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A connection-based analysis of networks using the position value: a computational approach

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ABSTRACT

In this paper, we introduce the position value as a centrality measure to evaluate the relevance of the edges and players in a network, with the additional advantage that this value integrates the degree measure of each player in it. In fact, in the real world, it is particularly important to consider the natural influence of connections of a player in a network. Its applications were very limited in real-world situations due to the high computational complexity of exactly obtaining this value. With the aim of solving this problem we provide a method, based on sampling theory, to estimate the position value, which is analyzed in terms of the theoretical properties of the resulting estimator. Moreover, we establish specific statistical results for bounding the absolute error in this approximation. It is important to emphasize that this approach allows for obtaining rankings not only of the nodes but also of the edges of the network. To illustrate the advantages and interest of the proposed methodology, as well as the variety of problems that can be analyzed in this framework, we applied it in three very different settings, the suburban train network of Madrid in the year 2000, the Spanish national team in a match against Portugal, and the Zerkani network responsible for the terrorist attacks of Paris (2015) and Brussels (2016).

1. Introduction

In this paper, we focus on the position value for communication situations as a centrality measure, solving by sampling methods the computational problems involved with it, and showing the advantages and variety of settings in which it can be applied. This value was introduced in Meesen (1988) and later studied by Borm et al. (1992), who provided a characterization of it for cycle-free communication situations. In Algaba et al. (2000), this value was generalized and characterized for a special class of union stable systems which extends the subclass of cycle-free communication situations and, in Algaba et al. (2004), its connection with hypergraphs was established. Slikker (2005) provided a first characterization of the position value for the whole class of communication situations. New characterizations of the position value as a special solution of the class of Harsanyi solutions in the settings of communication situations and union stable systems are obtained, respectively, in van den Brink et al. (2011) and Algaba et al. (2015). Recently, Manuel et al. (2023) have provided a characterization of the position value from a marginality properties-based approach. Also, relationships, in these contexts, between the Myerson value (Myerson, 1977) and the position value can be found in Gómez et al. (2004), Casajus (2007) and Algaba et al. (2012).

The literature about the position value shows interesting features about this value, specifically, in a communication situation, the position value gives a measure of the importance of a node in a graph, assessing firstly the relevance of the communication links between players, which is of big interest when applying to real data in network structures defined by an undirected graph. However, its exact computation is a very complicated computational task. In fact, the position value is defined from the Shapley value applied to the transferable utility game (TU game) defined on the set of edges that connect the different nodes, focusing on the role of the edges in the network.

Nevertheless, for certain class of TU games and/or graphs some efficient algorithms have been introduced for computing certain values based on the Shapley value, for instance, Fernández et al. (2002) proposed polynomial time algorithms for computing the Myerson value (Myerson, 1977), for weighted voting games restricted by a tree. Likewise, algorithms by means of the Harsanyi dividends for the Myerson value can be found in Algaba et al. (2007). Aadithya et al. (2010), Michalak et al. (2013), and Szczepanski et al. (2012, 2016) also gave polynomial time algorithms for computing the Shapley value and the so-called value betweenness centrality, for certain graphs, respectively. In the general framework of TU games, a common problem that has

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already been tackled was the computational drawbacks of using known solution concepts as allocation procedures in large-scale real-world problems. Namely, the study of the Shapley value (Shapley, 1953) and the Banzhaf value (Banzhaf, 1964), as well as their extensions the Owen value (Owen, 1977) and the Banzhaf-Owen value (Owen, 1982), respectively, to contexts with a priori unions, have been already dealt with. Specifically, sampling techniques were considered for the approximation of the Shapley value (Castro et al., 2009) and the Banzhaf value (Bachrach et al., 2010), and their extensions when assuming an a priori unions structure, such as the Owen value (Saavedra-Nieves et al., 2018) and the Banzhaf-Owen value (Saavedra-Nieves & Fiestras-Janeiro, 2021). More recently, Saavedra-Nieves (2023) used stratified sampling for the approximation of the Owen value and the Banzhaf-Owen value. For other families of games, such as games with externalities, their solutions have been also approximated (Saavedra-Nieves & Fiestras-Janeiro, 2022). However, to the best of our knowledge, the problem of studying the position value in communication situations from a computational point of view, and apply it as centrality measure to provide simultaneously rankings of both nodes and edges of a network defined by a graph, has not been addressed so far.

Along the last decades, social network analysis has focused on the identification of those key members within an internal organization, often without full knowledge of the numerous connections between these elements. For instance, in sports, the interactions of players of a team can be also described as a complex network (see Buldú et al., 2019). Similarly, from a socio-economic perspective, this approach allows us to quantify the relative importance of transfer stops in public transport systems (see Hadas et al., 2017). Terrorist networks are a case of networks, in which members use violence, receiving much attention because of the massive attacks in Western World along recent years. Examples include the 9/11 attacks in 2001, or the attacks carried out by the Zerkani network of Paris in 2015 and Brussels in 2016, among others. Koschade (2006), Sparrow (1991), Klerks (2001), Farley (2003), Guzman et al. (2014), and McGuire et al. (2015) are examples that use the conventional social network perspective to identify essential agents within this kind of organizational structure. However, recent works in graph analysis have incorporated a link-based perspective, for instance, for the detection of communities. We refer to Li et al. (2022), that use likelihood optimization; or Song et al. (2022) and Li et al. (2023), who consider a non-cooperative game scheme for the same purpose.

In network analysis described by a graph, Lindelauf et al. (2013) and Husslage et al. (2015) are the first in considering information about the communication between the members of the network. In fact, the heterogeneity of edges and nodes is incorporated for the first time through transferable utility games (TU games), in which cooperation of individuals is a vital aspect. An overall ranking of the nodes of the network, according to its importance, can be determined using solutions for TU games. For example, Hamers et al. (2019) rank the members of the Zerkani network using the Shapley value. Recently, Algaba, Prieto, Saavedra-Nieves, and Hamers (2023) and Algaba, Prieto, and Saavedra-Nieves (2023) assume the existence of an a priori union system modeling the affinities in the cooperation of agents, and Saavedra-Nieves and Casas-Méndez (2023) use games with externalities for this same purpose.

Unlike of the perspective analyzed in all these works, the link-based perspective in graph analysis has not been still considered, in practice, as basis for a ranking of the members of a network under cooperation. In this paper, we want to focus on the importance of a certain node in a network according to its position in it, namely, through of the relevance of the communication links between players. With this aim, we firstly analyze the position value for communication situations (cf. Borm et al., 1992) as a centrality measure for networks from a quantitative approach. This value offers additional advantages in comparison with all centrality measures proposed until now, even compared to those based on classical solution concepts for TU games considered so far. The position value, following the cooperative approach in the literature, is the unique solution concept that includes not only information related to the nodes, but also to the number of links incident on them and their strength. On one hand, the position value captures the topology of the network, independently of the initial TU game. On the other hand, it allows for providing a ranking not only of the influence of the nodes but also, it can be obtained a ranking of the strength of the connections among players, achieving a much more complete information and overview of the network than with the classical measures in social network analysis and the ones based on TU games. To illustrate and motivate it, we provide a first application to the suburban train network of Madrid in the year 2000, establishing the main stations and the most robust railway segments. However, many other networked situations can be represented by means of communication situations in which connections between network members and their individual weights have a strong influence on the study of the effectiveness of coalitions at different cooperation scenarios. A major drawback of considering the position value lies in the fact of that, due to the computational complexity involved, its exact calculation has so far been limited to academic examples with few nodes and edges not being possible to apply it for real world examples, in general. Thus, inspired by the sampling techniques that were considered for the Shapley value estimation (Castro et al., 2009), we propose a specific approximation methodology based on simple random sampling with replacement to estimate the position value for large-scale communication situations. In this context, we specifically analyze the problem of bounding the estimation error by providing some useful theoretical results. Moreover, to illustrate the applicability and relevance of the position value, we center on three very different scenarios that reflect the wide range of real situations in which its use is justified. First, we motivate its introduction as centrality measure in the setting of the suburban train network of Madrid in the year 2000. Second, we classify the players as well as the best performances of pairs of players corresponding to the Spanish national team in a match against Portugal. Third, we rank the members providing additionally the stronger connections between terrorists of the Zerkani network supporting the attacks of Paris and Brussels in 2015 and 2016, respectively. Finally, a comparison between the position value and the most well-known centrality measures in the literature, related to this value, is made, showing the interest and utility of this approach.

This paper is structured as follows. Section 2 briefly presents the position value and those notions required for the understanding of the paper. Section 3 introduces the position value as a new connection-based ranking mechanism of nodes and edges corresponding to a communication situation, motivating it with an application to the suburban train network of Madrid corresponding to the year 2000. The computational problems arising from its exact calculation in those situations with a large enough number of links in the network are addressed in Section 4 from a sampling perspective. Section 5 illustrates the performance of proposal of ranking on two different scenarios: we classify both the players and pairs of players of the Spanish national team in a match against Portugal, and we rank the terrorists and pairs of terrorists of the Zerkani network. Finally, taking account the nature of the position value and with the aim of showing the feasibility of our approach, the results are compared to those obtained for the main centrality measures related to the graph or the own definition of the position value. Hence, the position value is analyzed versus the main classical centrality measures in the literature as well as the Myerson value (Myerson, 1977), defined also from Shapley value (Shapley, 1953) of a TU game whose cooperation among the players is restricted by the connection of the nodes in the graph. Finally, Section 6 concludes.

2. Preliminaries

In this section, we formally present some theoretical terminology on cooperative game theory and communication situations, focusing on solution concepts such as the Shapley value (cf. Shapley, 1953) and the position value (cf. Meesen, 1988).

2.1. On transferable utility games and the Shapley value

A *transferable utility game*, or TU game, is a pair (N, v), where $N = \{1, 2, \dots, n\}$ is the set of players, called usually the grand coalition, and v is a map that assigns a real value $v(S)$ to each coalition $S \subseteq N$ such that $v(\emptyset) = 0$. The set of all cooperative games with player set N is denoted by $\mathcal{G}^N$.

For each $T \subseteq N$, with $T \neq \emptyset$, the *unanimity* game (N, u_T) is given by $u_T(S) = 1$, if $T \subseteq S$, and $u_T(S) = 0$, otherwise. It is well-known that the unanimity games form a basis for the vector space $\mathcal{G}^N$. For every $v \in \mathcal{G}^N$, it holds that $v = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \Delta_v(T)u_T$, where $\Delta_v(T) = \sum_{S\subseteq T}(-1)^{|T|-|S|}v(S)$ are the *Harsanyi dividends*, Harsanyi (1959).

A payoff vector $z = (z_i)_{i\in N} \in \mathbb{R}^n$ is a vector where z_i represents the payoff associated to player i by its collaboration in a given TU game (N, v). In general, a solution concept (in short a solution) is a map $\phi : \mathcal{G}^N \to \mathbb{R}^n$ that assigns to each TU game (N, v) a payoff vector.

One of the most appealing and well-known solution concepts for cooperative games is the Shapley value, introduced by Shapley (Shapley, 1953). Formally, the *Shapley value* for each $(N, v) \in \mathcal{G}^N$ assigns to each $i \in N$,

$$Sh_i(N, v) = \frac{1}{|\Pi(N)|}\sum_{\sigma\in\Pi(N)} m_v^\sigma(i), \tag{1}$$

with $\Pi(N)$ being the set of all permutations of N, and $m_v^\sigma(i)$ the marginal contribution of player i in a given $\sigma \in \Pi(N)$. Formally, it is defined as

$$m_v^\sigma(i) = v\left(P_i^\sigma \cup \{i\}\right) - v\left(P_i^\sigma\right),$$

being P_i^σ the set of predecessors of i in σ, i.e., $P_i^\sigma = \{k \in N : \sigma(k) < \sigma(i)\}$. Then, the Shapley value for (N, v) is interpreted, for each player i of N, as the expected value of i's marginal contributions over the set of all possible orders of N.

The popularity, attractiveness and versatility of this value is highlighted, in a wide collection of theoretical and applied results on it, in Algaba et al. (2019a). In fact, the Shapley value not only continue being so appealing as when it was first introduced in 1953 but its interest has even increased enormously in the last years, due not only to the fairness properties that this value satisfy, see Algaba et al. (2019b), but also to the numerous solution concepts derived from it.

In our opinion the magic and strength of the Shapley value is its endurance and at the same time its flexibility over time. It has been and keeps being analyzed from many different perspectives. For instance, for networked coalition structures when the TU game is focused on the edges of the graph, see Meesen (1988), the original idea behind the Shapley value, planed in 1953, on how assess the venture of playing a game remains as important as ever and the solution provided by Shapley solves the problem in a successful and desirable way, giving way to the position value (cf. Meesen, 1988 and Borm et al., 1992).

2.2. On communication situations and the position value

A *communication situation* is denoted by (N, v, L), being (N, v) a TU game and (N, L) an undirected graph without parallel edges nor loops connecting the members of N. Notice that the set of nodes N in the graph (N, L) coincide with the set of players in the TU game (N, v). Therefore, the set of edges L describes all relationships between pairs of players. A relationship between players i and j is denoted by ij, with $ij \in L$.

For a coalition $S \subseteq N$, the subgraph (S, L_S), where $L_S = \{ij \in L : i, j \in S\}$, consists of the players in S and their edges in L_S. A coalition $S \subseteq N$ is a connected coalition, if the subgraph (S, L_S) is connected, otherwise, S is called disconnected. Clearly, the set of maximal connected coalitions of N determines a partition on N, called the components of N, this set will be denoted as $C_L(N)$. Thus, the resulting partition for coalition S induced by the subgraph (S, L_S) is denoted by $C_L(S)$.

Given (N, v, L) a communication situation, Myerson (1977) defined a new game[1] as

$$v^L(S) = \sum_{T\in C_L(S)} v(T), \tag{2}$$

with $v^L(\emptyset) = 0$. In fact, the value $v^L(S)$ can be interpreted as the worth of the cooperation on the components of S under the communication edges in L_S. Notice that (N, v^L) focuses on the economic possibilities of the players in the game. By contrast, an alternative type of TU game can be introduced centering in the economic possibilities of the edges, and taking the edges as players. Formally, the *link game*, (L, r^L)[2] associated with a communication situation (N, v, L) is given, for every non-empty coalition of links $A \subseteq L$ by

$$r^L(A) = v^A(N), \tag{3}$$

and such that $r^L(\emptyset) = 0$. The link game (L, r^L) assigns to each possible coalition of links $A \subseteq L$ the worth of the cooperation of N specified by (N, v^A), by considering only those links in A on the network.[3]

A solution concept on the class of communication situations is given by a function γ that assigns a payoff vector $\gamma(N, v, L) \in \mathbb{R}^n$, to each communication situation (N, v, L). The *position value* $\pi(N, v, L)$ was introduced in Meesen (1988) and, later, studied in Borm et al. (1992). For every communication situation (N, v, L), the *position value* is defined from the Shapley value of the link game by

$$\pi_i(N, v, L) = \sum_{ij\in L_i} \frac{1}{2} Sh_{ij}(L, r^L), \text{ for all } i \in N, \tag{4}$$

where $L_i = \{ij \in L : j \in N\}$ denotes the set of edges with i as an endpoint, and $Sh_{ij}(L, r^L)$ is the allocation provided for edge ij by the Shapley value for the link game (L, r^L). Then, the position value assigns to each player i of N in the communication situation half as much as the Shapley value of the link game assigns to each of the incident edges in i. However, given a collection of individual weights $\{w_j\}_{j\in N}$, for each player $j \in N$, with $w_j > 0$, a weighted version of the position value can be naturally established. Thus, the *weighted position value* $\pi^\omega(N, v, L)$ is specified, for every communication situation (N, v, L) and for every $i \in N$, by

$$\pi_i^\omega(N, v, L) = \sum_{ij\in L_i} \frac{w_i}{w_i + w_j} Sh_{ij}(L, r^L). \tag{5}$$

Notice that if all individual weights w_j, for each player j, are equal, then the weighted position value is equal to the position value.

Example 2.1 illustrates the obtaining of the position value on a small network defined by a graph.

Example 2.1. Let (N, L) be the graph with the topology of nodes and links in Fig. 1. Thus, we have $N = \{1, 2, 3, 4\}$ and $L = \{a, b, c, d\}$. The weights associated with each of the players are detailed in Table 1.

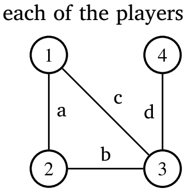


Fig. 1. Graph of the network (N, L).

Now, let (N, v, L) be the communication situation, where the TU game (N, v) is specified in Table 2 for every connected coalition $S \subseteq N$. Using (2), we display the associated TU game (N, v^L) in Table 3.

[1] The Shapley value of this game, i.e. $Sh(N, v^L)$, is called the *Myerson value* (Myerson, 1977).

[2] If coalition N and TU game v are fixed, we will denote the link game as r^L, if only coalition N is fixed, we will write it as $r^L_{v^L}$..

[3] Both games v^L and r^L have been considered in Algaba, Bilbao, Borm, and López (2001) and Algaba et al. (2000), respectively, for *union stable systems,* which generalize communication situations.

Table 1
List of weights for the nodes in (N, L).

Agent i	w_i
1	2
2	1
3	2
4	3

Table 2
The TU game (N, v) for connected coalitions.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$
$v(S)$	2	1	2	3	3	8
S	$\{2,3\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v(S)$	3	15	10	21	18	24

Table 3
The TU game (N, v^L).

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
$v^L(S)$	0	2	1	2	3	3	8	5
S	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v^L(S)$	3	4	15	10	6	21	18	24

From its characteristic function, we can obtain the link game (L, r^L) defined in (3). For each coalition of edges $A \subseteq L$, its associated characteristic function can be found in Table 4.

Table 4
The TU game (L, r^L).

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
$r^L(A)$	0	8	8	12	18	13	13	18
A	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
$r^L(A)$	13	20	22	13	24	24	24	24

Once the link game is determined, the position value can be computed for the communication situation (N, v, L). For this purpose, we firstly obtain the Shapley value of the considered link game, which gives us a measure of the relevance of the edges of the graph. That is, $Sh(L, r^L) = (2.750, 3.083, 4.750, 12.417)$. In view of these results, we can conclude that the stronger relation is given between players 3 and 4, as this component of the Shapley value associated with the link is the largest.

Next, we illustrate the obtaining of the position value and the weighted position value for Agent 1, i.e., $\pi_1(N, v, L)$ and $\pi_1^{\omega}(N, v, L)$. Using the formulas given in (4) and (5), as links a and c are adjacent on node 1 in the network considered, we have that

- $\pi_1(N, v, L) = \frac{1}{2} \cdot Sh_a(L, r^L) + \frac{1}{2} \cdot Sh_c(L, r^L) = \frac{1}{2} \cdot 2.750 + \frac{1}{2} \cdot 3.083 = 3.750$, and
- $\pi_1^{\omega}(N, v, L) = \frac{w_1}{w_1+w_2} \cdot Sh_a(L, r^L) + \frac{w_1}{w_1+w_3} \cdot Sh_c(L, r^L) = \frac{2}{2+1} \cdot 2.750 + \frac{2}{2+2} \cdot 4.750 = 4.208$.

Similarly, the other components of $\pi(N, v, L)$ and $\pi^{\omega}(N, v, L)$ can be easily obtained. Their numerical results are shown in Table 5.

Table 5
The position value $\pi(N, v, L)$ and the weighted position value $\pi^{\omega}(N, v, L)$.

Player	1	2	3	4
$\pi(N, v, L)$	3.750	2.917	10.125	6.208
$\pi^{\omega}(N, v, L)$	4.208	1.944	9.397	7.450

The fact that edge d receives the largest amount of the worth of the cooperation leads to players 3 and 4 receiving the largest allocations through the position value, although player 3 gets much more than others by the allocations received from the other edges adjacent to it in the graph, which underline the influence of the position of a node in a graph in relation with the degree measure of it.

3. A connection-based approach for network analysis

As mentioned, in a communication situation, (N, v, L), the players in the TU game (N, v) are represented by the nodes of the graph (N, L) and the links describe the interactions between each pair of players. Classical network measures, such as degree, betweenness, or closeness centrality (see, for furthers details, Koschade, 2006), provide initial methodologies for ranking their members. However, these lines of research only consider the structure of the network under study, without considering the possibilities of cooperation among its members. Lindelauf et al. (2013) or Husslage et al. (2015) solve this drawback by using cooperative game theory to include the heterogeneity of edges and nodes in the graph. In fact, this information is valued through transferable utility games (or TU games). From now on, we will center our study on two well-known TU games considered in the literature as representatives of a network defined by the graph (N, L).

As mentioned in preliminaries, any TU game can be expressed as a linear combination of unanimity games through the coefficients of Harsanyi, see Harsanyi (1959). Specifically, first, we consider the communication situation (N, u_N, L) derived from considering the unanimity game on the grand coalition. In this case, following Myerson (1977), the TU game (N, u_N^L) is obtained, which will be called *grand coalition connectivity game* and denoted by (N, v^{gcconn}), from now on. It assigns for each $S \subseteq N$ the worth of the cooperation according to the following expression:

$$v^{gcconn}(S) = \begin{cases} 1, & \text{if } S = N \text{ and connected,} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

In particular, it assigns a worth equal to 1 when the grand coalition N is formed, which means that all agents are connected in the graph, or equivalently that the grand coalition is connected. The other coalitions receive a value equal to zero. The grand coalition connectivity game (N, v^{gcconn}) underlines the relevance of the cooperation, agreement or communication of all agents. In other words, the key is the connectivity of the total network under coordination of its members. Moreover, other important aspect about this TU game is that it does not contain any other information about the influence of the nodes or edges in the graph under study.

Second, we consider the *additive weighted connectivity TU game (awconn)* (N, v^{awconn}), with respect to (N, L), following Myerson (1977) and Husslage et al. (2015). In order to define this game, a non-negative function f is defined, depending on coalition S, to quantify the effectiveness of any coalition in graph (N, L) according to the influence of the players in it, represented by a set of weights on N, i.e., $\mathcal{I} = \{w_j\}_{j \in N}$ with $w_j \geq 0$, and the relational strength between them in the network, given by a set of weights on the edges L, i.e., $\mathcal{R} = \{k_{lh}\}_{lh \in L}$ with $k_{lh} \geq 0$.

An example of effectiveness function f is the one considered in Husslage et al. (2015), that is defined, for each non-empty connected coalition $S \subseteq N$ in a given graph (N, L), as

$$f(S, \mathcal{I}, \mathcal{R}) = \begin{cases} \left(\sum_{j \in S} w_j\right) \cdot \max_{lh \in L_S} k_{lh}, & \text{if } |S| > 1, \\ w_i, & \text{if } S = \{i\}, \text{ with } i \in N. \end{cases} \tag{7}$$

This framework includes information about relationships between individuals as well as personal information about individuals. More specifically, for each possible connected coalition S with more than one player, this map specifies the sum of the individual weights of the members of S multiplied by the maximum weight over the set of edges connecting the subgraph induced by S. In the case of the unitary coalitions the value is its own worth and for the empty coalition, it will assign the value 0.

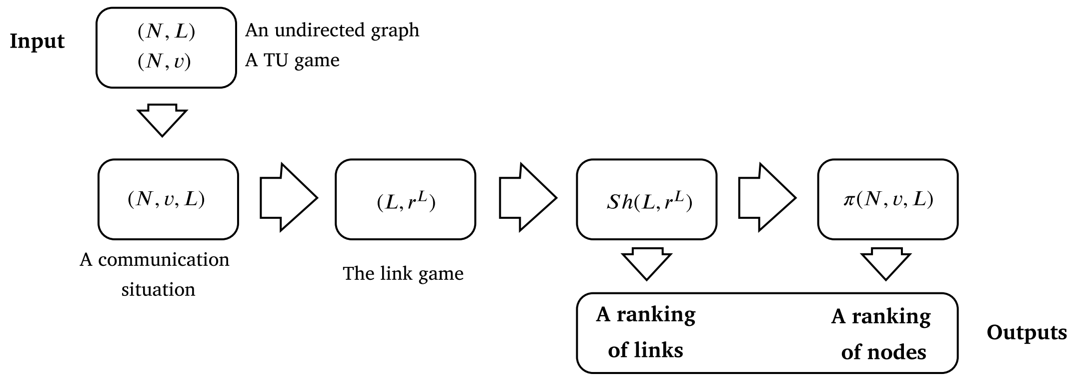


Fig. 2. Flowchart for the implementation of rankings of nodes and links based on the position value.

Given the communication situation (N, f, L), following Myerson (1977), the *additive weighted connectivity game* (N, v^{awconn}), is given for each $S \subseteq N$, by the following expression:

$$v^{awconn}(S) = \sum_{T \in C_L(S)} f(T, \mathcal{I}, \mathcal{R}). \tag{8}$$

Notice that the additive weighted connectivity game assigns to any connected coalition S the worth of its effectiveness prescribed by the function f considered. In the case of a disconnected coalition S, it assigns the aggregated effectiveness of all its maximal connected subsets (or components) in the subgraph induced by S. By contrast, with the approach provided by the TU game (N, v^{awconn}), we want to emphasize the worth of the cooperation of each coalition of players, that itself depends on the connectivity of the coalition and, therefore, on the strength of the relations given between players. The definition of this TU game is consistent and has been extensively utilized for analyzing networks derived from graphs (Myerson, 1977), as well as in more general networks with communication properties (Algaba, Bilbao, & López, 2001). Furthermore, it has also been applied to networks with both communication and hierarchical features, as discussed in Algaba et al. (2018). Note that unlike of the grand coalition connectivity game, the additive weighted connectivity game is considering implicitly the features of the network in it, which will have implications when applying the Shapley or the position values on them.

It is important to stress that unlike the classical game theory solutions, our proposal additionally integrates the degree measure and, moreover, it allows for obtaining rankings not only of the nodes of a graph but also of the edges or relations between individuals in a graph. In what follows, we focus on establishing a centrality node measure based on the capabilities of influence of links of the any network defined by the graph (N, L). Clearly, the number of links incident on each node (the degree) and the strength of these connections will mark the real influence of each node on the graph as a whole through the link game (L, r^L) associated. The flowchart of the steps of the position value-based procedure to rank the nodes and the links of a graph are summarized in Fig. 2.

For this purpose, the consideration of the *additive weighted connectivity link game* $(L, r^L_{v^{awconn}})$ obtained from the expression in (3) by using (N, v^{awconn}), and the *grand coalition connectivity link game* $(L, r^L_{v^{gcconn}})$, associated to the TU game (N, v^{gcconn}) when applying the expression in (3), as well as the computation of the position value, are required.

Now, Example 3.1 illustrates the usage of the position value as a mechanism of ranking the nodes and links of a real transportation network.

Example 3.1. Let (N, L) be the graph specified in Fig. 3. It specifies the topology of the nodes, representing the main stations of the suburban trains in Madrid in the year 2000, and of the links, detailing the lines that connect them. Thus, we have that $|N| = 18$ and $|L| = 19$. Each station is weighted 1, except for the intermodal train stations of Atocha

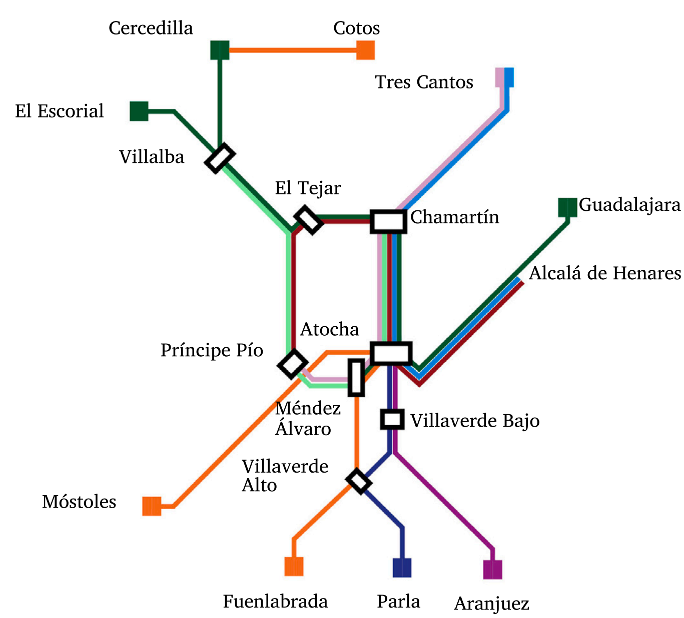


Fig. 3. Graph of the suburban train network (N, L) of Madrid in the year 2000.

and Chamartín, which are weighted 2, according to the volume of their passengers. Each link has a weight equal to the number of train lines passing through it and that are shown in different colors in the figure.

Now, let (N, u_N, L) and (N, f, L)[4] be the communication situations above-defined, which induce the TU games (N, v^{gcconn}) and (N, v^{awconn}), respectively. The last one is obtained by using the effectiveness function in (7). From their characteristic functions, we can obtain the grand coalition connectivity link game $(L, r^L_{v^{gcconn}})$ and the additive weighted connectivity link game $(L, r^L_{v^{awconn}})$, for each coalition of edges $A \subseteq L$.

Using the TU games $(L, r^L_{v^{gcconn}})$ and $(L, r^L_{v^{awconn}})$, several rankings can be obtained derived from the computation of the position value. For

[4] From now on, when dealing with the communication situations (N, u_N, L) and (N, f, L), we will denote $\pi(N, v^{gcconn}, L)$ and $\pi(N, v^{awconn}, L)$ instead of $\pi(N, u_N, L)$ and $\pi(N, f, L)$, to make clear the games used in the link game.

Table 6
Shapley value for the link games $(L, r^L_{v^{gcconn}})$ and $(L, r^L_{v^{awconn}})$.

Pair of stations	$Sh(L, r^L_{v^{gcconn}})$	$Sh(L, r^L_{v^{awconn}})$	Pair of stations	$Sh(L, r^L_{v^{gcconn}})$	$Sh(L, r^L_{v^{awconn}})$
El Escorial-Villalba	0.08256 (1–11)	2.41850 (18)	Atocha-Alcalá de Hen.	0.08256 (1–11)	6.76259 (4)
Villalba-Cercedilla	0.08256 (1–11)	3.43730 (13)	Atocha-Vill. Bajo	0.01066 (16)	7.28532 (3)
Cercedilla-Cotos	0.08256 (1–11)	2.07143 (19)	Atocha-Móstoles	0.08256 (1–11)	3.60263 (11)
Villalba-El Tejar	0.08256 (1–11)	7.30238 (2)	Méndez Álvaro-Vill. Alto	0.01066 (17)	4.17460 (8)
El Tejar-Príncipe Pío	0.01393 (12)	4.15913 (9)	Vill. Bajo-Vill. Alto	0.01066 (18)	3.51508 (12)
El Tejar-Chamartín	0.01393 (13)	4.83091 (7)	Vill. Alto-Fuenlabrada	0.08256 (1–11)	2.71573 (16)
Chamartín-Tres Cantos	0.08256 (1–11)	3.92208 (10)	Vill. Alto-Parla	0.08256 (1–11)	2.71573 (17)
Príncipe Pío-Méndez Álvaro	0.01393 (14)	5.12817 (6)	Vill. Bajo-Aranjuez	0.08256 (1–11)	2.86454 (14)
Méndez Álvaro-Atocha	0.00413 (19)	6.37539 (5)	Alcalá de Hen.-Guadalajara	0.08256 (1–11)	2.71930 (15)
Chamartín-Atocha	0.01393 (15)	23.9992 (1)			

Table 7
Position value for the nodes of the graph (N, L).

Station	$\hat{\pi}(N, v^{gcconn}, L)$	$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	$\hat{\pi}(N, v^{awconn}, L)$	$\hat{\pi}^{\omega}(N, v^{awconn}, L)$
El Escorial	0.04128 (9)	0.04128 (9)	1.20925 (17)	1.20925 (15)
Villalba	0.12384 (1)	0.12384 (2)	6.57909 (6)	6.57909 (5)
Cercedilla	0.08256 (4)	0.08256 (4)	2.75436 (10)	2.75436 (10)
Cotos	0.04128 (10)	0.04128 (10)	1.03571 (18)	1.03571 (17)
El Tejar	0.05521 (6)	0.05289 (7)	8.14621 (3)	7.34106 (3)
Chamartín	0.05521 (7)	0.07129 (5)	16.3761 (2)	17.8349 (2)
Tres Cantos	0.04128 (11)	0.02752 (15)	1.96104 (11)	1.30736 (14)
Príncipe Pío	0.01393 (18)	0.01393 (17)	4.64365 (9)	4.64365 (8)
Atocha	0.09692 (2)	0.12691 (1)	24.0126 (1)	28.0169 (1)
Méndez Álvaro	0.01436 (17)	0.01367 (18)	7.83908 (4)	6.77652 (4)
Villaverde Bajo	0.05194 (8)	0.05017 (8)	6.83246 (5)	5.61825 (7)
Villaverde Alto	0.09322 (3)	0.09322 (3)	6.56057 (7)	6.56057 (6)
Móstoles	0.04128 (12)	0.02752 (16)	1.80132 (12)	1.20088 (16)
Fuenlabrada	0.04128 (13)	0.04128 (11)	1.35786 (15)	1.35786 (12)
Parla	0.04128 (14)	0.04128 (12)	1.35786 (16)	1.35786 (13)
Aranjuez	0.04128 (15)	0.04128 (13)	1.43227 (13)	1.43227 (11)
Alcalá de Henares	0.08256 (5)	0.06880 (6)	4.74094 (8)	3.61385 (9)
Guadalajara	0.04128 (16)	0.04128 (14)	1.35965 (14)	1.35965 (12)

this purpose, we firstly obtain the Shapley value of the two considered link games, which gives us a ranking of the edges of the graph according to their relevance (see Table 6). When considering $(L, r^L_{v^{gcconn}})$, there are several edges that can be qualified as key to the connectivity of the network, as there are ties in the maximum allocations that the Shapley value specifies. However, the edge that contributes the least to the connectivity of the entire network is Méndez Álvaro-Atocha. On the other hand, under the consideration of $(L, r^L_{v^{awconn}})$, the link that contributes most to the network is the one that connects the two most important stations in the city, Atocha and Chamartín. The link that has the least contribution to the network under this approach is the one between Cotos and Cercedilla stations. Table A.1 in Appendix A of the Online resource section (ORS) gives the overall rankings.

The position value and the weighted position value are obtained for the main stations of the suburban train network under the TU games considered. These results are shown in Table 7.

Thus, such allocations prescribe different rankings for the nodes of the graph. For each station, we indicate in brackets its position in the associated ranking. Atocha occupies the first position in the rankings, except for the case of $\hat{\pi}(N, v^{gcconn}, L)$ that assigns it position 2 of the ranking. Although Chamartín station is ranked second when using v^{awconn}, it drops several positions in the rankings based on v^{gcconn}. As for the least influential stations, v^{gcconn} assigns to the centrally located stations of Mendez Álvaro and Príncipe Pío the last two positions. Conversely, the two least relevant stations according to v^{awconn} are stations on the outskirts of the central area of the city. The overall rankings are detailed in Table A.2 of Appendix A in the ORS.

The previous example illustrates the computation of rankings based on the position value in a "small" scheme. However, the exact computation of the position value becomes a challenging task when the number of links in the graph significantly increases. Notice that similar problems arise for the exact computation of the Shapley value in those contexts with a large number of players (see, for example, Fernández-García & Puerto-Albandoz, 2006 or Castro et al., 2009), for which sampling techniques provide an approximated solution as an alternative.

4. Estimating the position value

As mentioned, the main drawback concerning the (weighted) position value is computational, since its complexity exponentially increases with the number of players and the number of links connecting them. However, up to the authors' knowledge, the task of obtaining a procedure for the approximation of the position value has not been explored yet, in spite of the importance of taking into account the connections in real networks.

Let (N, r^L) be the link game associated with a given communication situation (N, v, L). Notice that the (weighted) position value can be obtained in terms of the Shapley value for the associated link game. Hence, its own definition justifies a proposal for its estimation, based on the ideas used by Fernández-García and Puerto-Albandoz (2006) and Castro et al. (2009) for the Shapley value approximation by using sampling techniques.

The steps of that sampling procedure for the estimation of the weighted position value are illustrated below:

1. The sampling population corresponds to all orders of the set of links L, i.e., $\Pi(L)$.
2. The vector of unknown parameters to be estimated is $\pi^{\omega} = (\pi^{\omega}_i)_{i\in N}$, with π^{ω}_i being $\pi^{\omega}_i(N, v, L)$, for all $i \in N$.
3. The feature to analyze is the vector

   $$(m^{\sigma}_{r^L}(a))_{a\in L} = (r^L\left(P^{\sigma}_a \cup \{a\}\right) - r^L\left(P^{\sigma}_a\right))_{a\in L},$$

   for each sampling unit $\sigma \in \Pi(L)$.
4. Each permutation $\sigma \in \Pi(L)$ is equally likely.
5. The average of the marginal contribution vectors over a sample S of permutations of L is the estimation of the Shapley value for

the link game (L, r^L), i.e., $\hat{Sh} = (\hat{Sh}_a)_{a\in L}$, such that

$$\hat{Sh}_a(L, r^L) = \frac{1}{\ell}\sum_{\sigma\in S} m^{\sigma}_{r^L}(a),$$

for all $a \in L$, where ℓ is the sample size.

6. The estimator for the *weighted position value* $\pi^{\omega}(N, v, L)$ is $\hat{\pi}^{\omega} = (\hat{\pi}^{\omega}_i)_{i\in N}$, where

$$\hat{\pi}^{\omega}_i(N, v, L) = \sum_{ij\in L_i} \frac{w_i}{w_i + w_j} \hat{Sh}_{ij}(L, r^L), \text{ for all } i \in N. \tag{9}$$

Upon applying this procedure, the resulting vector $\hat{\pi}^{\omega} = (\hat{\pi}^{\omega}_1, \ldots, \hat{\pi}^{\omega}_n)$ corresponds to the estimation of the weighted position value for all players involved in the communication situation (N, v, L). First, we examine the statistical properties of the estimator given in Eq. (9). Consider a fixed player $i \in N$. Thus, $\hat{\pi}^{\omega}_i$ is an unbiased estimator since it holds that

$$\begin{aligned}\mathbb{E}(\hat{\pi}^{\omega}_i) &= \mathbb{E}\Big(\sum_{ij\in L_i} \frac{w_i}{w_i + w_j}\hat{Sh}_{ij}(L, r^L)\Big)\\ &= \sum_{ij\in L_i} \frac{w_i}{w_i + w_j}\mathbb{E}\Big(\hat{Sh}_{ij}(L, r^L)\Big)\\ &= \sum_{ij\in L_i} \frac{w_i}{w_i + w_j} Sh_{ij}(L, r^L)\\ &= \pi^{\omega}_i.\end{aligned} \tag{10}$$

These equalities are satisfied due to the unbiased nature of the Shapley value estimator for a TU game, obtained when using simple random sampling with replacement (cf. Castro et al., 2009). Similarly, the consistency of the estimator $\hat{\pi}^{\omega}$ is also ensured.

Over a sample of permutations in L, given by S, the estimator $\hat{\pi}^{\omega}_i$ for the weighted position value of player i in (9) admits an alternative formulation, for every $i \in N$, that we detail below. Thus, we have that

$$\begin{aligned}\hat{\pi}^{\omega}_i(N, v, L) &= \sum_{ij\in L_i} \frac{w_i}{w_i + w_j}\hat{Sh}_{ij}(L, r^L) = \sum_{ij\in L_i} \frac{w_i}{w_i + w_j}\Big(\frac{1}{\ell}\sum_{\sigma\in S} m^{\sigma}_{r^L}(ij)\Big)\\ &= \frac{1}{\ell}\Big(\sum_{ij\in L_i}\sum_{\sigma\in S} \frac{w_i}{w_i + w_j} m^{\sigma}_{r^L}(ij)\Big)\\ &= \frac{1}{\ell}\sum_{\sigma\in S}\Big(\sum_{ij\in L_i} \frac{w_i}{w_i + w_j} m^{\sigma}_{r^L}(ij)\Big)\\ &= \frac{1}{\ell}\sum_{\sigma\in S} x(\sigma)_i,\end{aligned} \tag{11}$$

being $x(\sigma)_i = \sum_{ij\in L_i} \frac{w_i}{w_i+w_j} m^{\sigma}_{r^L}(ij)$, for all $i \in N$.

A fundamental issue in problems of solution approximating for TU games lies in bounding the estimation error, which refers to the difference between the approximated value and the exact value. Since it is often impractical to measure this error directly, a probabilistic bound is typically provided instead. Roughly speaking, the approximation of the weighted position value for player i is at a distance greater than ε of the real value with a probability α as maximum. Formally, it means

$$\mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \geq \varepsilon) \leq \alpha, \text{ with } \varepsilon > 0 \text{ and } \alpha \in (0, 1].$$

Therefore, as the sampling size enlarges, the estimated weighted position value tends to be a more accurate approximation of the real one.

Given the relevance of selecting an appropriate sampling size, we provide valuable results for effectively bounding such error. In order to accomplish this, we have thoroughly examined relevant literature concerning the approximation of coalitional values in cooperative games, as Fernández-García and Puerto-Albandoz (2006), Castro et al. (2009) and Maleki (2015) for the Shapley value estimation, and Bachrach et al. (2010) for the estimation of power indices for simple games. It is noteworthy that when estimating coalitional values, the population sizes considered in sampling procedures are typically large but always finite. Thus, the conservative bound provided by Castro et al. (2009) cannot be considered. The findings presented below enable the determination of the minimum sample size necessary to approximate the weighted position value with a desired maximum error of ε and a confidence level of $1-\alpha$. They follow the lines of research of Maleki (2015) and Bachrach et al. (2010), that make use of concentration bounds for the analysis of the error in estimating unknown parameters on finite populations.

To establish a bound on the absolute error in estimating the weighted position value, a statement relying on Hoeffding's concentration inequality is introduced. *Hoeffding's inequality* (Hoeffding, 1963) states that if $\sum_{j=1}^{k} X_j$ represents the sum of k observations $X_1, \ldots, X_k$ extracted with replacement, such that $a_j \leq X_j \leq b_j$ for all $j \in \{1, \ldots, k\}$, then

$$\mathbb{P}(|\sum_{j=1}^{k} X_j - \mathbb{E}(\sum_{j=1}^{k} X_j)| \geq t) \leq 2\exp\Big(\frac{-2t^2}{\sum_{j=1}^{k}(b_j - a_j)^2}\Big), \text{ for all } t \geq 0. \tag{12}$$

Proposition 4.1 specifically formalizes this result for the case of estimating the weighted position value.

Proposition 4.1. *Let (N, v, L) be a communication situation. Take $\varepsilon > 0$, $\alpha \in (0, 1)$ and denote the range of $x(\sigma)_i$ by*

$$r_i = \max_{\sigma,\sigma'\in \Pi(L)} (x(\sigma)_i - x(\sigma')_i).$$

Then,

$$\ell \geq \frac{\ln(2/\alpha) r_i^2}{2\varepsilon^2} \text{ implies that } \mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \geq \varepsilon) \leq \alpha.$$

Proof. Indeed, we have that $\hat{\pi}^{\omega}_i = \frac{1}{\ell}\sum_{\sigma\in S} x(\sigma)_i$ for a sample of ℓ elements. Thus,

$$\mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \geq \varepsilon) = \mathbb{P}(|\hat{\pi}^{\omega}_i - \mathbb{E}(\hat{\pi}^{\omega}_i)| \geq \varepsilon) = \mathbb{P}(|\sum_{\sigma\in S} x(\sigma)_i - \mathbb{E}(\sum_{\sigma\in S} x(\sigma)_i)| \geq \varepsilon\ell).$$

Using Hoeffding's inequality (12), it satisfies that

$$\mathbb{P}\Big(\Big|\sum_{\sigma\in S} x(\sigma)_i - \mathbb{E}(\sum_{\sigma\in S} x(\sigma)_i)\Big| \geq \varepsilon\ell\Big) \leq 2\exp\Big(\frac{-2\varepsilon^2\ell}{r_i^2}\Big) \leq \alpha,$$

and we conclude the proof. □

Below, we establish a general bound on the range of $x(\sigma)_i$ that may be useful in determining the sample sizes in the estimation of the weighted position value for the additive weighted connectivity TU game. For this purpose, the nature of the effectiveness function f considered is essential. Specifically, we consider the case in which the effectiveness function $f : 2^N \longrightarrow \mathbb{R}$ is superadditive, that is, if it holds that

$$f(S, \mathcal{I}, \mathcal{R}) + f(T, \mathcal{I}, \mathcal{R}) \leq f(S \cup T, \mathcal{I}, \mathcal{R}) \tag{13}$$

for all pair of disjoint coalitions $S, T \subseteq N$.

Proposition 4.2. *Let (N, v^{gcconn}) and (N, v^{awconn}) be the grand coalition connectivity TU game and the additive weighted connectivity TU game associated with the communication situations (N, u_N, L) and (N, f, L), with f a superadditive effectiveness function. For every $i \in N$, it is satisfied that,*

- *for the case of estimating $\pi^{\omega}(N, v^{gcconn}, L)$,*

$$r_i \leq |L_i| \frac{w_i}{w_i + \min_{j\in N\,:\, ij\in L_i} w_j}; \tag{14}$$

- *for the case of estimating $\pi^{\omega}(N, v^{awconn}, L)$,*

$$r_i \leq |L_i| \frac{w_i}{w_i + \min_{j\in N\,:\, ij\in L_i} w_j} f(N, \mathcal{I}, \mathcal{R}), \tag{15}$$

being $|L_i|$ the number of edges with player i as endpoint.

Proof. Take $i \in N$ and let σ and σ' be two different permutations of edges in $\Pi(L)$. Taking into account that $x(\sigma)_i = \sum_{ij \in L_i} \frac{w_i}{w_i+w_j} m^{\sigma}_{r^L}(ij)$, for all $i \in N$ and for every σ,

$$\begin{aligned} x(\sigma)_i - x(\sigma')_i &= \sum_{ij\in L_i} \frac{w_i}{w_i+w_j}\left(m^{\sigma}_{r^L}(ij) - m^{\sigma'}_{r^L}(ij)\right) \\ &\le \sum_{ij\in L_i} \frac{w_i}{w_i+w_j} m^{\sigma}_{r^L}(ij) \\ &\le |L_i| \max_{j\in N\,:\,ij\in L_i}\left\{\frac{w_i}{w_i+w_j}\right\} m^{\sigma}_{r^L}(ij) \\ &\le |L_i|\left(\frac{w_i}{w_i+\min_{j\in N\,:\,ij\in L_i} w_j}\right) m^{\sigma}_{r^L}(ij), \end{aligned} \tag{16}$$

being $|\cdot|$ the cardinal operator of a set. The fourth inequality is satisfied due to the decreasing character of the function $f(x) = \frac{C}{x+C}$ for all $x \ge 0$ and being $C > 0$.

From the last inequality in (16), for the case of estimating $\pi^{\omega}(N, v^{\text{gcconn}}, L)$, we can state, for all $i \in N$, that

$$r_i = \max_{\sigma,\sigma'\in\Pi(L)} (x(\sigma)_i - x(\sigma')_i) \le |L_i| \frac{w_i}{w_i + \min_{j\in N\,:\,ij\in L_i} w_j}, \tag{17}$$

since, under the consideration of the grand coalition connectivity TU game (N, v^{gcconn}), $m^{\sigma}_{r^L}(ij) \le 1$.

For the case of estimating $\pi^{\omega}(N, v^{\text{awconn}}, L)$, let $ij \in L$, and $A \subseteq L$, it holds that

$$\begin{aligned} m^{\sigma}_{r^L}(ij) = r^L(A\cup\{ij\}) - r^L(A) &\le r^L(A\cup\{ij\}) \\ &= v^{A\cup\{ij\}}(N) \\ &= \sum_{T\in C_{A\cup\{ij\}}(N)} v^{\text{awconn}}(T) \\ &= \sum_{W\in C_{A\cup\{ij\}}(N)} f(W,\mathcal{I},\mathcal{R}) \\ &\le f(N,\mathcal{I},\mathcal{R}), \end{aligned}$$

where the last inequality is satisfied by the superadditive character of the effectiveness function f. Then, we immediately have that, for all $i \in N$,

$$r_i \le |L_i|\left(\frac{w_i}{w_i+\min_{j\in N\,:\,ij\in L_i} w_j}\right) f(N,\mathcal{I},\mathcal{R}),$$

concluding the proof. □

The following two corollaries can be immediately obtained. First, we state a general bound for the case of the grand coalition connectivity TU game.

Corollary 4.3. *Consider $\varepsilon > 0$, $\alpha \in (0,1)$ and (N, v^{gcconn}) the grand coalition connectivity TU game associated with the communication situation (N, u_N, L). If ℓ satisfies that*

$$\ell \ge \frac{\ln(2/\alpha)}{2\varepsilon^2}|L_i|^2\left(\frac{w_i}{w_i+\min_{j\in N\,:\,ij\in L_i} w_j}\right)^2, \tag{18}$$

then, $\mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \ge \varepsilon) \le \alpha$, for each $i \in N$.

Proof. The proof is straightforward from Proposition 4.2 and the inequality in (14). □

Second, we establish a general bound for the additive weighted connectivity TU games, when considering a superadditive effectiveness function f.

Corollary 4.4. *Consider $\varepsilon > 0$, $\alpha \in (0,1)$ and (N, v^{awconn}) the additive weighted connectivity TU game associated with the communication situation (N, f, L), by using a superadditive effectiveness function f. If ℓ satisfies that*

$$\ell \ge \frac{\ln(2/\alpha)}{2\varepsilon^2}|L_i|^2\left(\frac{w_i}{w_i+\min_{j\in N\,:\,ij\in L_i} w_j}\right)^2 (f(N,\mathcal{I},\mathcal{R}))^2, \tag{19}$$

then, $\mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \ge \varepsilon) \le \alpha$, for each $i \in N$.

Proof. The proof directly follows Proposition 4.2 and the inequality in (15). □

The previous result can be directly formalized for the case of the additive weighted connectivity TU game associated with the communication situation (N, f, L) when using the effectiveness function in (7) of Husslage et al. (2015). Given $\varepsilon > 0$ and $\alpha \in (0,1)$, if

$$\ell \ge \frac{\ln(2/\alpha)}{2\varepsilon^2}|L_i|^2\left(\frac{w_i}{w_i+\min_{j\in N\,:\,ij\in L_i} w_j}\right)^2\left(\left(\sum_{j\in N} w_j\right)\cdot \max_{lh\in L} k_{lh}\right)^2, \tag{20}$$

then, $\mathbb{P}(|\hat{\pi}^{\omega}_i - \pi^{\omega}_i| \ge \varepsilon) \le \alpha$, for each $i \in N$.

In the particular case of the position value, obtained from the weighted position value when all edge weights are equal, we want to mention explicitly the bounds of the error that can be specifically established from the above results. In this case, the estimator for the position value $\pi(N, v, L)$ is directly given by $\hat{\pi} = (\hat{\pi}_i)_{i\in N}$, where

$$\hat{\pi}_i(N, v, L) = \frac{1}{2}\sum_{ij\in L_i} \hat{Sh}_{ij}(L, r^L), \tag{21}$$

for all $i \in N$. Immediately, the bounds of the error for the position value, derived from Corollaries 4.3 and 4.4, respectively, are stated below.

- First, we formalize a specific bound for r_i, with $i \in N$, for the case of estimating the position value. As in Proposition 4.2, it is easy to check that,
  - for the case of estimating $\pi(N, v^{gcconn}, L)$, we have that $r_i \le \frac{|L_i|}{2}$ for every $i \in N$;
  - for the case of estimating $\pi(N, v^{\text{awconn}}, L)$ with a superadditive effectiveness function f, it holds that $r_i \le \frac{|L_i|}{2} f(N,\mathcal{I},\mathcal{R})$, for every $i \in N$,

  being $|L_i|$ the number of edges with player i as endpoint.
- Using these last two inequalities, we can formalize the following two statements on sampling sizes. Take $\varepsilon > 0$ and $\alpha \in (0,1)$. Let (N, v^{gcconn}) be the grand coalition connectivity TU game associated with (N, u_N, L). If ℓ satisfies that

  $$\ell \ge \frac{\ln(2/\alpha)}{2\varepsilon^2}\left(\frac{|L_i|}{2}\right)^2, \tag{22}$$

  it holds $\mathbb{P}(|\hat{\pi}_i - \pi_i| \ge \varepsilon) \le \alpha$, for each $i \in N$.
  Similarly, let (N, v^{awconn}) be the additive weighted connectivity TU game associated with (N, f, L), by using a superadditive effectiveness function f. If ℓ satisfies that

  $$\ell \ge \frac{\ln(2/\alpha)}{2\varepsilon^2}\left(\frac{|L_i|}{2}\right)^2 (f(N,\mathcal{I},\mathcal{R}))^2, \tag{23}$$

  it holds $\mathbb{P}(|\hat{\pi}_i - \pi_i| \ge \varepsilon) \le \alpha$, for each $i \in N$.

Note that Castro et al. (2009) guaranteed polynomial complexity for the case of the Shapley value estimation as long as the characteristic function of the TU game considered is also obtained in polynomial time. Polynomial algorithms are also available to detect the connected components of any graph and thus to obtain the characteristic function of both TU games under consideration. Thus, when the link game is obtained in polynomial time, by the nature of our procedure, the approximation of the position value using simple random sampling with replacement has polynomial complexity by construction.

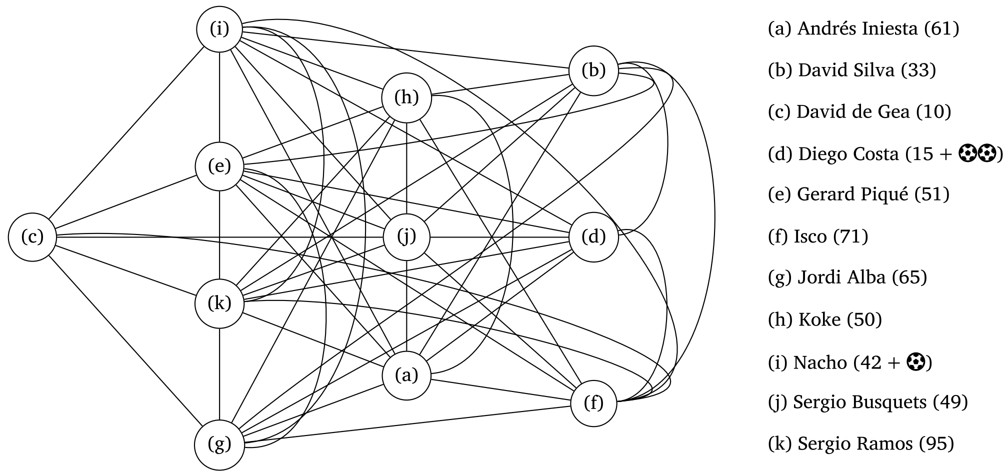


Fig. 4. Graph of the network (N, L) associated with the Spanish national team in the Portugal-Spain match at the 2018 World Cup in Russia.

5. Position value-based rankings

As mentioned, the importance of the connections between nodes in a graph can influence the results. Hence, the position value is of big interest when working with networks represented by a graph. In the first two subsections, we apply our sampling proposal on two very different scenarios modeled under the scheme of a network defined by a graph (N, L) and in which connections among agents are relevant enough. In both scenarios, we rank the nodes and edges of the graph according to their influence in the pursuit of the objectives of the network. The first case is devoted to the analysis of the network arisen from the passing structure of a football team. The second one analyzes the case of a well-known terrorist network: The Zerkani network. The last subsection studies the position value versus the most known centrality measures in the literature.

5.1. The Spanish national football team

First, we analyze the problem of identifying the leading player of a team by analyzing the network of passes of players in a football match. To this aim, we consider the approach provided by the position value where the connections between players (edges) take relevance. These situations involve a keen interest in determining the relative importance of network members in terms of their contribution to its operations.

Consider now the graph (N, L) arisen from the organization and the performance of football teams. In this sense, a team can be conceptualized as a complex network in which the nodes represent players who interact with the objective of outperforming the opposing network. Additionally, the associated graph is derived from the football passing networks, with the edges representing the interactions between the players.

Table 8
Distribution of passes between players of the Spanish national football team in the Portugal-Spain match of the 2018 World Cup.

Pair of players	Number of passes	Pair of players	Number of passes
Andrés Iniesta-David Silva	5	Diego Costa-Jordi Alba	7
Andrés Iniesta-Diego Costa	7	Diego Costa-Nacho	1
Andrés Iniesta-Gerard Piqué	6	Diego Costa-Sergio Busquets	4
Andrés Iniesta-Isco	28	Diego Costa-Sergio Ramos	2
Andrés Iniesta-Jordi Alba	27	Gerard Piqué-Isco	3
Andrés Iniesta-Koke	7	Gerard Piqué-Jordi Alba	2
Andrés Iniesta-Nacho	1	Gerard Piqué-Koke	21
Andrés Iniesta-Sergio Busquets	9	Gerard Piqué-Nacho	12
Andrés Iniesta-Sergio Ramos	38	Gerard Piqué-Sergio Busquets	14
David Silva-Diego Costa	3	Gerard Piqué-Sergio Ramos	26
David Silva-Gerard Piqué	5	Isco-Jordi Alba	44
David Silva-Isco	9	Isco-Koke	11
David Silva-Jordi Alba	4	Isco-Nacho	13
David Silva-Koke	17	Isco-Sergio Busquets	10
David Silva-Nacho	11	Isco-Sergio Ramos	26
David Silva-Sergio Busquets	6	Jordi Alba-Koke	1
David Silva-Sergio Ramos	7	Jordi Alba-Nacho	1
David de Gea-Gerard Piqué	5	Jordi Alba-Sergio Ramos	44
David de Gea-Isco	1	Koke-Nacho	21
David de Gea-Jordi Alba	4	Koke-Sergio Busquets	20
David de Gea-Nacho	3	Koke-Sergio Ramos	7
David de Gea-Sergio Busquets	1	Nacho-Sergio Busquets	11
David de Gea-Sergio Ramos	5	Nacho-Sergio Ramos	2
Diego Costa-Gerard Piqué	1	Sergio Busquets-Sergio Ramos	23
Diego Costa-Isco	6		

In this scenario, we examine the football match between Portugal and Spain during the 2018 World Cup in Russia, which resulted in a three-goal draw. Fig. 4 illustrates the passing pattern among the 11 players of the Spanish national football team during the match. On the right side, we present a list of the players involved, along with the number of passes they were involved in and the total number of goals they scored in the match (within brackets). For the sake of simplicity, the set of individual weights for each player in N is determined by the total amount of passes in which he has taken part divided by 10. Besides, the weights on the links in the network represent the relational strength between players and are calculated by the number of passes (in both directions) made between each pair of connected players divided by 10. Table 8 contains all the information about passes.

For each communication situation (N, u_N, L) and (N, f, L), when considering the effectiveness function given in Husslage et al. (2015) in (7), we approximate the position value of each player $i \in N$, using our sampling proposal for both TU games (N, v^{gcconn}) and (N, v^{awconn}), with $\ell = 10^7$. In practice, we only take a small size of the population of possible orders of the edges.

Table 9 shows the theoretical errors for the estimation of the weighted position value for $i = 11$, corresponding to Sergio Ramos and with $|L_i| = 10$, that Corollaries 4.3 and 4.4, respectively, establish for this sampling size and for several values of α.

Table 9
Theoretical errors for the weighted position value by using $\ell = 10^7$.

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
(N, v^{gcconn})	0.00350	0.00389	0.00466
(N, v^{awconn})	0.83507	0.92666	1.11056

Table 10, analogously, depicts the theoretical errors that inequalities in (22) and (23) ensure, when $\ell = 10^7$, for the estimation of the position value for $i = 11$ (Sergio Ramos) and for several values of α.

Table 10
Theoretical errors for the position value by using $\ell = 10^7$.

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
(N, v^{gcconn})	0.00194	0.00215	0.00257
(N, v^{awconn})	0.46149	0.51219	0.61373

Note that, for the case of the additive weighted connectivity TU game on the network of passes for the Spanish national football team, we have that $\left(\sum_{j\in N} w_j\right) \cdot \max_{lh\in L} k_{lh} = 238.48$.

Table 11 shows the positions of players in the rankings specified by the estimation of the position value, even in its weighted form with $\ell = 10^7$, for the 11 players belonging to the team. Recall that the usage of sampling is justified in this setting since the obtaining of the position value is based on the determination of the Shapley value for the link game. In this case, we have a total amount of 49 edges, which do not allow for computing it in an exact way, in spite of only having 11 players.

In view of the obtained results in Table 11, Sergio Ramos (k) usually occupies position 1 in the rankings, with the exception of the case of the position value for the grand coalition connectivity game (see rows 1–2). Under such perspective, he goes to position 5. Notice that this player counts with the larger amount of individual passes and take a key role as a defender. Jordi Alba (g) and Isco (f) occupy positions 2 and 3 (except for the case of the position value for the TU game v^{gcconn}), according to the scheme of individual passes that they do. In general, we check that under the usage of the additive weighted connectivity game, the position value and the weighted position value provides a very similar ranking although the numerical results are obviously different. Only four players differ, with a difference of one position. The weighted approach of the position value also provide a similar ranking for the players under the grand coalition connectivity game's approach. The main differences are that players (e), Gerard Piqué, and (i), Nacho, now exchange their positions with respect, for instance, to the mentioned ranking based on the additive weighted connectivity game with the weighted position value. From the three considered rankings, we can state that the strikers do not have a determining role to play when considering the position value (see the case of Diego Costa and Nacho, respectively). The ranking based on the position value for the grand coalition connectivity game (rows 1–2) deserves special consideration. Here, David de Gea (c), Sergio Busquets (j) and Jordi Alba (g) occupy the first three positions.

In a more general manner, Spearman's and Kendall's correlations allow us to analyze the possible association between pairs of variables, through a coefficient of association between the rankings. Table 12 numerically summarizes both measures. In view of such results, $\hat{\pi}(N, v^{gcconn}, L)$ prescribes the most different ranking, with a no clear association with the rest of the rankings, and being it slightly reversed with respect to the ranking based on $\hat{\pi}(N, v^{awconn}, L)$. Besides, the similarities between the rankings based on $\hat{\pi}(N, v^{awconn}, L)$ and $\hat{\pi}^{\omega}(N, v^{awconn}, L)$ are greater than those ones based on $\hat{\pi}(N, v^{gcconn}, L)$ and $\hat{\pi}^{\omega}(N, v^{gcconn}, L)$.

Table 12
Spearman's correlation matrix (upper triangular matrix) and Kendall's correlation matrix (lower triangular matrix) for the rankings of the Spanish national football team.

	$\hat{\pi}(N, v^{gcconn}, L)$	$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	$\hat{\pi}(N, v^{awconn}, L)$	$\hat{\pi}^{\omega}(N, v^{awconn}, L)$
$\hat{\pi}(N, v^{gcconn}, L)$	–	0.345	0.018	0.064
$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	0.273	–	0.836	0.882
$\hat{\pi}(N, v^{awconn}, L)$	−0.018	0.709	–	0.982
$\hat{\pi}^{\omega}(N, v^{awconn}, L)$	0.055	0.782	0.927	–

However, unlike the approaches in the literature on centrality measures, we highlight that this approach implicitly allows us to provide a measure of the influence of each of the edges in the graph to be analyzed at the same time, becoming a powerful tool in the network analysis. It is worth remembering that the position value is obtained, in practice, from the Shapley value for the link game. The Shapley value has already been widely considered in the game-theoretical literature as a measure of node centrality. Thus, this solution applied to the link game, innovatively provides an edge centrality measure in this context, which has not been used until now. Table B.1 in Appendix B of the ORS provides the estimation of the Shapley value for the TU games $(L, r^L_{v^{gcconn}})$ and $(L, r^L_{v^{awconn}})$ when using $\ell = 10^7$ permutations of edges, respectively. Next, by simplicity, we only focus on identifying the 10 most influential edges in our network, included in Table 13.

Table 11
Positions of players using the position value (π) and its weighted version (π^{ω}) under the approaches given by (N, v^{gcconn}) and (N, v^{awconn}), respectively.

	Players	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
(N, v^{gcconn})	$\hat{\pi}$	0.0821	0.0824	0.1207	0.0865	0.0889	0.0890	0.0927	0.0869	0.0889	0.0929	0.0890
	Pos.	11	10	1	9	6	4	3	8	7	2	5
	$\hat{\pi}^{\omega}$	0.0938	0.0675	0.0353	0.0377	0.1055	0.1190	0.1210	0.0827	0.0976	0.1102	0.1297
	Pos.	7	9	11	10	5	3	2	8	6	4	1
(N, v^{awconn})	$\hat{\pi}$	25.8204	16.7859	10.2529	13.0654	20.1048	27.7678	34.3119	20.1427	18.3471	17.2969	34.5839
	Pos.	4	9	11	10	6	3	2	5	7	8	1
	$\hat{\pi}^{\omega}$	27.3438	12.2241	2.7614	5.1719	19.8652	32.8996	39.7137	18.4597	16.3095	17.1850	46.5658
	Pos.	4	9	11	10	5	3	2	6	8	7	1

Table 13
Top-10 edges of the Spanish national team, according to the Shapley value of the link games.

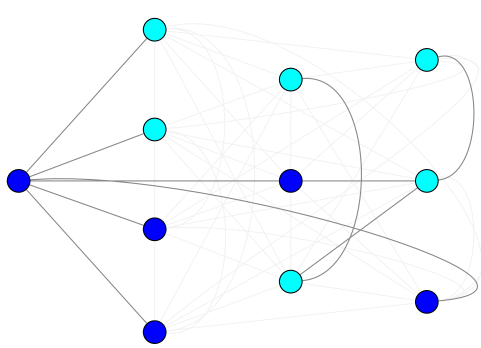
	$\hat{Sh}(L, r^L_{v^{gcconn}})$		$\hat{Sh}(L, r^L_{v^{awconn}})$	
	Pair of players	Alloc.	Pair of players	Alloc.
1	David de Gea-Jordi Alba	0.0425	Jordi Alba-Sergio Ramos	11.3596
2	David de Gea-Sergio Busquets	0.0421	Isco-Jordi Alba	10.2668
3	David de Gea-Isco	0.0393	Andrés Iniesta-Sergio Ramos	9.0507
4	David de Gea-Nacho	0.0393	Andrés Iniesta-Jordi Alba	8.4281
5	David de Gea-Sergio Ramos	0.0392	Isco-Sergio Ramos	7.4889
6	David de Gea-Gerard Piqué	0.0389	Jordi Alba-Koke	7.3670
7	David Silva-Koke	0.0235	Andrés Iniesta-Isco	6.9551
8	Andrés Iniesta-Diego Costa	0.0234	Gerard Piqué-Sergio Ramos	6.7874
9	Andrés Iniesta-Koke	0.0233	Sergio Busquets-Sergio Ramos	6.7686
10	David Silva-Diego Costa	0.0233	Gerard Piqué-Jordi Alba	6.7619

In the following, we briefly comment on the results, although we can already check at a glance the influence of the TU game under consideration in the resulting ranking of edges. Fig. 5 provides a more comprehensive illustration of these results and partially compare them with the ones in Table 11. The grand coalition connectivity link game $(L, r^L_{v^{gcconn}})$ highlights the influence of those edges involving the goalkeeper (David de Gea, (c)), defenders and midfielders, with those connecting to the goalkeeper being the most influential. However, the other link game considered, the additive weighted connectivity link game $(L, r^L_{v^{awconn}})$, gives greater weight in the network to those players on the right side of the football field, as Jordi Alba (g), Sergio Ramos (k), Andrés Iniesta (a) and Isco (f). Logically, the information contained in the TU game is vital in the outcomes, so a good choice will give key information or the possibility of studying the most probable situations. In any case, the choice of the TU game is up to the user criteria and his or her needs in the analysis.

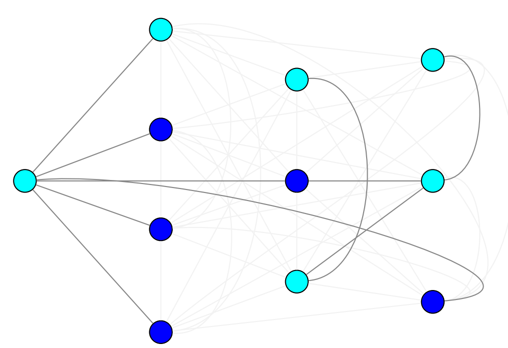
5.2. *The Zerkani network*

In this section, we change our perspective completely. The Zerkani network is a terrorist cell, supported by the Islamic State, that was considered as responsible for the attacks occurred in Paris in 2015 and Brussels in 2016. It is considered a representative example of terrorist group that leads to panic on European society. The Zerkani network provided personnel, training, planning, attack, escape and evasion.

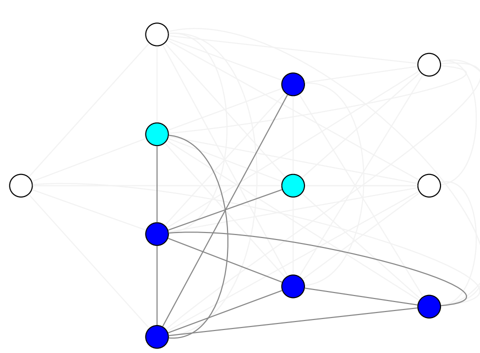
Fig. 6 shows its associated graph as well as the overall list of its members. The nodes denote the terrorists and the edges represent the connections among them. After conducting multiple analysis, it was determined that Abdelhamid Abaaoud and the recruiter Khalid Zerkani were identified as the individuals accountable for orchestrating the tactical operations behind the attacks in Paris and Brussels. Table 14 enumerates the 11 possible types of connections, that has associated a weight and that is associated to a corresponding edge in the graph.



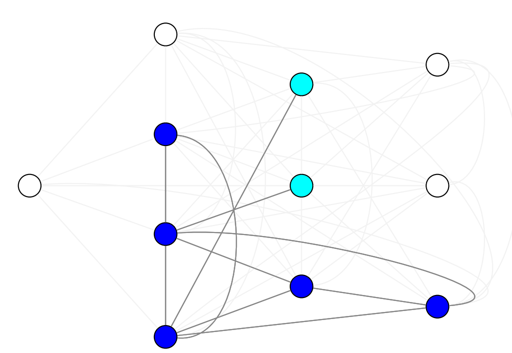
(a) Top-10 edges under $\hat{Sh}(L, r^L_{v^{gcconn}})$ and top-5 nodes under $\hat{\pi}(N, v^{gcconn}, L)$



(b) Top-10 edges under $\hat{Sh}(L, r^L_{v^{gcconn}})$ and top-5 nodes under $\hat{\pi}^{\omega}(N, v^{gcconn}, L)$

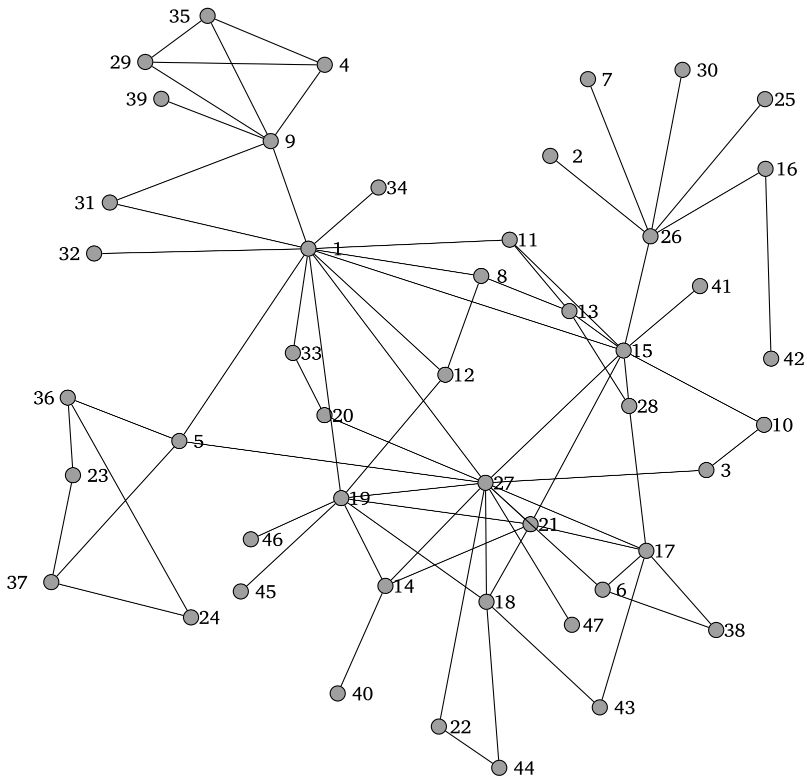


(c) Top-10 edges under $\hat{Sh}(L, r^L_{v^{awconn}})$ and top-5 nodes under $\hat{\pi}(N, v^{awconn}, L)$



(d) Top-10 edges under $\hat{Sh}(L, r^L_{v^{awconn}})$ and top-5 nodes under $\hat{\pi}^{\omega}(N, v^{awconn}, L)$

Fig. 5. Top-10 edges and their nodes (in cyan) under the Shapley value of the link games and top-5 nodes according to the position value (in dark blue) in the Spanish national football team.



(1) Abdelhamid Abaaoud	(13) Ilias Mohammadi	(25) Rabah M.	(37) Salzburg Refugee B
(2) Abderrahmane Ameroud	(14) Khaled Ledjeradi	(26) Reda Kriket	(38) Ibrahim Abdeslam
(3) Abid Aberkan	(15) Khalid Zerkani	(27) Salah Abdeslam	(39) AQI
(4) Adrien Guihal	(16) Miloud F.	(28) Souleymane Abrini	(40) Djamal Eddine Ouali
(5) Ahmed Dahmani	(17) Mohamed Abrini	(29) Thomas Mayet	(41) Soufiane Alilou
(6) Ali Oulkadi	(18) Mohamed Bakkali	(30) Y. A.	(42) AQIM
(7) Anis Bari	(19) Mohamed Belkaid	(31) Sid Ahmed Ghlam	(43) Ibrahim El Bakraoui
(8) Chakib Akrouh	(20) Mohammed Amri	(32) Ayoub el Khazzani	(44) Khalid El Bakraoui
(9) Fabien Clain	(21) Najim Laachraoui	(33) Mehdi Nemmouche	(45) Tawfik A.
(10) Fatima Aberkan	(22) Osama Krayem	(34) Reda Hame	(46) Identity Unknown
(11) Gelel Attar	(23) Paris Attacker A	(35) Macreme Abrougui	(47) Hamza Attou
(12) Hasna Ait Boulahcen	(24) Paris Attacker B	(36) Salzburg Refugee A	

Fig. 6. Graph of the Zerkani network.

Similarly, to remark their influence on the graph, we also consider extra weights on the end nodes of these edges according to the relationships. The third column of Table 14 shows them. Both criteria are considered in Hamers et al. (2019).

Abdelhamid Abaaoud, Fabien Clain, Khalid Zerkani, Miloud F., and Mohamed Belkaid emerged with the largest weights among the members of the Zerkani network, based on their direct involvement and active participation. These weights are considered as a measure of the danger of terrorism, in terms of the capability of attacking or receiving funding to support new actions. For the case of the edges, the weight between two members takes a large worth when their communication is more frequent than the communication of other members of the network.

Below, we evaluate the performance of our proposals for estimating the position value in this example. We approximate the position value of each player $i \in N$, using our sampling proposal with $\ell = 10^7$. In practice, we only take a small size of the population of possible orders of edges.

Table 14
Relations, weights for edges and for initial nodes.

Relationships	Weights on edges	Extra weight for initial nodes
"Associate of"	2	0
"Brother of"	1	0
"Commander of"	2	2
"Family relationship"	1	0
"Funded"	1	2
"Lived with"	2	0
"Nephew of"	1	0
"Recruiter of"	1	1
"Supporter of"	1	1
"Traveled to Syria with"	2	0
"Traveled with"	2	0
"Associate and traveled with"	4	0
"Traveled and lived with"	4	0

Table 15 shows the theoretical errors for the estimation of the weighted position value for player $i = 15$, that corresponds to Khalid Zerkani, with $|L_i| = 9$, when using $\ell = 10^7$. Specifically, we obtain the errors given by Corollaries 4.3 and 4.4 for several values of α.

Table 15
Theoretical errors for the weighted position value by using $\ell = 10^7$.

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
(N, v^{gcconn})	$2.903 \cdot 10^{-3}$	$3.221 \cdot 10^{-3}$	$3.860 \cdot 10^{-3}$
(N, v^{awconn})	0.871	0.966	1.158

Similarly, we include in Table 16, the theoretical errors that inequalities in (22) and in (23) establish for the estimation of the position value of player i, with $i = 15$, when $\ell = 10^7$.

Table 16
Theoretical errors for the position value by using $\ell = 10^7$.

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
(N, v^{gcconn})	$1.742 \cdot 10^{-3}$	$1.933 \cdot 10^{-3}$	$2.136 \cdot 10^{-3}$
(N, v^{awconn})	0.523	0.580	0.695

We distinguish the two approaches of TU games considered. Notice that, for obtaining such values for the case of the additive weighted connectivity TU game on Zerkani network, we have that $\left(\sum_{j\in N} w_j\right) \cdot \max_{lh\in L} k_{lh} = 300$.

Table 17 partially displays the top-10 of the terrorists belonging to the Zerkani network based on the position value estimation by using $\ell = 10^7$. More details can be found in the complete list of the members of the Zerkani network in Appendix C of the ORS. As the existing resources for surveillance of potential terrorists are limited, the task of identifying the most potentially dangerous members of the network is key to keeping them under surveillance.

From Table 17, some conclusions can be extracted from the top-10 of terrorists within the Zerkani network when the position value is considered.

Under the approach of the grand coalition connectivity game (N, v^{gcconn}), Khalid Zerkani now moves up from sixth to fourth position when the weights are considered in the position value. The top-3 is occupied by the same people: Reda Kriket, which moves up to the first position, and Adelhamid Abaaoud and Mohamed Belkaid, that go to the second and the third position, respectively. The main differences are the cases of Salah Abdeslam, Fabien Clain, Anis Bari and Y.A. (at fourth, fifth, seventh, and eighth positions in the non-weighted case), and of AQI and Miloud F., that are ranked at ninth and sixth positions in the weighted and non-weighted case, respectively. Hamza Attou is at tenth position under both perspectives. The remaining members of the ranking are the same, although they occupy different positions.

By considering the additive weighted connectivity game (N, v^{awconn}), Khalid Zerkani, the considered leader of the terrorist cell, occupies the fourth and the third position under the estimation of the non-weighted and weighted position value, respectively. The first position is occupied by Abdelhamid Abaaoud, with Salah Abdeslam and Mohamed Belkaid occupying the second positions. As a general comment, note that despite not occupying the same positions, the two rankings contain mostly the same terrorists. The exceptions are Khaled Ledjeradi, who is ranked tenth in the unweighted case and disappears in the weighted case, and Miloud F., who is ranked ninth in the weighted case.

In both cases, those terrorists in the first positions of the rankings are those ones in Fig. 6 that have the largest number of connections to other members of the network. In general, this statement can be justified by the definition of the position value in terms of the link game.

Next, we make an overall analysis of the obtained rankings by using Spearman's and Kendall's correlations, that are numerically shown in Table 18. Notice that the most similarities are found between the rankings prescribed by $\hat{\pi}(N, v^{gcconn}, L)$ and $\hat{\pi}^{\omega}(N, v^{gcconn}, L)$. However, the rankings associated with the TU games (N, v^{gcconn}) and (N, v^{awconn}) are barely associated with each other (correlations close to zero). Even so, the correlations between the rankings prescribed by $\hat{\pi}(N, v^{awconn}, L)$ and the two given rankings for the TU game (N, v^{gcconn}) indicate, by their sign, an inverse association.

Table 18
Spearman's correlation matrix (upper triangular matrix) and Kendall's correlation matrix (lower triangular matrix).

	$\hat{\pi}(N, v^{gcconn}, L)$	$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	$\hat{\pi}(N, v^{awconn}, L)$	$\hat{\pi}^{\omega}(N, v^{awconn}, L)$
$\hat{\pi}(N, v^{gcconn}, L)$	–	0.978	−0.095	0.018
$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	0.914	–	−0.072	0.083
$\hat{\pi}(N, v^{awconn}, L)$	−0.150	−0.156	–	0.851
$\hat{\pi}^{\omega}(N, v^{awconn}, L)$	−0.011	0.068	0.702	–

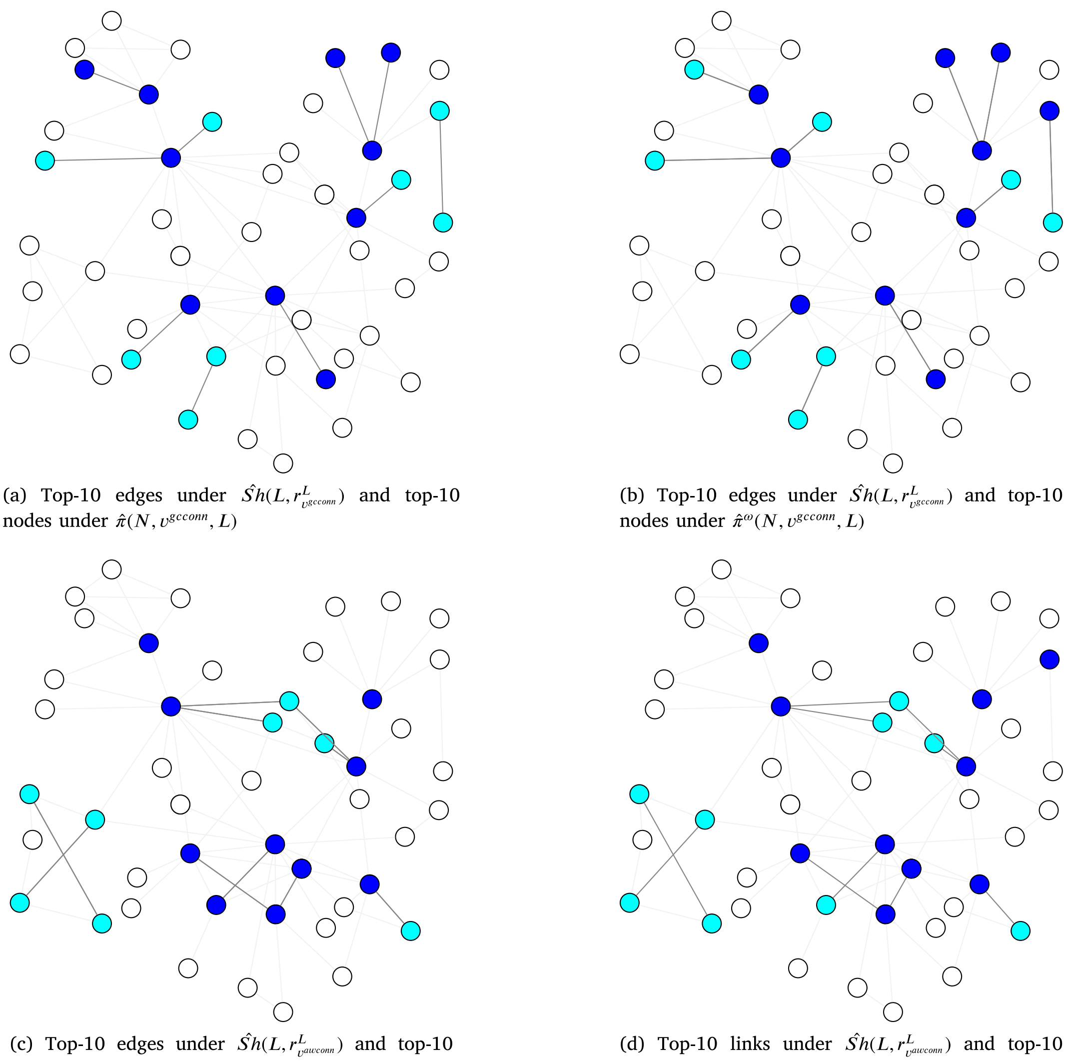
The node centrality analysis prescribed by the position value implicitly provides, as in the previous example, a ranking of the influence of the edges in the Zerkani network. Recall that the position value is based on the Shapley value of the link game. Table 19 depicts the 10 most influential edges in the Zerkani network based on the estimation of the Shapley value for the TU games $(L, r^L_{v^{gcconn}})$ and $(L, r^L_{v^{awconn}})$ with $\ell = 10^7$

Table 17
Top-10 terrorists in the Zerkani network, according to the position value.

	$\hat{\pi}(N, v^{gcconn}, L)$		$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	
	Terrorist	Alloc.	Terrorist	Alloc.
1	Reda Kriket	0.1397	Reda Kriket	0.1397
2	Abdelhamid Abaaoud	0.0740	Abdelhamid Abaaoud	0.1183
3	Mohamed Belkaid	0.0697	Mohamed Belkaid	0.1045
4	Salah Abdeslam	0.0404	Khalid Zerkani	0.0618
5	Fabien Clain	0.0376	Fabien Clain	0.0601
6	Khalid Zerkani	0.0371	Miloud F.	0.0466
7	Anis Bari	0.0352	Salah Abdeslam	0.0404
8	Y. A.	0.0351	Anis Bari	0.0352
9	AQI	0.0350	Y. A.	0.0350
10	Hamza Attou	0.0350	Hamza Attou	0.0350
	$\hat{\pi}(N, v^{awconn}, L)$		$\hat{\pi}^{\omega}(N, v^{awconn}, L)$	
	Terrorist	Alloc.	Terrorist	Alloc.
1	Abdelhamid Abaaoud	23.2243	Abdelhamid Abaaoud	33.5083
2	Salah Abdeslam	22.4601	Mohamed Belkaid	32.0624
3	Mohamed Belkaid	22.4579	Khalid Zerkani	28.6614
4	Khalid Zerkani	17.7802	Fabien Clain	20.7469
5	Reda Kriket	14.4410	Salah Abdeslam	19.0163
6	Fabien Clain	13.7486	Reda Kriket	12.5198
7	Mohamed Bakkali	10.7938	Mohamed Bakkali	9.5819
8	Najim Laachraoui	10.7547	Najim Laachraoui	8.3530
9	Mohamed Abrini	7.9700	Miloud F.	8.1037
10	Khaled Ledjeradi	7.7340	Mohamed Abrini	7.9700

Table 19
Top-10 edges of the Zerkani network, according to the Shapley value of the link games.

	$\hat{Sh}(L, r^L_{v^{gcconn}})$		$\hat{Sh}(L, r^L_{v^{awconn}})$	
	Pair of terrorists	Alloc.	Pair of terrorists	Alloc.
1	Anis Bari - Reda Kriket	0.0704	Mohamed Belkaid - Identity Unknown	7.6692
2	Fabien Clain - AQI	0.0702	Mohamed Belkaid - Tawfik A.	7.6504
3	Y. A. - Reda Kriket	0.0701	Miloud F. - AQIM	7.2270
4	Salah Abdeslam - Hamza Attou	0.0700	Fabien Clain - AQI	6.4286
5	Khalid Zerkani - Soufiane Alilou	0.0699	Khalid Zerkani - Soufiane Alilou	6.2271
6	Miloud F. - AQIM	0.0699	Abd. Abaaoud - Ayoub el Khazzani	5.6688
7	Abd. Abaaoud - Reda Hame	0.0698	Abd. Abaaoud - Reda Hame	5.5547
8	Khaled Ledjeradi - Djamal Eddine Ouali	0.0698	Mohamed Belkaid - Salah Abdeslam	5.5518
9	Mohamed Belkaid - Tawfik A.	0.0697	Salah Abdeslam - Hamza Attou	5.5228
10	Abd. Abaaoud - Ayoub el Khazzani	0.0695	Khaled Ledjeradi - Djamal Eddine Ouali	5.4095



(a) Top-10 edges under $\hat{Sh}(L, r^L_{v^{gcconn}})$ and top-10 nodes under $\hat{\pi}(N, v^{gcconn}, L)$

(b) Top-10 edges under $\hat{Sh}(L, r^L_{v^{gcconn}})$ and top-10 nodes under $\hat{\pi}^{\omega}(N, v^{gcconn}, L)$

(c) Top-10 edges under $\hat{Sh}(L, r^L_{v^{awconn}})$ and top-10 nodes under $\hat{\pi}(N, v^{awconn}, L)$

(d) Top-10 links under $\hat{Sh}(L, r^L_{v^{awconn}})$ and top-10 nodes under $\hat{\pi}^{\omega}(N, v^{awconn}, L)$

Fig. 7. Top-10 edges and their nodes (in cyan) under the Shapley value of the link games and top-10 nodes according to the position value (in dark blue) in the Zerkani network.

permutations of edges. Table C.3 of Appendix C.1 in the ORS provides the comprehensive estimation of the Shapley value for all edges in the network.

In view of the obtained rankings, 8 of the 10 edges in both rankings are the same although their positions differ. Only one of the 10 edges in each ranking involves Khalid Zerkani, who is considered to be the network's top leader. Fig. 7 provides a graphical comparison of these results and compare them with the ones in Table 17. Moreover, the most influential edges under the grand coalition connectivity link game, $(L, r^L_{v^{gcconn}})$, involved terrorists in the first 10 positions of the rankings prescribed by the position value (in its weighted and non-weighted version) and shown in Table 17. The exceptions are the edges Khaled Ledjeradi-Djamal Eddine Ouali (in both scenarios) and Miloud F.-AQIM for the case of $\hat{\pi}(N, v^{gcconn}, L)$. Even more, the most relevant edge involves Reda Kriket, who occupies the first position in the ranking of the most influential nodes. However, there exist some terrorists in the 10 most influential edges under the additive weighted connectivity link game $(L, r^L_{v^{awconn}})$ who do not belong to the top-10 of the most influential nodes. The most powerful edge deserves special mention, which corresponds to the Mohamed Belkaid - Identity Unknown edge.

This unidentified person plays a key role in the Zerkani network, under the focus on edges that brings us the position value. Therefore, with this approach, it is possible to obtain significant and complementary information about the relations between nodes in a network, which otherwise can go unnoticed with the existent centrality measures only based on ranking nodes in a graph.

5.3. The position value versus other centrality measures

To conclude our analysis, we study how our proposal performs with other classical centrality measures in the literature as well as the Shapley value of the TU games (N, v^{gcconn}) and (N, v^{awconn}).

Let (N, L) be an undirected graph and take an agent i of N. From the classical approach to social network analysis, we consider the following three centrality measures on the set of nodes of (N, L).

- The *normalized degree centrality* of i is specified by

$$degree(i) = \frac{d(i)}{|N|-1}, \tag{24}$$

  where $d(i)$ denotes the amount of direct connections impinging on i.
- The *normalized betweenness centrality* of i is prescribed by

$$betweenness(i) = \frac{2}{(|N|-1)(|N|-2)} \sum_{k,j\in N\setminus\{i\}:k<j} \frac{p_{kij}}{p_{kj}}, \tag{25}$$

  where p_{kj} prescribes the amount of shortest paths between agents k and j, and p_{kij} the amount of shortest paths between k and j that include agent i.
- The *normalized closeness centrality* of agent i is formally given by

$$closeness(i) = \frac{|N|-1}{\sum_{j\in N} s_{ij}}, \tag{26}$$

  where s_{ij} denotes the shortest distance between agents i and j.

Once determined, the decreasing order of each of these classical centrality measures in social network analysis also determines a ranking for the nodes. The corresponding rankings to these three indices for the examples of the Spanish team and the Zerkani network are included in Appendix D of the ORS.

By the Shapley value-based nature of the position value, it is interesting to compare our results with the ranking specified by the Shapley value for the two considered TU games, (N, v^{gcconn}) and (N, v^{awconn}).[5] These rankings can be found in Appendix E of the ORS.

According to these results, we study the similarities of the rankings in terms of the associated correlations. We omit the case of the Shapley value corresponding to the grand coalition connectivity game, i.e., $Sh(N, v^{gcconn})$, since as expected, it is constant in its components and its variance is zero. However, it is worth noting that unlike the position value for the grand coalition connectivity game, the Shapley value of this TU game does not capture the topology of the graph and as a result, it assigns the same value to each node. In the case of the Spanish national football team with respect to the three classical measures, from Table 20, the ranking specified by the weighted position value for the grand coalition connectivity game (N, v^{gcconn}) is most similar to those provided by the classical measures of centrality. In the case of normalized degree centrality, normalized betweenness centrality and normalized closeness centrality, Spearman's and Kendall's correlations are greater than 0.5, with the first being slightly higher. The same conclusions can be drawn for the two rankings based on the additive weighted connectivity game (N, v^{awconn}), where the Kendall's correlations are around 0.5 and the Spearman's correlations are greater than 0.55. As for the rankings based on the position value of the grand coalition connectivity game, $\hat{\pi}(N, v^{gcconn}, L)$, there are no similarities as their correlations are close to zero (or even negative).

Table 20
Spearman's correlation matrix (rows 1–3) and Kendall's correlation matrix (rows 4–6) for the rankings of the Spanish national football team.

		$\hat{\pi}(N, v^{gcconn}, L)$	$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	$\hat{\pi}(N, v^{awconn}, L)$	$\hat{\pi}^{\omega}(N, v^{awconn}, L)$
Spearman	degree	0.005	0.731	0.569	0.597
	betweenness	0.227	0.798	0.557	0.599
	closeness	0.005	0.731	0.569	0.597
	Shapley value	–	–	0.992	0.988
Kendall	degree	−0.042	0.624	0.499	0.499
	betweenness	0.081	0.650	0.488	0.488
	closeness	−0.042	0.624	0.499	0.499
	Shapley value	–	–	0.963	0.963

Next, we consider the case of the Zerkani network. The rankings using the TU game (N, v^{awconn}) indicate the largest correlations, being those ones associated with the position value greater than those prescribed by the weighted version of the position value. We even check that Spearman's correlations are always larger than Kendall's correlations when considering the TU game (N, v^{awconn}). However, under the grand coalition connectivity game (N, v^{gcconn}), the correlations are usually negative (except in the case of betweenness for the weighted position value) and close to zero, so no similarities can be justified (see Table 21).

Table 21
Spearman's correlation matrix (rows 1–3) and Kendall's correlation matrix (rows 4–6) for the rankings in the Zerkani network.

		$\hat{\pi}(N, v^{gcconn}, L)$	$\hat{\pi}^{\omega}(N, v^{gcconn}, L)$	$\hat{\pi}(N, v^{awconn}, L)$	$\hat{\pi}^{\omega}(N, v^{awconn}, L)$
Spearman	degree	−0.187	−0.165	0.944	0.871
	betweenness	0.047	0.084	0.845	0.883
	closeness	−0.060	−0.076	0.730	0.567
	Shapley value	–	–	0.969	0.965
Kendall	degree	−0.246	−0.224	0.851	0.740
	betweenness	−0.010	0.015	0.717	0.743
	closeness	−0.067	−0.083	0.549	0.424
	Shapley value	–	–	0.957	0.912

It is important emphasize that the largest correlations indicate a high degree of similarity between the rankings given by the Shapley value of the TU game (N, v^{awconn}) and the position value in both scenarios. It is due mainly to that the TU game (N, v^{awconn}) takes into consideration the main features of the graph. However, these differences between the rankings can be explained by the fact that the position value adds even more additional and relevant information about nodes and edges indicating that these variations must be taken into account since it may lead to big differences in the results. Even so, the information available and the potential user's own criteria will determine which ranking to use. This does not preclude their being obtained simultaneously, so that different scenarios can be evaluated at the same time.

6. Concluding remarks

In this work, we have introduced the position value as a useful tool to rank nodes and edges of a network defined by a graph. To achieve these rankings, we have provided a procedure to approximate the position value for communication situations based on simple random sampling with replacement. This approach is particularly valuable when handling communication situations with a large amount of connections, even if the number of players involved is not excessively large, as its exact computation becomes intractable. We have provided some theoretical results to ensure that our proposal correctly approximates the position value for any communication situation and, consequently, to determine the adequate sampling size in the estimation. The performance of our sampling proposal has been evaluated as application in

[5] Note that the Shapley value of the TU games (N, v^{gcconn}) and (N, v^{awconn}) corresponds to the Myerson value (Myerson, 1977) associated to the communication situations (N, u_N, L) and (N, f, L), respectively.

two real examples that can be modeled in terms of a communication situation, showing that this measure is specially advantageous when dealing with networks represented by a graph. In fact, the advantage and strength of this approach compared to other measures of centrality is revealed by integrating the particular and specific features of the network nature, independently whether it is taken into account in the original game and by allowing us to obtain, not only a ranking of the nodes but also of the edges of the graph. Therefore, with the position value as a new centrality measure, we can get significant and more realistic information about the most influential nodes and, at the same time, the stronger relations between them.

The theoretical analysis of the properties of the position value for communication situations (cf. Borm et al., 1992) has already had sufficient impact on cooperative game literature because of its interesting properties over the last decades. However, the difficulties of its exact calculation in general has not been addressed due to the enormous computational complexity associated. This fact has undoubtedly limited its practical application as a solution for many of the real-life situations that could be modeled in the form of a network structure represented by a graph. The position value is obtained from the Shapley value (Shapley, 1953) for the TU game defined over the set of edges communicating the existing nodes. Hence, in addition to the effort required to handle the corresponding link game, the computational problems arisen from the Shapley value computation can now be extended to this context as well. As in the estimation of solution concepts for TU games (Fernández-García & Puerto-Albandoz, 2006 and Castro et al., 2009; for the Shapley value approximation), the use of sampling methodologies (Cochran, 2007) to approximate the position value in this work solves these drawbacks. Its usage is justified by its formulation in terms of a population mean. From a purely statistical approach, a thorough analysis of the properties of the resulting estimator for the position value was covered. Moreover, the task of bounding the estimation error was also addressed through the establishment of specific results for the position value estimation. From a computational perspective, it is important to emphasize that the proposed procedure can be easily computed in parallel.

With this work, the scope of application of the position value estimation procedure is extended to any multi-agent situation that can be modeled as a communication situation, even with a very large number of edges involved. As illustration, we first focus on the analysis of football passing networks to establish rankings of the players of the Spanish national football team based on the estimated position value as well as a ranking of the pairs of players with the best performance during this football match according to the Shapley value of the link game. Second, we address the well-known problem of ranking terrorists in networks, in this case, under the approach given by the position value, obtaining, likewise, valuable information through the ranking of the stronger relations between the terrorists of the Zerkani network, which constitutes a big distinction with all the others existent centrality measures. Moreover, with these applications have been highlighted that independently of the initial TU game considered, the position value always takes into account the topology of the graph. Therefore, when dealing with networks defined by a graph, it is another important aspect to consider with respect to the centrality measures existent in the literature. Note that we have applied it in three very different fields such as transport, sports and security, although, as mentioned, communication situations also arise in other many contexts as far apart as economics, health or logistics, among others.

CRediT authorship contribution statement

Encarnación Algaba: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Alejandro Saavedra-Nieves:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Online resource section: supplementary material

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