Control Predictivo de sistemas lineales con restricciones para seguimiento de referencias

Model Predictive Control for Tracking Constrained Linear Systems

Thesis submitted for the Degree of Doctor of Philosophy

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Notation

Notation	Meaning
$T > 0$	Given a symmetric matrix T, $T > 0$ denotes that it is positive definite
$T > P$	$T - P > 0$
$\|x\|_P$	The weighted Euclidean norm of x, i.e.,$\|x\|_P = \sqrt{x^T P x}$, being $P > 0$ a symmetric matrix
(a, b)	$[a', b']'$
$Proj_a(\Gamma)$	Consider $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, and set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, then the projection operation is defined as the following set $Proj_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b},\ (a, b) \in \Gamma\}$
$\mathcal{U} \oplus \mathcal{V}$	Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$
$\mathcal{U} \ominus \mathcal{V}$	Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Pontryagin set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$
$\mathbf{t}$	Is a generic vector defined as $\mathbf{t} \triangleq \{t(0), t(1), \ldots\}$
$\mathcal{D}$ is a C set	If $\mathcal{D}$ is compact, convex and non empty.
$\mathbf{0}_{n,m}$	Denotes a matrix of zeros $\in \mathbb{R}^{n \times m}$.

Notation	**Meaning**
$M\mathcal{V} \subset \mathbb{R}^n$	For a given matrix $M \in \mathbb{R}^{n\times m}$ and a set $\mathcal{V} \subset \mathbb{R}^m$, the set $M\mathcal{V} \subset \mathbb{R}^n$ denotes the set $\{y = Mv,\, v \in \mathcal{V}\}$
$\lambda\mathcal{X}$	For a given positive λ, $\lambda\mathcal{X} = \{\lambda x : x \in \mathcal{X}\}$
$vert(\Gamma)$	Is the set of vertexes of a given C set Γ
$\mathcal{B}^N \subset \mathbb{R}^N$	Is the unitary ball $\mathcal{B}^N = \{b \in \mathbb{R}^N : \|b\|_\infty \leq 1\}$
$int(\mathcal{X})$	Denotes the interior of a set $\mathcal{X}$
$V_\perp$	Denotes a matrix such that $V^\top V\perp = 0$ and the matrix $[V, V\perp]$ is a non-singular square matrix.
$\bigoplus_{i=0}^{n} \mathcal{W}_i$	Denotes the Minkowski addition of the n sets $\mathcal{W}_i$ where $i = 0, \ldots, n$ $\bigoplus_{i=0}^{n} \mathcal{W}_i \triangleq \mathcal{W}_1 \oplus, \ldots, \oplus \mathcal{W}_n$
$\|A\|_p$	Denotes the p-norm of the matrix A
$\mathbf{1}$	Denotes a vector of apropriate dimension where each component is 1
I_p	Denotes the identity matrix of order p

Chapter 1

Introducción

Este capítulo tiene como fin poner en contexto la tesis desarrollada. Para ello, en primer lugar se presenta la relevancia que, en el campo de la industria, tiene el problema del control de sistemas sometidos a amplios cambios en el punto de funcionamiento. A continuación se hace un breve balance de las estrategias de control planteadas para la solucionarlo, enfocándose en la estrategia que se considera adecuada para abordarlo: el control predictivo. Seguidamente se hace un resumen de los controladores predictivos y se presenta de forma sucinta el problema de la estabilidad con restricciones y cómo el control predictivo soluciona dicha problemática. Para finalizar se presenta también el problema de las incertidumbres en el control predictivo y las formulaciones robustas del mismo.

1.1 Motivación y objetivos de la tesis

La forma de operar procesos en la industria ha experimentado avances significativos durante los últimos años, guiados por la necesidad de producir de forma segura, limpia y en condiciones competitivas, productos que satisfagan las necesidades del mercado, tanto en cuanto a demanda como en cuanto a calidad y uniformidad. Dos razones justifican este hecho: de un lado, la necesidad de dar respuesta a un mercado que en función de sus hábitos sociales y/o culturales se encuentra cada vez más diversificado y exige, además, productos sujetos a estrictos controles de seguridad, variedad y calidad, con lo que ello comporta en cuanto a ciclos de vida del producto cada vez más cortos.

De otro lado, la necesidad de propiciar un crecimiento sostenible minimizando tanto el impacto medioambiental como el consumo de recursos. Ambos factores contribuyen a que se desee producir de una más eficiente satisfaciendo las exigencias y límites impuestos a los productos.

Por lo tanto resulta deseable buscar técnicas que control que proporcionen leyes que optimicen ciertos criterios de eficiencia garantizando al mismo tiempo la satisfacción de los límites impuestos a los productos. Una de las pocas técnicas que permiten resolver este problema es el control predictivo (de Prada, 2004).

En la industria de procesos es habitual la existencia de un punto de operación óptimo o punto de funcionamiento en el cual el proceso debería permanecer con el fin de maximizar su eficiencia. Sin embargo, muchos procesos a lo largo de su normal funcionamiento se ven sometidos a frecuentes cambios en su punto de funcionamiento, de forma que para éstos no existe un punto de funcionamiento, sino más bien un rango de puntos de funcionamiento en cualquiera de los cuales el proceso puede operar durante un período de tiempo. La selección del punto de operación dentro de este rango se hará conforme a la diversidad de productos, lotes o situaciones en las que la planta se pueda encontrar.

Para ilustrar este tipo de procesos tómese por ejemplo un reactor por lotes. En este proceso se pretende obtener el producto de una reacción que se desarrolla en su interior de forma continua y sobre la que se puede influir típicamente manipulando el control de caudal de los reactivos y el caudal de refrigerante (véase la figura 1.1). Este proceso está sujeto a las restricciones sobre las variables manipulables que imponen las válvulas de entrada de reactivo y refrigerante. Por otro lado variables como la temperatura del reactor o la presión deben permanecer dentro de unos límites permitidos, y la concentración de producto debe cumplir ciertas especificaciones. Además es habitual que el funcionamiento de este reactor cambie su punto normal de operación de forma que durante un período se require un funcionamiento determinado por una ciertas condiciones de temperatura y concentración. Por lo tanto este proceso require un control que permita los cambios en el punto de funcionamiento garantizando el cumplimiento de los límites impuestos sobre ciertas variables.

Otro proceso en el que el funcionamiento varía en un determinado rango es el caso de una planta solar. En este sistema, el objetivo es calentar un determinado fluido gracias a la energía irradiada por el Sol. Para ello el fluido se hace pasar por unos paneles solares o bien unos colectores solares diseñados al efecto. El objetivo de control es situar la temperatura de salida en un determinado rango que permita la utilización

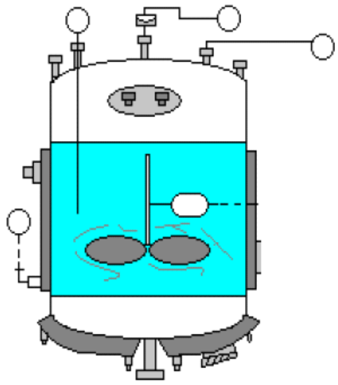

Figure 1.1: Esquema de un reactor continuo de tanque agitado.

provechosa de la energía ya sea para producir electricidad (como en el caso de la planta DISS que se muestra más adelante) o para otros usos, como producir frío mediante una máquina de absorción (como es el caso de la planta solar de refrigeración que se muestra a continuación). Obsérvese que las variables manipulables suelen ser válvulas que tienen unos límites de actuación determinados y generalmente con la velocidad de cambio también limitada. Además las temperaturas y los caudales a lo largo de los colectores o paneles deben permanecer dentro de unos límites para garantizar el buen funcionamiento de los mismos.

Figure 1.2: Fotografía de la planta solar DISS.

Debido a la evolución a lo largo del día de la potencia irradiada por el Sol[1] las tem-

[1]En efecto, esta potencia es periódica con una forma creciente por la mañana y decreciente a lo largo de la tarde.

peraturas de los colectores solares cambian, alejándose de los valores deseados. Por ello el sistema de control debe garantizar el buen funcionamiento de la planta cambiando su punto de operación en función de la potencia irradiada con el objeto de mantener la temperatura de salida dentro del rango permitido y además las temperaturas a lo largo de los colectores dentro de los límites. El sistema de control debe pues actuar sobre la planta para garantizar los límites de las temperaturas ante cada cambio de punto de funcionamiento.

En consonancia con las necesidades expuestas, este trabajo tiene por finalidad el desarrollo de una estrategia de control avanzado de procesos con puntos de operación cambiantes en presencia de restricciones que permitan una operación eficiente, flexible e integral de forma que, haciendo un uso racional de los recursos disponibles, se garantice de manera uniforme la seguridad y calidad del producto.

1.1.1 El control de plantas con puntos de operación cambiantes

A la hora de acometer el problema de control que se plantea es necesario tener en cuenta dos aspectos que los condicionan. El primer aspecto se deriva del amplio rango de operación que presentan las plantas, el cual acentúa la naturaleza no lineal de sus dinámicas (implícita en las ecuaciones constitutivas asociadas a los balances de materia, energía y cantidad de movimiento) y el grado de incertidumbre (estructural y paramétrica) asociado a sus representaciones en espacio de estados. Además en este tipo de plantas es habitual la distribución espacial de algunas, o todas sus variables de estado, lo que hace que su dinámica deba ser descrita por sistemas acoplados de ecuaciones algebraicas, diferenciales ordinarias y ecuaciones en derivadas parciales. Por ello, estos sistemas pueden considerarse como paradigmas de sistemas dinámicos complejos, con elevado acoplamiento y dimensión (fundamentalmente debido a la distribución espacial de sus variables de estado), y sujetos a restricciones de índole económica y medioambiental.

A la naturaleza compleja del sistema se añade la presencia de restricciones en su operación. Estas restricciones pueden ser límites en las variables que permiten manipular la plantas, así como límites impuestos sobre variables del proceso. Estas restricciones pueden derivar de límites físicos de las variables o bien de límites en las zonas de evolución de la planta por motivos económicos, medioambientales o de operación.

La presencia de restricciones condiciona de forma notable el comportamiento de los sistemas acentuando su aspecto no lineal, y pueden ser responsables de pérdidas de rendimiento, mal funcionamiento de la planta e incluso inestabilidad (Mayne, 2001).

La forma tradicional de resolver este problema consiste en el diseño de una estructura jerárquica de control en el que un control a bajo nivel se encarga del control regulatorio de la planta, generalmente realizada por PIDs o autómatas programables interconectados en red. Por encima de éste se encuentra el control de alto nivel en el que se implementa una estrategia avanzada de control, generalmente multivariable. El fin de este control es la determinación de las consignas de los controladores a bajo nivel para mantener el sistema en el punto de operación deseado. Este punto de operación se determina por un nivel de control superior en el que se implementa un optimizador de las consignas (generalmente calculadas en base a un modelo estático de la planta en su conjunto), de acuerdo con los datos de la planta y en base a criterios económicos, provenientes del sistema de integración de información de la planta (CIM). Esta estructura se ilustra en la figura 1.3.

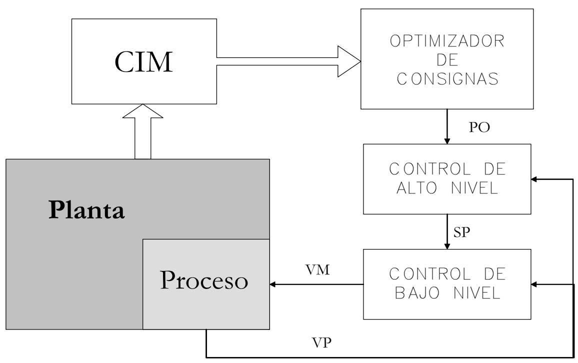

Figure 1.3: Estructura de control jerárquico.

El control de alto nivel se suele diseñar para regular el sistema en torno al punto de operación y evitar la violación de las restricciones. Una de las estrategias de control avanzado que más éxito han tenido en la industrial de procesos ha sido el control predictivo (Qin and Badgwell, 1997) pues incorpora un criterio óptimo y restricciones en la ley de control. Cuando el alto nivel indica un cambio de operación el controlador debe hacer frente a esta contingencia, conduciendo el sistema hacia el nuevo punto de operación. Cuando estos cambios son pequeños, entonces los controladores suelen acometer esta tarea con éxito, pero cuando los cambios de operación son amplios, pueden aparecer problemas ya sea por el cambio de dinámica en el nuevo punto o bien por garantizar la satisfacción de las restricciones en el transitorio al nuevo punto.

Con el fin de gestionar cambios significativos en los puntos de operación, el control a alto nivel se suele dividir en dos subniveles (Becerra, Roberts and Griffiths, 1998): un subnivel inferior encargado de regular el sistema y un subnivel superior encargado de la adaptación del controlador al nuevo punto, o bien de una forma general, de gestionar el control del transitorio ante grandes cambios en la operación. Este esquema se ilustra en la figura 1.4

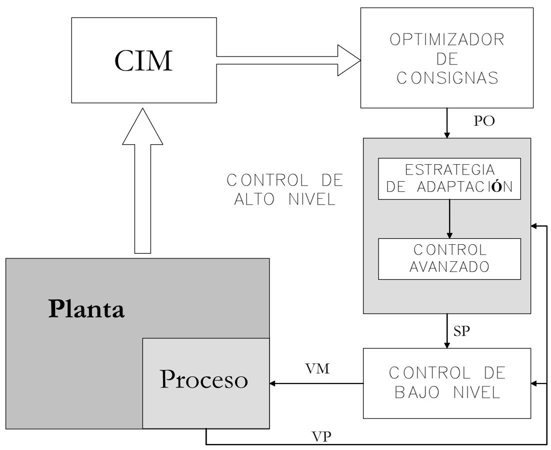

Figure 1.4: Estructura de control jerárquico con alto nivel adaptativo.

Dentro de este esquema se enmarcan por ejemplo los controladores adaptativos (como el clásico *gain scheduling* de los aviones al variar la altura de vuelo). Otros controladores de este tipo son los denominados controladores de referencias o *reference governors* (Gilbert, Kolmanovsky and Tan, 1994; Gilbert, Kolmanovsky and Tan, 1999). Estos controladores corresponden al subnivel superior y asumen que en el subnivel inferior se encuentra un controlador avanzado que estabiliza la planta. Los controladores de referencias tienen como fin gestionar de forma racional las referencias de un proceso con el fin de evitar la violación de restricciones cuando el valor deseado de la consigna cambia. Es de alguna forma una sofisticación del conocido filtro de referencias con el fin de evitar la violación de restricciones. El diseño de este tipo de controladores se hace sin atender a la eficiencia del proceso y con el único fin de evitar la violación de los límites. Este aspecto se trata en (Bemporad, Casavola and Mosca, 1997), en el que propone un método de incorporar cierto criterio de desempeño en el cálculo. Los controladores de referencias también se han extendido con éxito al caso de dinámicas no lineales (Bemporad, 1998*b*; Angeli and Mosca, 1999; Gilbert and Kolmanovsky, 2002).

En el caso del control predictivo existen formulaciones orientadas a gestionar grandes transiciones. Estos controladores permiten pues grandes cambios en el punto de operación y determinan las acciones de control en base a un criterio de desempeño. Sin embargo, la garantía de estabilidad se basa en una estructura jerárquica como la que se muestra en 1.4 en la que el subnivel superior se encarga de conmutar entre el controlador predictivo y el otro controlador orientado a recuperar al sistema en caso de pérdida de factibilidad.

Otra forma de abordar este problema es el llamado control integral en el que se estudian estrategias de control avanzado (generalmente control predictivo) en el que se incorporan objetivos económicos asociados a los cambios de operación. Por lo tanto, parte de la tarea de la optimización del proceso se traslada del optimizador de consignas, con el fin de incorporar de alguna forma el coste asociado a las transiciones en la determinación del punto de operación. En (Becerra and Roberts, 1996; Becerra, Roberts and Griffiths, 1997; Becerra et al., 1998) se plantean diferentes alternativas para la integración en el control predictivo con optimización en línea de objetivos económicos (como solución de un problema de control jerárquico, o solución de un problema de control multiobjetivo, donde se minimizan tanto los objetivos de regulación como los económicos, inclusión de consignas asociadas a costes, etc). En algunos de los trabajos que hacen uso de estructuras jerárquicas con dos capas, en los que se parte del supuesto de que los reguladores de las capas bajas mantienen la estabilidad. En esta tesis, se tiene en cuenta la diferencia entre el modelo y la realidad, planteando un método iterativo que permite al algoritmo converger incluso bajo la existencia de errores de modelado. En (Vesely, Kralova, Harsanyi and Hindi, 1998) se plantean los principios y propiedades básicas de un método factible para optimización de estado estacionario de sistemas complejos de los dos niveles de un controlador jerárquico, de modo que el problema se resuelve a través de la resolución de ecuaciones algebraicas.

Sin embargo, en la mayoría de estos trabajos no se lleva a cabo un estudio de la estabilidad, robustez y convergencia de los esquemas desarrollados, ni del efecto de la interacción entre componentes. Tampoco se aborda en profundidad el problema de estabilidad y transferencia suave entre consignas en aquellos sistemas que están sometidos a frecuentes cambios de especificaciones y a perturbaciones de carga, por lo que éstas deben modificarse con frecuencia durante la operación.

En consecuencia, las estructuras jerárquicas con garantía de estabilidad y satisfacción de restricciones producen un peor desempeño que un control integral debido a su diseño independiente. Por otro lado las estructuras de control integral adolecen de estudios de estabilidad y satisfacción de restricciones. Por ello resulta deseable diseñar estrategias de control que permitan unificar la solución de este problema de control integral para grandes transiciones en un sólo nivel que garantice la estabilidad en presencia de restricciones y tenga en cuenta la optimización de criterios de desempeño. El control predictivo es una de las pocas estrategias que permite el control de sistemas con restricciones atendiendo a un criterio óptimo y garantizando la estabilidad y convergencia al punto de equilibrio (Camacho and Bordons, 2004; Mayne, 2001). Por ello, se propone utilizar el control predictivo como estrategia para abordar el problema que se propone. En la figura 1.5 se ilustra la idea propuesta y en ella se observa que los dos niveles de control de la estructura jerárquica (véase la figura 1.4) se reemplazan por un sólo controlador que realiza simultáneamente la tarea de la estabilización y del control al nuevo punto de consigna.

El control predictivo basado en modelo ha concentrado el esfuerzo de numerosos investigadores en los últimos años, avanzando notablemente las bases teóricas, la comprensión del problema de control, el estudio de sus características y limitaciones y procedimientos de diseño estabilizante (Mayne, Rawlings, Rao and Scokaert, 2000; Limon, 2002). Además el control predictivo ha demostrado ser también una técnica efectiva para el control robusto con restricciones (Mayne et al., 2000; Limon, 2002; De Nicolao, Magni and Scattolini, 1996; Magni, Nijmeijer and van der Shaft, 2001; Fontes and Magni, 2003; Limon, Bravo, Alamo and Camacho, 2005; Limon, Alamo, Salas and Camacho, 2006). Como es bien sabido, el diseño estabilizante de los controladores se basa en el cálculo de regiones invariantes (Blanchini, 1999; Bertsekas, 1972).

En el caso de sistemas lineales con o sin incertidumbres, existen controladores eficientes que permiten controlar la planta con garantía de estabilidad y satisfacción de

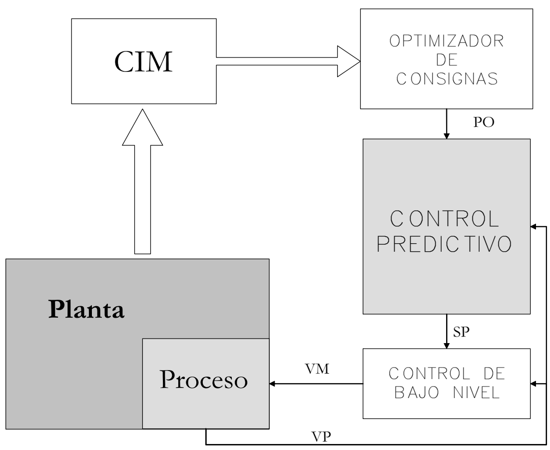

Figure 1.5: Estructura de control integral.

restricciones. En este caso se han propuesto técnicas para simplificar el problema de optimización a resolver (Alamo, Muñoz de la Peña, Limon and Camacho, 2005) y permita una implementación eficiente en línea. Por otro lado también se han desarrollado técnicas para el cálculo explícito de la ley de control vía la resolución de un problema multiparamétrico. En el caso de que el punto de equilibro cambie, el controlador debe ser rediseñado y la factibilidad del mismo puede perderse. Como ya se comentó anteriormente, en (Chisci and Zappa, 2003; Rossiter, Kouvaritakis and Gossner, 1996) se proponen técnicas para garantizar la factibilidad basados en un supervisor que conmuta entre dos controladores.

En el caso de sistemas no lineales, el problema es más complejo y requiere la solución de un problema de optimización no lineal (Camacho and Bordons, 2004). Para relajar la carga computacional se han establecido condiciones para garantizar la estabilidad en caso de soluciones subóptimas (Scokaert, Mayne and Rawlings., 1999). El control predictivo robustos para sistemas no lineales ha madurado mucho recientemente (Magni, Nijmeijer and van der Shaft, 2001; Limon et al., 2006), pero que su complejidad computacional hace que se considere un problema aún sin cerrar. En este sentido la aplicación de técnicas garantistas como (Limon, Bravo, Alamo and Camacho, 2005) resultan prometedoras.

1.2 Controladores predictivos en la industria.

El control predictivo ha tenido una evolución peculiar en la disciplina del control en tanto que ha sido una estrategia en la cual el campo industrial ha ido por delante de la comunidad investigadora. Si bien los controladores predictivos tienen su origen en el control óptimo (Propoi, 1963; Lee and Markus, 1967; Kwon and Pearson, 1977), nuevas y más avanzadas formulaciones surgieron en el seno de la industria, principalmente en la industria petroquímica y de procesos. La necesidad de controlar procesos en puntos de operación límites con el objetivo de optimizar el proceso productivo llevó a la aparición de controladores predictivos basados en modelos sencillos, orientados a la resolución de los problemas de control asociados, tales como la consideración de restricciones, incertidumbres y no linealidades. Entre otras formulaciones destacan las siguientes:

IDCOM o MPHC : (*Identification-Command* o *Model Predictive Heuristic Control*) propuesto en (Richalet, Rault, Testud and Papon, 1978), utiliza como modelo de predicción la respuesta impulsional (FIR), función de coste cuadrática, y restricciones en las entradas y salidas. El algoritmo de optimización es heurístico.

DMC : (*Dynamic Matrix Control*) propuesto en (Cutler and Ramaker, 1980), utiliza como modelo de predicción la respuesta ante escalón, lo cual limita su aplicación a plantas estables, considera un coste cuadrático penalizando el esfuerzo de control y no considera restricciones en la optimización.

QDMC : (*Quadratic Dynamic Matrix Control*) propuesto en (García and Morshedi, 1986), surge de la extensión del DMC al caso con restricciones. Este controlador forma parte de la denominada segunda generación de controladores predictivos, en los que el problema de optimización asociado se resuelve utilizando la programación matemática. Establece dos tipos de restricciones: duras y blandas, permitiendo la violación de estas últimas durante algún periodo de tiempo.

SMOC : (*Shell Multivariable Optimizing Control*) propuesto en (Marquis and Broustail, 1988), forma parte de la tercera generación de controladores predictivos. Permite la utilización de modelos en espacios de estados e incorpora observadores y modelos de perturbaciones. Introduce también restricciones duras, blandas y con niveles de prioridad.

GPC : (*Generalized Predictive Control*) propuesto en (Clarke, Mohtadi and Tuffs, 1987*a*; Clarke, Mohtadi and Tuffs, 1987*b*), utiliza como modelo de predicción la formulación CARIMA, que incorpora una perturbación modelada como ruido blanco. Incorpora restricciones y existen resultados asociados a la estabilidad.

Se han propuesto otras formulaciones de controladores predictivos tales como el RMPCT, el PCT o el PFC. Una lectura más profunda sobre todos estos controladores se puede encontrar en (Camacho and Bordons, 1999), donde se analizan tanto en aspectos prácticos, como en los relativos a la estabilidad y robustez.

En la mayoría de estos controladores, la estabilidad no está garantizada, requiriéndose un ajuste específico para cada sistema de una forma heurística y sin garantías de éxito. Por ello, se establecen reglas prácticas de ajuste, como la elección de un horizonte de predicción del orden del tiempo de establecimiento de la planta en sistemas estables.

El problema de la estabilidad no estaba resuelto en general y resultaba de hecho una suerte de barrera psicológica que los investigadores en control predictivo ni siquiera intentaban superar [2], produciéndose un vacío teórico que mermaba las características

[2] Morari en (Morari, 1994) hace la siguiente afirmación en relación a la estabilidad de los controladores predictivos *"the recent work has removed this technical and to some extent psychological barrier*

de estos controladores.

1.3 El problema de la estabilidad: optimalidad no implica estabilidad

La ley de control obtenida, en un controlador predictivo, surge de la optimización de un criterio relacionado con el comportamiento del sistema, en el que se penaliza tanto el error respecto al punto de equilibrio como el esfuerzo de control necesario para alcanzar dicho equilibrio. Contrariamente a lo que dicta el sentido común, el hecho de que la actuación aplicada sea óptima no garantiza que el sistema en bucle cerrado alcance el punto de equilibrio tal y como se desea. El problema de la estabilidad tiene su origen en el desarrollo propio de los controladores predictivos: la necesidad de utilizar un horizonte de predicción finito e invariante en el tiempo y la estrategia de horizonte deslizante.

El origen de los controladores predictivos está en el control óptimo en el cual se pretende calcular la ley de control $u = K_\infty(x)$ que minimiza el coste de regular el sistema al punto de equilibrio a lo largo de toda la evolución del mismo. Así, la función de coste a optimizar es:

$$J_\infty(x_k) = \sum_{i=0}^{\infty} L(x(k+i|k), K_\infty(x(k+i|k)))$$

y el problema de optimización a resolver viene dado por

$$\begin{aligned} \min_{K_\infty(x)} \quad & J_\infty(x_k) \\ s.a \quad & \\ & u(k+j|k) \in U \quad \forall j \geq 0 \\ & x(k+j|k) \in X \quad \forall j \geq 0 \end{aligned}$$

siendo $x(k+j|k)$ la predicción del estado del sistema en el instante $k+j$ a partir del

(people did not even try) and started wide spread efforts to tackle extensions of these basic problems with the new tools".

estado en x_k. Los conjuntos U y X definen las restricciones, de forma que X es un conjunto acotado, U compacto y ambos contienen el origen en su interior.

Este problema de control, bajo ciertas condiciones de observabilidad relacionadas con la función de coste de etapa [3], estabiliza asintóticamente todo estado en cual exista una solución con un coste asociado acotado. De hecho, todo punto asintóticamente estabilizable, se puede estabilizar por esta estrategia de control.

El problema del control óptimo se puede resolver utilizando dos técnicas: la primera se deriva del principio de optimalidad de Bellman (Bellman, 1957; Bryson and Ho, 1969), según el cual

$$J_\infty^*(x) = \min_{u \in U}\{L(x,u) + J_\infty^*(f(x,u)) |\, f(x,u) \in X_\infty\}$$

siendo la ley de control la solución de este problema de optimización en cada estado $K_\infty(x) = u^*(x)$. El conjunto X_∞ es el conjunto de estados asintóticamente estabilizables al origen de una forma admisible. Es en este conjunto en el que está definido $J_\infty^*(x)$.

La solución de este problema se puede obtener a partir de las ecuaciones de Hamilton-Jacobi-Bellman, cuya resolución es muy compleja, si no imposible, salvo en casos especiales como el problema de regulación de un sistema lineal sin restricciones con una función de coste de etapa cuadrática, que da lugar al regulador lineal cuadrático o LQR (Bryson and Ho, 1969).

Otro procedimiento para resolver este problema es la aplicación del cálculo variacional, que conduce a las ecuaciones de Euler-Lagrange. La gran diferencia entre ambas resoluciones es que en las ecuaciones de H-J-B la solución es la ley de control, conduciendo a soluciones globales y en bucle cerrado, mientras la formulación de E-L conduce a soluciones locales y en bucle abierto, si bien la resolución de estas ecuaciones es más sencilla que la de H-J-B. Un análisis más exhaustivo, pero con un carácter didáctico, sobre control óptimo puede encontrarse en (Nevistić, 1997).

La dificultad en la resolución de este problema llevó a adoptar soluciones prácticas que hiciesen más sencilla su realización. Éstas ideas son básicamente las siguientes:

[3]La condición de observabilidad consiste en $L(x,u) \geq l \cdot \|(h(x),u)\|^\sigma$ siendo $\sigma \geq 1$ y $h(x)$ una función detectable con el modelo, y garantiza que si $J_N^*(x_k) \to 0$ cuando $k \to \infty$, entonces $x_k \to 0$ (Keerthi and Gilbert, 1988).

Horizonte finito y fijo : considerando un horizonte finito, el problema de optimización toma la forma habitual del control predictivo:

$$\min_{u_F(k)} \quad J_N(x_k, u_F(k))$$

$$s.a$$

$$u(k+j|k) \in U \quad j = 0, \cdots, N-1$$

$$x(k+j|k) \in X \quad j = 0, \cdots, N-1$$

$$x(k+N|k) \in \Omega$$

donde el coste a optimizar

$$J_N(x_k, u_F(k)) = \sum_{i=0}^{N-1} L(x(k+i|k), u(k+i|k)) + V(x(k+N|k))$$

siendo $V(x)$ una función que penaliza el coste estado final de la predicción (estado terminal), denominada función de coste terminal. Al conjunto Ω al que se restringe dicho estado se denomina región terminal.

La principal ventaja de la adopción del horizonte finito reside en que el problema de optimización tiene la forma de un problema de programación matemática, el cual admite solución numérica gracias a los algoritmos existentes (Luenberger, 1989). Nótese que el coste computacional de la resolución de este problema puede ser muy elevado si el modelo es no lineal.

Estrategia de horizonte deslizante : según esta técnica, en cada periodo de muestreo se resuelve el problema de optimización y se aplica tan sólo la actuación obtenida para el siguiente periodo de muestreo. En el siguiente periodo de muestreo se toma un nuevo estado del sistema y se repite la operación. Esto dota de realimentación a la formulación basada en el problema de optimización en bucle abierto, lo cual le confiere cierto grado de robustez.

El problema de control óptimo con horizonte finito i se puede resolver mediante el problema de programación dinámica asociado:

$$J_i^*(x) = \min_u \{L(x,u) + J_{i-1}^*(f(x,u)) |\ f(x,u) \in X_{i-1}\}$$

siendo $J_0^*(x) = V(x)$ y $X_0 = \Omega$. De la solución de este problema se deriva la ley de control $K_i(x) = u^*$. El conjunto X_{i-1} es el conjunto de estados que pueden ser llevados por una ley de control admisible siguiendo una trayectoria admisible hasta el conjunto Ω en $i-1$ pasos. Este problema de optimización es factible en el conjunto X_i, siendo éste el dominio de definición del controlador $K_i(x)$ y por lo tanto de $J_i^*(x)$.

Considérese un estado inicial tal que el problema de optimización con horizonte N es factible, es decir, $x_0 \in X_N$. Entonces, aplicando sobre el sistema la actuación óptima $u_0 = K_N(x_0)$, el estado evoluciona a x_1. En ese instante, la actuación óptima viene dada por la ley de control óptima con un horizonte $N-1$, por tanto $u_1 = K_{N-1}(x_1)$. Esto se debe al principio de optimalidad de Bellman. Entonces, en el instante k, la actuación óptima vendrá dada por $u_k = K_{N-k}(x_k)$, que es el controlador óptimo para conducir al sistema en $N-k$ pasos al conjunto terminal Ω.

En consecuencia, el horizonte de predicción se va reduciendo en cada instante, hasta el instante N en el cual el sistema alcanza la región terminal Ω. En esta región, el problema de optimización dinámica no está definido, requiriéndose un controlador alternativo.

Sin embargo, en el control predictivo la estrategia de horizonte deslizante y horizonte finito e invariante hace que siempre se aplique el controlador con horizonte N. Por lo tanto, la ley de control del MPC es invariante en el tiempo y viene dada por

$$u_k = K_{\text{MPC}}(x_k) = K_N(x_k)$$

Esto hace que la convergencia del controlador óptimo con horizonte finito se pierda, pues no se reduce el horizonte y este controlador no garantiza que el sistema evolucione hacia el punto de equilibrio, ni siquiera que alcance la región terminal. Una segunda consecuencia es la posible pérdida de la factibilidad del problema. Por tanto, a pesar de que en cada instante se aplica una actuación óptima, en el sentido que optimiza un coste y satisface unas restricciones, esta actuación no garantiza ni la factibilidad ni la convergencia del sistema en bucle cerrado.

Esta pérdida de la estabilidad supuso un grave problema de los controladores predictivos, haciendo necesario un ajuste del controlador particular para cada sistema con el fin de garantizar la estabilidad, con el temor añadido sobre cómo podía influir la variación de un parámetro sobre ésta. Por ello los controladores predictivos contaban entre sus desventajas la dificultad del ajuste.

1.4 Controladores predictivos con estabilidad garantizada

Dar un soporte teórico bajo el cual se garantizase la estabilidad en los controladores predictivos supuso un reto en la comunidad investigadora, lo que favoreció el rápido desarrollo de los controladores predictivos. Esto dio lugar a una serie de formulaciones con estabilidad garantizada, cuyo denominador común es la utilización de la teoría de Lyapunov, y en particular, el coste óptimo como función de Lyapunov. Estas formulaciones se pueden agrupar de la siguiente forma:

MPC con restricción terminal de igualdad : fue propuesto para garantizar estabilidad del problema LQR con restricciones en (Kwon and Pearson, 1977) y extendido en (Keerthi and Gilbert, 1988) a sistemas no lineales con modelo en espacio de estados, en tiempo discreto, sujetos a restricciones. La estabilidad se garantiza imponiendo como restricción terminal

$$x(k+N|k) = 0$$

bajo ciertas condiciones de controlabilidad y observabilidad del sistema. En este caso, la función de coste óptimo es estrictamente decreciente con el tiempo, por lo que es una función de Lyapunov del sistema.

En (Mayne and Michalska, 1990), se formula este controlador para sistemas en tiempo continuo y se relajan las condiciones para garantizar la estabilidad.

En (Chisci, and Mosca, 1994; Bemporad, Chisci and Mosca, 1995) se extiende esta condición a sistemas lineales descritos por un modelo CARIMA, sin restricciones. En este caso, la restricción terminal se traduce en una condición sobre las salidas y las entradas del sistema.

MPC con coste terminal : la estabilidad se logra incorporando en la función de coste, un término que penalice el estado terminal mediante el denominado coste terminal. En (Bitmead, Gervers and Wertz, 1990) se propone, para sistemas lineales sin restricciones un coste terminal cuadrático cuya matriz de ponderación se obtiene de la resolución de una ecuación de Riccati.

En (Rawlings and Muske, 1993), en el caso de un sistema lineal estable con restricciones politópicas, se propone tomar como coste terminal el coste infinito resultante de aplicar la actuación nula.

En (Alamir and Bornard, 1995) se utiliza esta técnica para sistemas no lineales tomando como coste terminal el coste de un controlador localmente estabilizante durante un periodo suficientemente largo.

MPC con restricción terminal de desigualdad : los problemas computacionales que supone el cumplimiento de una restricción de igualdad, llevaron a relajar esta condición, extendiendo la restricción terminal a una vecindad del origen. Así, se establece una restricción terminal de desigualdad de la forma

$$x(k+N|k) \in \Omega$$

siendo el conjunto Ω el denominado conjunto terminal.

Esta estrategia fue propuesta en (Michalska and Mayne, 1993) para sistemas no lineales en tiempo continuo y sujeto a restricciones. En este trabajo, se elige como región terminal un invariante positivo del sistema no lineal controlado por un controlador local. Además, para garantizar la factibilidad se introduce como variable de decisión el horizonte de predicción. El controlador así formulado garantiza que conduce al sistema a la región terminal, donde el sistema pasa a regularse por el controlador local que lo estabiliza al origen. De ahí que este controlador se denomine controlador MPC dual. Las bondades de esta formulación son tan notables, que marcó las futuras líneas de investigación en estabilidad.

En (Chisci, Lombardi and Mosca, 1996) se extiende el control predictivo dual al caso de sistemas lineales con restricciones.

En esta misma línea, se enmarcan los denominados controladores predictivos con estabilidad forzada, en los que ésta se garantiza por la satisfacción de una restricción estabilizante. En (De Oliveira Kothare and Morari, 2000) se presenta el denominado control predictivo contractivo. Esta estrategia está basada en el trabajo anterior (Yang and Polak, 1993) e incorpora como restricción terminal una restricción que fuerza al estado terminal $x(k+N|k)$ a tener una norma inferior a la del estado actual x_k

$$\|x(k+N|k)\|_P < \|x_k\|_P$$

La secuencia obtenida se aplica en bucle abierto desde el instante k hasta hasta el $k+N$, en el que se vuelve a resolver el problema. Esta formulación tiene dos problemas: el funcionamiento en bucle abierto durante el horizonte de predicción, que se resuelve con un procedimiento de realimentación extra, y el hecho de que la factibilidad está garantizada tan sólo en una vecindad del origen, que puede ser pequeña y no se conoce a priori.

En (Primbs, Nevistić and Doyle, 2000), se garantiza la estabilidad forzando que una función de control de Lyapunov conocida a priori, sea estrictamente decreciente

$$V(x(k+1|k)) < V(x_k)$$

Esta restricción se impone en la actuación para el instante actual u_k, y garantiza estabilidad para todo horizonte de predicción $N \geq 1$.

MPC con coste y restricción terminal : esta es la estructura en la que se enmarcan las más recientes formulaciones del MPC. Es importante decir que en algunas de las formulaciones propuestas en las que se garantiza estabilidad con la adición únicamente de una función de coste terminal, implícitamente, impone que la predicción alcance una vecindad del origen. En consecuencia se deben considerar también como formulaciones con restricción terminal.

El primer trabajo en el que se garantiza estabilidad incorporando ambos ingredientes es en (Sznaier and Damborg, 1987) en el cual, para sistemas lineales sujetos a restricciones politópicas, se considera como controlador local el LQR y como región terminal un invariante asociado. En este trabajo se demuestra que para cada estado, existe un horizonte de predicción suficientemente largo, tal que la solución óptima garantiza la satisfacción de la restricción terminal, lo que permite eliminarla.

Esta misma línea se sigue en (Parisini and Zoppoli, 1995) para sistemas no lineales en tiempo discreto con restricciones. Se calcula un controlador lineal basado en la linealización del modelo en torno al punto de equilibrio (análogamente al procedimiento presentado en (Michalska and Mayne, 1993) para el cálculo de la región terminal) y se toma como coste terminal una función proporcional a la función de Lyapunov asociada al sistema linealizado en torno al origen en bucle cerrado $V(x) = a{\cdot}x^T{\cdot}P{\cdot}x$. La estabilidad se garantiza demostrando que existe una combinación de la constante a y del horizonte de predicción N tal que el estado terminal resultante del problema de optimización (sin restricción terminal) alcanza un invariante positivo del sistema y la función de coste óptimo es estrictamente decreciente.

En (De Nicolao, Magni and Scattolini, 1998) se propone como coste terminal el coste infinito incurrido por el sistema controlado por el controlador local. Esta opción es una aproximación razonable al coste óptimo en el estado terminal, por lo que el coste del MPC será próximo al del controlador óptimo. La imposición de que el estado terminal alcance la región terminal se impone implícitamente en la suposición de que la función de coste terminal sólo está definida en la región terminal, tomando un valor infinito fuera de ella. La región terminal es un invariante positivo del sistema controlado por el controlador local. En (Magni, De Nicolao, Magni and Scattolini, 2001), se propone una formulación implementable del controlador predictivo anterior. Se basa en considerar como función de coste terminal una aproximación truncada del coste infinito. Pero lo más destacable de este trabajo es que considera un horizonte de predicción mayor que el de control gracias a la incorporación del controlador local.

En (Jadbabaie, Yu and Hauser, 2001) se establece la estabilidad de un controlador predictivo para sistemas sin restricciones, tomando como coste terminal una función de Lyapunov de control y sin restricción terminal. En este trabajo se demuestra que existe una vecindad del origen en la cual la solución óptima del problema sin restricciones, garantiza la satisfacción de la restricción terminal. Así, de una forma implícita, se considera dicha restricción. La eliminación de la restricción terminal hace que el controlador se formule como un problema de optimización sin restricciones, lo que permite su implementación en sistemas rápidos, como sistemas aeronáuticos.

La formulación del MPC incluyendo explícitamente la restricción terminal y la función de coste terminal no se alcanza hasta el denominado MPC con horizonte quasi-infinito (Chen and Allgöwer, 1998). Para el cálculo de éstos, se propone un controlador local lineal con una función de Lyapunov cuadrática asociada tal que garantiza que el coste terminal es una cota superior del coste óptimo del estado terminal controlado por el controlador local. De ahí la denominación *horizonte cuasi-infinito*, pues el coste óptimo del MPC es una cota superior del coste óptimo (con horizonte infinito).

1.5 Formulación general del MPC: necesidad de la región terminal y el coste terminal

Como se ha mostrado anteriormente, las formulaciones del control predictivo con garantía de estabilidad han ido evolucionando hasta llegar a la necesidad de la región terminal y del coste terminal de una u otra forma. Sorprendentemente, todas las estrategias responden a unas condiciones generales de estabilidad. Este importante resultado es el que se propuso en (Mayne et al., 2000). Este trabajo constituye una piedra angular del control predictivo y una referencia obligada para futuros desarrollos en este campo.

En este trabajo se analizan las formulaciones existentes de controladores predictivos con estabilidad garantizada y se establece que el control predictivo con coste terminal y restricción terminal puede, bajo ciertas condiciones, estabilizar asintóticamente un sistema no lineal sujeto a restricciones. Además se establecen condiciones suficientes sobre la función de coste terminal y la región terminal para garantizar dicha estabilidad. Estas condiciones son las siguientes:

- La región terminal Ω debe ser un conjunto invariante positivo admisible del sistema. Es decir, que debe existir una ley de control local $u = h(x)$ tal que estabiliza el sistema en Ω y además la evolución del sistema y las actuaciones en dicho conjunto son admisibles.
- El coste terminal $V(x)$ es una función de Lyapunov [4] asociada al sistema regulado por el controlador local, tal que

  $$V(f(x, h(x))) - V(x) \leq -L(x, h(x))$$

  para todo $x \in \Omega$. Por lo tanto, la ley de control local estabiliza asintóticamente el sistema.

Considerando el análisis realizado en la sección 1.3, se puede ver cómo las hipótesis impuestas resuelven los problemas existentes y garantizan la estabilidad.

Necesidad de la región terminal invariante : Si la región terminal es un invariante positivo, entonces el conjunto de estados factibles es el conjunto de estados estabilizables en N pasos $X_N = S_N(X, \Omega)$. Considérese $x_k \in X_N$. Dada la ausencia de discrepancias entre el modelo de predicción y el sistema, se tiene que el estado al que evoluciona el sistema es el predicho $x_{k+1} = x(k+1|k)$. Este estado puede alcanzar la región Ω en $N-1$ pasos, luego $x_{k+1} \in X_{N-1}$. Gracias a que Ω es un conjunto invariante, este conjunto tiene la propiedad que $X_{N-1} \subseteq X_N$ y por lo tanto X_N es un conjunto invariante positivo del sistema en bucle cerrado, lo que garantiza la factibilidad del controlador en todo instante.

Necesidad del coste terminal como función de Lyapunov : bajo esta condición se garantiza que el coste óptimo es estrictamente decreciente, y por lo tanto es una función de Lyapunov del sistema. Esto garantiza la estabilidad asintótica del sistema en bucle cerrado con restricciones.

[4] En el artículo (Mayne, 2001), inspirado por (Jadbabaie et al., 2001), se extiende esta condición a funciones de Lyapunov de control (CLF), que son más generales que las funciones de Lyapunov.

La monotonía de la función de coste óptimo se basa en la existencia de una secuencia de actuaciones factibles $\bar{u}_F(k+1)$ basada en la solución óptima obtenida en el instante anterior $u_F^*(k)$. Esta secuencia no es más que los $N-1$ términos que restan de la secuencia anterior más la actuación obtenida de la ley de control local. Así, la diferencia entre el coste de esta secuencia, $\bar{J}_N(x_{k+1})$, y el coste óptimo anterior, $J_N^*(x_k)$, es

$$\begin{aligned}\bar{J}_N(x_{k+1}) - J_N^*(x_k) \;\; &= \;\; -L(x_k, u^*(k|k)) + \Big\{ L(x^*(k+N|k), h(x^*(k+N|k))) \\ &\quad + V(f(x^*(k+N|k), h(x^*(k+N|k)))) - V(x^*(k+N|k)) \Big\}\end{aligned}$$

La incorporación del coste terminal garantiza que el término entre llaves es negativo, y por lo tanto la secuencia factible tiene un coste menor que el óptimo anterior, por lo que la solución óptima también lo tendrá. En consecuencia

$$J_N^*(x_{k+1}) - J_N^*(x_k) \;\; \leq \;\; -L(x_k, K_{\text{MPC}}(x_k))$$

y por lo tanto el coste óptimo es una función de Lyapunov que decrece a lo largo de la evolución del sistema, lo que garantiza la estabilidad asintótica. Esta demostración se hace con más detalle en el capítulo siguiente, dedicado al análisis de estabilidad del MPC.

Es importante resaltar que, aun en el caso en el que el controlador garantice la estabilidad en bucle cerrado del sistema, la trayectoria seguida por el mismo no es óptima, en el sentido de que puede existir otra cuyo coste total a lo largo de la misma sea inferior. Esto se deriva del horizonte finito considerado en la formulación del problema. Así, la trayectoria del sistema difiere de la trayectoria resultante de la secuencia óptima calculada en un instante (consecuencia derivada del principio de optimalidad de Bellman).

Un resultado muy interesante relacionado con la estabilidad de los controladores MPC es el denominado MPC subóptimo que se presenta en (Scokaert et al., 1999). En este trabajo se demuestra que es la factibilidad de la solución la que garantiza la estabilidad, siempre que ésta garantice un decrecimiento de la función de coste, no siendo necesaria la optimalidad de la solución obtenida al problema de optimización. Esta consideración es trascendental para la implementación de controladores MPC en sistemas no lineales. En efecto, el problema de optimización implicado en el MPC es en general no convexo, y por lo tanto el problema puede presentar mínimos locales. La obtención del mínimo global es sumamente costosa comparada con la obtención de un mínimo local, y tanto más comparada con la obtención de una solución que simplemente presente un menor coste que la anterior.

1.6 Robustez de los controladores MPC

1.6.1 Introducción

La estabilidad de los controladores predictivos se garantiza bajo la hipótesis de que el modelo de predicción coincide con el modelo del sistema a controlar. Sin embargo, todo sistema tiene asociado un error con el modelo que representa su dinámica. Por ello, para que un controlador sea aplicable debe poseer ciertas características de robustez.

En el caso en que no hubiese incertidumbres, si se aplica la secuencia de actuaciones obtenida en bucle abierto, el sistema evoluciona de una manera admisible hasta alcanzar el conjunto terminal. Sin embargo, las posibles discrepancias existentes entre el modelo de predicción y el sistema real pueden hacer que su evolución viole las restricciones o bien que el controlador deje de ser factible o incluso que se pierda la convergencia del sistema en bucle cerrado. El hecho de que el MPC se aplique mediante la estrategia de horizonte deslizante hace que la actuación se recalcule en cada periodo de muestreo, lo que dota de realimentación al sistema y por lo tanto de cierta robustez.

El estudio de la robustez se puede realizar desde dos puntos de vista: el del análisis de robustez y el de la síntesis de controladores robustos. En el primero, se parte de un controlador MPC obtenido para un sistema sin considerar el efecto de las incertidumbres en su diseño y se determina qué grado de incertidumbres es capaz de soportar dicho controlador conservando la estabilidad del sistema.

El segundo enfoque es el de la síntesis, por el cual se establecen formulaciones del controlador que consideran en el cálculo de las actuaciones el efecto que tienen las incertidumbres sobre el sistema. El objetivo es por lo tanto garantizar, para cierto grado de incertidumbres, la estabilidad, la satisfacción de las restricciones y, a ser posible, alguna especificación sobre el desempeño.

A continuación se trata el primer aspecto, abordándose la síntesis de controladores robustos en el siguiente apartado.

1.6.2 Análisis de robustez de los controladores MPC

En el caso de sistemas lineales, existen numerosas técnicas de análisis de robustez de sistemas sin restricciones, pero pocas en el caso de sistemas con restricciones, pues, en este caso el sistema en bucle cerrado puede ser no lineal. En (Zafiriou, 1990) se presentan condiciones suficientes (y también necesarias) para garantizar la estabilidad nominal y robusta del MPC. En (Genceli and Nikolau, 1993) se dan condiciones suficientes de estabilidad robusta del DMC y se analiza el comportamiento del sistema en presencia de incertidumbres. En (Primbs et al., 2000) se presenta un procedimiento para comprobar la robustez de controladores predictivos de sistemas lineales con restricciones en las entradas, basado en la solución de una serie de desigualdades matriciales lineales (LMI).

La robustez de los controladores predictivos de sistemas no lineales se ha analizado siguiendo dos líneas: una en la que se explota la optimalidad del controlador, y otra basada en la teoría de Lyapunov.

En la línea basada en la optimalidad del controlador, en (Glad, 1987) se presenta un compendio de resultados de estabilidad robusta de controladores óptimos para sistemas no lineales en tiempo continuo, afines en la actuación y sin restricciones. En este trabajo se considera únicamente incertidumbres en las actuaciones de dos tipos: incertidumbres en la ganancia, de forma que la actuación real $u_r = \phi(u)$ es una función (estática) de la actuación que se aplica, e incertidumbres aditivas, de forma que $u_r = u + \psi(x)$. El principal resultado de este trabajo es la demostración de que el controlador óptimo estabiliza al sistema con incertidumbres de ganancia contenidas en el sector $(1/2, \infty)$, es decir tal que

$$\frac{1}{2} \cdot u^2 < u \cdot \phi(u) < \infty$$

Este resultado se demuestra para el caso de una entrada y se extiende al caso de múltiples entradas, garantizándose la robustez en otro sector.

Este trabajo se extendió en (Geromel and Da Cruz, 1987) al caso de sistemas discretos afines en la actuación, sin restricciones y con incertidumbres en las actuaciones, considerando también horizonte infinito. En este trabajo se analiza la estabilidad del sistema ante incertidumbres en la ganancia del controlador, obteniéndose bajo ciertas condiciones, un margen de estabilidad, que en el caso de una única entrada, se reduce al sector $(0.5, \infty)$, como en los sistemas continuos. En este trabajo también se analiza el caso de incertidumbres aditivas en la ganancia.

En (De Nicolao et al., 1996) se extiende el trabajo de (Geromel and Da Cruz, 1987) al análisis de robustez de sistemas discretos sin restricciones controlados con un MPC con restricción terminal. Si bien el trabajo original está formulado para el caso de restricción terminal nula, ésta se puede extender al caso general pues está basado en el principio de optimalidad (De Nicolao et al., 1998). Siguiendo un desarrollo prácticamente paralelo al desarrollado en (Geromel and Da Cruz, 1987), se obtienen resultados semejantes.

Otra forma de demostrar la robustez del MPC general basada en la optimalidad es la presentada en (Magni and Sepulchre, 1997). En este trabajo se demuestra la optimalidad inversa del MPC: el controlador obtenido de un MPC con horizonte finito y estabilidad garantizada para un sistema sin restricciones se puede considerar como un controlador óptimo con horizonte infinito considerando un coste de etapa modificado. En consecuencia, todo controlador MPC hereda las propiedades de robustez de los controladores óptimos, y en particular, el margen de incertidumbre en la ganancia de $(0.5, \infty)$.

El segundo enfoque para el análisis de robustez de los controladores es la teoría de Lyapunov, y se basa en la estabilidad asintótica (o bien, exponencial) que presentan estos controladores. La idea básica consiste en garantizar que el coste óptimo sigue siendo una función de Lyapunov estrictamente decreciente a pesar de las incertidumbres. Es por tanto una herramienta general y no aprovecha la optimalidad de los controladores, sin embargo permite el análisis en presencia de restricciones, no consideradas en el enfoque de la optimalidad. Es importante resaltar que la presencia de restricciones en el controlador, en especial sobre los estados, impone un grado de complejidad mayor a la robustez del controlador, pues se debe garantizar la satisfacción de las restricciones en presencia de las incertidumbres.

En (Scokaert, Rawlings and Meadows, 1997) se analiza la estabilidad robusta de los controladores MPC con horizonte finito para sistemas en tiempo discreto con restricciones bajo incertidumbres aditivas que decaen con el tiempo. Este análisis se orienta a la estabilidad del MPC cuando se conecta en cascada un observador para estimar los estados del sistema, suponiendo que el error en la estimación de los estados decae con el tiempo. Basándose en las propiedades de los sistemas exponencialmente estables y bajo la continuidad Lipschitz de la ley de control, se demuestra que el controlador MPC soporta cierto grado de incertidumbre.

En (De Nicolao et al., 1998) se presenta una formulación estable del MPC y además se analiza la estabilidad del controlador siguiendo la línea de (Scokaert et al., 1997), considerando incertidumbres aditivas que decaen y cierta condición de continuidad sobre el coste óptimo.

Todos estos trabajos demuestran que los controladores predictivos tienen cierto grado de robustez de forma inherente, es decir, que ante cierto grado de incertidumbres, el sistema mantiene la estabilidad. Otro problema distinto es la síntesis de controladores considerando las incertidumbres que presenta el sistema, el cual se trata en la siguiente sección.

1.6.3 Formulaciones robustas del MPC

Dado el alto grado de complejidad de los controladores predictivos (que incorporan optimalidad y satisfacción de restricciones en el control de sistemas no lineales), la incorporación de las incertidumbres en el diseño es muy costosa, por lo que la mayor parte de las formulaciones propuestas (con estabilidad garantizada) constituyen soluciones meramente teóricas.

La forma habitual de considerar las incertidumbres en predictivo es incorporando todas las posibles realizaciones de éstas en la solución del problema de optimización. Así, si el sistema incierto responde a un modelo

$$x_{k+1} = f(x_k, u_k, w_k)$$

siendo w_k el vector de incertidumbres tal que $w_k \in W \subset \mathbb{R}^p$. La predicción de la evolución del sistema incierto a lo largo del horizonte depende de la realización de las incertidumbres $w_F = \{w_i\}$ con $w_i \in W$ para $i = 0, \cdots, N-1$. Considerando esta, el controlador tiene la forma

$$\begin{aligned}
\min_{u_F(k)} \quad & J_N(x_k, u_F(k), W) \\
s.a \quad & \\
& u(k+j|k) \in U \qquad j = 0, \cdots, N-1 \\
& x(k+j|k) \in X \qquad j = 0, \cdots, N-1, \; \forall\, w_F \\
& x(k+N|k) \in \Omega \qquad \forall\, w_F
\end{aligned}$$

siendo el estado predicho

$$x(k+j+1|k) = f(x(k+j|k), u(k+j|k), w_{k+j})$$

con $x(k|k) = x_k$.

Nótese que las restricciones en la evolución de los estados se deben satisfacer de una forma robusta, es decir, para todas las posibles realizaciones de las incertidumbres. La incorporación de restricciones en el estado complica notablemente el problema, pero, aún en el caso en el que no haya restricciones sobre los estados, la restricción terminal siempre está presente en el problema de optimización pues se añade para garantizar la estabilidad del controlador.

El coste a optimizar $J_N(x_k, u_F(k), W)$ puede basarse en las predicciones nominales del sistema o bien considerar el efecto de las incertidumbres tomando, por ejemplo, la peor situación posible. Esto da lugar a la denominada formulación min-max

$$J_N(x_k, u_F(k), W) = \max_{w_F} \left\{ \sum_{i=0}^{N-1} L(x(k+i|k), u(k+i|k), w_{k+i}) + V(x(k+N|k)) \right\}$$

Otra formulación consiste en añadir un término en la función de coste de etapa que pondera la posible incertidumbre, como en la formulación H_∞.

Esta forma de considerar las incertidumbres es intuitiva y razonable, pero puede conducir a soluciones muy conservadoras. Este conservadurismo radica en la naturaleza misma del control predictivo: la predicción en bucle abierto.

En efecto, el control predictivo surge como solución práctica e implementable de controladores óptimos, evitando la resolución del problema de programación dinámica gracias a la resolución, en cada instante, del problema de optimización asociado. Eso hace que, en vez de obtener una ley de control $u_k = K_N(x_k)$, se obtiene una secuencia actuaciones asociada al estado actual, x_k. En el caso en que haya incertidumbres, la factibilidad del problema se debe garantizar para todas las posibles incertidumbres. Por tanto, para que la secuencia obtenida sea factible, debe garantizar que la evolución del sistema incierto satisfaga las restricciones para toda posible incertidumbre. En consecuencia, el conjunto de estados para los cuales existe una solución factible X_{ba}, puede ser muy reducido.

Entre los controladores predictivos que se pueden englobar en esta formulación está el controlador min-max propuesto en (Campo and Morari, 1987) para sistemas lineales, y el controlador dual robusto presentado en (Michalska and Mayne, 1993). En este trabajo se presenta el denominado MPC dual y se propone un procedimiento que garantiza robustez. Este procedimiento parte de un controlador local robusto con un invariante positivo robusto asociado Ω. Sin embargo, considera en el problema de optimización un conjunto $\phi \subset \Omega$, de forma que la secuencia óptima para la región conservadora garantice que el estado terminal incierto esté contenido en Ω. Esta idea permite garantizar la satisfacción robusta de la restricción terminal.

Si se considera, en vez de una secuencia de actuaciones, una ley de control en la optimización, la actuación a aplicar en cada estado predicho para una determinada realización de las incertidumbres, depende del estado en que se encuentra y en consecuencia, de la realización de las incertidumbres. Por tanto la actuación puede compensar el efecto de las mismas con el fin de satisfacer las restricciones. Así pues, el conjunto de estados que pueden satisfacer las restricciones de este modo X_{bc} es mucho mayor, de forma que

$$X_{ba} \subset\subset X_{bc}$$

La incorporación de esta idea en el controlador MPC da lugar a la denominada *formulación en bucle cerrado* y fue introducida en (Scokaert and Mayne, 1998; Lee and Yu, 1997) en el contexto del min-max. En esta formulación, el problema de control no está planteado en términos de una secuencia de actuaciones, sino de una secuencia de leyes de control

$$\pi_F = \{u, \pi_1(x) \cdots, \pi_{N-1}(x)\}$$

lo cual hace que el problema de optimización implicado sea infinito-dimensional. En consecuencia, estos controladores constituyen (por ahora) una herramienta meramente teórica (Mayne et al., 2000). Sin embargo, estas consideraciones han permitido enfocar mejor el problema de la robustez, dando un giro en la investigación en este campo.

Esta formulación se sigue también en (Magni, Nijmeijer and van der Shaft, 2001), donde se propone un controlador H_∞ con horizonte deslizante para un sistema no lineal, afín en la actuación y sin restricciones.

Dentro del control predictivo en bucle cerrado se pueden considerar otras formulaciones como por ejemplo el trabajo presentado en (Kothare, Balakrishnan and Morari, 1996). En éste se propone un controlador que estabiliza una planta incierta tal que se puede expresar en cada instante como una combinación convexa de una serie de plantas lineales y que presenta restricciones en los estados y en las actuaciones. En esta formulación se considera como variable de optimización un controlador lineal que estabiliza todas las plantas y se puede plantear como un LMI que se resuelve en cada instante.

También se pueden considerar dentro de los controladores en bucle cerrado los trabajos (Bemporad, 1998*a*) y (Chisci, Rossiter and Zappa, 2001) en los cuales se parametriza la ley de control como $u_k = K{\cdot}x_k + v_k$, siendo $K{\cdot}x$ una ley de control que estabiliza la planta nominal. El controlador predictivo se formula en términos de v_k. Las restricciones en los estados y en las actuaciones son politópicas, y por lo tanto, tras el cambio de variables siguen siendo politópicas. Esta técnica mejora el condicionamiento numérico del problema de optimización y en cierta forma, el controlador añadido reduce el efecto de las incertidumbres, pues añade cierta realimentación en las predicciones en bucle abierto. Recientemente (Mayne, Seron and Rakovic, 2005) se ha aprovechado esta idea para diseñar un controlador robusto prealimentado basado en la noción de tubo (Langson, Chryssochoos, Rakovic and Mayne, 2004). Este controlador se explicará de forma más detallada en el siguiente capítulo.

Además, en (Chisci and Zappa, 1999) se añade una restricción adicional con el fin de garantizar la satisfacción robusta de las incertidumbres. Esta idea se generaliza al caso no lineal en (Kerrigan, 2000), donde se analiza utilizando la teoría de conjuntos invariantes la satisfacción robusta de las restricciones.

Todas estos controladores reducen el conservadurismo de la formulación en bucle abierto, pero siguen siendo más conservadoras que la formulación en bucle cerrado. A su favor tienen que son controladores más fácilmente implementables.

1.6.4 El problema del cambio de referencia en el contexto de los controladores predictivos

Como se comentó previamente, un optimizador de consignas es el responsable de proveer el punto de trabajo de la planta, de tal forma que si el punto de trabajo cambia, el controlador a bajo nivel debe realizar dicho cambio. La solución clásica es trasladar el problema al nuevo punto de trabajo (Muske and Rawlings, 1993), esta solución es solo válida en ausencia de restricciones.

Un control óptimo con horizonte infinito que considere restricciones podría ser una solución válida, realizaría el cambio de referencia de forma admisible, pero este tipo de controladores no son implementables debido a que un horizonte infinito implica un número infinito de variables de decisión. Así que habrá que utilizar un control predictivo con horizonte finito para resolver el problema, el problema con este tipo de controladores es que un cambio de referencia puede producir una perdida de la factibilidad del problema por una de las siguientes causas: (1) la restricción terminal para el nuevo punto de equilibrio puede no ser un invariante admissible lo que puede provocar la perdida de factibilidad. (2) la region terminal para el nuevo punto de operación podría no ser alcanzable en N pasos, lo que haría de nuevo, perder la factibilidad del problema. Esto requeriría el recálculo del horizonte para recuperar la factibilidad, por lo que un cambio de referencia conllevaría el rediseño on-line del controlador, lo que no será siempre posible.

En la figura 1.6 se muestra el problema de la perdida de la factibilidad cuando se produce un cambio de referencia. El ejemplo escogido corresponde a un modelo real de una planta usado más adelante en la sección 3.8.1.

Supongamos que la planta se encuentra en el punto x_0 y el estado de referencia es r_1, $O_\infty(r_1)$ es el máximo invariante para el sistema controlado por la ley de control $u = K(x - x_1) + u_1$ ((x_1, u_1) es el estado y la acción de control del sistema en equilibrio en el estado de referencia r_1), es también la restricción terminal para nuestro MPC con horizonte $N = 3$, luego la región de atracción será $X_3(r_1)$. Supóngase que en este instante se cambia la referencia al punto r_2, lo primero de que se da uno cuenta es de que la region terminal ya no es válida puesto que $O_\infty(r_1)$ trasladado al punto de referencia r_2 es un invariante no admisible (las restricciones serían violadas claramente) lo que nos lleva a una posible perdida de factibilidad. Por otro lado el punto x_0 no pertenece a la región de atracción $X_3(r_2)$ con horizonte $N = 3$, habría que cambiar el horizonte a $N = 6$ para recuperar la factibilidad. Resumiendo, un cambio de referencia puede producir la perdida de la factibilidad debido a una región terminal no admisible o a un horizonte insuficiente.

Con objeto de superar estos problemas se han propuesto varias soluciones, en (Rossiter et al., 1996; Chisci and Zappa, 2003) se utiliza un controlador auxiliar que es capaz de recuperar la factibilidad en tiempo finito cuando esta de pierde por un cambio de referencia, es pues una estrategia de conmutación. Otra posible solución es la propuesta por (Pannocchia and Kerrigan, 2005; Pannocchia, 2004) considerando el cambio de referencia como una perturbación a rechazar, de esta forma este controlador es capaz de llevar el sistema al punto deseado pero solo cuando las variaciones en la referencia son pequeñas, es por lo tanto, una solución conservadora.

Un enfoque diferente es el que se le da a este problema en el contexto de los con-

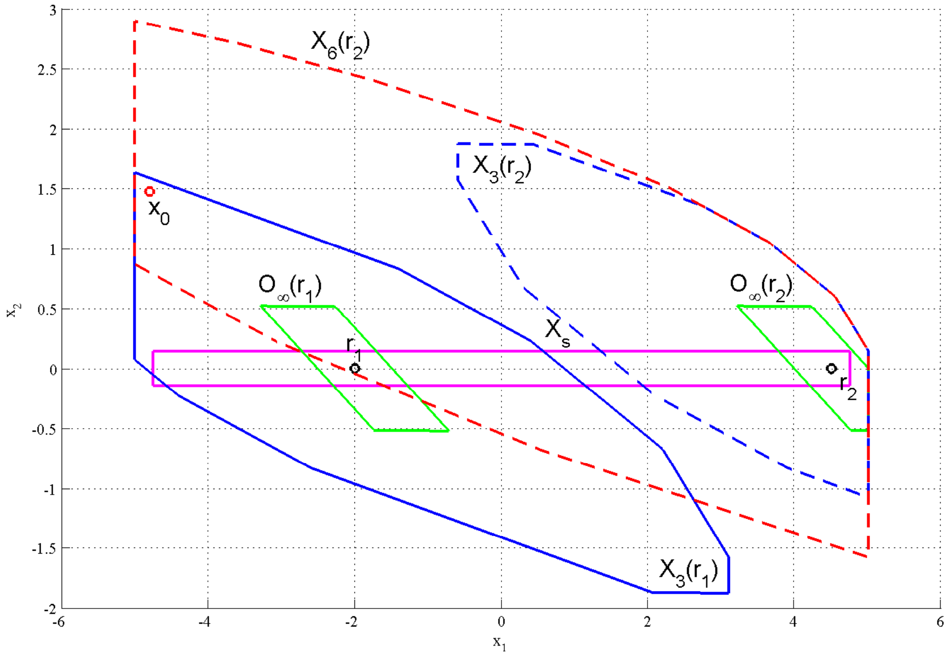

Figure 1.6: Pérdida de la factibilidad debido a una región terminal no admisible o a un horizonte insuficiente.

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troladores de referencia (Gilbert et al., 1999; Bemporad et al., 1997). Esta técnica de control asume que el sistema está estabilizado de forma robusta por un controlador local, y se diseña un filtro de referencia no lineal para la satisfacción robusta de las restricciones. Este tipo de controlador es capaz de seguir los cambios de referencia sin considerar el desempeño y la región de atracción de dicho controlador.

Este mismo problema en el contexto de sistemas no lineales ha sido tratado en (Magni, De Nicolao, Magni and Scattolini, 2001; Magni, 2002; Magni and Scattolini, 2005).

El objetivo de esta tesis es el estudio del problema de la perdida de factibilidad, en ella se propone un nuevo tipo de controlador para el seguimiento de referencias capaz de asegurar la evolución admisible del sistema y un desempeño óptimo para cualquier cambio de referencia admisible.

1.7 Contribuciones de la tesis.

1.7.1 Control predictivo para seguimiento de referencias constantes

En esta tesis se propone una nueva formulación de control predictivo para sistemas lineales con restricciones capaz de seguir cualquier valor de referencia constante satisfaciendo las restricciones.

Las principales características del controlador propuesto son

1. Considera, como variable de decisión, un valor de los estados y entradas de equilibrio como que juegan el papel de referencia artificial.

2. Se modifica la función de coste de forma que penalice el error con la referencia artificial.

3. Se añade un término adicional a la función de coste, que penaliza la desviación entre la referencia y la referencia artificial.

4. Se considera como restricción terminal en el estado y referencia artificial un invariante para tracking.

Este controlador presenta las siguientes ventajas:

- Es capaz de alcanzar prácticamente cualquier valor de referencia admisible para cualquier valor de horizonte de predicción
- El cálculo de la acción de control requiere la solución de un único problema de optimización QP
- La estructura del problema de optimización permite determinar de forma explícita la ley de control por medio del herramientas de programación multiparamétrica.
- La región de atracción es potencialmente mayor que la del MPC estándar.

1.7.2 Control predictivo robusto para seguimiento de referencias constantes

Otra novedad presentada en la tesis es el desarrollo de un controlador predictivo robusto para sistemas lineales con incertidumbres aditivas acotadas y restricciones. El controlador es capaz de conducir al sistema de una forma robusta a un punto de equilibrio admisible del sistema nominal. Este control se basa en el controlador para seguimiento presentado en el capítulo anterior. Incorporando la noción de tubo de trayectorias se consigue hacer el controlador robusto satisfaciendo las restricciones.

El controlador propuesto tan sólo requiere la solución de un problema de programación cuadrática QP en el cual aparecen como variables de decisión la secuencia nominal de acciones de control, la referencia artificial y el estado inicial del tubo.

Bajo ciertas condiciones, el controlador propuesto garantiza la convergencia robusta y admisible a una vecindad de un punto de equilibrio objetivo, y además mantiene las propiedades de convergencia ante cualquier cambio en la referencia.

Por ultimo, en el caso de perturbaciones persistentes, éstas producen un offset en las salidas que se podría compensar incorporando un estimador de perturbaciones y añadiendo un bucle externo que modifica las referencias.

1.7.3 Control predictivo robusto con realimentación de salidas para el seguimiento de referencias constantes

El controlador predictivo robusto para seguimiento anteriormente presentado, se adapta para acometer el problema de diseñar el controlador para el caso en que no todo el estado es medible, basándose en (Mayne, Rakovic, Findeisen and Allgöwer, 2006). Para ello, el estado real de la planta estará confinado en un tubo (tubo de estimación) cuyo centro es la trayectoria estimada y cuya sección un conjunto invariante para la dinámica del error de observación. El error de control también se encuentra confinado en un tubo de sección un conjunto invariante. La suma de ambos tubos, permite acotar el efecto de las incertidumbres en un tubo de sección la suma de las secciones de los tubos de error de control y observación.

Así, el controlador propuesto es capaz de garantizar la convergencia al punto de equilibro deseado a partir de las medidas de las salidas. Además se puede cancelar el offset en las salidas gracias a la estimación de estados y la variación del setpoint.

1.7.4 Síntesis de los controladores propuestos

El controlador predictivo propuesto logra sus propiedades siempre que los parámetros que definen el problema cumplan una ciertas condiciones. Dado que estas condiciones sobre los parámetros no los fijan de manera unívoca, estos se pueden ajustar de forma que se cumplan ciertas especificaciones de control.

Parte de los parámetros que se pueden ajustar son estándar en los controladores predictivos, tales como las matrices de ponderación del coste de etapa Q y R, la del coste terminal P, y fijan el desempeño del sistema en ausencia de incertidumbres.

Los parámetros más característicos del controlador propuesto son la ponderación del coste de offset T y la elección de la ganancia del controlador robusto. El parámetro T permite fijar la rapidez de la evolución de la referencia artificial y permite priorizar salidas para minimizar el offset, en caso de que exista. Por otro lado, la ganancia del controlador robusto permite acometer el rechazo a perturbaciones, que en esta tesis se propone como criterio la minimización del tamaño del mínimo invariante robusto. Esto además permitirá aumentar el dominio de atracción. Este controlador se puede diseñar mediante la resolución de LMIs.

Finalmente, se presenta métodos para calcular estimaciones del mínimo invariante robusto cuando las incertidumbres son paralelotopes.

1.7.5 Aplicaciones experimentales

Los controladores propuestos han sido aplicados a sistemas reales con objeto de probar su aplicabilidad. Estas plantas son las siguientes:

1. Un sistema de posicionamiento basado en un motor lineal.
2. Una gran instalación solar térmica llamada ACUREX perteneciente a la PSA (Planta Solar de Almería)
3. El proceso de los 4 tanques

1.7.5.1 Sistema de posicionamiento basado en un motor lineal

Una descripción exhaustiva de este sistema de puede encontrar en (Fiacchini, Alvarado, Limon, Alamo and Camacho, 2006). El objetivo es controlar la posición de un motor lineal actuando sobre la consigna del control de velocidad del mismo.

El modelo lineal con y sin incertidumbres ha sido identificado a partir de datos experimentales excitando al planta con una señal PRBS

El controlador ha sido implementado en una tarjeta de control dSpace que gobierna la planta. El tiempo de muestreo es de 30 ms lo que ha requerido el cálculo de la solución explícita de dicho controlador, esto se ha conseguido con técnicas de programación multiparamétricas. La búsqueda de la solución apropiada se ha acelerado mediante un árbol de búsqueda que también se ha implementado en la tarjeta dSpace.

Esta planta ha sido controlado con éxito por el controlador presentado en la sección (1.7.1)

Con objeto de probar la formulación robusta del mismo, sección (1.7.2), se le ha añadido sobre el motor lineal un péndulo invertido, que añade una perturbación adicional.

1.7.5.2 ACUREX

ACUREX es una instalación solar térmica de grandes proporciones del complejo PSA (Planta Solar de Almería) que produce energía eléctrica a partir de aceite caliente.

La principal característica de esta planta es que la fuente primaria de energía, la radiación solar, no se puede controlar. Además está sujeta a grandes variaciones durante el día, causando cambios en la dinámica de la planta y fuertes perturbaciones sobre le proceso. La planta real se modela como una planta lienal con incertidumbres en los estados. La planta, en función de la energía demandada, del rechazo de perturbaciones, o de los valores de la temperatura de entrada de colectores, varía el punto de trabajo, siendo necesario un controlador preparado para puntos de referencia cambiantes.

Bajo ciertas condiciones suaves, el controlador propuesto en la sección (1.7.2) es capaz de llevar al sistema incierto de forma factible a cualquier punto de trabajo admisible, permitiéndonos rechazar perturbaciones compensando su efecto cambiando el punto de trabajo.

1.7.5.3 El proceso de los 4 tanques

El proceso de los 4 tanques es una planta experimental basada en el conocido artículo de (Johansson, 2000*a*) que se ha desarrollado como parte de esta tesis y cuyos fines son educativos y de investigación.

El proceso de diseño, la instrumentación elegida y de más detalles del diseño se pueden encontrar en el apéndice (B).

El proceso de los 4 tanques consiste en cuatro tanques interconectados que pueden ser fácilmente configurados para mostrar el efecto de los ceros de transmisión de fase mínima y no mínima sobre el comportamiento del sistema. Otras características interesantes de la planta son:

- Es una planta MIMO.
- Todos los estados son accesibles.
- Es un proceso con restricciones

- Es una planta no lineal con incertidumbres (modelado como lineal con incertidumbres aditivas para nuestro controlador)

En la implementación de la planta real, la estructura original del proceso se ha modificado con el fin de ofrecer una amplia variedad de usos, tanto para uso educacional como para investigación. Así, se pueden configurar diferentes plantas, tales como un depósito aislado, 2 o 3 en cascada, un proceso de mezcla o bien un proceso híbrido. Además los parámetros que caracterizan la dinámica de cada tanque se pueden ajustar variando la sección de salida de los depósitos.

Merece la pena destacar que la instrumentación utilizada en la implementación de la planta es instrumentación industrial, lo que confiere a la planta un mayor parecido a un proceso industrial. El control a bajo nivel se realiza en un PLC y el control y supervisión a alto nivel se realiza en un PC, que permite la conectividad con otros equipos mediante el protocolo estándar OPC (LabView, Matlab, SIMATIC IT, etc.).

Los controladores propuestos en esta tesis se han probado con éxito sobre esta planta.

1.8 Lista de publicaciones

1.8.1 Publicaciones en revistas

1. D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of piecewise constant references for constrained linear systems. *Automatica*, Accepted for publication.

1.8.2 Publicaciones en revistas pendientes

1. D.Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust MPC for tracking of constrained linear systems with additive disturbances. *IEEE Transactions on Automatic Control*, 2007.

2. I. Alvarado, D.Limon, T. Alamo, and E.F. Camacho. Output feedback robust

tube based MPC for tracking of piece-wise constant references. *Automatica*, 2007.

3. I. Alvarado, D.Limon, T. Alamo, and E.F. Camacho. Robust control of the distributed solar collector field acurex using mpc for tracking. *IEEE Trans. Control Systems Technology*, 2007.

1.8.3 Publicaciones en congresos

1. D. Limon, I. Alvarado, T. Álamo, and E.F. Camacho. MPC for tracking of piece-wise constant references for constrained linear systems. In *Proceedings of the IFAC World Congress*, 2005.
2. I. Alvarado, D. Limon, W. Garcia-Gabyn, T. Alamo, and E.F. Camacho. An educational plant based on the quadruple-tank process. In *Proceedings of the ACE*, 2006.
3. M. Fiacchini, I. Alvarado, D. Limon, T. Alamo, and E.F. Camacho. Predictive control of a linear motor for tracking of constant references. In *Proceedings of the CDC*, 2006.
4. I. Alvarado, D. Limon, T. Alamo E.F. Camacho. Robust tube based output feedback MPC for tracking of piece-wise constant references for constrained linear systems with additive disturbances. In *Proceedings of the CDC*, 2007.
5. D. Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust tube based MPC for tracking of piece-wise constant references for constrained linear systems with additive disturbances. In *Proceedings of the CDC*, 2007
6. D. Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust control of the distributed solar collector field ACUREX using MPC for tracking. IFAC World Congress, 2008
7. I. Alvarado, D. Limon, T. Alamo, and E.F. Camacho. On the design of Robust tube-based MPC for tracking. In IFAC World Congress, 2008
8. I. Alvarado, D. Limon, A. Ferramosca, T. Alamo, and E.F. Camacho. Robust tubed-based MPC for tracking applied to the quadruple-tank process. In IEEE Multi-conference on Systems and Control, 2008

Chapter 2

Introduction

The aim of this chapter is to describe the motivation and objectives of this thesis and to introduce the research work done. For this, first the relevance in the industry of the control problem for systems subject to large setpoint changes is presented, defining the aims of this work. Then, it stock is taken of the control strategies used to solve this problem, focusing on the suitable strategy to approach it: predictive control. Later, a summary of different predictive controllers is considered and the stability problem in presence of restrictions is presented, focusing on the way predictive control solve this problem. Finally, the problem of uncertainties in predictive control and its robust formulations are presented.

2.1 Motivation of the thesis

In the last years, operation techniques in the process industry has made important progress, due to the need for production in a safe, clean, in competitive way and satisfying the necessities of the market, with respect to both demand and quality. Two reasons justify this fact: on the one hand, the need to satisfy the necessities of a market which is even more diversified because of its social and cultural habits and the need for strict safety controls on products as well as variety and quality, which all results in a shorter product life cycle. On the other hand, the need to favor sustainable growth, minimizing both environmental impact and resource consumption. Both factors contribute to the desire for the most efficient production which satisfies requirements and limits imposed on the products.

For all this, it is desirable to look for control techniques which provide control laws that optimize some efficiency criteria and guarantee the satisfaction of the limits imposed on the products. Model predictive control is one of the few techniques which permit this problem to be solved (de Prada, 2004).

In the process industry, the existence of an optimal operation point at which the process should remain in order to maximize its efficiency is habitual. However, many processes are subject to frequent changes of the operation point when working, in such a way that there is not one operation point, but a range of operation points at any of which the process is able to work for a time. The selection of a point from this range will be made with respect to the variety of product, sets or situations in which the plant could be.

To illustrate this kind on process, consider a lot reactor. This process is characterized by a continuous reaction in its interior which can be controlled by manipulating the reactant and refrigerant flows (see figure 2.1). This process is subject to restrictions on the manipulated variables, imposed by the reactant and refrigerant inflow valves. On the other hand, variables like the reactor temperature or pressure have to remain within a permitted range, and the product concentration has to fulfil certain specifications. Moreover, the reactor can changes its operation point due to the fact that, for a time, a particular way of working is required, depending on temperature and concentration conditions.

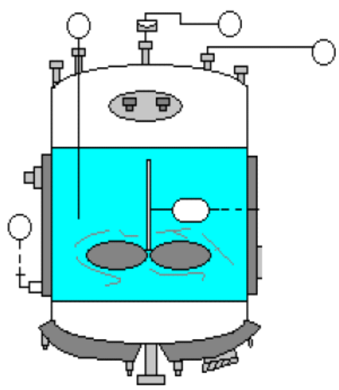

Figure 2.1: Continuous Stirred-Tank Reactor Scheme.

A solar plant is another process characterized by changes in its daily operation. The aim of this system is to heat a fluid by means of the energy radiated by the sun.

The fluid is pumped through solar panels, or solar collectors where the solar energy os collected. The control object is to raise the output temperature within a useful range in order to be used for different porpoises, such as producing electricity (as in the DISS plant case shown later) or, alternatively, produce cold air using an absorption machine (as in the solar refrigeration plant case, also shown later). Usually, the manipulated variables are valves with fixed actuation limits and limited opening/closing velocities. Moreover, the temperatures and flows through the collectors are also constrained because of safety and/or operating conditions.

Figure 2.2: DISS solar plant.

Because of the solar radiation evolution throughout the day, solar collector temperatures change, moving away from the desired values. The control system has to guarantee the correct operation of the plant, changing its manipulated variables with the aim of keeping the output temperature within the permitted range.

In relation to these needs, the aim of this work is to develop an advanced control strategy for processes with changing operation points, in the presence of constraints that permits efficient, flexible and integral operation in such a way that, using the available resources, the security and quality of products are guaranteed.

2.1.1 Control of Plants with Changing Operation Points

When undertaking the proposed control problem, it is necessary to consider two determining aspects. The first one derives from the large range of operation of the plant, which stresses the non linear nature of its dynamics (implicit in the equations associated to mass, energy and momentum balances) and the uncertainty level (structural and parametric) associated to its state space representation. Moreover, in this kind of plant, the spacial distribution of some or all of its state variables is usual, giving rise to dynamics composed of coupled algebraic, ordinary differential or partial differential equation systems. Therefore, this kind of system can be considered as a paradigm for complex dynamic systems, with high coupling and dimensions (basically due to the spacial distribution of its state variables), and subject to restrictions of an economical and environmental nature.

The presence of restrictions in the system operation is added to its complex nature. These restrictions can be limits on the control variables, or limits on the process variables. They can derive from the physical limits of the variables or from limits in the plant evolution zone due to economical, environmental or operational reasons.

The presence of restrictions conditions the systems behavior in a notable way, stressing its non linear nature. They can also cause performance loss, poor plant operations and instability(Mayne, 2001).

The traditional way to solve this problem consists of a hierarchical control structure, in which lower level control deals with the regulation of the plant, usually done by PID or by programmable automatons connected in a network. Higher level control is usually characterized by an advanced control strategy, generally multivariable. The aim of this control is to determine the input of the lower level control in order to keep the system at the desired operation point. This operation point is calculated by an upper level control, in which a set-point optimizer is implemented, according to certain data from the plant and according to economic criteria, coming from the plant information integration system. This structure is shown in figure 2.3.

Higher level control can be designed in order to keep the systems at the operation points avoiding constraint violation. Model predictive control (Qin and Badgwell, 1997) is one of the most successful advanced control strategies because it includes an optimum criterion and constraints in the control law. When the higher level indicates an operation change, the controller has to react to this event by moving the system to the new operation point. When these changes are slight, then the controllers are used to undertake this task successfully, but when the changes are great, then there could be a problem due to the change of dynamics at the new point, or to guarantee the constraints satisfaction during the transition to the new point.

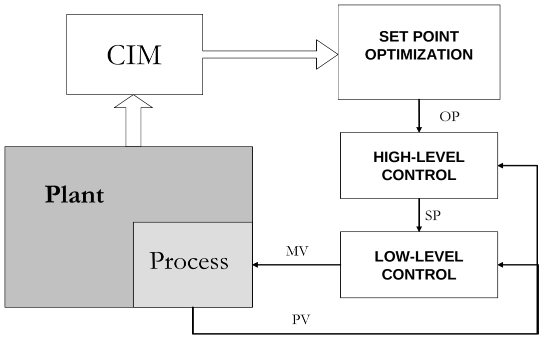

Figure 2.3: Hierarchical control structure.

In order to manage significant changes at the operation points, the higher level control is divided into two sublevels (Becerra et al., 1998): the lower one deals with the regulation of the system, and the upper one deals with the adaptation of the controller for the new point, or more generally, with the management of transitory control when faced with big changes at the operation points. This scheme is shown in figure 2.4.

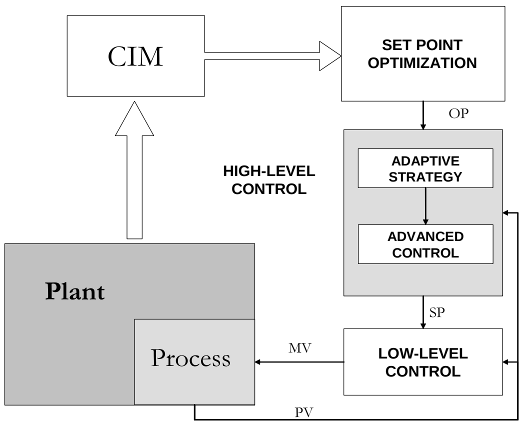

Figure 2.4: Hierarchical control structure with adaptive higher level.

Adaptive controllers such as the classic gain scheduling in case of changes in the flight height belongs to this scheme. Other controllers belonging to this category are the ones know as *reference governors* (Gilbert et al., 1994; Gilbert et al., 1999). This kind of controller corresponds to the higher sublevel, and assumes that an advanced controller which stabilizes the plant is at the higher level. The aim of the reference governors is to manage, in a rational way, the process reference, to avoid the violation of constraints when the setpoint changes. In this way, it is a sophistication of the know reference filter, in order to avoid the violation of constraints. The design of this kind of controller does not pay attention to process efficiency, but only aims at avoiding the violation of the limits. This aspect is discussed in (Bemporad et al., 1997), in which a minimization of a performance index of the predicted evolution of the system is proposed. Reference governors are also successfully used in case of non linear dynamics (Bemporad, 1998*b*; Angeli and Mosca, 1999; Gilbert and Kolmanovsky, 2002).

In the case of model predictive control, there are a lot of formulations oriented at the management of large transitions. These controllers permit large changes in the operation point and determine the control action on the basis of a performance criterion. However, the stability guarantee is based on a hierarchical structure, like the one shown in figure 2.4, in which the higher sublevel deals with the commutation between the predictive controller and the other controller oriented at the recuperation of the system in case of loss of feasibility.

Another way to approach the problem is by integral control, in which advanced control strategies (generally predictive control) are studied in which economical objectives associated to operation changes are considered. Hence, some of the optimization tasks move from the setpoint optimizer to the advanced controller, in order to incorporate the cost associated to the transitions into the operation point determination. In (Becerra and Roberts, 1996; Becerra et al., 1997; Becerra et al., 1998) a different way to integrate model predictive control with on-line optimization of economical objectives are considered such as the solution of a hierarchical control problem or the solution of a multi-objective control problem in which both the regulation objectives and the economical ones are minimized. In some works that consider a double-layer hierarchical structure, it is supposed that the lower layer controllers maintain the stability, although in these works the difference between model and reality is considered, defining an iterative method which permits the algorithm to converge, even in case of model errors. In (Vesely et al., 1998) the basics and properties of a feasible method for complex system steady state optimization is presented, in such a way that the problem can be solved by means of algebraic equations. However, in these works a stability, robustness and convergence study for the developed schemes does not appear not even for the interaction of the components. The stability and smooth transition between set-point problems for systems subjected to continuous specification changes and load perturbations are not considered either.

In consequence, the hierarchical structures with guaranteed stability and constraint satisfaction give worse performance than integral control due to their independent design. However, the integral control structures lack stability and constraint satisfaction demonstrated properties. Hence, it is suitable to design control strategies which permit the unification of the integral control for large transitions at only one level, guaranteeing stability in case of constraints and minimizing a performance index.

Model predictive control is one of the few techniques which permit the control of systems subjected to restrictions considering an optimum criterion and guaranteeing stability and convergence to the equilibrium point (Camacho and Bordons, 2004; Mayne, 2001). Hence, it is proposed as the strategy to approach the considered problem. In figure 2.5 the proposed idea is shown and it is clear that the two level control structure (see figure 2.4) is replaced by only one controller which carries out both tasks of stabilization and of new setpoint control simultaneously.

2.2 Model Predictive Control

One of the most successful control techniques for constrained systems is model predictive control (MPC). This control strategy is capable of ensuring an admissible evolution of the system while optimizing the closed-loop performance measured by a cost, function that takes into account the error with the desired set point. The cost function is based on the prediction of the future evolution of the system by means of the prediction model of the form

$$x(i+1) = f(x(i), u(i))$$

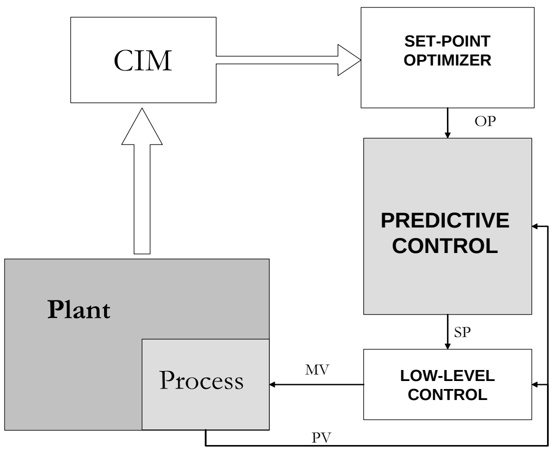

Figure 2.5: Integral Control Structure.

This allows to be considered the predicted cost of the system as follows

$$V_N(x_k, \mathbf{u}) = \sum_{i=0}^{N-1} L(x(i), u(i)) + F(x(N)),$$

where $u(i)$ is the future sequence of control action computed at the current sampling time k, and $x(i)$ is the predicted state at sampling time i, considering that $x(0) = x_k$. The function $L(x, u)$ is known as stage cost, while $F(x)$ is the terminal cost function.

The control action is derived from an optimization problem in which the future sequence of control actions is computed to minimize the predicted cost while satisfying the constraints. This optimization problem can be posed as the following mathematical programming problem:

$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N(x_k, \mathbf{u}) \\ s.a \quad & \\ & u(j) \in U \qquad j = 0, \cdots, N-1 \\ & x(j) \in X \qquad j = 0, \cdots, N-1 \\ & x(N) \in \Omega. \end{aligned}$$

The additional constraint on the terminal state is usually considered for stability reasons.

Once the optimal sequence of control actions $\mathbf{u}^*$ is computed, feedback is achieved by considering only the current control action and re-computing the optimization problem at each sample time. This is the so-called receding horizon technique. Thus, the control law is given by

$$u_k = u^*(0; x_k)$$

It would be desirable to consider the whole predicted evolution of the system at the predicted cost, but this leads to an infinite-horizon approach that cannot, in general, be solved. Thus, MPC considers a finite prediction horizon, which makes the problem tractable at the expense of the loss of the good properties of the optimal control problem, such as stability and inherent robustness. To overcome this problem, some additional conditions must be considered in the controller design (Mayne et al., 2000).

There exist different stabilizing formulations of MPC:

- MPC with terminal equality constraint (Kwon and Pearson, 1977): Stability is guaranteed by adding an additional constraint over the state at the end of horizon x_N (terminal state) called terminal constraint:

$$x_N = x_s$$

  where x_s is the desired steady state.

- MPC with terminal cost (Bitmead et al., 1990): Stability of the system is guaranteed by adding a new term to the cost function that penalizes the state and the end of the horizon.

- MPC with inequality terminal constraint (Michalska and Mayne, 1993): The terminal equality constraint was replaced by a set Ω that has to fulfil certain conditions.

$$x_N \in \Omega$$

  This approach provides a greater region of attraction and less numerical problems than the equality constraint approach.

- MPC with terminal cost and constraint (Sznaier and Damborg, 1987): This approach is the result of the union of the last 2 techniques, adding a terminal cost to the cost function and using an inequality terminal constraint.

In (Mayne et al., 2000) all these formations are analyzed and it is established that terminal conditions (a terminal cost together with a suitable terminal constraint) have resulted to be essential to the stabilizing design and nowadays, this approach is considered as standard. These conditions are as follows:

Let Ω be a set in $\mathbb{R}^n$, let $F(x)$ be a definite positive function, continuous at the origin and let $h(x)$ be a control law such that,

- For all $x \in \Omega \subseteq X$, then $f(x, h(x)) \in \Omega$ and $h(x) \in U$

- For all $x \in \Omega$, we have that

$$F(f(x,h(x)) - F(x) \leq -L(x,h(x))$$

The invariant condition on terminal set Ω ensures feasibility of the closed-loop evolution of the system, while the condition on terminal function $F(x)$ guarantees convergence, thanks the fact that the optimal cost can be considered as a Lypaunov function.

2.2.1 Robustness in Model Predictive Control

It is well known that under mild conditions, MPC is able to ensure some degree of robustness (Scokaert et al., 1997; De Nicolao et al., 1996; Limon, Álamo and Camacho, 2002). When the uncertainties are big enough, a robust design must be accomplished. To this aim, an uncertainty model must be considered; this is typically considered as an external disturbance acting on the dynamics in a parametric way or by means of an additive term in the model function.

This model is used to take into account the effect of the uncertainty on the predicted cost in order to minimize the worst case scenario. This leads to the so-called min-max approach. On the other hand, the constraints must be fulfilled for every possible uncertainty (that is, robust feasibility).

Nowadays, this problem has been studied in depth and it has been pointed out that the open-loop nature of the robust MPC formulations based on a sequence of control inputs as decision variables leads to conservative controllers (Bemporad, 1998*a*). To overcome this problem, a sequence of control laws should be considered as decision variables, allowing the effect of the predicted uncertainty to be compensated. This closed-loop approach produces larger domains of attractions and better performance at the expense of a large computational burden (Kerrigan and Maciejowski, 2004; Mayne, Rakovic, Vinter and Kerrigan., 2006).

A trade-off solution between the open and the closed loop formulations is to add a robustly pre-stabilized plant (Bemporad, 1998*a*; Chisci et al., 2001). This solution enhances robustness, while the optimization problem can be cast as a mathematical programming problem similar to the open-loop formulation. A recent novel robust MPC of this class based on the notion of a tube of trajectories has been proposed

(Langson et al., 2004)). Using this notion, an enhanced robust MPC controller has been proposed in (Mayne et al., 2005). This controller exploits the notion of invariant sets to obtain a robust control law based on nominal predictions.

2.2.2 The Problem of MPC when the Setpoint Changes

As has been mentioned before, a higher level real time optimizer is responsible for providing a target or desired setpoint to operate the plant. Therefore, if this operation point changes then the lower level control law must deal with the setpoint changes. The classic solution is to translate the system to the new steady state (Muske and Rawlings, 1993), which guarantees the set point tracking when there are no constraints. However, this simple translation may not solve the problem when the plant has constraints.

If an optimal control law derived from a constrained infinite horizon regulator can be used, any admissible set point can be tracked in an admissible way. However, infinite horizon optimal control laws are not computationally tractable and so finite prediction horizons must be considered. In this case, the change of set points can produce a loss of feasibility of the optimization problem throughout the system evolution. This can be a consequence of one or both of the two following causes: (i) the terminal set shifted to the new operating point may not be an admissible invariant set, which means that the all time feasibility property may be lost and (ii) the terminal region at the new setpoint could be unreachable in N steps, which means that the optimization problem is unfeasible, making a re-calculation of an appropriate value of the prediction horizon to ensure feasibility. Therefore, this would require an on-line re-design of the controller for each set point, which can be computationally unaffordable.

The loss of feasibility problem under a setpoint change is illustrated in figure 2.6. This corresponds to the predictive control of the model and constraints used in example 3.8.1. Consider that the current state is x_0,the current target is r_1, the set $O_\infty(r_1)$ is the maximal invariant set for the system controlled by $u = K(x - x_1) + u_1$ ((x_1, u_1) is the state and control action for the system in permanent regime at setpoint r_1) which is the terminal constraint for an MPC controller with horizon $N = 3$; in this case, the region of attraction is $X_3(r_1)$. Suppose that the setpoint is changed to r_2 at a certain sampling instant. The first consequence that is shown is that set $O_\infty(r_1)$ translated to the steady state corresponding to r_2 is not an admissible invariant set (the constraint would be clearly violated). Then this procedure would lead to a loss of feasibility. Another issue is that, due to the fact that x_0 is out of $X_3(r_2)$, the MPC with this

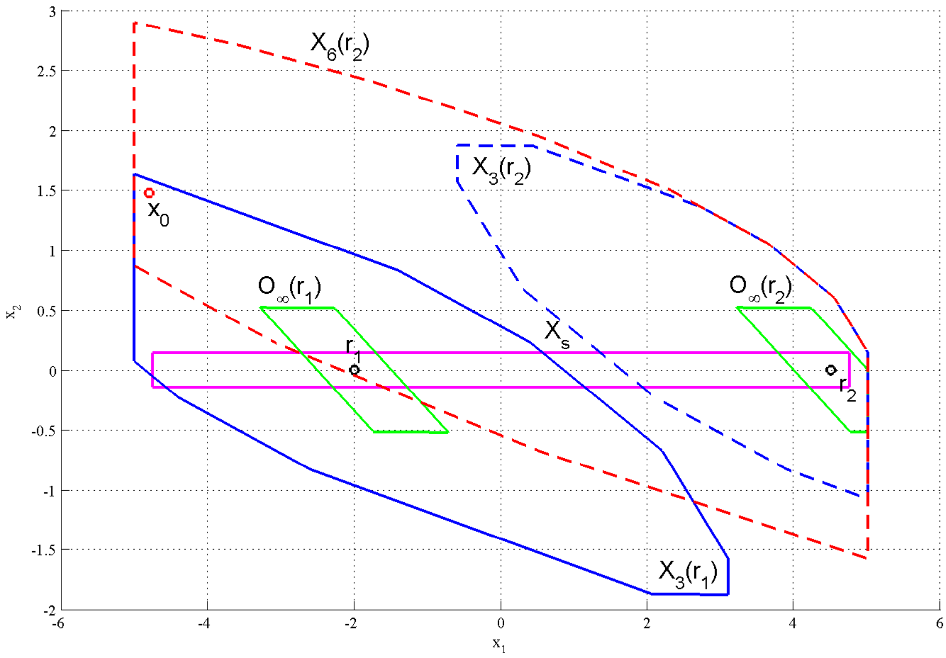

Figure 2.6: Loss of feasibility of the optimization problem throughout the system evolution, derived from a non-admissible terminal condition or a short horizon.

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horizon would be feasible; In order to recover feasibility, the prediction horizon should be enlarged to $N = 6$. In brief, a setpoint change can produce a loss of feasibility derived from a non-admissible terminal condition or a short horizon.

In order to overcome this problem several solutions have been proposed: in (Rossiter et al., 1996; Chisci and Zappa, 2003) an auxiliary controller that is able to recover feasibility in finite time is used leading to a switching strategy. The controllers proposed in (Pannocchia and Kerrigan, 2005; Pannocchia, 2004) consider the change of the set point as a disturbance to be rejected; thus, this technique is able to steer the system to the desired set point, but only when the variations of the set point are small enough; so this solution proves to be conservative.

A different approach has been proposed in the context of the reference governors (Gilbert et al., 1999; Bemporad et al., 1997). This control technique assumes that the system is robustly stabilized by a local controller, and a nonlinear filtering of the reference is designed to ensure the robust satisfaction of the constraints. These controllers ensure robust tracking without considering the performance of the obtained controller nor the domain of attraction.

The problem of tracking in the case of nonlinear MPC has been tackled in (Magni, De Nicolao, Magni and Scattolini, 2001; Magni, 2002; Magni and Scattolini, 2005).

The objective of this thesis is to study the loss of feasibility problem and to propose novel predictive controllers which ensure admissible evolution and optimal performance for any change in the target state.

2.3 Thesis Contributions

2.3.1 MPC for Tracking Piece-Wise Constant References for Constrained Linear Systems

In this thesis, a novel MPC is proposed to track any desired steady state ensuring feasibility and convergence for any prediction horizon. The main differences to the standard MPC for regulation are:

1. Considering an artificial steady state and input as decision variables (in a similar way to the reference governors).
2. Modifying the cost function to penalize the deviation to the artificial steady state.
3. Adding an additional term that penalizes the deviation between the artificial steady state and the desired one.
4. Considering an extended terminal constraint (an invariant set for tracking) on both the terminal state and the artificial steady state.

The main advantages of this controller are:

- It is able to reach any reference independently of the horizon.

- It solves the problem of admissible reference, constraint satisfaction and asymptotical stability in one shot, by just solving a single QP.

- The structure of the QP allows to apply multiparametric result to compute the explicit control law, and apply it to fast systems.

- Due to the fact that the change of reference is not considered as a disturbance, the range of reachable admissible setpoints is wider than the existing predictive controllers for tracking.

- The region of attraction is (potentially) greater than in the case of MPC for regulation because of the terminal region (the invariant set for tracking) is also (potentially) greater.

2.3.2 Robust MPC for Tracking Constrained Linear Systems with Additive Disturbances

In this thesis, a novel formulation of robust MPC for tracking for constrained linear systems subject to additive and bounded disturbances is proposed. This formulation is capable of leading the system to any robustly admissible set point in an admissible way. The proposed controller follows the novel MPC formulation presented in section (2.3.1) and in (Limon, Alvarado, Álamo and Camacho, 2005). This controller will lead the nominal system to reference and, by incorporating the notion of tube-based robust control, the real system will be steered to a neighborhood of the reference.

The robust controller obtained is based on the solution of a single Quadratic Programming problem and the decision variables are the initial state of the nominal system, the control sequence over a finite horizon and the artificial steady state.

Under mild conditions, the proposed controller ensures robust and admissible convergence to (a neighborhood of) the desired steady state, and maintains these properties under any change of reference. Moreover, offset free control can be achieved by means of a simple procedure if the disturbances converge to a constant value.

2.3.3 Enhanced Controller Design

The controller formulation has some parameters which provide an extra degree of freedom to the design procedure of the predictive controller. These allow the following control objectives to be dealt with: disturbance rejection, output offset priorization and enlargement of the domain of attraction.

Due to the fact that the stability conditions only impose bounds over these parameters, some criteria is given in order to help choose the correct value of the parameters. Moreover a tool based on LMIs has been developed to optimize the selection of one of the parameters. This tool tries to minimize an invariant that depends on a gain, $u = Kx$, and is based on the following ellipsoid, $\mathcal{E}(P, 1) = x \in \mathbb{R}^n \quad x^T Px \leq 1$, that should satisfy the following conditions:

1. The ellipsoid must be an invariant set for the system: $x^+ = (A + BK)x + v$.
2. The maximum value of u in the ellipsoid is bounded $\forall x \in \mathcal{E}(P, 1)$, $|K_i x| \leq \rho_i$, $i = 1, \ldots, m$ (where K_i denotes the i-row of the matrix K).
3. The size of the ellipsoid should be minimum.

This problem can be posed as an LMI problem.

Is also provided, in this thesis, a procedure to calculate the minimum robust invariant set where the constraints and disturbances set can be represented by a zonotope, and an efficient tool to calculate the polytope from the zonotope.

2.3.4 Output feedback Robust Tube Based MPC for Tracking Piece-Wise Constant References

The proposed controller for tracking is based on the results presented in section (2.3.1) and in (Mayne, Rakovic, Findeisen and Allgöwer, 2006). This paper is an extension of the concept of tubes to the robust output feedback MPC case. The real state is confined in a tube, the center of which is the estimated trajectory and its section is an invariant set (estimation tube). In an analogous way, the estimated state will be confined by the control law in another tube (control tube), the center of which is the nominal trajectory (the system without disturbances) and its section is an invariant set. Therefore, the real state will be confined in a tube the center of which is the nominal trajectory and its section is a larger invariant set that results in the Minkowsky addition of the other two. Then, using the tracking algorithm proposed in section (2.3.1), the nominal system is steered to the desired setpoint. Thus, the real state will be steered to the invariant set aforementioned, centered on the desired steady state.

The constraints for the nominal system will be those in which the tube satisfies the real constraints.

2.3.5 Experimental Applications

All of these controllers have been applied to real plants, in order to demonstrate their applicability. The real plants are the following:

1. A positioning system based on a linear motor
2. A distributed solar collector field of the Solar Power Plant at Almería (PSA) called ACUREX
3. The Four Tank Process

2.3.5.1 The Positioning System Based on a Linear Motor

An in depth description of this positioning plant can be found in (Fiacchini et al., 2006). The objective is to control the position of the linear motor actuating on the speed setpoint of the low-level controller.

The linear model of the plant and the uncertainties were identified from the experimental result stimulating the plant with a PRBS signal.

The controller was implemented on the dSpace Control Card to be tested at the real plant. Given that the sampling time is 30 ms, an explicit implementation of the proposed MPC was compulsory. This was achieved by using multiparametric programming techniques and calculating a search tree to express the control law (Tøndel, Johansen and Bemporad, 2002). The obtained search tree has been coded as an S-function in C language and compiled to run on the dSpace Card.

This plant was successfully controlled by the controllers introduced in section (2.3.1)

An inverted pendulum located over the positioning system was released in order to add some disturbances. The system with the additional disturbances was successfully controlled by the robust tracking formulation introduced in section (2.3.2).

2.3.5.2 ACUREX

The distributed collector field ACUREX, is located at the solar power plant of PSA (Solar Plant of Almería). The porpoise of this plant is to collect solar energy to be transformed in electricity by heating oil. The main characteristic of a solar power plant is that the primary energy source, solar radiation, cannot be manipulated. Solar radiation varies throughout the day, causing changes in the plant dynamics and strong disturbances in the process. The real plant is assumed to be modelled as a linear system with additive bounded uncertainties on the states.

This plant varies its setpoint in function of the energy demand, the disturbance rejection or depending on the values of the inlet temperature of the collectors, and this makes the use of a control for tracking necessary.

Under mild assumptions, the controller proposed in section (2.3.2) can steer the uncertain system in an admissible evolution to any admissible steady state; that is, under any change of the set point. This allows us to reject constant disturbances, compensating for the effect of then changing the setpoint.

2.3.5.3 The Four Tank Process

The four tank process is an experimental tank system developed at the University of Seville for process control education and research and it was designed and buildt by the authors of this thesis. This plant is based on the well known quadruple-tank process (Johansson, 2000*a*).

The designing procedure, the instrumentation and the physical skills can be found in the appendix (B).

The quadruple tank process consists of 4 interconnected tanks that can be easily configured to exhibit the effect of multivariable zero (minimum and non-minimum phase) on the system behavior. Other interesting features of the plant are:

- It is a MIMO plant.
- All the states are accesible.
- It is a constrained plant
- It is an uncertain non-linear plant (modelled as a linear with disturbances)

In the real plant implementation, the original structure of the process has been modified to offer a wide variety of uses for both educational and research purposes. Thus, different plants can be configured, such as one single tank, two or three cascaded tanks, a mixture process and hybrid dynamics. Moreover the dynamics parameters of each tank can be set up by tuning the cross-section of the outlet hole of the tank. Furthermore, the real plant has been implemented using industrial instrumentation and a PLC for the low level control. Supervision and control of the plant is carried out in a computer by means of OPC (Ole for Process Control) which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or industrial SCADA.

The controller introduced in section (2.3.2) was successfully implemented over this plant.

2.4 List of Publications

2.4.1 Journal Papers:

1. D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of piecewise constant references for constrained linear systems. *Automatica*, Accepted for publication.

2.4.2 Submitted Journal Papers:

1. D.Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust MPC for tracking of constrained linear systems with additive disturbances. *IEEE Transactions on Automatic Control*, 2007.

2. I. Alvarado, D.Limon, T. Alamo, and E.F. Camacho. Output feedback robust tube based MPC for tracking of piece-wise constant references. *Automatica*, 2007.

3. I. Alvarado, D.Limon, T. Alamo, and E.F. Camacho. Robust control of the distributed solar collector field acurex using mpc for tracking. *IEEE Trans. Control Systems Technology*, 2007.

2.4.3 Conference Papers:

1. D. Limon, I. Alvarado, T. Álamo, and E.F. Camacho. MPC for tracking of piece-wise constant references for constrained linear systems. In *Proceedings of the IFAC World Congress*, 2005.

2. I. Alvarado, D. Limon, W. Garcia-Gabyn, T. Alamo, and E.F. Camacho. An educational plant based on the quadruple-tank process. In *Proceedings of the ACE*, 2006.

3. M. Fiacchini, I. Alvarado, D. Limon, T. Alamo, and E.F. Camacho. Predictive control of a linear motor for tracking of constant references. In *Proceedings of the CDC*, 2006.

4. I. Alvarado, D. Limon, T. Alamo E.F. Camacho. Robust tube based output feedback MPC for tracking of piece-wise constant references for constrained linear systems with additive disturbances. In *Proceedings of the CDC*, 2007.

5. D. Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust tube based MPC for tracking of piece-wise constant references for constrained linear systems with additive disturbances. In *Proceedings of the CDC*, 2007

6. D. Limon, I. Alvarado, T. Alamo, and E.F. Camacho. Robust control of the distributed solar collector field ACUREX using MPC for tracking. IFAC World Congress, 2008

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Chapter 3

MPC for tracking of piece-wise constant references for constrained linear systems

3.1 Introduction

Among the existing results to deal with the tracking problem in presence of constraints, a remarkable approach is the so-called command governors (Gilbert et al., 1994); this technique is based on the addition of a nonlinear low-pass filter of the reference to guarantee the admissible evolution of the system to the reference. This can be seen as adding an artificial reference (the output of the filter) which is computed at each sampling time to ensure the admissible evolution of the system, converging on the desired reference. In (Bemporad et al., 1997) a command governor is designed to minimize a performance index of the predicted evolution of the system. In (Blanchini and Miani, 2000) it is proved that any control invariant set for the constrained system is a tracking domain of attraction and an interpolation-based control law is proposed.

In (Rossiter et al., 1996), a Constrained Stable Generalized Predictive Controllers (CSGPC) for SISO plants is presented; this ensures feasibility by adding an artificial reference as decision variable and convergence is ensured by means of a contractive constraint based on the closest reachable reference. In (Chisci and Zappa, 2003) a dual-mode strategy for tracking based on MPC is presented: if the MPC is not feasible, the controller switches to a feasibility recovery mode which steers the system to the feasibility region of the MPC. In (Pannocchia and Kerrigan, 2005), the offset-free tracking problem is analyzed for uncertain systems with unmeasured disturbances; the proposed MPC considers the time-varying set point as a disturbance to be rejected. This makes that the set of set points to be tracked is potentially small and the domain of attraction reduced.

In this chapter, a novel formulation of the MPC is proposed for general non-square linear systems to track any admissible target steady state in an admissible evolution. The main ingredients are: (i) an artificial steady state and input are considered as decision variables, (ii) the cost function penalizes the deviation between the artificial steady state and the desired one, and (iii) an extended stabilizing terminal conditions are considered, consisting of adding a tracking error penalty term in the cost function and adding a terminal constraint in both the terminal state and the artificial steady state and input.
This controller drives the system to any admissible target steady state, and if this is not admissible, the system is steered to the closest admissible steady state. Since the control law is derived from the solution of a single QP (for a given target steady state), the control law can be explicitly calculated by means of multiparametric programming tools (Bemporad, Morari, Dua and Pistikopoulos, 2002), and therefore, to be applied on fast systems. Compared with the standard MPC, the proposed controller provides a larger domain of attraction (for the same control horizon), but the local optimality property can not be ensured due to the extra degree of freedom added. Fortunately, the loss of optimality can be arbitrarily reduced by weighting the tracking error penalty term.

3.2 Problem description

Let a discrete-time linear system be described by:

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the current output and x^+ is the successor state. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively.

The system is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z}$$

for all $k \geq 0$. The set $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a compact convex polyhedron containing the origin in its interior:

$$\mathcal{Z} \;=\; \{z \in \mathbb{R}^{n+m} : A_z z \leq b_z\} \tag{3.2}$$

where z denotes the extended state of the system $z = (x, u)$.

Assumption 3.1 *The pair (A, B) is stabilizable.*

The problem we consider is the design of an MPC controller to track a piece-wise constant sequence of set points $s(k)$ in a way such that the constraints are satisfied for all the time.

3.3 Preliminary results

3.3.1 Characterization of the steady states of the linear system

The aim of this section is to determine the set of the steady states which produce a desired output; that is, for a given setpoint s, the set of steady states (x_s, u_s) such that $Cx_s + Du_s = s$.

Consider a given setpoint s, then any steady state of the system $z_s = (x_s, u_s)$ associated to this set-point must satisfy the following equation

$$\begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ s \end{bmatrix} \tag{3.3}$$

Denoting

$$E = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix}, \quad F = \begin{bmatrix} \mathbf{0}_{n,p} \\ I_p \end{bmatrix} \tag{3.4}$$

The equation (3.3) can be written as

$$Ez_s = Fs$$

The solution of this equation is characterized in the following lemma.

Lemma 3.1 *Assume that (A, B) is stabilizable. Let r be the rank of matrix E. Consider the minimal singular value decomposition of matrix E, i.e. $E = U\Sigma V^T$, where $U \in \mathbb{R}^{(n+p)\times r}$, $\Sigma \in \mathbb{R}^{r\times r}$ is a non-singular diagonal matrix and $V \in \mathbb{R}^{(n+m)\times r}$ such that $U^T U = I$ and $V^T V = I$. Then a pair (z_s, s) is a solution of equation (3.3), if and only if there exists a vector θ such that*

$$\begin{aligned} z_s &= M_\theta \theta \\ s &= N_\theta \theta \end{aligned} \tag{3.5}$$

where matrices M and N are given by

$$M_\theta = \begin{cases} \begin{bmatrix} V\Sigma^{-1}U^T FG & V_\perp \end{bmatrix} & \text{if} \quad r < n+m \\ V\Sigma^{-1}U^T FG & \text{if} \quad r = n+m \end{cases}$$

$$N_\theta = \begin{cases} \begin{bmatrix} G & \mathbf{0}_{p,n+m-r} \end{bmatrix} & \text{if} \quad r < n+m \\ G & \text{if} \quad r = n+m \end{cases}$$

where

$$G = \begin{cases} I_p & \text{if} \quad r = n+p \\ (F^T U_\perp)_\perp & \text{if} \quad r < n+p \end{cases}$$

The proof of this lemma can be found in the appendix.

Matrix G is a full column rank and its rank is $r_g \leq p$. The dimension of θ is $r_g + n + m - r$, it is the dimension of the subspace of equilibrium points, thus $n_\theta \leq n$. The first r_g components of θ are determined by the given set point s; if $r < n+m$, the rest of $n+m-r$ components can be considered as free variables. The set of references s that can be reached is the subspace spanned by the columns of G. Then this set depends on the rank of E according to the following two cases:

1. If $r = n + p$ then $r_g = p$. This implies that the system can be steered to any setpoint s.

2. If $r < n + p$ then r_g is smaller than p. This implies that equation (3.3) has a solution only for those setpoints s contained in the linear subspace spanned by the columns of G, and hence not every reference s can be reached. The usual way of overcoming this problem is re-defining the system: *new* controlled variables, $y_c \in \mathbb{R}^{p_c}$ with $p_c \leq r_g$ are taken; these *new* controlled variables are chosen as a linear combination of the actual outputs, i.e. $y_c = L_c y = L_c C x + L_c D u$. Matrix L_c must be such that the rank of the *new* matrix
$$E_c = \begin{bmatrix} A - I_n & B \\ L_c C & L_c D \end{bmatrix}$$
is full row rank, i.e. its rank is $n + p_c$.

Since the system is subject to constraints, the system should be steered to those steady states that satisfy the constraints. The set of these admissible steady states and inputs is defined as

$$\mathcal{Z}_s = \{z_s = (x_s, u_s) : \; z_s \in \mathcal{Z}, \text{and } (A - I_n)x_s + Bu_s = 0\}$$

Thus, the set of admissible steady states and the set of admissible inputs is defined as

$$X_s = Proj_x(\mathcal{Z}_s), \; U_s = Proj_u(\mathcal{Z}_s)$$

respectively. The set of all admissible set-points is denoted as S and it is given by

$$S = \{s \in \mathbb{R}^p : \; \exists z_s = (x_s, u_s) \in \mathcal{Z}_s \text{ such that } s = Cx_s + Du_s\}$$

For a given admissible set point $s \in S$, the steady state and input, i.e. z_s, such that the corresponding output is equal to s is unique if and only if the rank of E is equal to $n + m$. If the rank of E is less than $n + m$, then there exists infinite steady states and inputs z_s such that the associated output is equal to s.

3.3.2 Calculation of an invariant set for tracking

Consider the following controller

$$u = K(x - x_s) + u_s, \tag{3.6}$$

where x_s and u_s are the steady states and inputs that we want to reach. It is well known that if the controller gain K is such that $A+BK$ has all its eigenvalues inside the unit circle, then the system is steered to the desired steady state. Since the system is constrained, this controller leads to an admissible evolution of the system only in a neighborhood of the origin.

Substituting (3.5) in (3.6), it results that

$$\begin{aligned} u &= Kx + [-K\ I_m]z_s \\ &= Kx + [-K\ I_m]M_\theta\theta. \end{aligned}$$

That is,

$$u = Kx + L\theta, \tag{3.7}$$

where $L = [-K\ I_m]M \in \mathbb{R}^{m\times n_\theta}$. Consider the augmented state $x_a = (x,\theta)$, then the closed loop system can be posed as

$$\begin{bmatrix} x \\ \theta \end{bmatrix}^+ = \begin{bmatrix} A+BK & BL \\ 0 & I_{n_\theta} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{3.8}$$

that is, $x_a^+ = A_a x_a$.

Because of reasons that will be clearer later, define the following convex polyhedron

$$\mathcal{X}^a_\lambda = \{x_a = (x,\theta) : z = (x, Kx + L\theta) \in \mathcal{Z},\ z_s = M_\theta\theta \in \lambda\mathcal{Z}\}$$

It is clear that the set of constraints for system (3.8) is $\mathcal{X}^a = \mathcal{X}^a_{\lambda=1}$. That is, both $(x,u) = (x, Kx+L\theta)$ and $(x_s,u_s) = M_\theta\theta$ must belong to $\mathcal{Z}$.

We say that a set Ω^a_t is an admissible invariant set for tracking, for system (3.8) constrained to $\mathcal{X}^a$, if $\forall x_a \in \Omega^a_t$, then $A_a x_a \in \Omega^a_t$ and $\Omega^a_t \subseteq \mathcal{X}^a$. The maximal admissible invariant set for tracking is given by:

$$\mathcal{O}^a_\infty = \{x_a : A^i_a x_a \in \mathcal{X}^a, \forall i \geq 0\}$$

Due to the unitary eigenvalues of A_a, this set might be not finitely determined, i.e., described by a finite set of constraints (Gilbert and Tan, 1991). Consider the maximal admissible invariant set for tracking considering $\mathcal{X}^a_\lambda$ as constraint set, which is given by

$$\mathcal{O}^a_{\infty,\lambda} = \{x_a : A^i_a x_a \in \mathcal{X}^a_\lambda, \forall i \geq 0\}$$

Taking into account that the controller given by (3.6) guarantees that (x, u) converges asymptotically to (x_s, u_s) and following similar arguments to (Gilbert and Tan, 1991), it can be shown that for any $\lambda \in (0, 1)$, $\mathcal{O}^a_{\infty,\lambda}$ is finitely determined and $\lambda\mathcal{O}^a_\infty \subset \mathcal{O}^a_{\infty,\lambda} \subset \mathcal{O}^a_\infty$. Notice that because λ can be chosen arbitrarily close to 1, the obtained invariant set can be made arbitrarily close to the real maximal invariant set $\mathcal{O}^a_\infty$.

In what follows, superscript a denotes that set Ω^a_t is defined in the extended state, while no superscript denotes that set Ω_t is defined in the state vector space x, i.e., $\Omega_t = Proj_x(\Omega^a_t)$.

Hereafter $\mathcal{O}_\infty(x_s)$ denotes the maximal invariant set of states that can be steered to x_s in an admissible way by the control law (3.6). It is easy to see that the computed polyhedral set $\mathcal{O}_{\infty,\lambda}$ is such that

$$\mathcal{O}_{\infty,\lambda} = \bigcup_{x_s \in \lambda X_s} \mathcal{O}_\infty(x_s)$$

It is clear that set λX_s is contained in $\mathcal{O}_{\infty,\lambda}$, and the set of set points associated to the invariant set is λS.

3.4 Plant Operation Point to be tracked

The main objective of every controller is to steer and maintain the plant in given a plant operation point. This operating point is typically provided to the controller by an upper level optimizer called *Set-point optimizer*; this calculates the optimal operating point according to economics criteria such as market forecasting, expected prices, etc. and based on a complex model of the plant.

The controller must calculate the control action considering the current state of the plant and the target operating point. Thus, the control law depends on how the target operating point is provided by the set point optimizer. This is usually one of the following forms:

1. Output target y_t, i.e., the desired values of the (controlled) process variables.
2. State target x_s, which is an steady state of the linear model used by the controller.
3. Input target u_t, i.e., the desired values of the manipulable variables.

4. The parameter θ that characterizes the steady state and input of the linear model used by the controller.

5. State target x_t or augmented state target $z_t = (x_t, u_t)$, which are **not** an steady state of the lineal model used by the controller.

In the first case, it is desirable to ensure that every output target is potentially reachable, that is, the output target corresponds to a certain equilibrium point of the linear model. This property is guaranteed if the rank of matrix E in equation (3.3.1) equals to $n+p$. In the cases 2, 3 and 4, there exist a pair of steady state and input of the linear model which match to the operating point of the plant. The most compact way to characterize an equilibrium point for the linear model is the parameter θ; therefore, the operating point provided in this four cases can be described by a value of θ that can be easily calculated as a linear combination of the operation point data (see section 3.3.1). Therefore, the proposed control law will depend on the current state and the parameter θ, that is, $u = \kappa_N(x, \theta)$.

In the fifth case, the operating point is not an equilibrium point of the linear model used by the controller, and hence, this can not be characterized by θ. In this case, the control law must be given by $u = \kappa_N(x, x_t)$ or $u = \kappa_N(x, z_t)$.

It is worth remarking that the admissibility of the operating point to be tracked is not an issue on this stage because, as it will be proved, the proposed predictive controller ensures the admissible trajectory of the system despite the operating point, steering the system to the closest admissible equilibrium point of the linear model.

In the following section, the proposed model predictive controller for the case that the parameter θ characterize the operation point is proposed. In section 3.7, the proposed controller for fifth case is presented.

3.5 MPC for tracking

In this section the proposed MPC scheme for tracking is presented. It is assumed that the target operation point corresponds to an equilibrium point of the linear model and this point is provided by the *set point optimizer level* as a target value θ.

This predictive controller is based on the addition of an artificial steady state and the corresponding input as decision variables, the use of a modified cost function and an extended terminal constraint. The following assumption is considered.

Assumption 3.2

1. *Let* $Q \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{m\times m}$ *be positive definite matrices.*
2. *There exists a constant* $\sigma > 0$ *such that* $\sigma T \geq M_x^T M_x$*, where* $M_x = [I_n, \mathbf{0}_m]M_\theta$*.*
3. *Let* $K \in \mathbb{R}^{m\times n}$ *be a stabilizing control gain such that* $(A + BK)$ *is Hurwitz.*
4. *Let* $P \in \mathbb{R}^{n\times n}$ *be a positive definite matrix such that*

$$(A + BK)^T P(A + BK) - P = -(Q + K^T RK)$$

5. *Let* $\Omega^a_{t,K} \subseteq \mathbb{R}^{n+n_\theta}$ *be an admissible polyhedral invariant set for tracking for system (3.1) subject to (3.2) and a gain controller* K*.*[1]

In order to ensure the feasibility of the problem for any desired steady augmented state $z_s = (x_s, u_s) = M_\theta \theta$, an artificial steady augmented state $\bar{z}_s = (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}$ is introduced as a decision variable in the minimization of the performance index. Moreover, the convergence to the desired steady state is ensured by adding a term $\|\bar{\theta} - \theta\|_T^2$ in the cost function (*offset cost*) that penalizes the deviation between the desired steady state and the artificial one (see that any quadratic term weighting the offset, as $\|\bar{x}_s - x_s\|_H^2$ for instance, can be expressed in this form ($T = M_x' H M_x$)). Thus the proposed cost is

$$\begin{aligned} V_N(x, \theta, \mathbf{u}, \bar{\theta}) \;=\; & \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 \\ & + \|x(N) - \bar{x}_s\|_P^2 + \|\bar{\theta} - \theta\|_T^2 \end{aligned}$$

where $\mathbf{u}$ is a sequence of N future control inputs, i.e., $\mathbf{u} = \{u(0), \ldots, u(N-1)\}$, $x(i)$ is the predicted state of the system at time i given by $x(i+1) = Ax(i) + Bu(i)$, with $x(0) = x$. Note that this cost can be posed as a quadratic function of the decision variables.

[1]Hereafter the dependence of λ is not represented

The proposed MPC optimization problem $\mathcal{P}_N(x,\theta)$ is given by

$$\begin{aligned} V_N^*(x,\theta) \quad = \quad & \min_{\mathbf{u},\bar{\theta}} V_N(x,\theta,\mathbf{u},\bar{\theta}) \\ s.t. \quad & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j),u(j)) \in \mathcal{Z}, \qquad j = 0,\cdots,N-1 \\ & (\bar{x}_s,\bar{u}_s) = M_\theta\bar{\theta}, \\ & (x(N),\bar{\theta}) \in \Omega^a_{t,K}. \end{aligned} \tag{3.9}$$

Vectors $\mathbf{u}$ and $\bar{\theta}$ are the decision variables while x and θ are parameters of the proposed optimization problem $\mathcal{P}_N(x,\theta)$. Moreover, it turns out to be a standard (parametric) quadratic programming problem that can be efficiently solved by specialized algorithms. In the following we denote the optimal solution of the optimization problem by the superscript $*$, that is $\mathbf{u}^*, \bar{\theta}^*$.

The constraints of $\mathcal{P}_N(x,\theta)$ do not depend on θ. This implies that there exists a polyhedral region $\mathcal{X}_N \subset \mathbb{R}^n$ such that $\forall x \in \mathcal{X}_N$, $\mathcal{P}_N(x,\theta)$ is feasible for any $\theta \in \mathbb{R}^{n_\theta}$. The controller is implemented following the receding horizon strategy. At each time step k, $\mathcal{P}_N(x,\theta)$ is solved and $u = \kappa_N(x,\theta) = u^*(0)$ is applied, the rest of the optimal trajectory is discarded. The main result of this chapter is presented in the following theorem:

Theorem 3.1 (Stability) *Consider that assumptions 3.1 and 3.2 hold and* $\Omega^a_{t,K} = \mathcal{O}^a_{\infty,\lambda}$ *for a given* $\lambda \in (0,1)$*. Consider a target operation point given by* θ *such that* $(\mathrm{x}_s,\mathrm{u}_s) = M_\theta\theta \in \lambda\mathcal{Z}_s$*. Then for any feasible initial state* $x_0 \in \mathcal{X}_N$ *the proposed MPC controller* $\kappa_N(x,\theta)$ *asymptotically steers the system to* x_s *fulfilling the constraints all the time.*

Proof of theorem 3.1: It is assumed that the hypothesis 3.2 is satisfied.

The first part of the proof is devoted to prove the feasibility of the controlled system, that is, $x(k+1) \in \mathcal{X}_N$, for all $x(k) \in \mathcal{X}_N$, and θ. Consider the optimal solution of $\mathcal{P}_N(x(k),\theta)$, then the successor state is $x(k+1) = Ax(k) + B\kappa_N(x(k),\theta)$. Define the following sequences:

$$
\begin{aligned}
\mathbf{u}(x(k+1),\theta) &\triangleq [u^*(1;x(k),\theta),\cdots,u^*(N-1;x(k),\theta),\\
&\quad K(x^*(N-1;x(k),\theta)-\bar{x}_s^*(x(k),\theta))+\bar{u}_s^*(x(k),\theta)] \qquad (3.10)\\
\bar{\theta}(x(k+1),\theta) &\triangleq \bar{\theta}^*(x(k),\theta)
\end{aligned}
$$

Then, $(\mathbf{u},\bar{\theta})$ is a feasible solution for the optimization problem $\mathcal{P}_N(x(k+1),\theta)$ due to:

- Since $x(x(k+1),\theta) = x^*(1;x(k),\theta)$, then $x(i;x(k+1),\theta) = x^*(i+1;x(k),\theta)$ $i=0,1,\cdots,N-1$; then the first $N-1$ terms of the trajectory are admissible. Admissibility of $x(N;x(k+1),\theta)$ is derived from the fact that $(x(N-1;x(k+1),\theta),\bar{\theta}(x(k+1),\theta)) \in \Omega_{t,K}^a$ and hence the control action $u(N-1;x(k+1),\theta)$ ensures that $(x(N;x(k+1),\theta),\bar{\theta}(x(k+1),\theta)) \in \Omega_{t,K}^a$.
- Feasibility of $\mathbf{u}^*(x(k),\theta)$ and admissibility of set $\Omega_{t,K}^a$ ensures the feasibility of $\mathbf{u}(x(k+1),\theta)$.
- The terminal constraint satisfaction stems from the same arguments.

Convergence is derived proving that the optimal cost is a Lyapunov function. Consider the proposed feasible solution (3.10). Taking into account the properties of the feasible nominal trajectories for $x(k+1)$, the condition *(iv)* of Assumption 3.2 and using standard procedures in MPC (Mayne et al., 2000) it is possible to obtain

$$
\begin{aligned}
V_N(x(k+1),\theta;\mathbf{u},\bar{\theta}) - V_N^*(x(k),\theta) &\leq -\|x^*(x(k),\theta)-\bar{x}_s^*(x(k),\theta)\|_Q^2\\
&\quad -\|u^*(0;x(k),\theta)-\bar{u}_s^*(x(k),\theta)\|_R^2\\
&\leq -\|x^*(x(k),\theta)-\bar{x}_s^*(x(k),\theta)\|_Q^2 \qquad (3.11)
\end{aligned}
$$

By optimality, we have that $V_N^*(x(k+1),\theta) \leq V_N(x(k+1),\theta;\mathbf{u},\bar{\theta})$ and then:

$$
V_N^*(x(k+1),\theta) - V_N^*(x(k),\theta) \leq -\|x^*(x(k),\theta)-\bar{x}_s^*(x(k),\theta)\|_Q^2
$$

Taking into account that $Q>0$, we have that

$$
\lim_{k\to\infty} \|x^*(x(k),\theta)-\bar{x}_s^*(x(k),\theta)\|_Q^2 = 0
$$

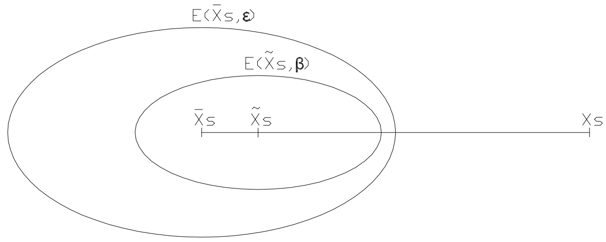

Figure 3.1: Ellipsoid $\mathcal{E}(\bar{x}_s^*, \epsilon)$ and $\mathcal{E}(\tilde{x}_s, \beta)$.

In virtue of lemma 3.2, we can deduce that if $\|x^*(x(k), \theta) - \bar{x}_s^*(x(k), \theta)\|$ tends to 0 then $\|\bar{x}_s^*(x(k), \theta) - x_s\|$ also tends to 0. Therefore, $\bar{x}^*(x(k), \theta)$ tends to x_s and then $x(k)$ tends to x_s. ■

The proof the theorem is based on the result provided in the following lemma.

Lemma 3.2 *Consider that the assumptions of theorem 3.1 hold. Consider a desired steady state* $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta \theta$ *such that* $(\mathrm{x}_s, \theta) \in \Omega_{t,K}^a$ *and assume that for a given state* x *the optimal solution of* $\mathcal{P}_N(x, \theta)$ *is such that* $\|x - \bar{\mathrm{x}}_s^*(x, \theta)\|_Q = 0$ *(i.e.* $x = \bar{\mathrm{x}}_s^*(x, \theta)$*), then* $\|x - \mathrm{x}_s\|_Q = 0$.

Proof: The proof is obtained by contradiction. Let $\bar{\theta}^*$ denote $\bar{\theta}^*(x, \theta)$ and consider $(\bar{\mathrm{x}}_s^*, \bar{\mathrm{u}}_s^*) = M_\theta \bar{\theta}^*$ and assume that $\bar{\mathrm{x}}_s^* \neq \mathrm{x}_s$.

Firstly it is proved that there exists a $\hat{\lambda} \in [0, 1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$, $\tilde{\theta} = \lambda \bar{\theta}^* + (1 - \lambda)\theta$ and $(\tilde{x}_s, \tilde{u}_s) = M_\theta \tilde{\theta}$, the state $\bar{\mathrm{x}}_s^*$ is contained in the maximal admissible invariant set (denoted as $\mathcal{O}_\infty(\tilde{x}_s)$) for the nominal system controlled by $u = K(x - \tilde{x}_s) + \tilde{u}_s$. To this aim, consider that P is a Lyapunov matrix of the closed loop system $x^+ = (A + BK) \cdot x$ and define the ellipsoid $\mathcal{E}(x_0, \tau) = \{x \in \mathbb{R}^n : \|x - x_0\|_P^2 \leq \tau\}$. Given that $(\bar{\mathrm{x}}_s^*, \bar{\mathrm{u}}_s^*) \in \lambda \mathcal{Z}_s \subset int(\mathcal{Z})$, there exists constants $\epsilon > 0$ and $\gamma \in (0, 1)$ such that for all $x \in \mathcal{E}(\bar{\mathrm{x}}_s^*, \epsilon)$, $(x, Kx + L\bar{\theta}^*) \in \gamma \mathcal{Z}$. Take a value of $\lambda \in (0, 1)$ sufficiently large such that $(0, L(\tilde{\theta} - \bar{\theta}^*)) = (0, (1 - \lambda)L(\theta - \bar{\theta}^*)) \in (1 - \gamma)\mathcal{Z}$. Find a $\beta > 0$ such that $\bar{\mathrm{x}}_s^* \in \mathcal{E}(\tilde{\mathrm{x}}_s, \beta) \subset \mathcal{E}(\bar{\mathrm{x}}_s^*, \epsilon)$ (see figure 3.1).

Then, for all $x \in \mathcal{E}(\tilde{\mathrm{x}}_s, \beta)$ we have that $(x, Kx + L\tilde{\theta}) = (x, Kx + L\bar{\theta}^*) + (0, L(\tilde{\theta} - \bar{\theta}^*))$. Given that for all $x \in \mathcal{E}(\tilde{\mathrm{x}}_s, \beta)$, $(x, Kx + L\bar{\theta}^*) \in \gamma \mathcal{Z}$ and $(0, L(\tilde{\theta} - \bar{\theta}^*)) \in (1 - \gamma)\mathcal{Z}$, we have that $(x, Kx + L\tilde{\theta}) \in \mathcal{Z}$. Therefore $\bar{\mathrm{x}}_s^* \in \mathcal{E}(\tilde{\mathrm{x}}_s, \beta) \subset \mathcal{O}_\infty(\tilde{x}_s)$. This proves the claim.

Defining **u** as the sequence of control actions derived from this control law, it is readily inferred that $(\mathbf{u}, \bar{\mathrm{x}}_s^*, \bar{\theta})$ is a feasible solution for $\mathcal{P}_N(\bar{\mathrm{x}}_s^*, \theta)$. Then from Assumption 3.2,

$$\begin{aligned} V_N^*(\bar{x}_s^*, \theta) &\leq V_N(\bar{x}_s^*, \theta; \mathbf{u}, , \tilde{\theta}) \\ &= \sum_{i=0}^{N-1} \overbrace{\|x(i) - \tilde{x}_s\|_Q^2 + \|K(x(i) - \tilde{x}_s)\|_R^2}^{\|x(i)-\tilde{x}_s\|^2_{(Q+K^TRK)}} + \|x(N) - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \\ &= \|\bar{\mathrm{x}}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \end{aligned}$$

Let M_x and σ be defined in Assumption 3.2. Since $\bar{\mathrm{x}}_s^* - \tilde{x}_s = (1-\lambda)M_x(\bar{\theta}^* - \theta)$ and $\tilde{\theta} - \theta = \lambda(\bar{\theta}^* - \theta)$ we can rewrite

$$\|\bar{\mathrm{x}}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 = \|\bar{\theta}^* - \theta\|_H$$

where $H = (1-\lambda)^2 M_x^T P M_x + \lambda^2 T$. Choosing a constant $\beta \geq \hat{\lambda}(1-\hat{\lambda})^{-1}$ such that the maximum eigenvalue of P is lower than β/σ, we have that $H \leq (\beta(1-\lambda)^2 + \lambda^2)T$. Taking $\lambda = \beta(1+\beta)^{-1}$, then $\lambda \geq \hat{\lambda}$ and $H \leq \lambda T < T$. Therefore, we have that $V_N^*(\bar{\mathrm{x}}_s^*, \theta) < \|\bar{\theta}^* - \theta\|_T^2$.

Given that the optimal solution of $\mathcal{P}_N(\bar{x}_s^*, \theta)$ is given by $\mathbf{u}^*(\bar{x}_s^*, \theta) = \{u_s^*, \cdots, u_s^*\}$ and the associated nominal state sequence is $\mathbf{x}^*(\bar{x}_s^*, \theta) = \{\bar{x}_s^*, \cdots, \bar{x}_s^*\}$, then the optimal cost is $V_N^*(\bar{x}_s^*, \theta) = \|\bar{\theta}^* - \theta\|_T^2$ yielding a contradiction and proving the lemma. ■

3.6 Properties of the proposed controller

In the previous section, a novel model predictive controller aimed to steer the system to a given operation point is presented. This predictive controller is derived from the solution of a quadratic programming problem as the standard MPC for regulation. However, the proposed formulation gives the controller some properties that make it very interesting and allows to be exploited in the design and implementation stages.

The first property shows that the controller can be used to track changing operation points or even set point trajectories maintaining the recursive feasibility and admissibility. Since this property holds for any value of the prediction horizon N, it can be derived that the proposed controller is able to track any admissible set point even for a prediction horizon $N = 1$, if the system starts from an equilibrium point.

Property 3.1 (Changing operation points) *: Consider any $x(0) \in \mathcal{X}_N$ and consider that the operating point is given by a sequence $\theta(k)$, then $x(k) \in \mathcal{X}_N$ for all $k \geq 0$.*

Proof: The recursive feasibility property ensures that if $x(0) \in \mathcal{X}_N$, then $x(k) \in \mathcal{X}_N$. This has been proved in theorem 3.1 for a given value of θ. Since the set of constraints of $\mathcal{P}_N(x, \theta)$ does not depend on θ, the optimization problem is feasible for any value of θ, and consequently, this property can be extended to a sequence of values of θ. ■

The second property states that for a given prediction horizon, the proposed controller provides a larger region of attraction than the one of the MPC for regulation. This remarkable property allows to extend the controllability of the predictive controller to a larger region at expense of n_θ additional decision variables. This increment of computational cost is similar to the derived from incrementing the prediction horizon by 1. this makes the proposed controller interesting even for regulation objectives. On the other hand, for a given region of initial states, the necessary prediction horizon to control the system is potentially smaller, which implies a lower computational cost.

Property 3.2 (Larger domain of attraction) *: Consider an operation point $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta \theta \in \lambda \mathcal{Z}_s$. Consider a MPC for tracking and an MPC for regulation with the same stabilizing ingredients to steer the system to x_s. Then the domain of attraction of the MPC for tracking is generally larger than the region of attraction of the MPC for regulation.*

Proof: Since $(\mathrm{x}_s, \mathrm{u}_s) \in \lambda \mathcal{Z}_s$, there exist the maximal admissible invariant set of the linear model controlled by $u = K(x - \mathrm{x}_s) + u_s$, which is denoted as $\mathcal{O}_\infty(x_s)$. On the other hand, the invariant set for tracking ensures that

$$\mathcal{O}_{\infty,\lambda} = \bigcup_{\mathrm{x}_s \in \lambda X_s} \mathcal{O}_\infty(\mathrm{x}_s)$$

and hence $\mathcal{O}_\infty(\mathrm{x}_s) \subset \mathcal{O}_{\infty,\lambda}$. Consequently, in virtue of the monotonicity of the size of feasibility set w.r.t the size of the terminal set (Limon, 2002), the domain of attraction of the MPC for tracking is larger. ■

The following property is probably the most remarkable one. This deals with the case when the operation point to be tracked is not admissible. In this case, the operation point can not be reached by the constrained system, and hence, the closed loop system will exhibit offset. This property demonstrates that in this case, the controller steers the system to an admissible equilibrium point such that the distance to the operating point is minimal. Moreover this distance measure is given by the offset cost function, which allows to choose this cost according to a given criteria. An illustrative example can be found in the subsection 3.8.2

It is worth remarking that this important property makes that a higher level setpoint optimizer, aimed to calculate the best admissible equilibrium point corresponding to a given operating point, is not necessary, since this feature is implicitly incorporated in the MPC for tracking.

Property 3.3 (Offset minimization) *:*

Consider an operation point given by θ such that $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta\theta \notin \lambda\mathcal{Z}_s$. Then for any $x(0) \in \mathcal{X}_N$, the system state fulfils $x(k) \in \mathcal{X}_N$ and converges to an equilibrium point $(\mathrm{x}_s^*, \mathrm{u}_s^*) = M_\theta\theta^* \in \lambda\mathcal{Z}_s$ such that

$$\theta^* = arg \quad \begin{array}{l} \min\limits_{\bar{\theta}} \|\bar{\theta} - \theta\|_T^2 \\ M_\theta\bar{\theta} \in \lambda\mathcal{Z}_s \end{array}$$

that is, the admissible equilibrium point that minimizes the offset cost function.

Proof: This is proved by contradiction. Consider that $x(k) = \mathrm{x}_s^*$, then $V_N^*(\mathrm{x}_s^*, \theta) = \|\theta^* - \theta\|_T^2$.
Assume that θ^* is not the one which minimizes the offset cost function. Then there exist a $\tilde{\theta} \neq \theta^*$ such that

$$\|\tilde{\theta} - \theta\|_T^2 < \|\theta^* - \theta\|_T^2$$

In virtue of lemma 3.3, there exist a $\bar{\theta}$ such that the sequence of control actions derived from the local control law $u = Kx + L\bar{\theta}$ and the parameter $\bar{\theta}$ is a feasible solution of $\mathcal{P}_N(\mathrm{x}_s^*, \theta)$, and the corresponding suboptimal cost function

$$\bar{V}_N(\mathrm{x}_s^*, \theta) = \|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta} - \theta\|_T^2$$

fulfils that

$$\bar{V}_N(\mathrm{x}_s^*, \theta) < V_N^*(\mathrm{x}_s^*, \theta)$$

This contradicts the optimality assumption of θ^*, and then the property is proved. ■

Lemma 3.3 *Let* $(\mathrm{x}_s^*, \mathrm{u}_s^*) = M_\theta \theta^*$ *be such that* $V_N^*(\mathrm{x}_s^*, \theta) = \|\theta^* - \theta\|_T^2$. *Let* $\tilde{\theta} \neq \theta^*$ *be such that* $\|\tilde{\theta} - \theta\|_T^2 < \|\theta^* - \theta\|_T^2$. *Consider* $\bar{\theta}$ *given by* $\bar{\theta} = \lambda\theta^* + (1-\lambda)\tilde{\theta}$.
Then, there exist a $\bar{\lambda}$ *such that for all* $\lambda \in [\bar{\lambda}, 1)$, *for the system controlled by the local control law* $u = Kx + L\bar{\theta}$, $\mathrm{x}_s^* \in \mathcal{O}_\infty(\bar{\mathrm{x}}_s)$ *and*

$$\|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta} - \theta\|_T^2 < \|\theta^* - \theta\|_T^2$$

Proof: Firstly, the following preliminary results are stated:

- For any θ and for all $\gamma > 1, \ \gamma \neq \gamma(\lambda)$,

$$\|\theta^* - \bar{\theta}\|_T^2 \leq \gamma \|\theta^* - \theta\|_T^2 + \frac{\gamma}{\gamma - 1}\|\bar{\theta} - \theta\|_T^2$$

- For a given $\tilde{\theta}$, there exist a $\alpha \in (0,1), \ \alpha \neq \alpha(\lambda)$ such that $\|\tilde{\theta} - \theta\|_T^2 = \alpha^2 \|\theta^* - \theta\|_T^2$
- For a given $\bar{\theta} = \lambda\theta^* + (1-\lambda)\tilde{\theta}$ and for every θ we have that

$$(\bar{\theta} - \theta) = \lambda(\theta^* - \theta) + (1-\lambda)(\tilde{\theta} - \theta)$$

  Hence, in virtue of the convexity of the norms, the following inequalities hold

$$\begin{aligned}
\|\bar{\theta} - \theta\|_T &\leq \lambda\|\theta^* - \theta\|_T + (1-\lambda)\|\tilde{\theta} - \theta\|_T \\
&= (\lambda + (1-\lambda)\alpha)\|\theta^* - \theta\|_T \\
&= (\alpha + (1-\alpha)\lambda)\|\theta^* - \theta\|_T
\end{aligned}$$

- In virtue of lemma 3.2, there exist a $\hat{\lambda} \in (0,1)$ such that for all $\lambda \in [\hat{\lambda}, 1)$,

$$\|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta} - \tilde{\theta}\|_T^2 \leq (\beta(1-\lambda)^2 + \lambda^2)\|\theta^* - \tilde{\theta}\|_T^2 < \|\theta^* - \tilde{\theta}\|_T^2$$

From these results, we derive that

$$\begin{aligned}
\|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta} - \theta\|_T^2 &= \|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta} - \tilde{\theta}\|_T^2 - \|\bar{\theta} - \tilde{\theta}\|_T^2 + \|\bar{\theta} - \theta\|_T^2 \\
&\leq (\beta(1-\lambda)^2 + \lambda^2)\|\theta^* - \tilde{\theta}\|_T^2 - \|\bar{\theta} - \tilde{\theta}\|_T^2 + \|\bar{\theta} - \theta\|_T^2 \\
&= (\beta(1-\lambda)^2)\|\theta^* - \tilde{\theta}\|_T^2 + \|\bar{\theta} - \theta\|_T^2 \\
&\leq (\beta(1-\lambda)^2)(\gamma\|\theta^* - \theta\|_T^2 + \frac{\gamma}{\gamma-1}\|\bar{\theta} - \theta\|_T^2) + \|\bar{\theta} - \theta\|_T^2 \\
&\leq \Big(\beta(1-\lambda)^2(\gamma + \frac{\gamma}{\gamma-1}\alpha^2) + (\alpha + (1-\alpha)\lambda)^2\Big)\|\theta^* - \theta\|_T^2
\end{aligned}$$

Taking $\hat{\beta} = \beta(\gamma + \frac{\gamma}{\gamma-1}\alpha^2)$, the lemma is proved if there exist a $\lambda \in [\hat{\lambda}, 1)$ such that $1-(\alpha+(1-\alpha)\lambda)^2 - \hat{\beta}(1-\lambda)^2 > 0$. Considering that

$$\begin{aligned}
1-(\alpha+(1-\alpha)\lambda)^2 - \hat{\beta}(1-\lambda)^2 &= (1-\alpha-(1-\alpha)\lambda)(1+\alpha+(1-\alpha)\lambda) - \hat{\beta}(1-\lambda)^2 \\
&= (1-\alpha)(1-\lambda)(1+\alpha+(1-\alpha)\lambda) - \hat{\beta}(1-\lambda)^2 \\
&= (1-\lambda)\Big((1-\alpha)(1+\alpha+(1-\alpha)\lambda) - \hat{\beta}(1-\lambda)\Big) \\
&= (1-\lambda)\Big((1-\alpha^2-\hat{\beta}+((1-\alpha)^2+\hat{\beta})\lambda)\Big)
\end{aligned}$$

it is derived that this condition is fulfilled for any

$$\lambda > \frac{\hat{\beta}-(1-\alpha^2)}{\hat{\beta}+(1-\alpha)^2} = \eta$$

Since $\alpha \in (0,1)$ there exist $1 \geq \lambda \geq \max\{\hat{\lambda}, \eta\}$ that satisfies the required conditions.

■

Model predictive controllers can be considered as suboptimal controllers since the cost function is only minimized for a finite prediction horizon. If an infinite prediction horizon is chosen, then the cost of the closed-loop system trajectory is the minimal. This desirable controller can be calculated for unconstrained linear systems, but when the constraints are present, the calculation of the control law is quite more involved. Fortunately, it has been proved that if the terminal cost function is the optimal cost of the unconstrained LQR, then the resulting finite horizon MPC equals to the constrained LQR in a neighborhood around the terminal region (Hu and Linnemann, 2002).
The proposed controller for tracking does not fulfil this property due to the artificial steady state and input and the chosen offset cost function. The following property states that the optimality gap can be made arbitrarily small by simply weighting the offset cost function.

Property 3.4 (Local optimality) *: Let $\kappa_N^r(x,\theta)$ be the Model predictive controller for regulation calculated for a steady state and input target $(x_s, \mathrm{u}_s) = M_\theta\theta$, let $V_N^r(x,\theta)$ be the corresponding optimal cost function and let $\mathcal{X}_N^r$ be its domain of attraction. Then for a given $\epsilon > 0$, there exist a $T > 0$ such that $|V_N^r(x,\theta) - V_N^*(x,\theta)| \leq \epsilon$ for all $x \in \mathcal{X}_N^r$. Besides, let $\Upsilon \subset \mathbb{R}^n$ be the region where $\kappa_N^r(x,\theta) = K_\infty^r(x,\theta)$ (i.e. $V_N^r(x,\theta) = V_\infty^r(x,\theta)$), then for any $\epsilon > 0$, there exist a $T > 0$ such that $|V_\infty^r(x,\theta) - V_N^*(x,\theta)| \leq \epsilon$ for all $x \in \Upsilon$.*

Proof: Assume that conditions of theorem 3.1 holds and define the problem $\mathcal{P}_N(x, \theta; \lambda)$ as

$$\begin{aligned} V_N^*(x,\theta;\lambda) \quad = \quad & \min_{\mathbf{u},\bar{\theta}} \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 + \|x(N) - \bar{x}_s\|_P^2 + \lambda\|\bar{\theta} - \theta\|_T^2 \\ s.t. \quad & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \qquad j = 0, \cdots, N-1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^a. \end{aligned}$$

On the other hand, the optimization problem of the MPC for regulation can be posed as

$$\begin{aligned} V_N^r(x,\theta) \quad = \quad & \min_{\mathbf{u},\bar{\theta}} \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 + \|x(N) - \bar{x}_s\|_P^2 \\ s.t. \quad & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \qquad j = 0, \cdots, N-1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^a. \\ & \|\bar{\theta} - \theta\|_T^2 = 0 \end{aligned}$$

The optimization problem $\mathcal{P}_N(x, \theta; \lambda)$ can be seen as the optimization problem $P_N^r(x, \theta)$, where the last constraint is posed as penalty function (Luenberger, 1989). Define an increasing sequence $\{\lambda_k\}$ such that $\lambda_{k+1} > \lambda_k$ and $\lim_{k\to\infty} \lambda_k = \infty$. Then, in virtue of the properties of penalty functions (Luenberger, 1989), the following property holds

$$\lim_{k\to\infty} V_N^*(x, \theta, \lambda_k) = V_N^r(x, \theta)$$

From the properties of the limits, we have that for a given $\epsilon > 0$ there exist a $k^*(\epsilon)$ such that for all $k \geq k^*(\epsilon)$,

$$V_N^r(x, \theta) - V_N^*(x, \theta, \lambda_k) \leq \epsilon$$

Then, first claim of the property is proved by considering the proposed MPC for tracking with the offset cost function $\|\bar{\theta} - \theta\|_{\lambda_k T}^2$, for any $k \geq k^*(\epsilon)$.

The second claim is derived from the local optimality property of the MPC for regulation (Hu and Linnemann, 2002). ■

In order to apply the proposed controller on a system, the optimal problem $\mathcal{P}_N(x,\theta)$ has to solved at each sampling time. This optimization problem results to be a quadratic problem that can be solved in polynomial time w.r.t the number of decision variables. The number of decision variables in the MPC optimization problems is affine in the prediction horizon N, and therefore the computational load grows with N.

Since the proposed controller provides a bigger region of attraction than the one of a MPC for regulation (as stated in property 3.2), the required prediction horizon to stabilize a given set of initial states is smaller, and hence also the computational burden. Moreover, if the initial state of the system is an admissible steady state, the proposed MPC for tracking is capable to control the system to any admissible steady state for any prediction horizon $N \geq 1$, and hence the number of decision variables could be $m + n_\theta$. This would yield to significant reduction of the computational cost.

In case that the sampling time requires a quicker solution of the optimization problem, the following property could be used. It is proved that optimization problem $\mathcal{P}_N(x,\theta)$ is a multiparametric quadratic optimization problem and hence the control law $\kappa_N(x,\theta)$ is a piecewise affine function of (x,θ) that can be computed off-line (Bemporad, Morari, Dua and Pistikopoulos, 2002). The piecewise affine function is defined in a set of critical regions which are a partition of the feasibility region.

This allows to reduce dramatically the computational cost of the on-line implementation of the control law, since it suffices to find the active critical region and apply the corresponding affine function. Moreover, the finding procedure can be efficiently implemented by means of a search-tree strategy (Tøndel et al., 2002).

Property 3.5 (Explicit solution) *: The control law $\kappa_N(x,\theta)$ derived from the optimization problem $\mathcal{P}_N(x,\theta)$ results to be a piecewise affine function of the pair (x,θ), that is, it can be described as*

$$u = K_x(j)x + K_\theta(j)\theta + c(j), \quad (x,\theta) \in \Gamma_j$$

where the set of n_r regions Γ_j is a partition of $\mathcal{X}_N \times \mathbb{R}^{n_\theta}$, i.e. $\Gamma_j \cap \Gamma_i = \emptyset$ for any $j \neq i$ and $\bigcup_{j=1}^{nr} \Gamma_j = \mathcal{X}_N \times \mathbb{R}^{n_\theta}$.

Proof: A multiparametric quadratic programming problem is given by (Bemporad, Morari, Dua and Pistikopoulos, 2002)

$$\begin{array}{ll} \min\limits_{v} & \frac{1}{2}v^T H v + v^T F \mu \\ s.t. & G_v v + G_\mu \mu \leq g \end{array}$$

where v is the vector of decision variables and μ the parameters of the optimization problem. It is assumed that H is semidefinite positive. In (Bemporad, Morari, Dua and Pistikopoulos, 2002) it is demonstrated that the solution of this class of problems can be calculated as a function of the parameters. This solution results to be a piece-wise function of the parameter defined over a partition of the feasible region.

The optimization problem $\mathcal{P}_N(x, \theta)$ defined in (3.9) is a quadratic programming depending on the parameters (x, θ) with $(\mathbf{u}, \bar{\theta})$ as decision variables. The constraints of $\mathcal{P}_N(x, \theta)$ are linear and do not depend on θ; hence the feasibility region is given by $\mathcal{X}_N \times \mathbb{R}^{n_\theta}$.
On the other hand, considering that the predicted states are a linear function of $\mathbf{u}$ and x, the artificial steady state and input are given by $(\bar{\mathrm{x}}_s, \bar{\mathrm{u}}_s) = M_\theta \bar{\theta}$ and the offset cost function is a quadratic function of $\bar{\theta}$ and the parameter θ, the cost function $V_N(x, \theta, \mathbf{u}, \bar{\theta})$ can be casted as a quadratic function of the decision variables that depends linearly on the parameters.

Therefore, the optimization problem $\mathcal{P}_N(x, \theta)$ is a parametric quadratic programming problem, and hence its solution is a piecewise affine function of (x, θ). ■

In the chapter 7 of this thesis, this property has been tested on a real positioning plant based on a linear motor. This system has been controlled with a sampling time of 10 ms by means of the calculation of the explicit control law and its implementation by means of a search-tree.

3.7 MPC for tracking of operating points inconsistent with the prediction model

In this section, the proposed predictive controller for tracking is extended to the case that the provided target operation point x_t or z_t are not consistent with the linear model considered for predictions. In this case, the operation point may be not an equilibrium point of the linear model, and hence this can not be expressed by a value of the parameter θ. Therefore, the proposed control law must calculate the control action from the current state and the target operation point x_t or z_t, i.e., $u = \kappa_N(x, x_t)$ or $u = \kappa_N(x, z_t)$. Moreover, the proposed offset cost function and the stability and convergence proof presented in the previous section can not be applied in this case, since it is based on the fact that the target operation point can be expressed by θ.

In this section, the following assumption is considered

Assumption 3.3

1. *Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ be positive definite matrices.*
2. *Let $H \in \mathbb{R}^{(n+m)\times(n+m)}$ a semidefinite positive matrix such that there exists a constant $\sigma > 0$ fulfilling*
$$\sigma H \geq \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$
3. *Let $K \in \mathbb{R}^{m\times n}$ be a stabilizing control gain such that $(A + BK)$ is Hurwitz.*
4. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix such that*
$$(A + BK)^T P(A + BK) - P = -(Q + K^T RK)$$
5. *Let $\Omega^a_{t,K} \subseteq \mathbb{R}^{n+n_\theta}$ be an admissible polyhedral invariant set for tracking for system (3.1) subject to (3.2) and a gain controller K.*

To cope with the proposed tracking problem, the following cost function with a modified offset cost function is considered

$$\begin{aligned} V_N(x, z_t, \mathbf{u}, \bar{\theta}) &= \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|^2_Q + \|u(i) - \bar{u}_s\|^2_R \\ &\quad + \|x(N) - \bar{x}_s\|^2_P + \|\bar{z}_s - z_t\|^2_H \end{aligned}$$

where $\bar{z}_s = (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}$.

Note that $\|\bar{z}_s - z_t\|_H^2 = \|\bar{x}_s - x_t\|_{Hx}^2$ where

$$H = \begin{bmatrix} Hx & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

being equivalent the formulation of the setpoint in terms of the augmented state z_t and in terms of the state x_t. [2]

Considering this cost function, the proposed MPC optimization problem $\mathcal{P}_N(x, z_t)$ is given by

$$\begin{aligned} V_N^*(x, z_t) \quad = \quad & \min_{\mathbf{u},\bar{\theta}} V_N(x, z_t, \mathbf{u}, \bar{\theta}) \\ s.t. \quad & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \qquad j = 0, \cdots, N-1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^a \end{aligned}$$

The resulting control law is given by $\kappa_N(x, z_t) = u^*(0; x, z_t)$. Notice that this optimization problem only differs from the one presented in the previous section in the cost function to minimize. Then the region of states where the optimization problem is feasible are the same in both problems and it is denoted as $\mathcal{X}_N$.

Theorem 3.2 (Stability) *Consider that assumptions 3.1 and 3.3 hold and* $\Omega_{t,K}^a = \mathcal{O}_{\infty,\lambda}^a$ *for a given* $\lambda \in (0, 1)$. *Consider a target operation point given by* $z_t = (x_t, u_t)$. *Then for any feasible initial state* $x_0 \in \mathcal{X}_N$ *the proposed MPC controller* $\kappa_N(x, z_t)$ *asymptotically steers the system to the equilibrium point* $\mathrm{z}_s^* = (\mathrm{x}_s^*, \mathrm{u}_s^*) \in \lambda\mathcal{Z}_s$ *fulfilling the constraints all the time. Moreover, this equilibrium point* z_s^* *is such that*

$$\mathrm{z}_s^* = \min_{z_s \in \lambda \mathcal{Z}_s} \|z_s - z_t\|_H^2$$

Proof: This proof is similar to the the proof of theorem 3.1. Firstly it can be seen that

$$V_N^*(x(k+1), \theta) - V_N^*(x(k), \theta) \leq -\|x^*(x(k), \theta) - \bar{x}_s^*(x(k), \theta)\|_Q^2$$

[2] Hereafter only the formulation in terms of the augmented state will be considered

and hence

$$\lim_{k\to\infty} \|x^*(x(k),\theta) - \bar{x}_s^*(x(k),\theta)\|_Q^2 = 0$$

and hence the system evolves to an admissible steady state $\bar{x}_s^* \in \lambda\mathcal{Z}_s$.

In order to prove that this steady state $\bar{x}_s^*$ is the minimizer of the offset cost function, we proceed by contradiction.
Consider that $x(k) = \bar{x}_s^*$, then $V_N^*(\bar{x}_s^*, \theta) = \|\bar{z}_s^* - z_t\|_H^2$, where $\bar{z}_s^* = (\bar{x}_s^*, \bar{u}_s^*)$.
Assume that $\bar{z}_s^*$ is not the one which minimizes the offset cost function. Then there exist a $\tilde{z}_s \neq z_s^*$ such that

$$\|\tilde{z}_s - z_t\|_H^2 < \|\bar{z}_s^* - z_t\|_H^2$$

The following statements must be considered

- There exist a $\alpha < 1$ such that $\|\tilde{z}_s - z_t\|_H^2 = \alpha^2 \|\bar{z}_s^* - z_t\|_H^2$.
- Taking $\hat{z}_s = \lambda\bar{z}_s^* + (1-\lambda)\tilde{z}_s$ for any $\lambda \in [0,1]$, we have that

  $$(\hat{z}_s - z_t) = \lambda(\bar{z}_s^* - z_t) + (1-\lambda)(\tilde{z}_s - z_t)$$

  and hence, from convexity of the norms, we have that

  $$\|\hat{z}_s - z_t\|_H^2 \leq (\lambda + (1-\lambda)\alpha)^2 \|\bar{z}_s^* - z_t\|_H^2$$

- Following similar arguments to lemma 3.2, and thanks to the existence of a constant $\sigma > 0$ fulfilling

  $$\sigma H \geq \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

  it can be proved that there exist a pair of constants $\hat{\lambda}$ and $\hat{\beta}$ such that for all $\lambda \in [\hat{\lambda}, 1)$ the steady state and input $\hat{z}_s = (\hat{x}_s, \hat{u}_s) = M_\theta\hat{\theta}$ is such that the sequence of control actions derived from the control law $u = Kx + L\hat{\theta}$ and $\hat{\theta}$ compound a feasible solution to $\mathcal{P}_N(\bar{x}_s^*, \tilde{z}_s)$ which suboptimal cost is denoted as $\hat{V}_N(\bar{x}_s^*, \tilde{z}_s)$. Since the feasibility region of $\mathcal{P}_N(\bar{x}_s^*, z_t)$ equals to the one of $\mathcal{P}_N(\bar{x}_s^*, \tilde{z}_s)$, the latter solution is also feasible for $\mathcal{P}_N(\bar{x}_s^*, z_t)$ and the suboptimal cost is denoted as $\hat{V}_N(\bar{x}_s^*, z_t)$.
  Moreover, in an analogous way to the lemma 3.2, it can be proved that for any $\beta \geq \hat{\beta}$, the following condition holds

  $$\hat{V}_N(\bar{x}_s^*, \tilde{z}_s) = \|\bar{x}_s^* - \hat{x}_s\|_P^2 + \|\hat{z}_s - \tilde{z}_s\|_H^2 \leq (\beta(1-\lambda)^2 + \lambda^2)\|\bar{z}_s^* - \tilde{z}_s\|_H^2$$

Considering these statements and taking into account the proof of lemma 3.3, we have that

$$\begin{aligned}
\hat{V}_N(\bar{x}_s^*, z_t) &= \|\bar{x}_s^* - \hat{x}_s\|_P^2 + \|\hat{z}_s - z_t\|_H^2 \\
&= \|\bar{x}_s^* - \hat{x}_s\|_P^2 + \|\hat{z}_s - \tilde{z}_s\|_H^2 - \|\hat{z}_s - \tilde{z}_s\|_H^2 + \|\hat{z}_s - z_t\|_H^2 \\
&\leq (\beta(1-\lambda)^2 + \lambda^2)\|\bar{z}_s^* - \tilde{z}_s\|_H^2 - \lambda^2\|\bar{z}_s^* - \tilde{z}_s\|_H^2 + \|\hat{z}_s - z_t\|_H^2 \\
&\leq \left(\beta(1-\lambda)^2\left(\gamma + \frac{\gamma}{\gamma-1}\alpha^2\right) + (\alpha + (1-\alpha)\lambda)^2\right)\|\bar{z}_s^* - z_t\|_H^2 \\
&\leq f(\lambda)\|\bar{z}_s^* - z_t\|_H^2
\end{aligned}$$

From lemma 3.3 we have that there exist a value of λ such that $f(\lambda) < 1$ and hence

$$\hat{V}_N(\bar{x}_s^*, z_t) < \|\bar{z}_s^* - z_t\|_H^2 = V_N^*(\bar{x}_s^*, z_t)$$

which contradicts the fact that $\bar{z}_s^*$ is not the minimizer of the offset cost function, proving the last statement of the theorem. ■

Remmark 3.1 (Properties of the controller) *The proposed controller for general targets fulfils the property 3.1, ensuring the admissible evolution under changing operating point, the property 3.2 which ensures that the domain of attraction of the proposed controller is larger than the one of the MPC for regulation, and the property 3.5 which states that the proposed control law is a piecewise affine function of (x, z_t).*

An example of that can be found on section 3.8.2.

3.8 Examples

3.8.1 Example 1: The double integrator

Consider a LTI system defined by the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The state and input of the system is constrained in $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 0.3$ respectively. The steady state and the corresponding input and output are characterized by the matrices

$$M_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad N_\theta = \begin{bmatrix} 0 & 0.4082 & 0.4082 & -0.8165 \end{bmatrix}^T.$$

The weighting matrices have been chosen as $Q = I_2$ and $R = I_2$. The local controller gain and the Lyapunov matrix P have been computed solving an LQR problem, obtaining the following matrices:

$$K = \begin{bmatrix} -0.0226 & -0.0610 \\ -0.0768 & -0.4387 \end{bmatrix} \quad P = \begin{bmatrix} 0.0552 & 0.1220 \\ 0.1220 & 0.5707 \end{bmatrix}$$

The maximal invariant set for tracking $\Omega_{t,K}$ has been computed taking $\lambda = 0.95$ and it is shown in figure 3.2.

The system is controlled by the proposed MPC with a prediction horizon $N = 3$. This controller is able to steer the system to any constant reference in $\lambda S = \{s \in \mathbb{R} : s \in [-4.75, 4.75]\}$ for all initial states contained in X_3 (see figure 3.2).

To illustrate the proposed controller, a piece-wise constant sequence of references has been tracked. The first value of the set point is $s_1 = 4.75$, the second $s_2 = -4.75$ and the third $s_3 = 0$. The evolution of the state is shown in figure 3.2. The state trajectory corresponding to the set point s_1 is depicted with circles, the one corresponding to the setpoint s_2 is depicted with dots and with squares the corresponding to the third setpoint s_3. Figure 3.3 shows the evolution of the output and inputs. As it can be seen, the system evolution is admissible for all times even for drastic changes of the set point. The controller steers the system to the desired set points. In this figure the artificial reference is drawn in dashed line. Note that, in the first samples the artificial reference takes values far away from the desired reference in order to ensure an admissible evolution (see that the control inputs are saturated).

The weighting matrix of the offset T, is a free design knob of the proposed controller that affects the closed-loop performance of the system in a significant manner. This matrix allows one to penalize the deviation between the desired steady state and the artificial one; hence if T penalizes more heavily this deviation, then the artificial steady state evolves more quickly to the desired steady state and the closed-loop system is

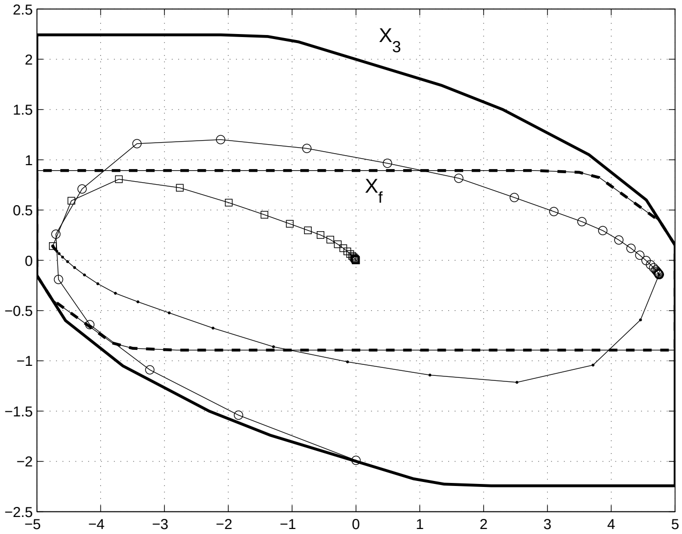

Figure 3.2: Domain of attraction and state portrait of the closed-loop system.

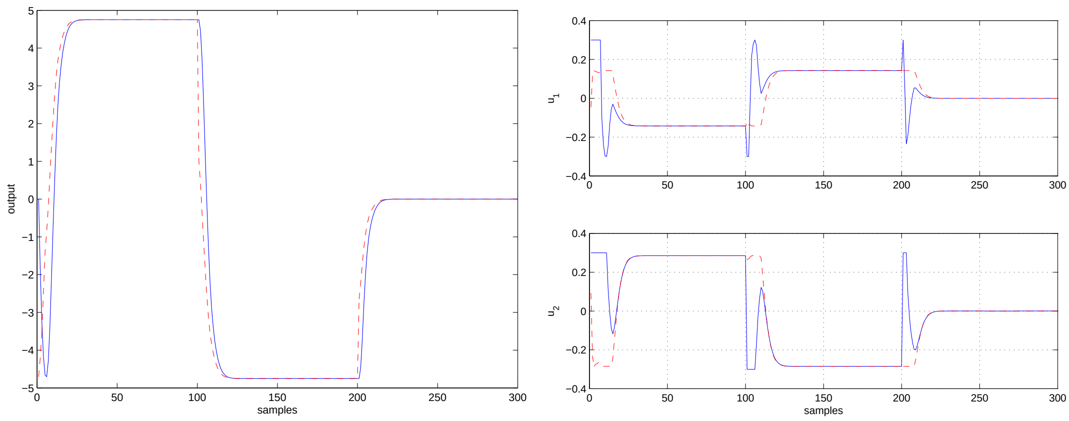

Figure 3.3: Evolution of the output and inputs of the system

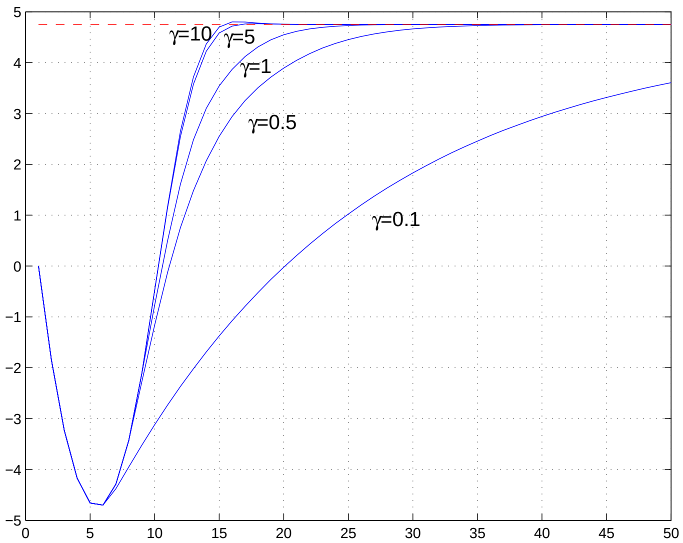

Figure 3.4: Evolution of the output changing the weighting matrix $T = \gamma P$

faster. In order to show this point, several simulations have been executed for the first step of figure 3.3, choosing $T = \gamma P$ with $\gamma \in \{0.1, 0.5, 1, 5, 10\}$. The closed-loop evolution of the output is shown in figure 3.4, where the aforementioned property is clearly illustrated. See that the evolution of the system is the same during the first samples; this is because all the controllers have to saturate the input to guarantee the admissibility of the state trajectories.

To compare the proposed MPC for tracking with the standard MPC to regulate the system at the origin, two aspects are shown. First, the maximal admissible invariant set $\mathcal{O}_\infty(0)$ for the local control law $u = Kx$ and the domain of attraction of the MPC for regulation using $\mathcal{O}_\infty(0)$ as terminal set have been computed and shown in figure 3.5 together with these sets computed for the MPC for tracking. As it can be seen, MPC for tracking provides a significant enlargement of the domain of attraction.

The second aspect analyzed is the local optimality of the proposed controller. As it was previously mentioned, the local optimality property of MPC controllers may be lost when the steady state is used as decision variables. However, the degree of optimality of the controller depends on the weighting matrix T. To demonstrate this point, the optimal cost has been computed at $x_0 = [2, -1]^T \in X_3(\mathcal{O}_\infty(0))$ which has been chosen because the optimal cost of MPC is infinite-horizon optimal (i.e., the MPC is the constrained LQR). In figure 3.6, the evolution of the infinite-horizon optimal cost (thick line) has been compared with the optimal cost of the MPC for tracking with $N = 3$ and $T = \gamma P$, for $\gamma = \{1, 5, 10, 100\}$. As it can be seen, the optimality degree of

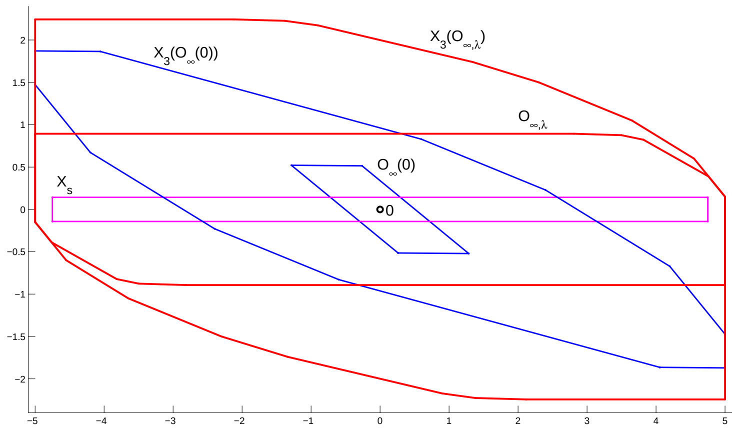

Figure 3.5: Comparison of the domain of attraction between MPC for tracking and regulation to the origin.

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the controller can be done arbitrarily high by increasing the weighting matrix T.

3.8.2 Example 2: A non-square process: Two Cascaded Tanks

A remarkable property of the controller is its natural capability to control non-square systems. This kind of processes may present offset if the setpoints are not appropriately chosen.

The objective of this example is double, first, is to illustrate that the proposed controller can deal with non-square system, the second aim, is to show the role of the offset cost function in the priorization of the outputs in the case of setpoint with offset. (property 3.3)

A scheme of the system is presented in the figure 3.7, the input is the flow and the outputs are the both levels of the tanks. The nonlinear model of the system is:

$$\begin{aligned} \frac{dh_1}{dt} &= -\frac{a_1}{A}\cdot\sqrt{2gh_1} + \frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{\gamma}{A}\cdot q \\ \frac{dh_3}{dt} &= -\frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{1-\gamma}{A}\cdot q \end{aligned} \tag{3.12}$$

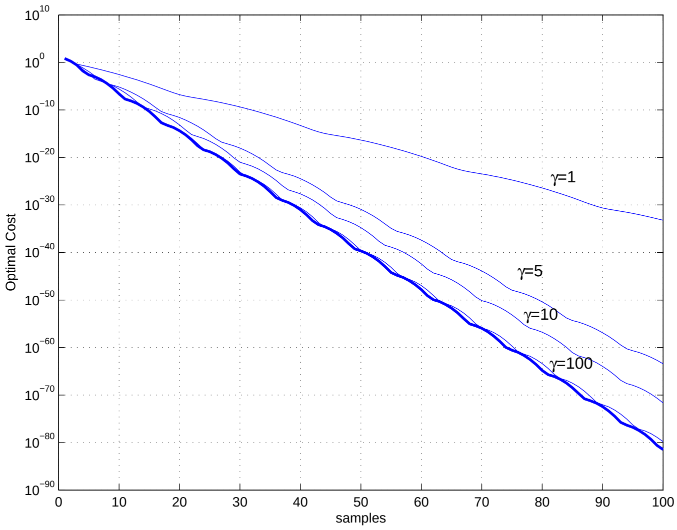

Figure 3.6: Evolution of the optimal cost for tracking versus the constrained LQR.

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where:

- A: Cross-section of tanks.
- a_i: Cross-section of the outlet hole of the tank i, $i = 1 \ldots 4$.
- h_i: Water level of the tank i, $i = 1 \ldots 4$.
- q: Pumped flow.
- g: The acceleration of gravity.
- q_i: Inflow of each tank.
- γ_a and γ_b: Parameters of the three-way valves.

Linearizing the model in an operating point given by h_i^0 and defining the variables $x_i = h_i - h_i^o$ and $u = q - q^o$ where $i = 1, 2$ we have that:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & \frac{1}{\tau_3} \\ 0 & \frac{-1}{\tau_3} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma}{A} \\ \frac{1-\gamma}{A} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \tag{3.13}$$

where $\tau_i = \frac{A}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0,\ i = 1, \cdots, 2$, are the time constants of each tank.

Table 3.1 shows the magnitudes and constraints are the ones of the 4 tanks process presented in 9 particularized for this example:

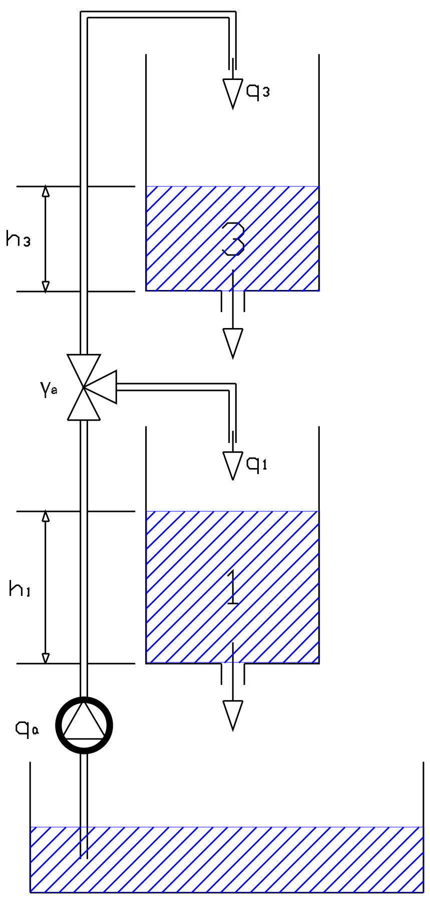

Figure 3.7: 2 Tanks scheme.

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Table 3.1: Parameters of the two cascaded tanks

Parameter	Value	Unit	Concept
A	0.06	m^2	Cross Section of Tanks
H_{1max}	1.36	m	Maximum Level of the Tank 1
H_{3max}	1.30	m	Maximum Level of the Tank 3
H_{1min}	0.3	m	Minimum Level of the Tank 1
H_{3min}	0.3	m	Minimum Level of the Tank 3
Q_{max}	4	m^3/h	Maximal Flow
h_1^0	0.68	m	Equilibrium level of Tank 1
h_3^0	0.65	m	Equilibrium level of Tank 3
q^0	2.0000	m^3/h	Equilibrium Flow
a_1	6.7371e-004	m^2	Cross-sections of the outlet hole 1
a_3	4.0423e-004	m^2	Cross-sections of the outlet hole 3

The parameter γ has a value of:

$$\gamma = 0.4$$

The steady states and inputs are characterized by the matrices:

$$M_\theta = \begin{bmatrix} 0.7150 & 0.6991 & 0.9405 \end{bmatrix}^T, \quad N_\theta = \begin{bmatrix} 0.7150 & 0.6991 \end{bmatrix}^T.$$

The weighting matrices have been chosen as $Q = I_2$ and $R = 100 \times I_1$. The local controller gain and the Lyapunov matrix P has been computed using an LQR, obtaining

$$K = \begin{bmatrix} -0.0023 - 0.0051 \end{bmatrix}$$

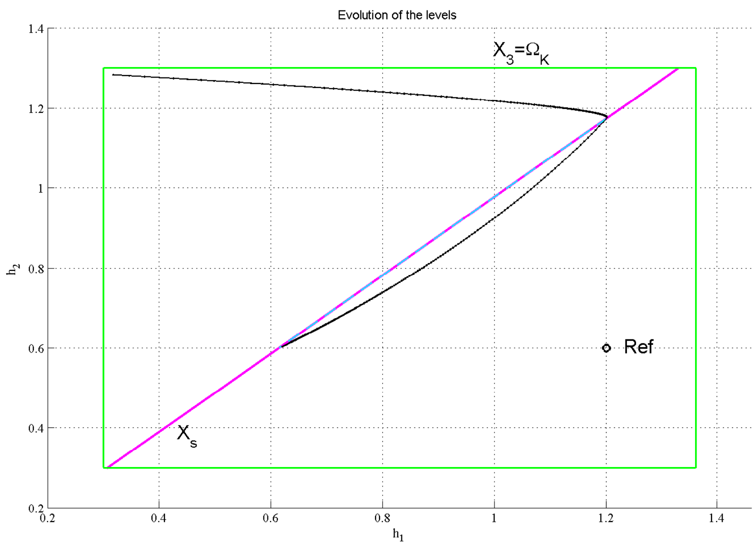

Figure 3.8: Evolution of the levels.

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and an associated Lyapunov matrix

$$P = \begin{bmatrix} 16.9174 & 6.2151 \\ 6.2151 & 33.4079 \end{bmatrix}$$

The system is controlled by the proposed MPC with a control horizon $N = 3$.

The maximal invariant set for tracking $\Omega_{t,K}$ has been computed taking $\lambda = 0.999$ and it is shown in figure 3.8, the region of attraction $\mathcal{X}_3$, the set of equilibrium levels $\mathcal{X}_s$ and the evolution of the levels are also shown in this figure. The idea of this test is illustrate the property 3.3, the system will be leaded to the closest equilibrium point in the sense that the distance, in the direction determined by H, were minimal ($T = M_x^T H M_x$).

In the sample time 300 the value of the matrix H change from:

$$H_1 = \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix}$$

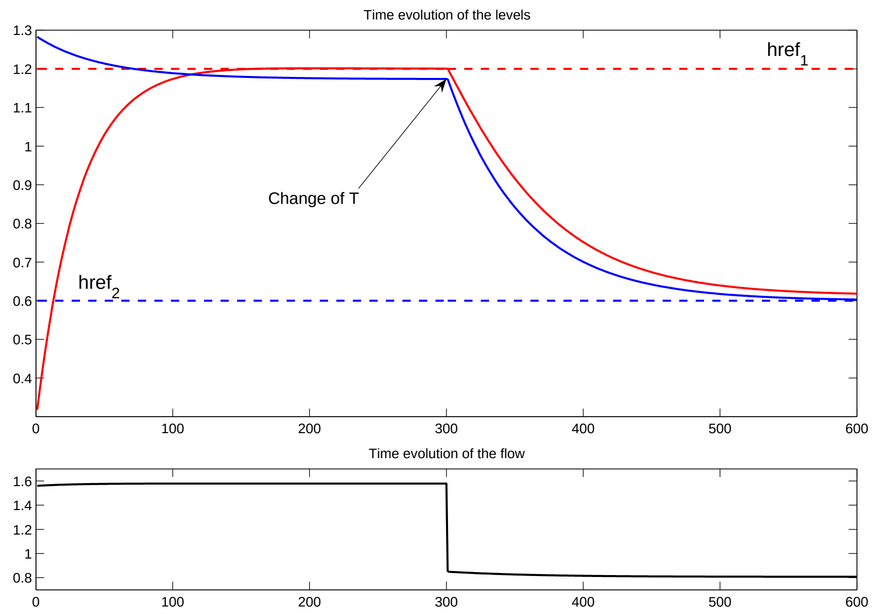

Figure 3.9: Time evolution of the plant.

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to

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1000 \end{bmatrix}$$

being, in both cases, the condition $\sigma T \geq M_x^T M_x$ satisfied. Then, the system will evolve to the closest equilibrium point that minimize the distance in the direction determined by T. Thus, with H_1, the first axis and with H_2, the second axis.

Figure 3.9 shows the time evolution of this test.

Chapter 4

Robust MPC for tracking of constrained linear systems with additive disturbances

4.1 Introduction

In this chapter is introduced a robust model predictive controller capable to robustly stabilize a constrained uncertain linear system maintaining the feasibility when the setpoint changes. On the one hand, a suitable controller should ensure stability despite the uncertainties and robust constraint satisfaction. On the other hand, the feasibility should be guaranteed under any set point change.

The proposed MPC for tracking based on nominal predictions introduced in the previous chapter, is able to control an uncertain system for a certain bound of the uncertainties. The optimal cost of the MPC for tracking is a Lipschitz continuous Lyapunov function. This is enough to ensure that the closed loop system is input-to-state stable with relation to the uncertainty, and hence robust.

However, when the uncertainty is large enough, specialized robust controller must be designed. Several solutions have been proposed: In (Rossiter et al., 1996; Chisci and Zappa, 2003) an auxiliary controller that is able to recover the feasibility in finite time is used leading to a switching strategy. The controllers proposed in (Pannocchia and

Kerrigan, 2005; Pannocchia, 2004) consider the change of the set point as a disturbance to be rejected; thus, this technique is able to steer the system to the desired set point, but only when the variations of the set point are small enough; so this solution results to be conservative.

A different approach has been proposed in the context of the reference governors (Gilbert et al., 1999; Bemporad et al., 1997). This control technique assumes that the system is robustly stabilized by a local controller, and a nonlinear filtering of the reference is designed to ensure the robust satisfaction of the constraints. These controllers ensure robust tracking without considering the performance of the obtained controller nor the domain of attraction.

In this chapter, a novel formulation of robust MPC for tracking is proposed. This is capable to lead the system to any robustly admissible set point in an admissible way. The proposed controller follows the novel MPC formulation presented in chapter 3 aimed to control constrained linear systems to track piecewise constant references in absence of uncertainties. This controller has been extended to control uncertain linear systems by incorporating the notion of tube-based robust control presented in (Mayne et al., 2005). The obtained robust controller is based on the solution of a single Quadratic Programming problem. Under mild conditions, the proposed controller ensures robust and admissible convergence to (a neighborhood of) the desired steady state, and maintains these properties under any change of reference. Moreover, offset free control can be achieved by means of a simple procedure.

4.2 Problem statement

Consider the following uncertain discrete-time linear time-invariant system

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + Du \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state of the system at the current time instant, x^+ denotes the successor state, that is, the state of the system at next sampling time, $u \in \mathbb{R}^m$ is the manipulated control input, w is an unknown but bounded state disturbance, and $y \in \mathbb{R}^p$ is the output or controlled variable. In what follows, $x(k)$, $u(k)$, $w(k)$ denote the state, the input and the disturbance, respectively, at sampling time k. It is worth to remark that the plant might be not square, that is, with a different number of inputs and outputs.

Assumption 4.1 *System (4.1) verifies the following assumptions:*

- (A, B, C, D) *are known and* (A, B) *are controllable.*
- *The uncertainty vector* w *is bounded and lies in a convex polyhedron containing the origin in its interior*

$$\mathcal{W} = \{w \in \mathbb{R}^n : A_w w \leq b_w\} \tag{4.2}$$

that is, $w(k) = \Big(x(k+1) - Ax(k) - Bu(k)\Big) \in \mathcal{W}$.

- *The state of the system is accessible.*

The state and input trajectories must satisfy the following constraints for any possible uncertain trajectory:

$$(x(k), u(k)) \in \mathcal{Z} \triangleq \{z = (x, u) \in \mathbb{R}^{n+m} : A_z\, z \leq b_z\} \tag{4.3}$$

The goal is to find a MPC control law $u(k) = \kappa(x(k), \theta)$ capable to steer the uncertain system to (a neighborhood of) any admissible steady state and input $(x_s, u_s) = M_\theta \theta$ (3.5) guaranteing robust constraints satisfaction.

Le us introduce the following definitions: x is the initial state , the predicted sequence of states is given by $\mathbf{x} = \phi(x, \mathbf{u}, \mathbf{w})$, where each component of this vector is given by $x(i) = \phi(i; x, \mathbf{u}, \mathbf{w})$. The predicted nominal sequence is calculated taking $\mathbf{w} = \mathbf{0}$, then $\bar{x}(i) = \bar{\phi}(i; x, \mathbf{u}) = \phi(i; x, \mathbf{u}, \mathbf{0})$ is the nominal predicted state sequence.

4.3 Preliminary results

4.3.1 Tube of trajectories and robust MPC for regulation

The proposed controller for tracking is based on the robust MPC for regulation proposed in (Mayne et al., 2005), which is based on nominal predictions and the notion of tube of trajectories. In this section these concepts are briefly introduced.

Consider a given sequence of control actions $\bar{\mathbf{u}}$, then the predicted nominal trajectory is given by $\bar{x}(i) = \bar{\phi}(i; \bar{x}(0), \bar{\mathbf{u}})$. Since the real system may be disturbed, the real trajectory may probably differ from this prediction. To counteract the disturbances it is desirable to force the trajectory to lie close to the nominal trajectory; this can be done by choosing the control action $u(i)$ as:

$$u(i) = \bar{u}(i) + K(x(i) - \bar{x}(i)) \tag{4.4}$$

which makes the system describe a trajectory $x(i) = \phi(i; x(0), \mathbf{u}, \mathbf{w})$ (which depends on the realization of the disturbance $\mathbf{w}$). Defining the signal $e \triangleq (x - \bar{x})$, its dynamics is given by

$$e^+ = A_K e + w; \quad A_K = (A + BK) \tag{4.5}$$

If the feedback control gain K is such that A_K is Hurwitz, then the evolution of $e(i)$ is bounded and hence the real trajectory $x(i)$ lies close to the predicted one. In order to bound the difference $e(i)$, the notion of robust positively invariant (RPI) set (Kolmanovsky and Gilbert, 1998; Rakovic, Kerrigan, Kouramas and Mayne, 2005) is used.

A set ϕ_K is called a robust positively invariant (RPI) set for the uncertain system (4.5) if $A_K\phi_K \oplus \mathcal{W} \subseteq \phi_K$. The minimum RPI is an RPI set F_∞ such that $F_\infty \subseteq \phi_K$ for every RPI ϕ_K. The calculation of F_∞ was analyzed in (Kolmanovsky and Gilbert, 1998), proving that this set is given by

$$F_\infty = \bigoplus_{k=0}^{\infty} (A_K)^k \mathcal{W}$$

The fact that A_K is Hurwitz ensures the existence of this set, but its calculation may require an infinite number of defining constraints. If a dead-beat control law is used, the calculation of F_∞ can be achieved in a finite number of steps. In the recent paper (Rakovic et al., 2005), a tractable method for computation of reliable outer robust positively invariant approximation of F_∞ is given.

Based on these results the notion of tube can be introduced (Mayne et al., 2005): consider that ϕ_K is a RPI for the system (4.5) and that $x(0)$ and $\bar{x}(0)$ are such that $e(0) = x(0) - \bar{x}(0) \in \phi_K$; then the trajectory of the uncertain system controlled by (4.4) is such that $x(i) \in \bar{x}(i) \oplus \phi_K$, for any possible realization of the disturbances. See that the sequence of sets $\bar{x}(i) \oplus \phi_K$ can be seen as a tube of trajectories. Moreover, if the nominal control sequence $\bar{\mathbf{u}}$ is such that $(\bar{x}(i), \bar{u}(i)) \in \bar{\mathcal{Z}}$

$$\bar{\mathcal{X}} \triangleq \mathcal{X} \ominus \phi_K \quad \bar{\mathcal{U}} \triangleq \mathcal{U} \ominus K\phi_K, \quad then \quad \bar{\mathcal{Z}} \triangleq \mathcal{Z} \ominus (\phi_K \times K\phi_K) \tag{4.6}$$

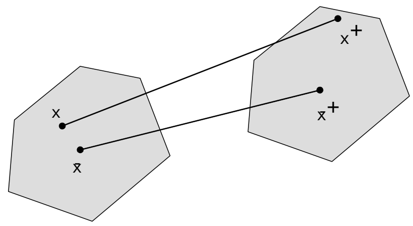


Figure 4.1: The successor state x^+ must be in $\bar{x}^+ \oplus \phi_K$.

The real trajectory and control actions derived from (4.5) satisfy $(x(i), K(x(i)-\bar{x}(i))+\bar{u}(i)) \in \mathcal{Z}$ for any possible realization of the disturbances. This implies that forcing a suitable tighter set of constraints for the nominal system, the evolution of uncertain system controlled by (4.5) is robustly admissible (Mayne and Langson, 2001).

This property is exploited in (Mayne et al., 2005) to derive a robust MPC for regulation; which is based on the notion of tube and forces that the current state x is in the interior of the tube for the current sampling time, i.e. $x \in \bar{x}(0) \oplus \phi_K$ considering $\bar{x}(0)$ as a decision variable.

4.3.2 Set point characterization and invariant set for tracking

As it was previously presented, the robust tube-based control law allows one to use the nominal system to obtain a bound on the uncertain trajectories. Therefore, the set points that can be robustly reached are those admissible steady states of the nominal system considering the following modified set of constraints:

$$\begin{aligned} \bar{\mathcal{X}} &= \mathcal{X} \ominus \Phi_K \\ \bar{\mathcal{U}} &= \mathcal{U} \ominus K\Phi_K \end{aligned}$$

Or in a compact form:

$$\bar{\mathcal{Z}} \triangleq \mathcal{Z} \ominus (\phi_K \times K\phi_K) \tag{4.7}$$

The procedure to determine the set of reachable desired steady states $\mathcal{X}_s$, the description of the artificial steady states in function of θ,

$$z_s = M_\theta \theta \quad s = N_\theta \theta \tag{4.8}$$

and the invariant set for tracking $\Omega^a_{t,\bar{K}}$ for the nominal system subject to the tighter constraints set (4.7) is described in section (3.3.1) and (3.3.2).

4.4 Plant Operation Point to be tracked

The operating point is typically provided to the controller by an upper level optimizer called *Set-point optimizer*, the way that this setpoint is provided to the controller is detailed in section 3.4.

When the provided setpoint it is not an equilibrium point of the linear model of the plant, there exist an alternative formulation of the controller that it is not going to be explained in this chapter. The formulation and the proof can be easily derived from the previous chapter and the following section.

4.5 Proposed robust MPC for tracking.

In this section the robust MPC scheme for tracking is presented. It is assumed that the target operation point corresponds to an equilibrium point of the linear model and this point is provided by the *set point optimizer level* as a target value θ.

Model predictive controllers are capable to steer the system to a given steady state by means of using shifted states and inputs, and hence a shifted set of constraints. If the desired steady state changes, the states and constraints must be shifted to the new steady conditions and hence the feasibility of the controller may be lost. This section is devoted to present a robust predictive controller which is able to track any set point maintaining the properties of MPC controllers: robust stability and admissible evolution.

The main idea is to use the algorithm introduced in the previous chapter to steer the nominal system to the desired steady state, then, applying the tube control law, the real state will confined to the the tube which center is the nominal trajectory. Consider so, a target steady state and input $(x_s, u_s) = M_\theta \theta$, the artificial steady state $(\bar{\mathrm{x}}_s, \bar{\mathrm{u}}_s) = M_\theta \bar{\theta}$ ($\bar{x}$ $\bar{u}$ $\bar{y}$ $\bar{\theta}$ denotes the variables of the nominal system). The offset cost has the form $\|\bar{\theta} - \theta\|_T^2$ and penalizes the deviation between the desired steady state and the artificial one. The extended terminal constraint forces the terminal state and the artificial reference (characterized by θ) to be in an invariant set for tracking.

Then, the cost function for a given state x and desired steady state θ is as follows

$$V_N(x,\theta;\bar{\mathbf{u}},\bar{x},\bar{\theta}) = \sum_{i=0}^{N-1}\left(\|\bar{x}(i)-\bar{\mathrm{x}}_s\|_Q^2 + \|\bar{u}(i)-\bar{\mathrm{u}}_s\|_R^2\right) + \|\bar{x}(N)-\bar{\mathrm{x}}_s\|_P^2 + \|\bar{\theta}-\theta\|_T^2 \quad (4.9)$$

where $\bar{x}(i) = \bar{\phi}(i;\bar{x},\bar{\mathbf{u}})$ and $(\bar{\mathrm{x}}_s,\bar{\mathrm{u}}_s) = M_\theta\bar{\theta}$. This cost penalizes the deviation between the predicted nominal trajectory and the artificial steady state along the horizon N and the deviation between the artificial and the desired θ. The current state x and the desired steady state θ are parameters, while the decision variables are (i) the sequence of the future actions of the nominal system $\bar{\mathbf{u}}$, (ii) the initial state of the nominal trajectory $\bar{x}$ and (iii) the parameter vector $\bar{\theta}$ that determines the artificial steady state $(\bar{\mathrm{x}}_s,\bar{\mathrm{u}}_s)$.

The formulation of the optimization problem $\mathcal{P}_N(x,\theta)$ to be solved is:

$$\begin{aligned}
\min_{\bar{\mathbf{u}},\bar{x},\bar{\theta}} \quad & V_N(x,\theta;\bar{\mathbf{u}},\bar{x},\bar{\theta}) & \\
s.t. \quad & \bar{x} \in x \oplus (-\Phi_K) & (4.10)\\
& (\bar{x}(i),\bar{u}(i)) \in \bar{\mathcal{Z}} & (4.11)\\
& (\bar{x}_s,\bar{u}_s) = M_\theta\bar{\theta}, & (4.12)\\
& (\bar{x}(N),\bar{\theta}) \in \Omega^a_{t,\bar{K}} & (4.13)
\end{aligned}$$

where

$$\bar{\mathcal{Z}} \triangleq \mathcal{Z} \ominus (\Phi_K \times K\Phi_K),$$

$\bar{x}(i) = \bar{\phi}(i;\bar{x},\bar{\mathbf{u}})$, Φ_K is a robust positive invariant set for the system (4.1) controlled by (4.4). The terminal constraint (4.13) forces the extended nominal state $(\bar{x}(N),\theta)$ to be contained in an invariant set for tracking $\Omega^a_{t,\bar{K}}$; this set is invariant for a control law gain $\bar{K}$, possibly different from the gain K used for the tube (i.e., the calculation of Φ_K). Since the set of constraints does not depend on the desired steady state $(\mathrm{x}_s,\mathrm{u}_s) = M_\theta\theta$, the optimization problem $\mathcal{P}_N(x,\theta)$ has a feasible solution for all x contained in $\mathcal{X}_N \subset \mathbb{R}^n$ (regardless of the choice of θ). In what follows, $V_N^*(x,\theta)$ denotes the optimal cost, $\bar{\mathbf{u}}^*(x,\theta)$, $\bar{x}^*(x,\theta)$ and $\bar{\theta}^*(x,\theta)$ the optimal value of the decision variables, $\bar{\mathbf{x}}^*(x,\theta)$ the nominal optimal trajectory and $(\bar{\mathrm{x}}_s^*(x,\theta),\bar{\mathrm{u}}_s^*(x,\theta)) = M_\theta\,\bar{\theta}^*(x,\theta)$ denotes the optimal artificial reference. The MPC control law is derived from the optimal solution as follows

$$\kappa_N(x,\theta) = K(x-\bar{x}^*(x,\theta)) + \bar{u}^*(0;x,\theta) \quad (4.14)$$

It is important to remark that all the defining sets of $\mathcal{P}_N(x,\theta)$ are convex polyhedra; therefore this optimization problem is a Quadratic Programming (QP) problem, that can be efficiently solved using specialized algorithms.

In the following section we show that under mild and standard assumptions, the proposed controller guarantees robust stability of the closed loop system.

4.6 Stability analysis

Consider that the proposed predictive controller $\mathcal{P}_N(x,\theta)$ is designed according to the following assumption:

Assumption 4.2 *The matrices Q, R, T, P, K, $\bar{K}$ and the sets Φ_K, $\Omega^a_{t,\bar{K}}$ satify:*

(i) $Q > 0$, $R > 0$.

(ii) There exists a constant $\sigma > 0$ such that $\sigma T \geq M_x^T M_x$, where $M_x = [I_n, \mathbf{0}_m] M_\theta$.

(iii) The eigenvalues $A+BK$ are in the interior of the unitary circle and there exists an admissible robust positively invariant (RPI) set Φ_K of (4.1) subject to constraints (4.3) controlled by $u = Kx$.

(iv) The eigenvalues $A + B\bar{K}$ are in the interior of the unit circle and $P > 0$ is such that

$$P - (A + B\bar{K})^\top P(A + B\bar{K}) = Q + \bar{K}^\top R\bar{K}. \tag{4.15}$$

(v) Set $\Omega^a_{t,\bar{K}}$ is the invariant set for tracking $\mathcal{O}^w_{\infty,\lambda}$ for a given $\lambda \in (0, 1)$ (arbitrarily close to 1) for system (4.1), calculated as in (3.3.2) subject to the tighter set of constrains $\bar{\mathcal{Z}}$ (4.7) using as control gain matrix $\bar{K}$.

Under these conditions, the following theorem can be formulated.

Theorem 4.1 *Consider system (4.1) subject to constraints (4.3). Suppose that matrices Q, R, T, P, K and $\bar{K}$, and sets Φ_K and $\Omega^a_{t,\bar{K}}$ satisfy Assumption 4.2. Consider the control law (4.14) resulting from the optimal solution of problem $\mathcal{P}_N(\cdot,\cdot)$. Then system (4.1) controlled by this law guarantees that:*

(i) For every initial condition $x(0) \in \mathcal{X}_N$ and for every θ, the evolution of the system is robustly feasible and admissible, that is, $x(i) \in \mathcal{X}_N$ and $(x(i), K_N(x(i),\theta)) \in \mathcal{Z}$, $\forall w(k) \in \mathcal{V}$, $k = 0, 1, \cdots, i-1$.

(ii) For every target steady state $\mathrm{x}_s \in \Omega_{t,\bar{K}}$*, the system is steered asymptotically to* $\mathrm{x}_s \oplus \Phi_K$.

Proof of theorem 4.1:

It is assumed that the hypothesis 4.2 is satisfied.

The first part of the proof is devoted to prove the feasibility of the controlled system, that is, $x(k+1) \in \mathcal{X}_N$, for all $x(k) \in \mathcal{X}_N$, $w(k) \in \mathcal{W}$ and θ. Consider the optimal solution of $\mathcal{P}_N(x(k), \theta)$, then the successor state is $x(k+1) = Ax(k) + B\kappa_N(x(k), \theta) + w(k)$, where $w(k) \in \mathcal{W}$. Define the following sequences:

$$\begin{aligned}
\bar{\mathbf{u}}(x(k+1), \theta) &\triangleq [\bar{u}^*(1; x(k), \theta), \cdots, \bar{u}^*(N-1; x(k), \theta), \\
&\quad \bar{K}(\bar{x}^*(N-1; x(k), \theta) - \mathrm{x}_s^*(x(k), \theta)) + \mathrm{u}_s^*(x(k), \theta)] \\
\bar{x}(x(k+1), \theta) &\triangleq \bar{x}^*(1; x(k), \theta) \\
\bar{\theta}(x(k+1), \theta) &\triangleq \bar{\theta}^*(x(k), \theta)
\end{aligned} \tag{4.16}$$

Then, $(\bar{\mathbf{u}}, \bar{x}, \bar{\theta})$ is a feasible solution for the optimization problem $\mathcal{P}_N(x(k+1), \theta)$ due to:

- From the notion of tube it is possible to deduce that $x(k+1) \in \bar{x}(x(k+1), \theta) \oplus \Phi_K$.
- Since $\bar{x}(x(k+1), \theta) = \bar{x}^*(1; x(k), \theta)$, then $\bar{x}(i; x(k+1), \theta) = \bar{x}^*(i+1; x(k), \theta)$ $i = 0, 1, \cdots, N-1$; then the first $N-1$ terms of the nominal trajectory are admissible. Admissibility of $\bar{x}(N; x(k+1), \theta)$ is derived from the fact that $(\bar{x}(N-1; x(k+1), \theta), \bar{\theta}(x(k+1), \theta)) \in \Omega^a_{t,\bar{K}}$ and hence the control action $\bar{u}(N-1; x(k+1), \theta)$ ensures that $(\bar{x}(N; x(k+1), \theta), \bar{\theta}(x(k+1), \theta)) \in \Omega^a_{t,\bar{K}}$.
- Feasibility of $\bar{\mathbf{u}}^*(x(k), \theta)$ and admissibility of set $\Omega^a_{t,\bar{K}}$ ensures the feasibility of $\bar{\mathbf{u}}(x(k+1), \theta)$.
- The terminal constraint satisfaction stems from the same arguments.

Convergence is derived proving that the optimal cost is a Lyapunov function. Let us consider the proposed feasible solution (4.16). Considering the properties of the feasible nominal trajectories for $x(k+1)$, the condition *(iv)* of Assumption 4.2 and using standard procedures in MPC (Mayne et al., 2000) it is possible to obtain

$$\begin{aligned} V_N(x(k+1),\theta;\mathbf{\bar{u}},\bar{x},\bar{\theta}) - V_N^*(x(k),\theta) &\leq -\|\bar{x}^*(x(k),\theta) - \bar{\mathrm{x}}_s^*(x(k),\theta)\|_Q^2 \\ &\quad -\|\bar{u}^*(0;x(k),\theta) - \bar{\mathrm{u}}_s^*(x(k),\theta)\|_R^2 \\ &\leq -\|\bar{x}^*(x(k),\theta) - \bar{\mathrm{x}}_s^*(x(k),\theta)\|_Q^2 \quad (4.17) \end{aligned}$$

By optimality, we have that $V_N^*(x(k+1),\theta) \leq V_N(x(k+1),\theta;\mathbf{\bar{u}},\bar{x},\bar{\theta})$ and then:

$$V_N^*(x(k+1),\theta) - V_N^*(x(k),\theta) \leq -\|\bar{x}^*(x(k),\theta) - \bar{\mathrm{x}}_s^*(x(k),\theta)\|_Q^2$$

Taking into account that $Q > 0$, we have that

$$\lim_{k\to\infty} \|\bar{x}^*(x(k),\theta) - \bar{\mathrm{x}}_s^*(x(k),\theta)\|_Q^2 = 0$$

In virtue of lemma 4.1, we can deduce that if $\|\bar{x}^*(x(k),\theta) - \bar{\mathrm{x}}_s^*(x(k),\theta)\|$ tends to 0 then $\|\bar{\mathrm{x}}_s^*(x(k),\theta) - \mathrm{x}_s\|$ also tends to 0. Therefore, $\bar{x}^*(x(k),\theta)$ tends to x_s and then $x(k)$ tends to $\mathrm{x}_s \oplus \Phi_K$. ■

Lemma 4.1 *Consider that the assumptions of theorem 4.1 hold. Consider a desired steady state* $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta\theta$ *and assume that for a given state* x *the optimal solution of* $\mathcal{P}_N(x,\theta)$ *is such that* $\|x - \bar{\mathrm{x}}_s^*(x,\theta)\|_Q = 0$ *(i.e.* $x = \bar{\mathrm{x}}_s^*(x,\theta)$*), then* $\|x - \mathrm{x}_s\|_Q = 0$.

Proof: The proof is obtained by contradiction. Let $\bar{\theta}^*$ denote $\bar{\theta}^*(x,\theta)$ and consider $(\bar{\mathrm{x}}_s^*, \bar{\mathrm{u}}_s^*) = M_\theta\bar{\theta}^*$ and assume that $\bar{\mathrm{x}}_s^* \neq \mathrm{x}_s$.

From continuity arguments it can be derived that there exists a $\hat{\lambda} \in [0,1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$, $\tilde{\theta} = \lambda\bar{\theta}^* + (1-\lambda)\theta$ and $(\tilde{x}_s, \tilde{u}_s) = M_\theta\tilde{\theta}$, the state $\bar{\mathrm{x}}_s^*$ is contained in the maximal admissible invariant set (denoted as $\mathcal{O}_\infty(\tilde{x}_s)$) for the nominal system controlled by $\bar{u} = \bar{K}(\bar{x} - \bar{x}_s) + \bar{u}_s$ (see (Limon, Alvarado, Alamo and Camacho, 2007*b*) for further details). Defining $\mathbf{\bar{u}}$ as the sequence of control actions derived from this control law, it is readily inferred that $(\mathbf{\bar{u}}, \bar{\mathrm{x}}_s^*, \bar{\theta})$ is a feasible solution for $\mathcal{P}_N(\bar{\mathrm{x}}_s^*, \theta)$. Then from Assumption 4.2,

$$\begin{aligned} V_N^*(\bar{\mathrm{x}}_s^*, \theta) &\leq V_N(\bar{\mathrm{x}}_s^*, \theta; \mathbf{\bar{u}}, \bar{\mathrm{x}}_s^*, \tilde{\theta}) \\ &= \sum_{i=0}^{N-1} \overbrace{\|\bar{x}(i) - \tilde{x}_s\|_Q^2 + \|\bar{K}(\bar{x}(i) - \tilde{x}_s)\|_R^2}^{\|\bar{x}(i)-\tilde{x}_s\|^2_{(Q+\bar{K}^T R\bar{K})}} + \|\bar{x}(N) - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \\ &= \|\bar{\mathrm{x}}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \end{aligned}$$

Let M_x and σ be defined in Assumption 4.2 (ii). Since $\bar{x}_s^* - \tilde{x}_s = (1-\lambda)M_x(\bar{\theta}^* - \theta)$ and $\tilde{\theta} - \theta = \lambda(\bar{\theta}^* - \theta)$ we can rewrite

$$\|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 = \|\bar{\theta}^* - \theta\|_H$$

where $H = (1-\lambda)^2 M_x^T P M_x + \lambda^2 T$. Choosing a constant $\beta \geq \hat{\lambda}(1-\hat{\lambda})^{-1}$ such that the maximum eigenvalue of P is lower than β/σ, we have that $H \leq (\beta(1-\lambda)^2 + \lambda^2)T$. Taking $\lambda = \beta(1+\beta)^{-1}$, then $\lambda \geq \hat{\lambda}$ and $H \leq \lambda T < T$. Therefore, we have that $V_N^*(\bar{x}_s^*, \theta) < \|\bar{\theta}^* - \theta\|_T^2$.

Given that the optimal solution of $\mathcal{P}_N(\bar{x}_s^*, \theta)$ is given by $\bar{\mathbf{u}}^*(\bar{x}_s^*, \theta) = \{\bar{u}_s^*, \cdots, \bar{u}_s^*\}$ and the associated nominal state sequence is $\bar{\mathbf{x}}^*(\bar{x}_s^*, \theta) = \{\bar{x}_s^*, \cdots, \bar{x}_s^*\}$, then the optimal cost is $V_N^*(\bar{x}_s^*, \theta) = \|\bar{\theta}^* - \theta\|_T^2$ yielding a contradiction and proving the lemma. ■

Besides robust stability, this controller also guarantees the same properties of the previous controller like:

- Changing operation points, section 3.1.
- Larger domain of attraction, section 3.2.
- Offset minimization, section 3.3.
- Local optimality, section 3.4.
- Explicit solution, section 3.5.

and following one:

Property 4.1 (Controller design) *The design of this controller is based on two local control gains K and $\bar{K}$. The former one allows one to optimize the RPI set Φ_K and hence the disturbance rejection response. The latter allows to optimize the terminal conditions, enhancing the optimality of the controller (improving the closed loop performance) and maximizing the terminal constraint set (enlarging the domain of attraction $\mathcal{X}_N$). This task is accomplished in the chapter 6*

4.7 Cancelation of the output tracking error

The proposed controller steers the uncertain system to a neighborhood of the desired steady state x_s given by $\mathrm{x}_s \oplus \Phi_K$. If the uncertainties tends to a steady state value $w(\infty)$, then the state of the closed-loop system reaches a steady state that differs from x_s producing an offset between the output $y(k)$ and the setpoint $s = C\mathrm{x}_s + D\mathrm{u}_s$. By straightforward calculations, it can be deduced that the new steady state and input are $x(\infty) = \mathrm{x}_s + (I_n - (A+BK))^{-1}w(\infty)$ and $u(\infty) = K(I_n - (A+BK))^{-1}w(\infty) + \mathrm{u}_s$, and hence the steady output is $y(\infty) = s + (C + DK)(I_n - (A + BK))^{-1}w(\infty)$.

From these calculations, it is easy to see that for a given desired setpoint s_d, if we provide to the proposed controller the following modified setpoint $s = s_d - (C + DK)(I_n - (A + BK))^{-1}w(\infty)$ then the offset is compensated, i.e., $y(\infty) = s_d$. To this end, the modified setpoint must be allowable, which is ensured if the rank of matrix

$$E = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix}$$

is $n + p$ and $\theta \equiv s$. Moreover an estimation of the disturbance must be calculated by means of an appropriate observer (Pannocchia and Kerrigan, 2005).

Therefore, for a given target setpoint s_d, the offset compensator is an outer loop composed of a disturbance estimator and a feedback loop given by

$$s(k) = s_d - (C + DK)(I_n - (A + BK))^{-1}\hat{w}(k)$$

where $\hat{w}(k)$ is the estimated disturbance. Notice that, because the estimator dynamics do not depend on the MPC controller, the estimator converges to the steady value of the disturbance and the whole system is stable (because of the separation principle). If $s(k)$ converges to an admissible set point s_d, then the real output converges to desired reference $y(\infty) = s_d$; if not, the proposed controller steers the nominal output to the closest admissible reference in an admissible trajectory and hence the real output may present offset.

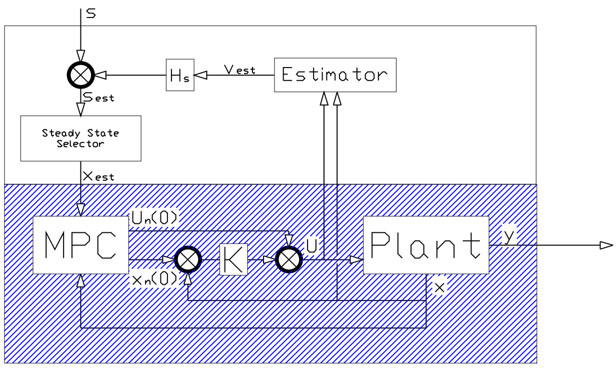


Figure 4.2: Block diagram for the cancelation of the tracking error

4.8 Example

Consider a constrained sampled double integrator (Mayne et al., 2005):

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

where the disturbances are bounded in $w \in \mathcal{W} = \{w : |w_1| \leq 0.2, |w_2| \leq 0.1\}$. The system must fulfil the following constraints: $|x_1| \leq 10$, $|x_2| \leq 2$, $|u| \leq 0.5$.

The chosen weighting matrices are $Q = I_2$ and $R = 100$. The LQ regulator gain has been used as the local control gain $\bar{K}$; matrix P has been derived from (4.15); matrix T has been taken as $T = 100 M_x^T P M_x$. The local controller for the tube, designed for disturbance rejection, is given by the gain matrix $K = [-0.1921, -1.0591]$.

The RPI set Φ_K, the set of reachable steady states X_s, the invariant set for tracking (for $\lambda = 0.999$) $\Omega_{t,\bar{K}}$ are shown in figure 4.3. The set of admissible setpoints $\mathcal{S}$ to be tracked is $s \in \mathcal{S} = \{s : |s| \leq 8.62\}$. The region of attraction of the robust MPC for $N = 2$ ($\mathcal{X}_2$) is also shown in figure 4.3.

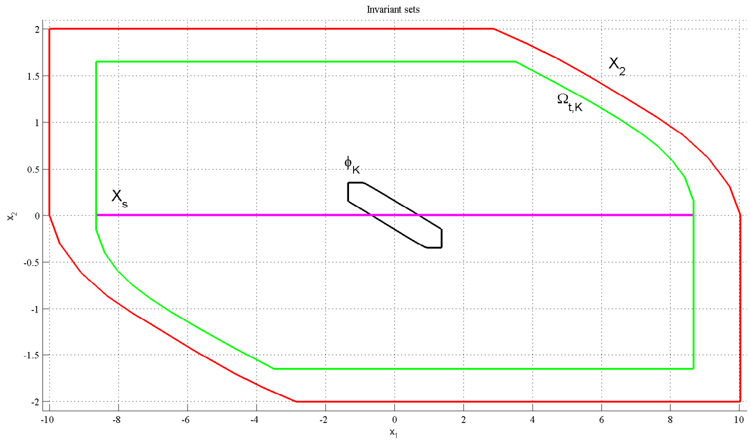


Figure 4.3: Regions of interest for the designed Robust MPC for tracking

In order to demonstrate the properties of the proposed controller, two simulations have been carried out starting from the same initial state $x(0) = (-7.5, -1)$ for a setpoint initially equals to 7.5 and changed to -7.5 and considering the same realization of the disturbances (shown in the bottom axes of figure 4.4). In the first one (Test #1) the proposed MPC is applied and the output trajectory together with the setpoint (solid line) and the artificial setpoint (dashed line) are depicted in figure 4.4. Notice how the system is able to track large changes of the setpoint (i.e., desired steady state and input) in an admissible way, thanks to the role of the artificial reference (even for a short prediction horizon). On the other hand, as it can be seen in the trajectory, the output exhibits an offset with respect to the desired setpoint due to the effect of the disturbances. In the Test #2, the controller with the offset cancelation loop is considered. As it can be seen in figure 4.4, the output reaches the desired setpoint (solid) thanks to the manipulation of the setpoint (dash-dot line); the artificial setpoint (dashed line) evolves converging to the corrected setpoint.

4.9 Conclusions

In this chapter a novel robust MPC to track piece-wise references has been presented. The robustness of the controller is achieved by using the notion of tubes proposed in (Mayne et al., 2005). Feasibility of the problem for any desired admissible steady state is guaranteed by adding an artificial steady state and considering an extended terminal constraint. Robust convergence is ensured by minimizing a performance index

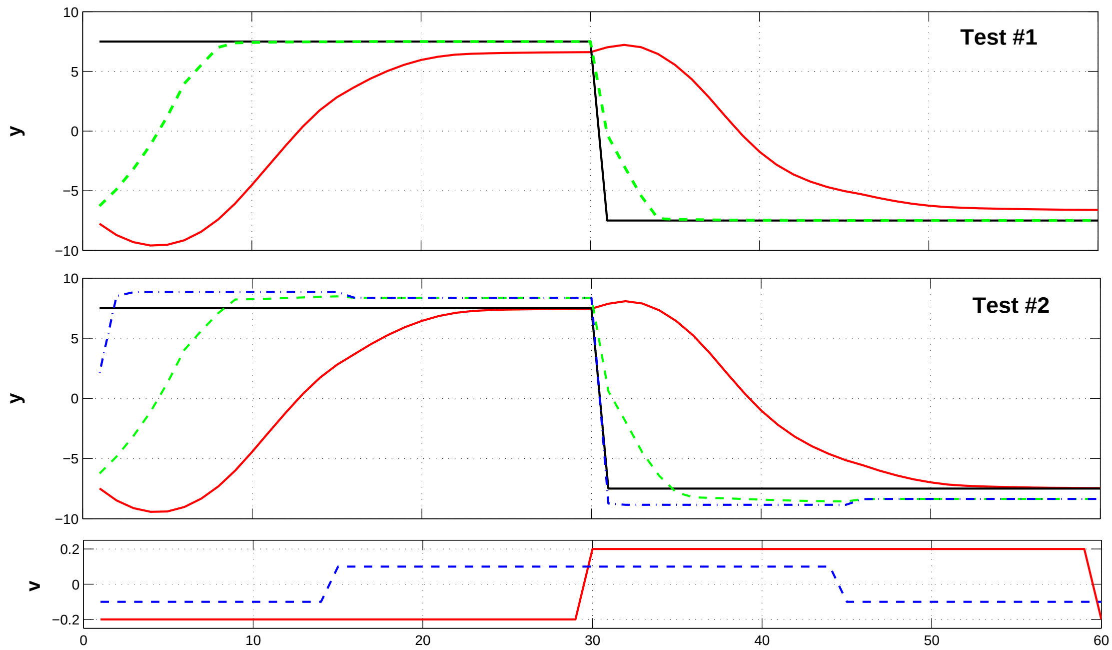


Figure 4.4: Output trajectories for test #1 and #2, and the considered realization of disturbances.

which penalizes the error with the desired steady state and the deviation between the desired steady state and the artificial one. The presented controller allows us to track any setpoint maintaining the feasibility under any change of the setpoint. The MPC controller is based on nominal predictions, requires the solution of a single QP problem and provides a larger domain of attraction. Moreover, the design allows to design the controller to accomplish specifications for tracking and for disturbance rejection.

Chapter 5

Robust output feedback MPC for tracking of constrained linear systems with additive disturbances

5.1 Introduction

In the last chapter, a robust model predictive controller capable to robustly stabilize a constrained uncertain linear system maintaining the feasibility when the setpoint changes is presented. This controller assumes that the full state is measured and ready to use for feedback. However, this hypothesis is not always fulfilled, which makes necessary to design a controller based on the measured outputs.

This is typically solved by adding a dynamic observer which provides a state estimation used for feedback. Separation principle ensures the stability of the closed loop system but the observer dynamics might produce a constraint violation. Therefore, the output feedback problem in presence of constraints requires a more careful design.

To overcome this problem several solutions have been proposed, in (Bemporad and Garulliy, 1997) A set-membership state estimator is used in the MPC for tracking framework for constrained linear systems in introduced. The estimation is approximately obtained solving a number of linear programming and then the control action just solving a QP, this makes not able to calculate an explicit solution of the controller, therefore it is not suitable to control fast systems.

A different approach has been proposed in the context of the reference governors (Angeli, Casavola and Mosca, 2001). This control technique assumes that the system is robustly stabilized by a controller, and a nonlinear filtering of the reference is designed to ensure the robust satisfaction of the constraints considering that the real state in confined in a set provided by a set-membership state estimator. These controllers ensure robust tracking without considering the performance of the obtained controller nor the domain of attraction. And, like in the previous case, the state estimator makes this controller not suitable to control fast systems

In this chapter, the previously presented robust predictive control for tracking has been extended to deal with the output feedback problem based on the results presented in (Mayne, Rakovic, Findeisen and Allgöwer, 2006). This controller maintains some ingredients of the state feedback case, but some modifications must be added to deal with the proposed problem. A nominal control problem is defined whose solution (a trajectory) defines the center of a tube, and where the 'cross section' of the tube is an invariant set. The constraints for the nominal system would be those that the tube satisfies the real constraints. The real state is inside of a tube which center is the estimated state, and it section is an invariant set (estimation tube). The estimated state is forced by the control to lie in another tube (control tube), which center is the nominal system (the system obtained by neglecting the disturbances) and it section is another invariant set so that, the state trajectory of the original (disturbed) system lies in a 'larger' tube with the nominal system as the center. At each sample time, a new tube is determined by solving an optimization problem in which the decision variables are the initial state of the nominal system, the control sequence over a finite horizon and the artificial steady state.

5.2 Problem description

Consider the following uncertain discrete-time linear time-invariant system:

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + v \end{aligned} \tag{5.1}$$

where x is the current state, u is the current control action, x^+ is the successor state, w is an unknown state disturbance, y is the current measured output, v is an unknown

output disturbance and $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$.
System 5.1 is subject to the following state and control constraints:

$$x(k) \in \mathcal{X}$$
$$y(k) \in \mathcal{U}$$

Or in a compact form:

$$(x(k), u(k)) \in \mathcal{Z} \tag{5.2}$$

The disturbances w and v are bounded in the following sets:

$$w(k) \in \mathcal{W}; \quad v(k) \in \mathcal{V} \tag{5.3}$$

where $\mathcal{W}$, $\mathcal{V}$ are C sets and $\mathcal{Z}$ is a polyhedral set.

Let $\phi(i; x, \mathbf{u}, \mathbf{w})$ denote the solution of (5.1) at time i if the initial state is x and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$.

The overall objective is to stabilize the system and steer the state to a neighborhood of the setpoint despite the fact that disturbances are present and that only (disturbed) output measurements are available. The following assumption is made:

Assumption 5.1 *The couple (A, B) is controllable and the couple (A, C) is observable.*

5.3 Preliminary results

The output feedback RMPCT combines the ideas of robust tube-based output feedback MPC and MPC for tracking introduced in chapter 3. In the following section these results are reviewed:

5.3.1 Tubes of trajectories

As it was mentioned in the introduction, the real state is confined in a tube (estimation tube) which center is the estimated trajectory and its section is a robust invariant set. This tube is described in the following section.

5.3.1.1 Estimation tube

To build an estimation of the state from output measurements a standard Luenberger type observer is considered:

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{5.4}$$

where $\hat{x} \in \mathbb{R}^n$ is the current observer state, $u \in \mathbb{R}^m$ is the current control action, $\hat{x}^+$ is the successor state of the observer system, $\hat{y} \in \mathbb{R}^p$ is the current observer output and $L \in \mathbb{R}^{n\times p}$ is the observer gain.

Let the state estimation error e_e be defined by:

$$e_e \triangleq x - \hat{x} \tag{5.5}$$

The dynamics of the estimation error is given by:

$$\begin{aligned} e_e^+ &= A_L e_e + w_{ee} \\ w_{ee} &= (w - Lv) \\ A_L &= (A - LC) \end{aligned} \tag{5.6}$$

The disturbance w_{ee} lies in the C set $\mathcal{W}_{ee}$ defined by:

$$\mathcal{W}_{ee} \triangleq \mathcal{W} \oplus (-L\mathcal{V}) \tag{5.7}$$

Assumption 5.2 *The matrix L is such that A_L is stable (Hurwitz).*

Since A_L is Hurwitz, there exists a robust positively invariant set (Kolmanovsky and Gilbert, 1998; Rakovic et al., 2005) Φ_{ee}[1] for the system (5.6) that satisfies:

$$A_L\Phi_{ee} \oplus \mathcal{W}_{ee} \subseteq \Phi_{ee}$$

There are several methods for constructing robust positively invariant sets (Blanchini, 1999; Rakovic et al., 2005)]. From the invariance of Φ_{ee} the following proposition is derived:

[1]Hereafter the dependence of RPI on the gain of the controller $\bar{K}$ is not presented

Proposition 5.1 (Estimation tube) *If the initial system and observer states, $x(0)$ and $\hat{x}(0)$ respectively, satisfy $e_e(0) = x(0) - \hat{x}(0) \in \Phi_{ee}$, then $e_e(i) \in \Phi_{ee}$ i.e. $x(i) \in \hat{x}(i) \oplus \Phi_{ee}$ for all $i \in \mathbb{N}$, and all admissible disturbance sequences* $\mathbf{w}$ *and* $\mathbf{v}$.

This proposition allows us to bound the real system state by the observer state and the disturbance set Φ_{ee}.

Remmark 5.1 *The assumption that $e_e(0) \in \Phi_{ee}$ can be essentially seen as a steady state assumption for the observer error, i.e., it is assumed that the observer has been running for a time large enough for guaranteing that the estimation error e_e lies in the set Φ_{ee}.*

If $\hat{x}$ is guaranteed to lie in the set $\mathcal{X} \ominus \Phi_{ee}$, then the real state x is guaranteed to lie in $\mathcal{X}$.

The estimated trajectory is confined, by the control law, in another tube (control tube) which center is the nominal trajectory and its section is a robust invariant set. This tube is detailed in the following section.

5.3.1.2 Control tube

The control methodology adopted is to control the observer system (5.4) in such a way that $x = \hat{x} + e_e$ satisfies the state constraint and the corresponding control input satisfies the control constraints. To achieve this, it is introduced a third dynamic system, the nominal system obtained from (5.1) by neglecting the disturbances w and v. The nominal system is described by:

$$\begin{aligned} \bar{x}^+ &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} \end{aligned} \tag{5.8}$$

Let $\bar{\phi}(i; x, \bar{\mathbf{u}}) \triangleq \phi(i; x, \bar{\mathbf{u}}, \mathbf{0})$ denote the state of the nominal system (5.8) at time i if its initial state is $\bar{x}$ and if the control sequence is $\bar{\mathbf{u}}$. To counteract the disturbances it is desirable to force the observer trajectory to lie close to the nominal trajectory; this can be done by choosing the control u to satisfy:

$$\begin{aligned} u &= \bar{u} + Ke_c \\ e_c &\triangleq (\hat{x} - \bar{x}) \end{aligned} \tag{5.9}$$

where e_c denotes the control error between the observer state and the state of the nominal system. The error e_c satisfies the difference equation:

$$\begin{aligned} e_c^+ &= A_K e_c + w_{ec} \\ w_{ec} &= (LCe_e + Lv) \\ A_K &= (A + KC) \end{aligned} \tag{5.10}$$

where (since e_e is bounded by Φ_{ee}) the disturbance w_{ec} lies in the set $\mathcal{W}_{ec}$ defined by

$$\mathcal{W}_{ec} \triangleq LC\Phi_{ee} \oplus L\mathcal{V} \tag{5.11}$$

Assumption 5.3 *The feedback control matrix K is such that A_K is stable (Hurwitz).*

This assumption ensures that there exists a robust positively invariant set Φ_{ec} for the system (5.10) that satisfies:

$$A_K \Phi_{ec} \oplus \mathcal{W}_{ec} \subseteq \Phi_{ec}$$

From the invariance of Φ_{ec} the following proposition is derived:

Proposition 5.2 (Tube) *If the initial observer and nominal system states satisfies that, $e_c(0)$ and $e_e(0)$ respectively, satisfy $e_c(0) \in \Phi_{ec}$ and $e_e(0) \in \Phi_{ee}$, then $\hat{x}(i) \in \bar{x}(i) \oplus \Phi_{ec} \quad \forall i \in \mathbb{N}$, and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$. Moreover, $x(i) \in \bar{x}(i) \oplus \Phi \quad \forall i \in \mathbb{N}$, and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$, where $\Phi = \Phi_{ee} \oplus \Phi_{ec}$. The proof can be found in (Mayne et al., 2005).*

Assumption 5.4 *There exist matrices K, L and a robust positively invariant set Φ_{ee} for the system $e_e^+ = A_L e_e + w_{ee}$ and and a robust positively invariant set Φ_{ec} for the system $e_c^+ = A_K e_c + w_{ec}$ such that $\mathcal{X} \ominus \Phi$ (being $\phi \triangleq \phi_{ee} \oplus \phi_{ec}$) and $\mathcal{U} \ominus K\Phi_{ec}$ are not empty sets.*

Theorem 5.1 ((Mayne, Rakovic, Findeisen and Allgöwer, 2006)) *Suppose that Assumption 5.4 holds, and suppose that the initial system, observer, and nominal system states $x, \hat{x}, \bar{x}$ lie all in $\mathcal{X}$ and satisfy that $e_e(0) \in \Phi_{ee}$ and $e_c(0) \in \Phi_{ec}$. If, in addition, the initial state $\bar{x}$ and control sequence $\bar{\mathbf{u}}$ of the nominal system satisfy the tighter constraints $\bar{x}(i) = \bar{\phi}(i; \hat{x}, \bar{\mathbf{u}}) \in \mathcal{X} \ominus \Phi$ and $\bar{u}(i) \in \mathcal{U} \ominus K\Phi_{ec}$ for all $i \in \mathbb{N}$, then the state and the control of the real system (5.1) $x(i)$ and $u(i) = \bar{u}(i) + K(\hat{x}(i) - \bar{x}(i))$ satisfy the original constraints $(x(i), u(i)) \in \mathcal{Z}$ for all $i \in \mathbb{N}$ and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$.*

5.3.2 Set point characterization and invariant set for tracking

As it was previously presented, the robust tube-based control law allows one to use the nominal system to obtain a bound on the uncertain trajectories. Therefore, the set points that can be robustly reached are those admissible steady states of the nominal system considering the following modified set of constraints:

$$\begin{aligned} \bar{\mathcal{X}} &= \mathcal{X} \ominus \Phi \\ \bar{\mathcal{U}} &= \mathcal{U} \ominus K\Phi_{ec} \end{aligned}$$

Or in a compact form:

$$\bar{\mathcal{Z}} \triangleq \mathcal{Z} \ominus (\phi \times K\phi_{ec}) \tag{5.12}$$

The procedure to determine the set of reachable desired steady states $\mathcal{X}_s$, the description of the artificial steady states in function of θ,

$$z_s = M_\theta \theta \quad s = N_\theta \theta \tag{5.13}$$

and the invariant set for tracking $\Omega^a_{t,\bar{K}}$ for the nominal system subject to the tighter constraints set (5.12) is described in section (3.3.2).

5.4 Plant Operation Point to be tracked

The operating point is typically provided to the controller by an upper level optimizer called *Set-point optimizer*, the way that this setpoint is provided to the controller is detailed in section 3.4.

When the provided setpoint it is not an equilibrium point of the linear model of the plant, there exist an alternative formulation of the controller that it is not going to be explained in this chapter. The formulation and the proof can be easily derived from the chapter 3 and the following section.

5.5 Robust output feedback model predictive control for tracking

In this section the output feedback robust MPC scheme for tracking is presented. It is assumed that the target operation point corresponds to an equilibrium point of the linear model and this point is provided by the *set point optimizer level* as a target value θ.

Taking into account the discussion in §5.3.1, if Assumption 5.4 holds, it is possible to propose a computationally tractable scheme for robust output feedback model predictive controller for tracking. In order to guarantee the feasibility of the problem for any desired steady state $(x_s, u_s) = M_\theta\theta$, an artificial steady state $(\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta}$ is introduced as a decision variable in the minimization of the performance index. Moreover, robust convergence to the desired steady state is guaranteed by adding a term $\|\bar{\theta} - \theta\|_T^2$ in the cost function (*offset cost*) that penalizes the deviation between the desired steady state and the artificial one (see that any quadratic term weighting the offset, as $\|\bar{x}_s - x_s\|_S^2$ for instance, can be expressed in this form), the initial state of the nominal system $\bar{x}$ is also a decision variable that must to be inside of $\hat{x} \oplus \Phi_{ec}$. Thus, if $\hat{x} - \bar{x} = e_c(0) \in \Phi_{ec}$, then $e_c(i) \in \Phi_{ec}$, $\forall i \in \mathbb{N}$.
Let the cost be defined by:

$$\begin{aligned} V_N(\hat{x}, \theta; \bar{x}, \bar{\mathbf{u}}, \bar{\theta}) &= \sum_{i=0}^{N-1} \|\bar{x}(i) - \bar{x}_s\|_Q^2 + \|\bar{u}(i) - \bar{u}_s\|_R^2 \\ &+ \|\bar{x}(N) - \bar{x}_s\|_P^2 + \|\bar{\theta} - \theta\|_T^2 \end{aligned}$$

where $\bar{x}(i) \triangleq \bar{\phi}(i; x, \bar{\mathbf{u}})$ and N is the horizon. The resulting optimization problem $\mathcal{P}_N(\hat{x}, \theta)$ is:

$$\begin{aligned} \min_{\bar{x}, \bar{\mathbf{u}}, \theta} \quad & V_N(\hat{x}, \theta; \bar{x}, \bar{\mathbf{u}}, \bar{\theta}) \\ s.t. \quad & \bar{x} \in \hat{x} \oplus (-\Phi_{ec}) \\ & \bar{x}(i) \in \bar{\mathcal{X}} = \mathcal{X} \ominus \Phi \quad (\Phi \triangleq \Phi_{ee} \oplus \Phi_{ec}) \\ & \bar{u}(i) \in \bar{\mathcal{U}} = \mathcal{U} \ominus K\Phi_{ec} \\ & (\bar{x}(N), \bar{\theta}) \in \Omega_{t,\bar{K}}^a \end{aligned} \tag{5.14}$$

$\mathcal{P}_N(\hat{x}, \theta)$ is solved online to yield an optimal initial state $\bar{x}^*(\hat{x}, \theta)$, an optimal control sequence $\bar{\mathbf{u}}^*(\hat{x}, \theta) = \{\bar{u}^*(0; \hat{x}, \theta), \bar{u}^*(1; \hat{x}, \theta), \ldots, \bar{u}^*(N-1; \hat{x}, \theta)\}$ for the nominal system and $\theta^*(\hat{x}, \theta)$ that determines the artificial steady state; then, the control applied to the plant is

$$k_N(\hat{x},\theta) = \bar{u}^*(0;\hat{x},\theta) + K(\hat{x} - \bar{x}^*(\hat{x},\theta)) \tag{5.15}$$

Let $\bar{\mathcal{X}}_N$ be the set of the admissible nominal initial states, then $\hat{\mathcal{X}}_N$ is the set of the admissible estimated states ($\hat{\mathcal{X}}_N = \bar{\mathcal{X}}_N \oplus \Phi_{ec}$) and $\mathcal{X}_N$ is the set of the admissible real states ($\mathcal{X}_N = \bar{\mathcal{X}}_N \oplus \Phi$).

In order to establish the main result of this chapter it is necessary to introduce the following assumption:

Assumption 5.5 *Matrices Q, R, T, P, $\bar{K}$, and the set $\Omega^a_{t,\bar{K}}$ satisfy:*

- *(i) $Q > 0$ and $R > 0$*
- *(ii) There exists a positive constant σ such that $\sigma T \geq M_x^t M_x$, where $M_x = [\, I_n \ \ \mathbf{0}_m \,] M_\theta$.*
- *(iii) Matrix $\bar{K}$ is such that $A + B\bar{K}$ is Hurwitz, $P > 0$ and the following equality holds:*

  $$P - (A + B\bar{K})^\top P (A + B\bar{K}) = Q + \bar{K}^\top R \bar{K}$$

  .
- *(iv) $\Omega^a_{t,\bar{K}}$ is an invariant set for tracking (as large as possible) for the nominal system (5.8) subject to the following constraints (5.12) and using as control gain matrix $\bar{K}$ for a given λ arbitrarily close to* 1.

Remmark 5.2 *Matrix $\bar{K}$ should be different from K because the first one is calculated to maximize the size of the set $\Omega^a_{t,\bar{K}}$ and the second to minimize the size of Φ_{ec} (Limon et al., 2007*b*).*

Now it is possible to establish the following result:

Theorem 5.2 *Consider system (5.1) subject to the constraints (5.2) such that the Assumptions 5.1, 5.2, 5.3, 5.4, 5.5 are satisfied and assume that $e_e(0) = x - \hat{x} \in \Phi_{ee}$. Consider $k_N(\hat{x},\theta)$ as the control law resulting from the solution of the optimal problem $\mathcal{P}_N(\hat{x},\theta)$ (5.15).*

The closed-loop system guarantees that: $\forall \hat{x} \in \hat{\mathcal{X}}_N$ and any desired steady state $x_s \in \lambda\mathcal{X}_s$ the proposed output feedback RMPCT controller steers asymptotically the system to the neighborhood of the desired steady state $x_s \oplus \Phi$ in an admissible way.

Proof: The proof is the same as in the previous chapter changing the dependence of the optimization problem on the state by the estimated state.

It is assumed that the assumptions 5.1, 5.2, 5.3, 5.4, 5.5 are satisfied.

The statement of this proof are shown by proving the following two facts:

Fact 1: Feasibility Assume that the current estimate state at the current sample time k is such that $\hat{x}(k) \in \hat{\mathcal{X}}_N$, assume also that the optimal solution of $\mathcal{P}_N(\hat{x}(k), \theta)$ is $(\bar{x}^*(\hat{x}(k), \theta), \bar{\mathbf{u}}^*(\hat{x}(k), \theta), \bar{\theta}^*(\hat{x}(k), \theta))$. Let $\hat{x}(k+1)$ be the estimated estate at the next sample time.
Let us consider so:

$$\begin{aligned}
\bar{\mathbf{u}}(\hat{x}(k+1), x_s) &= [\bar{u}^*(1; \hat{x}(k), \theta), \cdots, \bar{u}(k+N-1; \hat{x}(k), \theta), \\
&\quad \bar{K}(\bar{x}(k+N-1; \hat{x}(k), \theta) - \bar{x}_s(\hat{x}(k), \theta)) \\
&\quad +\bar{u}_s(\hat{x}(k), \theta)] \\
\bar{x}(\hat{x}(k+1), x_s) &= \bar{x}^*(1; \hat{x}(k), \theta) \\
\bar{\theta}(\hat{x}(k+1), x_s) &= \bar{\theta}^*(\hat{x}(k), \theta)
\end{aligned} \tag{5.16}$$

where $\bar{x}^*(i; \hat{x}(k), \theta) = \bar{\phi}(i; \bar{x}^*(\hat{x}(k), \theta), \bar{\mathbf{u}}^*(\hat{x}(k), \theta))$

Then, these values are a solution for the optimization problem $\mathcal{P}_N(\hat{x}(k+1), \theta)$ due to:

- From the proposition 5.2 we have that:

  $$\hat{x}(k+1) \in \bar{x}(\hat{x}(k+1), x_s) \oplus \Phi_{ec}$$

- Since $\bar{x}(\hat{x}(k+1), \theta) = \bar{x}^*(1; \hat{x}(k), \theta)$, then

  $$\bar{x}(i; \hat{x}(k+1), \theta) = \bar{x}^*(i+1; \hat{x}(k), \theta) \tag{5.17}$$

  So the first $N-1$ terms of the nominal trajectory are admissible. Admissibility of $\bar{x}(N; \hat{x}(k+1), \theta)$, is derived from the fact that $(\bar{x}(N-1; \hat{x}(k+1), \theta), \bar{\theta}(\hat{x}(k+1), \theta)) \in \Omega^w_{\bar{K}}$, and hence the control action $\bar{u}(N-1; x(k+1), \theta)$ ensures that $\bar{x}(N; \hat{x}(k+1), \theta, \bar{\theta}(\hat{x}(k+1), \theta)) \in \Omega^w_{\bar{K}}$

- The feasibility of $\bar{\mathbf{u}}^*(\hat{x}(k),\theta)$ an the admissibility of the set $\Omega^w_{\bar{K}}$ guarantees the feasibility of $\bar{\mathbf{u}}(\hat{x}(k+1),x_s)$.

- The terminal constraint satisfaction stems from the same arguments.

Fact 2: Convergence Is derived proving that the optimal cost is a Lyapunov function. Let us consider the proposed feasible solution (5.16). Considering the properties of the feasible nominal trajectories for $(\hat{x}(k+1),\theta)$, the condition (ii) of the Assumption (5.5) and using standard procedures in MPC (Mayne et al., 2000) it is posible to obtain:

$$\begin{aligned} V_N(\hat{x}(k+1),\theta) - V_N^*(\hat{x}(k),\theta) &\leq -\|\bar{x}^*(\hat{x}(k),\theta) - \bar{x}_s^*(\hat{x}(k),\theta)\|_Q^2 \\ &\quad -\|\bar{u}^*(\hat{x}(k),\theta) - \bar{u}_s^*(\hat{x}(k),\theta)\|_R^2 \\ &\leq -\|\bar{x}^*(\hat{x}(k),\theta) - \bar{x}_s^*(\hat{x}(k),\theta)\|_Q^2 \end{aligned}$$

Due to the solution proposed in (5.16) is suboptimal for $\hat{x}(k+1)$, the optimal solution should verify $V_N^*(\hat{x}(k+1),x_s) \leq \bar{V}_N(\hat{x}(k+1),x_s)$, then:

$$V_N^*(\hat{x}(k+1),x_s) - V_N^*(\hat{x}(k),\theta) \leq -\|\bar{x}^*(\hat{x}(k),\theta) - \bar{x}_s^*(\hat{x}(k),\theta)\|_Q^2$$

Taking into account that $Q > 0$

$$\lim_{k\to\infty} \|\bar{x}^*(\hat{x}(k),\theta) - \bar{x}_s^*(\hat{x}(k),\theta)\|_Q^2 = 0$$

In virtue of lemma (5.1), it is possible to deduce that if $\|\bar{x}^*(\hat{x}(k),\theta) - \bar{x}_s^*(\hat{x}(k),\theta)\|$ tends to 0 then $\|\bar{x}_s^*(\hat{x}(k),\theta) - x_s\|$ tends to 0. Therefore, $\bar{x}^*(\hat{x}(k),\theta)$ tends to x_s and $x(k)$ tends to $x_s \oplus \Phi$. ■

Lemma 5.1 *Consider that the assumptions of the theorem (5.2) hold. Consider a desired steady state* $(x_s,u_s) = M_\theta\theta$ *and assume that for a given estimated state* $\hat{x}$ *the optimal solution* $\mathcal{P}(\hat{x},\theta)$ *is such that* $\|\bar{x} - \bar{x}_s^*(\hat{x}(k),\theta)\| = 0$ *then* $\|\bar{x} - x_s\| = 0$

Proof: The proof is obtained by contradiction. Let $\bar{\theta}^*$ denote $\bar{\theta}^*(\hat{x},\theta)$ and consider $(\bar{x}_s^*,\bar{u}_s^*) = M_\theta\bar{\theta}^*$ and assume that $\bar{x}_s^* \neq x_s$.

From the continuity arguments it can be derived that exists a $\hat{\lambda} \in [0,1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$ there exists a $\tilde{\theta} = \lambda\bar{\theta}^* + (1-\lambda)\theta$ and $(\tilde{x}_s, \tilde{u}_s) = M_\theta \tilde{\theta}$ such that $\bar{x}_s^*$ is contained in the maximal invariant set (denoted as $\vartheta_\infty(\tilde{x}_s)$) for the nominal system controlled by $\bar{u} = \bar{K}(\bar{x} - \bar{x}_s) + \bar{u}_s$ (see (Limon, Alvarado, Álamo and Camacho, 2007*a*) for further details). Defining $\bar{\mathbf{u}}$ as the sequence of control actions derived from this control law, it is readily inferred that $(\bar{\mathbf{u}}, \bar{x}_s^*, \bar{\theta})$ is a feasible solution for $\mathcal{P}(\bar{x}_s^*, \theta)$. Then $V_N^*(x_s^*, \theta) \leq V_N(x_s^*, \theta; \bar{\mathbf{u}}, x_s^*, \tilde{\theta})$ and,

$$\begin{aligned} V_N(x_s^*, \theta; \bar{\mathbf{u}}, x_s^*, \tilde{\theta}) &= \sum_{i=0}^{N-1} \|\bar{x}(i) - \tilde{x}_s\|_Q^2 + \|\bar{u}(i) - \tilde{u}_s\|_R^2 \\ &+ \|\bar{x}(N) - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \\ &= \|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 \end{aligned}$$

Let M_x and σ be defined as in Assumption 5.5 (ii). Since $\bar{x}_s^* - \tilde{x}_s = (1-\lambda)M_x(\tilde{\theta} - \theta)$ and $\tilde{\theta} - \theta = \lambda(\bar{\theta}^* - \theta)$ it is possible to rewrite the equation as:

$$\|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|\tilde{\theta} - \theta\|_T^2 = \|\bar{\theta}^* - \theta\|_H^2$$

where $H = (1-\lambda)^2 M_x^t P M_x - \lambda^2 T$. Choosing $\beta > \hat{\lambda}(1-\hat{\lambda})$ such that the maximum eigenvalue of P is lower than β/σ, then $H < (\beta(1-\lambda^2) + \lambda^2)T$. Taking $\lambda = \beta(1+\beta)^{-1}$, then $H < \lambda T < T$. Therefore, $V_N^*(\bar{x}_s^*, \theta) < \|\bar{\theta}^* - \theta\|_T^2$.

Given that the optimal solution of $P_N(\bar{x}_s^*, \theta)$ is given by $\bar{\mathbf{u}}(\bar{x}_s^*, \theta) = \{\bar{u}_s^*, \ldots, \bar{u}_s^*\}$ and the associated nominal sequence is $\bar{\mathbf{x}}(\bar{x}_s^*, \theta) = \{\bar{x}_s^*, \ldots, \bar{x}_s^*\}$, then the optimal cost is $V_N^*(\bar{x}_s^*, \theta) = \|\bar{\theta}^* - \theta\|_T^2$ yielding a contradiction and proving the lemma. ■

This controller present the same properties as the previous ones, like:

- Changing operation points, section 3.1.
- Larger domain of attraction, section 3.2.
- Offset minimization, section 3.3.
- Local optimality, section 3.4.
- Explicit solution, section 3.5.
- Controller design, section 4.1

5.6 Cancelation of the tracking error

If the uncertainties tends to a steady value v_∞, w_∞ then the system reaches a steady state that differs from x_s producing an offset between the output y and the setpoint $s = Cx_s$. By straightforward calculations, it can be deduced that the value of the steady estimated error is $e_e(\infty) = w_\infty - Lv_\infty$; the steady control error can be obtained from (5.10) and its expression is:

$$e_c(\infty) = (I - AK)^{-1}L(Ce_e(\infty) + v_\infty)$$

Thus, the value of the output in permanent regime results to be:

$$\begin{aligned} y(\infty) &= Cx(\infty) + v_\infty = C(e_e(\infty) + e_c(\infty) + \bar{x}(\infty)) + v_\infty \\ &= (I + C(I - AK)^{-1}L)(Ce_e(\infty) + v_\infty) + s \end{aligned}$$

From these calculations, it is easy to see that for a given desired set point s_d, if it is provided to the controller the following modified setpoint

$$s = s_d - F(Ce_e(\infty) + v_\infty), \text{ where } F = I + C(I - AK)^{-1}L \tag{5.18}$$

then the offset is compensated ($y_\infty = s_d$). To this aim the modified setpoint must be allowable, which ensured if:

$$rank \begin{bmatrix} A - I_n & B \\ C & O_p \end{bmatrix} = n + p$$

Since the steady value $Ce_e(\infty) + v_\infty$ is not known, it should be estimated. This can be done by a simple procedure observing that the signal $y(k) - C\hat{x}(k)$ results to be equal to $Ce_e(k) + v(k)$. Then the following setpoint signal could be used to cancel the offset

$$s(k) = s_d - F(y(k) - C\hat{x}(k))$$

If $v(k) \to v_\infty$ and $w(k) \to w_\infty$ then $s(k) \to s_d - F(Ce_e(\infty) + v_\infty)$.

Because the output signal $y(k)$ is noisy, it is sensible the addition of an unbiased filter to avoid unnecessary variations on the setpoint signal. The proposed method adds an outer-loop which closed-loop dynamics results to be stable. In effect, see that the stable observer design ensures that the signal $y(k) - C\hat{x}(k)$ reaches a steady value, and hence the setpoint signal $s(k)$ is asymptotically convergent to an steady value. In virtue of Theorem 5.2, the closed-loop system remains stable and converges to an steady state.

5.7 Example

Let consider a discrete time double integrator:

$$\begin{aligned} x^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + v \end{aligned}$$

subject to the following constraints:

$$\begin{aligned} x \in \mathcal{X} &= \{x \in \mathbb{R}^2 : -50 \leq x \leq 10\} \\ u \in \mathcal{U} &= \{u \in \mathbb{R}^1 : -10 \leq u \leq 10\} \end{aligned}$$

The disturbances are bounded in the following sets:

$$\begin{aligned} \mathcal{W} &= \{w \in \mathbb{R}^2 | \, \|w\|_\infty \leq 0.5\} \\ \mathcal{V} &= \{v \in \mathbb{R}^1 | \, \|v\|_\infty \leq 0.5\} \end{aligned}$$

The feedback control matrix K and the output observer gain L are:

$$K = [-1, -1] \quad L = \begin{bmatrix} 1.6180 \\ 0.6180 \end{bmatrix}$$

The cost function is defined by (5.14) with $Q = I$, $R = 0.01$. Matrix P is the corresponding Lyapunov matrix for the system controlled by $u = \bar{K}x$, where $\bar{K}$ is calculated as the solution of the LQR with Q and R. The terminal constraint set is $\Omega^a_{t,\bar{K}}$ is calculated with $\bar{K}$. The prediction horizon is $N = 2$ and $\lambda = 0.999$.
Figure 5.1 shows the region of attraction $\mathcal{X}_2$, the terminal constraint $\Omega^a_{t,\bar{K}}$, the set of equilibrium states $\mathcal{X}_s$ and the minimal robust invariant sets Φ_{ee}, Φ_{ec} and Φ.

In order to show the controller properties, two simulations have been carried out starting from the same initial state and estimated state for setpoints changing from $s = 0$ to $s = 40$ and subject to the same realization of the disturbances (showed at the bottom the figure 5.2). In the $Test\#1$ corresponds to the plant controlled by the

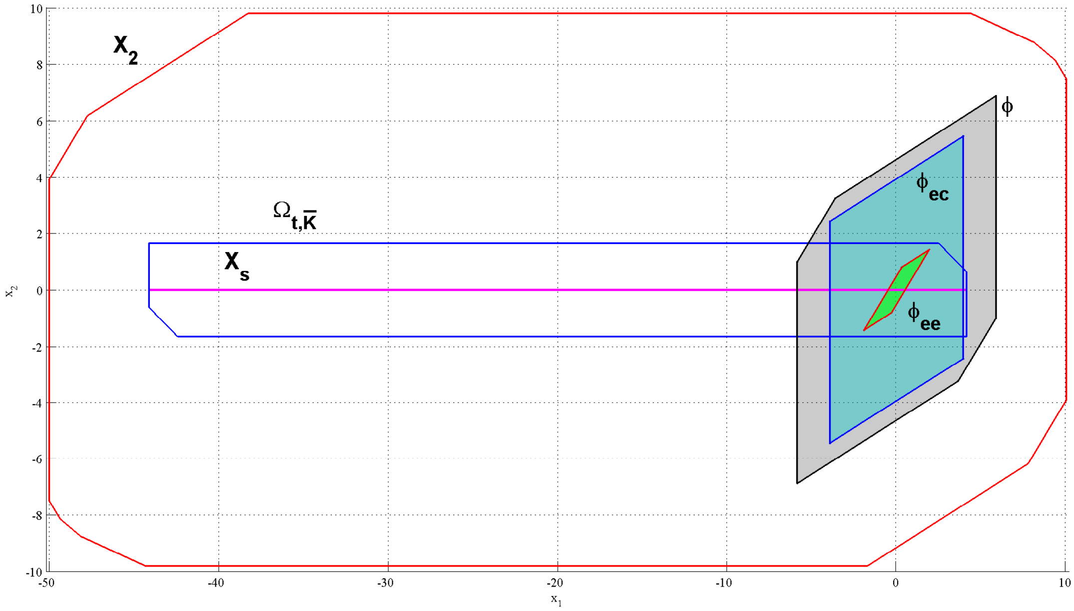


Figure 5.1: Regions and state trajectories.

output feedback RMPCT and it is possible to see the time evolution of the output (solid line), the reference (solid line) and the artificial reference (dashed line) (figure 5.2, at the top). In the $Test\#2$ corresponds to the plant controlled by the same controller with the offset cancelation loop proposed in the section 5.6 (figure 5.2, at the middle); it is possible to see the same evolution plus the the corrected reference (dash-dot line) as in the $Test\#1$. It is possible to see how the steady state error is removed using this scheme.

5.8 Conclusions

This chapter presents an output feedback model predictive controller for tracking of piece wise constant references for constrained linear systems with input and output disturbances. Assuming that the disturbances are bounded, it is possible to establish, for a given state observer, explicit bounds on the state estimation error (bounded by an invariant set Φ_{ee}). This allows us to consider for control, instead of the unknown system state, the nominal observer state subject to bounded additive disturbances and tightened state and input constraints. The input applied is defined via a novel robust model predictive tube based control for tracking scheme employing feed forward control

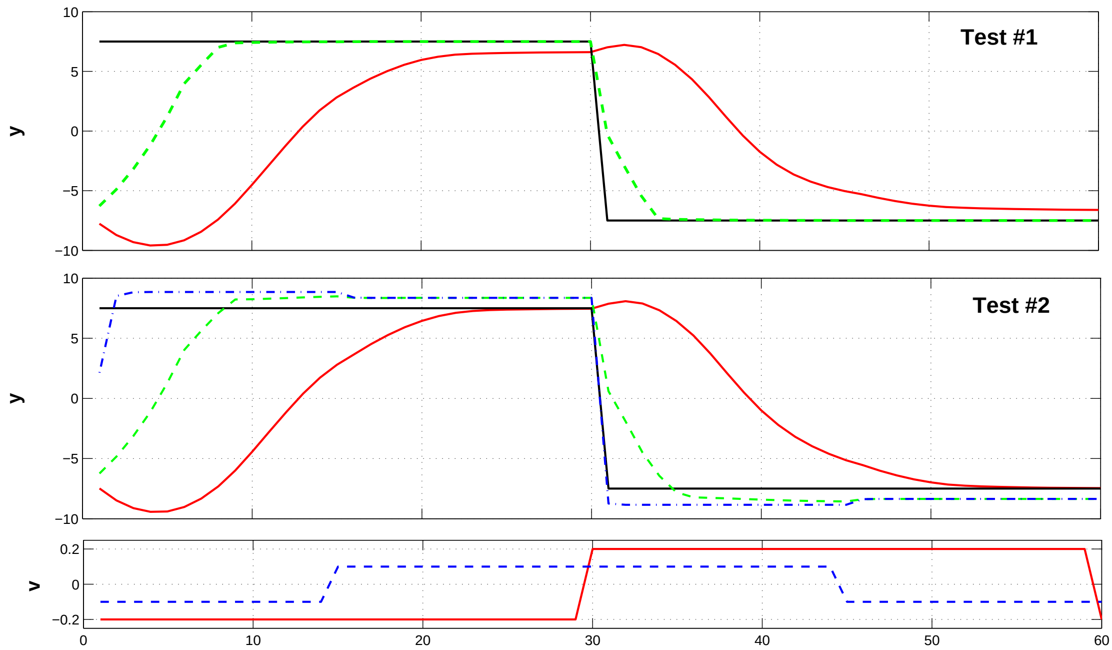


Figure 5.2: Closed loop evolution of the proposed controller.

and feedback control to counteract the effect of the disturbances. It has been probed that the invariant set Φ_{ec} is robustly exponentially stable for the resultant controlled observer and that the observer state converges robustly to this set exponentially. This implies that the state of the original system converges robustly and exponentially to the set $x_s \oplus \Phi$ while satisfying the control and state constraints. The output feedback model predictive controller for tracking requires online solution of a quadratic program. Moreover, under the constant disturbance rejection scheme, the steady state error can be removed under this class of disturbances. If the disturbances are characterized by a noise signal the mean value can be rejected just adding an appropriate filter.

Chapter 6

Synthesis of the proposed controller

6.1 Introduction

The proposed predictive controllers allow to steer an uncertain system to a neighborhood of a desired setpoint or output target fulfilling the constraints along its evolution. These controllers have some parameters to be tuned in the design stage. Some of these parameters are standard in model predictive controllers such as the prediction horizon N, and the stage cost weighting matrices Q and R; however, there also exist another parameters which add extra degree of freedom to the proposed controller that can be exploited according to the control objectives. This allows us to deal with (i) Disturbance rejection, (ii) Enlargement of the domain of attraction (iii) Output tracking prioritization when the set point is not admissible or reachable equilibrium point.

In this chapter, the effect of each parameter of the proposed controllers is analyzed and design techniques are presented. Firstly, in case that there exist outputs with offset, this can be minimized in some selected outputs by means of the offset cost function. On the other hand, the robust local controller is designed to minimize the effect of the disturbances on the state evolution and hence, to enlarge de domain of attraction of the MPC. The control gain is obtained by solving some LMIs.

Finally, existing results on the computation of an approximation to the minimal robust positively invariant set are specialized for disturbances that can be represented as a zonotope (an affine mapping of a unitary box). Practical procedures for the calculation of the robust invariant set are also presented.

6.2 Synthesis of the proposed controller

The MPC for tracking presented in the chapter 4 has to satisfy the conditions introduced in Assumption 4.2. The parameters that are not determined by the conditions introduced in this assumption can be chosen according to other control objectives such as closed-loop performance, disturbance rejection, domain of attraction, etc.

Matrices Q and R defines the performance index that the controller optimizes. The prediction horizon N is typically chosen as large as possible, since a large N provides a larger domain of attraction and, in general, a better closed-loop performance. However, the dimension of the MPC optimization problem grows with N, and hence larger prediction horizons require larger computation time. Therefore, a tradeoff must be achieved. In this section, we present a design method to obtain the matrices T, K, $\bar{K}$, P, the minimal robust invariant set ϕ_K and the invariant set for tracking $\Omega^a_{t,\bar{K}}$.

6.2.1 The offset cost weighting matrix T

This matrix defines the offset cost $\|\bar{\theta} - \theta\|_T^2$ in the cost function [1]. According to the properties of the proposed controllers, the effects of this parameter on the closed-loop system are the following :

- *Transient dynamics and setpoint filtering*: if matrix T is chosen to penalize more heavily the offset cost, then the convergence of $\bar{\theta}$ to θ is faster. This implies that the closed-loop system converges to the target setpoint in a shorter time.
- *Local optimality*: it is well known that MPC can be designed to be locally infinite horizon optimal (for the nominal system), however, the addition of the artificial steady state and input makes that this property does not hold in the proposed

[1] This formulation corresponds to the case when target operation point is an equilibrium point of the linear prediction model. If this is not the case, the function $\|\bar{z}_s - z_t\|_H^2$ has been proposed as offset cost function, which is defined by the matrix H. The results presented in this section are directly extended to the latter case.

controller. The optimality loss can be arbitrarily reduced by picking a large enough matrix T.

- *Offset minimization*: This issue is one of the most interesting properties of the proposed controller. In the case that the target operation point does not fulfil the constraints the closed-loop system will present offset in the outputs, that is, the system can not be driven to the operation point but to an admissible steady state and input. This is the admissible equilibrium point which minimizes the offset cost function, that is, $(x(k), u(k))$ tends to $(\bar{x}_s^*, \bar{u}_s^*) = M_\theta \tilde{\theta}$ such that

$$\tilde{\theta} = arg \quad \min_{\bar{\theta}} \|\bar{\theta} - \theta\|_T^2 \quad M_\theta \bar{\theta} \in \Omega_{t,\bar{K}} \times \bar{\mathcal{U}}$$

  Therefore, matrix T allows us to prioritize some of the outputs (by weighting more heavily its corresponding term in matrix T) to achieve a minimum offset on these outputs. See that this prioritization is not affected by the scaling of matrix T, and hence the effects previously presented can be simultaneously considered in the design of T.

According to this analysis the following design procedure is proposed: Firstly, the structure of matrix T must be fixed according to the offset minimization in case of offset. Then this matrix is scaled to achieve a quick transient with a small enough optimality loss. Finally it is worth remarking that, since the value of T is independent of the rest of the parameters, it can be tuned online without affecting the stabilizing properties of the controller.

6.2.2 The matrices $\bar{K}$, P and the set $\Omega_{t,\bar{K}}^a$

The terminal cost and terminal set strongly depend on the control gain $\bar{K}$. This control gain can be designed according to the following issues:

(i) From a performance point of view, it is desirable that the terminal cost is taken as the cost-to-go, which can be ensured if $\bar{K}$ is chosen as the LQ regulator gain

(ii) From a domain of attraction viewpoint, it is desirable to chose $\bar{K}$ to achieve a large terminal set $\Omega_{t,\bar{K}}^a$.

Since the size of the maximal invariant set for tracking is potentially large, the domain of attraction $\mathcal{X}_N$ of the proposed controllers is usually large (even for small values of the prediction horizon). This makes sensible to prioritize the first issue. Then $\bar{K}$ is proposed to be chosen as the linear quadratic regulator and the matrix P as the corresponding solution of the Riccatti equation.

Once the gain matrix $\bar{K}$ is fixed, the invariant set for tracking $\Omega^a_{t,\bar{K}}$ can be taken as a polyhedral set which approximates arbitrarily well to maximal invariant set for tracking.

6.2.3 The matrix K

This parameter has an important role in the proposed robust controllers. This control gain is used to compensate the deviation from the nominal predictions in case of disturbances by means of the control law ($u(i) = K(x(i) - \bar{x}(i)) + \bar{u}(i)$) (4.4). Therefore, this characterizes the dynamics of the closed-loop system in the presence of disturbances. Consequently, matrix K must be designed according to a robustness or disturbance rejection criterium.

In this thesis, we consider as robustness criterium the minimization of the size of the minimum admissible robust positively invariant set ϕ_K. This objective cope with the disturbance rejection problem, since the size of the minimum RPI set is a measure of the amplitude (with respect to the steady state and input) of the trajectory of the closed-loop system in presence of disturbances. On the other hand, a minimization of ϕ_K leads to achieve a larger domain of attraction, since the sets $\mathcal{X} \ominus \phi_K$ and $\mathcal{U} \ominus K\phi_K$ are larger. Thus, K is chosen to: (i) ensure the existence of an admissible robust positively invariant set ϕ_K such that the sets $X \ominus \phi_K$ and $U \ominus K\phi_K$ are not empty sets; (ii) minimize the size of ϕ_K.

Consider that $\mathcal{X} = \{x : |h_i^\top x| \leq 1,\ i = 1, \cdots, n_{rx}\}$ and $\mathcal{U} = \{u : |\ell_j^\top u| \leq 1,\ j = 1, \cdots, n_{ru}\}$. Then, the synthesis problem to solve is to calculate the control law $u = Kx$ such that the size of the ellipsoid $\mathcal{E}(P, 1) = \{\ x \in \mathbb{R}^n\ :\ x^\top P x \leq 1\ \}$ is minimized while guarantee that:

(i) $\mathcal{E}(P, 1)$ is a robust invariant set for system (4.5). This condition can be reformulated as follows:

$$(x^+)^\top P x^+ \leq 1,\ \ \forall x \in \mathcal{E}(P, 1),\ \forall w \in \mathcal{W} \tag{6.1}$$

Applying the S-procedure, and taking into account that x^+ is convex with respect to w, equation (6.1) is satisfied if exists $\lambda \geq 0$ such that:

$$\begin{array}{ll} (A_K x + w)^\top \;\; P \;\; (A_K x + w) + \lambda(1 - x^\top P x) < 1 & \\ A_K = A + BK \qquad\qquad \forall x \in \mathbb{R}^n,\ \forall w \in vert(\mathcal{W}) & (6.2) \end{array}$$

where $vert(\mathcal{W})$ denotes the set of vertexes of $\mathcal{W}$. This constraint can be rewritten as:

$$\begin{bmatrix} \lambda P - (A+BK)^\top P(A+BK) & -(A+BK)^\top P w \\ -w^\top P(A+BK) & 1 - \lambda - w^\top P w \end{bmatrix} > 0 \quad \forall w \in vert(\mathcal{W})$$

(ii) For all $x \in \mathcal{E}(P,1)$, $|\ell_j^\top K x| \leq \rho_j$ for all $j = 1, \cdots, n_{ru}$ and $\rho_j \in (0,1]$. The role of the parameter ρ_j is to restrict the set of admissible control inputs to guarantee a given control range of the the MPC controller i.e. the set $\bar{\mathcal{U}} = \mathcal{U} \ominus K\phi_K$ is not empty. This condition can be posed as:

$$\ell_j^\top K P^{-1} K^\top \ell_j \leq \rho_j^2,\ j = 1, \ldots, n_{ru}$$

Applying the Schur complement, we obtain the following inequality:

$$\begin{bmatrix} \rho_j^2 & \ell_j^\top K \\ K^\top \ell_j & P \end{bmatrix} > 0,\ j = 1, \ldots, n_{ru}$$

(iii) For all $x \in \mathcal{E}(P,1)$, $|h_i^\top x| \leq 1$. Considering similar arguments to the previous fact, this condition is equivalent to

$$\begin{bmatrix} 1 & h_i^\top \\ h_i & P \end{bmatrix} > 0,\ i = 1, \ldots, n_{rx}$$

In order to minimize the size of the ellipsoid $\mathcal{E}(P,1)$, a suitable measure of this set must be chosen. In this paper, we propose as measure a parameter $\gamma > 0$ such that $\mathcal{E}(P,1) \subseteq \sqrt{\gamma}\mathcal{X}$. Therefore, minimize the size of $\mathcal{E}(P,1)$ is posed as minimizing the parameter γ. Obviously, admissibility of the solution requires that $\gamma \leq 1$.

Applying standard operations of LMIs (Boyd, Ghaoui, Feron and Balakrishnan, 1994), the proposed synthesis procedure can be formulated as the solution of the following convex optimization problem:

$$
\begin{aligned}
&\min_{Y,W,\gamma} \gamma \\
&\quad s.a. \\
&\qquad \begin{bmatrix} \lambda W & * & * \\ 0 & 1-\lambda & * \\ AW+BY & w & W \end{bmatrix} > 0, \quad \forall w \in vert(\mathcal{W}) \\
&\qquad \begin{bmatrix} \rho_i^2 & * \\ Y^\top \ell_i & W \end{bmatrix} > 0, \; i = 1, \ldots, n_{ru} \\
&\qquad \begin{bmatrix} \gamma & * \\ Wh_i & W \end{bmatrix} > 0, \; i = 1, \ldots, n_{rx}
\end{aligned}
$$

for a given $\lambda \geq 0$. If feasible, the ellipsoid is given by $P = W^{-1}$ and the control gain is $K = YW^{-1}$. It is worth remarking that any robust criterium that can be posed as LMIs can be added to this synthesis problem.

6.2.4 Calculation of the robust invariant set ϕ_K

Once the control gain K is designed, an admissible robust positively invariant (RPI) set (as small as possible) must be calculated. Note that the proposed synthesis method of K guarantees the existence of this set. It would be desirable to compute the minimum robust positively invariant set (mPRI) F_∞ (Kolmanovsky and Gilbert, 1998), given by

$$F_s = \bigoplus_{k=0}^{s-1} (A+BK)^k \mathcal{W}$$

when s tends to infinity. Unfortunately, F_∞ can only be calculated for some special cases, such as when a dead-beat control law is used.

In the recent paper (Rakovic et al., 2005), a procedure for the determination of an invariant set that approximates F_∞ is presented. This allows one to compute a RPI ϕ_K such that $F_\infty \subseteq \phi_K \subseteq F_\infty \oplus \epsilon\mathcal{B}^n$ for a given bound of the absolute error ϵ. To this end, the following functions are calculated

$$\begin{aligned} \alpha(s) &= \min_{\alpha}\{\alpha : (A+BK)^s\mathcal{W} \subseteq \alpha\mathcal{W}\} \\ \beta(s) &= \min_{\beta}\{\beta : F_s \subseteq \beta\mathcal{B}^n\} \end{aligned}$$

These values are calculated by solving a number of linear programming problems. If a large enough value of s is taken such that $(1-\alpha(s))^{-1}\alpha(s)\beta(s) \leq \epsilon$ then the set $\phi_K = (1-\alpha(s))^{-1}F_s$ is an approximation of F_∞ with an error less than ϵ (Rakovic et al., 2005).

A problem that arises in the calculation of the RPI for a given absolute error bound ϵ is that this parameter must be chosen a priori, without knowing a measure of the size of the mRPI. This may cause a wrong selection of this parameter. Because of this, in this chapter we consider a relative error bound of the approximation to F_∞ instead. This is given by a $\lambda \in (0,1)$ such that

$$F_\infty \subseteq \phi_K \subseteq (1+\lambda)F_\infty$$

In the following lemma, the conditions to ensure this relative bound are presented.

Lemma 6.1 *Let s be a positive real number such that*

$$\alpha(s) \leq \frac{\lambda}{1+\lambda}$$

for a given relative error bound $\lambda \in (0,1)$. Then the set $\phi_K = (1-\alpha(s))^{-1}F_s$ is a RPI such that

$$F_\infty \subseteq \phi_K \subseteq (1+\lambda)F_\infty$$

Proof: From the definition of ϕ_K we can derive that

$$\frac{1}{1-\alpha(s)}F_s = F_s \oplus \frac{\alpha(s)}{1-\alpha(s)}F_s$$

Taking a λ such that

$$\frac{\alpha(s)}{1-\alpha(s)} \leq \lambda$$

we have that

$$F_\infty \subset \phi_K \subset F_s \oplus \lambda F_s = (1+\lambda)F_s \subseteq (1+\lambda)F_\infty$$

and the lemma is proved.

■

On the other hand, we specialize this result for a class of additive uncertainty bound set $\mathcal{W}$ widely used in the plant modelling. This is the case when the uncertainty set is a parallelope, i.e. an affine mapping of an unitary box given by

$$\mathcal{W} = H\mathcal{B}^n \oplus w_0$$

where $H \in \mathbb{R}^{n\times n}$ is non singular. This is motivated because, in practice, this class of set is frequently used to bound additive uncertainties.
For this class of set $\mathcal{W}$, the results presented in (Rakovic et al., 2005) can be specialized obtaining a simpler procedure avoiding the solution of LPs. Firstly, a preliminary useful lemma is presented.

Lemma 6.2 *Consider the sets* $Z_1 = F\mathcal{B}^{N_1} \oplus f$, $Z_2 = G\mathcal{B}^{N_2} \oplus g$ *and* $Z_3 = H\mathcal{B}^{N_3}$ *contained in* $\mathbb{R}^n$. *Then*

(i) $Z_1 \oplus Z_2 = [F\,G]\mathcal{B}^{N_1+N_2} \oplus (f+g)$

(ii) The minimum γ *such that* $Z_3 \subseteq \gamma\mathcal{B}^n$ *is given by* $\gamma = \|H\|_\infty$

(iii) Let H_i *denote the i-th row of matrix* H *and let* $D(H)$ *be a diagonal matrix such that* $D_{ii} = \|H_i\|_1$. *Then*

$$Z_3 = H\mathcal{B}^{N_3} \subseteq D(H)\mathcal{B}^n$$

(iv) Let the singular value descomposition of matrix H *be given by* (U, S, V), *i.e.* $H = USV^\top$. *Then*

$$Z_3 = H\mathcal{B}^{N_3} \subseteq USD(V^\top)\mathcal{B}^n$$

Proof: The first claim is immediate and the second claim yields directly from the definition of the induced norm of a matrix.

The third statement holds if for all $x = Hb_1$ with $b_1 \in \mathcal{B}^{N_3}$, there exists a $b_2 \in \mathcal{B}^n$ such that $Hb_1 = D(H)b_2$. Since $D(H)$ is non-singular we have that $b_2 = D(H)^{-1}Hb_1$. Since for a given matrix H, $\|H\|_\infty = max_i(\|H_i\|_1)$ we have that $\|D(H)^{-1}H\|_\infty = 1$. This yields

$$\|b_2\|_\infty = \|D(H)^{-1}Hb_1\|_\infty \leq \|D(H)^{-1}H\|_\infty \|b_1\|_\infty = \|b_1\|_\infty \leq 1$$

The fourth statement stems directly from the third one.

■

Lemma 6.3 *Consider a set* $\mathcal{W} = H\mathcal{B}^n \oplus w_0$ *where* H *is a non singular matrix. Define the matrix*

$$H_z(s) = [(A + BK)^{s-1}H, (A + BK)^{s-2}H, \cdots, H] \tag{6.3}$$

Then

$$\begin{aligned} \|H^{-1}(A + BK)^s H\|_\infty &= \min_\alpha \{\alpha : (A + BK)^s H\mathcal{B}^n \subseteq \alpha H\mathcal{B}^n\} \\ \|H_z(s)\|_\infty &= \min_\beta \{\beta : \bigoplus_{k=0}^{s-1} (A + BK)^k H\mathcal{B}^n \subseteq \beta\mathcal{B}^n\} \end{aligned}$$

Proof: The first claim is proved if for all $b1 \in \mathcal{B}^n$ there exist a $b_2 \in \mathcal{B}^n$ such that $(A + BK)^s Hb_1 = \alpha Hb_2$. Given that $b_2 = \frac{1}{\alpha}H^{-1}(A + BK)^s Hb_1$, we have that

$$\|b_2\|_\infty = \|\frac{1}{\alpha}H^{-1}(A + BK)^s Hb_1\|_\infty \leq \frac{1}{\alpha}\|H^{-1}(A + BK)^s H\|_\infty \|b_1\|_\infty \leq 1$$

The optimality of the proposed bound stems from the definition of the induced norm of a matrix.

The second claim proof is based on lemma 6.2. From the statement (i) it is immediate to see that

$$\bigoplus_{k=0}^{s-1} (A + BK)^k H\mathcal{B}^n = H_z(s)\mathcal{B}^{sn}$$

Applying the statement (ii), the result is proved. ■

Based on this lemma, an approximation to the mRPI is proposed in the following lemma.

Lemma 6.4 *Let s be such that*

$$\|H^{-1}(A+BK)^s H\|_\infty \leq \frac{\lambda}{1+\lambda}$$

for a given relative error bound $\lambda \in (0,1)$ and denote $\hat{\alpha}(s) = \|H^{-1}(A+BK)^s H\|_\infty$. Then the zonotope ϕ_K given by

$$\phi_K = (1-\hat{\alpha}(s))^{-1} H_z(s)\mathcal{B}^{sn} \oplus (I_n - (A+BK))^{-1} w_0$$

is a RPI such that $F_\infty \subseteq \phi_K \subseteq (1+\lambda)F_\infty$.

Proof: Firstly, it is proved that ϕ_K is a RPI, that is $(A+BK)\phi_K \oplus \mathcal{W} \subseteq \phi_K$. In order to simplify the exposition, define $(A+BK)$ as A_K, $(I_n-(A+BK))^{-1}$ as K_e and $\hat{\alpha}(s)$ as $\hat{\alpha}$.

$$\begin{aligned}
A_K\phi_K \oplus \mathcal{W} &= A_K((1-\alpha)^{-1}H_z(s)\mathcal{B}^{sn} \oplus K_e w_0) \oplus H\mathcal{B}^n \oplus w_0 \\
&= (1-\alpha)^{-1}[A_K^s H\mathcal{B}^n \oplus A_K H_z(s-1)\mathcal{B}^{(s-1)n}] \oplus A_K K_e w_0 \oplus H\mathcal{B}^n \oplus w_0 \\
&\subseteq (1-\alpha)^{-1}[\alpha H\mathcal{B}^n \oplus A_K H_z(s-1)\mathcal{B}^{(s-1)n}] \oplus A_K K_e w_0 \oplus H\mathcal{B}^n \oplus w_0 \\
&= (1-\alpha)^{-1}A_K H_z(s-1)\mathcal{B}^{(s-1)n} \oplus \frac{\alpha}{1-\alpha}H\mathcal{B}^n \oplus H\mathcal{B}^n \oplus (A_K K_e + I_n)w_0 \\
&= (1-\alpha)^{-1}A_K H_z(s-1)\mathcal{B}^{(s-1)n} \oplus (1-\alpha)^{-1}H\mathcal{B}^n \oplus (A_K K_e + I_n)w_0 \\
&= (1-\alpha)^{-1}[A_K H_z(s-1)\mathcal{B}^{(s-1)n} \oplus H\mathcal{B}^n] \oplus (A_K K_e + I_n)w_0
\end{aligned}$$

Given that $A_K H_z(s-1)\mathcal{B}^{(s-1)n} \oplus H\mathcal{B}^n = H_z(s)\mathcal{B}^{sn}$ and $A_K(I_n - A_K)^{-1} + I_n = (I_n - A_K)^{-1}$, we have that

$$(1-\alpha)^{-1}[A_K H_z(s-1)\mathcal{B}^{(s-1)n} \oplus H\mathcal{B}^n] \oplus (A_K K_e + I_n)w_0 = (1-\alpha)^{-1}H_z(s)\mathcal{B}^{sn} \oplus K_e w_0 = \phi_K$$

and consequently, $A_K\phi_K \oplus \mathcal{W} \subseteq \phi_K$, that is ϕ_K is a RPI.

Secondly, the property $F_\infty \subseteq \phi_K$ is proved

$$\begin{aligned}
F_\infty &= \bigoplus_{i=0}^{\infty} A_K^i \mathcal{W} = \bigoplus_{i=0}^{\infty} A_K^i (H\mathcal{B}^n \oplus w_0) \\
&= \bigoplus_{i=0}^{\infty} A_K^i H\mathcal{B}^n \oplus \sum_{j=0}^{\infty} A_K^j w_0
\end{aligned}$$

$$
\begin{aligned}
&= \bigoplus_{i=0}^{\infty} A_K^i H \mathcal{B}^n \oplus (I_n - A_K)^{-1} w_0 \\
&\subseteq (1-\hat{\alpha})^{-1} \bigoplus_{i=0}^{s-1} A_K^i H \mathcal{B}^n \oplus (I_n - A_K)^{-1} w_0 \\
&= (1-\hat{\alpha})^{-1} H_z(s) \mathcal{B}^{sn} \oplus (I_n - A_K)^{-1} w_0 \\
&= \phi_K
\end{aligned}
$$

The proof is completed by demonstrating that $\phi_K \subseteq (1+\lambda)F_\infty$.

$$
\begin{aligned}
\phi_K &= (1-\hat{\alpha})^{-1} H_z(s) \mathcal{B}^{sn} \oplus (I_n - A_K)^{-1} w_0 \\
&\subseteq (1+\lambda) H_z(s) \mathcal{B}^{sn} \oplus (I_n - A_K)^{-1} w_0 \\
&\subseteq (1+\lambda) [H_z(s) \mathcal{B}^{sn} \oplus (I_n - A_K)^{-1} w_0] \\
&= (1+\lambda) \left[\bigoplus_{i=0}^{s-1} A_K^i H \mathcal{B}^n \oplus (I_n - A_K)^{-1} w_0 \right] \\
&\subseteq (1+\lambda) \left[\bigoplus_{i=0}^{\infty} A_K^i H \mathcal{B}^n \oplus (I_n - A_K)^{-1} w_0 \right] \\
&= (1+\lambda) F_\infty
\end{aligned}
$$

■

6.2.4.1 Using the obtained ϕ_K on the optimization problem

From the previously presented results, a RPI set is characterized as a zonotope of dimension sn given by matrix $H_z(s)$, the center $(I_n - (A+BK))^{-1} w_0$. The zonotope expression of the invariant set makes easier the calculation of the linear mapping of ϕ_K and Pontryagin difference of a polytope with ϕ_K and consequently the calculation of the polytopes $\mathcal{X} \ominus \phi_K$ and $\mathcal{U} \ominus K\phi_K$. This property is proved in the following lemma

Lemma 6.5 *Consider* $X = \{x \in \mathbb{R}^n : Fx \leq f\}$ *and* $Y = H\mathcal{B}^M \oplus y_0$, *then the set* $Z = X \ominus Y$ *is given by:*

$$Z = \{z \in \mathbb{R}^n : Fz \leq f - Fy_0 - g\}$$

where the vector g *is such that* $g_i = \|F_i H\|_1$ *and* F_i *is the* i*-th row of* F.

Proof: $z \in Z$ if $x = z + y \in X$ for all $y \in Y$. This can be posed as $F(z+y) \leq f$ for all $y \in Y$. From the definition of set Y, this equals to $Fz + FHb + Fy_0 \leq f$ for all $b \in \mathcal{B}^M$. Since the worst value of $b \in \mathcal{B}^M$ for each linear inequality is $\max_{b \in \mathcal{B}^M} F_i Hb = \|F_i H\|_1$, the lemma is proved. ■

However, constraint on the nominal initial state $\bar{x} \in x \oplus (-\Phi_K)$ (being x the current state) requires the calculation of the set of inequalities (hyperplanes) that define ϕ_K, that is, the $\mathcal{H}$-representation of ϕ_K. A zonotope is a compact expression of a polytope with a number of facets that grows exponentially with the dimension of the unitary box (sn). Although there exists specialized algorithms to obtain the facets, these are only tractable for a reduced dimension of the zonotope. In order to calculate the RPI set as small as possible while tractable, the following procedures are proposed:

- This method is based on the availability of a procedure to calculate the $\mathcal{H}$-representation of a given zonotope. This procedure is assumed to be time-limited, that is, if the calculation time exceeds a given limit, the calculation is considered as not-affordable.
  Thus, an iterative procedure is used to compute the largest value of s such that the $\mathcal{H}$-representation of the zonotope $H_z(s)\mathcal{B}^{sn}$ is affordable. Then, the RPI set ϕ_K is calculated.

  Define a polytope $\Gamma_i = \{x \in \mathbb{R}^n : G_i x \leq g_i\}$. Then the following procedure to calculate the largest affordable value of s is proposed:

  1. Make $s = 1$ and $\Gamma_1 = \mathcal{W}$, where the defining matrix of the $\mathcal{H}$-representation are given by
  $$G_1 = \begin{bmatrix} H^{-1} \\ \\ -H^{-1} \end{bmatrix} \qquad g_1 = \mathbf{1} - G_1 w_0$$
  2. Make $\Gamma_{s+1} = \Gamma_s \oplus A^s H \mathcal{B}^n$ and calculate its $\mathcal{H}$-representation characterized by G_s and g_s.
  3. If the calculation finishes successfully, then make $s = s + 1$ and iterate to (2).
  4. If the calculation is not-affordable, then stop.

  Then, in virtue of lemma 6.4 the RPI ϕ_K, expressed in its $\mathcal{H}$-representation, is as follwos:
  $$\phi_K = \{G_{s-1}x \leq (1-\hat{\alpha})^{-1} g_{s-1} + G_{s-1}(I_n - (A+BK))^{-1} w_0\}$$

where $\hat{\alpha} = \|H^{-1}(A+BK)^{s-1}H\|_\infty$. This RPI set ensures that

$$F_\infty \subset \phi_K \subset (1-\alpha(s))^{-1}F_\infty$$

Since a larger value of s results in a more accurate approximation to the mRPI F_∞, the obtained RPI is the most accurate affordable approximation to F_∞.

- The previously presented method may lead to RPI sets with a large number of inequalities which is not desirable for the solution of the optimization problem. On the other hand, depending on the dimension of the problem and the affordable time, the largest affordable value of s may be not sufficient for a given error bound on the approximation.

  In order to cope with these problems, a second procedure based on the calculation of the maximal RPI contained in a given set is proposed. The maximal RPI contained in a set can be calculated in a finite number of iterations (Blanchini, 1999) (typically tractable).[2] Since the complexity of the bounding polytope is low, this method will probably provide simpler RPI sets.

  To this aim, a polytope containing F_∞ has to be calculated; this can be done by several methods. One method consists in choose a sufficiently large value of s and compute a low complexity polytopic hull of the zonotope $(1-\hat{\alpha}(s))^{-1}H_z(s)\mathcal{B}^{sn}$ where $\hat{\alpha}(s) = \|H^{-1}(A+BK)^s H\|_\infty$. This can be done by means of the methods proposed in lemma 6.2. All of these outer approximations can be intersected to derive a tighter approximation. Once this is obtained, the maximal RPI set contained in this set is calculated. Assume that the polytopic hull $\Upsilon = \{x \in \mathbb{R}^n : Hx \leq h\}$ is computed satisfying that $H_z(s)\mathcal{B}^{sn} \subset \Upsilon \subset (1+\lambda)H_z(s)\mathcal{B}^{sn}$. Then, in the following well-known iterative procedure a set given by $\Psi_j = \{x \in \mathbb{R}^n : P_j x \leq p_j\}$ is calculated yielding a RPI contained in Υ.

  1. Make $\Psi_0 = \Upsilon$ and $j = 0$.
  2. Calculate $\Psi_{j+1} = \Psi_j \cap (A+BK)^{-1}(\Psi_j \ominus H\mathcal{B}^n)$, yielding

  $$P_{j+1} = \begin{bmatrix} P_j \\ P_j(A+BK) \end{bmatrix}, \quad p_{j+1} = \begin{bmatrix} p_j \\ (p_j - \|P_j(A+BK)H\|_{\mathbf{1}}) \end{bmatrix}$$

  3. Remove redundant inequalities of the $\mathcal{H}$-representation of Ψ_j

[2] Any other procedure for the calculation of a RPI contained in a compact set could be used instead.

4. If $\Psi_{j+1} = \Psi_j$, then stop.
5. If not, make $j = j + 1$ and go to (2).

Once the algorithm is finished, the set $\phi_K = \Psi_j \oplus (I_n - (A + BK))^{-1} w_0$ is a RPI set such that $F_\infty \subset \phi_K \subset (1 + \lambda) F_\infty$. The invariance property is derived from the following lemma

Lemma 6.6 *Let $\hat{\Omega}$ a RPI for system $x^+ = Ax + w$ with $w \in \hat{W}$. Then $\Omega = \hat{\Omega} \oplus (I_n - A)^{-1} w_0$ is a RPI for all $w \in W$, with $W = \hat{W} \oplus w_0$.*

Proof: Since $\hat{\Omega}$ is a RPI for system $x^+ = Ax + w$ with $w \in \hat{W}$, we have that $A\hat{\Omega} \oplus \hat{W} \subseteq \hat{\Omega}$. Then

$$\begin{aligned} A\Omega \oplus W &= A\hat{\Omega} \oplus A(I_n - A)^{-1} w_0 \oplus \hat{W} \oplus w_0 \\ &= A\hat{\Omega} \oplus (A(I_n - A)^{-1} + I_n) w_0 \oplus \hat{W} \\ &= A\hat{\Omega} \oplus (I_n - A)^{-1} w_0 \oplus \hat{W} \\ &\subseteq \hat{\Omega} \oplus (I_n - A)^{-1} w_0 \\ &= \Omega \end{aligned}$$

■

6.3 Example

Consider a constrained sampled double integrator

$$\begin{aligned} x^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

where the disturbances are bounded in $w \in \mathcal{W} = 0.1\mathcal{B}^2$. The system must fulfil the following constraints: $|x_1| \leq 5$, $|x_2| \leq 5$, $|u_1| \leq 0.3$, $|u_2| \leq 0.3$.

The objective is to show the proposed procedures to find a suitable robust contoller gain K to ensure the admissibility of the tube-based predictive controller, i.e. ensuring that $\mathcal{X} \ominus \phi_K$ and $\mathcal{U} \ominus K\phi_K$ are not empty sets.

In order to demonstrate the proposed synthesis method, we design two matrices, $K1$ and $K2$. The first matrix is obtained as the LQR gain for $Q = I_2$ and $R = 10$. The second matrix $K2$ is designed using the proposed method with a value of $\rho = 0.48$ is considered in order to get the same set $K\phi_K$ as in the previous case. Figure 6.1 shows the approximated RPI for both gains. It can be seen that the proposed method provides a smaller set ϕ_K.

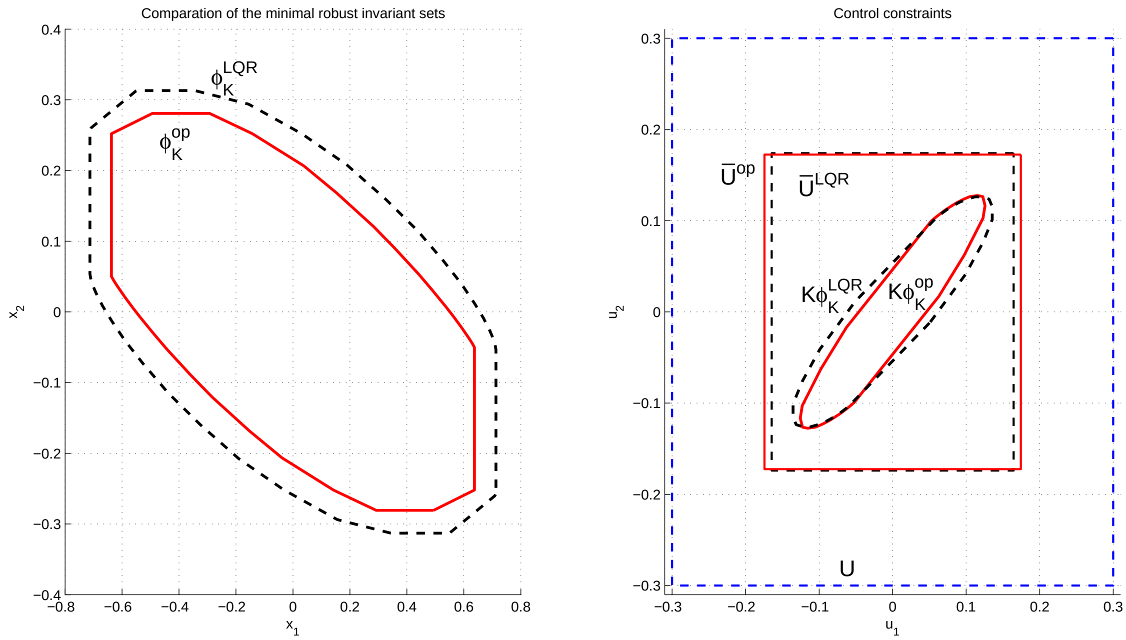


Figure 6.1: Comparative of the admissible minimal robust invariant sets

In order to demonstrate the role of the parameter ρ, let us try to calculate the minimum robust invariant set with the minimum size, for that, pick a $\rho = 1$ and consider a LQR gain with $Q = 1000 * I_2$ and $R = 1$. Figure 6.2 shows the RPI derived from LQR and the proposed method. It can be seen how the set $K\phi_K^{LQR}$ is not feasible due to violates the control constraints ($\bar{\mathcal{U}}^{LQR} = \mathcal{U} \ominus K\phi_K^{LQR}$ is an empty set). The proposed method provides a feasible RPI but, since $\rho = 1$, the set $\bar{\mathcal{U}}^{op} = \mathcal{U} \ominus K\phi_K^{op}$ is quite small, so smaller is the parameter ρ bigger is the robust minimal invariant set (worst disturbance rejection) but bigger is $\bar{\mathcal{U}}^{op}$ (The nominal system has a less restrictive control constraints, so the evolution of the nominal system is faster).

Thus, with the value of ρ it is possible to choose between disturbance rejection and performance, as bigger is ρ better disturbance rejection but worse is the performance.

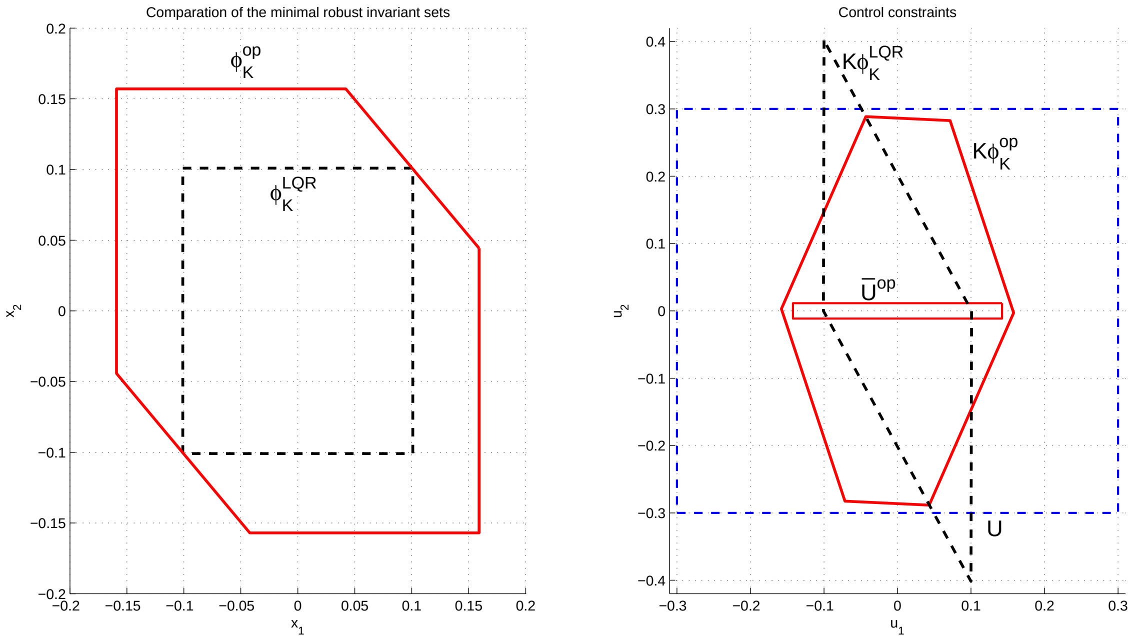


Figure 6.2: Comparative of the minimal robust invariant sets

6.4 Conclusions

This chapter deals with the design of the robust predictive controller for tracking (Limon et al., 2007*b*). This controller is a natural extension of predictive controllers for tracking. The proposed synthesis technique achieves the following objectives:

- Offset minimization
- Disturbance rejection by minimizing the RPI
- Guarantee of feasibility of the tube-based MPC and enlargement of the domain of attraction.

The first one depends on the offset cost weighting matrix T, which can be freely chosen (under a mild assumption) to penalize more heavily some outputs in order to minimize their offset. The last two objectives are achieved by means of a LMI which can be efficiently solved.

Finally, we present a method to estimate the RPI in the case that relative error bound is used, and the uncertainty is posed as a class of zonotopes. We also provide practical methods to derive reliable approximations in case the exact calculation is not affordable.

Chapter 7

Application to a linear motor

In this chapter we present experimental results of a positioning system driven by a linear motor in closed-loop with the MPC for tracking presented in Chapter 3 and 4. The experimental results demonstrate the properties of this control scheme.

7.1 Linear motors

The most extended class of electrical motors are motors that directly produce a rotatory movement of an axis. When the required movement is linear, is necessary to use systems that transform rotatory movement into linear movement. This is achieved with gears and straps among other elements. This implies that transforming rotatory into linear movement causes power and efficiency losses due to friction. Linear motors produce linear movement without using a mechanism to transform rotatory movement into linear.

The history of linear motors goes back to the last decade of the 19^{th} century. These mechanisms were forgotten during half a century to return in 1950. The principles in which the linear motor is based are the same as the rotatory motor, at least conceptually. A linear motor can be seen as a rotatory one that has been unrolled, see figure 7.1. As rotatory motors, linear motors are made of two parts, primary and secondary (figure 7.2). The primary consists of a set of copper coils where the intensity is applied to create an electromagnetic field. The number and length of the coils determine the

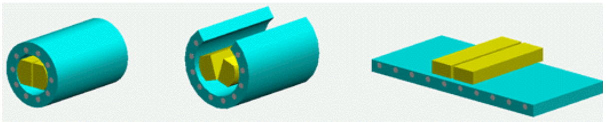


Figure 7.1: Equivalence between the linear and the conventional motor

force of the motor. The secondary is composed by a serie of permanent magnets that react to the electromagnetic force created by the primary, generating the movement. Both, the primary or the secondary can be the stationary part or the moving body.

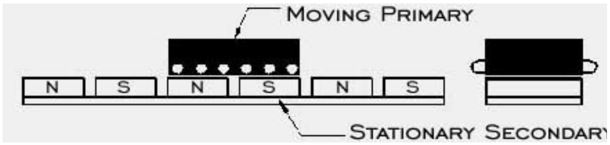


Figure 7.2: Primary and secondary

Linear motors can be:

- DC Motors.
- AC Motors.
- Steeping Motors.

In general, linear motors provide a better performance than rotatory motors in applications based on linear movements. This class of motors provide high measurement accuracy, fast response, high reliability, low maintenance and are not limited in length. Between the disadvantages of this class of motors are the high force of attraction between the primary and the secondary, the sensitivity to the temperature and the magnetic contamination produced.

7.1.1 Linear positioning system

The controlled plant is a positioning system driven by a linear motor with a length of $2,52$ meters. This plant plant has been built in the laboratory of the Institute of Automatics and Robotics of Seville to carry out real experiments with different control strategies. The system has the following elements:

Linear Motor: Linear motor Siemens 1FN3 050-2W00-0AA. This motor is an induction motor that provides, in nominal conditions, a force of 200 N, developing a maximal velocity of 373 m/min. The primary is the moving part of the motor and lodges the coils to create the magnetic field that interacts with the field created by the secondary (figure 7.4). Its dimensions are 254 × 67 × 47.7 mm. The secondary is composed by 21 permanent magnets with 4 pairs of poles each one. Its dimensions are 120 × 58 × 11.8 mm. The resulting length of the secondary is 2520 mm. A metallic base leans over the primary to carry an inverted pendulum.

Absolute position sensor: Optical sensor LC181 of Heidenhein (see the figure 7.4). This sensor provides the absolute position with a precision of $5\mu m$. The position is measured in an optical way from of seven graduated tracks, the first six tracks are used to measure the absolute position and seventh to measure the incremental position. Is important to note that this sensor can measure the speed correctly only if is smaller than 120 m/min.

Guides of bearings and skids: The guides are LAS20ALZ of NSK-RPH and its length are 2985 mm (figure 7.4).

Switchboard and control panel: The switchboard (figure 7.5) lodges the electrical protections, the power distribution, electrical security circuit and the unit of regulation SimoDrive 611 Universal of Siemens that implements a low level speed control loop based on a PI controller. In the front part of the switchboard the analogical and digital entrances and exits are accessible.

dSpace card in a PC: In the PC the high level control and the man-machine interface are implemented. The PC has a dSpace card DS1103 PPC, based on PowerPC processor 604e that works at 400 MHz. This processor is programmed in Simulink using *Real-Time Interface* and its own blocks library. Figure 7.3 shows the Simulink blocks distribution that allows the control of the plant.

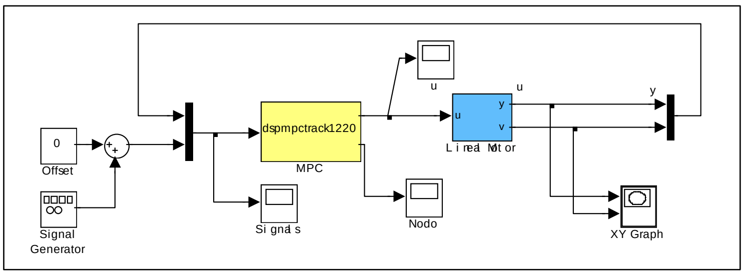


Figure 7.3: Simulink model.

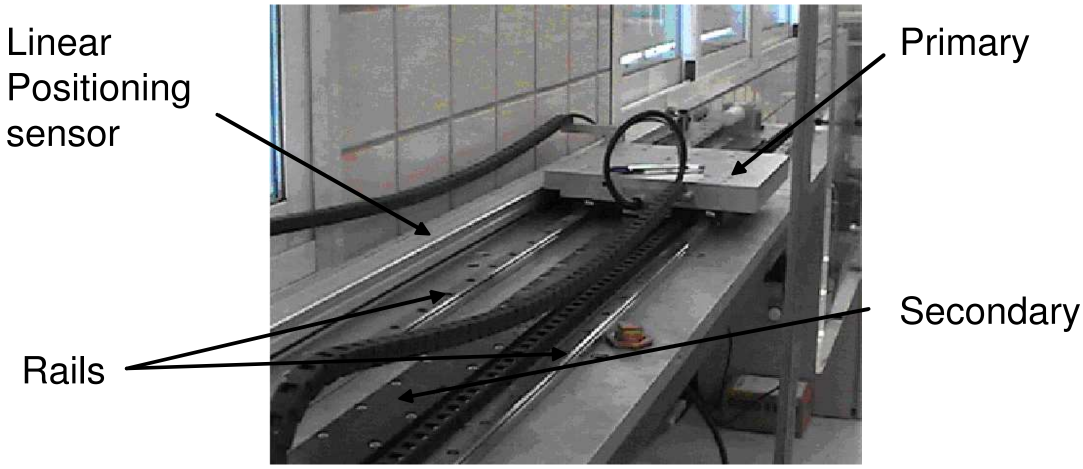


Figure 7.4: Parts of the linear motor.

The objective is to control the position of the linear motor actuating on the speed setpoint, the low level control is implemented in the Simodrive 611.

7.2 The linear motor controlled by MPC for tracking

In this section, the linear motor is controlled by MPC for tracking introduced in the chapter 3, for that it is necessary to obtain a discrete-time linear time-invariant state representation of the plant.

7.2.1 Plant identification

From the response of the system to PRBS (pseudo random binary sequence) input signal a linear discrete time model of the speed of the linear motor with respect to the input signal has been identified using least squares identification. We have chosen a sampling time of 10 ms (Note that the sampling time is very short and in general it is not possible to solve a QP problem online). This model results to be a first-order system with a delay of one sampling time. Considering that the position is the integral

Figure 7.5: Positioning System.

of the speed and the delay, the relation between the input signal and the position of the motor can be modeled by a third order system. The matrices identified are the following:

$$
\begin{aligned}
x^{+} &= \begin{bmatrix} 1 & 1.6667 \times 10^{-4} & 0 \\ 0 & 0.9686 & 5.1663 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u
\end{aligned}
$$

where the input is the voltage [V], the first state is the position $[m]$, the second state is the speed $[m/min]$ and the third state is the input at the previous time step $[V]$. It is important to remark that the whole state is measurable.

The real system has the following constraints in the input and the states:

$$U = \{u \in \mathbb{R} : |u| \leq 0.95\}, \quad X = \{x \in \mathbb{R}^3 : |x_1| \leq 0.8,\ |x_2| \leq 80,\ |x_3| \leq 0.95\}$$

See that the constraint on the states forces the system to be confined inside a security region to avoid the violating the physical limits of the linear motor.

7.2.2 Controller design

The linear model has been used to design the MPC for tracking. In this section we present the value of the different tuning parameters obtained by extensive simulations. The prediction horizon has been chosen as $N = 3$, the weighting matrices are $Q = C' \cdot C$, $R = 0.01$ and $T = 1$. These matrices have been chosen in order to obtain fast convergence of the closed-loop system to the desired steady state.

The local feedback controller K and the terminal cost P are the LQR optimal gain and the corresponding cost-to-go function obtained solving the Ricatti equation. For the computation of the invariant set for tracking, the parameter λ has been chosen as 0.95. Note that the limits os the set of reachable positions of the linear motor to the interval $[-0.76, 0.76]$.

In order to implement the MPC for tracking, the explicit solution of the corresponding optimization problem has been obtained using standard algorithms (Bemporad, Morari, Dua and Pistikopoulos, 2002; Bemporad, Borrelli and Morari, 2002). The resulting explicit MPC controller defined by 514 critical regions with a total of 4740 inequalities. In order to do the only region search to implement the explicit MPC controller a binary search tree has been computed as proposed in (Tøndel et al., 2002; ndel, Johansen and Bemporad, 2003). The corresponding search-tree is made of 417 nodes with a depth of 13. This implies that 12 inequalities must be checked to determine the critical region, instead of the 4740 inequalities required in the worst case if no search tree was computed off-line.

Figure shows 7.6 the region of attraction, the invariant set and the set of admissible steady states.

7.2.3 Controller implementation

In this section two experiments are shown. In experiment #1, a sequence of desired positions switching between -0.5 and 0.5 is tracked. Figures 7.7 and 7.8 show the state and input trajectories of the linear motor in closed-loop with the proposed MPC for tracking of Experiment #1. This experiment demonstrates that the the motor tracks successfully the references without violating any input or state constraints. Moreover, note that the results presented have been obtained from an experiment carried out using

Figure 7.6: Region of attraction $\mathcal{X}_3$, the invariant set for tracking Ω_K and the set of admissible steady states $\mathcal{X}_s$ of the system.

the real linear motor. This implies, that for this particular system, the proposed MPC for tracking is able to compensate for the model uncertainties and the measurement noise.

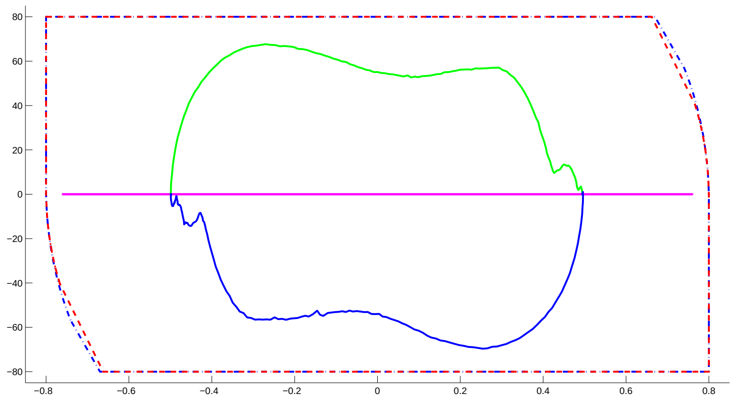


Figure 7.7: Space state evolution. Experiment #1.

In the second experiment, the role of the artificial reference in the proposed control algorithm is demonstrated. To this end, a reference of -1 m has been provided to the controller. It is impossible to reach this reference without violating the state constraints. Note that x=-1m is outside the set of admissible references $\mathcal{X}_s$. Figure 7.9 shows the state and input trajectories of the linear motor in closed-loop with the proposed MPC for tracking of Experiment #2. It can be seen how the controller lead

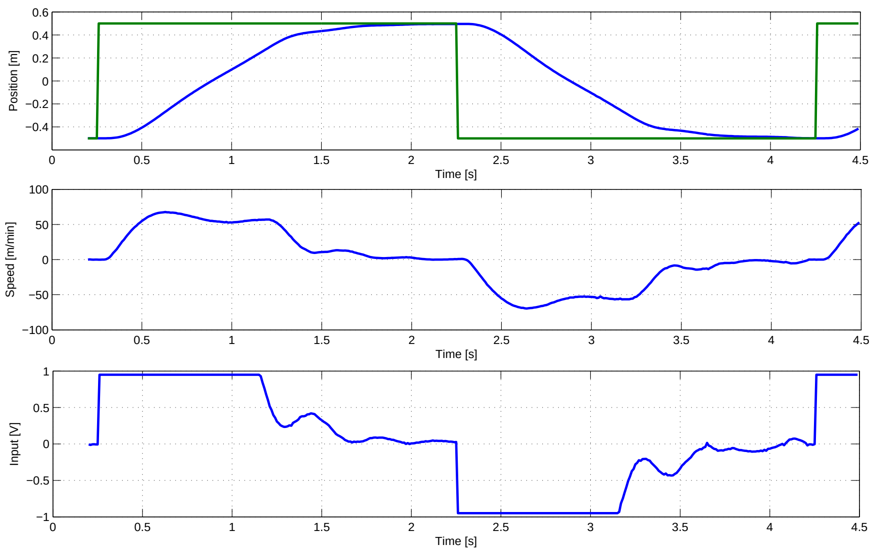


Figure 7.8: Evolution of the states and input. Experiment #1.

the system to the closest admissible setpoint which is -0.76.

Comparing the space state evolution of the Experiment #1 (figure 7.7) with the simulation of the same experiment (figure 7.10) it is clear that the model used for prediction in the MPC design differs from the real system model. Even being the MPC for tracking a robust controller, it is not possible to ensure that the controller is not going to fail due to this mismatch between the model and the real plant. In the following section, the robust formulation deals with task.

7.3 The linear motor controlled by robust MPC for tracking

In this section, the linear motor is controlled by robust MPC for tracking introduced in the chapter 4, for that it is necessary to obtain an uncertain discrete-time linear time-invariant state representation of the plant.

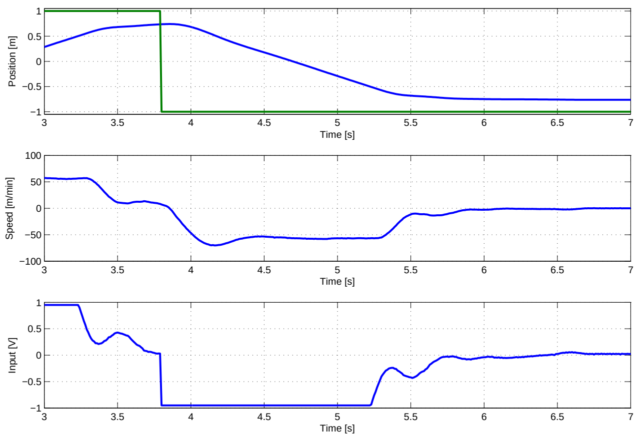


Figure 7.9: Evolution of the states and input. Experiment #2.

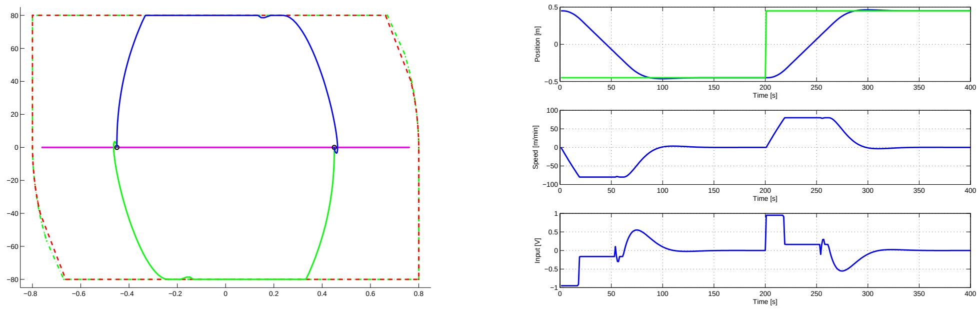


Figure 7.10: Simulation of the Experiment #1.

7.3.1 Plant identification

With the intention of identifying the plant and obtaining a linear model with bounded uncertainties, the plant was excited with a pseudo-random binary entrance (PRBS). The chosen sampling time was 30 milliseconds. This sample time is different from the sampling time used to control the system with the MPC for tracking, because with this sample time there is no delay, being appropriate for the plant, obtaining a simpler model.

The parameters of a several linear systems have been identified off-line from the identification experiment data using the minimum least square method. Different models have been identified in order to chose the one with a smaller set of disturbances $\mathcal{W}$. The best fitting was provided by a second order model, in which the two states are the position and the speed of the linear motor respectively. This model is defined by the following matrices:

$$
\begin{aligned}
x^+ &= \begin{bmatrix} 1.0000 & 0.0005 \\ 0.0000 & 0.8898 \end{bmatrix} x + \begin{bmatrix} 0.0036 \\ 15.4769 \end{bmatrix} u + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u
\end{aligned}
$$

The bound on the additive uncertainties has been evaluated off-line measuring the maximum deviation of the predicted state trajectories for each model from the real linear motor state trajectories for the same input signals. Thus, the following set has been obtained:

$$\mathcal{W} = \{w \in \mathbb{R}^2 : |w_1| \leq 0.0055, \ |w_2| \leq 10.3\}$$

The linear motor is subject to the following input and state constraints:

$$U = \{u \in \mathbb{R} : |u| \leq 0.95\}, \quad X = \{x \in \mathbb{R}^2 : |x_1| \leq 1, \ |x_2| \leq 80\}$$

Note that in this case, the security limits on the position are closer to the physical limits of the linear motor. The robust MPC for tracking allows us to work closer to the physical limits with guarantees because the model used to define the controller takes into account the model uncertainties and guarantees robust constraint satisfaction.

7.3.2 Controller design

The controller has been designed to guarantee that the conditions of Hypothesis(4.2) are satisfied. First the following stage cost matrices have been chosen:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = 0.01$$

The local controllers K and $\tilde{K}$ have been considered equal and have been determined by means of a LQR that stabilizes nominal model of the system. The obtained controllers are:

$$\bar{K} = K = [-7.6124 \;\; -0.0262]$$

The corresponding terminal cost matrix P is:

$$P = \begin{bmatrix} 8.732951 & 0.006480 \\ 0.006480 & 0.000027 \end{bmatrix}$$

The matrix that penalizes the difference between the artificial steady state and the desired one is:

$$T = 100P = \begin{bmatrix} 873.2951 & 0.6480 \\ 0.6480 & 0.0027 \end{bmatrix}$$

Finally, based on the previous parameters the regions Φ_K, $\bar{\mathcal{X}}_s$ and $\Omega_{\bar{K}}$ have been calculated . Figure 7.11 shows Φ_K, $\bar{\mathcal{X}}_s$, $\Omega_{\bar{K}}$, $\bar{\mathcal{X}}_3$ and $\mathcal{X}_3$.

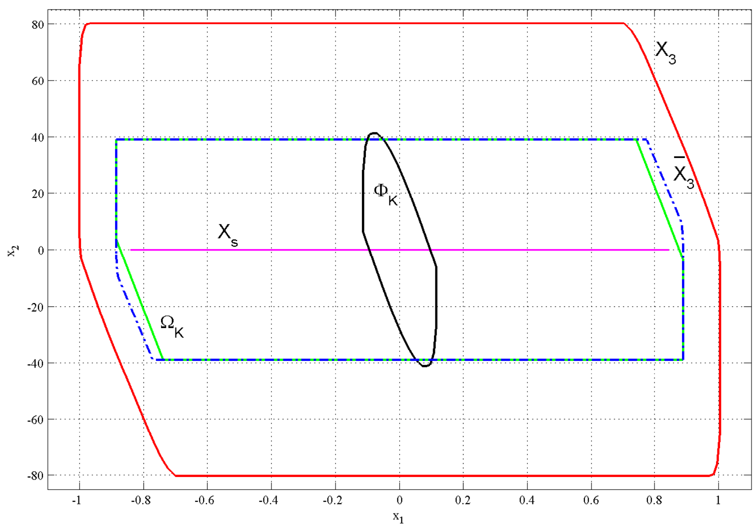


Figure 7.11: Φ_K, $\bar{\mathcal{X}}_s$, $\Omega_{\bar{K}}$, $\bar{\mathcal{X}}_3$ and $\mathcal{X}_3$.

The set of the admissible setpoint is $\mathcal{X}_s = \{x \in \mathbb{R}^2 : |x_1| \leq 0.84, \; x_2 = 0\}$. The control horizon has a value $N = 3$ the region of attraction associated to controller is $\mathcal{X}_3$ and the region of attraction for the nominal system is $\bar{\mathcal{X}}_3$. Therefore whenever the system leaves from a state in this region, it will evolve inside of it until it reach a neighborhood of the desired steady state.

7.3.3 Simulations results

In this section we present some simulations results to demonstrate that the robust MPC for tracking designed is appropriate to control the linear motor. These simulations have been carried out for complex scenarios characterized by initial states close to the feasibility region of the controller and extreme uncertainty trajectories. Two simulations have been carried out:

7.3.3.1 Simulation #1:

Starting from a very complicated initial state (the linear motor approaching the physical limits at high speed with a set point just in the other extreme), with the uncertainties pushing the system towards the boundary (at least in the first samples time), the MPC for tracking, thanks to the artificial steady state, leads the system to the desired steady state.

Figure 7.12 shows the state trajectory x_r and the nominal state trajectory x_n in the state space. Note the evolution of the artificial steady state to the desired steady state to guarantee the feasibility of the problem, and that all the constraints are satisfied despite of the disturbances. Figure 7.13 shows the state trajectory x and the nominal state trajectory $\bar{x}$ in the time space. It also possible to see the admissible trajectory of the inputs.

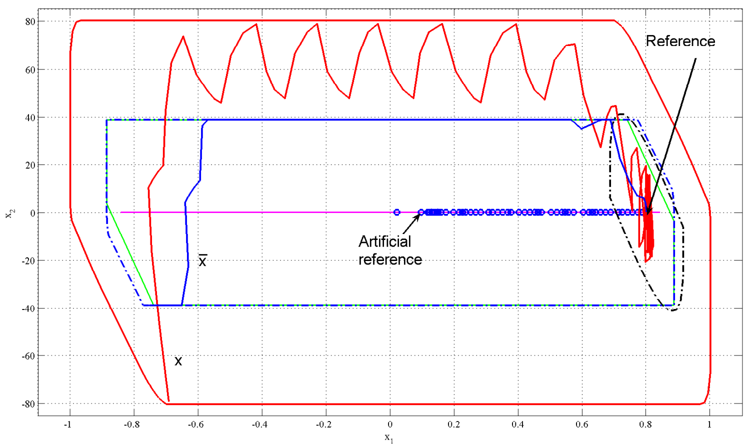


Figure 7.12: Evolution in the state space. Simulation #1.

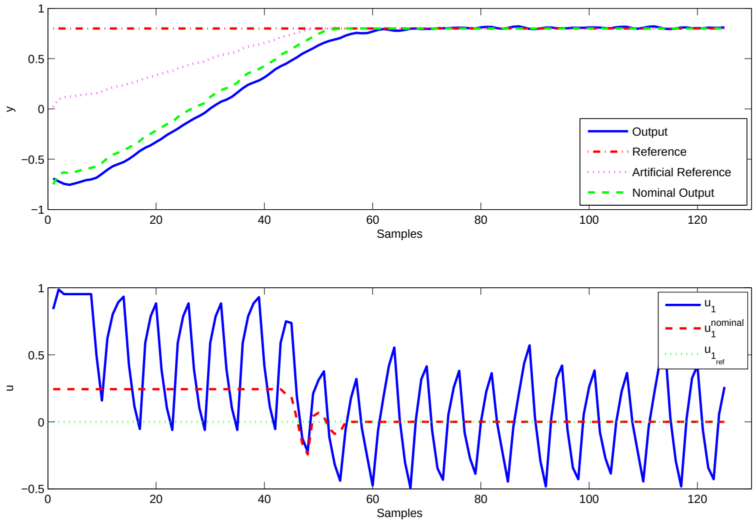


Figure 7.13: Time evolution. Simulation #1.

It can be seen in spite of the disturbances, the system is ultimately bounded in $x_s \oplus \Phi_K$.

7.3.3.2 Simulation #2:

The initial state for this simulation is $(0.7, 0.79)$. Note that this initial state is close to the state constraints. The uncertainty trajectory is given in figure 7.14 is space state, in this simulation it has been made a few changes in the set point, the first set point will be $(-0.8, 0)$, and then the opposite $(0.8, 0)$ (the trajectory in depicted with crosses) and finally $(0, 0)$, (the trajectory in depicted with circles). This simulation pretends to demonstrate how the system allows any change in the set point, keeping the trajectories inside of the region of attraction and satisfying the constraints.

Figure 7.15 shows the same test but in the time space. Note that this uncertainty trajectory is made up of extreme realizations of the uncertainty.

It can be seen in spite of the disturbances, the system is ultimately bounded in Φ_K centered in the desired steady state.

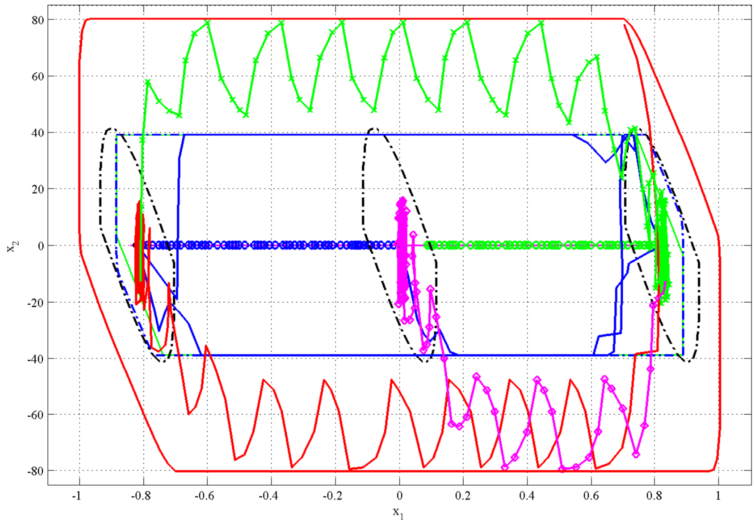


Figure 7.14: States trajectories in the state space. Simulation #2.

7.3.4 Controller implementation

Once the good operation of the controller is verified in simulation, It is going to be applied to the linear motor. Since the motor has a time of sampling of 30 ms, is not possible to implement the controller using an on-line QP solver because the solving time is approximately 100 ms. In order to implement the MPC for tracking, the explicit solution of the corresponding optimization problem has been obtained using standard algorithms (Bemporad, Morari, Dua and Pistikopoulos, 2002; Bemporad, Borrelli and Morari, 2002). In order to do the only region search to implement the explicit MPC controller a binary search tree has been computed as proposed in (Tøndel et al., 2002; ndel et al., 2003) as in the example of the linear motor controlled by the MPC for tracking.

7.3.5 Robust MPC for tracking applied to the Linear Motor

In this section experimental results of the linear motor in closed-loop with the robust MPC for tracking are presented. These results demonstrate the theoretical properties proved in chapter 4.

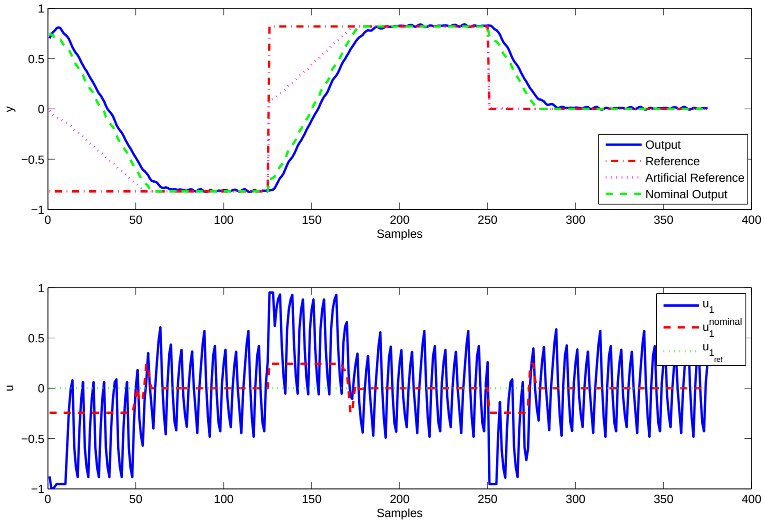


Figure 7.15: Time evolution. Simulation #2.

It is important to remark that, for all the experiments the initial speed is zero.

7.3.5.1 Test #1:

It is going to be provided changes of reference alternative between extreme positions, from 0.8 m to -0.8 m.

The trajectories of the state of the closed-loop system, the input signals and the reference signals are shown in figure 7.16 (state space portrait) and in figure 7.17 (time trajectories).

In this test is shown the admissible trajectories of the plant despite of disturbances and model mismatch verifying the control constraints.

7.3.5.2 Test #2:

To demonstrate the disturbance rejection capacity, a source of external disturbances has been added to the system adding a free inverted pendulum located on the base

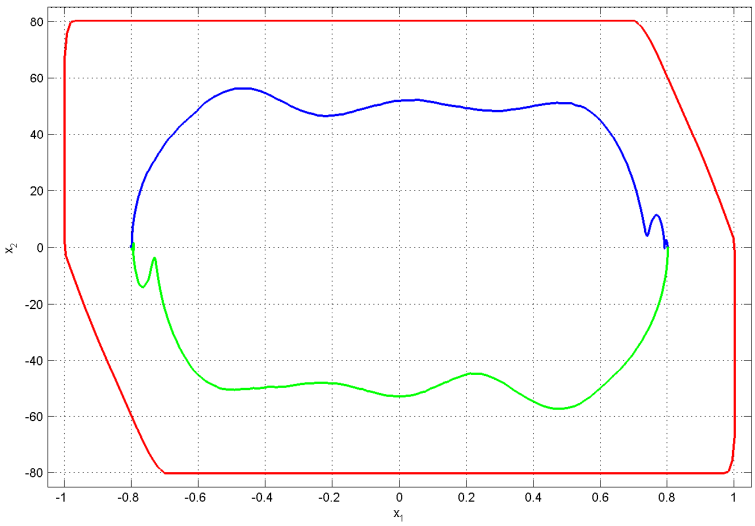


Figure 7.16: Space state evolution, Test #1.

of the motor. Thus during the movement of the motor, the free movement of the pendulum produces horizontal disturbances on the base. (Figure 7.18)

As in the previous test, It is going to be provided changes of reference alternative between extreme positions, from 0,8 m to -0,8 m.

The evolution of the system in state space and the temporary evolution of the signals can be observed in the figures 7.19 and 7.20.

In this test the disturbance rejection capacity of the controller is shown. Comparing the evolution of the system in this test with the previous one it is posible to appreciate how the trajectory is more oscillating because of the inverted pendulum. The applied control action is inside its limits but is more energetic trying to compensate the effect of this new disturbance.
It is important to remark also that at all moment the restrictions are satisfied.

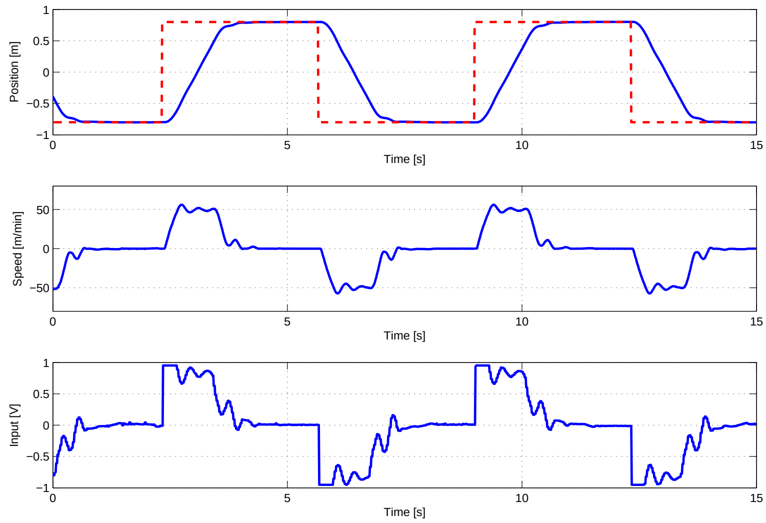


Figure 7.17: Temporary evolution, Test #1.

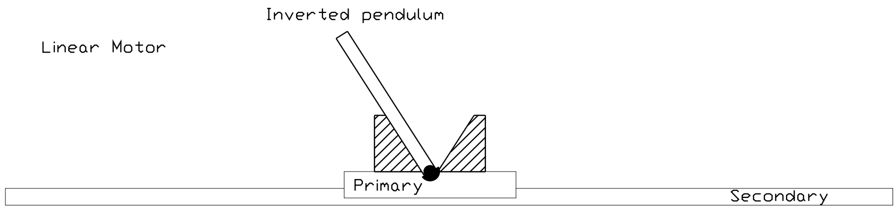


Figure 7.18: Linear motor plus the inverted pendulum.

7.4 Conclusions

In this chapter two solutions have been applied to the problem of controlling a positioning system based on a linear motor: the MPC for tracking and the robust MPC for tracking.

The MPC for tracking is capable to ensure the stability and feasibility in the case of any variation in the setpoint and no disturbances or model mishmashes, providing a better performance than the robust formulation. That is, because the proposed solution to the tracking problem is less conservative. The MPC for tracking guarantees the feasibility incorporating an artificial steady state as a decision variable. In order to guarantee the admissibility of the system and the convergence to the desired steady state, a term that penalize the distance between the artificial steady state and the

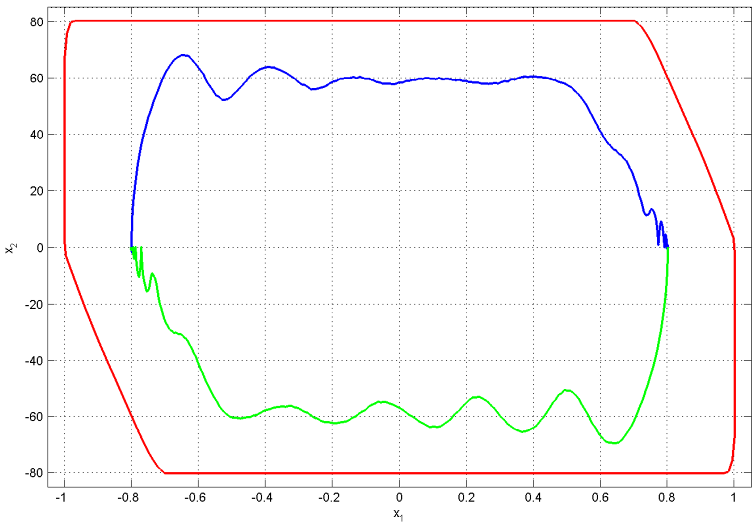


Figure 7.19: Space state evolution, Test #2.

desired one, and a terminal constraint, that is an invariant set for tracking, have been added. The MPC for tracking does not guarantee the feasibility in the case of disturbances or model mishmashes, even being a robust controller. For that, the robust MPC for tracking has been designed.

The robust MPC for tracking considers the possible model mishmashes or disturbances as an additional term in the model. The robust constraints are satisfied by introducing the notion of tubes. On the other hand, the convergence to a neighborhood of the desired steady state is guaranteed applying the MPC for tracking to the nominal system, which trajectory is the center of the tube.

The associate optimization problems result to be a QP in both cases. This allows its efficient implementation to a fast system by means of the use of well known algorithms for its explicit resolution. For that, techniques of multiparametric programming are used.

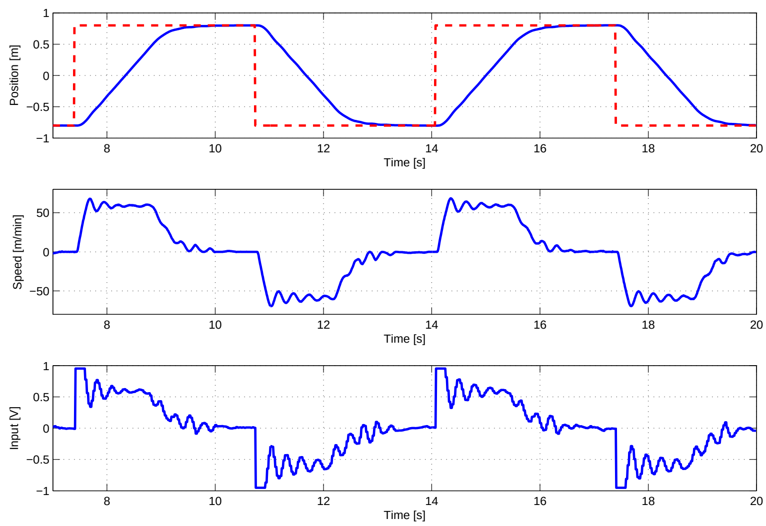


Figure 7.20: Temporary evolution, Test #2.

Chapter 8

Application to the ACUREX plant

8.1 Introduction

The results presented in this chapter were carried out during the period from 25^{th} to 29^{th} of September of 2006

ACUREX is one of the available experimental plants of the PSA complex (Solar plant of Almería) located at the Tabernas desert (Spain). The ACUREX plant consists of a series of parabolic mirrors that reflect solar radiation onto a pipe where oil gets heated while circulating (Camacho, Berenguel and Rubio, 1997). The objective of the control system in a distributed collector field is to maintain the outlet oil temperature at a desired level despite of disturbances such as changes in the solar irradiance level (caused by clouds), mirror reflectivity or inlet oil temperature. Since solar radiation cannot be adjusted, this can only be achieved by adjusting the oil flow (Camacho, Berenguel and Rubio, 1994; Camacho, Berenguel and Bordons, 1994).

During the five days that the plant was available, the identification of the model, its validation and the design of the controller was carried out before testing the robust MPC.

8.2 Description of the solar power plant ACUREX

The distributed collector field involves the collection of solar energy and its transfer to a fluid piped through the system. The energy collected is transferred to a storage tank, which can be tapped when conditions demand, on to either a steam generator for electrical power generation or the heat exchanger of a desalination plant. It mainly consists of a pipeline through which oil is flowing and onto which the sun's rays are concentrated by means of parabolic mirrors, in order to heat the oil. It consists of 480 modules arranged in 20 lines which form 10 parallel loops. Figure 8.1 shows a simplified diagram of the solar collector field. The field is also provided with a sun-tracking mechanism which causes the mirrors to revolve around an axis parallel to that of the pipeline. Each of the loops mentioned above is formed by four 12-module collectors, suitably connected in series. The loop is 172 m long, with an active part of the loop (exposed to concentrated radiation) measuring 142 m and a passive part of 30 m long.

A fundamental feature of a solar power plant is that the primary energy source, while

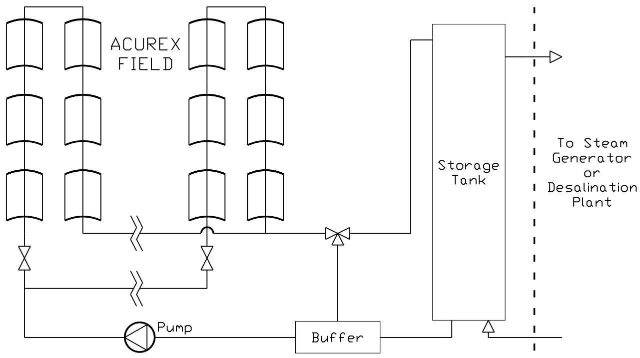


Figure 8.1: Acurex scheme.

it is variable, cannot be manipulated. The intensity of solar radiation, in addition to its seasonal and daily cyclic variations, is also dependent on atmospheric conditions such as cloud cover, humidity and air transparency. To compensate this effect, the disturbances are measured and used as an input for a series feedforward controller. Due to this feedforward controller, in order to obtain suitable experimental data for identification, the experiments must be executed in a period of time in which the environmental variables are constant, i.e., the changes in the output are only produced by the changes in the control input. The dynamical characterization of the field has been done in both the time and frequency domains (Camacho et al., 1997). Open-loop step responses have been obtained at several operating points. One of these responses is shown in Figure 8.2 (a change in oil flow from 8 to 7 $\frac{l}{s}$ was performed). As can be seen, the step

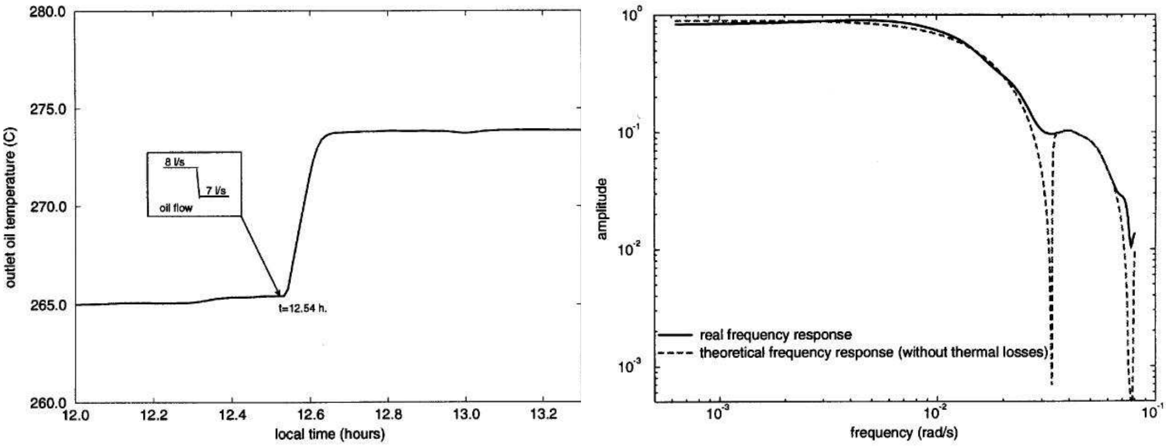


Figure 8.2: Dynamical characterization of the field.

response shows that the behavior of the field can be modelled by a low-order model as shown in previous works done in the plant (Camacho, Berenguel and Bordons, 1994), although the plant dynamics are more complex (presence of antiresonance modes) as can be seen in the frequency response plot shown in the same figure.

The use of high-order models with standard parameter estimation algorithms may provide wrong results because of the divergence of the estimated parameters (Camacho, Berenguel and Rubio, 1994). Using low-order models with MPC controllers, acceptable behavior can be obtained.

8.2.1 Series feedforward compensation

The outlet temperature of the plant is also influenced by changes in system variables such as solar radiation and fluid inlet temperature. In order to account for these disturbances, a series feedforward controller has been introduced. The feedforward has been calculated from steady state relationships, which make an adjustment in the fluid flow input, aimed to eliminate the change in outlet temperature caused by the variations in solar radiation and inlet temperature (Berenguel, Camacho and Rubio, 1993; Camacho, Berenguel and Bordons, 1994; Camacho, Berenguel and Rubio, 1994). The calculation employed is:

$$flow = \frac{a_1 Irr - a_2(FF_{ref} - a_3) - a_4}{FF_{ref} - T_{in}}$$

where a_1, a_2, a_3, a_4 are parameters, $flow$ is the oil flow, FF_{ref} is the temperature setpoint, T_{in} is the inlet oil temperature and Irr is the effective solar radiation over the mirrors. The feedforward is considered as a part of the model and thus the input signal for the plant model is the setpoint temperature for the feedforward term FF_{ref} and not the oil flow (that is calculated by the feedforward controller) as shown in the figure 8.5. Although exact elimination cannot be achieved, this term helps to preserve the validity of the assumed system model and provides control benefits when disturbances in solar radiation and fluid inlet temperature occur.

8.3 Identifying the plant plus the feedforward

In order to apply the RMPCT controller, a state space linear model with additive bounded disturbances of the plant must be obtained. The control input of the system to be identified (u) is FF_{ref} and the output (y) is T_{out}. To identify the plant, as it was aforementioned, a period where the environmental variables were constant is chosen, thus, the changes in the output are only produced by the changes in the input (see an example in figure 8.3).

Several low order linear models with different delays have been identified using the least square method. Finally the model with the smaller identification error set $\mathcal{W}$ was a first order model without delay:

$$A = 0.8656 \quad B = 0.1251$$

$$C = 1 \qquad D = 0$$

The sample time was provided by (Berenguel et al., 1993), in this document is deep study of the plant is done and also provide a nonlinear model of the plant used to tune the controller by simulation. The value of the sample time is $39s$.

The set of possible values of w has been calculated computing the maximum error between the real state trajectory and the modeled one. Figure 8.4 shows an example used to identify the maximum error. The obtained set that the ACUREX plant has to satisfy is the following $\mathcal{W} = \{w \in^n : |w| \leq 5\}$

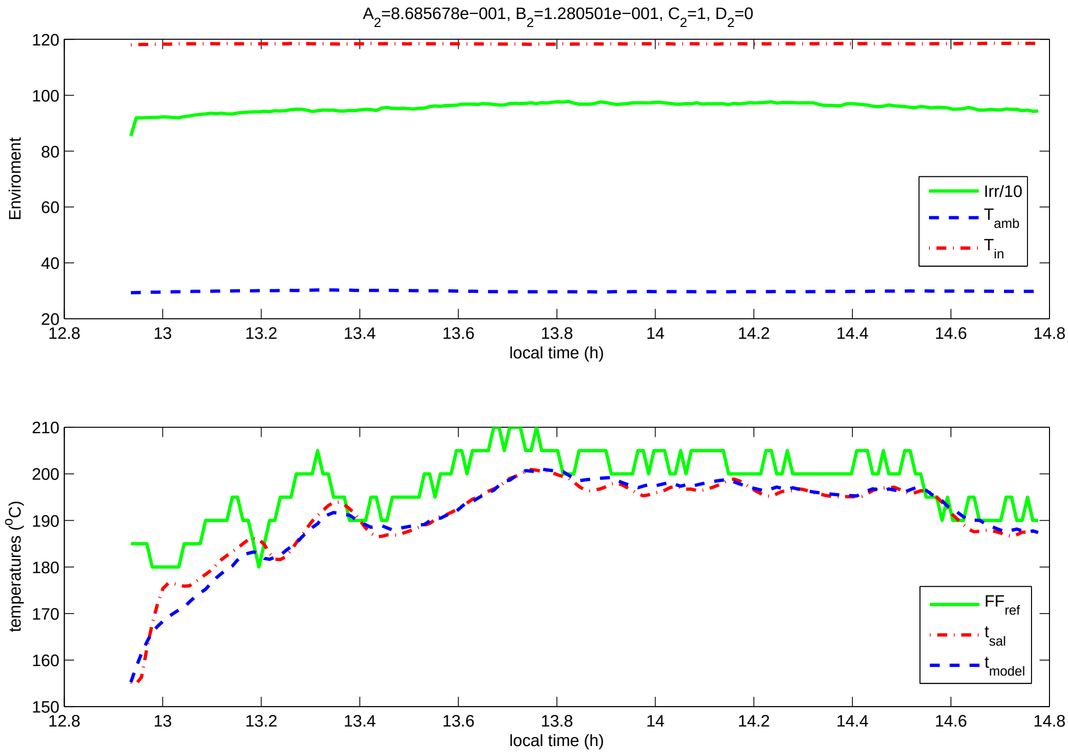


Figure 8.3: Plant identification.

The constraints set for that system are:

$$\mathcal{U} = \{u \in \mathbb{R}^m : 100 \leq u \leq 350\} \quad u \triangleq FF_{ref}$$

$$\mathcal{X} = \{x \in \mathbb{R}^n : 0 \leq x \leq 300\} \quad x = y \triangleq T_{out}$$

In addition to these constraints on the input and the output of the ACUREX plant there are a couple of constraints that have to be considered by the operator of the plant that constraint the rank of admissible setpoints:

- The maximal difference between the temperatures of each collector is 80^0,

$$T_{out} - T_{in} \leq 80$$

- The constraint set for the flow is

$$2 \leq flow \leq 10 \ \frac{l}{s}$$

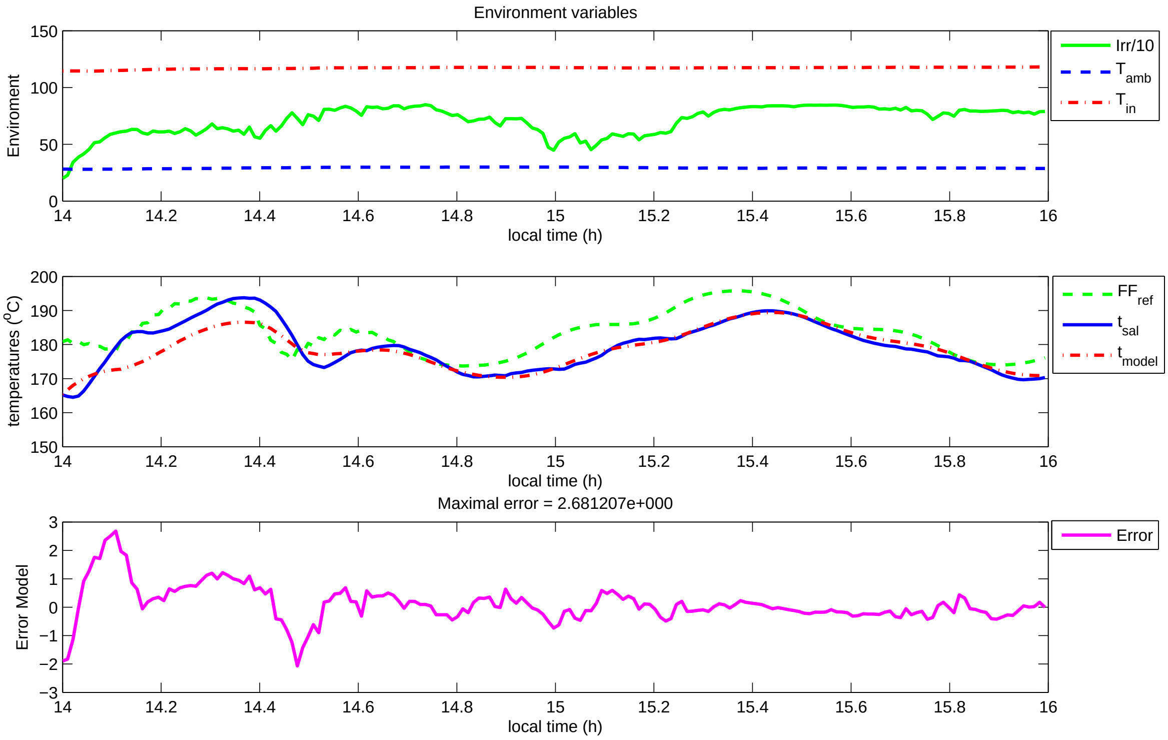


Figure 8.4: Maximal error w_{max}.

8.4 Applying Robust MPC for tracking

The proposed RMPCT is used to manipulate FF_{ref} ensuring robust admissible stability, disturbance rejection and setpoint tracking. Figure 8.5 shows the general scheme of the controlled plant. The tuning of the MPC controller has been done using a model of

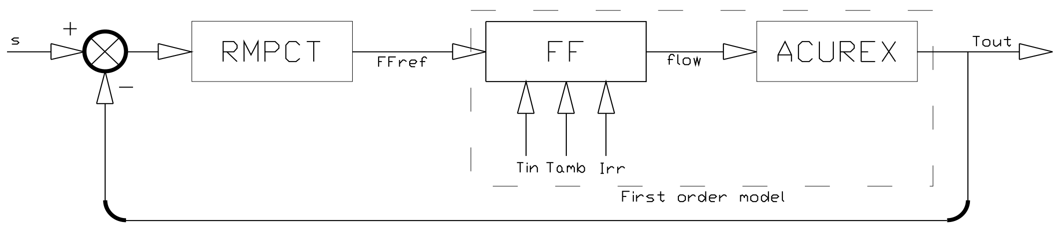


Figure 8.5: RMPCT scheme.

the plant based on partial difference equations developed in (Berenguel et al., 1993), to control the plant in order to be as fast as possible without overshooting. The controller parameters are:

MPC#1

$Q_\phi = 1,\ R_\phi = 1,\ K = -0.36726$

$\phi_K = \{e \in \mathbb{R}^n : |e| < 28.0174\}$

$Q = 1000,\ R = 1,\ \bar{K} = -6.40848$

$\mathcal{S} = \{s \in \mathbb{R}^p : (107.34 \leq s \leq 271.71)\}$

$\bar{\mathcal{U}} = \mathcal{U} \ominus K\phi_K = \{\bar{u} \in \mathbb{R}^m : (110.29 \leq \bar{u} \leq 339.71)\}$

$\mathcal{X}_n = \{x \in \mathbb{R}^n : (0 \leq x \leq 300)\}$

where:

- $e \triangleq x - \bar{x}$ is the control error.
- The local controller gain K to determine ϕ_K is the one resulting of the LQR with Q_ϕ and R_ϕ .
- The controller gain $\bar{K}$ to determine the invariant set for tracking used as the terminal constraint, is the one resulting of the LQR with the same weighting matrices used for the stage cost Q and R .
- $\mathcal{S}$ is the set of the admissible setpoints.
- $\bar{\mathcal{U}}$ is the set of the admissible control actions for the nominal system.
- $\mathcal{X}_n$ is the region of attraction.
- The prediction horizon in $N = 3$.

Remmark 8.1 *It is important to remark that the set ϕ_K is the section of the tube, so it is the maximum error between the nominal state and the real state, thus, in steady state, is the maximum error between the reference state and the real state (if the reference state is admissible).*

In the following figures, the top shows, the reference (solid line), the artificial reference (dashed-dot line) and the output (dashed line). The bottom shows the real disturbances (the radiation, the corrected radiation (both in dashed line), the temperature of the input T_{in} (dashed-dot line)) and the estimation of the disturbance w_{est} (solid line).

Figure 8.6 shows the evolution of the plant controlled by $MPC\#1$ under good environmental conditions. It can be seen how the output of the plant is following the

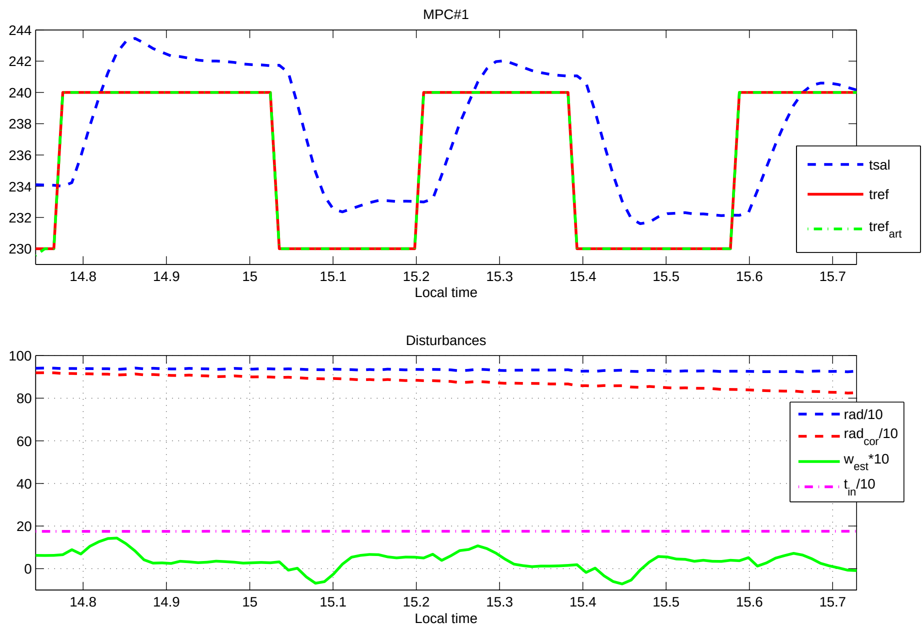


Figure 8.6: $MPC\#1$ without disturbances.

reference with a certain offset.

In the following subsection the experimental results of the controlled plant under different disturbances will be presented.

8.4.1 Radiation disturbance

Figure 8.7 shows the evolution of the plant controlled by $MPC\#1$ when there are clouds crossing over the plant. It can be seen, the controlled system follows the setpoint in an admissible way despite the disturbances. Note that although the control error is high,

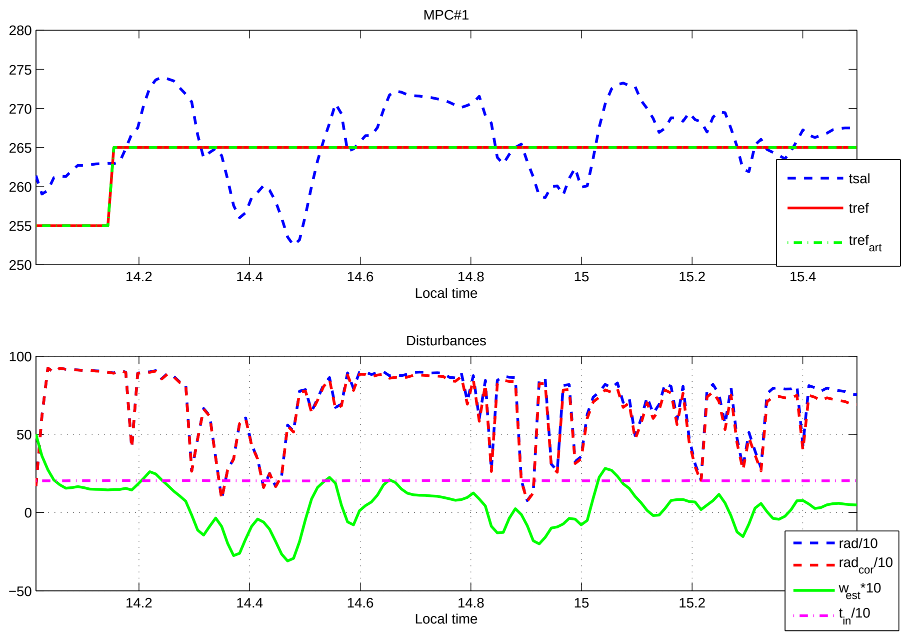


Figure 8.7: $MPC\#1$ with radiation disturbance.

the state trajectories remain inside the predicted limits, i.e., $\pm 28^0$ (the size of Φ_K). Unfortunately the weather conditions became worse and the plant had to be stopped. This happens when the radiation is under $200\frac{W}{m^2}$.

8.4.2 Pyrometer sensor error

This experiment demonstrates the robustness of the proposed controller to an unmodelled disturbance: an error on the pyrometer sensor. This error was artificially produced locating a convex lens between the sensor and the sun. In this case the controller has been designed to be more aggressive. The parameters are:

MPC#2

$Q_\phi = 10,\ R_\phi = 1,\ K = -1.81804$

$\phi_K = \{e \in \mathbb{R}^n : |e| < 13.7275\}$

$Q = 1000,\ R = 1,\ \bar{K} = -6.40848$

$\mathcal{S} = \{s \in \mathbb{R}^p : (121.62 \leq s \leq 258.89)\}$

$\bar{\mathcal{U}} = \mathcal{U} \ominus K\phi_K = \{\bar{u} \in \mathbb{R}^m : (124.96 \leq \bar{u} \leq 325.04)\}$

$\mathcal{X}_n = \{x \in \mathbb{R}^n : (0 \leq x \leq 300)\}$

Figure 8.8 shows the evolution of the plant controlled by $MPC\#2$. It can be seen how the evolution remains inside the predicted limits when the disturbances satisfy the bound used in the model. If the disturbance becomes bigger than its bound ($\|w\| < 5$), like in this test, the error between the nominal state and the real state can be bigger than 13.7275 (the size of ϕ_K), and the constraints satisfaction is not guaranteed. If the system is not close to the boundary, and the disturbance disappears, like in this case, the controller maintains it good performance and steers the system again inside the tube.

Anyway, the maximum error can be 13.7275, depending on the application this maximum error between the setpoint and the output can be admissible or not, in the following section a procedure to avoid this error if the disturbance is constant is introduced.

8.5 Cancelation of the tracking error

As it was shown in 4.7 the tracking error can be compensated changing the setpoint in the following way

$$s(k) = s_d - [C + DK](I_n - (A + BK))^{-1}\hat{w}(k) = s - H_s\hat{w}(k)$$

where $s(k)$ is the setpoint that should be provided to make the real system converge to the desired s_d and $\hat{w}(k)$ is an estimation of the disturbance w. Figure 8.9 shows the

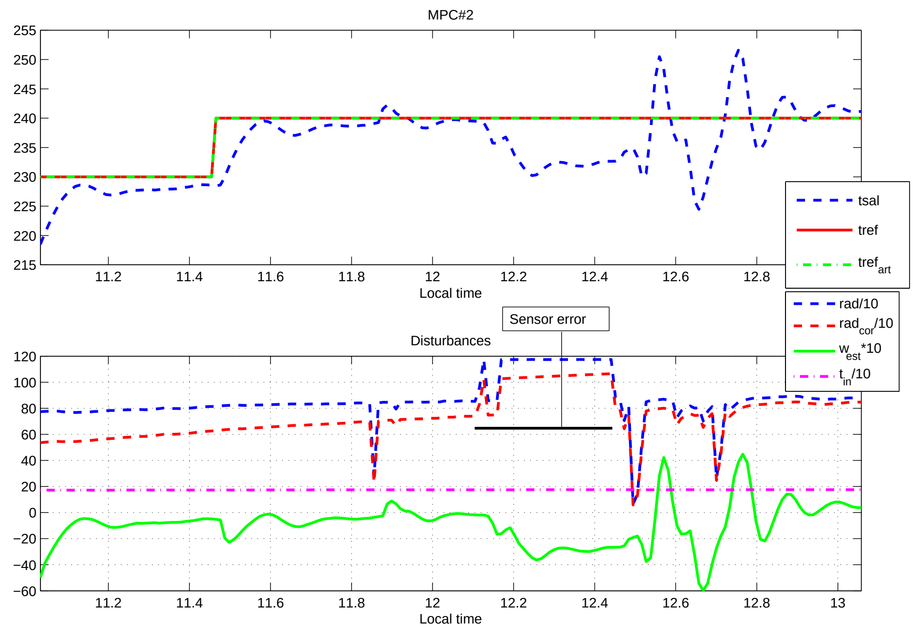


Figure 8.8: $MPC\#2$ with an error in the pyrometer.

general control scheme. In this case, the disturbance estimator proposed in (Normey-Rico and Camacho, 2007) has been used:

$$w^+ = b \cdot w - A \cdot \eta + \frac{(1-b)\cdot(y^+ - y^+_{model})}{C}$$

$$\eta = (1-b) \cdot \frac{(y - y_{model})}{C}$$

$$x^+_{model} = Ax_{model} + Bu, \quad y_{model} = Cx_{model} + Du$$

where $b = 0.6$ is the constant of the filter, it was tuned online to make filtered fast enough avoiding the overshooting. The offset cancelation loop with $MPC\#1$ has been tested in the following different scenarios.

8.5.1 T_{in} disturbance

Figure 8.10 shows how the reference is tracked in a day with a very good environmental conditions.

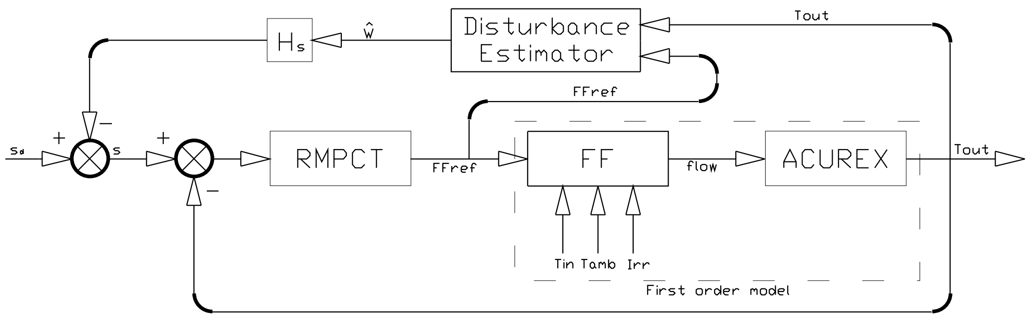


Figure 8.9: Cancelation of the tracking error scheme.

In order to see the disturbance rejection properties of the controller, an unmodelled disturbance in the temperature at the input of the collectors was introduced. It clear

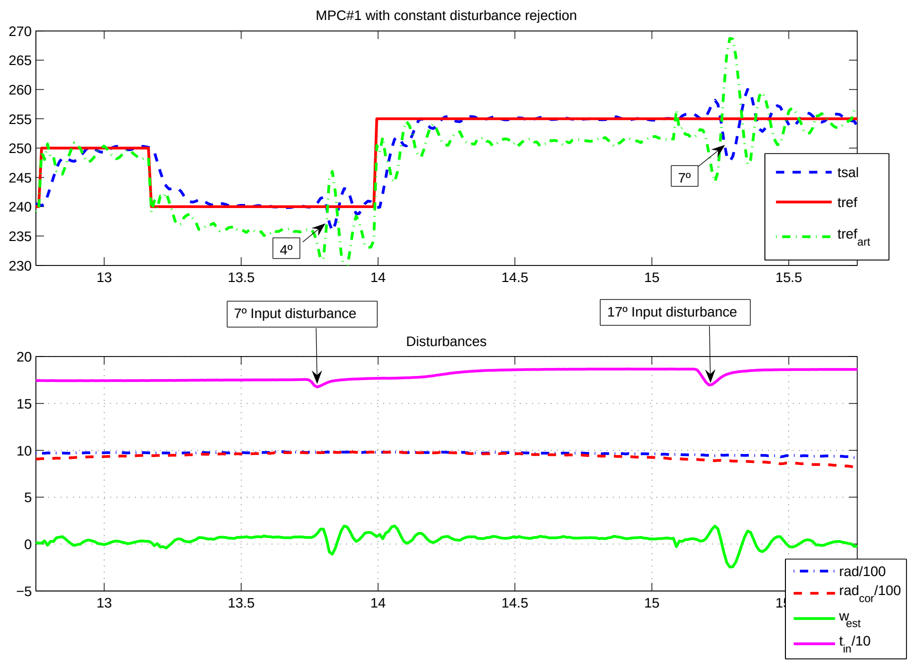


Figure 8.10: $MPC\#1$ with constant disturbance rejection.

how under this scheme the offset is removed and the disturbances are attenuated. See how a disturbance of -7^0 produces 4^0 of overshooting and a disturbance of -17^0 produces 7^0 of overshooting.

8.5.2 Pyrometer sensor disturbance

As in the section 8.4.2 a convex lens was located between the sun and the sensor for a period of time, under this scheme the disturbance estimator detect this disturbed radiation measurement allowing the controller to reject the effect over the system (this radiation measurement is used by the feedforward to calculate the flow). Figure 8.11 shows the closed-loop state trajectories of the ACUREX plant controlled the $MPC\#1$ using the offset cancelation loop demonstrating the disturbance rejection (Compare this figure with figure 8.8 where the effect of the disturbance was not rejected).

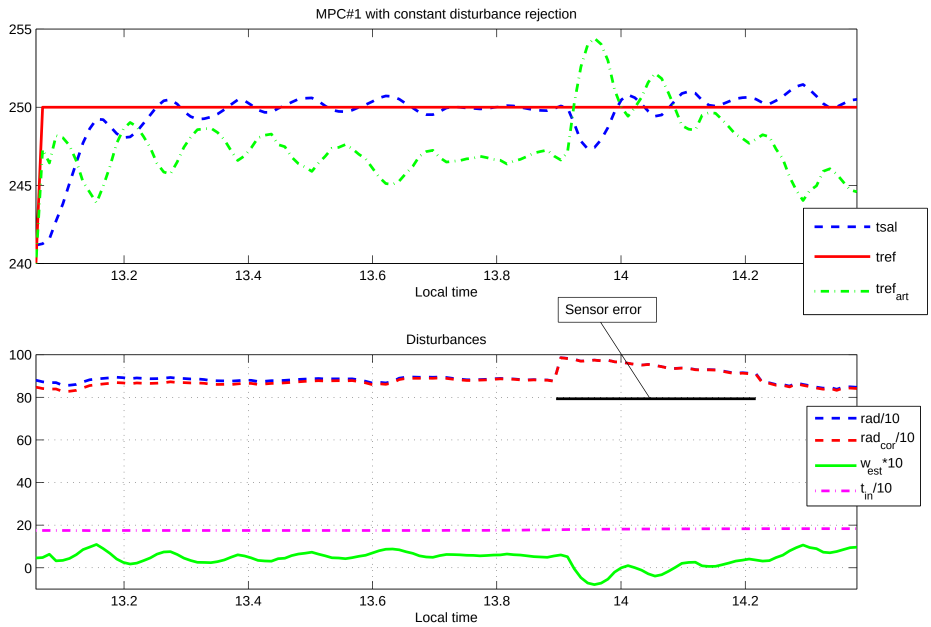


Figure 8.11: $MPC\#1$ with constant disturbance rejection.

8.5.3 Radiation disturbance

This experiment was carried out under a large disturbance produced by a cloud. Figure 8.12 shows the closed-loop state trajectories of the ACUREX plant controlled the $MPC\#1$ using the reference tracking error compensating scheme, demonstrating its robustness and the tracking capability. Due to the disturbance is varying the effect over the system is not rejected by the proposed scheme. however it is worth remarking how the artificial reference evolves trying to compensate the effect of the disturbance.

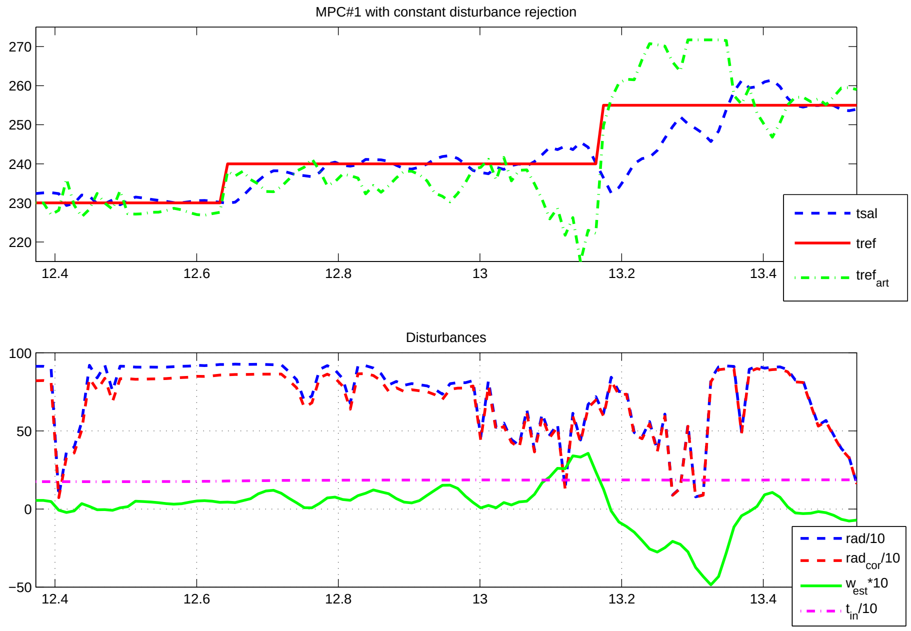


Figure 8.12: $MPC\#1$ with constant disturbance rejection.

8.6 Conclusions

In this chapter we have seen the ACUREX plant, which models a process characterized by large perturbations and main changes in dynamics caused by clouds and operating conditions. In despite of this, the proposed robust MPC for tracking controls successfully the plant. The trajectory of the controlled plant is admissible but presents offset due to the persistent nature of the disturbances. An offset cancelation loop has been used to avoid the constant offset. this scheme depends strongly on the estimator; if the measurements provided by the estimator are wrong, the control scheme will fail.

Due to the lack of time to work with the plant that we had, another problem of the controller was exhibited, it is that it is not possible to tune it online so it is needed a long time to do it.

Chapter 9

Application to the quadruple tank process

9.1 Introduction

In this chapter experimental results of the application of the MPC for tracking to an experimental tank system developed at the University of Seville are presented. This plant is based on the well known quadruple-tank process (Johansson, 2000*b*). In the quadruple tank process of the University of Seville several modifications have been done in order to obtain a wide range of different applications and dynamics. The quadruple tank process is a multivariable laboratory plant of interconnected tanks that can be modeled by a nonlinear dynamic system subject to state and input constraints. One important property of this plant is that it can be configured to work at operation points characterized by multivariable zeros (minimum and non-minimum phase). Figure 9.1 shows the original plant consists of four interconnected tanks. The inputs are the voltages of the two pumps and the outputs are the water levels in the lower two tanks.

The real plant can be modified to offer a wide variety of configurations such as one single tank, two or three cascaded tanks, a mixture process and hybrid dynamics. Moreover the parameters that defined the dynamics of each tank can be modified by tuning the cross-section of the outlet hole of each tank. The real plant has been implemented using industrial instrumentation and a PLC for the low level control.

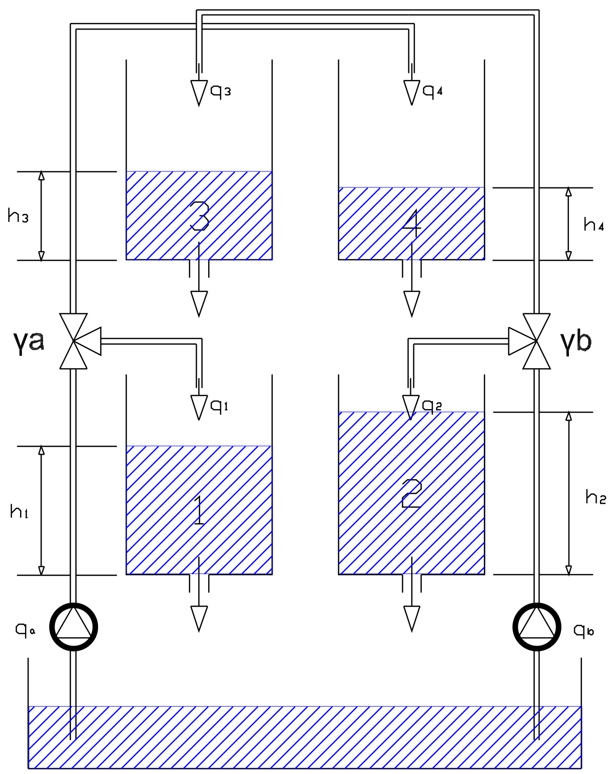


Figure 9.1: The quadruple tank process.

Supervision and control of the plant is carried out in a computer by means of OPC (ole for process control) which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or an industrial SCADA.

Additional information on the quadruple tank process of the University of Seville are given in Appendix B.

9.2 The quadruple tank process

A state space continuous time model of the system (Johansson, 2000*a*) can be derived from first principles as follows

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_a}{A_1}q_a \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_b}{A_2}q_b \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A_3}q_b \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A_4}q_a
\end{aligned} \tag{9.1}$$

where the parameters of the plant are:

- A_i: Cross-section of tank i.
- a_i: Cross-section of the outlet hole of the tank i.
- h_i: Water level of the tank i (state of the system).
- q_a, q_b: Flow produced by the the pumps a and b.
- g: The acceleration of gravity.
- q_i: Inflow of each tank.
- γ_i: Parameters of the three-way valves.

Linearizing the model in an operating point given by h_i^0 and defining the deviation variables $x_i = h_i - h_i^o$ and $u_j = q_j - q_j^o$ where $j = a, b$ and $i = 1, \cdots, 4$ we have that:

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_a)}{A_4} & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x
\end{aligned} \tag{9.2}
$$

where $\tau_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1, \cdots, 4$, are the time constants of each tank.

The system is open-loop stable with two multivariable zeros at any given operation point. The nature of these zeros is determined by the parameters γ_a and γ_b as follows

- If $0 \leq \gamma_a + \gamma_b < 1$, the system has right half plane transmission zeros (RHPZ).
- If $1 < \gamma_a + \gamma_b \leq 2$, the system has left half plane transmission zeros (LHPZ)

It is worth remarking that the sign of the real part of the zeros does not depend on the operating point. In addition to this remarkable property, the following interesting features that make the plant appropriate to be used for both educational and research purposes:

1. The linearized model of the quadruple-tank process has multivariable zeros, which can be located in either the left or the right half-plane by simply changing a couple of 3-ways valves.

2. All the states are measurable.

3. The outputs are strongly coupled.

4. The system is nonlinear.

5. The states and inputs of the plant are constrained.

6. The plant is easily harmlessly handled.

Thus this plant presents challenging and interesting control problems. Among these problems, the following ones can be highlighted:

- Control of multivariable systems.
- Control of systems with RHPZ and limits of performance.
- Robust control.
- Tracking of references.
- Control of systems subject to input constraints.
- Control of systems subject to state constraints.

The design and implementation of the plant has been carried out to maximize the potential educational and research interest. This is detailed in the appendix B.

9.3 Implementation of the laboratory plant

One of the main objectives of the implementation of the quadruple tank process has been to provide flexibility to the plant in the following aspects:

- Capability to setup different processes in the same plant.
- Capability to tune some parameters which allows us to configure the plant dynamics.

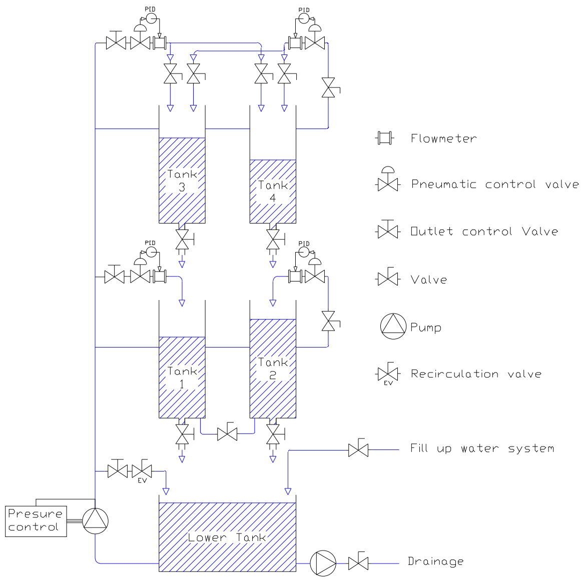


Figure 9.2: Real plant scheme.

- Wide range of operating points.

The plant layout has been designed to meet these specifications (see figure 9.2). The designed plant differs from the quadruple tank process proposed in (Johansson, 2000*a*) in the following items:

- The three-way valve has been replaced by two pneumatic valves that control the flow of the pipes. This allows us to fix a desired flow ratio between the two pipes (that is, the parameters γ_a and γ_b) and hence obtain an *ideal* three-way valve. Moreover, given that all the flows of the inlet pipes of the tanks can be modified, different processes can be configured.
- Extra pipes, valves and tank interconnections have been added to setup the different processes (See figure 9.2).
- A valve that can be opened and closed manually with a position display has been placed in the outlets of the tanks in order to manipulate the cross-section of the outlet hole a_i. This allows us to configure the dynamics of each tank of the process.
- The tanks are transparent, with a rectangular cross section and can be easily removed. This allows one to put in some *ad hoc* element to change the cross section, and hence obtain tanks with different dynamics.

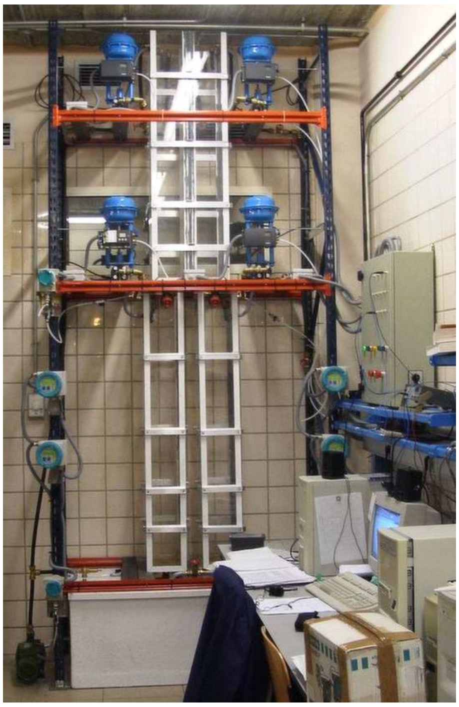


Figure 9.3: The implemented laboratory plant.

- A safety circuit has been added to avoid the possibility of overflow.
- The two pumps of the original 4 tank process have been replaced by a bigger pump with a frequency control that ensures a constant pressure at the output of the pump ensuring the decoupling of the flows across the pipes.
- A recirculation circuit controlled by an electronic valve that ensures that the flow across the pump is always over the minimum.

Another relevant aspect of the plant is that it has been implemented using industrial measurement devices and industrial control valves, which provides a realistic framework to test controllers. These, together with the low-level control system, will be shown later on. Figure 9.3 shows a photograph of the plant.

As it can be seen, this plant is larger that the typical scaled lab plants (Rusli, Ang and Braatz, 2004; Johansson, Horch, Wiljk and Hansson, 1999). Tables 9.1 and 9.2 show the real constraints of the plant.

Table 9.1: State constraints.

Parameter	Value	Unit	Description
H_{1max}	1.36	m	Maximum level of the tank 1
H_{2max}	1.36	m	Maximum level of the tank 2
H_{3max}	1.30	m	Maximum level of the tank 3
H_{4max}	1.30	m	Maximum level of the tank 4
H_{min}	0.3	m	Minimum level in all cases

The tanks have a minimum level under which appears an eddy that changes the value of the effective cross-section of the outlet hole. This implies that the nonlinear model introduced is only valid for states that satisfy this constraint.

Table 9.3 shows the cross-section of the tanks.

9.4 Model identification of the plant

In this section we are going to present the identification of the unknown parameters of the state space model (9.2). The dynamics of the plant are mainly characterized by the value of the cross-sections of the outlet holes a_i, this parameters have been designed to provide a maximum range of possible states.

Remmark 9.1 *In table 9.2 is possible to see the maximum physical inflow of each*

Table 9.2: Control constraints.

Parameter	Value	Unit	Description
Q_{1max}	2.8	m^3/h	Maximal inflow of tank 1
Q_{2max}	2.45	m^3/h	Maximal inflow of tank 2
Q_{3max}	2.3	m^3/h	Maximal inflow of tank 3
Q_{4max}	2.4	m^3/h	Maximal inflow of tank 4
Q_{min}	0	m^3/h	Minimal inflow in all cases

Table 9.3: Cross-section of the tanks.

Parameter	Value	Unit	Description
A	0.06	m^2	Cross section of all tanks

tank that can be provided by the pump. Due to the inflows of the tanks have to verify:

$$q_1 = \gamma_a q_a,$$

$$q_2 = \gamma_b q_b,$$

$$q_3 = (1 - \gamma_b) q_b,$$

$$q_4 = (1 - \gamma_a) q_a$$

q_1*,* q_4 *and* q_2*,* q_3 *are not independent, thus, it must be considered the maximum flow that respect the 3-ways relation between the flows, and this quantity depends on* γ_a *and* γ_b*.*
For this configuration of the 3-way valves, the maximum flows that respect the relation of the 3-ways valve are:

$$q_{a\,\max} = \min(\tfrac{Q_{1\,\max}}{\gamma_a}, \tfrac{Q_{4\,\max}}{1-\gamma_a})$$

$$q_{b\,\max} = \min(\tfrac{Q_{2\,\max}}{\gamma_b}, \tfrac{Q_{3\,\max}}{1-\gamma_b})$$

being $q_a = q_1 + q_4$ *and* $q_b = q_2 + q_3$ *(see figure 9.1).*

On order to obtain a system characterized by a non-minimum phase multivariable zeros, γ_a and γ_b have the following values:

$$\begin{aligned} \gamma_a &= 0.3 \\ \gamma_b &= 0.4 \end{aligned} \tag{9.3}$$

The value of the cross-sections of the outlet holes a_i have been adjusted to provide a maximum range of possible states depending on the values of γ_a and γ_b, thus, the value of a_i are the corresponding to have in equilibrium a in medium level when the inlet flows have a medium value. The values of a_i are measured from the real plant in permanent regime trying to achieve the previous condition.

Table 9.4 shows the values of the cross section of the outlet holes and the equilibrium levels and flows.

Table 9.4: Values of the cross section of the outlet holes and the equilibrium levels and flows

Parameter	Value	Unit	Description
h_1^0	0.627	m	Equilibrium level of tank 1
h_2^0	0.636	m	Equilibrium level of tank 2
h_3^0	0.652	m	Equilibrium level of tank 3
h_4^0	0.633	m	Equilibrium level of tank 4
Q_a^0	1.6429	m^3/h	Equilibrium flow $(Q_1 + Q_4)$
Q_b^0	2.0000	m^3/h	Equilibrium flow $(Q_2 + Q_3)$
a_1	1.341241e-004	m^2	Cross-sections of the outlet hole 1
a_2	1.533957e-004	m^2	Cross-sections of the outlet hole 2
a_3	9.322457e-005	m^2	Cross-sections of the outlet hole 3
a_4	9.061679e-005	m^2	Cross-sections of the outlet hole 4
γ_a	0.3		Parameter of the 3-ways valve
γ_b	0.4		Parameter of the 3-ways valve

For applying the robust MPC for tracking introduced in chapter (4), we need a linear model subject to additive bounded disturbances.

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx \end{aligned} \tag{9.4}$$

where $w(i) \in \mathcal{W}$, $\forall i = 1, 2, \ldots$ To identify the sets $\mathcal{W}$ the output and state of the model have been compared with the real ones from some experiments.

The main sources of disturbances are the following:

- The linearization approximation error.
- The parameters a_i are not constant and depend weakly on the levels of the tank.
- The actuator dynamics have been neglected. Recall that the output of the controller is the reference of the PID that control the flow of each pipe.

To identify $\mathcal{W}$, a worst case scenario input signal made of a series of step changes was applied to the plant. The level trajectories of this experiment allowed us to obtain a worst case bound on the disturbances. Figure 9.4 shows one of the test that have been done to determine the set $\mathcal{W}$.

The set $\mathcal{W}$ identified from the experimental data is defined by the following inequalities:

$$|w_1| \leq 5 \times 10^{-3}$$

$$|w_2| \leq 5 \times 10^{-3}$$

$$|w_3| \leq 5 \times 10^{-3}$$

$$|w_4| \leq 5 \times 10^{-3}$$

9.5 Application of robust MPC for tracking to the quadruple tank process

The controller has been designed following the method proposed in chapter 6.

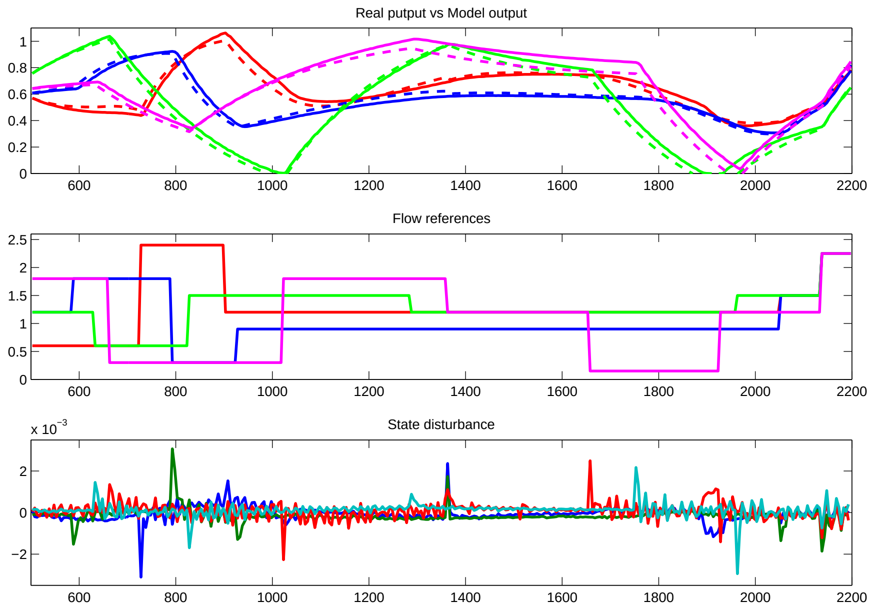


Figure 9.4: Real state evolution Vs Model state evolution.

The following stage cost matrices have been chosen. These matrices define the performance criterion that the MPC controller optimizes.

$$Q = C' \times C \quad R = 1 * 10^{-4} \times I_4 \tag{9.5}$$

For the calculation of K we have used the design method proposed in section (6.2.3) for a $\rho = 1$ (the minimum admissible invariant set). The value of K is:

$$K = \begin{bmatrix} -5.9997 & -18.7429 & 6.2544 & -37.0666 \\ -20.6413 & -12.8487 & -29.7042 & -3.0337 \end{bmatrix}$$

Figure 9.5 shows:

- the sets $C\phi_K$ and $C\mathcal{X}$ that are the projection over the outputs h_1 and h_2 of the minimal robust invariant set ϕ_K and the state constraints set $\mathcal{X}$
- the sets $\mathcal{U}$ that is the control constraints set and $K\phi_K$.
- the set of admissible setpoints $\mathcal{S}$ and $C\Omega_{t,\bar{K}}$ the projection over the outputs h_1 and h_2 of the invariant set for tracking $\Omega_{t,\bar{K}}$,

- and the control constraints set for the nominal system $\bar{\mathcal{U}} \triangleq \mathcal{U} \ominus K\phi_K$.

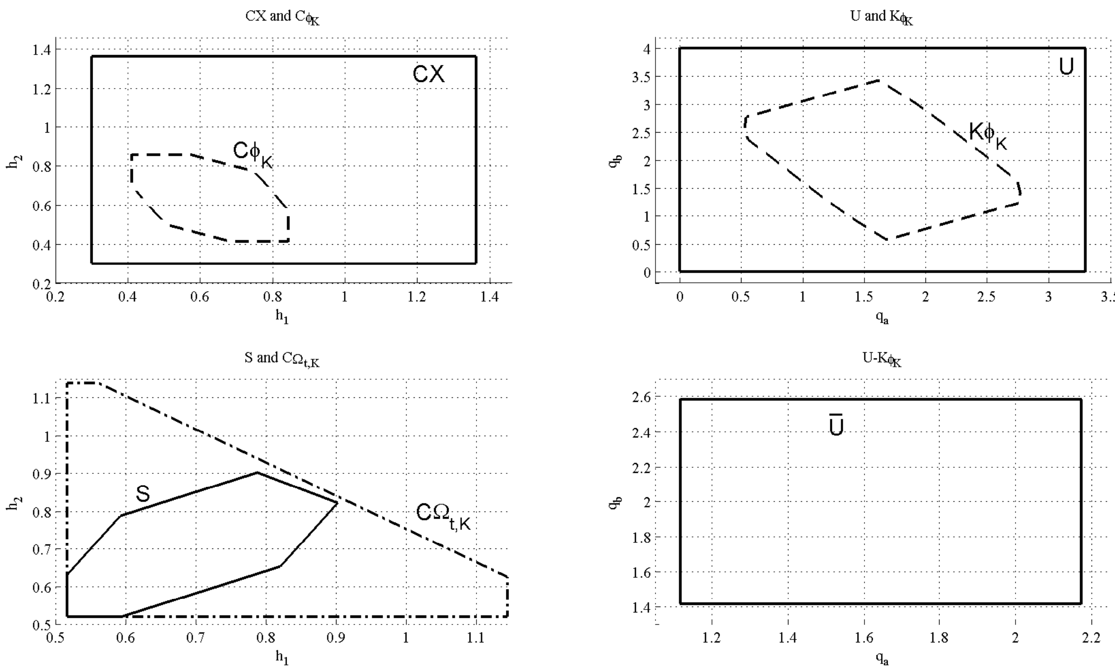


Figure 9.5: Different sets of the MPC for tracking applied to the quadruple tank process

The parameter $\bar{K}$ has been determined by means of a LQR that stabilizes the uncertain system in an admissible way with the same stage cost parameters as before (9.5). The controller is:

$$\bar{K} = \begin{bmatrix} -38.7806 & -22.3117 & 13.8988 & -13.3518 \\ -22.0209 & -42.5509 & -13.2292 & 8.4645 \end{bmatrix}$$

The matrix that penalizes the difference between the artificial steady state and the desired one is:

$$T = 100 \times P = 1 \times 10^4 \times \begin{bmatrix} 1.3107 & 1.0743 & 1.4670 & 0.7893 \\ 1.0743 & 1.7371 & 0.5064 & 2.4905 \\ 1.4670 & 0.5064 & 2.5572 & -0.8392 \\ 0.7893 & 2.4905 & -0.8392 & 4.9117 \end{bmatrix}$$

The control horizon has a value $N = 3$. The region of attraction associated to controller is $\mathcal{X}_3$ and the region of attraction for the nominal system is $\bar{\mathcal{X}}_3$. Therefore all the trajectories with a initial state inside $\mathcal{X}_3$, will remain inside this set and will be ultimately bounded inside a region that contains the desired steady state.

9.5.1 Simulation Results

The designed MPC for tracking has been tested in simulation to demonstrate its properties. The same experiments have been carried out in the real plant in order to compare the both sets of results.

Figure 9.6 shows output trajectories of the simulation experiment (solid line), the set of the admissible setpoints $\mathcal{S}$ and the references (circles). The top part of the figure 9.7 shows the time trajectories of the output (solid lines) and the references (dashed lines). The bottom part shows the time evolution of the control actions.

Figure 9.7 shows the classical step response of systems that have right half plane transmissions zeros. It can be seen how the levels react in an opposite direction to all reference step changes.

9.5.2 Controller Implementation

Once the operation of the controller is verified by simulation, It is applied to a real system.

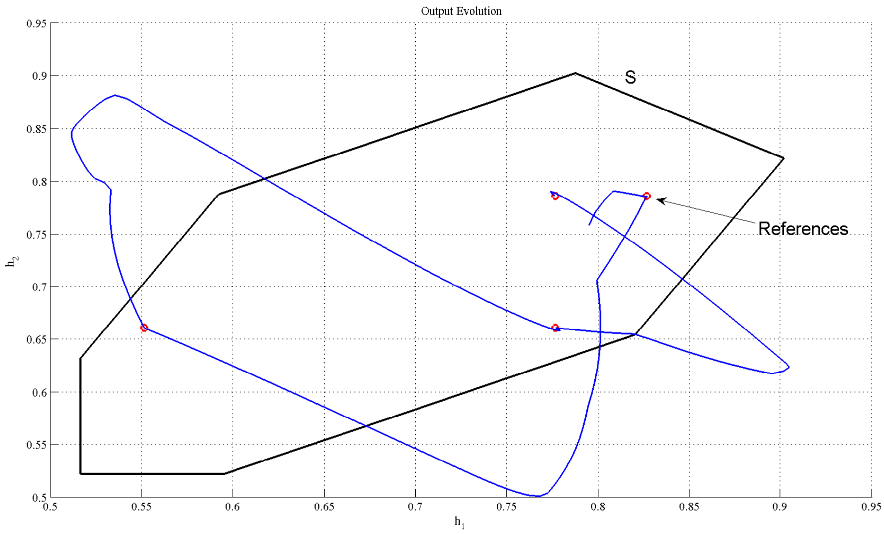


Figure 9.6: Simulation of the evolution of the outputs

The same test it is performed in order to compare the results by simulation with the real ones. Figure 9.8 shows the real evolution of the outputs, the set of the admissible setpoints $\mathcal{S}$ and the references. It can be appreciated that there is an error between the model and the real model, due to the reasons aforementioned, even with this disturbances and/or model discrepancies the evolution are quite similar.
The upper part of figure 9.9 shows the time evolution of the output (solid lines) and the references (dashed lines). The bottom shows the time evolution of the control actions.

The method proposed in section 4.7 is used to deal with tracking errors.

The following estimator is used:

$$\hat{w}_k = \lambda_f \hat{w}_{k-1} + (1-\lambda_f)(x_k - Ax_{k-1} - Bu_{k-1})$$

which guarantees that $\hat{w}_k$ converges to w_∞ with a rate of convergence fixed by λ_f.

The value of λ_f is $\lambda_f = 0.98$

Figures 9.10 and 9.11 show the time evolution of the plant with the disturbance rejection.

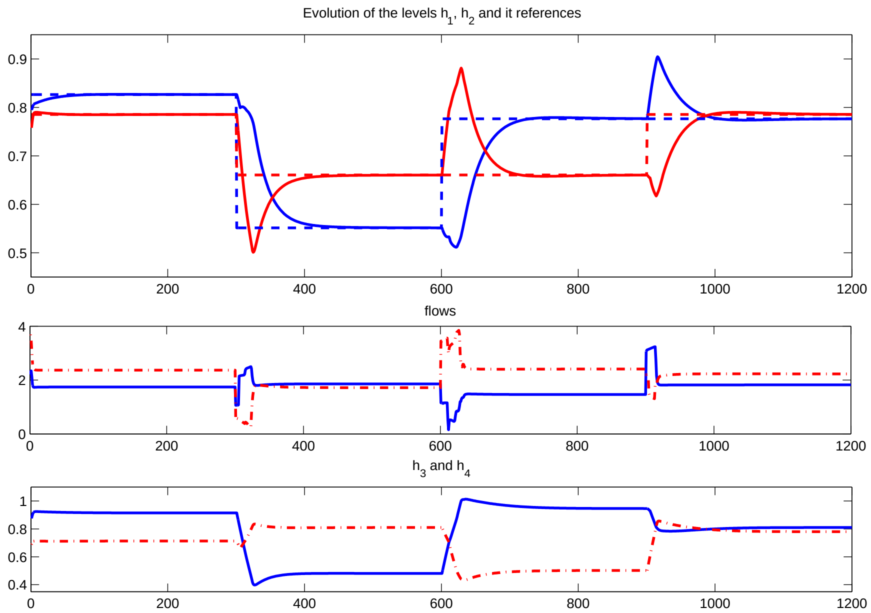


Figure 9.7: Simulation of the time evolution of the plant

9.6 Conclusions

In this chapter we have analyzed the quadruple tank process, which is a nonlinear uncertain multivariable process configured to work at operation points characterized by non-minimum phase multivariable zeros. The proposed robust MPC for tracking solves successfully this issue.

The main problem to apply this controller is the calculation of the invariants sets due to the dimension of the problem. To overcome this problem, the introduced procedure in section 6.2.4 was successfully applied.

The trajectory of the controlled plant is admissible but presents offset due to the persistent nature of the disturbances, in order to avoid this offset, the constant disturbance rejection scheme is successfully used.

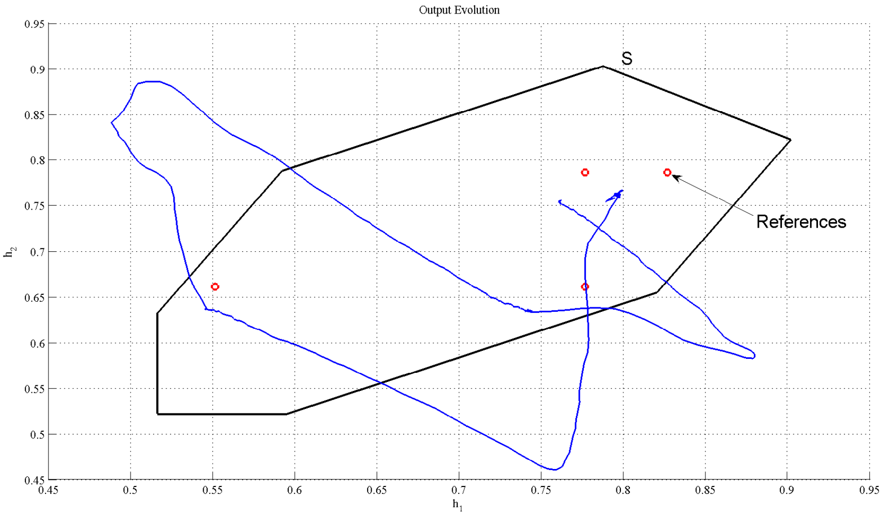


Figure 9.8: Evolution of the outputs

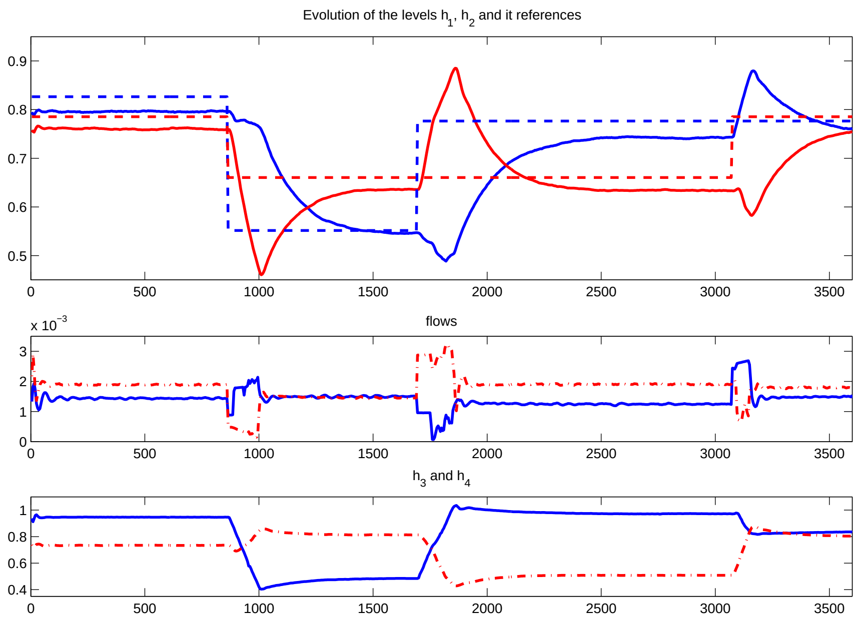


Figure 9.9: Time evolution of the plant

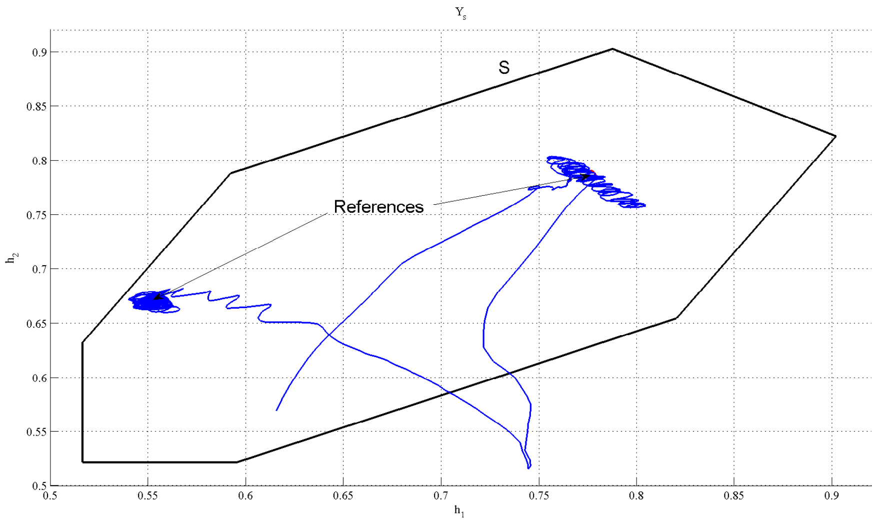


Figure 9.10: Evolution of the outputs

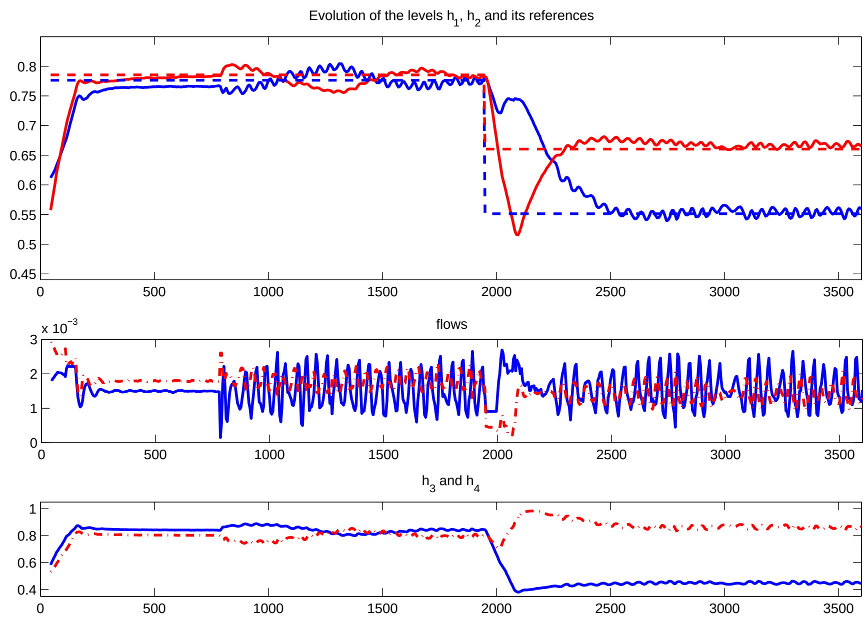


Figure 9.11: Time evolution of the plant

Chapter 10

Conclusions

10.1 MPC for tracking of piece-wise constant references for constrained linear systems

In this thesis, a novel (not switching) MPC is proposed to track any desired steady state ensuring feasibility and convergence for any prediction horizon. The main differences with the standard MPC for regulation are:

1. Considering an artificial steady state and input as decision variables (in a similar way as the reference governors).
2. Modifying the cost function to penalize the deviation to the artificial steady state.
3. Adding an additional term that penalizes the deviation between the artificial steady state and the desired one.
4. Considering an extended terminal constraint (an invariant set for tracking) on both the terminal state and the artificial steady state.

The main advantages of this controller are:

- It is able to reach any reference independently of the horizon.

- Solves the problem of admissible reference, constraints satisfaction and asymptotical stability in one shot, just only solving a single QP.

- The structure of the QP allow to apply multiparametric result can to compute the explicit control law, and apply it to fast systems.

- Due to the change of reference is not considered as a disturbance, the rank of reachable admissible setpoint is wider than existing predictive controllers for tracking.

- The region of attraction is (potentially) bigger than in the case of MPC for regulation due to the terminal region (the invariant set for tracking) is also (potentially) bigger.

10.2 Robust MPC for tracking of constrained linear systems with additive disturbances

In this thesis, a novel formulation of robust MPC for tracking for constrained linear system subject to additive and bounded disturbances is proposed. This is capable to lead the system to any robustly admissible set point in an admissible way. The proposed controller follows the novel MPC formulation presented in section (2.3.1) and in (Limon, Alvarado, Álamo and Camacho, 2005). This controller is going to lead the nominal system to reference and, by incorporating the notion of tube-based robust control, the real system is going to be steered to a neighborhood of the reference.

The obtained robust controller is based on the solution of a single Quadratic Programming problem and the decision variables are the initial state of the nominal system, the control sequence over a finite horizon and the artificial steady state.

Under mild conditions, the proposed controller ensures robust and admissible convergence to (a neighborhood of) the desired steady state, and maintains these properties under any change of reference. Moreover, offset free control can be achieved by means of a simple procedure if the disturbances converge to a constant value.

10.3 Enhanced Controller Design

There is a 2 tuning gains of this controller, the first one is chose to make the invariant set for tracking as big as possible, the second is calculated to make minimal the robust invariant set which is the section of the tube. In this section the problem of the choice of this second gain is accomplished.

The procedure to calculate this gain $u = Kx$ is based on the following ellipsoid $\mathcal{E}(P, 1) = x \in \mathbb{R}^n \quad x^T P x \leq 1$ that should satisfies the following conditions:

1. The ellipsoid must be an invariant set for the system: $x^+ = (A + BK)x + v$
2. The maximum value of u in the ellipsoid is bounded $\forall x \in \mathcal{E}(P, 1)$, $|K_i x| \leq \rho_i$, $i = 1, \ldots, m$ (where K_i denotes the i-row of the matrix K).
3. The size of the ellipsoid should be minimum.

This problem can be posed as an LMI problem.

10.4 Output feedback Robust tube based MPC for tracking of piece-wise constant references

The proposed controller for tracking is based on the results presented in section (2.3.1) and in (Mayne, Rakovic, Findeisen and Allgöwer, 2006). This paper is an extension of the concept of tubes to the robust output feedback MPC case. The real state is confined in a tube, whose center is the estimated trajectory and its section is an invariant set (estimation tube). In an analogous way, the estimated state is going to be confined by the control law in another tube (control tube), which center is the nominal trajectory (the system without disturbances) and its section is an invariant set, therefore, the real state is going to be confined in a tube with center is the nominal trajectory and its section is a 'larger' invariant set that results of the Minkowsky addition of the other 2. Then using the tracking algorithm proposed in section (2.3.1) the nominal system is steered to the desired setpoint. Therefore the real state will be steered to the invariant set aforementioned 'centered' in the desired steady state.

The constraints for the nominal system would be those that the tube satisfies the real constraints.

10.4.1 Experimental Applications

All of this controllers have been applied to real plants, in order to remonstrate its applicability. The real plants are the following:

1. A positioning system based on a linear motor
2. A distributed solar collector field of the Solar Power Plant at Almería (PSA) called ACUREX
3. The Four Tank Process

10.4.2 The positioning system based on a linear motor

A deeper description of this positioning plant can be found in (Fiacchini et al., 2006). The objective is control the position of the linear motor actuating on speed set-point of the low-level controller.

The linear model of the plant and the uncertainties were identified from the experimental result exciting the plant with a PRBS signal.

The controller was implemented on the dSpace Control Card to be tested on the real plant. Given that the sampling time is 30 ms, an explicit implementation of the proposed MPC results to be compulsory. This was achieved by using multiparametric programming techniques and calculating a search tree to express the control law (Tøndel et al., 2002). The obtained search tree has been coded as an S-function in C language and compiled to run on the dSpace Card.

This plant was successfully controlled by the controllers introduced in section (2.3.1)

With the porpoise to add some disturbances a inverted pendulum located over the positioning system was released. The system with the additional disturbance was successfully controlled by robust tracking formulation introduced in section (2.3.2).

10.4.3 ACUREX

The distributed collector field ACUREX, is located at the solar power plant of PSA (Solar Plant of Almería). This plant is aimed to collect solar energy to be transformed in electricity by heating oil. The main characteristic of a solar power plant is that the primary energy source, solar radiation, cannot be manipulated. Solar radiation varies throughout the day, causing changes in plant dynamics and strong disturbances in the process. The real plant is assumed to be modelled as a linear system with additive bounded uncertainties on the states.

This plant in function of the energy demand, the disturbance rejection or depending on the values of the inlet temperature of the collectors, varies its setpoint, witch makes necessary the use a control for tracking.

Under mild assumptions, the controller proposed in section (2.3.2) can steer the uncertain system in an admissible evolution to any admissible steady state, that is, under any change of the set point. This allows us to reject constant disturbances compensating the effect of then changing the setpoint.

10.4.4 The Four Tank Process

The four tank process is an experimental tank system developed at the University of Seville for process control education and research. This plant is based on the well known quadruple-tank process (Johansson, 2000*a*).

The quadruple tank process consist on 4 interconnected tanks that can be easily configured to exhibit the effect of multivariable zero (minimum and non-minimum phase) on the system behavior. Another interesting features of the plant are:

- It is a MIMO plant.
- All the states are accesible.
- It is a constrained plant
- It is a uncertain non-linear plant (modelled as a linear with disturbances for our controllers)

In the real plant implementation, the original structure of the process has been modified to offer a wide variety of uses for both educational and research purposes. Thus, different plants can be configured such as one single tank, two or three cascaded tanks, a mixture process and hybrid dynamics. Moreover the dynamics parameters of each tank can be set up by tuning the cross-section of the outlet hole of the tank.

Furthermore, the real plant has been implemented using industrial instrumentation and a PLC for the low level control. Supervision and control of the plant is carried out in a computer by means of OPC (Ole for Process Control) which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or industrial SCADA.

Over this plant was successfully implemented the controller introduced in section (2.3.2).

Chapter 11

Conclusiones

El problema del cambio de referencia en el ámbito de los controladores predictivos es, que se puede producir una perdida de la factibilidad del problema por una de las siguientes causas:

- La restricción terminal para el nuevo punto de equilibrio puede no ser un invariante admissible lo que puede provocar la perdida de factibilidad.
- La region terminal para el nuevo punto de operación podría no ser alcanzable en N pasos,

El objetivo de esta tesis es el estudio del problema de la perdida de factibilidad, en ella se propone un nuevo tipo de controlador para el seguimiento de referencias capaz de asegurar la evolución admisible del sistema y un desempeño óptimo para cualquier cambio de referencia admisible.

11.1 Control predictivo para seguimiento de referencias constantes

En esta tesis se propone una nueva formulación de control predictivo para sistemas lineales con restricciones capaz de seguir cualquier valor de referencia constante satisfaciendo las restricciones.

Las principales características del controlador propuesto son:

1. Considera, como variable de decisión, un valor de los estados y entradas de equilibrio como que juegan el papel de referencia artificial.
2. Se modifica la función de coste de forma que penalice el error con la referencia artificial.
3. Se añade un término adicional a la función de coste, que penaliza la desviación entre la referencia y la referencia artificial.
4. Se considera como restricción terminal en el estado y referencia artificial un invariante para tracking.

Este controlador presenta las siguientes ventajas:

- Es capaz de alcanzar prácticamente cualquier valor de referencia admisible para cualquier valor de horizonte de predicción
- El cálculo de la acción de control requiere la solución de un único problema de optimización QP
- La estructura del problema de optimización permite determinar de forma explícita la ley de control por medio del herramientas de programación multiparamétrica.
- La región de atracción es potencialmente mayor que la del MPC estándar.

11.2 Control predictivo robusto para seguimiento de referencias constantes

Otra novedad presentada en la tesis es el desarrollo de un controlador predictivo robusto para sistemas lineales con incertidumbres aditivas acotadas y restricciones. El controlador es capaz de conducir al sistema de una forma robusta a un punto de equilibrio admisible del sistema nominal. Este control se basa en el controlador para seguimiento presentado en el capítulo anterior. Incorporando la noción de tubo de trayectorias se consigue hacer el controlador robusto satisfaciendo las restricciones.

El controlador propuesto tan sólo requiere la solución de un problema de programación cuadrática QP en el cual aparecen como variables de decisión la secuencia nominal de acciones de control, la referencia artificial y el estado inicial del tubo.

Bajo ciertas condiciones, el controlador propuesto garantiza la convergencia robusta y admisible a una vecindad de un punto de equilibrio objetivo, y además mantiene las propiedades de convergencia ante cualquier cambio en la referencia.

Por ultimo, en el caso de perturbaciones persistentes, éstas producen un offset en las salidas que se podría compensar incorporando un estimador de perturbaciones y añadiendo un bucle externo que modifica las referencias.

11.3 Control predictivo robusto con realimentación de salidas para el seguimiento de referencias constantes

El controlador predictivo robusto para seguimiento anteriormente presentado, se adapta para acometer el problema de diseñar el controlador para el caso en que no todo el estado es medible, basándose en (Mayne, Rakovic, Findeisen and Allgöwer, 2006). Para ello, el estado real de la planta estará confinado en un tubo (tubo de estimación) cuyo centro es la trayectoria estimada y cuya sección un conjunto invariante para la dinámica del error de observación. El error de control también se encuentra confinado en un tubo de sección un conjunto invariante. La suma de ambos tubos, permite acotar el efecto de las incertidumbres en un tubo de sección la suma de las secciones de los tubos de error de control y observación.

Así, el controlador propuesto es capaz de garantizar la convergencia al punto de equilibrio deseado a partir de las medidas de las salidas. Además se puede cancelar el offset en las salidas gracias a la estimación de estados y la variación del setpoint.

11.3.1 Síntesis de los controladores propuestos

El controlador predictivo propuesto logra sus propiedades siempre que los parámetros que definen el problema cumplan una ciertas condiciones. Dado que estas condiciones sobre los parámetros no los fijan de manera unívoca, estos se pueden ajustar de forma que se cumplan ciertas especificaciones de control.

Parte de los parámetros que se pueden ajustar son estándar en los controladores predictivos, tales como las matrices de ponderación del coste de etapa Q y R, la del coste terminal P, y fijan el desempeño del sistema en ausencia de incertidumbres.

Los parámetros más característicos del controlador propuesto son la ponderación del coste de offset T y la elección de la ganancia del controlador robusto. El parámetro T permite fijar la rapidez de la evolución de la referencia artificial y permite priorizar salidas para minimizar el offset, en caso de que exista. Por otro lado, la ganancia del controlador robusto permite acometer el rechazo a perturbaciones, que en esta tesis se propone como criterio la minimización del tamaño del mínimo invariante robusto. Esto además permitirá aumentar el dominio de atracción. Este controlador se puede diseñar mediante la resolución de LMIs.

Finalmente, se presenta métodos para calcular estimaciones del mínimo invariante robusto cuando las incertidumbres son paralelotopes.

11.4 Aplicaciones experimentales

Los controladores propuestos han sido aplicados a sistemas reales con objeto de probar su aplicabilidad. Estas plantas son las siguientes:

1. Un sistema de posicionamiento basado en un motor lineal.
2. Una gran instalación solar térmica llamada ACUREX perteneciente a la PSA (Planta Solar de Almería)
3. El proceso de los 4 tanques

11.4.1 Sistema de posicionamiento basado en un motor lineal

Una descripción exhaustiva de este sistema de puede encontrar en (Fiacchini et al., 2006). El objetivo es controlar la posición de un motor lineal actuando sobre la consigna del control de velocidad del mismo.

El modelo lineal con y sin incertidumbres ha sido identificado a partir de datos experimentales excitando al planta con una señal PRBS

El controlador ha sido implementado en una tarjeta de control dSpace que gobierna la planta. El tiempo de muestreo es de 30 ms lo que ha requerido el cálculo de la solución explícita de dicho controlador, esto se ha conseguido con técnicas de programación multiparamétricas. La búsqueda de la solución apropiada se ha acelerado mediante un árbol de búsqueda que también se ha implementado en la tarjeta dSpace.

Esta planta ha sido controlado con éxito por el controlador presentado en la sección (1.7.1)

Con objeto de probar la formulación robusta del mismo, sección (1.7.2), se le ha añadido sobre el motor lineal un péndulo invertido, que añade una perturbación adicional.

11.4.2 ACUREX

ACUREX es una instalación solar térmica de grandes proporciones del complejo PSA (Planta Solar de Almería) que produce energía eléctrica a partir de aceite caliente.

La principal característica de esta planta es que la fuente primaria de energía, la radiación solar, no se puede controlar. Además está sujeta a grandes variaciones durante el día, causando cambios en la dinámica de la planta y fuertes perturbaciones sobre le proceso. La planta real se modela como una planta lienal con incertidumbres en los estados. La planta, en función de la energía demandada, del rechazo de perturbaciones, o de los valores de la temperatura de entrada de colectores, varía el punto de trabajo, siendo necesario un controlador preparado para puntos de referencia cambiantes.

Bajo ciertas condiciones suaves, el controlador propuesto en la sección (1.7.2) es capaz de llevar al sistema incierto de forma factible a cualquier punto de trabajo admisible, permitiéndonos rechazar perturbaciones compensando su efecto cambiando el punto de trabajo.

11.4.3 El proceso de los 4 tanques

El proceso de los 4 tanques es una planta experimental basada en el conocido artículo de (Johansson, 2000*a*) que se ha desarrollado como parte de esta tesis y cuyos fines son educativos y de investigación.

El proceso de diseño, la instrumentación elegida y de más detalles del diseño se pueden encontrar en el apéndice (B).

El proceso de los 4 tanques consiste en cuatro tanques interconectados que pueden ser fácilmente configurados para mostrar el efecto de los ceros de transmisión de fase mínima y no mínima sobre el comportamiento del sistema. Otras características interesantes de la planta son:

- Es una planta MIMO.
- Todos los estados son accesibles.
- Es un proceso con restricciones
- Es una planta no lineal con incertidumbres (modelado como lineal con incertidumbres aditivas para nuestro controlador)

En la implementación de la planta real, la estructura original del proceso se ha modificado con el fin de ofrecer una amplia variedad de usos, tanto para uso educacional como para investigación. Así, se pueden configurar diferentes plantas, tales como un depósito aislado, 2 o 3 en cascada, un proceso de mezcla o bien un proceso híbrido. Además los parámetros que caracterizan la dinámica de cada tanque se pueden ajustar variando la sección de salida de los depósitos.

Merece la pena destacar que la instrumentación utilizada en la implementación de la planta es instrumentación industrial, lo que confiere a la planta un mayor parecido a un proceso industrial. El control a bajo nivel se realiza en un PLC y el control y supervisión a alto nivel se realiza en un PC, que permite la conectividad con otros equipos mediante el protocolo estándar OPC (LabView, Matlab, SIMATIC IT, etc.).

Los controladores propuestos en esta tesis se han probado con éxito sobre esta planta.

Appendix A

Appendix of the chapter 3

A.1 Proof of lemma 3.1

Assume that $r < n+m$ and consider the following two cases:

- The rank of E is $n+p$:
  In this case the minimal singular value decomposition is given by $E = U\Sigma V^T$ where U is a full rank square matrix. Then

$$\begin{aligned} EM\theta &= U\Sigma V^T[V\Sigma^{-1}U^T FG \quad V_\perp]\theta \\ &= [U\Sigma V^T V\Sigma^{-1}U^T FG \quad U\Sigma V^T V_\perp]\theta \end{aligned}$$

  From the properties of matrices U and V we have that $U\Sigma V^T V\Sigma^{-1}U^T = I$ and $V^T V_\perp = \mathbf{0}$. Then

$$\begin{aligned} EM\theta &= [FG \quad \mathbf{0}_{n+p,n+m-r}]\theta \\ &= F[G \quad \mathbf{0}_{p,n+m-r}]\theta = Ft \end{aligned}$$

- The rank of E is lower than $n+p$:
  In this case the minimal singular value decomposition is given by $E = U\Sigma V^T$ where matrix U is not full rank. Consider the following square full rank matrix $\Pi = [U \ U_\perp]^T$, then equation (3.3) holds if and only if $\Pi E z_s = \Pi F t$. Taking into

account that $E = U\Sigma V^T$, the left hand side of this equation is

$$\begin{bmatrix} U^T \\ \\ U_\perp^T \end{bmatrix} U\Sigma V^T z_s \;=\; \begin{bmatrix} U^T U\Sigma V^T z_s \\ \\ U_\perp^T U\Sigma V^T z_s \end{bmatrix} = \begin{bmatrix} \Sigma V^T z_s \\ \\ \mathbf{0} \end{bmatrix}$$

Then z_s is solution if and only if the following inequalities hold

$$\Sigma V^T z_s = U^T F t \quad \text{(A.1)}$$
$$\mathbf{0} = U_\perp^T F t \quad \text{(A.2)}$$

Evaluating the left hand side of equation (A.1) we have that

$$\begin{aligned} \Sigma V^T z_s &= \Sigma V^T [V\Sigma^{-1} U^T F G \quad V_\perp]\theta \\ &= [U^T F G \quad \mathbf{0}]\theta = U^T F [G \quad \mathbf{0}]\theta = U^T F t \end{aligned}$$

which proves equation (A.1). On the other hand, given that $t = [G \quad \mathbf{0}]\theta$, the right hand side of equation (A.2) is given by

$$U_\perp^T F t = U_\perp^T F [G \quad \mathbf{0}]\theta = [U_\perp^T F G \quad \mathbf{0}]\theta$$

From definition of matrix G we have that $U_\perp^T F G = \mathbf{0}$ and hence (A.2) holds.

In the case that $r = n + m$ then $V_\perp$ there does not exist and the proof it is inferred directly from the previous case.

Appendix B

Designing of the Four Tank Process

B.1 Introduction

One of the difficulties in control education consists in providing a theoretical foundation maintaining the practicality. To this aim, experimental labs provide a powerful tool to fill this gap. An experimental lab should be designed to show interesting and industrially relevant control problems which require not too skilled control solutions and real tools, such as instrumentation, control programs, etc.

The quadruple tank process has proved to be a very interesting system for control education in advanced control courses as well as in research courses (Johansson, 2000*a*; Johansson et al., 1999; Rusli et al., 2004; Long, Holland, and Gatzke, 2005). The main property of this process is that it is appropriate to illustrate the importance of multivariable zeros since these can be located either at the right or at the left half plane. Furthermore, there exist other interesting properties, such as the coupled nature of the plant, the measurable states or the nonlinear behavior for instance.

To this objective, a laboratory plant based on the quadruple tank process has been designed and developed at University of Seville. This plant is used for both educational and research purposes. The objective of the design has been to provide flexibility to the plant in such a way the plant can be easily configured to obtain different processes and the different control implementation procedures. Thus, the plant has been implemented to be controlled from the PLC, from an external device or from a computer by means of an open and standard protocol OPC (OLE for Process Control). Thus, any control software with OPC connectivity (such as MATLAB, LAbView or commercial SCADAs) can be used to control the plant.

B.2 Description of the ideal system

The four tanks system is a control benchmark proposed in (Johansson, 2000a) and composed of four interconnected tanks as shown in figure B.1. A flow of water q_a is pumped by pump A from the reservoir tank and poured in tanks 1 and 4 with flows q_1 and q_4 respectively, depending on the position of a 3-ways valve. In a similar way, pump B pours the water (q_b) in tanks 2 and 3. All tanks are discharged by gravity as follows: tank 3 discharges in 1, 4 in 2, and both 2 and 1 discharges in the reservoir tank.

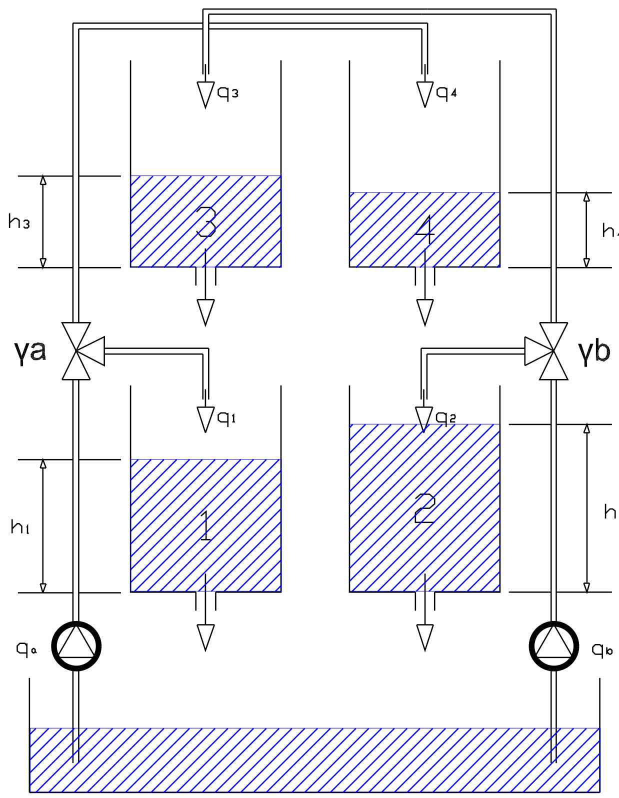


Figure B.1: The four tank process

The objective of the plant is to control the level of lower tanks 1 and 2 (denoted as h_1 and h_2 respectively) manipulating the pumps. This is not an easy task due to variables h_1 and h_2 are strongly coupled. Note that an increase in the level of tank 1 gives rise to a level increase in tank 4, and therefore in tank 2.

This plant can be used to show very interesting control problems. Among these problems, the following ones can be highlighted:

- Control of multivariable systems.
- Control of systems with RHPZ and limits of performance.
- Robust control.
- State estimation.
- Tracking of constant references.
- Control under saturating actions.
- Control of systems subject to constraints.

Because of the previous features, the quadruple tank plant has been considered as a very interesting system to be used as benchmark for design control strategies. In fact, this plant has been chosen as one of the control benchmarks in the Network of Excellence HYCON of the the IV Framework Program of the European Union.

B.2.1 Model of the system

In order to carry out the analysis and control of the plant, a mathematical description of the plant dynamics is required. The model has been derived under following simplifying hypothesis:

(i) The 3-way valves are ideal. Thus, the input flows to each tank satisfy the following equations:

$$q_1 = \gamma_a \cdot q_a$$

$$
\begin{aligned}
q_2 &= \gamma_b \cdot q_b \\
q_3 &= (1-\gamma_b) \cdot q_b \\
q_4 &= (1-\gamma_a) \cdot q_a
\end{aligned}
\tag{B.1}
$$

where $0 \leq \gamma_a \leq 1$ and $0 \leq \gamma_b \leq 1$ are the characteristics parameters of the 3-way valve depending on their positions.

(ii) The tanks are discharged by gravity. Applying the mass balance and Bernoulli equations, the speed of the water through the outlet hole is

$$v_1 = k \cdot a_h \cdot \sqrt{2 \cdot g \cdot h}\ \frac{m}{s}$$

where a_h is the cross-section of the outlet hole, k models the friction and it is measured in a experimental way, g is the gravity constant, h is the tank level and A is the tank section.
The output flow of the tank is:

$$q_s = a_h \cdot v_1$$

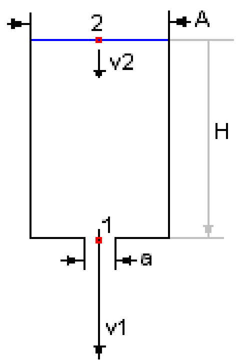


Figure B.2: Model of one tank.

The dynamics equation of one tank is derived from the application of a mass balance to the tank, yielding

$$\frac{d(volume)}{dt} = q_{in} - q_{out},$$

Considering that the cross-section of the tank is constant and the mode of the output flow,

$$A \cdot \frac{d(h)}{dt} = q_{in} - a_h \cdot k \cdot \sqrt{2 \cdot g \cdot h}$$

Once a model of an isolated tank and the 3-ways valves have been derived, the model of the whole plant is obtained taking into account the interconnection of the tanks. This leads to the following model:

Nonlinear model of the quadruple tank plant

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\cdot\sqrt{h_1} + \frac{a_3}{A_1}\cdot\sqrt{h_3} + \frac{\gamma_a}{A_1}\cdot q_a \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\cdot\sqrt{h_2} + \frac{a_4}{A_2}\cdot\sqrt{h_4} + \frac{\gamma_b}{A_2}\cdot q_b \\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\cdot\sqrt{h_3} + \frac{1-\gamma_b}{A_3}\cdot q_b \\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\cdot\sqrt{h_4} + \frac{1-\gamma_a}{A_4}\cdot q_a\end{aligned} \tag{B.2}$$

where:

- γ_a and γ_b are no dimensional parameters to set the flow proportion of the two outlets of the 3-ways valve V_a and V_b respectively. These parameters must be limited to the interval $[0, 1]$.
- q_a and q_b are the pump flows.
- $a_j = a_{hj}\cdot k\cdot\sqrt{2\cdot g}$ is called the discharge rate of the tank j.

Notice that $\{h_1 \cdots h_4\}$ are the state variables of the system.

B.2.2 Analysis of the model

In order to analyze the behavior of the system, the model has been linearized in a equilibrium point given by $(h_1^0, h_2^0, h_3^0, h_4^0)$. The resulting linear model is the following

$$\frac{dx_1}{dt} = \frac{-a_1}{A_1}\cdot\sqrt{\frac{g}{2\cdot h_1^0}}\cdot x_1 + \frac{a_3}{A_1}\cdot\sqrt{\frac{g}{2\cdot h_3^0}}\cdot x_3 + \frac{\gamma_a}{A_1}\cdot u_a$$

$$\begin{aligned}
\frac{dx_2}{dt} &= \frac{-a_2}{A_2} \cdot \sqrt{\frac{g}{2 \cdot h_2^0}} \cdot x_2 + \frac{a_4}{A_2} \cdot \sqrt{\frac{g}{2 \cdot h_4^0}} \cdot x_4 + \frac{\gamma_b}{A_2} \cdot u_b \\
\frac{dx_3}{dt} &= \frac{-a_3}{A_3} \cdot \sqrt{\frac{g}{2 \cdot h_3^0}} \cdot x_1 + \frac{1-\gamma_b}{A_3} \cdot u_b \\
\frac{dx_4}{dt} &= \frac{-a_4}{A_4} \cdot \sqrt{\frac{g}{2 \cdot h_4^0}} \cdot x_4 + \frac{1-\gamma_a}{A_4} \cdot u_a
\end{aligned} \tag{B.3}$$

where

- $x_i = h_i - h_i^0 \quad i \in \{1, 2, 3, 4\}$.
- $u_j = q_j - q_j^0 \quad j \in \{a, b\}$.

The model can be posed as follows

$$\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A_3}{A_1 \cdot \tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2 \cdot \tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_1)}{A_4} & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x
\end{aligned} \tag{B.4}$$

where the constants τ_i are

$$\tau_i = \frac{A_i}{a_i} \sqrt{\frac{2 \cdot h_i^0}{g}} \geq 0 \quad i \in \{1, 2, 3, 4\}$$

Notice that each constant τ_i is positive and depends on the operating point.
From this model, it is possible to infer that the eigenvalues of the system are $\frac{-1}{\tau_i}$.
Therefore, the system is stable and it exhibit a exponential behavior.

The transfer function of the linearized system is as follows

$$G(s) = \begin{bmatrix} \frac{\gamma_1 \cdot c_1}{(1+s\cdot\tau_1)} & \frac{(1-\gamma_2)\cdot c_1}{(1+s\cdot\tau_1)(1+s\cdot\tau_3)} \\ \frac{(1-\gamma_a)\cdot c_2}{(1+s\cdot\tau_2)(1+s\cdot\tau_4)} & \frac{\gamma_b \cdot c_2}{(1+s\cdot\tau_2)} \end{bmatrix} \tag{B.5}$$

where $c_1 = \frac{\tau_1}{A_1}$ and $c_2 = \frac{\tau_2}{A_2}$. The determinant of the transfer matrix is:

$$\det\left(G(s)\right) = \frac{\gamma_a \cdot \gamma_b \cdot c_1 \cdot c_2}{\prod_{i=1}^{4}(1+s\cdot\tau_i)} \left((1+s\cdot\tau_3)\cdot(1+s\cdot\tau_4) - \frac{(1-\gamma_a)\cdot(1-\gamma_b)}{\gamma_a \gamma_b} \right)$$

which is a transfer function with four negative real poles and two zeros. As it was previously stated, the poles are in $s = -\tau_i$ and hence the system is locally stable. In order to analyze the nature of the zeros, the roots of the numerator polynomial

$$\tau_3 \tau_4 s^2 + (\tau_3 + \tau_4)\cdot s + 1 - \frac{(1-\gamma_a)(1-\gamma_b)}{\gamma_a \gamma_b} = 0$$

are studied, deriving the following conclusions:

- If $0 \leq \gamma_a + \gamma_b < 1$, the system has two real zeros. One of them has the real part positive, and therefore, instable (non minimum phase).
- If $1 < \gamma_a + \gamma_b \leq 2$, the system has two zeros with real part negative (minimum phase).
- If $\gamma_a + \gamma_b = 1$, the system has a double zero in the origin.
- The four poles are stable for all the values of γ_a and γ_b.
- The sign of the zeros and poles is operating point independently and the opposite for the module.

B.3 Description of the real plant

B.3.1 Plant design

The main objective in the design of the real plant is the implementation of the quadruple tank system. However, additional objectives have been coped with, leading to modified configuration that can be seen in the figure B.3. These modifications are the following:

1) In order to allow several configurations, hydraulics circuits and extra valves have been added.

2) The original hydraulic circuit with two controlled pumps and two 3-way valves has been replaced by a conventional pump and four pneumatic valves and four magnetics flow meters. This allows us to control the inflow of each tank by manipulating the pneumatic valves. For a given total flow q_a or q_b and the desired ratio of flows γ_a and γ_b, the desired flows that feed each tank is derived.

   Another obtained advantage it that the capability to set up the plant with different configurations. Some of them are shown in figure B.4. This allows us to use the plant to implement different systems with different dynamics, number of inputs, etc.

3) The dynamics of the plant can be adjusted by means of the manipulable valves located in the drainage of each tank.

4) A filled up and drainage circuits of the reservoir tank have been added.

B.3.2 Instrumentation

One of the main property of the plant is that this has been implemented using industrial instrumentation. This makes that the benchmark suitable for testing advances controllers for process industry. The instruments of the plant are the following:

Pump system: This consists of a centrifugal pump with a pressure regulator in the outlet. This regulator also avoid hydraulic and electric damage on the pump. Moreover, in order to avoid that the pump works below its minimum flow, a by-pass pipe has been installed.

Magnetic Flowmeters: These are Siemens Sitrans FM. They are split in two parts:

- *The tube*: *Flow Sensor 711/S*, appropriate for water and heavy water. A conductivity of the fluid higher than 3 S/cm is required for its use. For the present application, given that the water flow is low, a sensor with a

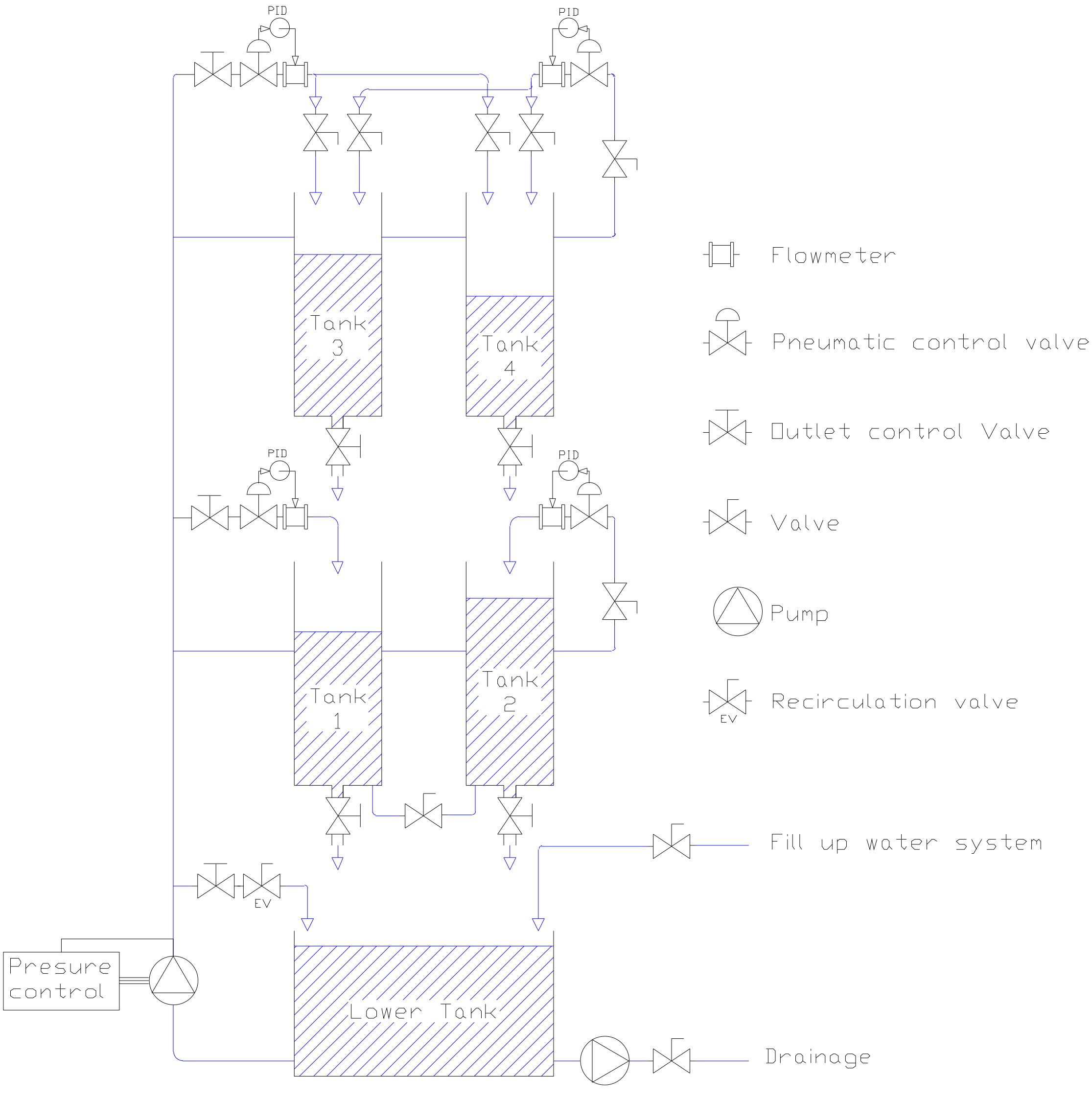
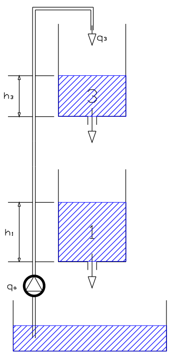
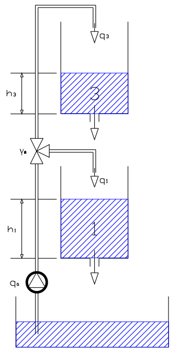


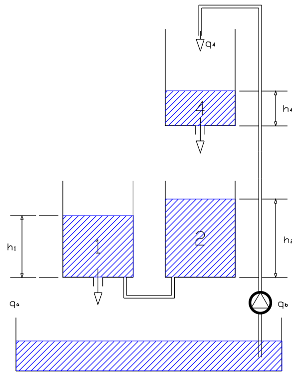
Figure B.3: Real Plant.



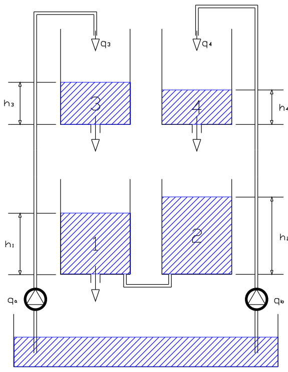
(a) Two cascaded tanks



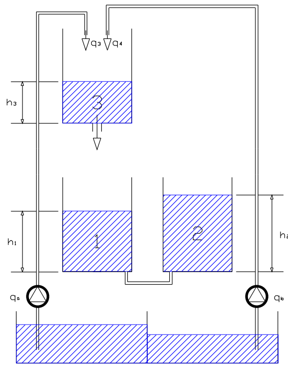
(b) Two cascaded tanks, two inlets.



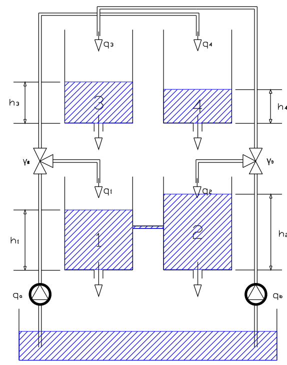
(c) Three cascaded tanks



(d) Four coupled tanks.



(e) Mixture process.



(f) Hybrid process.

Figure B.4: Some configurations of the plant.

nominal size of DN15 has been chosen. It is able to measure flows between 0.16-240 m3/h at speeds ranging between 0.25-12 m/s. This minimum flow value has meant another restriction for the present work.

- *The transmitter*: *Flowmeters Sitrans FM. Transmitters Intermag/Transmag.* It uses an electric supply of 220 V, the outlet signal ranges from 4 to 20 mA and its precision is 0.1% of the conversion error.

Pneumatic Valves: The four valves are *Siemens VC 101* with a pneumatic actuator. The chosen nominal diameter is DN 15 and the valve characteristic is equal percentage. The value of the flow capacity Kvs is 4.472. The valves have the positioning system *Siemens Sipart PS2 PA*, which provides a precision of 0.05%.

Pressure Sensors: They can be of two types, both with similar characteristics, the models *Siemens 7MF4032* and *Siemens 7MF4020*. The measure range is from 0.03 bar to 1 bar, the voltage supply: 11-30 VDC and the outlet signal: 4-20 mA. The precision is 0.1% of the maximum limit of the selected range of operation.

The PLC: The model is *Siemens S7 200* with 10 digital inputs and 14 digital outputs. Some elements have been added to the basic model:

- 2 modules with 4 analogue inputs of 12bits (EM231) for the Flowmeters and pressure sensors.
- 2 modules with 2 analog outputs of 12bits (EM232) for the pneumatic valves.

The PLC includes 4 PIDs to control the flows. It has a direct communication with the PC, reporting the flows and the levels, for instance. The PC provides references for the flows. The PLC jointly with the security circuit and supplying devices are located in the control panel, as shown in figure B.5. Figure B.6 shows

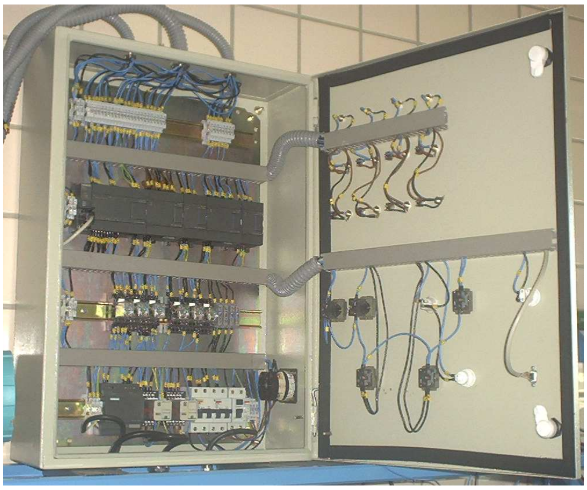


Figure B.5: Electric panel

the wiring circuit in the plant.

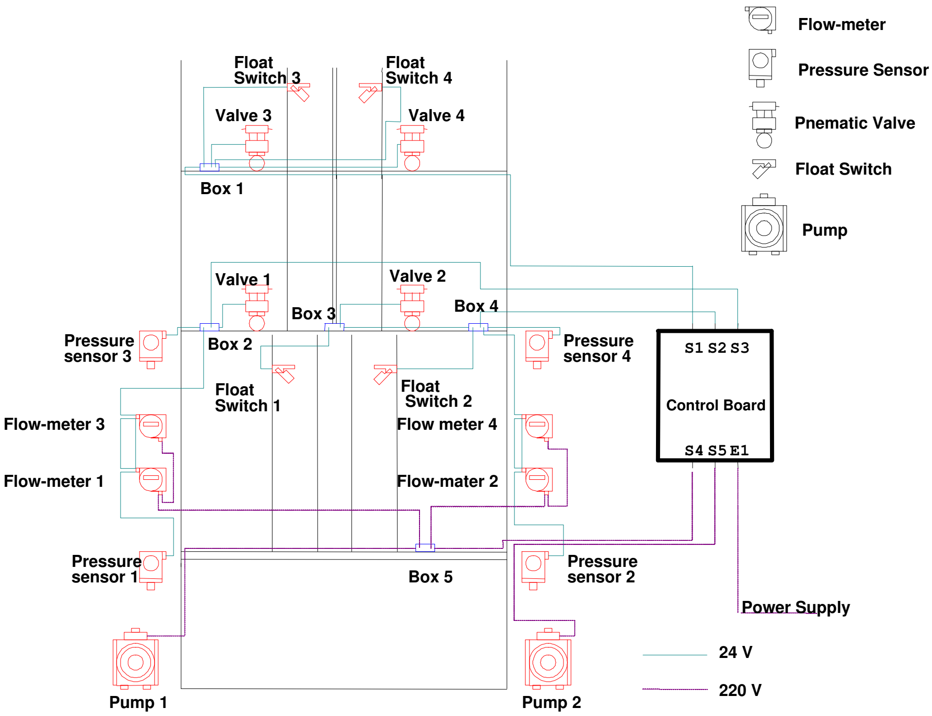


Figure B.6: Instrumentation and wiring in the plant

B.3.3 Control structure of the plant

The objective of the plant is the control of the water level of the tanks 1 and 2. It is done by actuating over the inflows of the four tanks. There is a flow control loop which manipulates the valves based on the information taken from the flowmeters. Moreover, there is a high level control which decides the needed flow to get the desired evolution of the tank levels. In this control structure, the flow control (low level) is done by the PLC. Moreover the PLC receives all the measures of the instruments and send the setpoints to the actuators; the PLC store this data in a real-time data base, which can be accessed from the PC. The high level control is carried out in the PC, which consults the PLC database and defines the proper inflow values of each tank (see figure B.7).

If required, the PLC can be by-passed in such a way that the signals (from or to the instruments) can be directly accessed through plugs located in the front of the control panel.

To guarantee the synchronous access to the database it is necessary to create a

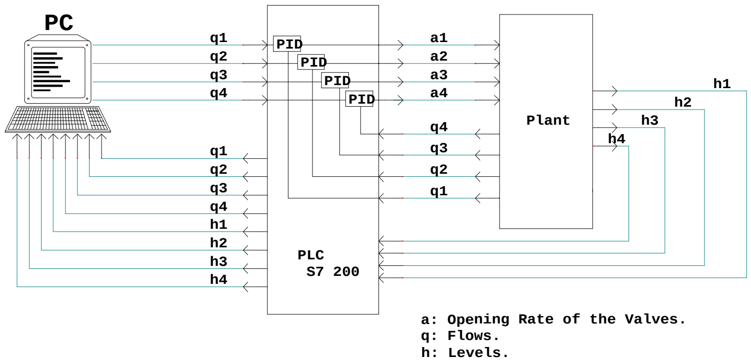


Figure B.7: The computer-based cascaded control structure

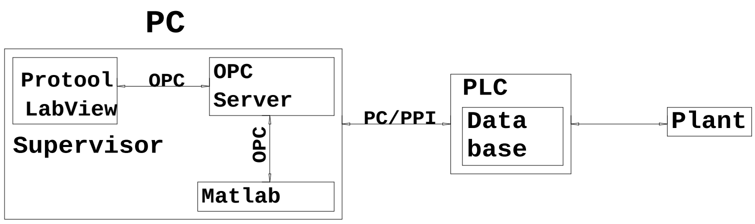


Figure B.8: Computer controlled system by means of OPC

resident application able to administer the database in real time. This requires that the communication with the PLC database must be synchronous, what can be done through a PC/PPI wire and using a free communication protocol, based on RS232. The PC/PPI wire converts the RS232 protocol into the RS485 protocol, suitable for the PLC. On the other hand, another method is based on a resident application which become the PLC in a OPC server. This allows the communication of control applications, which have to be OPC clients, with the database. In this way, via OPC, SCADAS as *Win CC, Protool Pro, Simatic IT* or applications as *LabView* or *Matlab* can be connected for the supervision, historical data management, etc. and for the high level control application (see figure B.8).

A possible inconvenient of the use of this structure could be the fact that the communication through the free port protocol is character by character and it might be slow. Fortunately the communication band width is larger than the plant band-width, and hence the communication method is appropriate.

B.3.4 Communications

The chosen structure to test the proposed controller is shown in figure B.7, where the low-level flow control is implemented in the PLC while the high-level controllers are executed in the PC. To deal with this configuration, a Kepware OPC server is used; this can communicate with the PLC by using the Siemens PC/PPI protocol. Notice that this method allows several clients to connect to this server at the same time, for example using *Matlab* for controller implementation and *Labview* for HMI.

The OPC server must be configured about the kind of PLC connected to or the media used, the memory addresses to be read or written. These addressed are assigned a label, which will be the identifiers of the OPC item. Besides some fictitious variables without any real support can be created, for example the level references given to the plant controller. This server allows the creation of a variable to keep the value written by the client *Matlab*, which can be read by any other client, for example the *Labview* client. Figure B.9 shows the configuration screen of the OPC server.

Moreover, the variables sampling time in the PLC can also be configured, ensuring that the communication is in real time. This can not be guaranteed when the OPC is used, given that, it is a protocol that works over TCP/IP.

B.3.5 Calibration of the PIs

Because of the flow is affected by a certain noise caused by the control that governs the opening of the valves, PI and not PID controllers have been used. But, as it can be seen in figure B.10, the opening-flow characteristic is not linear, what makes the PI controller design more involved. To cope with this problem, the static non-linear characteristic has been approximated by two or three straight lines, so that, it can be used a PI controller using this lines to pseudo-invert this static nonlinearity.

Figure B.10 shows in solid line the flow dependence of the opening rate of the valve, in dashed line the lines used to pseudo linearized that dependence and in dashed-dot line the pseudo linearized dependence.

The resultant control scheme can be seen in figure B.11.

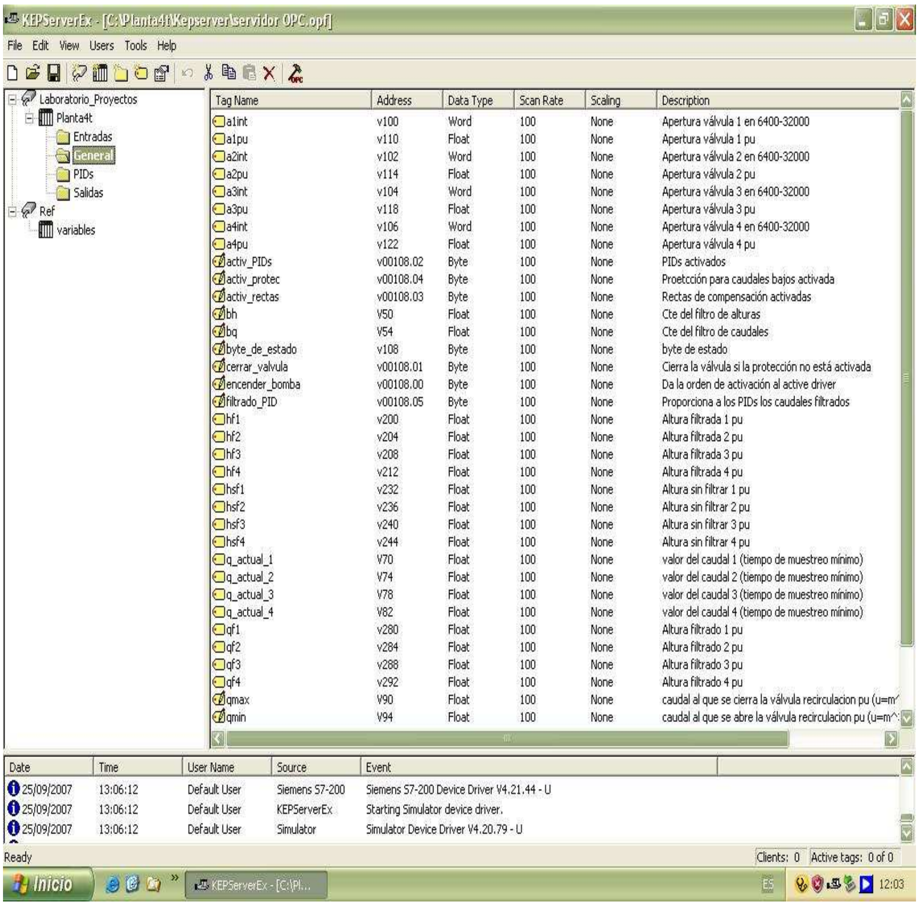


Figure B.9: OPC server

B.4 Startup of the plant

Tuning of the PLC's PIs.

The objective of the PI is controlling the inflow of the tanks by acting on the pneumatic valve (once the opening-flow characteristic have been corrected to get a more linear).

The tuning of the PIs has been designed based on *Labview*. The application allows to see the evolution of the flows and the openings and, at the same time, the manipulation of all the parameters that determine the operation of the four PIs. This screen is shown in figure B.12.

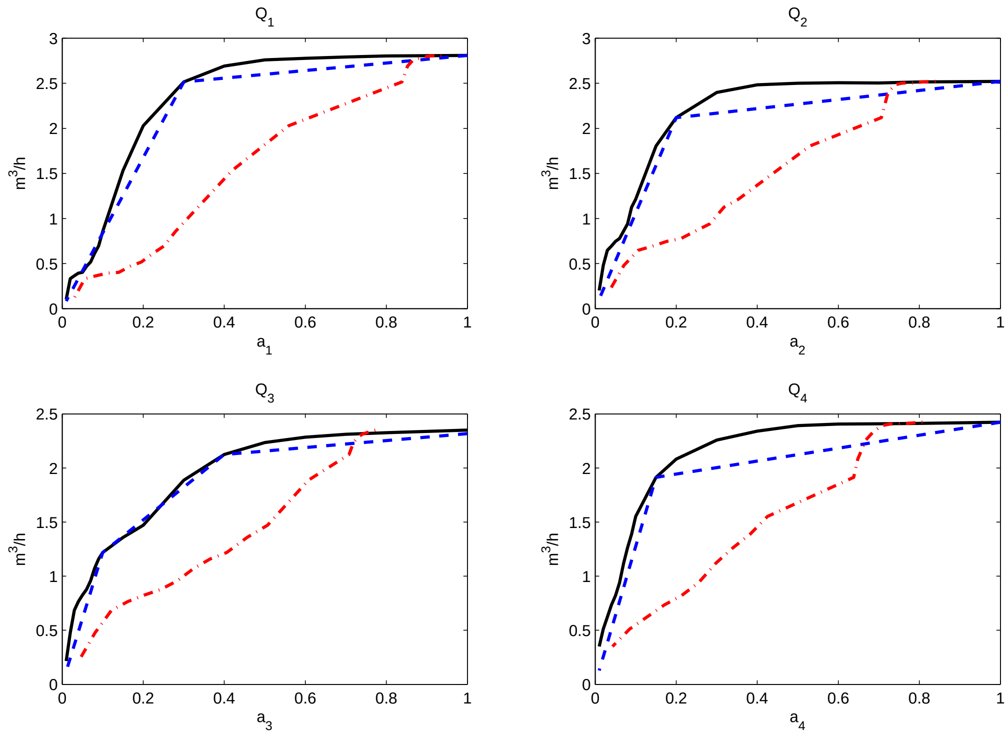


Figure B.10: Flow dependence of opening rate of the valve.

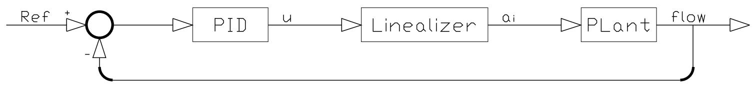


Figure B.11: Control flow scheme

The open loop Ziegler-Nichols method has been used as initial guess of the parameters of the PIs. The system was excited by a step signal and from the answer of it the values of the parameters can be obtained. The constants are:

- τ: Time constant (it is obtained from the time spent to reach the 63% of the final value).
- τ_d: System delay.
- K: Gain.

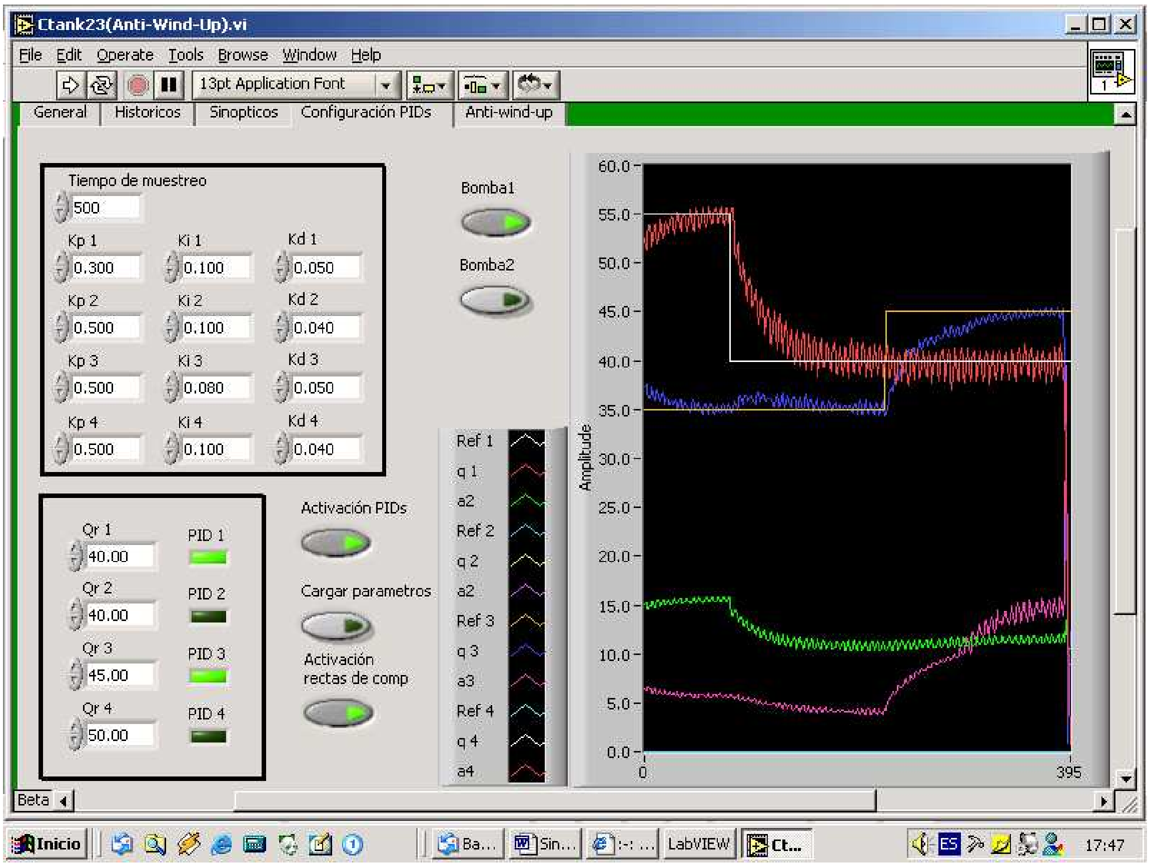


Figure B.12: Screen for the configuration of the PIs.

	K	T_i	T_d
P	$\frac{\tau}{K \cdot \tau_d}$		
PI	$\frac{0.9 \cdot \tau}{K \cdot \tau_d}$	$3 \cdot \tau_d$	
PID	$\frac{1.2 \cdot \tau}{K \cdot \tau_d}$	$2 \cdot \tau_d$	$0.5 \cdot \tau_d$

The answer of the step signal for some different points is shown below. (Figures B.13 and B.14)

For instance, the step signal shown in figure B.13 has been identified with the

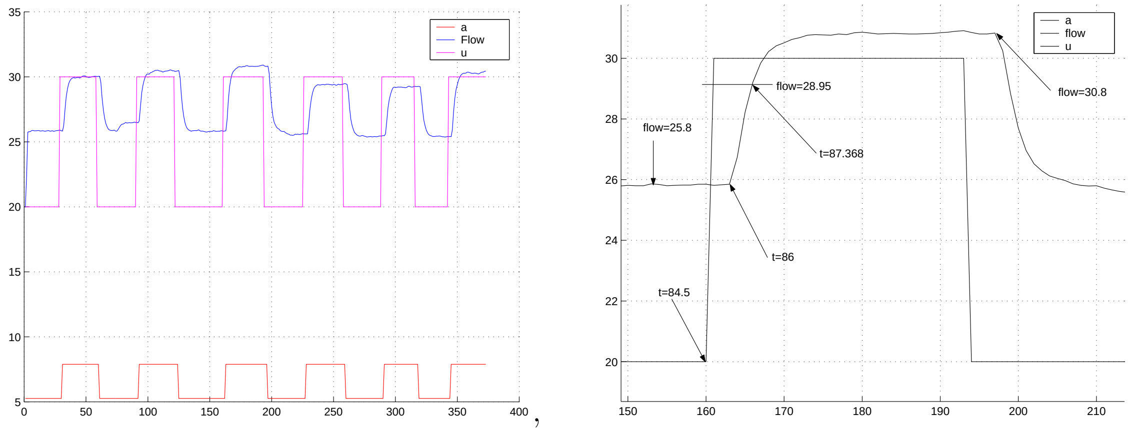


Figure B.13: (a) Step Answer (b)Detail.

constants:

$$K = \frac{30.8 - 25.8}{10} = 0.5$$

$$\tau_d = 86 - 84.5 = 2.5s$$

$$\tau = 87.368 - 86 = 1.368s$$

The procedure is repeated for several steps providing the following values of the constants (see figure B.14):

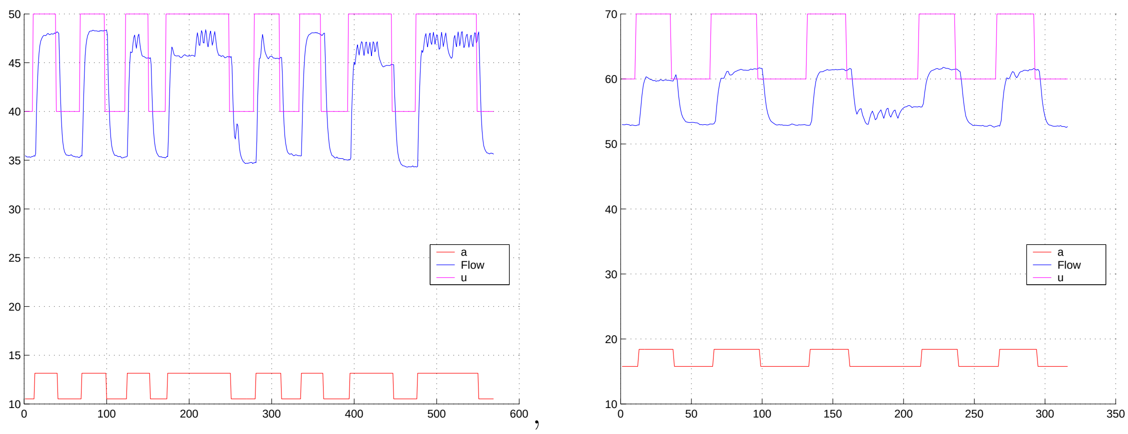


Figure B.14: Steps response.

Opening range	K	τ_d	τ
20%-30%	0.5	2.5	1.368
20%-30%	0.148	1.5	1.495
20%-30%	0.383	2	1.277
20%-30%	0.382	1.5	1.65
40%-50%	1.255	1.5	1.664
40%-50%	1.297	1.5	1.42
40%-50%	1.049	1.5	1.03
40%-50%	1.01	1.5	0.9
60%-70%	0.685	1.5	0.96
60%-70%	0.85	1.5	1.668
60%-70%	0.86	1.5	1.53
60%-70%	0.58	1.5	1.466

The mean values are:

$$K = 0.8$$

$$\tau_d = 1.5s$$

$$\tau = 1.37s$$

and the constants for the PI are:

$$Kc = 1.0275$$

$$Ki = \frac{Kc*Ts}{Ti} = 0.11416$$

$$Kd = \frac{Kc*Td}{Ts} = 0$$

where T_s is the sampling time. Figure B.15 shows an example of how the PI controls the flow.

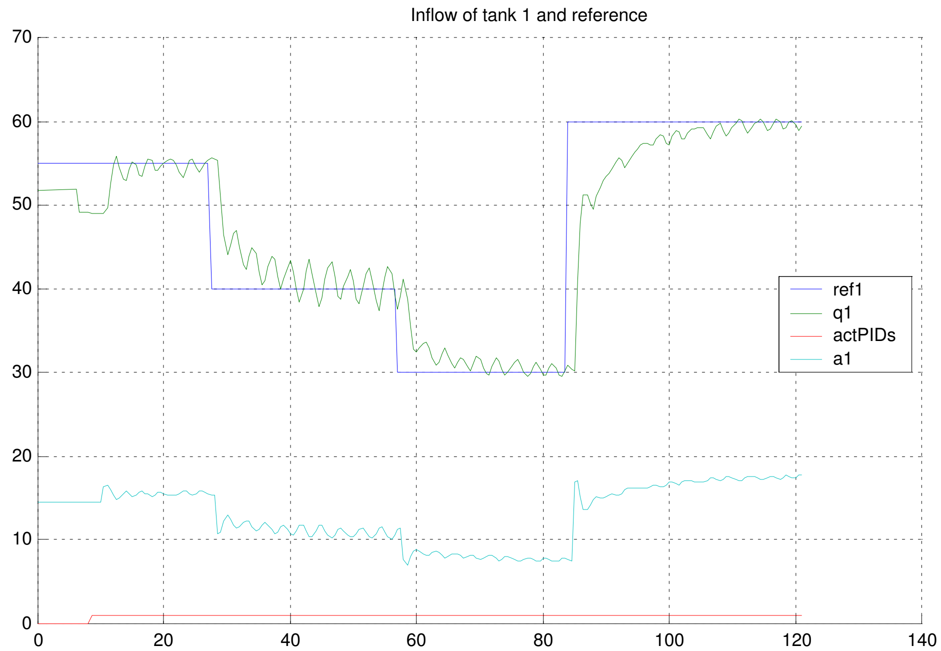


Figure B.15: PIs controlling the pseudo-linearized system.

It can be observed that the flow follows the reference once the PI has been activated. There is a certain ripple in the flow due to the PI attempt (especially its proportional term) of correcting the oscillations introduced by the pneumatic valve controller while it is trying to reach the opening value provided by the PI. A feedback between these two phenomena appears creating this ripple. Some ways have been tried to attenuate it:

1) The mean value of the position (calculated by an appropriate low pass filter) is chosen as process variable. Notice that the constant of the filter affect the closed-loop dynamics.

2) Using a pure integrator as controller. This choice makes the system behaves slower, the frequency noise decreases and it becomes more random and with a higher amplitude.

3) Another option is to increase the value of the flowmeters parameter that filters the flow at the exit. It is currently established in five seconds and it could be increased till ten, but it would make the system slower.

4) An increase of the dead zone amplitude of the valves (which can be configured). It would provide bigger amplitudes and less frequency oscillations. It has been tried with success in open loop but this does not happen in close loop because the reference changes and it is not static.

B.4.1 Identification of the plant

1) The first task before doing any test is the calibration of the pressure sensors in the following way:

    a) The tank is fulfilled with the discharge valve closed (the alarm will go off and the pump will stop). In the configuration of the pressure sensor this point is the maximum value.

    b) Then the tank is emptied by opening the discharge valve. Now this point must the minimum value in the configuration of the pressure sensor.

2) The second step is the determination of the maximum flows for certain values of γ.

    The maximum flows that respect the relation of the 3-ways valve are:

$$q_{a\,\max} = \min(\tfrac{Q_{1\max}}{\gamma_a}, \tfrac{Q_{4\max}}{1-\gamma_a})$$

$$q_{b\,\max} = \min(\tfrac{Q_{2\max}}{\gamma_b}, \tfrac{Q_{3\max}}{1-\gamma_b})$$

3) The dynamics of the plant are characterized by the cross-sections of the outlet holes a_i this parameters have been chosen, in function of the parameters of the

3-ways valves γ_a and γ_b, to have the maximum rank of available levels. This is achieved adjusting them to have a medium level at the tank for the medium flow (respecting the relation imposed by the 3-way valve).

The values of a_i are measured from the real plant in steady conditions trying to achieve the condition aforementioned.

In the following table the real values of the cross-sections of the outlet holes are presented.

Parameter	Value	Unit	Concept
a_1	1.341241e-004	m^2	Cross-sections of the outlet hole 1
a_2	1.533957e-004	m^2	Cross-sections of the outlet hole 2
a_3	9.322457e-005	m^2	Cross-sections of the outlet hole 3
a_4	9.061679e-005	m^2	Cross-sections of the outlet hole 4

B.4.2 Validation of the plant

In order to validate the plant an application in Simulink is created (figure B.16). The same reference in flow is given to the real plant an the model till the permanent regime is reached.

Once the values in permanent regime are obtained, the areas of the model are corrected. Then the test are repeated with the following results (figure B.17, B.18

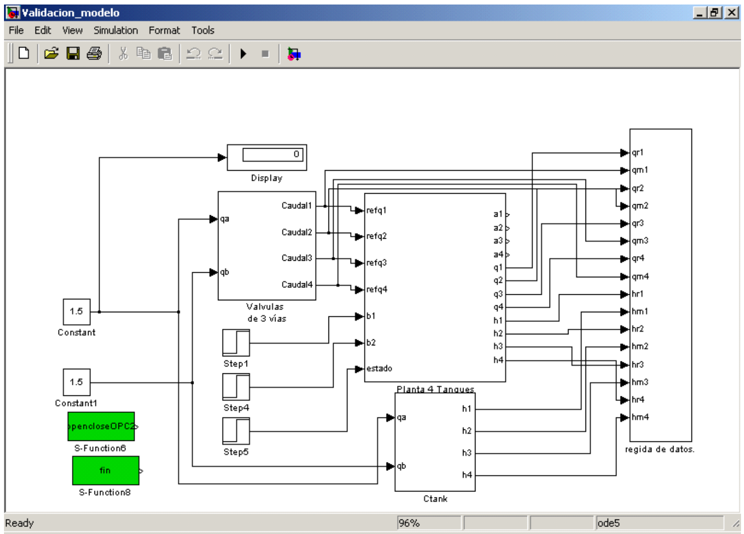


Figure B.16: Screen in Simulink for the validation of the model

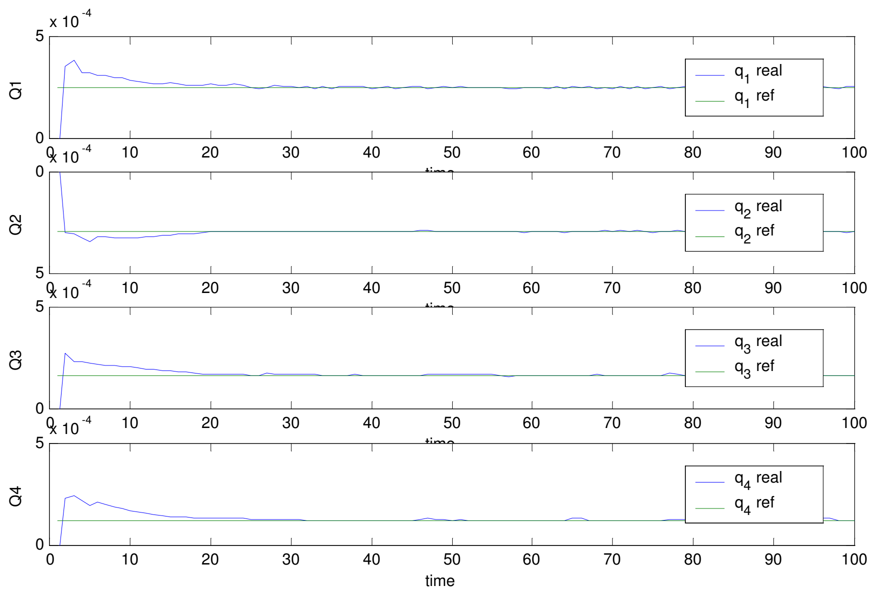
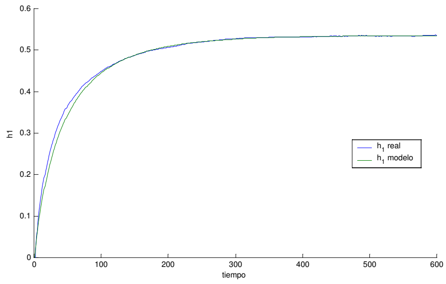


Figure B.17: Flows and their references during the validation test



,

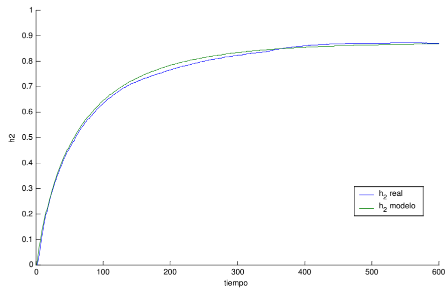


Figure B.18: The real level and modeled one during the validation test

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