(\sum_{\{\theta\in\Theta(3^N):\alpha[\theta(-i)]=(S,T)\}} p_\theta \right) [b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i])] \Bigg]. \end{aligned}$$

To prove that $\Phi_i = \Psi_i$ we only need to show that the coefficients satisfy

$$\overline{p}^i_{(S,T)} = \sum_{\{\theta \in \Theta(3^N):\, \alpha[\theta(i)\setminus i]=(S,T)\}} p_\theta,$$

$$\underline{p}^i_{(S,T)} = \sum_{\{\theta \in \Theta(3^N):\, \alpha[\theta(-i)]=(S,T)\}} p_\theta,$$

for every $(S, T) \in 3^{N\setminus i}$.

We next prove the first equality. The second one is similarly obtained. Let $(S, T) \in 3^{N\setminus i}$ with $|S| = s$ and $|T| = t$. Then we consider a chain $\theta_1 = (i_1, \ldots, i_k)$, where $k = s + n - t$, from $(\emptyset, N)$ to (S, T) and a chain $\theta_2 = (i_{k+2}, \ldots, i_{2n})$ from $(S \cup i, T)$ to $(N, \emptyset)$. These chains can be concatenated with i to make a maximal chain $\theta = (i_1, \ldots, i_k, i, i_{k+2}, \ldots, i_{2n})$.

Now we compute

$$\begin{aligned}
\sum_{\{\theta:\alpha[\theta(i)\setminus i]=(S,T)\}} p_\theta &= \sum_{i_k\in(S,T)} \sum_{i_{k-1}\in(S,T)\setminus\{i_k\}} \cdots \sum_{i_1\in(S,T)\setminus\{i_k,\ldots,i_2\}} \sum_{i_{k+2}\notin(S\cup i,T)} \\
&\quad \cdots \sum_{i_{2n}\notin(S\cup i,T)\cup\{i_{k+2},\ldots,i_{2n-1}\}} p_{(i_1,\ldots,i_k,i,i_{k+2},\ldots,i_{2n})} \\
&= A(i,(S,T)) \sum_{i_k\in(S,T)} A(i_k,(S,T)\setminus\{i_k\}) \\
&\quad \cdots \sum_{i_1\in(S,T)\setminus\{i_k,\ldots,i_2\}} \underline{p}^{-i_1}_{(\emptyset,N\setminus -i_1)} \cdots \sum_{i_{k+2}\notin(S\cup i,T)} A(i_{k+2},(S\cup i,T)) \\
&\quad \cdots \sum_{i_{2n}\notin(S\cup i,T)\cup\{i_{k+2},\ldots,i_{2n-1}\}} A(i_{2n},(S\cup i,T)\cup\{i_{k+2},\ldots,i_{2n-1}\}),
\end{aligned}$$

where $i_k \in (S, T)$ if $i_k \in S$ or $-i_k \notin T$ and $(S, T) \setminus \{i_k\} = (S \setminus i_k, T)$ if $i_k > 0$ and $(S, T) \setminus \{i_k\} = (S, T \cup -i_k)$ if $i_k < 0$. Also, $(S \cup i, T) \cup \{i_{k+2}\} = (S \cup i \cup i_{k+2}, T)$ if $i_{k+2} > 0$ and $(S \cup i, T \setminus -i_{k+2})$ otherwise.

First we prove that the last $2n - k - 1$ sums each, in turn, have value 1. Indeed, if $H_p = (S \cup i, T) \cup \{i_{k+2}, \ldots, i_p\}$, $k + 2 \le p \le 2n - 1$ and $H_{k+1} = (S \cup i, T)$, then

$$\begin{aligned}
&\sum_{\{i_p\notin(S\cup i,T)\cup\{i_{k+2},\ldots,i_{p-1}\}\}} A(i_p,(S\cup i,T)\cup\{i_{k+2},\ldots,i_{p-1}\}) \\
&= \sum_{\{i_p\notin H_{p-1}:H_{p-1}\cup\{i_p\}\in 3^N,\, i_p>0\}} A(i_p,H_{p-1}) + \sum_{\{i_p\notin H_{p-1}:H_{p-1}\cup\{i_p\}\in 3^N,\, i_p<0\}} A(i_p,H_{p-1}) \\
&= \sum_{\{i_p\notin H_{p-1}:H_{p-1}\cup\{i_p\}\in 3^N,\, i_p>0\}} \frac{\overline{p}^{i_p}_{H_{p-1}}}{A(H_{p-1})} + \sum_{\{i_p\notin H_{p-1}:H_{p-1}\cup\{i_p\}\in 3^N,\, i_p<0\}} \frac{\underline{p}^{-i_p}_{H_{p-1}\cup\{i_p\}}}{A(H_{p-1})},
\end{aligned}$$

and this expression is equal to 1 since $A(H_{p-1})$ is given by

$$A(H_{p-1}) = \sum_{\{j\notin H_{p-1}:H_{p-1}\cup\{j\}\in 3^N,\, j>0\}} \overline{p}^j_{H_{p-1}} + \sum_{\{j\notin H_{p-1}:H_{p-1}\cup\{j\}\in 3^N,\, j<0\}} \underline{p}^{-j}_{H_{p-1}\cup\{j\}}.$$

If we now consider the first $k+1$ sums, we see that each numerator of one factor is equal to the previous denominator by the efficiency axiom. Indeed, if we define $L_p = (S, T) \setminus \{i_k, \ldots, i_p\}$, where $2 \le p \le k$, and $L_{k+1} = (S, T)$, we claim that

$$\begin{aligned}
&A(i,(S,T)) \sum_{\{i_k\in L_{k+1}:L_{k+1}\setminus\{i_k\}\in 3^N\}} A(i_k,L_k) \cdots \sum_{i_1\in L_2} \underline{p}^{-i_1}_{L_2} \\
&= \frac{\overline{p}^i_{(S,T)}}{\sum_{j\notin S\cup T} \overline{p}^j_{(S,T)} + \sum_{j\in T} \underline{p}^j_{(S,T\setminus j)}} \sum_{\{i_k\in L_{k+1}:L_{k+1}\setminus\{i_k\}\in 3^N\}} A(i_k,L_k) \cdots \sum_{i_1\in L_2} \underline{p}^{-i_1}_{L_2} \\
&= \overline{p}^i_{(S,T)}.
\end{aligned}$$

Note that the numerator of each term $\sum_{i_{p-1}\in L_p} A(i_{p-1}, L_{p-1})$ is given by

$$\sum_{\{j\in L_p : j>0\}} \overline{p}^{\,j}_{L_p\setminus\{j\}} + \sum_{\{j\in L_p : j<0\}} \underline{p}^{-j}_{L_p},$$

and it is preceded by a factor with denominator

$$A(L_p) = \sum_{\{j\notin L_p : L_p\cup\{j\}\in 3^N,\ j>0\}} \overline{p}^{\,j}_{L_p} + \sum_{\{j\notin L_p : L_p\cup\{j\}\in 3^N,\ j<0\}} \underline{p}^{-j}_{L_p\cup\{j\}}.$$

Since these expressions are equal, applying the equations of Theorem 8, the entire expression simplifies to $\overline{p}^{\,i}_{(S,T)}$. □

Note that a particular case of compatible-order value is the value $\Phi = (\Phi_1, \ldots, \Phi_n)$ on $\mathcal{BG}^N$, with the same probability for all possible maximal chains, that is,

$$p_\theta = \frac{1}{|\Theta(3^N)|} = \frac{2^n}{(2n)!} = \frac{1}{n!(2n-1)!!}.$$

This value is an extension of the Shapley value for bicooperative games. Thus, we introduce the following definition.

Definition 7. Let $b \in \mathcal{BG}^N$. The Shapley value for the bicooperative game b is given by

$$\Phi_i(b) = \frac{1}{n!(2n-1)!!} \sum_{\theta\in\Theta(3^N)} p_\theta a_i^\theta(b),$$

for each $i \in N$.

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