Análisis y Síntesis de Sistemas con No-Linealidades del Tipo Afín a Trozos

Analysis and Synthesis of Systems with Piecewise Affine Nonlinearities

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Tesis doctoral

Departamento de Ingeniería de Sistemas y Automática
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Analysis and Synthesis of Systems with Piecewise Affine Nonlinearities

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A mis padres,

To my parents,

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Agradecimientos

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Sobre el autor

Alfonso Cepeda nació en España en 1976. Terminó sus estudios de ingeniero industrial en Sevilla en 2001. Desde entonces ha estado trabajando en varios proyectos de investicación, primero en redes neuronales y posteriormente en modelado de sistemas. Desde el 2001 hasta el 2005 ha estado vinculado al departamento de Ingeniería de Sistemas y Automática de la Universidad de Sevilla, con una estancia como estudiante de doctorado en el Imperial College de Londres donde ha trabajado en sistemas cantizados. Ha participado en varios proyectos europeos y sus campos de investigación actual incluyen análisis de estabilidad, control de sistemas con restricciones, sistemas lineales a trozos y conjuntos invariantes.

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About the author

Alfonso Cepeda was born in Spain in 1976. He finished his grade studies on Industrial Engineering in Seville in 2001. Since then he has been working in several researching projects, first on neural networks and later on system modelling. Since 2001 until 2005, he has been afiliated to System and Automatic Engineering in the University of Seville. He has also been working as Phd student at the Imperial College on quantized systems. He has participated in several European Projects and his current research interests include stability analysis, control of constrained systems, piecewise affine systems and invariant sets.

1

Introducción

1.1 Introducción

La mayor parte de los sistemas de control que están tratados en la literatura son los pertenecientes a la familia de sistemas lineales, bien porque son intrínsecamente lineales, o bien porque se han linealizado para poder trabajar mejor con ellos, en la mayoría de los casos, las variaciones respecto a esta linealización se toman como errores del modelo.

Este tipo de controladores tienen notables propiedades, pero en la realidad no existe ningún controlador que se comporte de forma totalmente lineal, ya que al ser un componente físico, no puede proporcionar una salida arbitrariamente alta, siempre debe tener un límite tanto superior como inferior. Por lo tanto, cuando el sistema se comporte de forma que la realimentación lineal esté en torno a su punto de operación, ésta realimentación controlará correctamente al sistema. Sin embargo, si el sistema genera una realimentación que esté fuera de las especificaciones del controlador, la salida será menor, y el sistema será impredecible. Esta no linealidad se denomina *saturación*.

La saturación en la realimentación, es decir, en el controlador, es la más común de todas las no linealidades,y aunque se puede tratar dentro de la teoría de control no lineal genérica, existe literatura específica para tratarla (ver [21, 47, 29], en [5] se describe una bibliografía cronológica). El control de sistemas lineales puede considerarse un campo maduro, donde existen numerosos métodos para diseñar los controladores más apropiados. La saturación entra dentro de los controladores que tienen restricciones en la actuación, y este tipo de controladores es un área de investigación muy dinámico, ver por ejemplo [20, 38, 37] y sus referencias.

Además, cuando el controlador satura, el rendimiento del sistema en bucle cerrado sin considerar la saturación disminuye seriamente, incluso se puede perder la estabilidad. Un ejemplo bien conocido es la disminución de rendimiento al utilizar un compensador PID en un sistema en bucle cerrado. Durante el tiempo en el que el controlador satura, el error se integra continuamente incluso cuando el control no responde lo que debería responder, y por tanto, el controlador produce valores del controlador mayores del límite del controlador. Este efecto se conoce como *windup* [18]. Debido a la forma en que el error afecta a los integradores, la salida del controlador puede no ser la deseada, por lo que a veces es necesario modificar la referencia del integrador.

A grandes rasgos se puede decir que existen dos estrategias para tratar con esta no linealidad. La primera es no tener en cuenta la realimentación en el proceso de diseño del controlador, y posteriormente añadir técnicas específicas para mitigar los efectos adversos producidos por la saturación.

Este tipo de técnicas se llama de *anti-windup*. La idea principal utilizada en estas técnicas es introducir realimentaciones adicionales de forma que el actuador permanezca dentro de los límites de linealidad. La mayoría de estas técnicas consiguen una gran eficiencia pero con una zona de estabilidad pequeña. Recientemente algunos investigadores han desarrollado técnicas sistemáticas para manejarlos (ver [11, 10, 25]).

La segunda estrategia consiste en tener en cuenta la saturación en el diseño del controlador. Esta es la estrategia que se utilizará en esta tesis.

Dentro de los sistemas lineales con realimentación lineal, se puede dividir en los que son globalmente controlables al origen y los que no lo son. Se ha demostrado [43, 48, 50] que los sistemas estabilizables linealmente con todos sus polos en el semiplano izquierdo son globalmente estabilizantes al origen. Por lo tanto, un sistema estabilizable linealmente con los polos en el semiplano izquierdo es estabilizable al origen con un controlador saturado en al menos una zona de operación del sistema. Este tipo de sistemas se denominan asintóticamente controlable al origen con controles saturados (*asymptotically null controllable with bounded controls* o *ANCBC*). Para este tipo de sistemas se pueden ver resultados en [42] y las referencias allí indicadas. Los sistemas que tienen algún polo en el semiplano derecho no es globalmente controlable al origen con controles saturados, por lo que los sistemas de control diseñados para ese tipo de sistemas no funcionarán globalmente. En este trabajo solamente se tratarán sistemas *ANCBC*.

Por otro lado, de los conceptos relacionados con la estabilidad de un sistema, dos tienen gran importancia. Éstos son el *dominio de atracción* y el *conjunto invariante*. El *dominio de atracción* es la región del espacio de estados desde los que el sistema converge al origen, por lo tanto el sistema es estable dentro de ese *dominio de atracción*. El *conjunto invariante* representan los estados del sistema controlado en los que el sistema no evoluciona fuera de ese *conjunto invariante*.

El dominio de atracción es muy importante ya que es una zona de operación en la que sabemos que existe al menos un controlador que hace converger el sistema al origen, por lo que podemos elegir un controlador distinto, siguiendo criterios de optimalidad con la seguridad de que convergerá al origen una vez utilicemos el controlador nominal [19, 15, 20, 17].

Este dominio de atracción se puede también utilizar dentro de las técnicas de control predictivo basado en modelo (MPC) [9]. El MPC consiste en predecir según el modelo del sistema cómo va a evolucionar en función de las entradas y así elegir dichas entradas en función de los objetivos que se quieran conseguir. En este tipo de técnicas es muy importante encontrar una zona en la que se tenga la seguridad que existe un controlador que converge al origen, así se puede utilizar como región terminal, es decir, obligarle a que en un número determinado de pasos el sistema evolucione a esa zona, y de esta forma garantizar que el sistema es estable.

En esta tesis se estudiarán formas de calcular conjuntos invariantes y estimaciones del dominio de atracción para sistemas en los que afecta, de una forma u otra, limitaciones en la realimentación. Este tipo de limitaciones aparecen en todos los sistemas físicos. Por ejemplo, la potencia del motor que mueve un brazo robot está limitada, o el aditivo a añadir a una reacción química que también está limitada.

El objetivo principal de la tesis es el estudio de las propiedades y el desarrollo de métodos de implementación de técnicas para el cálculo de estimaciones de dominios de atracción y de los máximos invariantes aplicado a sistemas lineales, o lineales a trozos, con una realimentación que de una u otra forma se contemple la saturación.

1.2 Sistema con realimentación saturada

Tal y como se ha comentado un sistema lineal es un modelo de procesos que pueden describir su dinámica a través de ecuaciones diferenciales o a través de ecuaciones en diferencias.

La diferencia entre un tipo de ecuaciones u otros es si el sistema es un sistema en tiempo continuo o discreto respectivamente.

Todo sistema lineal continuo se puede expresar de la siguiente forma

$$\frac{dx}{dt} = Ax + Bu, \tag{1.1}$$

donde $x \in \mathbb{R}^n$ corresponde a los estados, $u \in \mathbb{R}^m$ corresponde a la actuación y $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ son matrices que definen la dinámica del sistema.

Por otro lado, si el sistema es en tiempo discreto, se puede formular en la forma

$$x^+ = Ax + Bu, \tag{1.2}$$

donde x, A, B y u representa lo mismo que para tiempo continuo, y x^+ se denomina sucesor del estado y representa el estado en el tiempo de muestreo siguiente, es decir, despues de T segundos.

La actuación u se suele calcular realimentando el sistema. La realimentación más utilizada es la lineal, que se puede expresar

$$u = Kx, \tag{1.3}$$

donde $K \in \mathbb{R}^{n\times m}$ es la matriz de realimentación y se utiliza tanto para sistemas en tiempo continuo como en tiempo discreto.

En este caso u está realimentado con el estado x, en caso de que el controlador no pueda acceder a los valores del estado, se necesitará un observador, sin embargo, en esta tesis se considerarán sistemas observables donde los valores de x están accesibles.

Las realimetaciones lineales no son reales, ya que ningún aparato real puede dar una salida arbitrariamente grande, por lo que un modelo más realista a la realimentación 1.3 sería

$$u = \sigma(Kx), \tag{1.4}$$

donde la función multivariable $\sigma(s) = [\sigma_1(s_1)\ \sigma_2(s_2)\ \ldots\ \sigma_i(s_i)\ \ldots\ \sigma_m(s_m)]^\top$ es la saturación cuyas componentes están definidas por la expresión

$$\sigma_i(x) = \begin{cases} x^i_{min} & \text{if } x < x^i_{min}, \\ x & \text{if } x^i_{min} \leq x \leq x^i_{max}, \\ x^i_{max} & \text{if } x > x^i_{max}. \end{cases} \tag{1.5}$$

En esta tesis nos restringiremos a las saturaciones simétricas, y sin pérdida de generalidad, en forma normalizada, por lo que se puede utilizar la definición de $\sigma_i()$ siguiente

$$\sigma_i(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{1.6}$$

La figura 1.1 muestra el diagrama que representa un sistema realimentado. El sistema puede ser en tiempo continuo 1.1 como en tiempo discreto 1.2.

1.3 Técnicas tradicionales

El efecto de la saturación en la realimentación provoca un efecto de *windup*. Este efecto es indeseable ya que en caso de tener un controlador con una

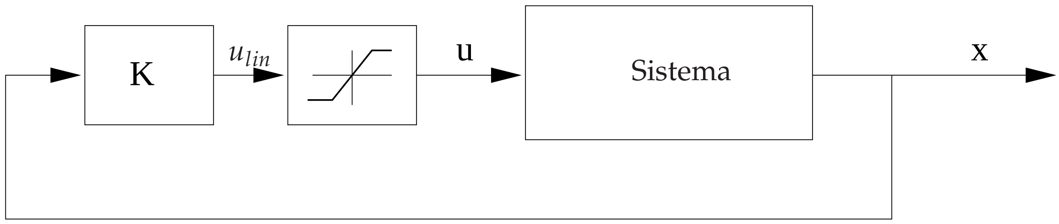

Figura 1.1: Diagrama representando un sistema realimentado.

parte integradora se está integrando un error que hace que la señal de control no sea la deseada.

Para evitar el *windup* el controlador debe de tener en cuenta a través de una realimentación del error en la entrada. La figura 1.2 muestra como se gestiona el efecto.

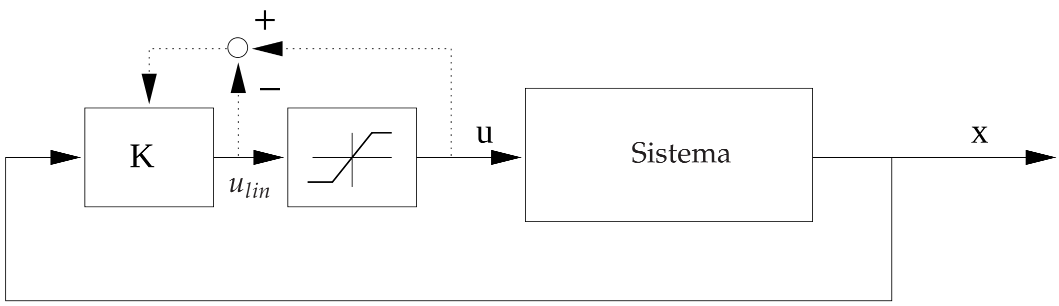

Figura 1.2: Diagrama representando un sistema realimentado con corrección de la saturación en el controlador.

La idea básica es controlar la diferencia entre la entrada y la salida del bloque de saturación. Cuando sea cero significa que está actuando dentro del rango de operación lineal, y vale distinto de cero fuera, donde la acción integral u otros estados del controlador produciría el efecto *windup*. Por lo tanto, el valor $error = u - u_{lin}$ se utilizará para conseguir que la acción integral no tome valores en exceso.

Estas técnicas se pueden utilizar tanto para sistemas en tiempo discreto como en tiempo continuo (ver [25, 11] y sus referencias).

1.4 Estabilidad

Supongamos que se tiene un sistema como 1.1 con la realimentación ya definida. Es importante saber si existe o no estabilidad garantizada. En este trabajo se estudiarán los sistemas que se pueden estabilizar al origen. Este tipo de sistemas son los que el controlador dado sin la saturación, es decir el sistema lineal, converge globalmente al origen. Al converger el sistema lineal, se puede decir que al menos existe una región alrededor del punto de equilibrio en el que el sistema saturado converge al origen.

La región, dentro del espacio de estados, que converge al origen se denomina dominio de atracción. El dominio de atracción es difícil de calcular exactamente debido a que es en general una región no convexa y consume mucho tiempo de proceso en el ordenador, por lo que es conveniente tener al menos, de forma rápida, una aproximación de dicho dominio de atracción.

Para calcular las estimaciones al dominio de atracción, existen técnicas tradicionales como el criterio del círculo o el criterio de Popov, en este trabajo se han diseñado otras técnicas para obtener esas estimaciones.

1.4.1 Cálculo iterativo del dominio de atracción

Aunque tal y como hemos comentado anteriormente el cálculo exacto del dominio de atracción de forma iterativa es una tarea que requiere mucho tiempo de proceso computacional es también interesante poder calcularla. Este método se utiliza únicamente en sistemas en tiempo discreto ya que se basa en propiedades dentro de cada iteración.

El proceso de cálculo es el siguiente, se parte de una región en el espacio de estados estable C_0 que converge al origen, y se calculan los estados que en un paso alcanzan esa región. Dicho subconjunto del espacio de estados se denota C_1. Aplicamos iterativamente este proceso hasta conseguir el conjunto C_∞. Esa región es el dominio de atracción.

1.5 Cálculos iterativos del dominio de atracción

Este tipo de métodos se aplica en sistemas en tiempo discreto, que una vez realimentados quedan en la forma

$$x^+ = Ax + B\sigma(u), \tag{1.7}$$

donde x es el vector de estados de cualquier dimensión, x^+ es el sucesor del estado y A es una matriz de dimensiones adecuadas. $\sigma(\cdot)$ es la función de saturación ya comentada. En este apartado se explicará a groso modo una técnica para obtener el dominio de atracción para este sistema.

Los sistemas de los que versará este trabajo son los que se pueden estabilizar al origen localmente, que significa que se puede conseguir una región que contenga el origen donde todos los puntos convergen a él. Note que el origen es un punto de equilibrio.

Existen varias técnicas para conseguir este conjunto inicial, en el capítulo 3 se muestran algunas que serán utilizadas en otros capítulos. Una posible técnica es encontrar una función de Lyapunov en la forma $f(x) = x^t P x$ con P definida positiva que sea contractiva, esto nos define una elipse que es invariante dentro de la zona en la que el sistema se comporta con régimen lineal.

Dado que nos interesa encontrar no solo un conjunto inicial, sino un conjunto inicial poliédrico y convexo, elegimos uno aleatoriamente incluido dentro de la elipse invariante. Ese conjunto es una estimación interna al dominio de atracción, y lo llamaremos C_0.

Posteriormente definimos el operador a un paso $Q(\cdot)$ como sigue

$$Q(\Omega) = \{\ x\ :\ \exists u : Ax + B\sigma(u) \in \Omega\ \}$$

En la definición 9 del capítulo 2 se define formalmente para realimentación lineal.

También podemos definir la iteración $C_{i+1} = Q(C_i)$, para $i = 0, 1, 2, \ldots$. C_i representa el conjunto de estados que en un número de pasos i alcanzan el conjunto C_0, y dado que todos los puntos de C_0 convergen al origen, todos los puntos de C_i también convergen al origen. Es más, el límite de esta sucesión C_∞ es el dominio de atracción del sistema que se intenta obtener.

1.5.1 Sistemas lineales con restricciones en la actuación

Normalmente los sistemas se controlan de forma lineal, ya que es mucho más fácil de realizar de forma industrial y existen tablas para ajustarlas que funcionan muy bien en la práctica, sin embargo para sistemas en los que es más importante la eficiencia del sistema que la sencillez del controlador se utilizan otro tipo de técnicas. En este tipo de casos lo normal es que el controlador esté integrado por un ordenador o un controlador digital programable, por lo que se suelen tratar sistemas en tiempo discreto.

Este es el caso de sistemas lineales con actuación saturada, la formulación de este tipo de sistemas

$$x^+ = Ax + B\sigma(u),$$

donde se puede observar que la realimentación u es una variable saturada.

Para este tipo de sistemas se puede obtener el máximo dominio de atracción de la siguiente forma.

Obtengamos una región C_0 polihédrica que pertenezca al dominio de atracción del sistema tal y como se mostró en la subsección 1.4.1, o más extensamente en el capítulo 3.

El sistema que se utilizará para mostrar como calcular este dominio de atracción será el de aplicar la función a un paso $Q(\cdot)$. Se utilizará esta función genéricamente, pero su significado depende del contexto. De la forma mas general está definida por

$$Q(\Omega) = \{\ x\ :\ Ax + Bu \in \Omega\ \}.$$

Esta función $Q(\cdot)$ se aplicará a la región C_0 inicialmente para obtener la región C_1, posteriormente se utilizará la iteración $C_{i+1} = Q(C_i)$, para $i = 0, 1, 2 \ldots$. El límite de esta recursión C_∞ será el dominio de atracción buscado.

Hay que hacer notar que cada uno de las regiones C_i son poliedros. Esto se puede fundamentar en que al ser C_0 un poliedro convexo que incluye al origen está definido como,

$$C_0 = \{\ x\ :\ H_0 x \preceq g_0\ \}.$$

supongamos que el conjunto C_i está definido por

$$C_i = \{\ x\ :\ H_i x \preceq g_i\ \}.$$

a este conjunto se aplica el operador a un paso $Q(\cdot)$ para este sistema que viene definido por

$$Q(C_i) = \{\ x\ :\ Ax + Bu \in C_i\ \}$$

con u saturado.

El hecho de que u esté saturado implica que está comprendido en el poliedro definido por

$$u \in \{\ u\ :\ H_u u \preceq g_u\ \}.$$

y con esta restricción $C_{i+1} = Q(C_i)$ se puede definir como

$$C_{i+1} = Q(C_i) = \{\ x\ :\ H_i(Ax + Bu) \preceq g_i\ ;\ H_u u \preceq g_u\ \}$$

que es un poliedro en (x,u). Proyectando dicho poliedro en x obtendremos el poliedro C_{i+1}. Es decir, si C_i es un poliedro convexo que contiene al origen $C_{i+1} = Q(C_i)$ también es un poliedro convexo que contiene al origen.

Utilizando este sistema se puede obtener el dominio de atracción buscado,

Un ejemplo de este sistema es el mostrado en la figura 1.3. En esa figura se muestra el dominio de atracción del sistema

$$x^+ = Ax + B\sigma(u)$$

con

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

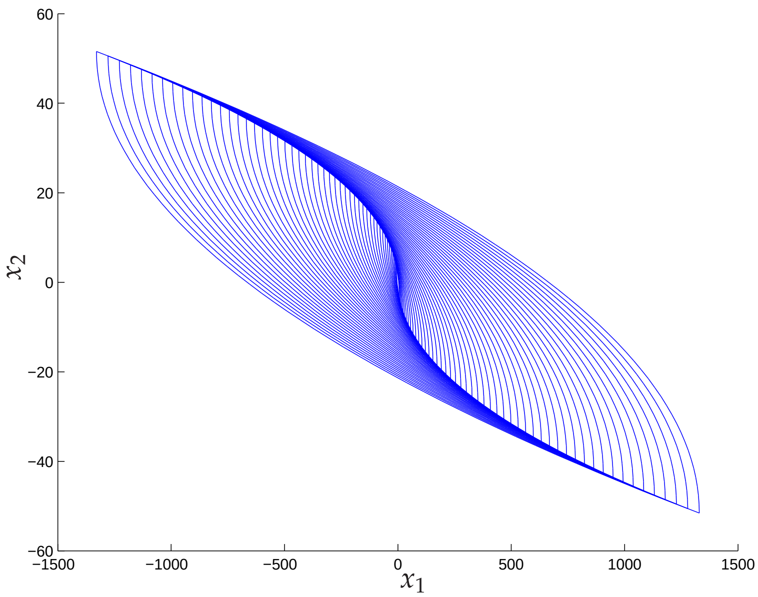

Figura 1.3: Cálculo del dominio de atracción para sistemas con actuadores limitados

El ejemplo mostrado en la figura 1.3 no representa el dominio de atracción total, sino una aproximación a 50 pasos (C_{50}) que es la región obtenida con la potencia de cálculo disponible. Dicha figura es comparable con el mostrado en la figura 1.4. En esa figura el dominio de atracción es mucho más pequeña que en la figura 1.3, pero en un caso la función de realimentación es una saturación lineal, y en el otro es una realimentación con actuadores limitados genérica. En un caso sabemos cual es la ley de control y en el otro está indeterminada.

En general, este tipo de dominio de atracción tiene dos problemas principales

- Es extremadamente costoso computacionalmente hablando el proceso de proyectar un poliedro sobre unas variables, y este proceso hay que realizarlo en cada iteración del sistema.
- Aun cuando consigamos calcular este dominio de atracción, no conocemos el valor de u para cada estado, es decir, no conocemos la función de realimentación que produce dicho dominio de atracción. Esta función se puede calcular en linea calculando cuales son los valores de u que dentro de la saturación lo mantiene dentro de C_∞. Este tipo de controlador obliga al sistema a permanecer dentro de un conjunto, pero no lo obliga a converger al origen, por lo que puede no ser adecuado.

Sin embargo hay una ventaja, y es que el dominio de atracción calculado es en general más grande que el real para leyes de control lineales saturadas. En caso de tener no solo el dominio de atracción sino la secuencia $\{C_1, C_2, \ldots, C_\infty\}$, podemos crear la ley de control siguiente,

$$\mathcal{U} = \{ \; u \; : \; Ax + B\sigma(u) \in C_{i-1} \; \}$$
$$f(x) = \begin{cases} \sigma(u) \;\; u \in \mathcal{U} & \text{si } x \notin C_1, \\ Kx \;\; \text{si } x \in C_1. \end{cases}$$

donde i viene definido por el valor tal que $x \in C_i, x \notin C_{i+1}$, y K es la solución LQR al sistema lineal sin saturación.

Esta ley de control converge al origen, pero tiene como inconvenientes que solo puede ser utilizada en linea, requiere tener toda la secuencia C_i, necesita un gran cálculo computacional fuera de linea y requiere gran cálculo computacional en linea.

Por estas razones es conveniente calcular el dominio de atracción para una función de control fija, y una de las más utilizadas es la función de control lineal saturada ya utilizada

$$u = \sigma(Kx),$$

que veremos seguidamente.

1.5.2 Sistemas lineales con realimentación lineal saturada

Los sistemas lineales se realimentan normalmente de forma lineal, es el sistema de realimentación con mejores propiedades y su simplicidad hace que se utilice mucho en la industria. Sin embargo, el sistema cambia cuando se trabaja en zonas de saturación. El sistema con realimentación se puede expresar en la forma

$$x^+ = Ax + B\sigma(Kx), \tag{1.8}$$

donde K es una matriz de las dimensiones adecuadas.

La función $Ax + B\sigma(Kx)$ es una función lineal a trozos, y por tanto no lineal, lo que significa que en caso de que un conjunto Ω sea un poliedro, $Q(\Omega)$ no tiene por que ser un poliedro y en general será un conjunto de poliedros, que en el caso más desfavorable puede llegar a 3^m poliedros siendo m la dimensión de la actuación.

La estimación del dominio de atracción conseguido será también una estimación del dominio de atracción el sistema con actuadores saturados y además con estabilidad garantizada ya que existe una ley de control conocida $u = \sigma(Kx)$ para la cual el sistema en ese dominio de atracción converge al origen.

El valor de K es en principio indeterminado, y se puede calcular de forma que la estimación del dominio de atracción conseguido sea el mejor posible.

Sin embargo este cálculo es extremadamente largo computacionalmente hablando, pero aún cuando podemos obtener dicho dominio de atracción, éste estará formado en general por un conjunto de poliedros que puede ser inmanejable. Es interesante obtener una aproximación poliédrica y convexa de dicho conjunto, que cumpliera dos propiedades principales, ser una estimación interna del dominio de atracción y ser un invariante del sistema. Estas propiedades nos indican una región segura de operación y además estará caracterizada por un poliedro que es computacionalmente manejable.

Cuanto mayor se pueda conseguir para un sistema dado el poliedro invariante y/o estimador del dominio de atracción, mayor será esa región segura de operación en la que el sistema se podrá mover.

En este trabajo se estudian métodos para conseguir esa región, y se obtienen regiones mejores que otros mostrados en la literatura.

El problema de la complejidad computacional y de la complejidad en la representación se puede observar con un ejemplo. Suponemos el sistema

$$x^+ = Ax + B\sigma(Kx)$$

con

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},\ K = [-0.6167 \ -1.2703].$$

el dominio de atracción de este sistema se puede observar en la figura 1.4.

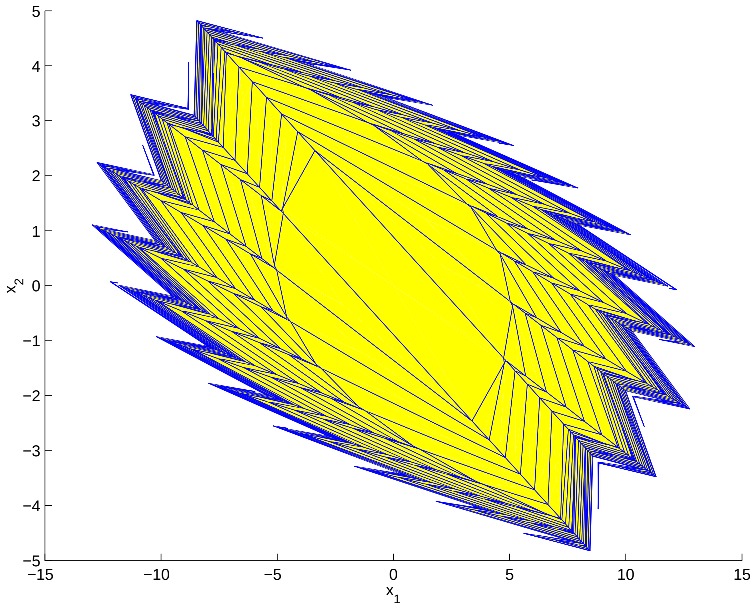

Figura 1.4: Cálculo exacto del dominio de atracción

La representación del dominio de atracción es muy compleja. La unión de cada uno de los poliedros interiores de la figura 1.4 produce ese dominio. Por lo tanto, para ver si un punto pertenece al dominio de atracción hay que comprobar si pertenece a alguno de los poliedros. De esta figura se puede justificar la utilización de alguno de los métodos mostrados en este trabajo para la obtención de aproximaciones del dominio de atracción de forma que la representación resultante sea un único poliedro y por tanto una región convexa.

Para evitar estos problemas en este trabajo se mostrarán estimaciones conservativas de la función $Q(\cdot)$ que proporciona una estimación del dominio de atracción convexa.

1.6 Objetivos de la tesis doctoral

Tal y como se ha comentado frecuentemente en esta introducción, el cálculo de aproximaciones de dominios de atracción convexos y de conjuntos invariantes es muy importante en la teoría de control. Esta tesis tiene por objetivos la obtención de dichas estimaciones de dominios de atracción para sistemas saturados, tanto en tiempo continuo como en tiempo discreto y la extensión de los métodos de calculo de dominio de atracción a sistemas lineales a trozos y también a sintesis de controladores.

Los objetivos se estructuran en:

- Cálculo de estimaciones de dominio de atracción tipo H para sistemas lineales con realimentación saturada. Los dominios de atracción tipo H son unas estimaciones interiores del dominio de atracción utilizando una función a un paso $Q(\cdot)$ aproximada. Este método de cálculo de estimaciones de dominio de atracción está extensamente detallado en la literatura para sistemas en tiempo continuo donde se obtienen regiones elipsoidales. Uno de los objetivos de esta tesis es la utilización de este tipo de métodos para la generación de regiones convexas poliédricas para sistemas en tiempo discreto. Este método de cálculo de regiones convexas ha sido publicado en [2, 14].

- Cálculo de estimaciones de dominio de atracción tipo SNS poliédricas para sistemas lineales con realimentación saturada. Los dominios de atracción tipo SNS son unas estimaciones interiores del dominio de atracción utilizando una función a un paso $Q(\cdot)$ aproximada. Este método de cálculo de dominio de atracción es original en la literatura y engloba a los dominios de atracción tipo H. Uno de los objetivos de la tesis es la aplicación de estos métodos para el cálculo de estimaciones de dominio de atracción poliédricas para sistemas discretos.

- Cálculo de estimaciones de dominio de atracción tipo SNS para sistemas lineales con realimentación saturada en tiempo continuo. Los métodos SNS se pueden aplicar a sistemas en tiempo continuo para la obtención de estimaciones de dominio de atracción elipsoidales. Este método de cálculo es original en este trabajo.

- Cálculo de controladores lineales para sistemas con realimentación saturada. Los métodos SNS pueden utilizarse también para sintesis, y así obtener el controlador lineal que una vez saturado obtiene los mejores estimadores del dominio de atracción. En este trabajo se muestra de forma original el cálculo de estos controladores lineales para sistemas en tiempo discreto con perturbaciones.

- Cálculo de estimaciones de dominio de atracción tipo LNL para sistemas de Lur'e en tiempo discreto. Los métodos SNS se pueden extender de sistemas saturados a sistemas de Lur'e. Para ello se crea el concepto LNL y se utiliza para obtener estimadores de dominio de atracción poliédricos. Este sistema es original en este trabajo y ha sido publicado en [13]

- Cálculo de estimaciones de dominio de atracción para sistemas afines a trozos en tiempo discreto. Se muestra unos métodos originales para la obtención de estimaciones de dominio de atracción para este tipo de sistemas. Este método ha sido publicado en [1].

Por lo tanto en este trabajo se espera generar un nuevo sistema de obtención de estimaciones de dominios de atracción y la extensión de dicho método para sistemas afines a trozos.

1.7 Estructura de la tesis doctoral

En esta tesis se obtendrán unos conjuntos por un lado invariantes y por otro estimaciones del dominio de atracción que sean tratables computacionalmente hablando para los sistemas saturados mostrados anteriormente. Se calcularán dichos conjuntos para sistemas en tiempo continuo y en tiempo discreto. Estas estimaciones proporcionarán un resultado mejor que los obtenidos por técnicas LDI utilizadas en la literatura. Las técnicas LDI utilizan la propiedad de que si dos sistemas lineales son estables, también lo es, bajo ciertas condiciones, una sistema no lineal con una dinámica entre esos dos sistemas lineales será también estable.

Por otro lado se tratará la sintesis de controladores para un sistema dado, y un tratamiento a sistemas de tipo lineales a trozos (*piecewise affine (PWA)*).

Por lo tanto los objetivos de este trabajo se centran en

- Generalizar el uso de técnicas $LDIs$ a la obtención de invariantes polihédricos para sistemas en tiempo discreto.

- Desarrollar una técnica de calculo de regiones invariantes (a la que se denominará SNS) que mejora las técnicas $LDIs$, y mostrarla tanto para sistemas en tiempo continuo como para sistemas en tiempo discreto.

- Cálculo de controladores lineales con saturación robustos utilizando técnicas SNS.
- Generalización de estas técnicas a sistemas más complejos como los de L'ure.
- Cálculo de regiones invariantes y estimaciones de dominio de atracción para sistemas lineales a trozos (PWA).

La estructura es la siguiente.

1.7.1 Notación

En el capítulo 2 se muestra la notación a utilizar en el resto de los capítulos. Este capítulo está dividido en sistemas en tiempo continuo y sistemas en tiempo discreto ya que esos tipos de sistemas van a tener un tratamiento totalmente distinto.

En cada uno de estos tipos sistemas se caracteriza la familia de sistemas con realimentación lineal saturada, que son los que más se utilizarán durante todo este trabajo, en una familia de sistemas mayor. Se definen formalmente funciones como $\sigma(\cdot)$ ya utilizada anteriormente y se muestran ejemplos de los efectos de la saturación en los sistemas.

Además se muestran teoremas generales que serán utilizadas en el resto del trabajo sobre estabilidad local y se definen otras variables.

Adicionalmente, para sistemas en tiempo discreto se define formalmente el operador a un paso $Q(\cdot)$ y se muestra un ejemplo de su utilización.

Este capítulo es un capítulo general donde se pone la base sobre la que se trabajará en los siguientes capítulos.

1.7.2 Dominio de atracción H

En la literatura se trata el cálculo de invariantes y de estimadores del dominio de atracción a través de $LDIs$ tanto para sistemas en tiempo continuo como en discreto. Los conjuntos obtenidos son invariantes [24, 33]. Sin embargo, para sistemas en tiempo discreto se pueden conseguir también regiones poliédricas conteniendo al origen utilizando estas técnicas.

En el capítulo 3 se muestra la utilización de estas técnicas para la obtención de regiones poliédricas. Estos conjuntos están incluidos en los conjuntos que se obtienen técnicas *SNS* mostradas en el capítulo 4, pero se muestran aquí para generalizar el uso de *LDIs* a este caso.

Se denominan conjuntos H (invariantes H ó dominios de atracción H) porque se restringe artificialmente la región a obtener a una región que viene dada por una matriz H a calcular, que será también utilizada para el *LDI*.

En este capítulo, después de mostrar sobre qué sistemas se va a tratar, se define el *LDI* en función de la anteriormente comentada matriz H, y se muestran las funciones a un paso hacia adelante $G_H(\cdot,\cdot)$ y hacia atrás $Q_H(\cdot,\cdot)$ en función del *LDI*.

Seguidamente se explican las características de los conjuntos invariantes y las estimaciones de los dominios de atracción que se van a conseguir con este sistema y se obtiene la mejor matriz H a través de un problema de maximización.

El capítulo continua con la propuesta de tres algoritmos distintos para obtener las regiones deseadas, y concluye con ejemplos con los distintos algoritmos para un sistema dado.

1.7.3 *SNS* Discreto

Una vez definido la utilización de técnicas *LDI* para la obtención de invariantes para sistemas en tiempo discreto, se desarrollará una técnica que mejora dichos conjuntos. Esa técnica se denomina *SNS* que proviene de *Saturado y No Saturado* haciendo referencia a que hay que considerar tanto el sistema con la saturación, como el sistema lineal sin tener en cuenta la saturación.

En el capítulo 4 se presenta la técnica *SNS* para sistemas en tiempo discreto. En él, después de mostrar el problema que se pretende resolver, se definen los conceptos de invariancia y dominio de atracción utilizando esta técnica *SNS*.

Posteriormente se define el operador a un paso $Q_{SNS}(\cdot,\cdot)$, que es más conservativa que el operador no convexo $Q(\cdot)$, pero menos que el operador $Q_H(\cdot,\cdot)$ mostrado en el capítulo 3. Esta relación entre el concepto *SNS* y el concepto H es analizada.

Por último se muestra unos ejemplos utilizando los algoritmos realizados.

1.7.4 *SNS* Elipsoidal

El concepto *SNS* presentado en el capítulo 4 puede ser también aplicado a los sistemas en tiempo continuos.

En el capítulo 5 se muestra esta aplicación de *SNS* a sistemas en tiempo continuo para la obtención de invariantes y estimadores del dominio de atracción elipsoidales.

Se estructura de la forma siguiente. Después de fijar el tipo de problema a resolver se definen la caracterización de los elipsoides que se pretenden obtener. Seguidamente se analiza el concepto de contractividad, y cómo se introduce dentro de la formulación *SNS*.

Posteriormente se compara la relación entre el concepto *SNS* y los métodos *LDIs* utilizados en la literatura para obtener invariantes elipsoidales de sistemas con realimentación saturada. Se estudia el problema del coste computacional de la resolución del problema de maximización para obtener el elipsoide invariante y se concluye con ejemplos numéricos de este sistema.

1.7.5 Síntesis de controladores saturados robustos

En los capítulos 4 y 5 se definió el concepto *SNS* a sistemas con realimentación lineal saturada en tiempo continuo y discreto. Sin embargo, el controlador para dichos sistemas estaba definido a priori, es decir,

$$u = \sigma(Kx)$$

con K constante. Sin embargo, es posible que nos interese calcular K para obtener una mayor estimación del dominio de atracción o un mayor invariante. Es decir, es posible que interese realizar la síntesis de un controlador más que realizar el análisis de un controlador dado. Es más, generalmente los sistemas no son lineales puros sino que tienen perturbaciones aleatorias, por lo que un modelo del sistema en tiempo discreto con perturbaciones es

$$x^+ = Ax + Bu + E\theta \tag{1.9}$$

donde θ es una variable aleatoria con restricciones que representa las perturbaciones.

En el capítulo 6 se muestra una técnica para calcular K para sistemas en la forma 1.9.

En dicho capítulo se presenta el operador $Q(\cdot,\cdot)$ para este tipo de sistemas y se muestra un algoritmo para conseguir el dominio de atracción conocido K.

Seguidamente se definen los conceptos SNS para este tipo de sistemas, obteniéndose el operador $Q_{SNS}(\cdot,\cdot)$ en función de K para sistemas con perturbaciones y se propone un algoritmo para iterativamente modificar K para conseguir un mayor conjunto invariante.

El capítulo finaliza con la aplicación de estas técnicas a un ejemplo numérico.

1.7.6 Aplicación de LNL a sistemas L'ure

En los capítulos anteriores se ha mostrado técnicas para conseguir conjuntos invariantes y estimadores del dominio de atracción para sistemas en tiempo continuo y discreto lineales con realimentación saturada.

Este tipo de sistemas es una particularización de un tipo de sistemas más amplio llamado sistemas L'ure. Un tipo de sistemas L'ure que se tratará en este trabajo es el sistema lineal con realimentación cóncava-positiva lineal a trozos. Es decir, el sistema a controlar viene definido por la fórmula

$$x^+ = Ax + B\phi(kx)$$

donde $\phi(\cdot)$ en lugar de ser una saturación simple como $\sigma(\cdot)$, es una función lineal a trozos, con la característica de que es simétrica por el origen y en su parte positiva es cóncava. El sistema a tratar solo considerará $kx \in \mathbb{R}$. Es decir, la realimentación ha de ser unidimensional.

En el capítulo 7 se estudiarán este tipo de sistemas. Inicialmente, después de indicar el problema a tratar, se divide la no linealidad definida por $\phi(\cdot)$ en un conjunto de no linealidades que se puede representar como saturaciones $\sigma(\cdot)$. Posteriormente se mostrará el concepto de invariancia con el método SNS, al que se llamará LNL (que proviene de *Lineal y No Lineal*), por la forma de $\phi(\cdot)$. Seguidamente se definirán el operador a un paso $Q_{LNL}(\cdot)$ y se utiliza este operador dentro de un algoritmo para obtener tanto una estimación del dominio de atracción como un invariante para el sistema dado.

El capítulo termina con un ejemplo de aplicación.

1.7.7 Invariante de control robusto para sistemas lineales a trozos

Los métodos mostrados en la literatura para la obtención de máximo conjunto invariante robusto para un sistema lineal a trozos requieren una gran complexidad computacional.

En el capítulo 8 se presentará un algoritmo que produce estimaciones del conjunto invariante con menor consumo computacional. El algoritmo está formado por dos partes, la primera parte se utilizará para la obtención de una estimación exterior del conjunto invariante, y será esta estimación escalada la que se usará como conjunto inicial en la segunda parte del algoritmo.

Ejemplos de dicho algoritmo terminan el capítulo.

1

Introduction

1.1 Introduction

Most of control system analysed in the literature belong to lineal system family. Some of them are indeed linear systems, and others are non-linear systems linearised to be able to work with, and differences regarding this linearisation are considered as model errors.

This family of controllers has important characteristics, but actually there are no globally linear controller because, since controllers are physical items, their output can not be as big as required, there are always upper and lower limits. Therefore, when the system behaves in a way that the linear feedback is near the operation point, the feedback will correctly control the system. But if the system works outside the controller specifications, the output will be smaller, and the system will be unpredictable. This non-linearity is known as *saturation*.

The feedback saturation, that is, controller saturation, is the most common of non-linearities, and although it can be analysed in the general non-linear control theory, there are specific literature to deal with (see [21, 47, 29], in [5] there is a chronological bibliography). Control of linear systems theory has been widely studied, where numerous techniques provide well designed controllers. Saturation belongs to controllers with input constraints, and this is a very dynamic research work, see for example [20, 38, 37] and their references.

Moreover, when the controller works in the saturation point, the yield of the closed loop system with a controller that does not consider saturation decreases, it can even lose stability. A well known example is the decrease of the yield in the use of a PID in a closed loop system. When the controller work in the saturation point of operation, the error is integrated even when the control does not answer what it should does, and therefore, the controller provides values higher than the controller limits. This is known as *windup* [18]. Due to the way that integrators behave with this effect, the controller output can be undesirable, and sometimes the reference of the controller should be reseted.

Roughly speaking, there are two strategies to work with this non-linearity. The first one is not to take into account the feedback when designing the controller, and later to add some specific techniques to decrease the undesired effects of the saturation. This kind of techniques are called *anti-windup*. The main idea used is to introduce additional feedbacks so that the actuator works within the linearity limits. Most of these techniques get a good efficiency but with a small stability set. Recently some researchers have developed systematic techniques to work with them (see [11, 10, 25]).

The second strategy consists in taking into account the saturation in the controller design. This is the strategy that will be used in this thesis.

Linear systems with linear feedback can be divided into those who are globally controlled at the origin and those who are not. It has been proved [43, 48, 50] that globally controlled to the origin systems with all poles in the left semiplane are stabilizeable at the origin with a saturated controller at least in an operation set of the system. This family of systems are known *asymptotically null controllable with bounded controls* o *ANCBC*). Results from this family of systems are analysed in [42] and it references. Systems with any pole in the right semiplane are not globally null controllable with bounded controls, and therefore control systems designed to work with this family of systems will not work globally. In this document only *ANCBC* systems will be considered.

On the other hand, there are two important concepts related to systems stability. These are the *domain of attraction* and the *invariant set*. The *domain of attraction* is the state space set such that the system working in a point in this set converges to the origin, so the system is stable within this domain of attraction set. The *invariant set* is the space state set that if the system is in the domain of attraction. The invariant set represents the controlled system states where the system does not evolves outside this invariant set.

The domain of attraction is very important because it is a safe operation set in which at least a controller can be used to make that system converge to the origin, so a different controller can be selected with different optimization criteria, being sure that it will converge to the origin when the nominal controller is used [19, 15, 20, 17].

This domain of attraction can also be used under *model predictive control* techniques (*MPC*) [9]. *MPC* consists in the prediction of the system evolution, according to its model, depending on the input and this way it can be chosen with the minimisation of a cost function. This kind of techniques need a terminal set, that is, a region to wich the system can be forced to converge in a finite number of steps. This guarantees that the feedback system is stable.

In this thesis it will be analysed different ways to calculate invariant sets and estimations of the domain of attraction for input constraint systems. This constraints appear in every physical system. For example, the power of the motor that moves a robot is limited, or the additive in a chemical reaction is also limited.

The main purpose of this thesis is the study of properties and the development of implementation methods of calculation techniques for the estimation of domains of attraction and maximal invariant sets applied to linear systems, piecewise affine systems or systems with a saturation feedback systems.

1.2 Saturation feedback systems

As previously commented, a linear system is a model for processes whose dynamics can be described by means of differential or differences equations.

The difference between both equation types is that in the first case the system is a continuous time system and in the second it is a discrete time system.

Every linear system can be expressed as,

$$\frac{dx}{dt} = Ax + Bu, \tag{1.1}$$

where $x \in \mathbb{R}^n$ are the states, $u \in \mathbb{R}^m$ are the input of the system and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are matrixes that define the system dynamic.

On the other hand, if the system is a discrete time system, it can be expressed as,

$$x^+ = Ax + Bu, \tag{1.2}$$

where x, A, B and u are similar to continuous time systems, and x^+ is the successor of the state and is the state in the next step time, that is, after T seconds.

The actuator variable u is calculated with the feedback of the system. The most used one is linear, and it can be expressed as,

$$u = Kx, \tag{1.3}$$

where $K \in \mathbb{R}^{n\times m}$ is the feedback matrix and it is used for continuous time systems and for discrete time systems.

In this case, u is a function of x. If the controller can not access to state values, it will need an observer, however, in this work only observable systems will be considered, where values of x are accessibles.

The linear feedback does not exists in real life, because no physical device can provide an output as large as needed, therefore, a most realistic model to feedback 1.3 is,

$$u = \sigma(Kx), \tag{1.4}$$

where the multivariable function $\sigma(s) = [\sigma_1(s_1)\ \sigma_2(s_2)\ \dots\ \sigma_i(s_i)\ \dots\ \sigma_m(s_m)]^\top$ is the saturation whose components are defined as,

$$\sigma_i(x) = \begin{cases} x^i_{min} & \text{if } x < x^i_{min}, \\ x & \text{if } x^i_{min} \le x \le x^i_{max}, \\ x^i_{max} & \text{if } x > x^i_{max}. \end{cases} \tag{1.5}$$

In this work symmetric saturation will be considered, and without loss of generality, they will be normalized, therefore, $\sigma_i()$ can be defined as,

$$\sigma_i(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } -1 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{1.6}$$

Figure 1.1 shows the design of a feedback system. This system can be defined in continuous time 1.1 or in discrete time 1.2.

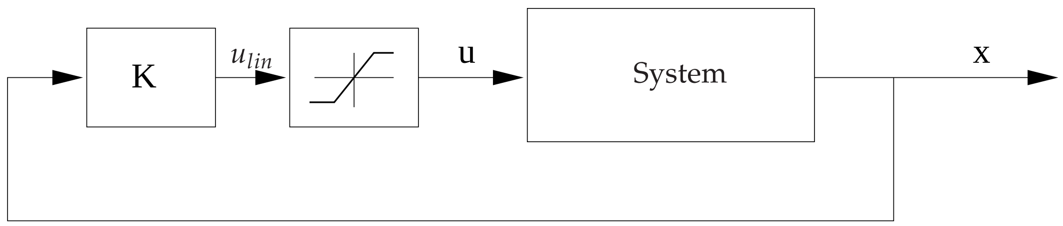

Figure 1.1: Design of a feedback system.

1.3 Traditional techniques

The feedback saturation generates an undesired effect called *windup*. This effect is negative because if the controller has an integrator, the error is being integrated and the control signal is not the desirable.

In order to avoid the *windup*, the controller should taking into account the input error by means of a feedback method. Figure 1.2 shows this design.

The main idea is to control the difference between the input and the output in the saturation block. When it is zero, it means that it is working in the linear operation range, and it is different from zero when saturation

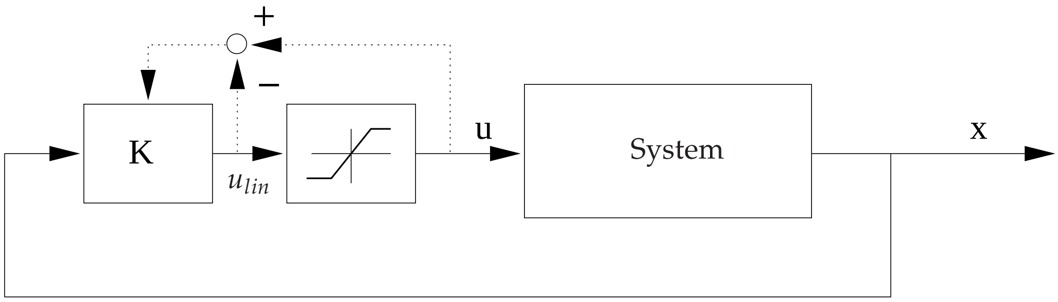

Figure 1.2: Design of the feedback of the system with a modified controller saturation feedback.

is reached. In this case, the integral part will produce the *windup* effect. Therefore the $error = u - u_{lin}$ value will be used to manage the integral action does not over.

These techniques can be used as well for continuous time systems as for discrete time systems (see [25, 11] and the references therein).

1.4 Stability

Let us suppose that the system is 1.1 with a defined feedback. In this work null controlled systems will be considered. This kind of systems are such that the controller without the saturation, i.e. the linear feedback system converges to the origin. The convergence of the linear system can be extended to the saturated system at least in a set near to the origin.

The state space set, wich converges to the origin is called domain of attraction. This domain of attraction is difficult to be exactly determinated because it is generally a non-convex set and it takes much computer process time. Therefore, it is necessary to have a method to obtain a fast approximation of that domain of attraction.

In order to calculate this estimations of the domain of attraction, there are some traditional techniques like the circle criteria or the Popov criteria. In this work, it will be designed some techniques to obtain this estimations.

1.4.1 Iterative computation of the domain of attraction

The exact computation of the domain of attraction in a iterative way is a task that needs much computational process time but it can be useful and it is important to have a method to determine it. This method is used only in discrete time systems due to it uses iteration properties.

The calculation process is the following, it begins with a stable set C_0 in the state space that converge to origin, and it is determined the set that evolves to this initial set in one step time. This new subset of the state space is denoted C_1. This method is used again in an iterative way in order to get the C_∞ set. This set is called domain of attraction.

1.5 Iterative computation of the domain of attraction

This type of methods are applied to discrete time systems, which once they have the feedback can b defined as follows

$$x^+ = Ax + B\sigma(u), \tag{1.7}$$

where x is the state vector of any dimension, x^+ is the successor of the state and A is a matrix. $\sigma(\cdot)$ is the saturation function. In this section a technique will be shown to obtain the domain of attraction of the system.

In this system null controlled systems with bound input constraint will be considered.

There are some techniques to obtain this initial set, in chapter 3 some of them will be analysed and they will be used in next chapters. A technique is to obtain a Lyapunov function like $f(s) = x^\top Px$ with a contractive positive definite P. It defines an ellipse that it is invariant in the linear behaviour set.

Note that it is important to obtain an initial set, but it is also important to obtain a convex polyhedral set, therefore an arbitrary polyhedral set inside the invariant ellipse is selected. This set is an inner estimation of the domain of attraction and it will be called C_0.

The one-step operator $Q(\cdot)$ is defined as,

$$Q(\Omega) = \{\ x\ :\ \exists u : Ax + B\sigma(u) \in \Omega\ \}$$

This definition will be completed in definition 9 on chapter 2 for linear feedback.

Iteration can also be defined $C_{i+1} = Q(C_i)$ for $i = 0, 1, 2, \ldots$. C_i represents the space state set that in i steps reach to C_0 set, and as C_0 converge to the origin, all states in C_i converge to the origin. Moreover, the limit in this succession C_∞ is the domain of attraction of the system that will be obtained.

1.5.1 Linear systems with input bound constraint

Usually, systems are linear controlled, since it is easier to build in the industry field and there are tables to set up that are widely used in practice. However, critical systems such their efficiency is more important than the simplicity of the controller can use different approaches. In this case, the controller is usually implemented in a computer or in a digital programable controller, and therefore systems are commonly analysed in discrete time.

This is the case of linear systems with bounded input constraints, this systems can be expressed as,

$$x^+ = Ax + B\sigma(u),$$

where the input u is bounded.

The maximal domain of attraction of this type of systems can be obtained in the following way.

Firstly an initial C_0 polihedrical set that belongs to the domain of attraction of the system in the way that has been shown in subsection 1.4.1, or in a more extended way in chapter 3 should be obtained.

The one-step operator function $Q(\cdot)$ will be used to obtain the domain of attraction of the system. This function is generic, and it depends on the context. In the most general way, it is defined as,

$$Q(\Omega) = \{\ x\ :\ Ax + Bu \in \Omega\ \}.$$

This operator will be applied to C_0 set to obtain the C_1 set, and in an iterative way it will be used $C_{i+1} = Q(C_i)$, for $i = 0, 1, 2 \ldots$. The limit of this recursion, C_∞ set will be the domain of attraction of the system.

Note that each C_i set is a polyhedral set, therefore convex. This can be explained because C_0 is a polyhedral set that includes the origin and it is defined as,

$$C_0 = \{\ x\ :\ H_0 x \preceq g_0\ \}.$$

Let us suppose that C_i is defined by

$$C_i = \{\ x\ :\ H_i x \preceq g_i\ \}.$$

The one step operator $Q(\cdot)$ is applied to this set for this system and it is defined as

$$Q(C_i) = \{\ x\ :\ Ax + Bu \in C_i\ \}$$

where u is saturated.

As u is saturated, constraints of u are polihedrical and they can be defined as,

$$u \in \{\ u\ :\ H_u u \preceq g_u\ \}.$$

and this restriction shows that $C_{i+1} = Q(C_i)$ can be defined as,

$$C_{i+1} = Q(C_i) = \{\ x\ :\ H_i(Ax + Bu) \preceq g_i\ ;\ \ H_u u \preceq g_u\ \}.$$

Note that this is a polihedrical set in (x,u). If this polihedrum is projected to x the C_{i+1} set is obtained, therefore C_{i+1} is a convex set. That is, if C_i is a polihedrical convex set that includes the origin, then $C_{i+1} = Q(C_i)$ also is a polihedrical convex set that includes the origin.

This method can be applied to obtain the domain of attraction of the system.

Figure 1.3 shows an example of the domain of attraction. This figure shows the domain of attraction of the system

$$x^+ = Ax + B\sigma(u)$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

Example shown in figure 1.3 does not represent all the whole domain of attraction, but an approximation in 50 steps (C_{50}) this is the set obtained with the available computational power.

This figure can be compared with figure 1.4. In that figure the domain of attraction is smaller than in this one, but in that case the feedback function is a linear saturation and in the case of figure 1.3, a input bounded constraint actuator is used. Note that in 1.4 the control law is known but in 1.3 is unknown.

This type of domain of attraction has two main problems in general,

- The projection of a polihedral set from one dimension to another needs a lot of computational time, and this process must be used each iteration of the system.

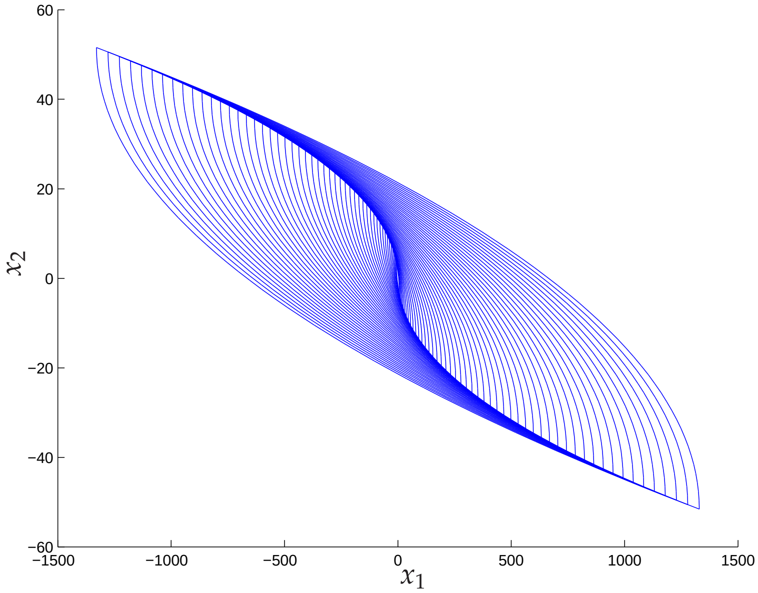

Figure 1.3: Domain of attraction for a system with input bounded constraints.

- Moreover, when this domain of attraction is obtained, the value of the control law u is unknown, that is, the function $u = f(x)$ that provides this domain of attraction is unknown. This control law can be obtained online just by obtaining values of u such that it maintains the system in C_∞. This type of controllers make the system remain inside a set, but do not make it converge to the origin.

However, there is an advantage, and this is that the obtained domain of attraction is in general larger that the real one for saturated linear control laws. In case of getting not only the domain of attraction but the sequence $\{C_1, C_2, \ldots, C_\infty\}$, the following control law can be used,

$$\mathcal{U} = \{\ u\ :\ Ax + B\sigma(u) \in C_{i-1}\ \}$$

$$f(x) = \begin{cases} \sigma(u) \quad u \in \mathcal{U} & \text{if } x \notin C_1, \\ Kx & \text{if } x \in C_1. \end{cases}$$

where i is defined by the value such that $x \in C_i, x \notin C_{i+1}$, and K is the LQR solution of the linear system without saturation.

This control law converges to the origin, but one important disadvantage is that it can only be used online, it needs all the sequence C_i, it needs much offline computational time and it also needs much online computational time.

These disadvantages show that it is important to be able to calculate the domain of attraction of a fixed control law. One of the most saturated control law function used is the control law

$$u = \sigma(Kx),$$

that will be seen in the following point.

1.5.2 Linear systems with bounded input

Usually, linear system feedback is linear, this feeback system has very important properties and this simplicity is important in the design. Nevertheless, system changes when saturation zone is reached. The feedback system can be shown as,

$$x^+ = Ax + B\sigma(Kx), \tag{1.8}$$

where K is a matrix of suitable dimension.

Function $Ax + B\sigma(Kx)$ is a piecewise affine system, and therefore it is non-lineal, that is if an Ω set is a polyhedral, $Q(\Omega)$ can be a non-polyhedral set. Actually, this set is an union of polyhedral in general, and in the most unfavourable case, it will be the union of 3^m polyhedral set, where m is the saturation dimension.

The estimation of the domain of attraction obtained by means of this feedback control will also be an estimation of the domain of attraction of the bounded actuation system since there exists a known control law $u = \sigma(Kx)$ with which the system in this domain of attraction converges to the origin.

The value of K is undetermined, and can be calculated in a way that the estimation of the domain of attraction to be obtained optimises a size measure.

Nevertheless, this calculus spends much computation time, but although this domain of attraction can be obtained, it will be defined as an union of polihedrical sets that can be unmanageable. It is important to obtain a polihedrical and convex approximation of this set, provided that it has got two main properties, to be an inner estimation of the domain of attraction and to be an invariant set of the system. These properties show a safe operational set wich will be defined by a polyhedral so computational definition is simple.

The larger the polihedrical invariant set is for a given system, the larger this safe operational set will be.

In this work new methods to obtain this set will be analyzed, and set obtained by means of these methods provides a larger estimation of the domain of atraction than shown in the literature.

The computational complexity problem, and the complexity in the representation of this set can be explained with an example. Let the following system be

$$x^+ = Ax + B\sigma(Kx)$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \ K = [-0.6167 \ -1.2703] \,.$$

The domain of attraction of this system can be shown in figure 1.4.

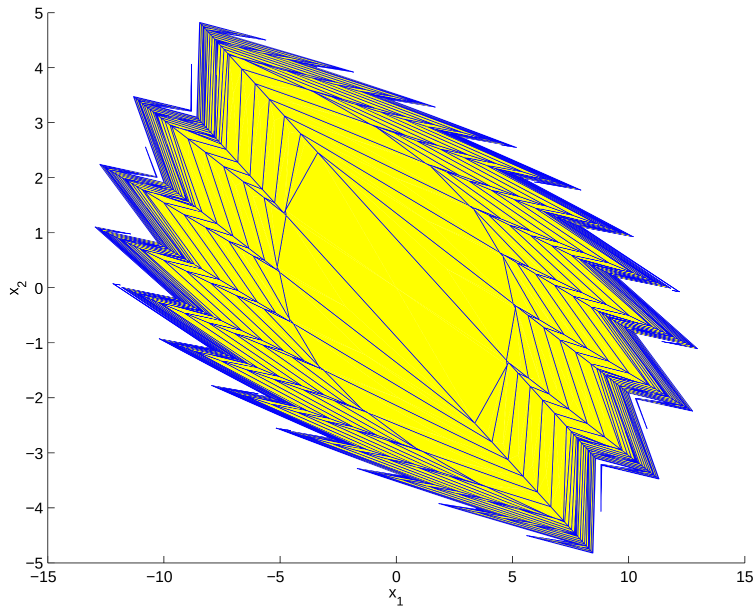

Figure 1.4: Exact domain of attraction set

This domain of attraction representation is very complex. The union of every inner polihedrical set in figure 1.4 provides this domain of attraction. Therefore, just to know if a state belongs to the domain of attraction of the system, you must test every set of the representation. This figures justify the use of some of the shown methods in this work to obtain the approximation of the domain of attraction of the way that the final representation will be only one polihedrical set and therefore a convex set.

In order to avoid these problems, this work will show conservative estimations of the $Q(\cdot)$ function that provides a convex estimation of the domain of attraction.

1.6 Objectives of this thesis

Calculus of convex approximations of the domain of attraction for a system and convex invariant set for a system is very important for control theory. This work will be used to obtainning this estimation of the domain of attraction for saturated systems, as well continuous time systems as discrete time systems, and the extension of this calculation methods to piecewise affine systems and the design of controllers.

The objectives of this thesis are,

- Determination of estimations of H-domain of attraction for linear systems with saturated feedback. H-domain of attractions are inner estimations of the domain of attraction using an approximate $Q(\cdot)$ function. This method is widely analyzed in the literature for continuous time systems where target sets are ellipsoidal. One of the objectives is this work is the utilisation of this type of methods in order to generate polihedrical convex set for discrete time systems. This method has been published in [2, 14].

- Determination of estimations of polihedrical SNS-domain of attraction for linear system with saturation feedback. SNS-domain of attractions are inner estimation of the domain of attraction using an approximate $Q(\cdot)$ function. this method is new in the literature and enlarge other estimations of the domain of attractions. One of the objectives of this work is the use of this methods in order to determine a polihedrical convex estimation of the domain of attraction. This method has been analyzed in [3]

- Determination of estimation of SNS-domain of attraction for linear system with saturation feedback in continuous time. SNS methods can be applied in continuous time systems in order to get ellipsoidal estimations of the domain of attraction. This method is new in this work.

- Determination of linear controllers for linear systems with saturation feedback. SNS methods can also be used in synthesis and to get linear controllers that can be used with the saturation in order to get estimations of the domain of attraction. In this work a new approach to the determination of linear controllers for discrete time systems with disturbances is shown.

- Determination of LNL-domain of attraction for Lur'e systems in discrete time. SNS methods can be extended to Lur'e saturated systems. This extension is called LNL methods, and can be used for the determination of polihedrical estimation of the domain of attraction. This method is new in this work and has been published in [13]

- Determination of estimation of domain of attraction for *piecewise affine* systems in discrete time. This method has been published in [1].

Therefore a new system in order to determinate estimation of the domain of attraction and the extension of that method in *piecewise affine* systems.

1.7 Structure of this thesis

In this work invariant sets and estimations of the domain of attraction for linear systems with a low computational cost will be analyzed. Target systems will be linear systems with saturation feedback and piecewise affine systems in both continuous time and discrete time. These estimations will be larger than those obtained by means of LDI methods shown in the literature. LDI methods use the property that if two different linear systems are stable, it is also stable, under some conditions, a nonlinear system with a dynamic between this two linear systems.

Therefore objectives of this work are

- Use of LDI methods in the determination of polyhedrical invariant set for discrete time systems.

- A new method called SNS to determinate invariant sets that oversize those obtained by LDI methods, for both discrete and continuous time systems.

- Determination of robust linear controllers with saturation using SNS methods.

- Determination of this methods to L'ure systems

- Determination of invariant sets and estimations of the domain of attraction for piecewise affine systems.

The structure is the following,

1.7.1 Notation

Chapter 2 contains the notation used in other chapters. This chapter is divided into continuous time systems and discrete time systems since methods to be applied to both family of systems are different.

Linear systems with saturation feedback are ones of the most used systems in this work, and therefore they are defined in this chapter. Saturation function and one-step operator are also defined.

Some theorems related with local stability are also shown.

This chapter is general and will be referenced in the following chapters.

1.7.2 H-domain of attraction

Invariant set and estimations of the domain of attraction by means of $LDIs$ techniques has been shown in the literature for both continuous time systems and discrete time systems, see [24, 33]. However these methods can also be used for determination of polyhedral sets.

In chapter 3 they will be applied to the determination of polihedrical set. These sets are inner to those obtained by means of SNS methods shown in chapter 4, but they will be shown here in order to extend $LDIs$ methods.

This methods are called H methods (that is, H invariant set or H domain of attraction) because the obtained set will be in a larger set defined by matrix H. This matrix should be calculated by means of this method, that will also be used for the LDI.

In this chapter the LDI is defined and depends on matrix H and the H-one step function is defined that also depends on the LDI.

Properties of invariant sets and estimation of the domain of attraction defined by means of this method are also commented.

The chapter continues with three different algorithms that provides these sets and ends with different examples.

1.7.3 Discrete SNS

LDI methods have been used for the determination of invariant sets for discrete time systems. Set provided for this method are smaller to other obtained by means of a new technique called SNS. SNS stands for *Saturated and Non Saturated* due to both systems are considered, the real saturated system and the extension of the linear system no taking into account the saturation.

In chapter 4 this SNS method is presented for discrete time system. SNS invariance and SNS domain of attraction are defined there after introducing the problem to solve.

The one step operator $Q_{SNS}(\cdot,\cdot)$, which is more conservative than non convex operator $Q(\cdot)$ but less conservative than convex $Q_H(\cdot,\cdot)$ shown in chapter 3 is presented. Relationship between SNS and H methods is also commented.

Some examples using the algorithms ends the chapter.

1.7.4 Ellipsoidal SNS

SNS concept presented in chapter 4 can also be applied to continuous time systems.

Chapter 5 shows this extension of SNS method to continuous time systems for the determination of ellipsoidal invariant sets and ellipsoidal estimations of the domain of attraction.

In this chapter contractivity concept is defined and inserted in SNS methods.

Relationship between SNS concepts and LDI methods used in the literature in order to get ellipsoidal invariant set for linear systems with saturation feedback is analysed. The computational cost of the solution of the maximisation problem to obtain the ellipsoidal invariant set is commented. It ends with numerical examples of this method.

1.7.5 Synthesis of robust saturated controllers

In chapters 4 and 5, SNS concept has been defined for linear system with saturation feedback for continuous time and discrete time. However, the controller of the system was known, that is,

$$u = \sigma(Kx)$$

where K is constant. Nevertheless K can be determined to maximise a measure of the size of the estimation of the domain of attraction that can be got by means of SNS methods. This means that it is possible to be more interesting to achieve a synthesis of the controller than make an analysis of it. Actually perturbation in system model can also be added to the formulation, that is the target system is

$$x^+ = Ax + Bu + E\theta \tag{1.9}$$

where θ is a variable that is bounded. This boundary is known.

Chapter 6 shows a method to determine K for systems like 1.9.

In this chapter, the operator $Q(\cdot,\cdot)$ for this family of systems is defined, and an algorithm to get the domain of attraction of the system if the feedback matrix K is known. A first approximation of K method is also presented.

Next, the SNS concepts are defined for this type of problems, and operator $Q_{SNS}(\cdot)$ as a function of K for systems with perturbations is obtained. This value of K is used in a new algorithm to optimise the invariant set obtained.

A numerical example is shown.

1.7.6 Application of LNL invariance for Lur'e systems

SNS method works in linear systems with saturation feedback. These systems belong to a larger family of systems called L'ure systems. In this work an specific family of L'ure system is analyzed, which is the family of systems that the controller can be written as

$$x^+ = Ax + B\phi(kx)$$

where $\phi(\cdot)$ is an even piecewise affine function where it is concave in $\mathbb{R}^+$. In this chapter $kx \in \mathbb{R}$ will be considered.

Chapter 7 analyzes this family of systems. Non-linearity $\phi(\cdot)$ is divided into a set of non-linearities that can be represented as saturations $\sigma(\cdot)$. SNS invariant set applied for this set of saturations will be presented and it will be called LNL that stands for *Linear and Non Linear* due to the shape of $\phi(\cdot)$. Also $Q_{LNL}(\cdot)$ operator is presented and it will be used in order to get an estimation of the domain of attraction and an invariant set of the target system.

The chapter ends with an example.

1.7.7 Robust control invariant set

Methods shown in the literature in order to compute the maximal robust control invariant set for a piecewise affine system require a large computational complexity.

Chapter 8 presents an algorithm that generates estimations of the maximal invariant set with a smaller computational time. The algorithm consists in two parts, the first part is used to obtain an outer estimation of the maximal invariant set, and it this set (scaled) will be used as the input for the second part of the algorithm.

Examples of this algorithm end the chapter.

2

Notation

2.1 Introduction

The saturation is probably the most commonly encountered nonlinearity in control engineering.Most of the nonlinear systems and nonlinear actuators are modeled as saturated functions. Linear systems with saturated actuation are widely used. They have the simplicity of linear systems and are able to cope with the most common nonlinearities.

In this chapter, some notation will be introduced to work with systems with saturated feedback. This will be divided in continuous time systems and discrete time systems although some of the notation can be used in both types of systems.

2.2 Continuous time systems

Definition 1 *A differential equation*

$$\dot{x} = \frac{dx}{dt} = f(x), \quad x = [x_1, x_2, \dots, x_n]^t,$$

is called a three zone piecewise affine system (denoted $3CPL_n$), if there exist three vectors b_0,b_1,b_2, a vector $v \neq 0 \in R^n$, two numbers $\delta_1 < \delta_2$ and three matrices A_0, A_1, $A_2 \in R^{n\times n}$ such that:

$$\dot{x} = f(x) = \begin{cases} A_0 x + b_0 & \text{if } v^t x < \delta_1, \\ A_1 x + b_1 & \text{if } \delta_1 \leq v^t x \leq \delta_2, \\ A_2 x + b_2 & \text{if } v^t x > \delta_2, \end{cases} \tag{2.1}$$

where for all x such that $v^t x = \delta_i$, $i = 1, 2$,

$$A_{i-1}x + b_{i-1} = A_i x + b_i.$$

Definition 2 *A $3CPL_n$ is called simetric and denoted $S3CPL_n$ if $f(x) = -f(-x)$, for all $x \in R$.*

Note that a $3CPL_n$ system like 2.1 is $S3CPL_n$ if and only if $A_0 = A_2$, $b_0 = -b_2$, $b_1 = 0$ and $\delta_1 = \delta_2$. Let us define $A = A_0 = A_2$, $b = b_2 = -b_0$, $k = v/\delta_1$, Continuity property of $3CPL_n$ systems is obtained if and only if

$$A_1 = A + bk^t.$$

Therefore, $S3CPL_n$ systems can be written as

$$\dot{x} = f(x) = \begin{cases} Ax - b & \text{if } k^t x < -1, \\ (A + bk^t)x & \text{if } -1 \leq k^t x \leq 1, \\ Ax + b & \text{if } k^t x > 1. \end{cases} \tag{2.2}$$

Definition 3 *The normalized saturation function $\sigma(\cdot)$ is a continuous function defined by*

$$\sigma(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x > 1, \end{cases} \tag{2.3}$$

where $x \in \mathbb{R}$.

This function is represented in figure 2.1

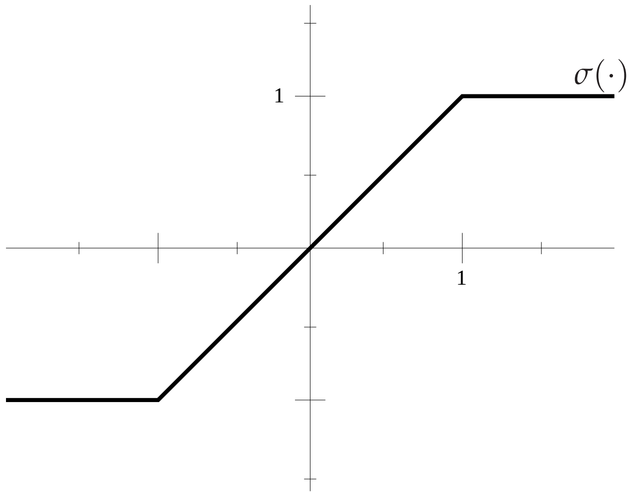

Figure 2.1: The normalized saturation function $\sigma(\cdot)$.

This new definition allows us to write $S3CPL_n$ functions like 2.2 as

$$\dot{x} = Ax + b\sigma(k^t x), \tag{2.4}$$

and it will be the notation used in this work.

Note that 2.4 systems are a particular case of a more extended linear control systems,

$$\dot{x} = Ax + bu, \tag{2.5}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $u \in \mathbb{R}$ is the control signal that is defined by the control law

$$u = \sigma(k^t x). \tag{2.6}$$

This control law is the most widely used control strategy (linear plants or linearized plants controlled by a linear control law). Linear systems have interesting properties that make them suitable for many aplications. Saturation restrictions arise mainly because to most control actuators are limited.

System 2.5 has a monodimensional control law, i.e, $u \in \mathbb{R}$, therefore they are a particular case of the more general family of systems

$$\dot{x} = Ax + Bu, \tag{2.7}$$

where $B \in \mathbb{R}^{n\times m}$, and $u = [u_1 u_2 \dots u_m]^t \in \mathbb{R}^m$. Let us suppose that this system is controlled by a linear control law $u = Kx$, where $K \in \mathbb{R}^{m\times n}$, and that components of u are saturated. That is,

$$u = \begin{bmatrix} \sigma(k_1 x) \\ \sigma(k_2 x) \\ \vdots \\ \sigma(k_m x) \end{bmatrix}, \tag{2.8}$$

where k_1, k_2, ... k_n are columns of matrix K.

This notation can be simplificated with the definition of normalized saturation function for vectors, that, with an abuse of notation, will also be denoted as $\sigma(\cdot)$.

Definition 4 *The normalized saturation function $\sigma(\cdot)$ is a continuous function defined by*

$$\sigma(x) = \begin{bmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{bmatrix}, \tag{2.9}$$

where $x \in \mathbb{R}^n$ and x_1, x_2, ... x_n are components of x.

Note that this formulation is not ambiguous because equation 2.9 becomes equation 2.3 when $x \in \mathbb{R}$

Therefore control law 2.8 is now defined as

$$u = \sigma(Kx). \tag{2.10}$$

The system given by system 2.7 with control law 2.10 includes descriptions 2.5-2.6 and is one of the most important descriptions analyzed in this work.

Due to the saturation, closed loop system 2.7-2.10 is non linear. For example, figure 2.2 shows value of u_1 and 2.3 shows value of u_2 for $x \in \mathbb{R}^2$ and

$$K = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Related to saturated control law 2.10 is the following non-saturated control law.

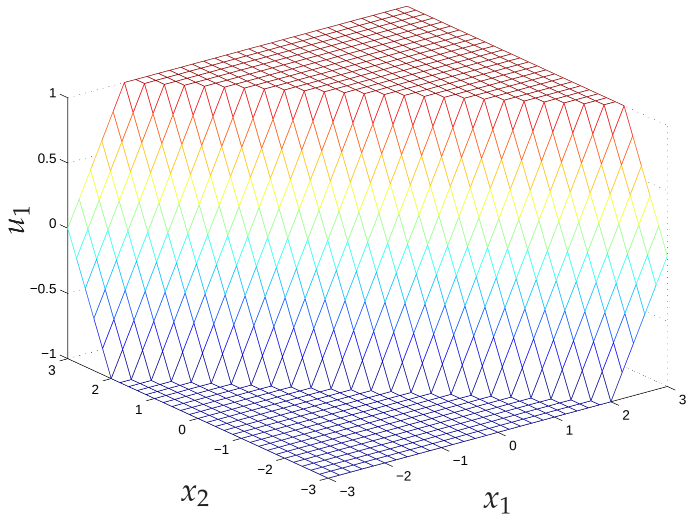

Figure 2.2: Value of the first component of a saturated control law.

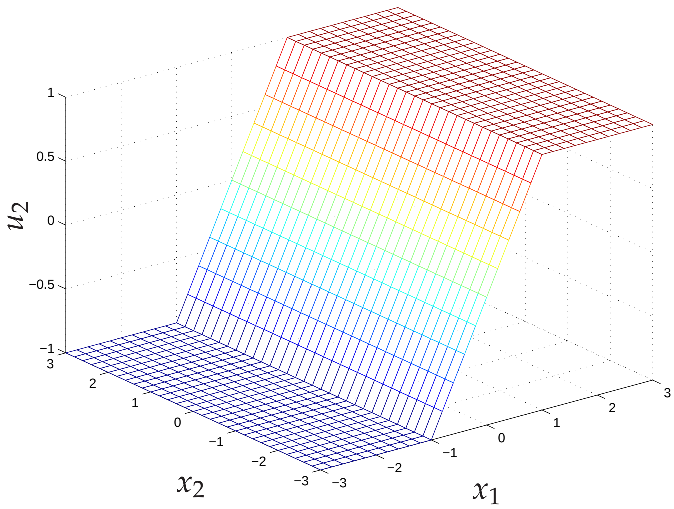

Figure 2.3: Value of the second component of a saturated control law.

Definition 5 *The non-saturated control law associated to a saturated control law $u = \sigma(Kx)$ is defined by*

$$u = Kx, \tag{2.11}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $K \in \mathbb{R}^{m \times n}$.

Property 1 *A saturated control law 2.10 is equal to a non-saturated control law 2.11 in a set that includes the origin.*

Proof :

The proof is trivial hence omited. ■

This property can be observed in figure 2.4.

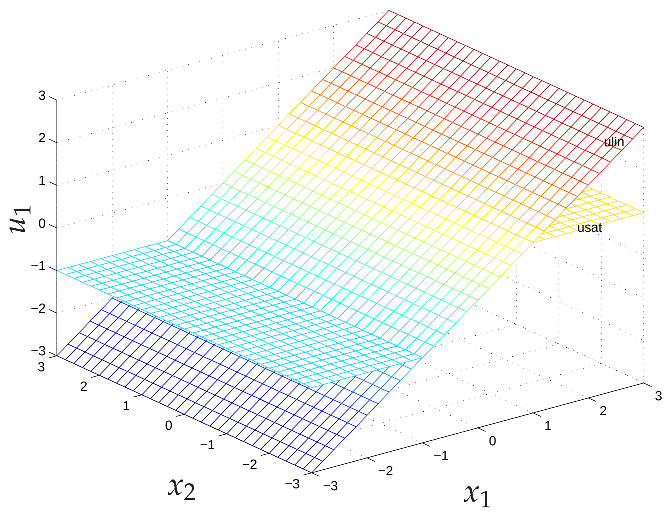

Figure 2.4: Example of the first component of a saturated and the asociated not saturated control law.

The previous property leads to the following theorem.

Theorem 1 *Let ϕ_s be a closed loop system like 2.7-2.10 and ϕ_{ns} be the associated non-saturated closed loop system defined as 2.7-2.11. Then there exists $\Omega_s \in \mathbb{R}^n$ inside the domain of attraction of ϕ_s if and only if there exists $\Omega_{ns} \in \mathbb{R}^n$ that converges to the origin for system ϕ_{ns}.*

PROOF :

Let us suppose that there exists $\Omega_s \in \mathbb{R}^n$ such that for all $x \in \Omega_s$ converges to the origin for the closed-loop system 2.7-2.10. That is, for all $x(t_0) \in \Omega_s$, and for all $\epsilon > 0$ there exists t_1 such that for all $t > t_1$, $|x(t)|_\infty < \epsilon$. Moreover, let suppose that $\epsilon = 1$, and let call $\Omega_s(t_1) = \{x(t_1) : x(t_0) \in \Omega_s\}$. Note that $\Omega_s(t_1)$ is also a set that converges to the origin for the closed-loop system 2.7-2.10. Then, for all $x(t)$, $t > t_1$ where $x(t_1) \in \Omega_s(t_1)$, $\sigma(Kx) = Kx$, therefore $\Omega_s(t_1)$ is also a set that converges to the origin for the closed-loop system 2.7-2.11.

The opposite is proved in the same way. ■

Throughout this text, some auxiliary notation will be used to denote closed-loop system 2.7-2.10. This system can be rewritten as

$$\dot{x} = Ax + B\sigma(Kx). \tag{2.12}$$

That can also be expressed as

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i \sigma(K_i x)$$

where B_i for $i = 1, 2, \ldots, m$ are columns of B and K_i for $i = 1, 2, \ldots, m$ are rows of K.

Let $\mathcal{M} = \{1, 2, \ldots, m\}$. the system 2.12 can also be rewritten as:

$$\dot{x} = Ax + \sum_{i \in \mathcal{M}} B_i \sigma(K_i x).$$

Definition 6 *Given $\mathcal{M}$, the set $\mathcal{V}$ is the set of all subsets of $\mathcal{M}$ including the empty set. That is,*

$$\mathcal{V} = \{ \, S \; : \; S \subseteq \mathcal{M} \, \}$$

Example: If $m = 2$, then $\mathcal{M} = \{1, 2\}$ and $\mathcal{V} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Notation 1 *Given $\mathcal{M}$ and $S \in \mathcal{M}$, S^c denotes the complementary of S in $\mathcal{M}$. That is, $S^c = \{ \, i \in \mathcal{M} \; : \; i \notin S \, \}$.*

In the previous example, $m = 2$, $\mathcal{M} = \{1, 2\}$, and if $S = 1$ then $S^c = 2$. If $S = \{1, 2\}$ then $S^c = \{\varnothing\}$.

This notation allows to express the family

$$\dot{x} = Ax + \sum_{i \in S} B_i \sigma(K_i x) + \sum_{i \in S^c} B_i K_i x. \tag{2.13}$$

Note that system 2.12 is a particular case of family 2.13. That is if $S = \mathcal{M}$, $S^c = \{\emptyset\}$ and leads to the same formulation.

Note also that the non-saturated control system 2.7-2.11 is a particular case of family 2.13. That is if $S^c = \{\emptyset\}$, $S = \mathcal{M}$.

In the following chapters it will be presented some properties that will use the system family 2.13. As far as all properties applies for all systems in this system family, they also can be used for saturation systems.

2.3 Discrete time systems

Many of the system problems in engineering are discrete time systems. Many of the previous concepts for continuous time systems can be applied to discrete time systems.

Definition 7 *A discrete time equation*

$$x^+ = f(x), \quad x = [x_1, x_2, \dots, x_n]^t,$$

is called a three zone discrete piecewise affine system (denoted $3DCPL_n$), if there exists three vectors b_0,b_1,b_2 and a vector $v \neq 0 \in R^n$, two numbers $\delta_1 < \delta_2$, three matrices A_0, A_1, $A_2 \in R^{n \times n}$ such that:

$$x^+ = f(x) = \begin{cases} A_0 x + b_0 & \text{if } v^t x < \delta_1, \\ A_1 x + b_1 & \text{if } \delta_1 \leq v^t x \leq \delta_2, \\ A_2 x + b_2 & \text{if } v^t x > \delta_2, \end{cases} \tag{2.14}$$

where for all x such that $v^t x = \delta_i$, $i = 1, 2$,

$$A_{i-1} x + b_{i-1} = A_i x + b_i.$$

In this formulation x is the state vector and x^+ is the successor of x. That is, if $x = x(k)$, $x^+ = x(k+1)$.

Definition 8 *A $3DCPL_n$ is called simetric and denoted $S3DCPL_n$ if $f(x) = -f(-x)$, for all $x \in R$.*

Note that a $3DCPL_n$ system as 2.14 is $SD3CPL_n$ if and only if $A_0 = A_2$, $b_0 = -b_2$, $b_1 = 0$ and $\delta_1 = \delta_2$. Let us define $A = A_0 = A_2$, $b = b_2 = -b_0$, $k = v/\delta_1$, Continuity property of $3DCPL_n$ systems is obtained if and only if

$$A_1 = A + bk^t.$$

Therefore, $SD3CPL_n$ systems can be written (using the normalized saturation function $\sigma(\cdot)$ as

$$x^+ = Ax + b\sigma(k^t x), \tag{2.15}$$

2.15 systems are a particular case of a more extended linear control systems,

$$x^+ = Ax + bu, \tag{2.16}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $u \in \mathbb{R}$ is the control signal that is defined by the control law defined in 2.6.

System 2.16 has a monodimensional control law, i.e, $u \in \mathbb{R}$, therefore they are a particular case of the most general systems

$$x^+ = Ax + Bu, \tag{2.17}$$

where $B \in \mathbb{R}^{n\times m}$, and $u = [u_1 u_2 \dots u_m]^t \in \mathbb{R}^m$. Let us suppose that this system is controlled by a linear control law $u = Kx$, where $K \in \mathbb{R}^{m\times n}$, and that the components of u are saturated. That is, $u = \sigma(Kx)$ as shown in 2.10.

Note that property 1 can be also applied for discrete time systems because definition of u does not depend on the system. That property leads to the following theorem.

Theorem 2 *Let ϕ_s a discrete closed loop like 2.17-2.10 and ϕ_{ns} the associated non-saturated discrete closed loop system defined as 2.17-2.11. Then there exists $\Omega_s \in \mathbb{R}^n$ that converges to the origin for system ϕ_s if and only if there exists $\Omega_{ns} \in \mathbb{R}^n$ that converges to the origin for system ϕ_{ns}.*

PROOF :

The proof is similar to proof of theorem 1. ■

Using the same notation that shown for continuous time systems, the closed loop system 2.17-2.10 can be written as

$$x^+ = Ax + B\sigma(Kx). \tag{2.18}$$

That, using the $\mathcal{M}$ notation, can also be expressed as

$$x^+ = Ax + \sum_{i \in \mathcal{M}} B_i \sigma(K_i x). \tag{2.19}$$

where B_i for $i = 1, 2, \ldots, m$ are columns of B and K_i for $i = 1, 2, \ldots, m$ are rows of K.

This notation allows to write the following system family,

$$x^+ = Ax + \sum_{i \in S} B_i \sigma(K_i x) + \sum_{i \in S^c} B_i K_i x. \tag{2.20}$$

Note that, in the same way that shown for continuous time systems, system 2.18 and the non-saturated control system 2.17-2.11 are a particular case of family 2.20.

In the context of set invariance theory, the one-step set plays an important role [17]. A general definition of the one-step set is given.

Definition 9 *Given a set Ω and a discrete-time system*

$$x^+ = f(x)$$

where $x \in \mathbb{R}^n$ is the state vector and x^+ is the successor, it is called one-step set of set Ω to

$$Q(\Omega) = \{\ x\ :\ f(x) \in \Omega\ \}$$

where $Q(\cdot)$ is denoted one-step operator.

With an abuse of notation it will be referred $Q(\cdot)$ in general when saturation system 2.18 is considered. That is,

$$Q(\Omega) = \{\ x\ :\ Ax + B\sigma(Kx) \in \Omega\ \}$$

When different systems are considered it will be shown explicitaly.

The one step set is the set that in one step reach Ω

For example, consider the matrices

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, K = \begin{bmatrix} 0.4 & 0.8 \end{bmatrix},$$

and $\Omega = \{ x : |x|_\infty \leq 1 \}$. Then $Q(\Omega)$ referred to $x^+ = Ax + BKx$ is the set shown in figure 2.5. In this figure, Ω set is represented as the inner yellow set, and $Q(\Omega)$ is the outer cyan set. One important property is that if Ω is a polyhedrical convex set, and $f(x)$ is a linear function then $Q(\Omega)$ is also a polyhedrical convex set. All states that belongs to $Q(\Omega)$ in the figure will reach Ω in one step by $x^+ = Ax + BKx$ system.

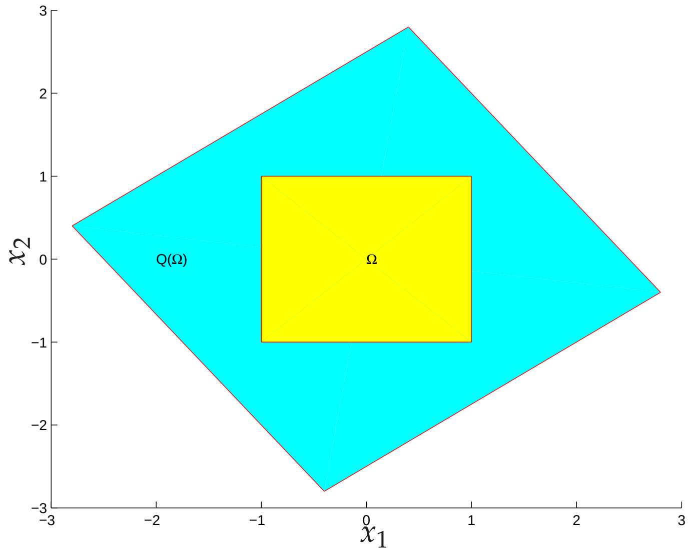

Figure 2.5: Example of $Q(\Omega)$ for a discrete linear system.

Figure 2.6 represents $Q(\Omega)$ referred to saturated system $x^+ = Ax + B\sigma(Kx)$. In this figure Ω is also represented as the inner yellow set and $Q(\Omega)$ is the outer cyan set. Note that as $\sigma(\cdot)$ is a non-linear function, $Q(\Omega)$ can be a non convex set although Ω is a polyhedrical convex set. In this example, $Q(\Omega)$ is represented as the union of polyhedrical sets due to the piece-wise affine nature of non-linearity $\sigma(\cdot)$.

It can be useful to use function $Q(\cdot)$ recursively, that it, if $C_0 = \Omega$, apply the recursion $C_{i+1} = Q(C_i)$ for $i = 0, 1, 2, \ldots$. This kind of iterations will be used in most of the following chapters. In this case it is important to be noted that if system is non-linear or C_0 is non-convex, C_i is non-convex in general. System $x^+ = Ax + B\sigma(Kx)$ produces sets C_i of the type of union of convex polihedra, that are in general non-convex.

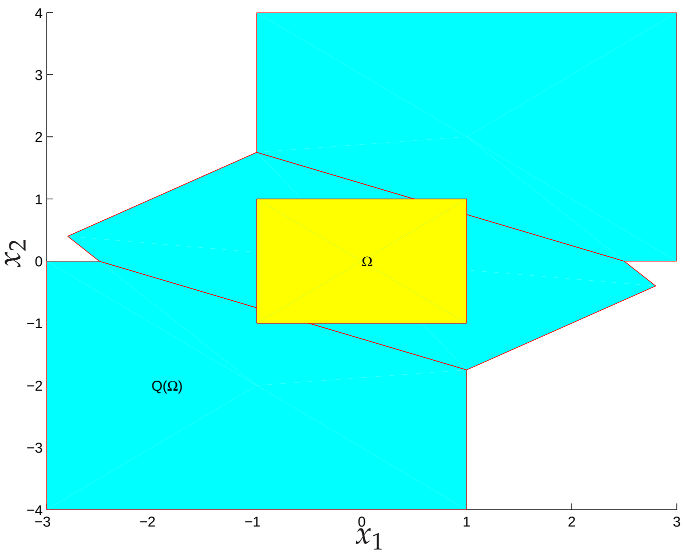

Figure 2.6: Example of $Q(\Omega)$ for a discrete saturated system.

3

H-domain of attraction

3.1 Introduction

The estimation of stability regions of nonlinear systems is important for many fields in engineering. Regions of asymptotic stability are zones of safe operation that can avoid unnecessary operational restrictions if they are non-conservative [15, 20, 17].

The estimation of the domain of attraction of linear systems subject to control saturation has received the attention of many authors in the last years (see, for example, [23, 4, 29, 33] and references therein).

One of the most relevant approaches to the analysis of saturated systems is based on a *Linear Difference Inclusion* (LDI) of the saturation nonlinearity. For example, in [24, 33], an invariant ellipsoid for the saturated system is obtained by means of an LDI. This approach has also been used in [41] to obtain a polyhedral invariant set for a saturated system.

The domain of attraction of a given saturated system can be approximated by means of an ellipsoid. In [33] and [29] an LDI for a linear saturated systems is presented. Based on that LDI, the authors propose how to choose simultaneously both the matrix H, that characterizes the LDI, and the greatest ellipsoid that is invariant under the corresponding LDI.

This chapter presents an approach to the polyhedric estimation of the domain of attraction of a saturated linear system. The polyhedric estimation is less conservative but at the expense of an increased representation complexity. For that purpose, given a system with m saturated control inputs, a *Linear Matrix Inequality* (LMI) problem with $2^m + m$ constraints must be solved. Moreover, given the obtained LDI, The maximum domain of attraction provided by the LDI (denoted H-domain of attraction) is characterized. It is also provided an algorithm that estimates the domain of attraction of the nonlinear system. Under mild conditions, the proposed algorithm obtains the exact H-domain of attraction of the system.

3.2 Problem Statement

Only discrete-time saturated systems will be considered in this chapter. It will be obtained an LDI that provides an estimation of the domain of attraction and an invariant set of the saturated system. Therefore, the following system is considered,

$$x^+ = Ax + B\sigma(Kx) \tag{3.1}$$

where $x \in \mathbb{R}^n$ denotes the state vector, x^+ the successor state vector and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $K \in \mathbb{R}^{m \times n}$ are matrices. Function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the normalized saturation function and it is defined in

2.9.

Note that this system is the discrete closed loop system defined in 2.17-2.10 of chapter 2.

Using the notation provided in chapter 2, this formulation can be expressed as

$$x^+ = Ax + \sum_{i \in \mathcal{M}} B_i \sigma(K_i x) \tag{3.2}$$

System 3.2 is a non-linear system due to the saturation function, hence the domain of attraction and the corresponding largest invariant set can be non-convex. In this chapter it will be provided a convex conservative estimation of the domain of attraction and a convex invariant set of the saturated system by means of an LDI. In order to obtain the desired convex sets the H-domain of attraction and H-invariant concepts will be used.

3.3 Linear difference inclusion

In the following, the linear difference inclusion (LDI) that is going to be used in the H-domain of attraction notion will be presented. This LDI is the one adopted in recent works like [33, 12] and it is a generalization that improves the one presented in [22] (see also, [24, 41]).

In order to introduce the LDI, some auxiliary definitions will be used.

Notation 2 *Given matrix $H \in \mathbb{R}^{m \times n}$, and set $S \in \mathcal{V}$, $G_H(x, S)$ is defined as follows,*

$$G_H(x, S) = (A + \sum_{i \in S^c} B_i K_i + \sum_{i \in S} B_i H_i)x. \tag{3.3}$$

Note that with these definitions, $x^+ = (A + BH)x = G_H(x, \mathcal{M})$. Also, $x^+ = G_H(x, \emptyset) = (A + BK)x$ represents the evolution of the system without saturation.

Notation 3 *Given matrix $H \in \mathbb{R}^{m \times n}$, $\mathcal{L}(H)$ denotes the following symmetric polyhedron:*

$$\mathcal{L}(H) = \{ \, x \in \mathbb{R}^n \; : \; \|Hx\|_\infty \leq 1 \, \}$$

For example, if

$$H = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \\ -0.5 & 0.5 \end{bmatrix}, \tag{3.4}$$

$\mathcal{L}(H)$ is shown in figure 3.1.

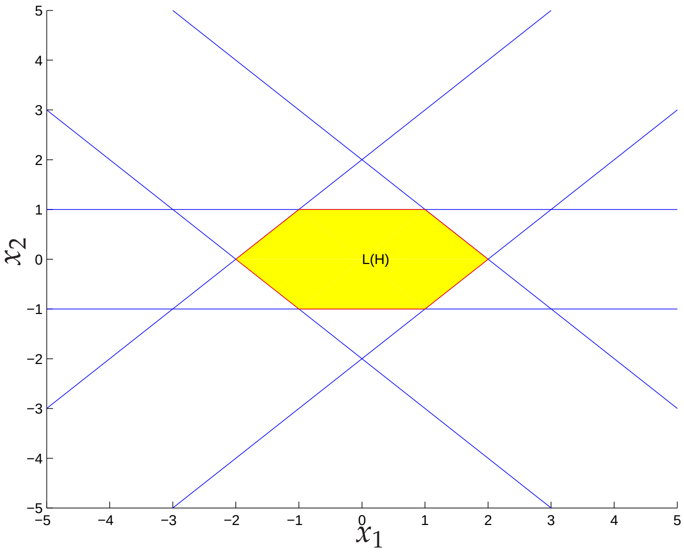

Figure 3.1: Graphical representation of $\mathcal{L}(H)$.

Note that $\mathcal{L}(H)$ can be a non-bounded set. Moreover, boundedness of $\mathcal{L}(H)$ is determined by rank of H.

Recall that co $\{v_1, v_2, \ldots, v_p\}$ denotes the convex hull of the vectors $v_1, v_2, \ldots, v_p$. That is, $x \in$ co $\{v_1, v_2, \ldots, v_p\}$ if and only if there exists scalars λ_i, $i = 1, \ldots, p$ such that $\lambda_i \geq 0$, $i = 1, \ldots, p$, $\sum_{i=1}^{p} \lambda_i = 1$ and $x = \sum_{i=1}^{p} \lambda_i v_i$.

For example, $\mathcal{L}(H)$ for H defined like 3.4 can also be defined as

$$\mathcal{L}(H) = \{ \, x \, : \, x \in \text{co} \, \{ \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \} \, \}.$$

This notation can not be used only if $\mathcal{L}(H)$ is a non-bounded set.

The following lemma [33, 29] provides, given matrix H, an LDI that is valid in $\mathcal{L}(H)$:

Lemma 1 *Let $H \in \mathbb{R}^{m\times n}$ be given. If $x \in \mathcal{L}(H)$ then*

$$Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) \in co\ \{\ G_H(x,S)\ :\ S \in \mathcal{V}\ \},$$

where $\mathcal{V}$ is defined in 6.

3.3.1 One step operator for the linear difference inclusion

The one step operator has been defined in definition 9 of chapter 2. In order to obtain an estimation of the domain of attraction of the saturated nonlinear system, the one-step operator for the LDI is presented in the following definition.

Definition 10

- *Given a set Ω and $S \in \mathcal{V}$:*

$$Q_H(\Omega, S) = \{\ x\ :\ G_H(x,S) \in \Omega\ \}$$

- *Given a set Ω:*

$$\hat{Q}_H(\Omega) = \bigcap_{S\in\mathcal{V}} Q_H(\Omega, S)$$

One of the most important properties of $\hat{Q}(\cdot)$ is that given a convex polyhedral set Ω and $S \in \mathcal{V}$, $Q_H(\Omega, S)$ is a convex polyhedron, hence $\hat{Q}(\Omega)$ is also a convex polyhedron.

When $Q_H(\Omega, S)$ operator is applied to an Ω set, it is obtained the set that evolves to Ω for system $G_H(\cdot, S)$. Moreover, all states included in the set given by $\hat{Q}_H(\Omega)$ evolves to Ω for all $S \in \mathcal{V}$. Note that if Ω is a convex set by direct application of Lemma 1, $\hat{Q}(\Omega)$ is a conservative convex set that evolves to Ω for system 3.2.

3.4 H-domain of attraction

H-domain of attraction concept is introduced in this section. H-domain of attraction is a convex set that estimates the domain of attraction of the system. Therefore prior to define H-domain of attraction, the definition of domain of attraction must be given.

Definition 11 *It is said that the initial condition x_0 belongs to the domain of attraction of system $x^+ = Ax + B\sigma(Kx)$ if the recursion*

$$x_{k+1} = Ax_k + B\sigma(Kx_k)$$

converges to the origin.

The domain of attraction for a given saturated systems is a set (bounded or not) that shows what are the states that converge to the origin using a linear saturated controller. It is a caracterization of the stability of the system, because only states included in the domain of attraction converge to the origin.

Prior to the definition of H-domain of attraction, the notion of admissible sequence is introduced.

Definition 12 *It is said that a sequence* $\{S_0, S_1, S_2, \ldots\}$ *is admissible if all the elements of the sequence belong to* $\mathcal{V}$.

Note that an admissible sequence can be open or closed.

Definition 13 *Given matrix* H*, it is said that the initial condition* $x_0 \in \mathcal{L}(H)$ *belongs to the* H*-domain of attraction of system* $x^+ = Ax + B\sigma(Kx)$ *if the recursion*

$$x_{k+1} = G_H(x_k, S_k)$$

satisfies the following two conditions:

1. $x_k \in \mathcal{L}(H)$*, for all* $k \geq 0$ *and for every admissible sequence* $\{S_0, S_1, \ldots, S_{k-1}\}$.
2. *The recursion converges to the origin for every admissible infinite sequence* $\{S_0, S_1, S_2, \ldots\}$.

It is clear from the linear difference inclusion provided by means of lemma 1 that any H-domain of attraction constitutes a conservative estimation of the domain of attraction of the non-linear system. See, for example, [41]. That is, any H-domain of attraction of a system is a subset of the domain of attraction of the saturated system.

3.5 *H*-invariant sets

This section presents the notion of H-invariant set which is used to obtain a conservative approximation of the maximal invariant set.

Definition 14 *It is said that a set Ω is an invariant set for system $x^+ = f(x)$ if*

$$f(x) \in \Omega, \ \ \forall x \in \Omega.$$

Note that the invariance is a concept related to stability. If an invariant set Ω is obtained then for all $x \in \Omega$, system evolves to a state included in Ω and the system remains in Ω for all the following sample times.

Given a matrix H and a set Ω, It is said that $\Omega \subseteq \mathcal{L}(H)$ is an H-invariant set if it is an invariant set for all the systems that compounds the *Linear Difference Inclusion* corresponding to H. This notion will be precisely stated in the following definition.

Definition 15 *Given a matrix H, It is said that a set $\Omega \subseteq \mathcal{L}(H)$ is an H-invariant set for system $x^+ = Ax + B\sigma(Kx)$ if*

$$G_H(x, S) \in \Omega, \ \ \forall x \in \Omega, \forall S \in \mathcal{V}.$$

The notion of H-invariance is a stronger notion of invariance. That is, by means of lemma 1, it is easily inferred that H-invariance implies invariance for the saturated system. However, the reverse is not true. The largest H-invariant set depends on the value of matrix H. Next section provides a method to obtain matrix H by means of a maximization problem restricted to an *LMI*.

In order to obtain an estimation of the domain of attraction of the saturated nonlinear system, the one-step operator for the linear difference inclusion presented in definition 10 will be used [6, 17].

From the definition of $\hat{Q}_H(\cdot)$, the following geometrical characterization of H-invariance is obtained:

Property 2 *Given matrix H, the set Ω is an H-invariant set for the system $x^+ = Ax + B\sigma(Kx)$ if and only if $\Omega \subseteq \hat{Q}_H(\Omega) \cap \mathcal{L}(H)$.*

Proof :

It derives directly of the definition of $\hat{Q}_H(\cdot)$.

Let $x \in \Omega$ and $\Omega \subseteq \hat{Q}_H(\Omega) \cap \mathcal{L}(H)$. By definition 10, $G_H(x, S) \in \Omega, \ \forall S \in \mathcal{V}$, therefore Ω is an H-invariant set.

In the other hand, if $\Omega \subseteq \mathcal{L}(H)$ is an H-invariant set, by notation 2, for all $x \in \Omega$, $G_H(x, S) \in \Omega$, for all $S \in \mathcal{V}$. That is $x \in Q_H(\Omega, S)$ for all $S \in \mathcal{V}$, and by definition 10, $x \subseteq \hat{Q}_H(\Omega)$. ■

Definition 16 *Given a matrix H, it is said that a set $\Omega \subseteq \mathcal{L}(H)$ is an H-contractive set for system $x^+ = Ax + B\sigma(Kx)$ (with a contraction factor $\alpha \in [0,1)$) if:*

$$G_H(x,S) \in \alpha\Omega, \quad \forall x \in \Omega, \forall S \in \mathcal{V}.$$

By linearity this is equivalent to say that $\Omega \subseteq \hat{Q}_H(\alpha\Omega) \cap \mathcal{L}(H)$.

3.6 Obtaining matrix H

In section 3.4 it was shown the definition of H-domain of attraction and in section 3.5 it was shown the definition of an H-invariant set. Both definitions depend strongly on matrix H. Note that one of the contributions of this work is a procedure to obtain an estimation of the domain of attraction and an invariant set of the saturated system 2.18.

Calculation of matrix H is presented in this section. In [33] and [29], ellipsoidal estimations of the domain of attraction of a saturated system are given. In order to maximize the size of the ellipsoidal sets, the authors propose an LMI maximization problem in which both matrix H and the parameters of the ellipsoid are simultaneously obtained. Inspired in the afore mentioned work, in this chapter it is proposed to generalize the results of [33] and [29] to the case of polyhedral invariant sets. The LDI of lemma 1 is a generalization of the one used previously in the literature. Therefore, less conservative results are provided.

The greatest H-contractive ellipsoidal set can be obtained by means of an LMI optimization, and will be obtained in theorem 3. This choice allows us to maximize the size of a polyhedral H-contractive set that contains the obtained ellipsoid.

Definition of $G_H(\cdot,\cdot)$ given by notation 2 can be rewritten using matrices.

Definition 17 *Given $i \in \mathcal{M}$ and $S \in \mathcal{V}$, the S membership function $d_i(S)$ is defined as*

$$d_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Definition 18 *Given $S \in \mathcal{V}$, the diagonal matrices E_S and E_S^c are defined as:*

$$E_S = \begin{bmatrix} d_1(S) & & & \\ & d_2(S) & & \\ & & \ddots & \\ & & & d_m(S) \end{bmatrix}, \quad E_S^c = I - E_S$$

For example, if $m = 2$:

$$E_{\emptyset} = \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}, \; E_{\{1\}} = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix},$$

$$E_{\{2\}} = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}, \; E_{\{1,2\}} = E_{\mathcal{M}} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

With this definitions, an alternative definition of $G_H(\cdot,\cdot)$ can be shown:

$$G_H(x, S) = Ax + B(E_S^c K + E_S H)x$$

Notation 4 *Given a positive definite matrix P, and a positive scalar ρ, $\mathcal{E}(P,\rho)$ represents the following ellipsoid,*

$$\mathcal{E}(P,\rho) = \{\ x\ :\ x^\top Px \le \rho\ \}.$$

Definition 19 *Given matrix $H \in \mathbb{R}^{m\times n}$, it is said that the ellipsoid $\mathcal{E}(P,1)$ is H-contractive if $\mathcal{E}(P,1) \subset \mathcal{L}(H)$ and there is $0 < \alpha < 1$ such that for every $S \in \mathcal{V}$,*

$$G_H^\top(x,S)PG_H(x,S) < \alpha x^\top Px.$$

Note that matrix H determines the H-domain of attraction of the system and the maximal H invariant set. Moreover, a bad election of H can determine that the maximal H-domain of attraction for systems with an stable linear part is the empty set. Hence the election of H is determinant. Next property shows that there exists a value of H for systems with a stable linear part that can be used to obtain a non-empty H domain of attraction set.

Property 3 *If the linear system $x^+ = (A + BK)x$ is asymptotically stable then there exists a matrix $H \in \mathbb{R}^{m\times n}$ and a matrix P such that $\mathcal{E}(P,1)$ is an H-contractive ellipsoid.*

PROOF :

Note that the asymptotically stability of $x^+ = (A + BK)x$ implies that there is $\hat{P} > 0$ and $\alpha \in (0,1)$ such that

$$(A + BK)^\top \hat{P}(A + BK) < \alpha\hat{P}$$

Note that if matrix H is equal to K then $G_H(x,S) = (A + BK)x$ for every set $S \in \mathcal{V}$. Therefore:

$$G_H^\top(x,S)\hat{P}G_H(x,S) = x^\top(A+BK)^\top\hat{P}(A+BK)x \leq \alpha x^\top\hat{P}x \tag{3.5}$$

It is always possible to find a scalar ζ *such that* $\mathcal{E}(\zeta\hat{P},1) \subset \mathcal{L}(H) = \mathcal{L}(K)$. *Therefore, if* P *is equal to* $\zeta\hat{P}$ *then* $\mathcal{E}(P,1) \subset \mathcal{L}(H)$. *Moreover, the inequality (3.5) is also satisfied for the scaled matrix* $P = \zeta\hat{P}$. ■

This property demonstrates that the LDI representation of saturated systems can always be quadratically stabilized, inhering this property from the not saturated closed-loop system. Moreover, it proves that an H-contractive ellipsoid can always be found for the saturated system.

The following result (see [33, 29]) states, by means of linear matrix inequalities ($LMIs$), a characterization of the H-contractive ellipsoids of a given system.

Theorem 3 *Let us suppose that given* $\alpha \in (0,1)$, *matrices* $W \in \mathbb{R}^{n\times n}$ *and* $Y \in \mathbb{R}^{m\times n}$ *satisfy the following linear matrix inequalities (LMIs):*

$$\begin{bmatrix} \alpha W & ((A+\sum\limits_{i\in S^c} B_iK_i)W + \sum\limits_{i\in S} B_iY_i)^\top \\ (A+\sum\limits_{i\in S^c} B_iK_i)W + \sum\limits_{i\in S} B_iY_i & W \end{bmatrix} > 0,\ \forall S \in \mathcal{V} \tag{3.6}$$

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0,\quad i = 1,\dots,m \tag{3.7}$$

where Y_i *denotes the i-th row of* Y. *Then, denoting* $H = YW^{-1}$ *and* $P = W^{-1}$ *it results that* $\mathcal{E}(P,1)$ *is an* H*-contractive ellipsoid.*

In order to obtain the greatest H-contractive ellipsoid, different approaches can be considered. In [33, 29], the concept of reference set is applied to give a measure of a given set.

An approach that has a good behaviour in experiments and it is proposed in this work is the maximization of the trace of matrix W. That is, the following LMI problem can be solved in order to obtain W and $H = YW^{-1}$:

$$\begin{array}{c} \max\limits_{W,Y} \quad \mathrm{tr}\, W \\ \text{subject to LMIs (3.6) and (3.7)} \end{array} \tag{3.8}$$

Note that the above LMI problem has $2^m + m$ constraints. Therefore, the computational burden associated to the LMI problem becomes unmanageable by solving the previous optimization problem when m grows beyond a certain limit.

A measure of the size of the H-contractive ellipsoid is maximized. Note that the ellipsoid is included in the H-domain of attraction. Therefore, the size of the obtained ellipsoid is a lower bound of the size of the H-domain of attraction. Denote H^* the matrix obtained from the solution of the proposed maximization problem. It will be used the linear difference inclusion corresponding to matrix H^*. Note that with this choice a lower bound of the size of the H-domain of attraction is maximized.

3.7 Proposed algorithms

In the previous section a technique to calculate matrix H has been described. This matrix H defines a set, $\mathcal{L}(H)$, of possible application of lemma 1.

This matrix H is the first step to obtain an estimation of the H-domain of attraction or an H-invariant set as large as possible. In order to get this purpose two different algorithms can be applied.

1. Outer-inner algorithm. The objective of this algorithm is to obtain the largest H-invariant set of the system. This algorithm is applied to obtain a sequence of sets that converges to the maximal H-invariant set. This algorithm also converges to the H-domain of attraction of the system.

   The most important disadvantage of this algorithm is that in general, intermediate sets provided by it are not invariant nor conservative estimations of the domain of attraction. If the complexity of these sets is big enough that they excess computational resources all work done is useless.

   Algorithm 3 proposed will overcome this problem.

2. Inner-outer algorithm. The objective of this algorithm is to obtain estimations of the H-domain of attraction of the system. The H-domain of attraction is an H-invariant set, however, estimations obtained by means of this systems are not H-invariant sets in general.

3. Two-phase algorithm. This algorithm works in tho phases. Firstly it obtains an H-invariant set that it is also a conservative estimation of the H-domain of attraction and in a second phase it enlarges this set to converge to the H-domain of attraction.

3.7.1 Outer-inner algorithm

This subsection presents an algorithm that, using concepts of H-domain of attraction and H-invariant set, computes an estimation of the domain of attraction and an invariant set of the nonlinear system: $x^+ = Ax + B\sigma(Kx)$. Moreover, this algorithm converges to the maximal H-invariant set and the H-domain of attraction of the system. This algorithm is based on the optimal computation of the matrix H previously presented. The obtained set is a bounded convex polytope under mild assumptions.

Prior to show algorithm, following theorem states some important properties.

Theorem 4 *Given matrix H and a given scalar $\alpha \in (0, 1]$ provided the conditions of theorem 3 hold for system $x^+ = Ax + B\sigma(Kx)$. Set $C_0 = \mathcal{L}(H)$ and consider the following recursion,*

$$C_{k+1} = \alpha \hat{Q}_H(C_k) \bigcap \mathcal{L}(H).$$

Then, the following properties hold for each obtained set C_k:

(i) *C_k is a convex polyhedron that can be obtained by means of definition (10).*

(ii) *There exists a matrix R_k such that $C_k = \mathcal{L}(R_k)$.*

(iii) *$C_{k+1} \subseteq C_k$ for all $k \geq 0$.*

(iv) *In the particular case than $\alpha < 1$, if there exists a $T = H_i$ or $T = K_i$ such that for all $S \in \mathcal{V}$ the pair $(T, A + B(E_S^c K + E_S H))$ is observable, then the recursion is finitely determined.*

(v) *If $\hat{x}$ noes not belong to the H-domain of attraction, then $\hat{x} \notin C_j$, where j is the smallest integer that satisfies:*

$$j \geq \frac{\ln(\hat{x}^\top P \hat{x})}{\ln(\frac{1}{\alpha})}.$$

(vi) *The H-domain of attraction is included in every set C_k.*

(vii) *C_∞ is the maximal H-invariant set for the saturated system.*

(viii) *The sequence $C_0, C_1, C_2, \ldots,$ converges to the H-domain of attraction.*

PROOF :

The proof of this theorem can be derived from [16, 41].

(i) *The first point stems directly from the fact that* $C_0 = \mathcal{L}(H)$ *is a polyhedron and the definition of* $\hat{Q}_H(\cdot)$.

(ii) *This point can be proved if the following can be asserted*

(a) *For all matrix* T *there exists matrix* Q *such that* $\mathcal{L}(Q) = \hat{Q}_H(\mathcal{L}(T))$.

(b) *For all matrix* T *and* $\alpha > 0$ *there exists matrix* Q *such that* $\mathcal{L}(Q) = \alpha\mathcal{L}(T)$.

(c) *For all matrixes* T, V *there exists matrix* Q *such that* $\mathcal{L}(Q) = \mathcal{L}(T) \bigcap \mathcal{L}(H)$.

And this is true because

(a) *Note that by definition 10,* $\hat{Q}_H(\mathcal{L}(T)) = \bigcap_{S\in\mathcal{V}} Q_H(\mathcal{L}(T), S)$, *and* $Q_H(\mathcal{L}(T), S) = \{\ x\ :\ G_H(x, S) \in \mathcal{L}(T)\ \}$. *Note also that by notation 2,* $G_H(x, S) = (A + \sum_{i\in S^c} B_iK_i + \sum_{i\in S} B_iH_i)x$ *where* B_i, K_i *and* H_i *are defined in that notation. Then, matrixes* $Q_i = T(A + \sum_{i\in S^c} B_iK_i + \sum_{i\in S} B_iH_i)$ *are such that* $\mathcal{L}(Q_i) = Q_H(\mathcal{L}(T), S)$. *This and the fact that intersections of sets in the form* $\mathcal{L}(\cdot)$ *can be described in the form* $\mathcal{L}(\cdot)$ *proves this paragraph.*

(b) Q *exists indeed and it is defined as* $Q = \frac{Q}{\alpha}$, *see notation 3. Note that* α *is positive.*

(c) Q *exists indeed and it is defined as* $Q = \begin{bmatrix} T \\ V \end{bmatrix}$, *see notation 3.*

Therefore, and by the definition of the recursion, C_k *can be expresed as* $\mathcal{L}(R_k)$.

(iii) *This property stems by definition 10 of* $\hat{Q}_H(\cdot)$. *Note that the recursion* C_k *is defined as the set where all states remain in* $\mathcal{L}(H)$ *for* k *steps. Therefore all states that remain in* $\mathcal{L}(H)$ *for* $k+1$ *steps, also remain for* k *steps.*

(iv) *Note that* C_{n-1} *exists and it is the admisible set in* $n-1$ *steps, that is, the set of states from which the system evolution remains in* $\mathcal{L}(H)$ *for the next* $n-1$ *steps.*

Note that restrictions

$$C^S_{n-1} = \{x \in \Re^n : |T(A + B(E^c_S K + E_S H))|^i \preceq \rho, \forall i = 0, \dots, n-1\}$$

is such that $C_{n-1} \subseteq C^S_{n-1}$.

Taking into account that the observability matrix of $(T, A + B(E^c_S K + E_S H))$ *is full rank then* C^S_{n-1} *and therefore* C_{n-1} *is compact.*

This and the fact that C_k is contractive proves that the recursion is finitely determined.

(v) *Denote $x_0 = \hat{x}$ where $\hat{x}$ does not belong to the H-domain of attraction of the system. Then, from the definition of H-domain of attraction, the recursion $x_{i+1} = G_H(x_i, S_i)$ does not remain in $\mathcal{L}(H)$ for every admissible sequence $\{S_0, S_1, \ldots\}$. Let us suppose that x_0 does belong to the H-domain of attraction of the system and x_{i+1} remains in $\mathcal{L}(H)$. Therefore$\mathcal{E}(P,1)$ is an H-contractive ellipsoid, guarantees that $x_{i+1}^\top P x_{i+1} \le \alpha x_i^\top P x_i$, for all $i \ge 0$ and all admissible sequence $\{S_0, S_1, \ldots\}$. From this it is inferred that:*

$$x_i^\top P x_i \le \alpha^i x_0^\top P x_0.$$

It can be easily seen that if

$$j \ge \frac{\ln(\hat{x}^\top P \hat{x})}{\ln(\frac{1}{\alpha})}$$

then $x_j^\top P x_j \le 1$. This implies that $x_j \in \mathcal{E}(P,1)$ or that x_j in the H-domain of attraction. Therefore, or $\hat{x}$ belongs indeed to the H-domain of attraction of the system, or for some $i < j$ and some addmisible sequence $\{S_0, S_1, \ldots, \S_i\}$, , $x_i \notin \mathcal{L}(H)$ and therefore $\hat{x} \notin C_i$.

(vi) *It can be proved as an extension of the proof of the previous point. If $\hat{x}$ belongs to the H domain of attraction, and it is defined $x_0 = \hat{x}$, then $x_j \in \mathcal{E}(P,1)$. Moreover, $\mathcal{E}(P,1)$ is an H-contractive set inside $\mathcal{L}(H)$, and therefore $\mathcal{E}(P,1) \supseteq \alpha \hat{Q}_H(\mathcal{E}(P,1)$. That is, for all $i > j$, $x_i \in \mathcal{E}(P,1)$. If $\hat{j}$ is defined as the maximum j for all $\hat{x}$ in the H-domain of attraction, then the H-domain of attraction is included in all sets C_i for all $i > \hat{j}$. This and the fact that $C_{k+1} \subseteq C_k$ for all $k \ge 0$, proves the claim.*

(vii) *This is derived from the two previous properties.*

(viii) *This is derived from the two previous properties.*

■

The finitely determination of the previous recursion is an important property and the calculation of the determination index is interesting from a practical point of view. Note that if the H-domain of attraction is not finitely determined, then the obtained sets C_k, $k = 0, 1, \ldots$, do not belong to the H-domain of attraction.In the following property a method for its computation is provided.

Property 4 *Let us suppose that $C_k = \mathcal{L}(R_k)$ is such that* rank R_k *is equal to n. Suppose also that LMI of theorem 3 is satisfied. Denote $U\Sigma V^\top$ the singular value decomposition of matrix R_k. Denote v_i, $i = 1,\ldots,n$, a set of n orthogonal eigenvectors of matrix $P = W^{-1}$ and λ_i, $i = 1,\ldots,n$, their corresponding eigenvalues. Under the previous assumptions, $C_{k+j} = C_\infty$, where j is the smallest integer that satisfies:*

$$j \geq \frac{\ln\left(\sum_{i=1}^{n} \lambda_i \|U\Sigma^{-1}V^\top v_i\|_1^2\right)}{\ln \frac{1}{\alpha}} \tag{3.9}$$

Proof :

Note that $x^\top P x$ is equal to $\sum_{i=1}^{n} \lambda_i |v_i^\top x|^2$. Make $\tau_i = U\Sigma^{-1}V^\top v_i$, $i = 1,\ldots,n$, then it results that: $|\tau_i^\top R_k x| = |v_i^\top V\Sigma^{-1}U^\top R_k x| = |v_i^\top x|$. Therefore, $|v_i^\top x| = |\tau_i^\top R_k x| \leq \|\tau_i\|_1 \|R_k x\|_\infty \leq \|\tau_i\|_1 = \|U\Sigma^{-1}V^\top v_i\|_1$. Thus, it is inferred that: $|v_i^\top x|^2 \leq \|U\Sigma^{-1}V^\top v_i\|_1^2$, $\forall x \in \mathcal{L}(C_k)$. The following bound is then obtained:

$$\max_{x \in C_k} x^\top P x \leq \sum_{i=1}^{n} \lambda_i \|U\Sigma^{-1}V^\top v_i\|_1^2$$

From the H-contractiveness of the ellipsoid $\mathcal{E}(P,1)$ is inferred that the recursion $x_{j+1} = G_H(x_j, S_j)$, $x_0 = x$ is such that $x_j^\top P x_j \leq \alpha^j x_0^\top P x_0 = \alpha^j x^\top P x \leq \alpha^j \sum_{i=1}^{n} \lambda_i \|U\Sigma^{-1}V^\top v_i\|_1^2$. Inequality 3.9 is equivalent to: $\alpha^j < \frac{1}{\sum_{i=1}^{n} \lambda_i \|U\Sigma^{-1}V^\top v_i\|_1^2}$. Therefore, if $x \in C_k$ then $x_j \in \mathcal{E}(P,1)$. This is equivalent to $C_{k+j} \subseteq \mathcal{E}(P,1)$. It can be concluded that the maximal H-invariant set is finitely determined. ■

Theorem 4 justifies the use of the following algorithm to obtain the maximal H-invariant set and an estimation of the domain of attraction of a saturated linear system.

Algorithm

1. Obtain matrix H solving the LMI problem proposed in section (3.6) for some $\alpha \in (0,1]$.

2. Set the initial region C_0 equal to $\mathcal{L}(H)$.

3. $C_{k+1} = \hat{Q}_H(C_k) \bigcap \mathcal{L}(H)$.

4. Obtain a polyhedral representation of C_{k+1} without redundant inequalities.

5. If $C_{k+1} = C_k$ then C_k is the H-domain of attraction. Stop.
   Else, set $k = k + 1$ and return to step (3).

This is a standard procedure for polytopic systems [16] and it has been applied for the computation of polyhedral invariant sets for saturated systems [41]. See also [45] for a related work. However, this algorithm is used in the context of the LDI proposed in[33]. In this way, it is guaranteed the quadratic stabilization of all the linear systems that take part in the *Linear Difference Inclusion*. Moreover, a lower bound of the size of the obtained H-invariant set is maximized.

The algorithm is finitely determined if there exists a $k^* < \infty$ such that $C_{k^*+1} = C_{k^*} = C_\infty$. In this case the obtained H-invariant set C_∞ is a polytope and it is the maximal polyhedral H-invariant set (see property 4).

Note that each iteration of the proposed algorithm requires to remove redundant inequalities of a polyhedron and subset testing. The complexity of these calculations grows with the number of constraints of the obtained polyhedra. Thus the number of constraints of the maximal H-invariant is related with the determination index. If the determination index is high, then the number of constraints of the maximal H-invariant may be large and the complexity of its calculation, representation and storage may be too demanding.

In subsection 3.7.2, it is presented another method to compute H-domain of attraction. This algorithm can also be used to obtain the largest H-invariant set because it is the H-domain of attraction, however, intermediate set are not H-invariant. Algorithm provided in subsection 3.7.3 provides an H-invariant set from the first stages of the algorithm and converges to the maximal H-invariant set. Consequently, if the number of constraints of the maximal H-invariant is too high, an H-invariant set (contained in the maximal one) can be obtained with a lower complexity.

3.7.2 Inner-Outer algorithm

In this subsection it is proposed an algorithm that provides an estimation of the domain of attraction of a saturated system. As claims property 3, if system $x^+ = (A + BK)x$ is asymptotically stable then it is possible to find an H-contractive ellipsoid for the saturated system. Based on the existence of such ellipsoid, it is proposed an algorithm that converges to the H-domain of attraction. Moreover, it is shown that if the H-domain of attraction is bounded then the algorithm is finitely determined.

The main advantage of this algorithm is that the sequence of sets obtained in the execution of the algorithm belongs to the H-domain of attraction of the system.

In order to present the algorithm, it will be used the auxiliary result,

Lemma 2 *Let us consider the ellipsoid $\mathcal{E}(P,1) \subset \mathbb{R}^n$. Suppose that v_i, $i = 1,\ldots,n$, are the orthonormal eigenvectors of matrix P and λ_i, $i = 1,\ldots,n$ their corresponding eigenvalues. Denote*

$$\Gamma(P) = \mathcal{L}(\begin{bmatrix} \sqrt{n\lambda_1}v_1^\top \\ \sqrt{n\lambda_2}v_2^\top \\ \vdots \\ \sqrt{n\lambda_n}v_n^\top \end{bmatrix}). \tag{3.10}$$

Then,

$$\mathcal{E}(P,\frac{1}{n}) \subseteq \Gamma(P) \subseteq \mathcal{E}(P,1).$$

PROOF :

Suppose that v_i, $i = 1,\ldots,n$, are the orthonormal eigenvectors of matrix P and λ_i, $i = 1,\ldots,n$ their corresponding eigenvalues. Then,

$$P = [v_1\ v_2\ \ldots\ v_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

From this equality it is inferred that

$$x^\top P x = \sum_{i=1}^{n} \lambda_i (v_i^\top x_i)^2.$$

The lemma is proved if it is shown that $\mathcal{E}(P,\frac{1}{n}) \subseteq \Gamma(P)$ and $\Gamma(P) \subseteq \mathcal{E}(P,1)$.

- $\mathcal{E}(P,\frac{1}{n}) \subseteq \Gamma(P)$*:*

Let us suppose that $x \in \mathcal{E}(P, \frac{1}{n})$. *That is,*

$$x^T P x = \sum_{i=1}^{n} \lambda_i (v_i^\top x_i)^2 \leq \frac{1}{n}$$

This can be rewritten as:

$$\sum_{i=1}^{n} (\sqrt{n\lambda_i} v_i^T x)^2 \leq 1.$$

This implies that

$$|(\sqrt{n\lambda_i} v_i^T x)| \leq 1, \;\; i = 1, \ldots, n.$$

Therefore it is inferred that $x \in \Gamma(P)$.

- $\Gamma(P) \subseteq \mathcal{E}(P, 1)$:
  Let us suppose that $x \in \Gamma(P)$. *That is,*

$$|(\sqrt{n\lambda_i} v_i^T x)| \leq 1, \;\; i = 1, \ldots, n.$$

Thus,

$$n\lambda_i (v_i^T x)^2 \leq 1, \;\; i = 1, \ldots, n$$

From this:

$$\sum_{i=1}^{n} n\lambda_i (v_i^T x)^2 \leq n,$$

and finally,

$$\sum_{i=1}^{n} \lambda_i (v_i^T x)^2 = x^T P x \leq 1,$$

that is, $x \in \mathcal{E}(P, 1)$.

■

The following theorem establishes the theoretical support of the algorithm proposed to obtain an estimation of the domain of attraction of the saturated system.

Theorem 5 *Let us suppose that $\mathcal{E}(P,1) \subset \mathbb{R}^n$ is an H-contractive ellipsoid for a given matrix H and a given scalar $\alpha \in (0,1)$. Set $\hat{C}_0 = \Gamma(P)$ and consider the following recursion:*

$$\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \cap \mathcal{L}(H),$$

Each obtained set $\hat{C}_k$ has the following properties:

(i) $\hat{C}_k$ is a convex polyhedron that can be obtained by means of definition (10).

(ii) $\hat{C}_k$ belongs to the H-domain of attraction of system $x^+ = Ax + B\sigma(Kx)$.

(iii) If $\hat{x}$ belongs to the H-domain of attraction, then $\hat{x} \in \hat{C}_j$, where j is the smallest integer that satisfies:

$$j \geq \frac{\ln(n\hat{x}^\top P\hat{x})}{\ln(\frac{1}{\alpha})}$$

(iv) The sequence $\hat{C}_0, \hat{C}_1, \hat{C}_2, \ldots,$ converges to the H-domain of attraction.

(v) If the H-domain of attraction is bounded then the H-domain of attraction is finitely determined. That is, there is a finite integer j^* such that $\hat{C}_{j^*}$ equals the H-domain of attraction.

Proof :

(i) The first point stems directly from the fact that $\hat{C}_0 = \Gamma(P)$ is a polyhedron and the definition of $\hat{Q}_H(\cdot)$.

(ii) From the properties of the one-step operator, it is inferred that every x_0 in $\hat{C}_k$ satisfies that the recursion

$$x_{i+1} = G_H(x_i, S_i)$$

is such that $x_i \in \mathcal{L}(H)$, $i = 0, \ldots, k$ and $x_k \in \Gamma(P)$ for every admissible sequence $S_0, S_1, \ldots, S_{k-1}$. Due to the fact that $\Gamma(P)$ belongs to the H-domain of attraction of the system, it is concluded that $\hat{C}_k$ belongs to the H-domain of attraction.

(iii) Suppose that $x_0 = \hat{x}$ belongs to the H-domain of attraction of the system. Then, from the definition of H-domain of attraction, the recursion $x_{i+1} = G_H(x_i, S_i)$ remains in $\mathcal{L}(H)$ for every admissible sequence $\{S_0, S_1, \ldots\}$. Moreover, the fact that $\mathcal{E}(P,1)$ is an H-contractive ellipsoid, guarantees

that $x_{i+1}^\top P x_{i+1} \leq \alpha x_i^\top P x_i$, *for all* $i \geq 0$ *and all admissible sequence* $\{S_0, S_1, \ldots\}$. *From this it is inferred that:*

$$x_i^\top P x_i \leq \alpha^i x_0^\top P x_0.$$

It can be easily seen that if

$$j \geq \frac{\ln(n\hat{x}^\top P\hat{x})}{\ln(\frac{1}{\alpha})}$$

then $x_j^\top P x_j \leq \frac{1}{n}$. *This implies, by means of lemma 2 that*

$$x_j \in \mathcal{E}(P, \frac{1}{n}) \subset \Gamma(P).$$

Note that $x_j \in \Gamma(P)$ *implies that* $\hat{x} = x_0 \in \hat{C}_j$.

(iv) It has been proved that if x_0 *belongs to the H-domain of attraction then there is j such that* $x_0 \in \hat{C}_j$. *This proves the claim.*

(v) If the H-domain of attraction is bounded then the maximum value of $x^\top P x$ *in the H-domain can be bounded by a finite constant. Suppose that* ρ *is such a constant. Then, using similar arguments that the ones used in the proof of claim (iii), it is obtained that the H-domain of attraction is equal to* $\hat{C}_{j^*}$, *where* j^* *is the smallest integer that satisfies:*

$$j^* \geq \frac{\ln(n\rho)}{\ln(\frac{1}{\alpha})}.$$

■

The previous theorem justifies the use of the following algorithm to obtain an estimation of the domain of attraction of a saturated linear system.

Algorithm

1. Obtain matrix H and the corresponding H-contractive ellipsoid $\mathcal{E}(P, 1)$ solving the LMI problem proposed in theorem 3.

2. Set the initial region $\hat{C}_0$ equal to $\Gamma(P)$ (see lemma 2).

3. $\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \bigcap \mathcal{L}(H)$.

4. Obtain a polyhedral representation of $\hat{C}_{k+1}$ without redundant inequalities.

5. If $\hat{C}_{k+1} = \hat{C}_k$ then $\hat{C}_k$ is the H-domain of attraction. Stop. Else, set $k = k + 1$ and return to step (3).

Note that one of the main advantages of this algorithm with respect to other existing ones (see, for example, [41]) is that it is not necessary the finite determinedness of the H-domain of attraction. That is, every obtained set $\hat{C}_k$ constitutes an estimation of the domain of attraction of the non-linear system. This allows us to obtain estimations of the domain of attraction with a given limit of computational burden and complexity of the polyhedral set representation.

3.7.3 Two-phase algorithm

In this section it is proposed an algorithm that provides an invariant set that is also an estimation of the domain of attraction of a saturated system. As claims property 3, if system $x^+ = (A + BK)x$ is asymptotically stable then it is possible to find an H-contractive ellipsoid for the saturated system. Based on the existence of such an ellipsoid, it is proposed an algorithm that converges to the maximal H-invariant set. The main advantage of this algorithm is that all the intermediate sets are invariant, hence the algorithm can be stopped at any iteration and an H-invariant set that it is also an estimation of the H-domain of attraction is found. However, the number of iterations to obtain the H-domain of attraction is larger than algorithm shown in 3.7.1.

This algorithm is divided into two parts: in the first one, an H-invariant set is computed by means of an enhanced procedure to reduce its determination index. Based on this, an iterative procedure is used in the second part.

In the first part of the algorithm, and in order to obtain the most contractive Lyapunov matrix $\tilde{P}$, the following minimization problem is solved,

$$\begin{array}{ll} \min\limits_{\alpha,P} & \alpha \\ s.t. & \alpha P - A_H^\top(S)\, P\, A_H(S) > 0 \quad \forall S \in \mathcal{V} \\ & P > 0 \end{array} \tag{3.11}$$

where $A_H(S) = A + B(E_s^c K + E_s H)$. Note that the family $A_H(S)$ admits a conve Lyapunov function. This is because it has been conveniently chosen. Based on the obtained matrix, some well known results are applied:

Lemma 3 *Let $\tilde{P}$ and $\hat{\alpha}$ be the optimal solution of (3.11). Denote $\tau = \max_{1 \leq j \leq m} H_j \tilde{P}^{-1} H_j^\top$, then the ellipsoid $\mathcal{E}(\hat{P}, 1)$, where $\hat{P} = \tau\tilde{P}$ is an H-contractive set for system (3.2) with contractiveness degree $\hat{\alpha}$.*

Lemma 4 *Let us consider the ellipsoid $\mathcal{E}(\hat{P}, 1) \subset \mathbb{R}^n$. Denote v_i, $i = 1, \ldots, n$ a set of n orthogonal eigenvectors of matrix $\hat{P}$ and λ_i, $i = 1, \ldots, n$, their corresponding eigenvalues. Denote*

$$\Gamma(\hat{P}) = \mathcal{L}(\frac{1}{\sqrt{n}} \begin{bmatrix} \sqrt{\lambda_1} v_1^\top \\ \sqrt{\lambda_2} v_2^\top \\ \vdots \\ \sqrt{\lambda_n} v_n^\top \end{bmatrix})$$

Then $\mathcal{E}(\hat{P}, \frac{1}{n}) \subset \Gamma(\hat{P}) \subset \mathcal{E}(\hat{P}, 1)$.

The proofs of the the previous two lemmas can be obtained by means of standard algebraic manipulations.

Theorem 6 *Let us suppose system (3.2) and a given matrix H. Let $\mathcal{E}(\hat{P}, 1)$ be an H-contractive ellipsoid with a contraction factor $\hat{\alpha}$. Set $\hat{C}_0 = \Gamma(\hat{P})$ and consider the following recursion: $\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \cap \hat{C}_0$. Then $\hat{C}_k$ is an H-invariant set for all $k \geq \hat{k} > \frac{ln\, n}{ln\, \frac{1}{\hat{\alpha}}}$.*

Proof :

Let us suppose that $x \in \hat{C}_{\hat{k}}$. From the H-contractiveness of $\mathcal{E}(\hat{P}, 1)$ it is inferred that the recursion $x_{i+1} = G_H(x_i, S_i)$, $x_0 = x$ is such that $x_{\hat{k}}^\top P x_{\hat{k}} \leq \hat{\alpha}^{\hat{k}} x_0^\top \hat{P} x_0 = \hat{\alpha}^{\hat{k}} x^\top P x$, for every possible sequence $\{S_0, S_1, \ldots, S_{\hat{k}-1}\}$. Taking into account that $x \in \Gamma(\hat{P}) \subset \mathcal{E}(\hat{P}, 1)$, it is concluded that $x_{\hat{k}}^\top P x_{\hat{k}} \leq \hat{\alpha}^{\hat{k}} x^\top \hat{P} x \leq \hat{\alpha}^{\hat{k}}$. From the inequality $\hat{k} > \frac{ln\, n}{ln\, \frac{1}{\hat{\alpha}}}$ it is inferred that $\hat{\alpha}^{\hat{k}} \leq \frac{1}{n}$. Therefore, $x \in C_{\hat{k}}$ implies $x_{\hat{k}} \in \mathcal{E}(\hat{P}, \frac{1}{n}) \subset \Gamma(\hat{P})$. The ellipsoid $\mathcal{E}(\hat{P}, \frac{1}{n})$ is an H-invariant set. Therefore, $x \in C_{\hat{k}}$ implies $x \in C_{\hat{k}+1}$. This is equivalent to say that $C_{\hat{k}} \subseteq \hat{Q}_H(C_{\hat{k}}) \cap C_0$, which proves the claim. ■

The H-invariant set proposed in theorem 6 can be computed by means of the following algorithm:

Algorithm I

1. Obtain matrix H and the corresponding H-contractive ellipsoid $\mathcal{E}(P,1)$ solving the LMI problem used in theorem 3.

2. Obtain the H-invariant ellipsoid $\mathcal{E}(\tilde{P},1)$ and the contraction factor $\hat{\alpha}$ solving the LMI problem proposed in 3.11.

3. Scale the previous H-invariant ellipsoid to obtain $\mathcal{E}(\hat{P},1)$ by the results of lemma 3.

4. Set the initial region $\hat{C}_0 = \Gamma(\hat{P})$

5. For $k = 1$ to $\hat{k} = \dfrac{ln\ n}{ln\ \frac{1}{\hat{\alpha}}}$,

6. $\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \bigcap \hat{C}_0$.

7. Obtain a polyhedral representation of $\hat{C}_{k+1}$ without redundant inequalities.

8. End for.

Note that the previous for loop is defined to an upper limit of iterations. This loop can be stopped when an H-invariant set is reached. The previous algorithm provides a polihedrical invariant set for system (3.2).

In the following algorithm it is shown how to take advantage of the obtained H-invariant set to obtain a sequence of H-invariant sets that converges to the maximal one:

Algorithm II

1. Obtain matrix H and a polyhedral invariant set $\hat{C}_{\hat{k}}$ by meaning of Algorithm I.

2. Find the maximum value β such that $\beta\hat{C}_{\hat{k}} \in \mathcal{L}(H)$.

3. $C_0 = \beta\hat{C}_{\hat{k}}$.

4. $C_{k+1} = \hat{Q}_H(C_k) \bigcap \mathcal{L}(H)$.

5. Obtain a polyhedral representation of C_{k+1} without redundant inequalities.

6. If $C_{k+1} = C_k$ then C_k is the maximal H-invariant set. Stop. Else, set $k = k+1$ and return to step (4).

Every obtained set C_k is an H-invariant set by construction. Moreover, $C_k \subseteq C_{k+1}$. If the maximal H-invariant set is bounded then the algorithm obtains the maximal H-invariant set in a finite number of steps. Note that the linear nature of the systems that compose the *Linear Difference Inclusion* and the H-invariance of $\hat{C}_{\hat{k}}$ guarantees the H-invariance of the initial set $C_0 = \beta \hat{C}_{\hat{k}}$.

Note also that one of the main advantages of this algorithm with respect to other existing ones is that the algorithm provides an H-invariant set also if the maximal H-invariant set is not finitely determined. This allows us to obtain invariant sets of the nonlinear system with a given limit on the computational burden required. Moreover, the complexity of the polyhedral set representation is also reduced.

3.8 Examples

3.8.1 A family of single input systems

Let us consider the discrete time caracterization of the following family of systems:

$$G(s) = \frac{1}{s^n}, \quad n = 2, \ldots, 5.$$

Let the sample time be equal to one. This family of systems has been used frequently in the literature.

In order to have a representation of algorithms shown earlier, firstly dimension two of the family of system will be analized and later a general analysis of all systems will be performed.

For $n = 2$, the discrete-time state space representation of the system (sample time equal to one) is given by $x^+ = Ax + Bu$ where:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

The closed-loop system is given by $x^+ = Ax + B\sigma(Kx)$ where gain matrix

$$K = [-0.6167 \ -1.2703]$$

corresponds to the discrete LQR controller with $Q = \mathrm{I}$ and $R = 0.1$.

Note that the closed-loop system corresponds to a double integrator controlled by a saturated linear controller.

This system will be used to be applied algorithms shown in section 3.7.

The first step for all three algorithms is to obtain matrices H and P by means of solving the maximization problem proposed in 3.8 with a contraction factor α. Matrix P defines an inner ellipsoidal set, $\mathcal{E}(P,1)$, that is a contractive set. An inner polyhedral set is obtained and finally it is used algorithm shown in 5.

Figure 3.2 shows $\mathcal{L}(H)$ set and the ellipsoidal set defined by matrix P. This matrixes P and H have been obtained by solving the maximization problem proposed in 3.8. That is,

$$\max_{W,Y} \quad \mathrm{tr}\, W$$
$$\text{subject to LMIs:}$$
$$\begin{bmatrix} \alpha W & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} W + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} Y\right)^\top \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} W + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} Y & W \end{bmatrix} > 0,$$

$$\begin{bmatrix} \alpha W & \left(\begin{bmatrix} 0.6916 & 0.3649 \\ -0.6167 & -0.2703 \end{bmatrix} W\right)^\top \\ \begin{bmatrix} 0.6916 & 0.3649 \\ -0.6167 & -0.2703 \end{bmatrix} W & W \end{bmatrix} > 0,$$
$$\begin{bmatrix} 1 & Y \\ Y^\top & W \end{bmatrix} > 0,$$

where $P = W^{-1}$ and $H = YP$.

In this figure $\mathcal{L}(H)$ is represented by the dash lines and $\mathcal{E}(P,1)$ is the solid line ellipsoid. Values of P and H result

$$P = \begin{bmatrix} 0.0397 & 0.0531 \\ 0.0531 & 0.1397 \end{bmatrix}$$

$$H = \begin{bmatrix} -0.1142 & -0.3674 \end{bmatrix}$$

Note that $\mathcal{L}(H)$ is unbounded due to $rank(H) = 1 < n$.

H-domain of attraction and the largest H-invariant sets obtained by means of these algorithms will be included in $\mathcal{L}(H)$. This limitation is less restrictive than traditional limitation to be included in $\mathcal{L}(K)$.

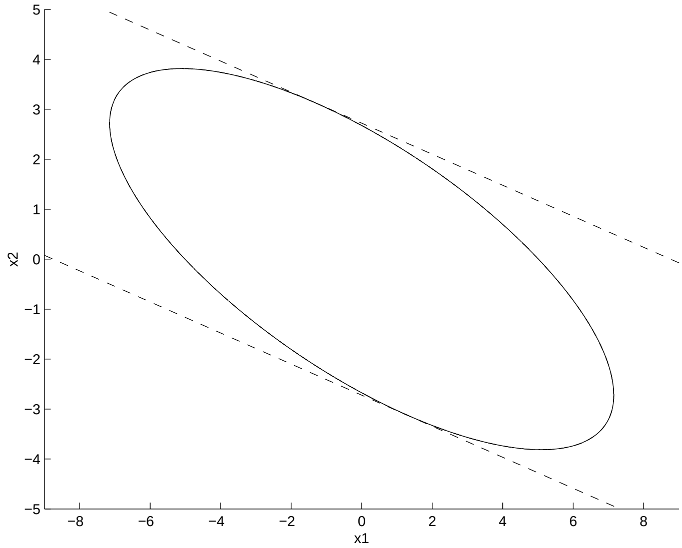

Figure 3.2: $\mathcal{L}(H)$ and the ellipsoidal set

Outer-inner algorithm.

This algorithm uses $\mathcal{L}(H)$ as the initial set, and then it applies recursively the operator $\hat{Q}(\cdot)$.

Figure 3.3 shows the H-domain of attraction obtained by means of this algorithm. Initial set $C_0 = \mathcal{L}(H)$ is represented by the dash lines, C_1 is represented by the outer red polyhedra, and C_2 is the inner red polyhedra filled in yellow. Note that this algorithm for this specific example converges in two steps, and note that as $\mathcal{E}(P,1)$ (shown as a dashed ellipsoid) is an estimation of the H-domain of attraction, it is included in C_2.

The disadvantage of this procedure is that the intermediate sets are not H-invariant sets nor estimations of the H-domain of attraction. Therefore if computation time is finished when obtained C_1, no result can be obtained.

Inner-outer algorithm.

This algorithm uses $\hat{C}_0 = \Gamma(P)$ set defined in 3.10 as initial set.

Figures 3.4 shows the initial set $\Gamma(P)$ in solid red lines obtained by application of lemma 2.

Note that $\hat{C}_0 \subset \mathcal{E}(P,1)$, and as $\mathcal{E}(P,1)$ converges to the origin, $\hat{C}_0$ is also an estimation of the H-domain of attraction.

Note also that $\mathcal{E}(P,1)$ is an H-invariant set, but $\hat{C}_0$ as a subset of $\mathcal{E}(P,1)$ is not an H-invariant set in general.

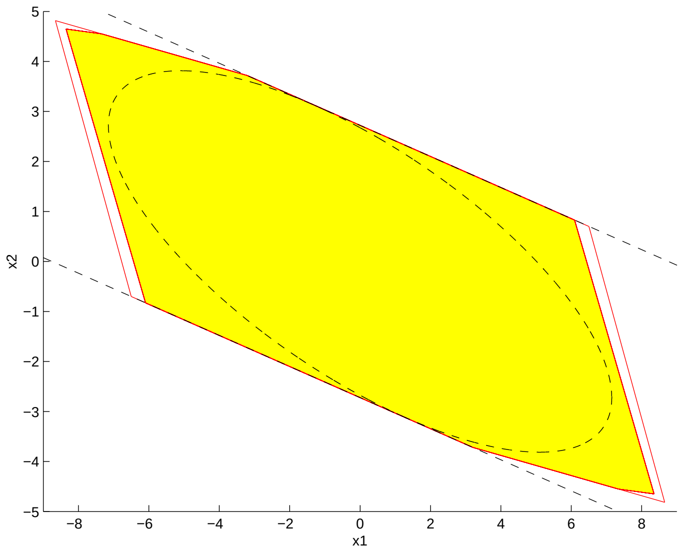

Figure 3.3: Maximal H-invariant set by Algorithm I

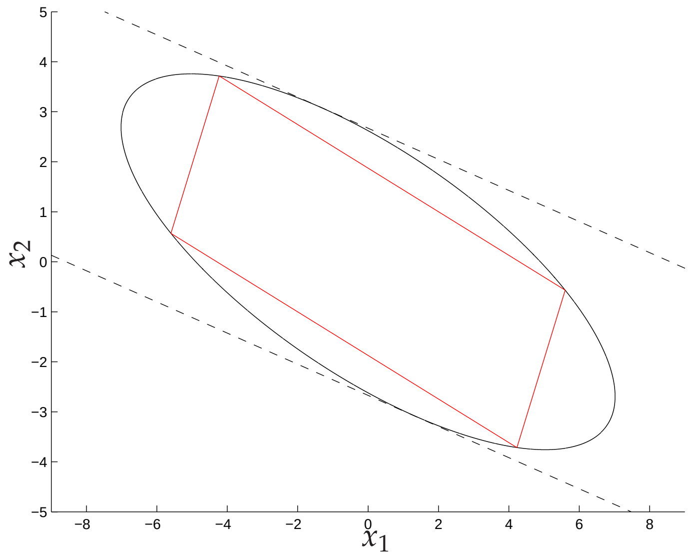

Figure 3.4: H-Contractive ellipsoid.

Figure 3.5 shows the sequence obtained by means of the algorithm. $\hat{C}_0 = \Gamma(P)$ is represented as the dashed red set, and sequence $\hat{C}_1, \hat{C}_2, \ldots$ are represented in solid black lines.

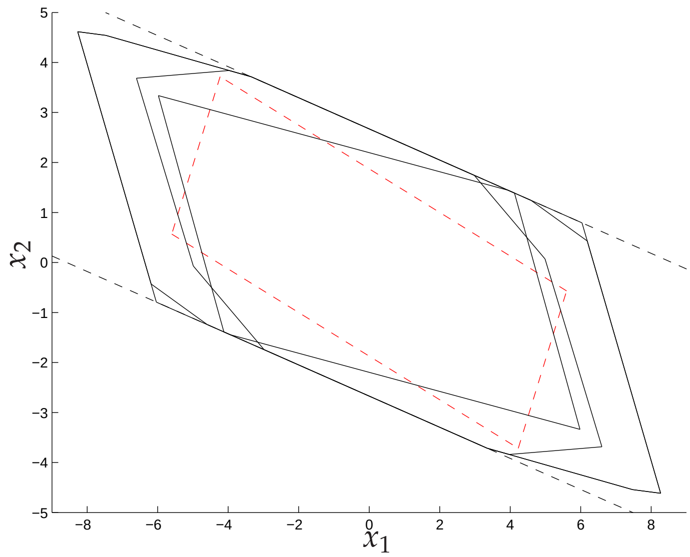

Figure 3.5: *H*-domain of attraction.

Note that $\hat{C}_k$ sequence is obtained by recursion of operator $\hat{Q}(\cdot)$ to $\hat{C}_0$, and taking into account that $\hat{C}_0$ is a conservative estimation of the *H*-domain of attraction, sequence $\hat{C}_k$ is also an estimation of the *H*-domain of attraction.

The outer black-lined set is $\hat{C}_7$ and represents the *H*-domain of attraction of the system, and therefore it is the maximal *H*-invariant set. This set results the same that obtained by means of the outer-inner algorithm explained before, but more iterations are needed.

2 phase algorithm. The first step in this algorithm is to determine H and P as other algorithm. Therefore, $\mathcal{L}(H)$, and $\mathcal{E}(P,1)$ can be shown in figure 3.2.

For the first phase of the algorithm, H is maintained and a most contractive Lyapunov matrix $\tilde{P}$ is calculated.

Figure 3.6 shows the process applied in this first phase. Original $\mathcal{E}(P,1)$ is represented as the dashed ellipsoid, and $\mathcal{L}(H)$ is the non-bounded dashed set.

New $\mathcal{E}(\tilde{P},1)$ is shown as the solid black ellipsoid. Note that $\mathcal{E}(\tilde{P},1)$ is smaller than $\mathcal{E}(P,1)$ because it is a more contractive set.

First phase of the algorithm uses $\hat{C}_0 = \Gamma(\tilde{P})$. This set is represented in this figure as the blue solid set inside $\mathcal{E}(\tilde{P},1)$. The first recursion of the

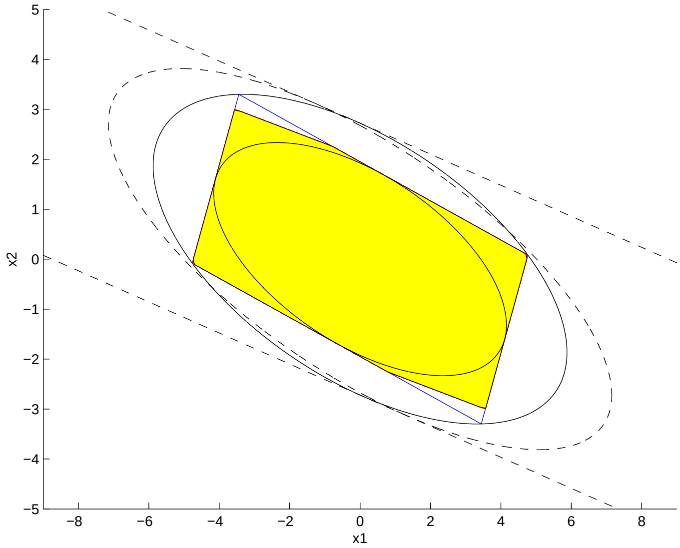

Figure 3.6: Inner H-invariant set by Algorithm II

algorithm is $\hat{C}_1$ and it is represented as the set filled in yellow.

Note that this first phase of the algorithm for this example converges in just 1 iteration, however, this number of iteration is limited by $\hat{\alpha}$.

$\hat{C}_1$ set will be used to start the second phase of the algorithm.

Figure 3.7 shows the process used in the second phase of the algorithm.

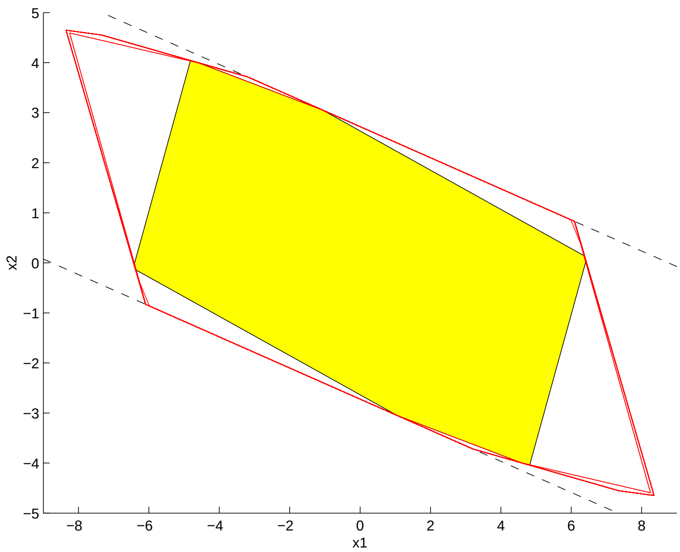

Figure 3.7: Maximal H-invariant set by Algorithm III

The initial set of the second phase of the algorithm C_0 is the last set of the previous phase $\hat{C}_1$ scaled to the maximal set included in $\mathcal{L}(H)$. In this figure it is shown as the yellow-filled set.

Recursion of the algorithm provides C_1 that it is the inner red-lined set, and C_2 that it is the outer red-lined set. C_2 is the H-domain of attraction of the system obtained by means of this algorithm.

The most important difference between this algorithm and the previous ones is that C_0, C_1 and C_2 are H-invariant sets and conservative estimations of the H-domain of attraction. Therefore if computational time is finished, or complexity of the caracterization of sets is unmanageable, an approximation can be taken.

Note that the H-domain of attraction obtained by means of this three different algorithms obtain the same set.

The domain of attraction of this system has been estimated by means of a saturation-dependent Lyapunov function in [12]. In that paper, the authors propose how to obtain matrix H in such a way that a saturation-dependent Lyapunov function is strictly decreasing for every system $x^+ = G_H(x, S)$, $S \in \mathcal{V}$. Therefore, the authors are obtaining an estimation of the H-domain of attraction of the system.

Note that the matrices of this example corresponds to the discretization of a double integrator (sample time equal to one). The matrix K corresponds to the discrete LQR controller with $Q = 1$, $R = 0.1$. Some results are shown for the class of systems: $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^q}$.

Table 3.1 shows a comparison between the volume of the H-contractive ellipsoid regions and the polyhedral H-domains of attraction for the family $G(s) = \frac{1}{s^n}$, $n = 2, \ldots, 5$. The corresponding discrete-time closed loop systems $x^+ = A_n x + B_n \sigma(K_n x)$, $n = 2, \ldots, 5$ are obtained with sample time equal to one and $u = \sigma(K_n x)$, where K corresponds to the discrete-time LQR controller ($Q = \mathrm{I}$, $R = 0.1$). As it can be observed, the volume of the H-domain of attraction is considerably greater than the one corresponding to the H-contractive ellipsoid. The last column of the table shows the number of non-redundant linear constraints required to represent each of the H-domain of attraction.

Note that the number of constraints of the H-domain of attraction obtained increases exponentially with the dimension of the system, therefore solutions obtained by means of the inner-outer and the outter-inner algorithms may not be computationally obtained. Table 3.2 shows the number of constraints of the H-domain of attraction for each dimension, and the number of constraints of the last iteration of the first phase of the 2-phase algorithm. Note that this set is an H-invariant set and an estimation of the H-domain of attraction of the system.

	Volume ellipsoid	Volume H-Domain	Volume increment	Number of constraints
n= 2	60.15	77.10	28.18%	8
n= 3	150.87	238.98	58.39%	26
n=4	791.28	1680.9	112.42%	66
n=5	6868.21	19884.17	189.51%	188

Table 3.1: Comparison between the volume of the obtained H-contractive ellipsoid and the polyhedral H-Domain of attraction (single input systems).

q	2	3	4	5	6
Maximal	8	26	66	188	453
Final	6	18	28	42	88

Table 3.2: Comparison between the inner H-invariant set and the maximal one.

Let us now consider a family of two-input systems ($m = 2$) :

$$Y(s) = \frac{U_1(s) + sU_2(s)}{s^n}, \quad n = 2, \ldots, 5$$

The inputs u_1 and u_2 are supposed to be saturated. The corresponding closed loop systems $x^+ = A_n x + B_n \sigma(K_n x)$, $n = 2, \ldots, 5$ are obtained with sample time equal to one and $[u_1, u_2]^\top = \sigma(K_n x)$, where K_n corresponds to the discrete-time LQR controller ($Q = \mathrm{I}$, $R = 0.1\mathrm{I}$). Table 3.3 shows a comparison between the volume of the H-contractive ellipsoid regions and the polyhedral H-Domains of attraction for this family of two-input systems.

3.8.2 A multiple input system

Consider system $x^+ = Ax + B\sigma(Kx)$, where:

	Volume ellipsoid	Volume H-Domain	Volume increment	Number of constraints
n= 2	15429	19367	25.52%	10
n= 3	222.63	384.56	72.74%	24
n=4	372.69	1055.6	183.24%	82
n=5	1208.6	6094.7	404.28%	288

Table 3.3: Comparison between the volume of the obtained H-contractive ellipsoid and the polyhedral H-Domain of attraction (two-input systems).

$$A = \begin{bmatrix} 1.2 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.475 & 0 \\ 0.55 & 0.075 \end{bmatrix}$$

This example has been used in [23]. In that paper, the authors use an initial contractive polyhedral set $Y \subseteq \mathcal{L}(K)$ and show how to enlarge it in such a way that the contractiveness of the enlarged polyhedral set is not lost. That is, they obtain the maximum value of the scalar α such that αY is a contractive polyhedron for the saturated system. In that paper the authors obtained the box $\|x\|_\infty < 10$ as an estimation of the domain of attraction of the saturated system.

Figure 3.8 shows the application of the outer-inner algorithm to the proposed system.

In this case, the obtained value for matrix H is:

$$H = \begin{bmatrix} -0.1099 & -3.68 \cdot 10^{-9} \\ 7.72 \cdot 10^{-9} & -2.69 \cdot 10^{-9} \end{bmatrix}$$

Set $\mathcal{L}(H)$ is represented with dashed lines. Note that the dimension of the input is the same that the dimension of the system and $\mathcal{L}(H)$ set is bounded. The intermediate sets C_k are shown on solid lines. In eight steps the maximal invariant set is obtained (shaded region).

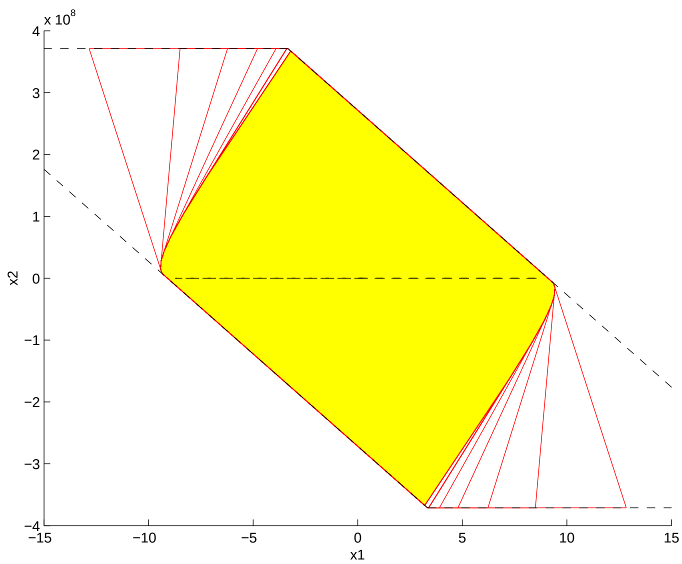

Figure 3.8: Maximal H-invariant by means of the outer-inner algorithm

This algorithm performs an estimation of the domain of attraction that does not depend on the initial elected set. Therefore it can obtain a good estimation for degenerated systems like this.

Figure 3.9 shows the application of the inner-outer algorithm to the proposed system. Central line represents the estimation of the domain of attraction of the saturated system obtained in [23]. Initial elected set perform a good estimation for x_1, but as the process maintain the shape of the initial elected set, it has a lack of estimation on x_2. Dashed lines represent $\mathcal{L}(H)$ set and red sets represents each iteration of the algorithm.

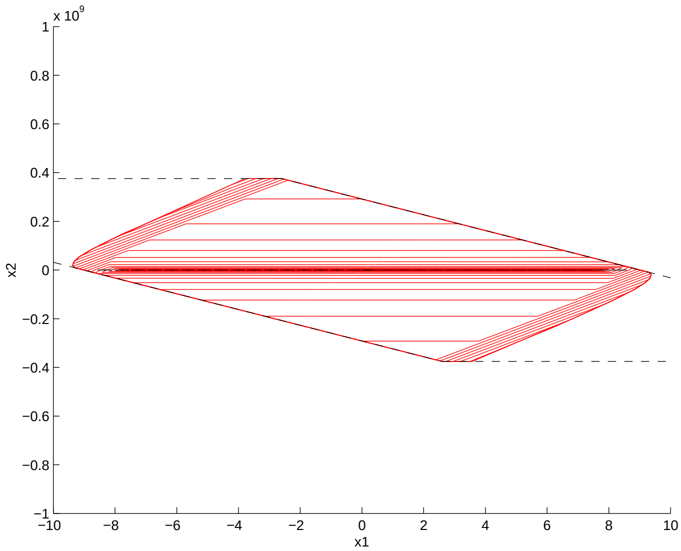

Figure 3.9: H-Domain of attraction obtained by means of the inner-outer algorithm.

Figure 3.10 shows the sequence of intermediate H-invariant sets to get the maximal H-invariant set (two-phase algorithm). The initial H-invariant set has six nonredundant constraints. The number of required steps to obtain the maximal H-invariant set is forty-two.

The domain of attraction of this saturated system can be obtained by analytical means. It results that the domain of attraction is an unbounded set: $\{ (x_1, x_2) \ : \ -10 < x_1 < 10 \}$.

3.9 Conclusions

It has been seen that a saturated system is a non-linear system and therefore, the domain of attraction of the system might be non-convex. Convexity is a very important property and a convex estimation of the domain of attraction is needed in many cases.

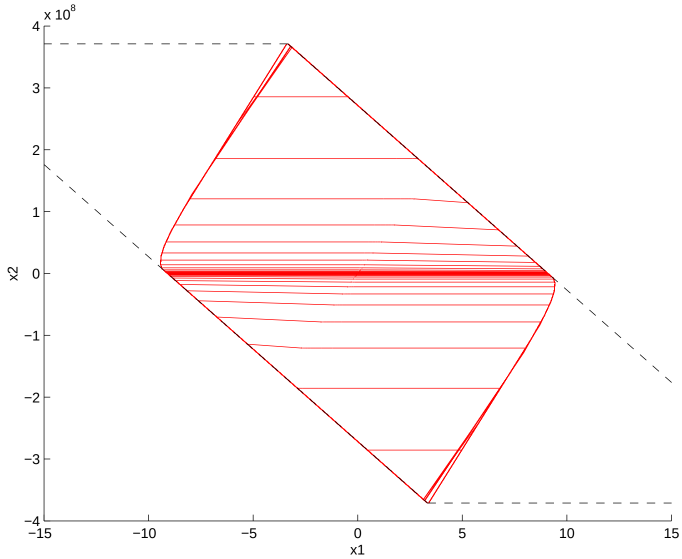

Figure 3.10: Maximal H-invariant set by means of the 2-phase algorithm

In this chapter, an alternative approach to the estimation of the domain of attraction of a saturated linear system is presented. It is shown how to choose a linear difference inclusion (LDI) in such a way that the conservativeness in the estimation is reduced. It is provided an algorithm that estimates the domain of attraction of the non-linear system. This estimation is called H-domain of attraction and it is a polyhedrical convex set.

Under mild assumptions, the proposed algorithm obtains the greatest domain of attraction for the linear difference inclusion. It is also shown how to obtain an invariant set for the saturated system in a finite number of steps. In this way, the complexity of the obtained invariant set is reduced.

In this chapter an estimation of the domain of attraction based in an LDI for discrete time systems has been obtained. In [24, 33] an analysis for continuous time systems can be seen. In the following chapters a new method to obtain an estimation of the domain of attraction is presented. As long as that method includes this one, estimations of the domain of attraction based in this method are more conservative than others shown in following chapters and therefore these are prefered.

4

Discrete SNS

4.1 Introduction

As it has already been seen in chapter 3, the domain of attraction of a saturated system can be estimated by means of a linear difference inclusion (LDI) of the system. The politopic representation provided by the LDI simplifies the analysis of the non linear system. In [24], an LDI is used to obtain invariant ellipsoids for saturated systems.

The linear difference inclusion presented in [24] is generalized in [29] and [33]. Based on this generalization, the authors propose how to choose an auxiliary matrix, that characterizes the LDI, in order to obtain the greatest ellipsoid that is invariant under the corresponding LDI. This LDI representation has also been used in [12] to obtain a saturation-dependent Lyapunov function that leads to a less conservative estimation of the domain of attraction.

This chapter shows a new notion of invariance, denoted SNS-invariance that provides less restrictive geometrical properties compared to the H-invariance concept.

Based on its geometrical properties, a simple algorithm to estimate the domain of attraction of a saturated linear system is proposed. It is shown that in case of single input systems, any contractive set is a SNS-invariant set. Moreover, any domain of attraction obtained by means of an LDI representation of the system is included in the estimation provided by the proposed algorithm.

4.2 Problem Statement

In this chapter, only discrete time linear systems with saturated feedback will be addressed, that is, the same family of systems analized in chapter 3.

Recall 2.18 where the following system defines the target system and can be written like

$$x^+ = Ax + B\sigma(Kx) \tag{4.1}$$

where $x \in \mathbb{R}^n$ denotes the state vector.

The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the vector-valued standard saturation function defined in definition 4 of chapter 2.

Using the definition of $\mathcal{M}$, system 4.1 can be rewritten as

$$x^+ = Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) = Ax + \sum_{i\in\mathcal{M}} B_i\sigma(K_i x).$$

A more detailed explication can be found in equation 2.19 of chapter 2.

In this chapter only discrete time system 4.1 will be addressed. For an analysis of continuous time system see chapter 5.

4.3 Defining the notion of SNS-invariance

In this section, the concept of SNS-invariance is presented. For this purpose, some auxiliary notation is required.

Specifically, $\mathcal{M}$, $\mathcal{V}$, S and S^c concepts shown in definition 6 of chapter 2 will be used.

SNS concepts are related with the family of system shown in 2.20. In order to examinate this family the following definition is used.

Definition 20 *Given a set $S \in \mathcal{V}$, $F(x, S)$ is defined as follows,*

$$F(x, S) = Ax + \sum_{i \in S^c} B_i K_i x + \sum_{i \in S} B_i \sigma(K_i x).$$

Note that with these definitions, $x^+ = Ax + B\sigma(Kx) = F(x, \mathcal{M})$. Also, $x^+ = F(x, \emptyset) = (A + BK)x$ represents the evolution of the system without saturation.

The notion of SNS-invariance is introduced in the following definition,

Definition 21 *A set Ω is said to be SNS-invariant for system $x^+ = Ax + B\sigma(Kx)$ if $x \in \Omega$ implies $F(x, S) \in \Omega$ for every $S \in \mathcal{V}$.*

It is clear from the previous definition that if Ω is a SNS-invariant set then Ω is an invariant set for the saturated system $x^+ = Ax + B\sigma(Kx)$. That is, if Ω is SNS-invariant then $Ax + B\sigma(Kx) = F(x, \mathcal{M}) \in \Omega$, for all $x \in \Omega$.

For single input systems ($m = 1$), the SNS-invariance of a given set Ω is equivalent to the invariance of Ω for the *Saturated* and *Non Saturated* systems: $x^+ = Ax + B\sigma(Kx)$ and $x^+ = Ax + BKx$. Note that SNS stands for *Saturated* and *Non Saturated*.

This definition is related with H-invariant definition shown in equation 15 of chapter 3. One of the conservativeness of H-invariance is that it is limited to $\mathcal{L}(H)$, and SNS-invariance has not such a priori limitation.

4.3.1 Geometric condition of SNS-invariance

In chapter 3 it has been seen that the one-step set plays an important role in the context of set invariance theory, see also [17]. In that chapter, it has been extended that concept to the $Q_H(\cdot)$ operator shown in definition 10 in chapter 3.

Recall $Q(\Omega)$ in definition 9 for the saturated system 4.1,

Definition 22 *Given system $x^+ = Ax + B\sigma(Kx)$ and set Ω, the one-step set $Q(\Omega)$ is defined as*

$$Q(\Omega) = \{\ x \ : \ Ax + B\sigma(Kx) \in \Omega\ \}$$

It is well known that Ω is an invariant set for system $x^+ = Ax + B\sigma(Kx)$ if and only if $\Omega \subseteq Q(\Omega)$ [17]. Given a convex set Ω, the one-step set $Q(\Omega)$ is not necessarily convex due to the nonlinear nature of the saturation function. The non convex nature of $Q(\Omega)$ makes it difficult to use of operator $Q(\cdot)$ in the computation of invariant sets for saturated systems.

In order to provide a geometric condition of SNS-invariance, the following definitions are introduced.

Definition 23 *Given a set Ω and $S \in \mathcal{V}$:*

$$Q_S(\Omega) = \{\ x \ : \ F(x, S) \in \Omega\ \},$$

and given a set Ω:

$$Q_{SNS}(\Omega) = \bigcap_{S \in \mathcal{V}} Q_S(\Omega)$$

From the definition of $Q_{SNS}(\cdot)$, the following property is directly inferred:

Property 5 *A set Ω is SNS-invariant if and only if $\Omega \subseteq Q_{SNS}(\Omega)$.*

The most remarkable property of $Q_{SNS}(\cdot)$ is that given a polyhedral set Ω, $Q_{SNS}(\Omega)$ is a convex polyhedron. This property will be proved in the following section.

4.4 Polyhedral representation of $Q_{SNS}(\Omega)$

In this section, a polyhedral representation of $Q_{SNS}(\Omega)$ is provided. Given a polyhedral set Ω and a set S in $\mathcal{V}$, $Q_S(\Omega)$ is not necessarily a polyhedral set. Surprisingly,

$$Q_{SNS}(\Omega) = \bigcap_{S \in \mathcal{V}} Q_S(\Omega)$$

is a polyhedral set that can be obtained in a direct way from polyhedron Ω as it is claimed in the following theorem.

Theorem 7 *Let us suppose that Ω is a convex polyhedron in $\mathbb{R}^n$ given by $\Omega = \{ x \ : \ Rx \preceq g \}$. Then:*

$$Q_{SNS}(\Omega) = \bigcap_{S \in \mathcal{V}} \{ x \ : \ R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq g \}$$

where S^c denotes the complementary set of S in $\mathcal{M}$ and $|RB_i|$ is the vector with entries equal to the absolute value of the entries of vector RB_i.

Proof :

Denote

$$P_{SNS} = \bigcap_{S \in \mathcal{V}} \{ x \ : \ R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq g \}$$

In what follows, it will be proven that $P_{SNS} \subseteq Q_{SNS}(\Omega)$. Let us suppose that there is $\hat{x} \in P_{SNS}$ such that $\hat{x} \notin Q_{SNS}(\Omega)$. That is, there is $\hat{S} \in \mathcal{V}$ such that $\hat{x} \notin Q_{\hat{S}}(\Omega)$. In this case, there must be j such that denoting R_j and g_j the j-esime row of R and j-esime component of g respectively:

$$R_j F(\hat{x}, \hat{S}) = R_j \left((A + \sum_{i \in \hat{S}^c} B_i K_i)\hat{x} + \sum_{i \in \hat{S}} B_i \sigma(K_i \hat{x}) \right) > g_j \tag{4.2}$$

It will be shown that the above inequality contradicts the fact that $\hat{x} \in P_{SNS}$. In effect, taking into account that $a\sigma(y) \leq \max \{ay, -|a|\}$ (see appendix A, lemma 10 for a proof):

$$R_j F(\hat{x}, \hat{S}) = R_j (A + \sum_{i \in \hat{S}^c} B_i K_i)\hat{x} + \sum_{i \in \hat{S}} R_j B_i \sigma(K_i \hat{x})$$

$$\leq R_j (A + \sum_{i \in \hat{S}^c} B_i K_i)\hat{x} + \sum_{i \in \hat{S}} \max \{R_j B_i K_i \hat{x}, -|R_j B_i|\}$$

Denote:

$$T = \{\ i \in \hat{S}\ :\ R_j B_i K_i \hat{x} < -|R_j B_i|\ \}$$

From this definition and the previous inequality:

$$\begin{aligned} R_j F(\hat{x}, \hat{S}) &\leq R_j(A + \sum_{i \in \hat{S}^c} B_i K_i)\hat{x} + \sum_{i \in \hat{S}, i \notin T} R_j B_i K_i \hat{x} - \sum_{i \in T} |R_j B_i| \\ &= R_j(A + \sum_{i \in \mathcal{M}, i \notin \hat{S}} B_i K_i + \sum_{i \in \hat{S}, i \notin T} B_i K_i)\hat{x} - \sum_{i \in T} |R_j B_i| \\ &= R_j(A + \sum_{i \in \mathcal{M}} B_i K_i - \sum_{i \in \hat{S}} B_i K_i + \sum_{i \in \hat{S}} B_i K_i - \sum_{i \in T} B_i K_i)\hat{x} - \sum_{i \in T} |R_j B_i| \\ &= R_j(A + \sum_{i \in \mathcal{M}} B_i K_i - \sum_{i \in T} B_i K_i)\hat{x} - \sum_{i \in T} |R_j B_i| \\ &= R_j(A + \sum_{i \in T^c} B_i K_i)\hat{x} - \sum_{i \in T} |R_j B_i| \end{aligned}$$

It is clear that $T \in \mathcal{V}$*. Thus* $\hat{x} \in P_{SNS}$ *implies:*

$$R(A + \sum_{i \in T^c} B_i K_i)\hat{x} - \sum_{i \in T} |RB_i| \preceq g$$

In particular,

$$R_j(A + \sum_{i \in T^c} B_i K_i)\hat{x} - \sum_{i \in T} |R_j B_i| \leq g_j$$

This inequality contradicts equation (4.2). Therefore, it is inferred that $R_j F(\hat{x}, \hat{S}) \leq g_j$ *and consequently:* $P_{SNS} \subseteq Q_{SNS}(\Omega)$.

To conclude the prove, it will be shown that $Q_{SNS}(\Omega) \subseteq P_{SNS}$*. In effect, due to the fact that* $-|RB_i| \preceq RB_i \sigma(K_i x)$*, it results that, for every* $S \in \mathcal{V}$*:*

$$R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq R(A + \sum_{i \in S^c} B_i K_i)x + \sum_{i \in S} RB_i \sigma(K_i x) = RF(x, S) \tag{4.3}$$

Suppose now that $x \in Q_{SNS}(\Omega)$*, that is,* $RF(x, S) \preceq g$*,* $\forall S \in \mathcal{V}$*. Then, taking into account equation (4.3), it results that*

$$R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq RF(x, S) \preceq g, \ \ \forall S \in \mathcal{V}$$

It is concluded that $x \in Q_{SNS}(\Omega)$ *implies* $x \in P_{SNS}$*. This proves the claim.* ■

4.5 SNS-domain of attraction

This section introduces the notion of SNS-domain of attraction. It is shown that the SNS-domain of attraction is included in the domain of attraction of the saturated system. Taking into account the results of the previous section, a simple algorithm that converges to the SNS-domain of attraction of the system is proposed. It is also shown that in case of single input systems any contractive set belongs to the SNS-domain of attraction.

In the SNS-invariance concept, the admissible sequence is similar that used in the H-invariance concept in definition 12,

Definition 24 *A sequence $\{S_0, S_1, S_2, \ldots\}$ is admissible if all the elements of the sequence belong to $\mathcal{V}$.*

The following definition will define a more conservative concept than domain of attraction,

Definition 25 *The initial condition x_0 belongs to the SNS-domain of attraction of system $x^+ = Ax + B\sigma(Kx)$ if the recursion*

$$x_{k+1} = F(x_k, S_k)$$

converges to the origin for any admissible sequence $\{S_0, S_1, S_2, \ldots\} = \{S_k\}_0^\infty$.

It is clear from the previous definition that the SNS-domain of attraction is included in the domain of attraction of the saturated system. The following theorem states that it is possible to obtain the SNS-domain of attraction by means of a simple recursion.

Note that an invariant set Φ for the linear system is also an invariant set for the saturated system if it is included in the region of linear behaviour, that is, if $\Phi \subseteq \mathcal{L}(K) = \{\ x \in \mathbb{R}^n\ :\ \|Kx\|_\infty \leq 1\ \}$.

Theorem 8 *Denote $\mathcal{L}(K)$ the region of linear behaviour of the saturated system, that is, $\mathcal{L}(K) = \{\ x \in \mathbb{R}^n\ :\ \|Kx\|_\infty \leq 1\ \}$. Suppose that $\Phi \subseteq \mathcal{L}(K)$ is a convex polyhedron with non zero volume. Suppose also that Φ is an invariant set for the asymptotically stable system $x^+ = (A + BK)x$. Denote now $C_0 = \Phi$ and consider the following recursion:*

$$C_{k+1} = Q_{SNS}(C_k).$$

Then:

(i) C_k *is a convex polyhedron that can be obtained by means of theorem 7,* $\forall k \geq 1$.

(ii) C_k *is a SNS-invariant set for system* $x^+ = Ax + B\sigma(Kx)$, $\forall k \geq 0$.

(iii) $C_k \subseteq C_{k+1}$, $\forall k \geq 0$.

(iv) C_k *belongs to the SNS-domain of attraction of system* $x^+ = Ax + B\sigma(Kx)$, $\forall k \geq 0$.

(v) *The sequence* $\{C_0, C_1, C_2, \ldots\}$ *converges to the SNS-domain of attraction of system* $x^+ = Ax + B\sigma(Kx)$.

(vi) *The SNS-domain of attraction of the saturated system* $x^+ = Ax + B\sigma(Kx)$ *is a convex set.*

PROOF :

(i) *Theorem 7 states that if* Ω *is a convex polyhedron then* $Q_{SNS}(\Omega)$ *is also a convex polyhedron. This, and the fact that* C_0 *is a convex polyhedron, prove that the recursion* $C_{k+1} = Q_{SNS}(C_k)$ *always yields convex polyhedrons.*

(ii) *As* C_0 *belongs to* $\mathcal{L}(K)$ *it results that* $F(x,S) = (A + BK)x$, *for all* $x \in C_0$ *and for all* $S \in \mathcal{V}$. *From this and the invariance of* C_0 *it is inferred that* $F(x,S) \in C_0$ *for all* $x \in C_0$ *and for all* $S \in \mathcal{V}$; *that is to say,* C_0 *is SNS-invariant.*

Let now suppose that C_{k-1} *is SNS-invariant, then* $C_{k-1} \subseteq Q_{SNS}(C_{k-1}) = C_k$ *(see property 5). Therefore, if* $x \in C_k = Q_{SNS}(C_{k-1})$ *then* $F(x,S) \in C_{k-1} \subseteq C_k$, *for all* $S \in \mathcal{V}$.

(iii) *From the geometric condition of SNS-invariance (see property 5):* $C_k \subseteq Q_{SNS}(C_k) = C_{k+1}$

(iv) *From the SNS-invariance of* $C_0 \subseteq \mathcal{L}(K)$ *and the asymptotically stability of the non saturated system it is inferred that* C_0 *belongs to the SNS-domain of attraction of the system. Note that if* C_{k-1} *belongs to the SNS-domain of attraction then* $C_k = Q_{SNS}(C_{k-1})$ *also belongs to the SNS-domain of attraction. This is due to the fact that* $F(x,S) \in C_{k-1}$, *for all* $x \in C_k$ *and for all* $S \in \mathcal{V}$. *Therefore, the recursion* $C_{k+1} = Q_{SNS}(C_k)$ *with* $C_0 = \Phi$ *yields SNS-invariant sets that belong to the SNS-domain of attraction.*

(v) *Suppose now that x belongs to the SNS-domain of attraction of the system. As Φ is an invariant set with nonzero volume, there exists p such that the recursion $x_{k+1} = F(x_k, S_k)$ with $x_0 = x$ satisfies $x_p \in \Phi = C_0$ for all admissible sequence $S_0, S_1, \ldots, S_p$. This is equivalent to say that x is included in C_p and, consequently, x belongs to the SNS-domain of attraction.*

(vi) *It suffices to show that given two points x_1 and x_2 belonging to the SNS-domain of attraction, $\lambda x_1 + (1-\lambda)x_2$ belongs to the SNS-domain of attraction for every $\lambda \in [0,1]$. If x_1 and x_2 belong to the SNS-domain of attraction then it is clear from the previous claim that there exists p_1 and p_2 such that $x_1 \in C_{p_1}$, $x_2 \in C_{p_2}$. Denote now $p = \max\{p_1, p_2\}$, taking into account that $C_k \subseteq C_{k+1}$, $\forall k \geq 0$, it is inferred that $x_1 \in C_p$ and $x_2 \in C_p$. From the fact that C_p is a convex set contained in the SNS-domain of attraction of the system it is concluded that $\lambda x_1 + (1-\lambda)x_2$ belongs to C_p and therefore to the SNS-domain of attraction for every $\lambda \in [0,1]$.*

■

The recursion presented in theorem 8 requires an invariant set of the linear system $x^+ = Ax + BKx$, included in $\mathcal{L}(K)$. Note that this admissible invariant set can be obtained by standard algorithms (see [20, 17]).

Let us consider any set C_k obtained from the recursion presented in theorem 8; any set included in C_k belongs to the *SNS*-domain of attraction. For example, an ellipsoidal inner approximation of set C_k serves as an estimation of the domain of attraction of the saturated system. From the convexity of the *SNS*-domain of attraction it is inferred that the convex hull of a given collection of sets belonging to the *SNS*-domain of attraction also belongs to the *SNS*-domain of attraction.

The following property states, for single input systems, that any contractive set of the saturated system belongs to the *SNS*-domain of attraction. This means that the maximal contractive set for a given single input system is characterized, in a non conservative way, by the recursion proposed in theorem 8.

Property 6 *Let us suppose that $m = 1$ (single input case), and that Ω is a contractive set for system (4.1). That is, there is $\lambda \in [0,1)$ such that $x \in \epsilon\Omega$ implies $Ax + B\sigma(Kx) \in \lambda\epsilon\Omega$, $\forall \epsilon \in [0,1]$. Then Ω is a SNS-invariant set that belongs to the SNS-domain of attraction of the system.*

PROOF :

It will be first shown that if $x \in \epsilon\Omega$ then $F(x,S) \in \lambda\epsilon\Omega$ for all $S \in \mathcal{V} = \{\emptyset, 1\}$. That is, $Ax + B\sigma(Kx) \in \lambda\epsilon\Omega$ and $Ax + BKx \in \lambda\epsilon\Omega$. The first inclusion is clear from the assumptions of the property. It is now shown that $Ax + BKx \in \lambda\epsilon\Omega$. If $x \in \epsilon\Omega$ then there exists $\gamma \in (0,1]$ such that $|K\gamma x| \leq 1$. Moreover, as $x \in \epsilon\Omega$, it results that $\gamma x \in \gamma\epsilon\Omega$. From the assumptions of the property, it can be now concluded that :

$$A\gamma x + BK\gamma x = A\gamma x + B\sigma(K\gamma x) \in \lambda\gamma\epsilon\Omega$$

Note that $A\gamma x + BK\gamma x \in \lambda\gamma\epsilon\Omega$ implies $Ax + BKx \in \lambda\epsilon\Omega$. It has then been proved that

$$x \in \epsilon\Omega,\ \epsilon \in [0,1] \quad \Rightarrow \quad F(x,S) \in \lambda\epsilon\Omega,\ \ \forall S \in \mathcal{V} = \{\emptyset, 1\}. \tag{4.4}$$

It is clear that this implies that Ω is SNS-invariant.

In what follows it is shown that Ω belongs to the SNS-domain of attraction of the system. Let us consider the recursion: $x_{k+1} = F(x_k, S_k)$ with $x_0 = x \in \Omega$. From the previous discussion it is clear that $x_k \in \lambda^k\Omega$, for every admissible sequence $S_0, S_1, \ldots, S_{k-1}$. Therefore, $\lim\limits_{k\to\infty} x_k = 0$, for every admissible sequence $\{S_k\}_0^\infty$. This proves the claim. ■

4.6 Relationship with LDI approaches

The domain of attraction of a saturated system can be estimated by means of a linear difference inclusion (LDI) of the system [24, 29]. The LDIs are used to obtain invariant ellipsoids for saturated systems [24, 33]. It has been shown in [30] that the greatest invariant ellipsoid for a single input continuous-time saturated system is characterized by means of an LDI. In this section it is proved that the estimation provided by the SNS-domain of attraction is less conservative than the estimations provided by $LDIs$ approaches.

The following notation and lemmas have been shown in chapter 3 and are only repeated here for the sake of clarity.

Recall notation 2,

Notation 5 *Given matrix $H \in \mathbb{R}^{m\times n}$, and set $S \in \mathcal{V}$, $G_H(x,S)$ is defined as follows:*

$$G_H(x,S) = (A + \sum_{i\in S^c} B_iK_i + \sum_{i\in S} B_iH_i)x.$$

The following lemma, shown in lemma 1 (see [29, 33]), provides an LDI representation of a given saturated system.

Lemma 5 *Let* $H \in \mathbb{R}^{m\times n}$ *be given. If* $x \in \mathcal{L}(H)$ *then*

$$Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) \in co\ \{\ G_H(x,S)\ :\ S \in \mathcal{V}\ \}.$$

where $co\ \{\cdot\}$ *denotes the convex hull of a set.*

Taking into account the previous lemma, it is clear that a given set Ω is invariant for the saturated system if it is invariant for the family of linear plants: $\{\ G_H(x,S)\ :\ S \in \mathcal{V}\ \}$. The estimation of the domain of attraction of the LDI approaches are based on this property. The notion of H-contractive set is introduced (see definition 16),

Definition 26 $\Omega \subseteq \mathcal{L}(H)$ *is an H-contractive set if it is a convex set containing the origin and there is* $\lambda \in [0,1)$ *such that for every* $\epsilon \in [0,1]$, $x \in \epsilon\Omega$ *implies:*

$$G_H(x,S) \in \lambda\epsilon\Omega,\ \ \forall S \in \mathcal{V}$$

It is clear that every H-contractive set constitutes an estimation of the domain of attraction of a saturated system. The following theorem states that any H-contractive set is included in the SNS-domain of attraction of the system. That is, the estimation of the domain of attraction of the saturated system given by theorem 8 is less conservative than the one obtained by means of H-contractive sets.

An auxiliary lemma is used,

Lemma 6 *Let* $H \in \mathbb{R}^{m\times n}$ *be given. If* $x \in \mathcal{L}(H)$ *then*

$$F(x,T) \in co\ \{\ G_H(x,S)\ :\ S \in \mathcal{V}\ \},\quad \forall T \in \mathcal{V} \tag{4.5}$$

PROOF :

Suppose that $T \in \mathcal{V}$ *is given. Taking into account the definition of both* $G_H(x,S)$ *and* $\mathcal{L}(H)$*, equation (4.5) can be rewritten as:*

$$(A + \sum_{i\in T^c} B_iK_i)x + \sum_{i\in T} B_i\sigma(K_ix) \in co\ \{\ (A + \sum_{i\in M, i\notin S} B_iK_i + \sum_{i\in S} B_iH_i)x\ :\ S \in \mathcal{V}\ \}.$$

Denote $A_T = A + \sum_{i \in T^c} BK_i$. *Then lemma 6 implies that if* $x \in \{ x : |H_i x| \leq 1, \forall i \in T \} \subseteq \mathcal{L}(H)$ *then*

$$F(x,T) = A_T x + \sum_{i \in T} B_i \sigma(K_i x) \in co\ \{ (A_T + \sum_{i \in T, i \notin S} B_i K_i + \sum_{i \in S} B_i H_i) x\ :\ S \in \mathcal{V}_T \} =$$

$$co\ \{ (A + \sum_{i \in M, i \notin T} B_i K_i + \sum_{i \in T, i \notin S} B_i K_i + \sum_{i \in S} B_i H_i) x\ :\ S \in \mathcal{V}_T \} =$$

$$co\ \{ (A + \sum_{i \in M} B_i K_i - \sum_{i \in T} B_i K_i + \sum_{i \in T} B_i K_i - \sum_{i \in S} B_i K_i + \sum_{i \in S} B_i H_i) x\ :\ S \in \mathcal{V}_T \} =$$

$$co\ \{ (A + \sum_{i \in M, i \notin S} B_i K_i + \sum_{i \in S} B_i H_i) x\ :\ S \in \mathcal{V}_T \} =$$

$$co\ \{ G_H(x,S)\ :\ S \in \mathcal{V}_T \} \subseteq co\ \{ G_H(x,S)\ :\ S \in \mathcal{V} \}$$

■

Theorem 9 *Let us suppose that* Ω *is an H-contractive set for a given matrix H. Then* Ω *is a SNS-invariant set that belongs to the SNS-domain of attraction of the system.*

Proof :

It will be first proved that if $x \in \epsilon\Omega$ *then* $F(x,S) \in \lambda\epsilon\Omega$ *for all* $S \in \mathcal{V}$. *From the hypothesis of the theorem, there exists* $\lambda \in [0,1)$ *such that:*

$$G_H(x,S) \in \lambda\epsilon\Omega,\ \ \forall S \in \mathcal{V} \tag{4.6}$$

According to lemma 6,

$$F(x,T) \in co\ \{ G_H(x,S)\ :\ S \in \mathcal{V} \},\ \ \ \forall T \in \mathcal{V}$$

Therefore, it is concluded from equation 4.6 that

$$F(x,T) \in co\ \{ G_H(x,S)\ :\ S \in \mathcal{V} \} \in \lambda\epsilon\Omega,\ \ \forall T \in \mathcal{V}.$$

It has then been proved that

$$x \in \epsilon\Omega,\ \epsilon \in [0,1]\ \ \Rightarrow\ \ F(x,T) \in \lambda\epsilon\Omega,\ \ \forall T \in \mathcal{V}$$

This proves the SNS invariance of Ω. In what follows it will be shown that Ω belongs to the SNS-domain of attraction of the system. Let us consider the recursion

$$x_{k+1} = F(x_k, S_k) \text{ with } x_0 = x \in \Omega.$$

From the previous discussion it is clear that $x_k \in \lambda^k \Omega$, for every admissible sequence $S_0, S_1, \ldots, S_{k-1}$. Therefore, $\lim_{k\to\infty} x_k = 0$, for every admissible sequence $\{S_k\}_0^\infty$. This proves the claim. ■

4.7 Numerical examples

4.7.1 A single input system

Let us consider the system $x^+ = Ax + B\sigma(Kx)$ with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \; K = [-0.6167 \; -1.2703] \, .$$

This system represents a discrete time caracterization of integrator system

$$Y(s) = \frac{1}{s^2} X(s)$$

for sample time equal to one. This system has been used frequently in the literature (see [29] for an aplication of this system to saturated input systems).

Figure (4.1) shows the sequence $\{C_0, C_1, \ldots, C_{27}\}$ provided by the recursion of theorem (8): $C_{k+1} = Q_{SNS}(C_k)$. The sequence starts with an invariant set $\Phi = C_0$ contained in the region of linear behaviour of the system (shadowed in the figure). The sequence leads to the SNS-domain of attraction of the system. The SNS-invariant sets C_k, $k = 1, \ldots, 27$, are displayed in the figure (note that, as it is claimed in theorem 8, $C_k \subseteq C_{k+1}$, $\forall k \geq 0$).

The domain of attraction of this system has been estimated by means of a saturation-dependent Lyapunov function in [12]. In that paper, the authors propose how to obtain matrix H in such a way that a saturation-dependent Lyapunov function is strictly decreasing for every system $x^+ =$

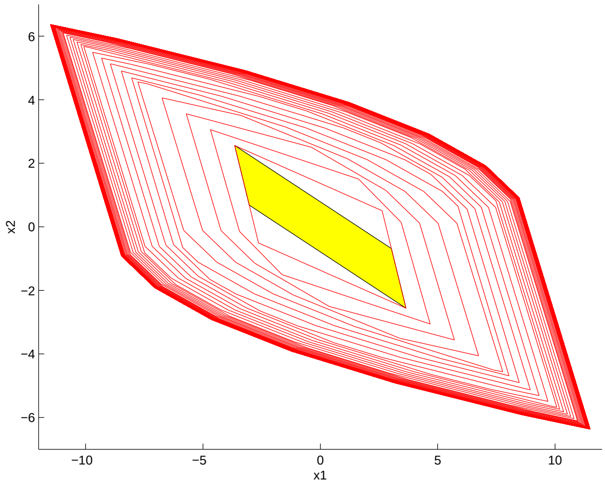

Figure 4.1: Sequence C_k leading to the SNS-domain of attraction.

$G_H(x, S)$, $S \in \mathcal{V}$. Therefore, the authors are obtaining an estimation of the domain of attraction of the system by means of the concept of LDI (see lemma 5).

Figure (4.2) compares the SNS-domain of attraction with the estimation obtained by means of a saturation-dependent Lyapunov function [12]. This figure shows that the SNS-domain of attraction contains the estimation provided by the LDI approach (see theorem 9).

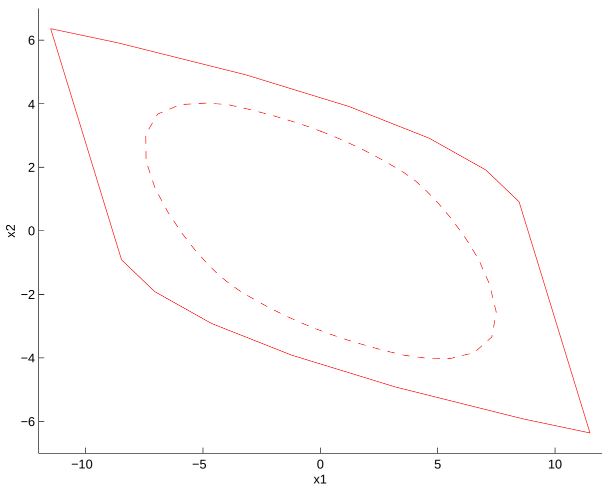

Figure 4.2: SNS-domain of attraction (solid line) and estimation of the domain of attraction obtained by means of a saturation-dependent Lyapunov function (dotted line).

4.7.2 A multiple input system

Consider the system $x^+ = Ax + B\sigma(Kx)$, with:

$$A = \begin{bmatrix} 1.2 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} -0.475 & 0 \\ 0.55 & 0.075 \end{bmatrix}$$

This example was introduced in [23] (and it has been used in chapter 3). In that paper, the authors use an initial contractive polyhedral set $\Upsilon \subseteq \mathcal{L}(K)$ and show how to enlarge it in such a way that the contractiveness of the enlarged polyhedron is maintained. That is, the maximum value of the scalar α such that $\alpha\Upsilon$ is a contractive polyhedron for the saturated system is obtained. Using this approach, the authors showed that the region $\{ x : \|x\|_\infty \leq 10 \}$ is included in the domain of attraction of the system.

Figure (4.3) shows the sequence $\{C_0, C_1, \ldots, C_{25}\}$ provided by the recursion of theorem (8). The domain of attraction of the system of this example is $\Gamma = \{ x \in \mathbb{R}^2 : |x_1| < 10 \}$. It can be shown that, in this particular case, the sequence $\{C_k\}$ converges not only to the SNS-domain of attraction of the system but also to the actual domain of attraction.

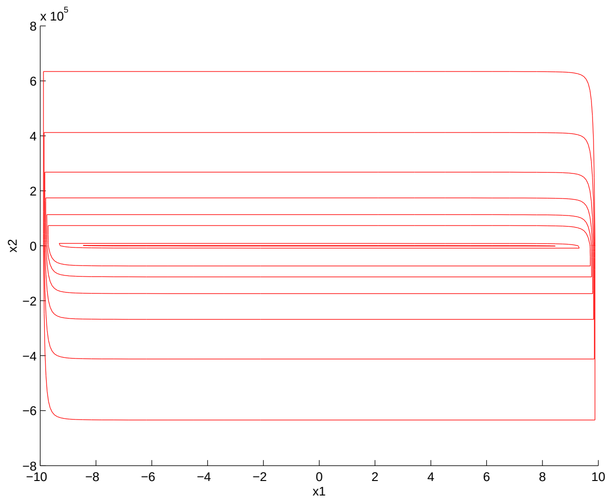

Figure 4.3: Sequence C_k leading to the SNS-domain of attraction of the system.

4.8 Conclusions

In this chapter a new notion of *SNS* invariance is introduced. The geometrical properties of the *SNS*-invariance concept leads to the definition of the *SNS*-domain of attraction of a given saturated system. A recursive algorithm that converges to the *SNS*-domain of attraction is presented. One of the most remarkable properties of the *SNS*-domain of attraction is that any contractive set for a saturated single input system is included in the *SNS*-domain of attraction. Moreover, it has also been shown that any estimation of the domain of attraction obtained by means of a linear difference inclusion is included in the *SNS*-domain of attraction. Numerical examples demonstrate the effectiveness of the new approach.

In the next chapter, the *SNS* technique is applied to continuous time systems.

5

Ellipsoidal *SNS*

5.1 Introduction

Chapters 3 and 4 proposed algorithms to obtain an approximation of the domain of attraction of a discrete linear system with input saturation.

Chapter 3 presented an *LDI* method, it has been proved that if a set Ω belongs to the domain of attraction of a set of linear systems, Ω also belongs to the domain of attraction of the determined saturated system. Additionally, due to the linear property of this systems, the largest subset that belongs to the domain of attraction of this set of linear systems is a convex set, and an algorithm to obtain this convex set was shown.

Chapter 4 used the a new method to obtain a less conservative approximation of the domain of attraction. It has been proved that a convex set Ω that belongs to the domain of attraction of a set of systems, that includes the target saturation system and other linear and saturated systems, can be obtained. It also has been shown that the largest set that belongs to the domain of attraction of all systems of this set, is a convex set and an algorithm to obtain this set has been shown. Note that this set is a polihedral convex approximation of the domain of attraction.

Previous algorithms use the $Q(\cdot)$ operator that is very related with the discretization of the system.

Invariant and approximation of the domain of attraction sets for continuous time systems can not be obtained by means of this operator $Q(\cdot)$, and different approaches must be used.

One of the most relevant approaches to the analysis of saturated systems is based on a *linear difference inclusion* (LDI) of the saturation nonlinearity (see [24, 52, 14]). In the literature, invariant ellipsoids have been used to estimate the domain of attraction for nonlinear systems [17, 7, 27, 28]. The domain of attraction of a given saturated system can be approximated by means of an ellipsoid. In [52] and [29] a linear difference inclusion for a linear saturated system is presented. Based on that LDI, the authors propose how to choose simultaneously both the matrix H, that characterizes the LDI, and the greatest ellipsoid that is invariant under the corresponding LDI.

A new algorithm to obtain an estimation of the domain of attraction of a continuous time saturation system will be presented in this chapter. The sufficient condition used is less conservative than the one obtained when a LDI approach is adopted.

5.2 Problem Statement

This chapter will show some methods to obtain approximations of the domain of attraction and less conservative invariant sets for continuous time systems. The family of systems under consideration is

$$\dot{x} = Ax + B\sigma(Kx) \tag{5.1}$$

where $x \in \mathbb{R}^n$ denotes the state vector. The function $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the vector-valued standard saturation function defined in definition 4.

The following notation defined in chapter 2 will be used. Denote $\mathcal{M} = \{1, 2, \ldots, m\}$. Denote also B_i, $i = 1, \ldots, m$ the columns of matrix B and K_i, $i = 1, \ldots, m$ the rows of matrix K. With this notation, system (5.1) can be rewritten as:

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i \sigma(K_i x) = Ax + \sum_{i \in \mathcal{M}} B_i \sigma(K_i x)$$

In this chapter it will be presented an *LMI* approach to the computation of ellipsoidal estimations of the domain of attraction for this class of saturated control systems.

In order to present the *LMI* approach to obtain estimations of the domain of attraction, the following notationis introduced.

Notation 6 *Given a positive definite matrix P, and a positive scalar ρ, $\mathcal{E}(P, \rho)$ represents the following ellipsoid:*

$$\mathcal{E}(P, \rho) = \{ \ x \ : \ x^\top P x \leq \rho \ \}.$$

This notation is in concordance with notation 4 of chapter 3.

Let $\mathcal{V}$ be defined as in definition 6, and S^c as notation 1, both in chapter 2.

5.3 *SNS* contractiveness

In this section, sufficient conditions for the contractiveness of a given ellipsoid are presented. The notion of contractiveness is given in the following definition,

Definition 27 *An ellipsoidal set $\mathcal{E}(P, \rho)$ is said to be contractive for system $\dot{x} = Ax + B\sigma(Kx)$ if for every $x \in \mathcal{E}(P, \rho), x \neq 0$:*

$$\frac{d}{dt}(x^\top P x) = 2x^\top P(Ax + B\sigma(Kx)) < 0$$

Note that contractiveness is a stronger property than invariance. Therefore, all contractive sets are invariant sets for the target system. Moreover, contractiveness is also a stronger property than domain of attraction and all contractive sets belong to the domain of attraction of the system.

Prior to the presentation of the theorem that shows an *LDI* that can be used to obtain a contractive ellipsoid, some properties will be shown.

Property 7 *Suppose that $Y \in \mathbb{R}^{1\times n}$ and matrix $W = W^\top \in \mathbb{R}^{n\times n}$ are such that:*

$$\begin{bmatrix} 1 & Y \\ Y^\top & W \end{bmatrix} > 0$$

then

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha W - \frac{\bar{B}\bar{B}^\top}{\alpha}, \quad \forall \alpha > 0, \forall \bar{B} \in \mathbb{R}^n$$

Proof :

Given $\bar{B} \in \mathbb{R}^n$ and $\alpha > 0$:

$$\begin{aligned} 0 &\leq (\sqrt{\alpha}Y^\top + \frac{\bar{B}}{\sqrt{\alpha}})(\sqrt{\alpha}Y^\top + \frac{\bar{B}}{\sqrt{\alpha}})^\top \\ &= \alpha Y^\top Y + \frac{\bar{B}\bar{B}^\top}{\alpha} + \bar{B}Y + Y^\top \bar{B}^\top \end{aligned}$$

It is then concluded that:

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha Y^\top Y - \frac{\bar{B}\bar{B}^\top}{\alpha} \tag{5.2}$$

Operating with the assumption

$$\begin{bmatrix} 1 & Y \\ Y^\top & W \end{bmatrix} > 0,$$

$$\begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & Y \\ Y^\top & W \end{bmatrix} \begin{bmatrix} 0 & 1 \\ I_n & 0 \end{bmatrix} > 0$$

$$\begin{bmatrix} W & Y^\top \\ Y & 1 \end{bmatrix} > 0,$$

where I_n is the $n \times n$ unit matrix. Therefore,

$$\begin{bmatrix} W & Y^\top \\ Y & 1 \end{bmatrix}^{-1} > 0.$$

Applying Schur's complement 13 to this matrix,

$$\begin{bmatrix} I_n & 0 \\ -Y & 1 \end{bmatrix} \begin{bmatrix} (W - Y^\top Y)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_n & -Y^\top \\ 0 & 1 \end{bmatrix} > 0$$

and it results

$$(W - Y^\top Y)^{-1} > 0$$

$$W > Y^\top Y.$$

From this and inequality (5.2) it is inferred that:

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha W - \frac{\bar{B}\bar{B}^\top}{\alpha}, \ \ \forall \alpha > 0$$

■

Property 8 *Suppose that $\bar{\epsilon} > 0$. Then, for every $a \in \mathbb{R}$:*

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} > -2|a|$$

PROOF :

Two cases should be taken into account

1. $a = 0$*: In this case:*

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} = \sup_{\bar{\alpha}>0} -\bar{\alpha} + \bar{\epsilon} > 0 = -2|a|$$

2. $a \neq 0$*: It is clear that $-\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon}$ is a concave differentiable function on α in $\mathbb{R}^+$. Thus, at the supremum:*

$$0 = \frac{d}{d\bar{\alpha}}(-\alpha - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon}) = -1 + \frac{a^2}{\bar{\alpha}^2}$$

It is then concluded that the maximal is obtained at $\bar{\alpha} = |a|$. Thus:

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} = -|a| - \frac{a^2}{|a|} + \bar{\epsilon} = -2|a| + \bar{\epsilon} > -2|a|$$

■

Property 9 *Given $z \in \mathbb{R}^m$:*

$$z^\top \sigma(Kx) \leq \max_{S \in \mathcal{V}} \{ \sum_{i \in S^c} z_i K_i x - \sum_{i \in S} |z_i| \}$$

where z_i denotes the i-th component of vector z.

Proof :

Note that according to the definition of $\sigma \cdot$ shown in definition 4 on chapter 2,

$$z^\top \sigma(Kx) = \sum_{i=1}^{m} z_i \sigma(K_i x).$$

Taking now into account that $a\sigma(y) \leq \max \{ay, -|a|\}$ (see lemma 10 in appendix A for a proof),

$$\sum_{i=1}^{m} z_i \sigma(K_i x) \leq \sum_{i=1}^{m} \max \{z_i K_i x, -|z_i|\}$$

,

and finally,

$$\sum_{i=1}^{m} \max \{z_i K_i x, -|z_i|\} =$$
$$\max_{S \in \mathcal{V}} \{ \sum_{i \in S^c} z_i K_i x - \sum_{i \in S} |z_i| \}$$

■

The following theorem present an LDI that can be used to obtain a contractive ellipsoid, which, as it will be seen in section 5.4, improves previous results from the literature.

Theorem 10 *The ellipsoid $\mathcal{E}(W^{-1}, 1)$ is contractive if for every $S \in \mathcal{V}$ there exists $Y^S \in \mathbb{R}^{m \times n}$ such that*

$$AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i^S +$$
$$(AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i^S)^\top < 0$$
$$\begin{bmatrix} 1 & Y_i^S \\ (Y_i^S)^\top & W \end{bmatrix} > 0, \forall i \in S$$

where Y_i^S denotes the i-th row of Y^S.

PROOF :

Note that the assumptions of the theorem guarantee that for every $S \in \mathcal{V}$ there is Y^S and $\epsilon > 0$ such that

$$AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i^S +$$
$$(AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i^S)^\top < -\epsilon \mathrm{I}$$

That is,

$$(A + \sum_{i \in S^c} B_i K_i)W + W(A + \sum_{i \in S^c} B_i K_i)^\top$$
$$+ \sum_{i \in S} (B_i Y_i^S + (Y_i^S)^\top B_i^\top) < -\epsilon \mathrm{I} \tag{5.3}$$

From the assumptions of the theorem:

$$\begin{bmatrix} 1 & Y_i^S \\ (Y_i^S)^\top & W \end{bmatrix} > 0, \forall i \in S$$

and property 7 it is inferred that for every $i \in S$:

$$B_i Y_i^S + (Y_i^S)^\top B_i^\top \geq -\alpha_i W - \frac{B_i B_i^\top}{\alpha_i}, \ \forall \alpha_i > 0$$

From the previous inequality and equation (5.3):

$$(A + \sum_{i \in S^c} B_i K_i)W + W(A + \sum_{i \in S^c} B_i K_i)^\top$$
$$- \sum_{i \in S} (\alpha_i W + \frac{B_i B_i^\top}{\alpha_i}) < -\epsilon \mathrm{I}, \forall \alpha > 0$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]^\top > 0$ denotes that each of the components of α is greater than zero. Denoting $P = W^{-1}$ and pre-multiplying and post-multiplying both sides of previous inequality by $x^\top P$ and Px respectively:

$$x^\top P(A + \sum_{i \in S^c} B_i K_i)x + x^\top (A + \sum_{i \in S^c} B_i K_i)^\top Px$$

$$-\sum_{i\in S}(\alpha_i x^\top Px + \frac{x^\top PB_iB_i^\top Px}{\alpha_i}) < -\epsilon x^\top P^2 x$$
$$\forall x \neq 0, \forall \alpha > 0$$

Taking into account that $x^\top Px \leq 1$ for every $x \in \mathcal{E}(P,1) = \mathcal{E}(W^{-1},1)$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$-\sum_{i\in S}(\alpha_i + \frac{x^\top PB_iB_i^\top Px}{\alpha_i}) < -\epsilon x^\top P^2 x$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0, \forall \alpha > 0$$

Note that $\epsilon x^\top P^2 x \geq \sum_{i\in S} \epsilon \frac{x^\top P^2 x}{m}$. Thus, denoting $\bar{\epsilon} = \epsilon \frac{x^\top P^2 x}{m}$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$-\sum_{i\in S}(\alpha_i + \frac{x^\top PB_iB_i^\top Px}{\alpha_i} - \bar{\epsilon}) < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0, \forall \alpha > 0$$

Taking into account that the previous inequality is satisfied for every $\alpha > 0$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$+\sum_{i\in S}\sup_{\bar{\alpha}}(-\bar{\alpha} - \frac{x^\top PB_iB_i^\top Px}{\bar{\alpha}} + \bar{\epsilon}) < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0$$

Note that $\bar{\epsilon} = \epsilon \frac{x^\top P^2 x}{m} > 0$ for every $x \neq 0$. This and property 8 guarantees that:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$-2\sum_{i\in S}|x^\top PB_i| < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x < 0$$

Denote $z = B^\top Px \in \mathbb{R}^m$. With this notation, the i-th component of vector z is equal to $B_i^\top Px$. Using this notation, the previous inequality can be rewritten as:

$$2x^\top PAx + 2\sum_{i\in S^c} z_i K_i x - 2\sum_{i\in S} |z_i| < 0$$

$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0$$

This last inequality is satisfied for every $S \in \mathcal{V}$. Therefore:

$$2x^\top PAx + 2\max_{S\in\mathcal{V}} \left\{ \sum_{i\in S^c} z_i K_i x - \sum_{i\in S} |z_i| \right\} < 0$$

$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0$$

Taking into account property 9

$$2x^\top PAx + 2z^\top \sigma(Kx) < 0$$

$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0$$

Recalling that $z = B^\top Px$:

$$2x^\top PAx + 2x^\top PB\sigma(Kx) = 2x^\top P\dot{x} = \frac{d}{dt}(x^\top Px) < 0$$

$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0$$

This proves the theorem.

5.4 Comparison with the linear difference inclusion approach

One of the most efficient ways of computing ellipsoidal estimations of the domain of attraction of a saturated control systems relies in the use of a Linear Differential Inclusion (LDI) of the saturated system. In this section it will be shown that theorem 10 yields less conservative ellipsoidal estimations than the ones provided by the LDI approach.

By means of the concept of LDI, the following sufficient condition for the contractiveness of a given ellipsoid is obtained (see [29] for a proof):

Theorem 11 *The ellipsoid $\mathcal{E}(W^{-1}, 1)$ is contractive if there exists $Y \in \mathbb{R}^{m\times n}$ such that*

$$
\begin{aligned}
&AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i + \\
&(AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i)^\top < 0 \\
&\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \; i = 1, \ldots, m
\end{aligned}
$$

where Y_i denotes the i-th row of Y.

The sufficient condition for an ellipsoid to be invariant provided by theorem 11 has been shown to be less conservative than the existing conditions resulting from the circle criterion or the vertex analysis [29, 52]. Moreover, as it is shown in [30], theorem 11 provides not only a sufficient but also a necessary condition for an ellipsoid to be invariant for the single input case ($m = 1$).

Note that the above result (theorem 11) can be obtained directly from theorem 10. It suffices to make $Y = Y^S$ for every $S \in \mathcal{V}$. Therefore, it is concluded that the results presented in this chapter provide an alternative proof of theorem (11) (in this case without using the concept of Linear Difference Inclusion).

The sufficient conditions provided by the main result of this chapter are less conservative than the ones corresponding to theorem 11 (this is due to the greater number of decision variables considered in theorem 10). It is concluded then that the approach proposed in this chapter improves the results obtained when a linear differential approach is adopted. The computational complexity of the ellipsoidal estimation of the domain of attraction presented in this chapter is greater than the one corresponding to the linear differential approach. This is due to the greater number of matrices involved in theorem 10. An analysis of the computational complexity will be presented in the following section.

5.5 Computational complexity

Theorem 10 can be applied to the computation of ellipsoidal estimation of the domain of attraction of a saturated system. However, the direct application of theorem 10 implies the solution of a convex optimization problem with $2^{(m+1)}$ constraints and $2^m + 1$ decision variables. Although the exponential number of constraints does not imply an excessive computational burden for practical values of m (there are convex algorithms in which the computational burden grows only linearly with the number of constraints), the same can not be affirmed for the number of decision variables: if m grows beyond a certain limit, the direct application of theorem 10 can be limited because of the exponential number of decision variables.

Fortunately, theorem 10 can be recast into an equivalent form in which the number of decision variables is reduced to only one, W. In this section it will be proved that the result provided in theorem 10) can be applied to the estimation of the domain of attraction by means of the solution of a convex problem with a reduced number of variables. For that purpose, the following definition is introduced.

Definition 28 *Given $W > 0$ and $S \in \mathcal{V}$, the function $\gamma_S(W)$ is defined as,*

$$\gamma_S(W) = \min_{Y \in \mathbb{R}^{m \times n}} \bar{\lambda} \Bigg(AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i + (AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i)^\top \Bigg)$$

$$s.t. \quad \begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \quad \forall i \in S$$

where Y_i denotes the i-th row of Y and $\bar{\lambda}(\cdot)$ denotes the matrix function greatest eigenvalue.

In what follow, it is shown that $\gamma_S(W)$ is a convex function on W for every $S \in \mathcal{V}$. The function

$$g(W, Y) = \bar{\lambda} \Bigg(AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i + (AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i)^\top \Bigg)$$

is cleary a convex function on W and Y, as $AW + \sum_{i \in S^c} B_i K_i W + \sum_{i \in S} B_i Y_i)^\top$ is an affine function hence convex and $\bar{\lambda}(\cdot)$ is a convex function. Moreover, the constraint

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \quad \forall i \in S$$

can be rewritten as

$$h(W,Y) = \max_{i \in S} \bar{\lambda} \left(- \begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} \right) < 0$$

Therefore, $\gamma_S(W)$ can be rewritten as

$$\gamma_S(W) = \begin{array}{l} \min_{Y \in \mathbb{R}^{m \times n}} g(W,Y) \\ \text{s.t.} \quad h(W,Y) < 0 \end{array}$$

As both $g(W,Y)$ and $h(W,Y)$ are jointly convex in W and Y, it is inferred that $\gamma_S(W)$ is convex with respect W (see [8]). For this class of optimization problems it is possible to find a subgradient of $\gamma_S(W)$ with respect W at any given W_0 (see also [8]).

Note that with the definition of $\gamma_S(W)$, theorem 10 can be rewritten as,

Theorem 12 *The ellipsoid $\mathcal{E}(W^{-1},1)$ is contractive if*

$$\gamma_S(W) < 0, \forall S \in \mathcal{V}$$

As the trace of W is a measure of the size of ellipsoid $\mathcal{E}(W^{-1},1)$, the maximization problem

$$\max_{W>0} \ \text{trace}\,(W)$$

$$\text{s.t.} \ \ \gamma_S(W) < 0, \ \ \forall S \in \mathcal{V}$$

yields to the maximization of the ellipsoidal estimation of the domain of attraction of the saturated system $\dot{x} = Ax + B\sigma(Kx)$. The proposed maximization problem has the following properties.

- It is a convex optimization problem. This convexity stems from the already proved fact that $\gamma_S(W)$ is convex on W.
- The evaluation of $\gamma_S(W)$ can be achieved solving an LMI problem with a unique decision variable: $Y \in \mathbb{R}^{n \times m}$.
- The computation of a subgradient of $\gamma_S(W)$ with respect W can also be done solving an LMI problem. This makes it possible the application of any cutting plane algorithm to the solution of the proposed optimization problem [8].

5.6 Numerical examples

In this section two different examples are presented. The first example shows the application of the presented approach to a two dimensional system. In the second example higher dimensional systems are considered.

5.6.1 Two dimensional system

Let us consider the system $\dot{x} = Ax + B\sigma(Kx)$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix},$$

K is obtained as the solution of the LQR problem with $Q = I$ and $R = 0.1 \cdot I$. That is,

$$K = \begin{bmatrix} -2.0506 & -5.9715 \\ -3.1458 & 2.1906 \end{bmatrix}.$$

In order to maximize the size of the ellipsoidal estimation of the domain of attraction of the system, the trace of matrix W is maximized.

Figure 5.1 shows how the conservativeness in the computation of the ellipsoidal estimation of the domain of attraction is reduced by means of the main result of the chapter. In this figure, two different contractive ellipsoids are drawn. The ellipsoid represented by means of a dotted line corresponds to the ellipsoid obtained when an LDI approach is adopted (theorem 11). The outer ellipsoid represented by a solid line corresponds to the application of the result of theorem 10. In both cases, the trace of matrix W is maximized.

It can be seen in the figure that the ellipsoid obtained by the sufficient condition presented in this chapter is greater than the other one. This is not surprising because it has been proved in section 5.4 that theorem (10) provides less conservative results of the estimation of the size of the domain of attraction ellipsoid than provided by theorem (11).

5.6.2 Higher dimensional systems

In order to provide some measure of the improvement obtained when using the proposed sufficient condition, a family of random systems have

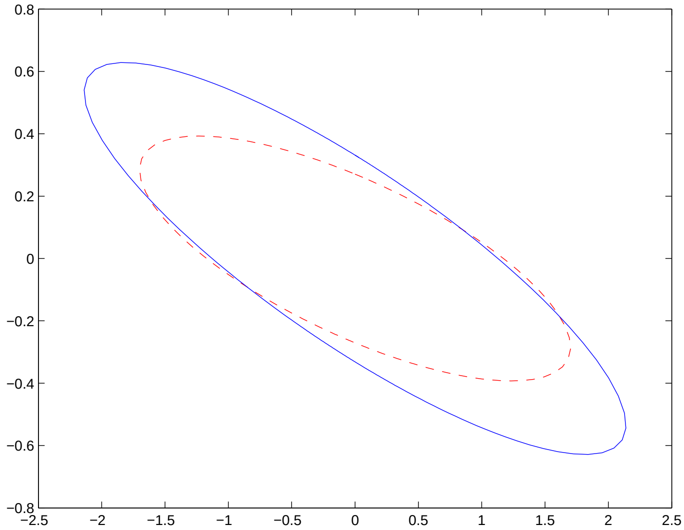

Figure 5.1: Contractive ellipsoidal sets.

been considered. Each of the considered examples belong to the following class of systems:

$$\dot{x} = Ax + B\sigma(Kx)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. Given A and B, matrix gain K corresponds to the solution of the LQR problem with $Q = I$ and $R = I$. The entries of matrices A and B are obtained from a uniform distribution in the interval $[0,1]$. That is, $A_{ij} \in [0,1]$, $B_{ij} \in [0,1]$ for every $1 \leq i \leq n$, $1 \leq j \leq n$. In order to obtain a bounded domain of attraction, only matrices A that satisfy that all their eigenvalues have positive real part are considered. In this way, a total of 1200 (open-loop unstable) systems have been obtained in a randomized way for different values of n ($n = 1, 2, \ldots, 6$).

A comparison between the sufficient condition provided in this chapter (denoted in what follows method A) and the one corresponding to the LDI approach (denoted method B) have been done. Table 5.1 shows the mean improvement both in terms of trace of W and volume of the obtained ellipsoids when using the results of this chapter with respect the ones obtained with the LDI approach. Dimensions $n = 1, 2, \ldots, 6$ have been considered. For each dimension, 200 examples are considered and only the mean value of the improvement is displayed. Dimension 1 shows the same results using both methods. In this case both methods provide the same sufficient condition and therefore the results are identical. The observed improvement depends on the dimension of the system. It increases with the dimension (more decision variables are considered

Dimension	Mean improvement Trace W	Mean improvement Volume ellipsoid	Mean computational time method A (seconds)	Mean computational time method B (seconds)
n=1, m=1	0%	0%	0.14	0.14
n=2, m=2	9.9 %	38.8 %	0.11	0.10
n=3, m=3	20.7 %	148.5 %	0.19	0.15
n=4, m=4	42.0 %	492.3 %	0.63	0.34
n=5, m=5	48.7 %	1225.3 %	5.21	1.22
n=6, m=6	57.2%	3615.7 %	94.59	5.72

Table 5.1: Improvement and computational time of the proposed method (method A) with respect to the one corresponding to the LDI approach (method B)

in the proposed approach). It is worth noting than in dimension 6, the obtained ellipsoids are, in average, 36.1 times greater in volume than the ones corresponding to the LDI approach.

The LMI problems corresponding to both methods A and B have been solved using the LMI-toolbox of $MATLAB$. The computational time required in the obtainment of the ellipsoidal estimation of the domain of attraction is greater when using the proposed approach. Note that the computational burden depends on the number of decision variables and constraints of the $LMIs$. The number of decision variables grows in an exponential way in the proposed approach. From the computational time corresponding to method A (shown in table 5.1) it is inferred that when the system has few control inputs the required time is moderate. However, for larger dimensions, the use of alternative approaches to the solution of the optimization problem (like the one presented in section 5.5) are required. The great enlargement of the obtained estimation of the domain of attraction justifies the greater computational time of the proposed approach.

5.7 Conclusions

In previous chapters, algorithms to obtain an estimation of the domain of attraction for discrete time systems have been presented.

In this chapter, an approach to the estimation of the domain of attraction of a saturated linear system is presented. A new sufficient condition for the contractiveness of a given ellipsoid has been presented and it provides estimations of the domain of attraction of a continuous time saturated system. This approach is less conservative than the one corresponding to the use of the concept of LDI. The computational complexity of the characterization of the proposed ellipsoidal estimation of the domain of attraction has been analized.

The adventadges of the proposed ellipsoidal estimation have been illustrated by means of a family of randomly chosen systems.

6

Synthesis of robust saturated controllers

6.1 Introduction

One of the most relevant approaches to the estimation of the domain of attraction of a saturated system consists in the use of a linear difference inclusion (LDI) of the system. This method has been presented in chapter 3 for discrete time systems. Polytopic representation provided by the LDI simplifies the analysis of the non linear system [51]. In [24, 29], an LDI is used to obtain invariant ellipsoids for saturated systems.

In chapter 4 (see also [3]), a new notion of invariance, denoted as SNS-invariance, is presented. Based on its geometrical properties, a simple algorithm to estimate the domain of attraction of a saturated linear system is proposed. It is shown that in case of single input systems, any contractive set for the saturated system is an SNS-invariant set. Moreover, any domain of attraction obtained by means of an LDI representation of the system is included in the estimation provided by the proposed algorithm.

In this chapter, the analysis of saturated control laws for an uncertain linear system is addressed. It is well known that, under certain assumptions, the greatest invariant ellipsoidal set for a given system can be obtained by means of a control law that does not saturate in the corresponding ellipsoid. For example, for single nominal input systems (m=1), the greatest invariant ellipsoid is obtained by means of a control law of the form: $u = K_L x$, where $|K_L x| \leq 1$ for every x in the ellipsoid. The synthesis of such a controller can be recast as an LMI problem [30]. An algorithm that improves the results obtained when only ellipsoidal sets are considered will be proposed in this chapter. This algorithm uses a generalization of the SNS-stability notion that allows to deal with systems with additive uncertainty. This generalization yields to polyhedric estimations of the domain of attraction of an uncertain system. Based on this, an algorithm suitable for the synthesis of saturated control laws is presented. The results improves previous results in the following sense, given a robust control ellipsoidal invariant set obtained by means of a non saturated control law, the algorithm yields an improved controller that contains the afore-mentioned ellipsoid.

6.2 Problem Statement

Consider the following discrete-time system

$$x^+ = Ax + Bu + E\theta \tag{6.1}$$

where $x \in \mathbb{R}^n$ denotes the state vector and x^+ is the successor state. Vector $u \in \mathbb{R}^m$ denotes the control input to the system and $\theta \in \mathbb{R}^{n_w}$

denotes an additive uncertainty to the system. It is assumed that $\theta \in \Theta = \{ \theta \in \mathbb{R}^{n_w} : \|\theta\|_\infty \leq 1 \}$. Note that Θ can be described as $\Theta = \mathcal{L}(E_{n_w})$ where E_i is the identity matrix in dimension i. The control input u is supposed to satisfy the following amplitude constraint, $\|u\|_\infty \leq 1$. Moreover, the state vector should be confined in the convex and bounded polyhedron X, which is assumed to contain the origin.

Note that equation 6.1 is a robust generalization of system family 2.17.

The purpose of this chapter is to obtain a matrix gain K such that the saturated control law $u = \sigma(Kx)$ maximizes the robust domain of attraction of the closed loop system.

The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the vector-valued normalized saturation function defined in 4.

6.3 Geometric condition of robust invariance

In the context of set invariance theory, the one-step set plays an important role [17]. This concept has been used in 3 and 4 for determinating an estimation of the domain of attraction of the system. In order to take account of the dependence on K, an extension of the one-step operator is defined here.

Definition 29 *Given system $x^+ = Ax + B\sigma(Kx)$ and set Ω, the one-step set $Q(\Omega, K)$ is defined as*

$$Q(\Omega, K) = \{ x : Ax + B\sigma(Kx) \in \Omega \}$$

It is well known that a geometric condition for robust invariance can be expressed by means of the one-step set and the Pointriagin difference.

Notation 7 *Given set Ω and set Θ, set $\Omega \sim E\Theta$ denotes the Pointriagin difference of Ω and $E\Theta$, that is:*

$$\Omega \sim E\Theta = \{ x \in \Omega : x + E\theta \in \Omega \text{ for all } \theta \in \Theta \}$$

It is well known that Ω is a robust invariant set for system $x^+ = Ax + B\sigma(Kx) + E\theta$ if and only if $\Omega \subseteq Q(\Omega \sim E\Theta, K)$ [17].

Note that the domain of attraction corresponding to a given gain matrix K can be obtained (theoretically) by means of the following algorithm,

1. $C_0(K) = X$

2. $C_{j+1}(K) = C_j \bigcap Q(C_j(K) \sim E\Theta, K)$

3. $j = j + 1$. Go to step 2

It is easy to see that if $x \in C_j(K)$ and the sequence of disturbances $\{\theta_0, \theta_1, \ldots, \theta_{j-1}\}$ satisfies that $\theta_k \in \Theta$, $k = 0, \ldots, j-1$ then the recursion:

$$x_{k+1} = Ax_k + B\sigma(Kx_k) + E\theta_k, \quad x_0 = x$$

is such that $x_k \in X$, $k = 0, \ldots, j$. Therefore, the set of initial conditions that correspond to trajectories always remaining in X (in spite of the disturbances) is equal to $C_\infty(K) = \lim_{k\to\infty} C_k(K)$.

Given a convex set Ω, the one-step set $Q(\Omega, K)$ is not necessarily convex due to the nonlinear nature of the saturation function. The non convex nature of $Q(\Omega, K)$ makes it difficult to use operator $Q(\cdot)$ in the analysis and synthesis of saturated control laws for the system. Therefore, the computation of $C_j(K)$ can be extremely difficult for sufficiently large j.

6.4 Greatest robust ellipsoidal set obtained with a non-saturated control law

Under certain assumptions, the greatest invariant ellipsoidal set for a given system can be obtained by means of a control law that does not saturate in the corresponding ellipsoid.

In case of systems with additive uncertainties, it is possible to characterize the non-saturated control laws that guarantee that a given ellipsoid is a robust invariant set for the the closed-loop system. This caracterization is an ellipsoidal set, that are defined in notation 6, but as all the ellipses considered in this chapter have an unitary radius, notation changes to use only one parameter,

Notation 8 *Given a definite positive matrix P,*

$$\mathcal{E}(P) = \{\ x\ :\ x^\top Px \leq 1\ \}.$$

Note that $\mathcal{E}(P) = \mathcal{E}(P, 1)$, and $\mathcal{E}(P, \rho) = \mathcal{E}(P/\rho)$.

The following property provides the characterization commented above,

Property 10 *Suppose that there exists a scalar $\gamma \in \mathbb{R}_+$ and matrices $Y \in \mathbb{R}^{m\times n}$ and $W = W^\top \in \mathbb{R}^{n\times n}$ such that the following matrix inequalities are satisfied,*

$$\begin{bmatrix} 1-\gamma & 0 & \theta^\top E^\top \\ 0 & \gamma W & WA^\top + Y^\top B^\top \\ E\theta & AW + BY & W \end{bmatrix} > 0, \ \ \forall \theta \in \mathrm{vert}(\Theta)$$

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \ \ i = 1, \ldots, m$$

where vert(Θ) *denotes the vertices of* Θ *and* Y_i*,* $i = 1, \ldots, m$ *denotes the i-th row of matrix* Y*. Denote* $K = YW^{-1}$*, then*

- $\|Kx\|_\infty \leq 1$ *for all* $x \in \mathcal{E}(W^{-1})$*.*
- *The ellipsoid* $\mathcal{E}(W^{-1})$ *is a robust invariant set of the system* $x^+ = Ax + B\sigma(Kx) + E\theta$*.*

Proof :

It will be proved by items,

- $\|Kx\|_\infty \leq 1$ *for all* $x \in \mathcal{E}(W^{-1})$*, Note that previous statement can be written as,*

  $$\|Kx\|_\infty \leq 1 \textit{ for all } \{ \ x \ : \ x^\top W^{-1} x \leq 1 \ \},$$

  and if it is defined $y = W^{-\frac{1}{2}}$*.*

  $$\|KW^{\frac{1}{2}} y\|_\infty \leq 1 \textit{ for all } \{ \ y \ : \ y^\top y \leq 1 \ \},$$

  $$\|YW^{-\frac{1}{2}} y\|_\infty \leq 1 \textit{ for all } \|y\|_2 \leq 1.$$

  Previous inequality can be written as

  $$|Y_i W^{-\frac{1}{2}} y| \leq 1 \textit{ for all } \|y\|_2 \leq 1, \ \ i = 1, \ldots, m.$$

  The worst situation for previous statement is if y *is in direction of* $Y_i W^{-\frac{1}{2}}$*. Note that as all inequalities must hold, worst condition of* y *can change among inequalities.*

  $$\left| \frac{Y_i W^{-\frac{1}{2}} W^{-\frac{1}{2}} Y_i^\top}{\|W^{-\frac{1}{2}} Y_i^\top\|_2} \right| \leq 1, \ \ i = 1, \ldots, m.$$

taking squares,

$$\frac{(Y_i W^{-1} Y_i^\top)^2}{Y_i W^{-1} Y_i^\top} \leq 1, \ \ i = 1, \ldots, m,$$

that is

$$Y_i W^{-1} Y_i^\top \leq 1, \ \ i = 1, \ldots, m.$$

And aplying Schur lemma (see 13 in appendix A),

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \ \ i = 1, \ldots, m.$$

- *The ellipsoid* $\mathcal{E}(W^{-1})$ *is a robust invariant set of the system* $x^+ = Ax + B\sigma(Kx) + E\theta$.

  The proof consists in the determination of a Lyapunov function, $L(x) = x^\top W^{-1} x$, *and to prove that for all* x *in* $\mathcal{E}(W^{-1})$, *the following statement hold*

$$L(x^+) - L(x) < 0.$$

The first assumption claims that,

$$\begin{bmatrix} 1-\gamma & 0 & \theta^\top E^\top \\ 0 & \gamma W & WA^\top + Y^\top B^\top \\ E\theta & AW + BY & W \end{bmatrix} > 0, \ \ \forall \theta \in \mathrm{vert}(\Theta)$$

Therefore, aplying Schur lemma 13 in appendix A,

$$\begin{bmatrix} 1-\gamma & 0 \\ 0 & \gamma W \end{bmatrix} - \begin{bmatrix} \theta^\top E^\top \\ WA^\top + Y^\top B^\top \end{bmatrix} W^{-1} [E\theta \ \ AW + BY] > 0, \ \ \forall \theta \in \mathrm{vert}(\Theta),$$

this inequality can be extended,

$$\begin{bmatrix} 1-\gamma & 0 \\ 0 & \gamma W \end{bmatrix} -$$

$$\begin{bmatrix} \theta^\top E^\top W^{-1} E\theta & \theta^\top E^\top W^{-1}(AW + BY) \\ (WA^\top + Y^\top B^\top) W^{-1} E\theta & (WA^\top + Y^\top B^\top) W^{-1} (AW + BY) \end{bmatrix} > 0, \ \ \forall \theta \in \mathrm{vert}(\Theta),$$

and joining both terms,

$$\left[\begin{array}{cc} (1-\gamma)-\theta^\top E^\top W^{-1}E\theta & -\theta^\top E^\top W^{-1}(AW+BY) \\ -(WA^\top+Y^\top B^\top)W^{-1}E\theta & \gamma W-(WA^\top+Y^\top B^\top)W^{-1}(AW+BY) \end{array}\right] > 0,$$

$$\forall\theta \in \text{vert}\Theta.$$

This inequality can be pre and post multiplied by

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & W^{-1} \end{array}\right],$$

and taking into account that $K = YW^{-1}$, it is obtained,

$$\left[\begin{array}{cc} (1-\gamma)-\theta^\top E^\top W^{-1}E\theta & -\theta^\top E^\top W^{-1}(A+BK) \\ -(A^\top+K^\top B^\top)W^{-1}E\theta & \gamma W^{-1}-(A^\top+K^\top B^\top)W^{-1}(A+BK) \end{array}\right] > 0,$$

$$\forall\theta \in \text{vert}\Theta.$$

A matrix inequality means that this matrix multiplied by any vector before and after the matrix must be greater than zero, that is,

$$[1 \;\; x^\top]\left[\begin{array}{cc} (1-\gamma)-\theta^\top E^\top W^{-1}E\theta & -\theta^\top E^\top W^{-1}(A+BK) \\ -(A^\top+K^\top B^\top)W^{-1}E\theta & \gamma W^{-1}-(A^\top+K^\top B^\top)W^{-1}(A+BK) \end{array}\right]\left[\begin{array}{c} 1 \\ x \end{array}\right] > 0,$$

$$\forall x \in \mathbb{R}^n, \forall\theta \in \text{vert}(\Theta).$$

Multiplying,

$$x^\top(A^\top+K^\top B^\top)W^{-1}(A+BK)x + x^\top(A^\top+K^\top B^\top)W^{-1}E\theta +$$
$$+\theta^\top E^\top W^{-1}(A+BK)x + \theta^\top E^\top W^{-1}E\theta < 1-\gamma(1-x^\top W^{-1}x),$$
$$\forall\theta \in \text{vert}\Theta, \forall \text{x} \in \mathbb{R}^{\text{n}}.$$

Note also that $x^\top W^{-1}x \leq 1$, therefore $\gamma(1-x^\top W^{-1}x) \geq 0$ and it implies

$$(x^\top(A^\top+K^\top B^\top)+\theta^\top E^\top)W^{-1}((A+BK)x+E\theta) < 1,$$
$$\forall\theta \in \text{vert}(\Theta), \forall \text{x}: \;\; \text{x}^\top \text{W}^{-1}\text{x} = 1.$$

Previous inequality is a linear inequality in θ, and taking into acount that it holds for all $\theta \in \text{vert}(\Theta)$ and the convex nature of linear functions, it is inferred that

$$(x^\top(A^\top + K^\top B^\top) + \theta^\top E^\top)W^{-1}((A+BK)x + E\theta) < 1,$$
$$\forall\theta \in \Theta, \forall x:\ x^\top W^{-1}x = 1.$$

Moreover, $x^+ = (A+BK)x + E\theta$ *for all* $x \in \mathcal{E}(W^{-1})$*, note that first claim of this property shows that* $\|Kx\|_\infty \leq 1$ *for all* $x \in \mathcal{E}(W^{-1})$*, and therefore* $\sigma(Kx) = Kx$*,*

$$x^{+\top}W^{-1}x^+ - 1 < 0, \forall x:\ x^\top W^{-1}x = 1,$$

that is $L(x^+) - L(x) < 0$ *for all* $x \in$ *bound*$(\mathcal{E}(W^{-1}))$*.*

■

6.5 Defining the notion of robust SNS-invariance

The concept of (nominal) SNS invariance, which was first proposed in [3] and shown in chapter 4, is extended here to encompass the possibility of additive uncertainties. This concept, along with its geometrical properties, allows to obtain a convex estimation of the (robust) domain of attraction of a given saturated system. This avoids the complexity associated to the computation of sets $C_j(K)$. As it will be seen in the following sections, it is possible not only to analyze the domain of attraction of a system using the SNS concept but also to obtain an appropriate matrix gain K maximizing the domain of attraction of the system.

Let consider the following closed-loop system,

$$x^+ = Ax + B\sigma(Kx) + E\theta \tag{6.2}$$

Following the notation presented in chapter 2, denote $\mathcal{M} = \{1, 2, \ldots, m\}$. Denote also B_i, $i = 1, \ldots, m$ the columns of matrix B and K_i, $i = 1, \ldots, m$ the rows of matrix K. With this notation, system 6.2 can be rewritten as,

$$x^+ = Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) + E\Theta = Ax + \sum_{i\in\mathcal{M}} B_i\sigma(K_i x) + E\theta.$$

Definition 6 on chapter 2 defines $\mathcal{V}$ for a given $\mathcal{M}$. However, a generalization of this set is needed to introduce the concept of robust SNS-invariance,

Definition 30 *Given a set of integers T, set $\mathcal{V}_T$ is the set of all subsets of T. That is,*

$$\mathcal{V}_T = \{ S : S \subseteq T \}$$

Example: If $T = \{1,2\}$ then $\mathcal{V}_T = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. Note that the empty set $\emptyset$ belongs to $\mathcal{V}_T$. Note also that in this work, S^c denotes the complement of S in the reference set, in this case T. That is, $S^c = \{ i \in T : i \notin S \}$. Throughout this text, when no subset is defined, $\mathcal{M}$ is assumed. That is, $\mathcal{V} = \mathcal{V}_{\mathcal{M}}$.

Definition 31 *Given a set $S \in \mathcal{V}$, $F(x,S)$ is defined as follows:*

$$F(x,S) = Ax + \sum_{i \in S^c} B_i K_i x + \sum_{i \in S} B_i \sigma(K_i x)$$

The notion of robust SNS invariance is introduced in the following definition. Note that this is an extension of the SNS invariance shown in chapter 4 for robust invariance.

Definition 32 *A set Ω is said to be robust SNS-invariant set for system $x^+ = Ax + B\sigma(Kx) + E\theta$ if $x \in \Omega$ implies $F(x,S) + E\theta \in \Omega$ for every $S \in \mathcal{V}$ and for every $\theta \in \Theta$.*

For single input systems ($m = 1$), the SNS invariance of a given set Ω is equivalent to the robust invariance of Ω for the *Saturated* and *Non Saturated* systems, $x^+ = Ax + B\sigma(Kx) + E\theta$ and $x^+ = Ax + BKx + E\theta$.

In order to provide a geometric condition of robust SNS invariance, the one step function is defined. This function has been used in chapters 3 and 4 to obtain an estimation of the domain of attraction (and an invariant set). Definition shown here is an extension of that definition that allows to explicitily consider K as a design parameter.

Definition 33 *Given a set Ω, K and $S \in \mathcal{V}$:*

$$Q_{SNS}(\Omega, K) = \{ x : F(x,S) \in \Omega \text{ for all } S \in \mathcal{V} \}$$

where $F(x,S)$ depends on K and it is defined in 31.

From the definition of $Q_{SNS}(\cdot)$, the following property is directly inferred,

Property 11 *A set Ω is robust SNS invariant set for system 6.2 if and only if $\Omega \subseteq Q_{SNS}(\Omega \sim E\Theta, K)$.*

The most remarkable property of $Q_{SNS}(\cdot)$ is that given a polyhedral set Ω, $Q_{SNS}(\Omega, K)$ is a convex polyhedron. Given a convex polyhedral set Ω, $Q(\Omega, K)$ is not necessarily convex. However, $Q_{SNS}(\Omega, K)$ is a polyhedral set that can be obtained in a direct way from polyhedron Ω as claimed in the following theorem.

Theorem 13 *Let us suppose that Ω is a convex polyhedron in $\mathbb{R}^n$ given by $\Omega = \{ \; x \; : \; Rx \preceq g \; \}$. Then:*

$$Q_{SNS}(\Omega, K) =$$

$$\bigcap_{S \in \mathcal{V}} \{ \; x \; : \; R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq g \; \}$$

where S^c denotes the complement of S in $\mathcal{M}$ and $|RB_i|$ is the vector with entries equal to the absolute value of the entries of vector RB_i.

Proof :

This is the same theorem than 7 in chapter 4, with the only modification of explicit dependence on K. Proof can be seen there.

The following algorithm can be used to obtain an estimation of the domain of attraction corresponding to a given gain matrix K. Given a contracting factor $\lambda \in (0, 1)$, consider the following recurrence,

1. $C_0^{SNS}(K, \lambda) = X$.
2. $C_{j+1}^{SNS}(K, \lambda) = Q_{SNS}(\lambda C_j^{SNS}(K, \lambda) \sim E\Theta, K)$
3. If $C_{j+1}^{SNS}(K, \lambda) \subseteq Q_{SNS}(C_{j+1}^{SNS}(K, \lambda) \sim E\Theta, K)$ then $C_{j+1}^{SNS}(K, \lambda)$ is a robust SNS-invariant set. Else, $j = j + 1$. Go to step (2)

Note that $\lambda\Omega$ denotes the set $\{ \; x \; : \; \frac{x}{\lambda} \in \Omega \; \}$. It is clear that $Q_{SNS}(\lambda\Omega, K) \subseteq Q_{SNS}(\Omega, K) \subseteq Q(\Omega, K)$, $\forall \lambda \in (0, 1)$. Therefore, $C_j^{SNS}(K, \lambda) \subset C_j(K)$, $\forall j \geq 0$, $\forall K$, $\forall \lambda \in (0, 1)$. It can be concluded that $C_\infty^{SNS}(K, \lambda) = \lim_{j \to \infty} C_j^{SNS}(K, \lambda)$ serves as an estimation of the robust domain of attraction of the closed loop system, $x^+ = Ax + B\sigma(Kx) + E\theta$. Moreover, as the size of the elements of the sequence $C_j^{SNS}(K, \lambda)$ is monotonically decreasing with j ($C_{j+1}^{SNS}(K, \lambda) \subseteq C_j^{SNS}(K, \lambda)$, $\forall j \geq 0$) it can be concluded that there is j^* such that $C_{j^*+1}^{SNS}(K, \lambda) \subseteq Q_{SNS}(C_{j^*+1}^{SNS}(K, \lambda) \sim E\Theta, K)$. This fact can be proved using the same arguments as in [6] (and assuming that $C_\infty^{SNS}(K, \lambda)$ is not the empty set).

6.6 Capturing the geometry of the greatest robust control invariant set

Definition 34 *Given system 6.1 and constraints $x \in X$, $u \in U$, we say that Ω is a robust control invariant set if for every $x \in \Omega$ there is $u = u(x) \in U$ such that $Ax + Bu(x) + E\theta \in \Omega$, for all $\theta \in \Theta$.*

Throughout this chapter the greatest control invariant set is denoted X_∞. The objective of this chapter is to find a saturated control law $u = \sigma(Kx)$ in such a way that the domain of attraction corresponding to the application of such a saturated control law is maximized. If possible, gain matrix K should be obtained in order to obtain a domain of attraction equal to X_∞. As this is not always possible, a measure of the size of the obtained domain of attraction should be maximized. In this chapter, it will be maximized the size of the greatest ellipsoid contained in the domain of attraction.

One of the theoretical approaches to the synthesis problem consists in trying to obtain K in such a way that the size of $C_\infty(K, \lambda)$ is maximized. However, as $C_\infty(K)$ is not necessarily a convex set the dependence of the size of $C_\infty(K)$ with respect to K is generally non-convex. In order to simplify the synthesis problem, consider the following operator:

Definition 35 *Given set $\Omega = \{\ x\ :\ Rx \preceq g\ \}$, set $Q^{\mathcal{M}}(\Omega)$ is defined as the following polyhedral set:*

$$Q^{\mathcal{M}}(\Omega) = \{\ x\ :\ RAx - \sum_{i \in \mathcal{M}} |RB_i| \preceq g\ \}$$

From theorem 13 it is clear that $Q_{SNS}(\Omega, K) \subseteq Q^{\mathcal{M}}(\Omega)$ for every K, note that $Q^{\mathcal{M}}(\Omega)$ restrictions are in $Q_{SNS}(\Omega, K)$ when $S = \mathcal{M}$. Consider now the following algorithm,

1. $C_0^{\mathcal{M}}(\lambda) = X$.
2. $C_{j+1}^{\mathcal{M}}(\lambda) = Q^{\mathcal{M}}(\lambda C_j^{\mathcal{M}}(\lambda) \sim E\Theta)$
3. If $C_{j+1}^{\mathcal{M}}(\lambda) = Q^{\mathcal{M}}(C_{j+1}^{\mathcal{M}}(\lambda) \sim E\Theta)$ then $\Gamma^{\mathcal{M}}(\lambda) = C_{j+1}^{\mathcal{M}}(\lambda)$, stop. Else, $j = j + 1$. Go to step (2)

In this algorithm, $\lambda \in (0, 1)$ constitutes a contracting factor.

It is clear that $C_\infty^{SNS}(K, \lambda) \subseteq \Gamma^{\mathcal{M}}(\lambda)$ for every K. That is, set $\Gamma^{\mathcal{M}}(\lambda)$ yields an outer bound of the maximal SNS robust invariant set that can be obtained by means of any gain matrix K. In this sense, it captures the geometry of the greatest robust control invariant set. Thus, it can be said that $\Gamma^{\mathcal{M}}(\lambda)$ is a good starting point when considering the synthesis problem.

6.7 Proposed algorithm

Suppose that $\mathcal{E}(P_L)$ is a robust control invariant set (obtained by means of a control law $u = \sigma(K_L x)$ that does not saturate in $\mathcal{E}(P_L)$). This ellipsoid can be obtained by means of property 10 maximizing a measure of the size of $\mathcal{E}(P_L)$ (for example the trace of P_L^{-1}). Suppose also that $\lambda \in (0,1)$ (λ can be chosen arbitrarily close to 1). In this chapter we proposed the following algorithm to obtain gain matrix K.

1. $\hat{C}_0 = \Gamma^{\mathcal{M}}(\lambda)$, $j = 0$,
2. Obtain $K(j)$ from the following optimization problem:

$$\begin{array}{cl} \max\limits_{K(j),P} & \text{trace}\, P^{-1} \\ s.t. & \mathcal{E}(P) \subset Q_{SNS}(\lambda \hat{C}_j \sim E\Theta, K(j)) \\ & \mathcal{E}(P) \subset \hat{C}_j \\ & \mathcal{E}(P_L) \subseteq \mathcal{E}(P) \\ & P - (A + BK(j))^\top P(A + BK(j)) > 0 \end{array}$$

3. $\hat{C}_{j+1} = \hat{C}_j \bigcap Q_{SNS}(\lambda \hat{C}_j \sim E\Theta, K(j))$
4. If $\hat{C}_{j+1} \subseteq Q_{SNS}(\hat{C}_{j+1} \sim E\Theta, K(j))$ then $K = K(j)$, stop. Else, $j = j + 1$. Go to step 2.

The following property states that the optimization problem required for the implementation of the algorithm can be recast as an LMI optimization problem,

Property 12 *Let us consider that $\hat{C}_j = \{\ x\ :\ Rx \preceq g\ \}$ where $R \in \mathbb{R}^{n_r \times n}$ and $g \in \mathbb{R}^{n_r}$. Then*

1. *The constraint $\mathcal{E}(P) \subset \hat{C}_j$ is equivalent to,*

$$\begin{bmatrix} g_l^2 & R_l W \\ W R_l^\top & W \end{bmatrix} \geq 0, \quad l = 1, \dots, n_r$$

where $W = P^{-1}$, R_l are rows of R and g_l are components of g.

2. $\lambda\hat{C}_j \sim E\Theta = \{\ x\ :\ Rx \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i|\ \}$

3. $Q_{SNS}(\lambda\hat{C}_j \sim E\Theta, K(j)) = \bigcap_{S\in\mathcal{V}} \{\ x\ :\ R(A + \sum_{i\in S^c} B_i K_i(j))x - \sum_{i\in S} |RB_i| \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i|\ \}$

4. *The constraint* $\mathcal{E}(P) \subset Q_{SNS}(\lambda\hat{C}_j \sim E\Theta, K(j))$ *is equivalent to:*

$$\begin{bmatrix} (\lambda g_l + \sum_{i\in S} |R_l B_i| - \sum_{i=1}^{n_w} |R_l E_i|)^2 & R_l AW + \sum_{i\in S^c} R_l B_i Y_i(j) \\ * & W \end{bmatrix} > 0,$$

$$l = 1, \ldots, n_r, \quad \forall S \in \mathcal{V}$$

where $Y_i(j) = K_i(j)W$, $i = 1, \ldots, m$.

5. *The constraint* $\mathcal{E}(P_L) \subseteq \mathcal{E}(P)$ *is equivalent to*

$$\begin{bmatrix} P_L & \mathrm{I} \\ \mathrm{I} & W \end{bmatrix} > 0$$

6. *The constraint* $P - (A + BK(j))^\top P(A + BK(j)) \geq 0$ *is equivalent to*

$$\begin{bmatrix} W & WA^\top + \sum_{i=1}^{m} Y_i^\top(j) B_i^\top \\ * & W \end{bmatrix} > 0$$

PROOF :

The proof will be shown in items, consider $\hat{C}_j = \{\ x\ :\ Rx \preceq g\ \}$,

1. *Constraint* $\mathcal{E}(P)$ *is defined as* $\mathcal{E}(P) = \{\ x\ :\ x^\top Px \leq 1\ \}$*, aplying the change of variable*

$$y = P^{\frac{1}{2}}x,$$

$\mathcal{E}(P)$ *is defined as* $\mathcal{E}(P) = \{\ x\ :\ y = P^{\frac{1}{2}}x, y^\top y < 1\ \}$*, this change of variable can be also applyed in definition of* $\hat{C}_j$*, therefore,* $\hat{C}_j = \{\ x\ :\ y = P^{\frac{1}{2}}x, RW^{\frac{1}{2}}y \preceq g\ \}$*, where* $W = P^{-1}$*. Actually the constraint* $\mathcal{E}(P) \subset \hat{C}_j$ *is equivalent to,*

$$RW^{\frac{1}{2}}y \preceq g \qquad \textit{for all} \ \ \|y\|_2 \leq 1$$

previous restriction can be divided in rows,

$$R_l W^{\frac{1}{2}} y \leq g_l \qquad \textit{for all} \ \ \|y\|_2 \leq 1, \ l = 1, \dots, n_r.$$

The worst case of previous inequality is when y is in the direction of $W^{\frac{1}{2}} R_l^\top$, and therefore inequalities can be written like,

$$0 \leq \frac{R_l W^{\frac{1}{2}} W^{\frac{1}{2}} R_l^\top}{|W^{\frac{1}{2}} R^\top|} \leq g_l, \qquad l = 1, \dots, n_r,$$

note that as $W > 0$, left part of the previous inequality is greater than cero. Taking squares,

$$\frac{R_l W R_l^\top R_l W R_l^\top}{R_l W R_l^\top} \leq g_l^2, \qquad l = 1, \dots, n_r,$$

and finally,

$$g_l^2 - R_l W R_l^\top \geq 0, \qquad l = 1, \dots, n_r,$$

aplying Schur lemma (see 13 in appendix A),

$$\begin{bmatrix} g_l^2 & R_l W \\ W R_l^\top & W \end{bmatrix} \geq 0, \ \ l = 1, \dots, n_r.$$

2. $\lambda \hat{C}_j \sim E\Theta = \{ \ x \ : \ Rx \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i| \ \}.$

   Note that $\lambda \hat{C}_j = \{ \ x \ : \ Rx \preceq \lambda g \ \}$, therefore $\lambda \hat{C}_j \sim E\Theta = \{ \ x \ : R(x + E\theta) \preceq \lambda g, \ \|\theta\|_\infty \leq 1 \ \}$. That is

$$\lambda \hat{C}_j = \{ \ x \ : \ Rx + RE\theta \preceq \lambda g, \quad \forall \|\theta\|_\infty \leq 1 \ \},$$

   $RE\theta$ can be written as sumatory of componets,

$$\lambda \hat{C}_j = \{ \ x \ : \ Rx + \sum_{i=1}^{n_w} RE_i \theta_i \preceq \lambda g, \quad \forall \|\theta\|_\infty \leq 1 \ \},$$

   where E_i are columns of E and θ_i are components of θ. This inequalities can be separated as in previous item,

$$\lambda \hat{C}_j = \{ \ x \ : \ R_l x + \sum_{i=1}^{n_w} R_l E_i \theta_i \leq \lambda g_l, \quad \forall \|\theta\|_\infty \leq 1, \ l = 1, \dots, n_r \ \},$$

where R_l are rows of R. Note that as all inequalities must hold, value of θ that provides the worst condition is different for each inequality. Note also that $R_l E_i \in \mathbb{R}$ and $\theta_i = \pm 1$, therefore worst condition for each inequality is such that $R_l E_i \theta_i = |R_l E_i|$, that is,

$$\lambda \hat{C}_j = \{ \ x \ : \ R_l x + \sum_{i=1}^{n_w} |R_l E_i| \leq \lambda g_l, \quad l = 1, \dots, n_r \ \},$$

and joinning all inequalities,

$$\lambda \hat{C}_j \sim E\Theta = \{ \ x \ : \ Rx \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i| \ \}.$$

3. $Q_{SNS}(\lambda \hat{C}_j \sim E\Theta, K(j)) = \bigcap_{S \in \mathcal{V}} \{ \ x \ : \ R(A + \sum_{i \in S^c} B_i K_i(j))x - \sum_{i \in S} |RB_i| \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i| \ \}$

   In theorem 13 it have been shown that

$$Q_{SNS}(\Omega, K) = \bigcap_{S \in \mathcal{V}} \{ \ x \ : \ R(A + \sum_{i \in S^c} B_i K_i)x - \sum_{i \in S} |RB_i| \preceq g \ \}$$

   and by subtitution of definition of $\lambda \hat{C}_j \sim E\Theta$ of previous item,

$$Q_{SNS}(\lambda \hat{C}_j \sim E\Theta, K(j)) =$$
$$= \bigcap_{S \in \mathcal{V}} \{ \ x \ : \ R(A + \sum_{i \in S^c} B_i K_i(j))x - \sum_{i \in S} |RB_i| \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i| \ \}.$$

4. *Previous item shows that the constraint $\mathcal{E}(P) \subset Q_{SNS}(\lambda \hat{C}_j \sim E\Theta, K(j))$ is equivalent to*

$$\mathcal{E}(P) \subset \bigcap_{S \in \mathcal{V}} \{ \ x \ : \ R(A + \sum_{i \in S^c} B_i K_i(j))x - \sum_{i \in S} |RB_i| \preceq \lambda g - \sum_{i=1}^{n_w} |RE_i| \ \}.$$

   This constraint is similar to the first item one, hence, with similar analysis it is obtained that this constraint is equivalent to

$$\begin{bmatrix} (\lambda g_l + \sum_{i \in S} |R_l B_i| - \sum_{i=1}^{n_w} |R_l E_i|)^2 & R_l A W + \sum_{i \in S^c} R_l B_i Y_i(j) \\ * & W \end{bmatrix} > 0,$$

$$l = 1, \dots, n_r, \quad \forall S \in \mathcal{V}$$

   where $Y_i(j) = K_i(j) W$, $i = 1, \dots, m$.

5. *$\mathcal{E}(P_L) \subseteq \mathcal{E}(P)$ if and only if $P_L \geq P$. Using the Schur complement,*

$$\begin{bmatrix} P_L & \mathrm{I} \\ \mathrm{I} & W \end{bmatrix} \geq 0.$$

6. *The constraint $P - (A + BK(j))^\top P(A + BK(j)) \geq 0$ is equivalent to $WPW - W(A + BK(j))^\top P(A + BK(j))W \geq 0$, hence, $W - (AW + BY(j))^\top P(AW + BY(j)) \geq 0$ and by application of Schur lemma 13 in appendix A,*

$$\begin{bmatrix} W & WA^\top + \sum_{i=1}^{m} Y_i^\top(j)B_i^\top \\ * & W \end{bmatrix} > 0.$$

■

Previous property shows that the sequence $\hat{C}_j$ is monotonically decreasing. Moreover, the LMI problem is always feasible (setting λ arbitrarily close to 1) as $K = K_L$ and $P = P_L$ constitutes a feasible solution to the problem. Therefore, the sequence converges and a domain of attraction containing $\mathcal{E}(P_L)$ is obtained. In this way, the proposed algorithm potentially yields a better control.

6.8 Numerical example

To illustrate the proposed method, the following uncertain system is considered

$$x^+ = Ax + Bu + E\theta,$$

where

$$A = \begin{bmatrix} -0.9554 & -1.2250 \\ 0.6014 & -1.0125 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$$

and $\theta \in \Theta = \{\ \theta\ :\ \|\theta\|_\infty \leq 1\ \}$ and $u \in U = \{\ u\ :\ \|u\|_\infty \leq 1\ \}$. The states of the system must be confined in the polyhedron $X = \{\ x \in \mathbb{R}^n\ :\ \|x\|_\infty \leq 3\ \}$.

First, the maximal ellipsoidal robust invariant set for a non-saturating linear control law is calculated using the result of property 1. Thus the following matrices are obtained

$$P_L = \begin{bmatrix} 1.6662 & -1.0249 \\ -1.0249 & 1.9645 \end{bmatrix}$$

$$K_L = \begin{bmatrix} 0.7830 & 0.4364 \end{bmatrix}$$

The ellipsoid $\mathcal{E}(P_L)$ is therefore robust invariant under the control law $u = \sigma(K_L x)$.

The proposed algorithm to design the saturated controller begins with the calculation of set $\Gamma^{\mathcal{M}}(\lambda)$, where the contraction factor is taken as $\lambda = 0.98$. The obtained sequence of sets $C_j^{\mathcal{M}}(\lambda)$ is shown in figure 6.1 where it can be seen that it converges to a polyhedral set $\Gamma^{\mathcal{M}}(\lambda)$.

Using this polyhedron as initial set, the iterative procedure to compute the gain matrix K is executed. This requires the solution of an *LMI* optimization problem at each iteration. In two iterations the algorithm converges to the solution. The obtained Lyapunov matrix P and controller gain matrix K are the following:

$$P = \begin{bmatrix} 0.6957 & -0.3541 \\ -0.3541 & 1.1287 \end{bmatrix}$$

$$K = \begin{bmatrix} 1.2154 & 0.7872 \end{bmatrix}$$

The sequence of sets $\hat{C}_j$ and ellipsoids $\mathcal{E}(P)$ for each iteration are shown in figure 6.2 in solid line; the maximal robust invariant ellipsoid corresponding to the ellipsoid-based controller law $u = \sigma(K_L x)$ is depicted in dashed line.

Figure 6.3 shows the robust *SNS*-invariant set obtained by means of the proposed algorithm. This domain of attraction is compared with the maximal robust invariant ellipsoid obtained with a non-saturated linear control law. As it can be observed, the new synthesis approach clearly improves the results obtained when the controller is calculated by means of non-saturated control laws.

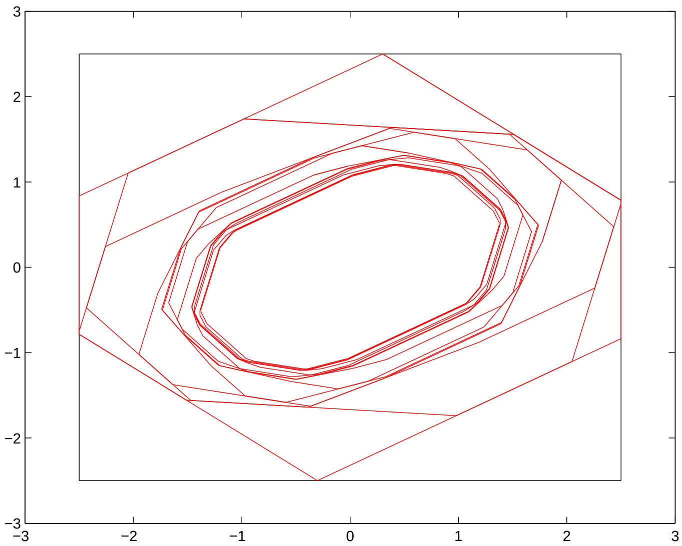

Figure 6.1: Sequence of sets $C_j^{\mathcal{M}}(\lambda)$

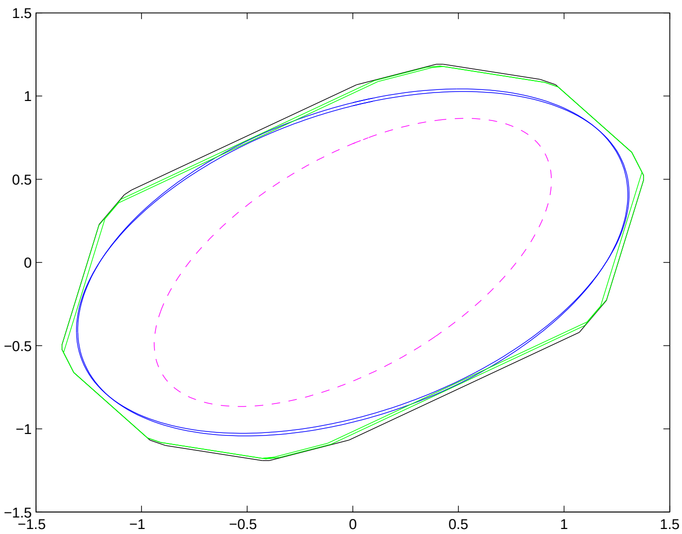

Figure 6.2: Sequence of sets $\hat{C}_j$ and ellipsoids $\mathcal{E}(P)$

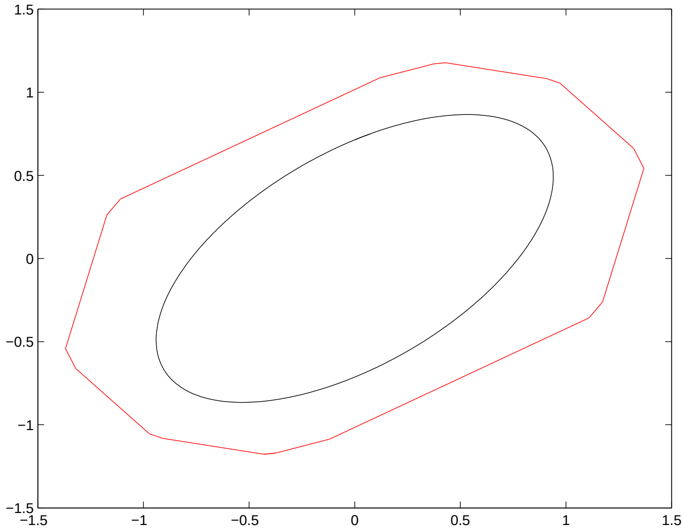

Figure 6.3: Robust SNS-invariant set

6.9 Conclusions

In chapters 4, a new method to obtain a conservative aproximation of the domain of attraction was presented. This approximation includes the estimation obtained by means of an LDI that it was shown in chapter 3. In this chapter it is shown how saturated control laws yield to greater domain of attractions when polyhedric invariant sets are considered. That is, the algorithm proposed in this chapter provides a controller with a domain of attraction that contains any pre-specified ellipsoidal control invariant set obtained by means of a non saturated control law.

7

Application of LNL invariance for Lur'e systems

7.1 Introduction

The importance of Lur'e systems in the context of control theory stems from the fact that different control schemes appearing in practical applications can be formulated using the Lur'e systems structure [49].

In this chapter, Lur'e systems in which the non-linearity appearing in the feedback path has a piecewise affine nature will be analized.

The stability analysis of a Lur'e system can be done, for example, by means of Popov and circle criterions (see [53]). A novel approach to this problem can be found in [31] where a procedure to compute invariant ellipsoids for Lur'e systems with piecewise affine nonlinearity is given.

In this chapter, a new notion of invariance (*LNL* invariance) very related to the notion of *SNS* invariance presented in chapters 4 and 5 is used. Based on its geometrical properties, a simple algorithm to obtain the largest *LNL*-invariant set is proposed. *LNL*-invariance is a more conservative concept than traditional invariance, but its geometrical properties allows us to obtain a polyhedric estimation of the domain of attraction of the non-linear system. It is shown that any contractive domain of attraction for the Lur'e system is included into the domain of attraction provided by the application of the algorithm proposed in this chapter.

7.2 Problem statement

In chapter 4, the discrete-time system envolved was 4.1. This system can be extended to the more general Lur'e systems.

Consider the following discrete-time Lur'e system:

$$\begin{cases} x_{k+1} &= Ax_k - B\phi(y_k) \\ y_k &= Fx_k \end{cases} \tag{7.1}$$

where $x_k \in \mathbb{R}^n$ represents the state vector and $y_k = Fx_k \in \mathbb{R}$ the output of the system. The nonlinear function $\phi(\cdot)$ is assumed to satisfy the following conditions

(i) $\phi(y)$ is piecewise-affine.

(ii) $\phi(y)$ is a continuous odd function.

(iii) $\phi(y)$ is concave in $\mathbb{R}^+$ (convex in $\mathbb{R}^-$).

The following property characterizes all the functions $\phi(.)$ that satisfy the previous assumptions.

Property 13 *[31] The piecewise-affine function $\phi(y)$ is a continuous odd func-*

tion, concave in $\mathbb{R}^+$ *if and only if it can be expressed as*

$$\phi(y) = \begin{cases} k_N y - c_N & \text{if } y \in (-\infty, -b_N) \\ \vdots & \\ k_1 y - c_1 & \text{if } y \in [-b_2, -b_1] \\ k_0 y & \text{if } y \in [-b_1, b_1) \\ k_1 y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_N y + c_N & \text{if } y \in [b_N, \infty) \end{cases} , \quad \forall y \geq 0 \tag{7.2}$$

where the scalars k_i, $i = 0, \ldots, N$, b_i, $i = 1, \ldots, N$ *and* c_i, $i = 1, \ldots, N$ *satisfy,*

$$0 < b_1 < b_2 < \ldots < b_N$$

$$k_0 > k_1 > k_2 > \ldots > k_N$$

$$c_i = \begin{cases} (k_0 - k_1)b_1 & \text{if } i = 1 \\ c_{i-1} + (k_{i-1} - k_i)b_i & \text{if } 2 \leq i \leq N \end{cases}$$

See figure 7.1 for an example of piecewise-affine concave in $\mathbb{R}^+$ function with $N = 3$.

Note that the results presented in this chapter can also be applied to systems of the form:

$$x_{k+1} = \hat{A}x_k - \hat{B}\hat{\phi}(y_k)$$

where $\hat{\phi}(\cdot)$ is an odd piecewise-affine function convex in $\mathbb{R}^+$ (it suffices to define $\phi(\cdot) = -\hat{\phi}(\cdot)$, $A = \hat{A}$ and $B = -\hat{B}$).

7.3 Analysis of the non-linear function

In the following some properties of function $\phi(\cdot)$ are presented. For that purpose the following definition is introduced.

Definition 36 *Given the piecewise-affine odd function:*

$$\phi(y) = \begin{cases} k_0 y & \text{if } y \in [0, b_1) \\ k_1 y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_N y + c_N & \text{if } y \in [b_N, \infty) \end{cases} , \quad \forall y \geq 0,$$

the odd functions $\phi_i(y)$, $i = 1, \ldots, N$ *are defined as:*

$$\phi_i(y) = \begin{cases} k_0 y & \text{if } y \in [0, d_i) \\ k_i y + c_i & \text{if } y \in [d_i, \infty) \end{cases} , \quad \forall y \geq 0, \tag{7.3}$$

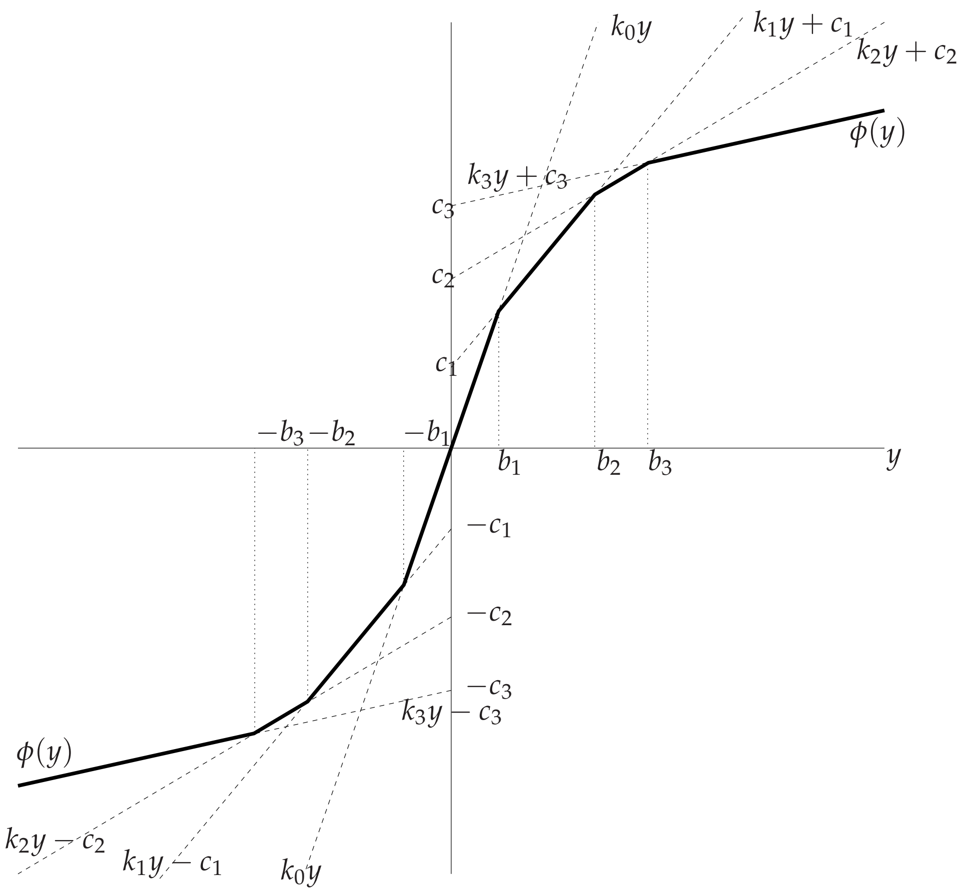

Figure 7.1: An example of a piecewise-affine concave in $\mathbb{R}*+$ function $\phi(\cdot)$.

where $d_i = \dfrac{c_i}{k_0 - k_i}$, $i = 1, \ldots, N$.

Note that they are define in $\mathbb{R}^+$ for clarity, but as long as they are odd functions, they are defined in all $\mathbb{R}$. That is, $\phi(x) = -\phi(-x)$ and $\phi_i(x) = -\phi_i(-x)$ for $i = 1, \ldots, N$.

Figure 7.2 depicts functions $\phi_i(\cdot)$, $i = 1, \ldots, 3$ for the function $\phi(\cdot)$ corresponding to figure 7.1.

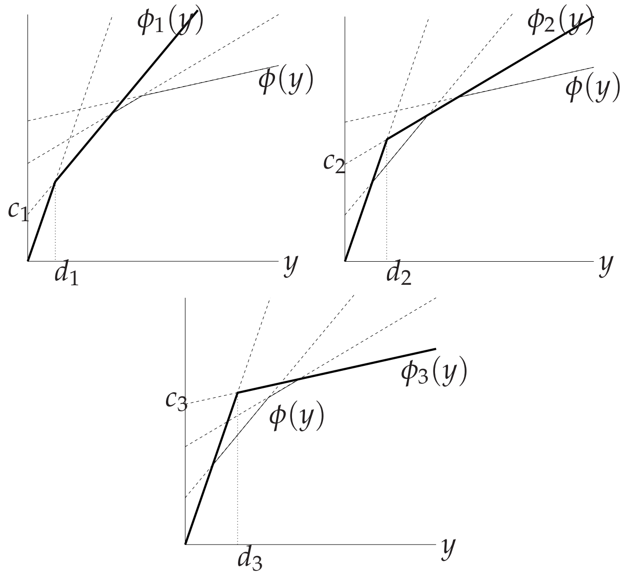

Figure 7.2: Three $\phi_i(\cdot)$ functions related to the previous $\phi(\cdot)$ function.

The following lemma shows how function $\phi(\cdot)$ can be expressed in terms of functions $\phi_i(\cdot)$, $i = 1, \ldots, N$.

Lemma 7 *[31] Suppose that $\phi(\cdot)$ is an odd piecewise-affine function concave in $\mathbb{R}^+$. Then:*

$$\phi(y) = \min_{1 \leq i \leq N} \phi_i(y), \quad \forall y \geq 0$$
$$\phi(y) = \max_{1 \leq i \leq N} \phi_i(y), \quad \forall y < 0$$

PROOF :

The proof comes from the convex property of the $\phi(\cdot)$ function. Firstly the proof will show that $\forall y \geq 0$, $\phi(y) = \phi_i(y)$ for some $i \leq N$, later it will be shown that there is no j such that $\phi_j(y) < \phi(y)$.

Let $x \geq 0 \in \mathbb{R}$, $j \leq N$ such that $b_j \leq x \leq b_{j+1}$, then, taking in notice that $b_i \geq d_i$, the equation $\phi(y) = k_i u + c_i = \phi_j(y)$ holds.

In the other hand, let us suppose that there exists $x \in$ IR and $j \leq N$ such that $b_j \leq x \leq b_{j+1}$, and $\phi_j(x) < \phi(x)$. If $x \leq d_j$, by the definition of $\phi_j(\cdot)$, $\phi_j(x) = k_0 x$, that is a line tangent to $\phi(y)$ function for all $y \in [0, b_1]$, therefore, by the convexity property of $\phi(\cdot)$, $\phi(x) \leq \phi_j(x)$. Otherwise, if $x > d_j$, the value of $\phi_j(\cdot)$ is defined as $\phi_j(x) = k_j x + c_j$, that is a line tangent to $\phi(y)$ function for all $y \in [b_j, b_{j+1}]$ (or $y \in [b_N, \infty]$ if $j = N$), and, by the convexity property of $\phi(\cdot)$, $\phi(x) \leq \phi_j(x)$.

In other words, $\phi(x) \leq \phi_j(x)$, which contradicts $\phi_j(x) < \phi(x)$. Therefore, $\phi(x) \leq \phi_j(x)$ for all $j = 1, \ldots, N$.

Similar analysis can be done if $y < 0$ and this proves the claim. ■

The previous lemma can also be justified from a graphical point of view (see figure 7.2). It can be observed in that figure that $\phi(\cdot)$ can be obtained from the minimum of $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$.

7.4 The LNL-invariance notion

The SNS-invariance notion has been presented in chapter 4. This concept can be extended to LNL-invariance. This new notion of invariance is stronger than the classical one. However, the LNL-invariance enjoys from a number of geometrical properties that makes it possible the computation of the greatest LNL-invariance set by means of a simple algorithm. Moreover, as it will be shown in this later in this chapter, every contractive convex set for the non-linear system under study is contained into the greatest LNL-invariant set provided by the proposed algorithm.

Definition 37 *Consider system $x_{k+1} = Ax - B\phi(Fx)$ and let function $\phi(\cdot)$ be defined as in equation (7.2), $f(x)$ and $f_L(x)$ are defined as:*

$$\begin{array}{ll} f(x) & = Ax - B\phi(Fx) \\ f_L(x) & = Ax - Bk_0 Fx \end{array} \tag{7.4}$$

The notion of LNL-invariance is introduced in the following definition:

Definition 38 *A set Ω is said to be LNL-invariant for system $x_{k+1} = Ax - B\phi(Fx)$ if $x \in \Omega$ implies:*

$$\begin{array}{lll} f(x) & = Ax - B\phi(Fx) & \in \Omega \\ f_L(x) & = Ax - Bk_0 Fx & \in \Omega \end{array}$$

This concept is stronger than simple invariance, that is, if Ω is LNL-invariant it is also invariant, but a invariant set may not be LNL-invariant.

LNL stands for $Linear$ and $Non - Linear$. Note that the new constraint $f_L(x) \in \Omega$ added to LNL-invariance is not a very strong condition as there is a neighbourhood of the origin where $f(x)$ equals $f_L(x)$.

The next definition shows the admissible sequence concept. This concept is similar to one defined on definition 12 on chapter 3 or definition 24 on chapter 4 but adapted to this special case of $m = 1$.

Definition 39 *It is said that $S_0, S_1, \ldots, S_k$ is an admissible sequence if $S_i \in \{1, -1\}$, $i = 0, \ldots, k$.*

Definition 40 *Given x and $S \in \{1, -1\}$, function $G(x, S)$ is defined as follows,*

$$G(x,S) = \begin{cases} f(x) & \text{if} \quad S = 1 \\ f_L(x) & \text{if} \quad S = -1 \end{cases}$$

Definition 41 *It is said that x belongs to the LNL-domain of attraction of system $x_{k+1} = Ax - B\phi(Kx)$ if the recursion:*

$$x_{k+1} = G(x_k, S_k), \quad x_0 = x$$

converges to the origin for every admissible infinite sequence $\{S_0, S_1, S_2, \ldots\}$.

7.4.1 The one step function

Definition 42 *Given a set Ω, the one step function for system $x_{k+1} = Ax_k - B\phi(Fx_k)$ is defined as,*

$$Q(\Omega) = \{ \ x \ : \ Ax_k - B\phi(Fx_k) \in \Omega \ \}. \tag{7.5}$$

Given functions $\phi_i(\cdot)$, $i = 1, \ldots, N$, defined as in equation (7.3), sets $Q_i(\cdot)$, $i = 1, \ldots, N$ are defined as:

$$Q_i(\Omega) = \{ \ x \ : \ Ax - B\phi_i(Fx) \in \Omega \ \}.$$

$Q(\cdot)$ is a non-convex operator, hence, it is very difficult to operate due to the computational caracterization method. To avoid this problem, a conservative operator $Q_{LNL}(\cdot)$ will be used,

$$Q_{LNL}(\Omega) = \{x : \text{ such that} \quad Ax - B\phi(Fx) \in \Omega \quad \text{and} \quad Ax - Bk_0Fx \in \Omega\}.$$

Although this operator is non-convex at first glance due to the non convexity function $\phi(\cdot)$, the following theorem will show that it is, in fact, convex.

The convexity theorem consists in showing that $Q_{LNL}(\cdot)$ operator is in fact the same than a new convex operator $P(\cdot)$.

Definition 43 *Let Ω be a polyhedrical set, i.e. $\Omega = \{x : \quad Hx \leq g\}$. The operators $P_i(\cdot)$ and $P(\cdot)$ are defined as,*

$$
\begin{aligned}
P_i(\Omega) &= \{x : \textit{ such that} \quad \begin{array}{l} H(A - Bk_0F)x \leq g \\ H(A - Bk_iF)x \leq g + |c_iHB|\} \end{array} \\
P(\Omega) &= \bigcap_{i=1}^{N} P_i(\Omega).
\end{aligned}
$$

The following lemma will be used to prove that property,

Lemma 8 *Every $\phi_i(\cdot)$ defined like 7.3 has the following property,*

$$a\phi_i(b) \leq max(ak_0b, ak_ib - |ac_i|)$$

PROOF :

There are two different possibilities.

If $|b| \leq d_i$, then the value of $\phi_i(b) = k_0b$, hence, the lemma holds.

Otherwise, if $|b| > d_i$, it can be obtained $\phi_i(b) = k_ib + sign(b)c_i$ where $k_i < k_0$ and $c_i > 0$. Note that $c_i > 0$ is a direct consequence of continuous condition of 7.3 and that $d_i \geq 0$.

There are now four different possibilities:

- $a > 0$ and $b > d_i$. The value is $a\phi_i(b) = ak_ib + ac_i < ak_0b$. Note that $k_0b \geq k_ib + c_i$, $\forall b > d_i$ due to concave definition of 7.3.
- $a > 0$ and $b < -d_i$. The value is $a\phi_i(b) = ak_ib - ac_i = ak_ib - |ac_i|$.
- $a < 0$ and $b > d_i$. The value is $a\phi_i(b) = ak_ib + ac_i = ak_ib - |ac_i|$.
- $a < 0$ and $b < -d_i$. The value is $a\phi_i(b) = ak_ib - ac_i < ak_0b$.

For all cases the lemma holds. ■

Following theorem shows operators $Q_{LNL,i}(\cdot)$ and $P_i(\cdot)$ are equivalent with some conditions.

Theorem 14 *Let Ω be a polyhedrical set in the form $\Omega = \{x : \; Hx \leq g\}$. x belongs to $Q_{LNL,i}(\Omega)$ if and only if x belongs to $P_i(\Omega)$.*

Proof :

Let us suppose that there exists a value of $x \notin Q_{LNL,i}(\Omega)$ such that $x \in P_i(\Omega)$, that is, there exists a value of j such that $H_j(Ax - B\phi_i(Fx)) \leq g_j$. The direct part of the proof consists in showing that $x \notin P_i(\Omega)$.

Using the lemma 8,

$$H_j(Ax - B\phi_i(Fx)) \leq \\ \leq H_jAx + max(H_jBk_0Fx, H_jBk_iFx - |H_jBc_i|).$$

There are two possible cases. If $H_jBk_0Fx > H_jBk_iFx - |H_jBc_i|$, then

$$g_j < H_j(Ax - B\phi_i(Fx)) \leq \\ \leq H_jAx + H_jBk_0Fx = H_j(A - Bk_0F)x.$$

This contradicts that $H(A - Bk_0F)x \leq g$ in the definition of $P(\Omega)$.

In case that $H_jBk_0Fx < H_jBk_iFx - |H_jBc_i|$, the previous expression leads to,

$$g_j < H_j(Ax - B\phi_i(Fx)) \leq H_jAx + H_jBk_1Fx - |H_jBc_i|,$$

that contradicts that $H(A - Bk_1F)x \leq g + |c_iHB|$ in the definition of $P_i(\Omega)$.

There is no $x \notin Q_i(\Omega)$ such that $x \in P_i(\Omega)$. That proves the first part of the claim.

To conclude the proof it will be shown that $Q_{LNL,i}(\Omega \subseteq P_i(\Omega)$. That is, due to the fact that $-|c_iHB| \preceq HB\phi(Fx)$,

$$HAx - |c_iHB| \preceq H(Ax - B\phi(Fx)) \preceq g.$$

■

The equality between $Q_{LNL,i}(\cdot)$ and $P_i(\cdot)$ operators has been shown. This property is also hold for $Q_{LNL}(\cdot)$ and $P(\cdot)$ operators, and will be shown in the theorem 15. Prior to this, a new lemma are needed.

Lemma 9 *Let Ω be a polyhedrical set in the form $\Omega = \{x : \ Hx \leq g\}$, then*

$$Q_{LNL}(\Omega) = \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$$

PROOF :

First it will be shown that $Q_{LNL}(\Omega) \subseteq \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$. Let us suppose that $x \in Q_{LNL}(\Omega)$ and $Fx \geq 0$. Then

$$\begin{array}{l} Ax - B\phi(Fx) \in \Omega \\ Ax - Bk_0Fx \in \Omega. \end{array}$$

Note that by lemma 7, $\phi(Fx) \leq \phi_i(Fx) \leq k_0Fx$ for all i in $1, \ldots, n$. From this and the fact that Ω is a convex set, it can be shown that

$$Ax - B\phi_i(Fx) \in \Omega,$$

for all $i = 1, \ldots, n$. That is, $x \in Q_{LNL,i}(\Omega)$.

To prove that $\bigcap_{i=1}^{N} Q_{LNL,i}(\Omega) \subseteq Q_{LNL}(\Omega)$ let us suppose that $x \in \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$ and $Fx \geq 0$. Let j be such that

$$\phi_j(Fx) = min(\phi_i(Fx) : \ \ i \in 1, \ldots, N).$$

then as $x \in Q_{LNL,j}(\Omega)$, it is obtained that

$$Ax - B\phi_i(Fx) = Ax - B\phi(Fx) \in \Omega.$$

Therefore this and the fact that $Ax - Bk_0Fx \in \Omega$ leads to $x \in Q_{LNL}(\Omega)$, that proves the claim.

Similar analysis can be made if $Fx < 0$. ∎

Theorem 15 *Let Ω be a polyhedrical set in the form $\Omega = \{x : \ Hx \leq g\}$. x belongs to $Q_{LNL}(\Omega)$ if and only if x belongs to $P(\Omega)$.*

PROOF :

The proof is direct application of theorem 14, previous lemma and definition of $P(\cdot)$. ∎

7.5 LNL-domain of attraction

In this section, it is proposed a recursion and shown it properties that allows to create an algorithm to obtain an LNL-invariant set that it is also LNL-domain of attraction. This invariant set is cartacterized by a polyhedral.

Theorem 16 *Denote $L(F)$ the region of linear behaviour of system 7.1, that is, $L(F) = \{x \in \mathbb{R}^n : |Fx| \leq b_1\}$. Suppose that $\Phi \in L(F)$ is an invariant set for the asymptotically stable system $x^+ = (A - BF)x$ with non zero volume. Denote now $C_0 = \Phi$ and consider the following recursion:*

$$C_{k+1} = Q_{LNL}(C_k).$$

Then:

1. *C_k for all $k \geq 1$ is a convex polyhedron.*
2. *C_k is a LNL-invariant set and belongs to the LNL-domain of attraction of system 7.1, $\forall k \geq 0$.*
3. *The sequence $\{C_0, C_1, \ldots\}$ converges to the LNL-domain of attraction of system 7.1.*
4. *The LNL-domain of attraction of system 7.1 is a convex set.*

PROOF :

1. This is inferred to the fact that C_0 is a convex polyhedron and theorem 16. Note that in the recursion law, the $Q_{LNL}(\cdot)$ operator is used, $Q_{LNL}(\cdot) = P(\cdot)$ by theorem 16 and $P(\cdot)$ is a convex operator.

2. Suppose that C_k is an LNL-invariant set and belong to the LNL-domain of attraction. Note that, by LNL-invariant set properties, $C_k \subseteq Q_{LNL}(C_k)$, that is, $C_k \subseteq C_{k+1}$. Therefore, C_{k+1} is an LNL-invariant set.

   Let us suppose that $x \in C_{k+1}$, using the recursion it is obtained that $x^+ \in C_k$, for systems 7.1 and

   $$\left\{ \begin{array}{c} x_{k+1} = Ax_k - Bmin_i(\phi_i(y_k)) = Ax + B\phi(y_k) \\ y_k = Fx_k \end{array} \right. .$$

   Therefore, by the supposition that C_k belongs to the LNL-domain of attraction, it is inferred that C_{k+1} belongs to the LNL-domain of attraction.

   This and the fact that C_0 is an LNL-invariant set and belongs to the LNL-domain of attraction holds the claim.

3. Suppose that x belongs to the LNL-domain of attraction. Then as C_0 has a non zero volume, there exists p such that all recursions $x_{k+1} = G_k(x_k)$ where G_k defined in 41 are such that $x_p \in C_0$. Therefore, $x \in C_j, \forall j \geq p$.

4. This is directly inferred of the fact that $C_k, \forall k \geq 0$ is a convex set, and that the sequence $\{C_0, C_1, \ldots,\}$ converges to the LNL-domain of attraction.

■

The recursion presented in the previous theorem, requires an invariant set of the linear system $x^+ = (A - Bk_0F)x$, included in $L(F)$. This admissible invariant set can be obtained by standard algorithms (see [20, 17]).

7.6 Numerical example

In this section a LNL-invariant set for a numerical example is obtained. This set will be compared with a new approach shown in [32].

Let us consider the system $x^+ = Ax - B\phi(Fx)$ with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \qquad F = [-0.6167 \; -1.2703]. \tag{7.6}$$

and $\phi(\cdot)$ function defined as

$$\phi(y) = \begin{cases} -1, & \text{if } y \in (-\infty, -1.5) \\ 0.5y - 0.25, & \text{if } y \in [-1.5, 0.5) \\ y, & \text{if } y \in [-0.5, 0.5) \\ 0.5y + 0.25, & \text{if } y \in [0.5, 1.5) \\ 1, & \text{if } y \in [1.5, \infty) \end{cases} . \tag{7.7}$$

Figure 7.3 shows the $\phi(\cdot)$ function.

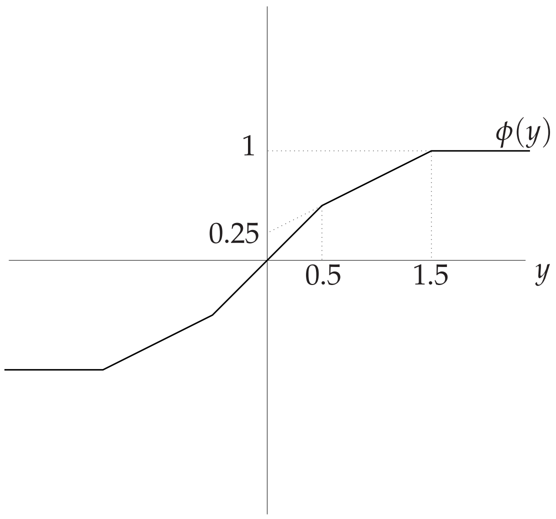

Figure 7.3: $\phi(\cdot)$ function of the example

Theorem 7.5 shows a sequence of polihedrals that are both *LNL*-invariant set and *LNL*-domain of attraction. This sequence has been calculated for system 7.6 and it is shown on figure 7.4.

In that figure, the inner set is a invariant set of the linear system that it is in the linear behaviour of the control law, and the outside set is the *LNL*-domain of attraction of the system.

This is not the only method to determinate invariant sets for piecewise-affine feedback systems. In [32], the authors propose an algorithm to determinate ellipsodial invariant sets for saturated feedback systems that had been generalized in [12] to this type of systems. The polyhedrical set

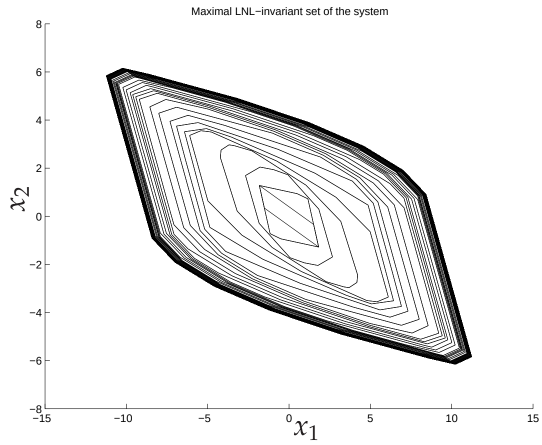

Figure 7.4: *LNL*-invariant set of the system

shown in this chapter is always greater than the ellipsodial, but at the risk of a greater caracterization complexity.

Figure 7.5 shows the comparison between the ellipsodial invariant set proposed in [32] (the inner region), and the polyhedrical *LNL*-invariant set proposed (the outer polyhedrum).

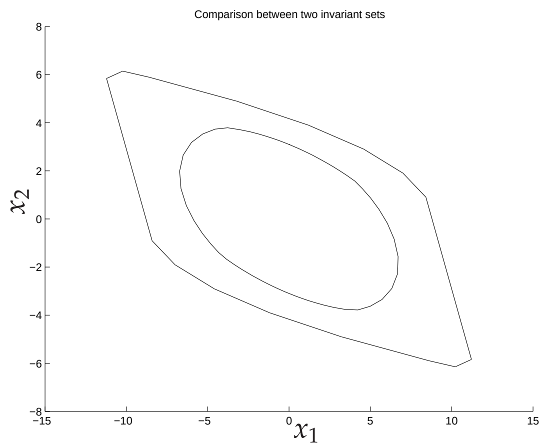

Figure 7.5: *LNL*-invariant set and ellipsodial invariant set.

In order to obtain an invariant set for this class of Lur'e systems, it will be firstly considered the simplified scenario in which the nonlinearity of the feedback path is given by means of an odd piecewise-function, concave in $\mathbb{R}^+$, consisting of only two different slopes in $\mathbb{R}^+$. That is, it will be considered systems of the form:

$$\begin{cases} x_{k+1} & = Ax_k - B\phi_i(y_k) \\ y_k & = Fx_k \end{cases} \tag{7.8}$$

where

$$\phi_i(y) = \begin{cases} k_0 y & \text{if } y \in [0, d_i) \\ k_i y + c_i & \text{if } y \in [d_i, \infty) \end{cases}, \quad \forall y \geq 0,$$

and $\phi_i(-y) = -\phi_i(y)$.

Theorem 17 *A convex set Ω scaled-invariant for system 7.1, that is, $\alpha\Omega$ for all $0 < \alpha \leq 1$ is an invariant set for system 7.1, if and only if Ω is also a scaled invariant set for systems 7.8 for all $i = 1, \ldots, N$.*

PROOF :

Let Ω be an convex scaled-invariant set for system 7.1. Let also β be

$$\beta = max(Fx : \quad \forall x \in \Omega).$$

and α be

$$\alpha = min(\frac{b_1}{\beta}, 1).$$

The definition of α is such that $0 < \alpha \leq 1$, therefore, $\alpha\Omega$ is an invariant set, and $|Fx| \leq b_1$ for all $x \in \alpha\Omega$. This means that $\phi(x) = k_0 x$ for all $x \in \alpha\Omega$. This linear feedback function can be extended thoughout a change of variable to Ω, that is, Ω is an invariant set for closed loop system

$$\begin{cases} x_{k+1} = Ax_k - Bk_0 y_k \\ y_k = Fx_k \end{cases}$$

(with similar analysis $\gamma\Omega$ for all $0 < \gamma \leq 1$ is also an invariant set) .

As it has been shown in this section, $k_0 x \geq \phi_i(x) \geq \phi(x)$ for $x \in \mathbb{R}$, and due to the convexity and invariance of Ω set, Ω (and therefore $\gamma\Omega$ for all $0 < \gamma \leq 1$) is also invariant for system 7.1, that holds the right part of the claim.

The left part of the proof is simple, let Ω be an scaled invariant set for all systems 7.8 for all $i \in 1, \ldots, N$, then Ω is also a scaled invariant for the system

$$\begin{cases} x_{k+1} = Ax_k - Bmin_i(\phi_i(y_k)) = Ax - B\phi(y_k) \\ y_k = Fx_k \end{cases},$$

that holds the claim. ∎

This invariant set is widely used in order to use the Minkowski function of Ω as a Lyapunov function of the system [6]. In this case a new λ-contractive condition must be assumed, that can be easily added to this formulation.

7.7 Conclusions

In this chapter the stabilization of a piecewise-affine Lur'e system has been considered. A simple algorithm for determining an estimation of the domain of attraction and an invariant set of the system is provided. It has also been exposed that invariant sets obtained by means of this algorithm belong to an special and more conservative class of invariants denoted LNL-invariants. This invariants are convex and the conservativeness can be reduced with some assumptions on the system.

8

Robust control invariant set

8.1 Introduction

In the context of nonlinear *Model Predictive Control* (*MPC*), the stable and admissible closed-loop behavior is typically based on the addition of a terminal constraint and cost [44, 39]. The terminal constraint is chosen to be an admissible robust control invariant set of the system. The size of this control invariant set determines, in much cases, the feasibility region of the nonlinear *MPC* controller [40]. It is shown in [40] that the domain of attraction of *MPC* controllers can be enlarged by means of a sequence of controllable (not necessarily invariant) sets.

The stability analysis of piecewise affine systems plays an important role in the context of hybrid systems control. This is mainly due to the fact that piecewise affine systems can model a broad class of hybrid systems (see [26, 46]). Therefore, it is of paramount relevance in the context of hybrid *MPC* the computation of controllable sets for this class of nonlinear systems.

The estimation of the domain of attraction of piecewise affine systems has been addressed by a number of authors. Piecewise quadratic Lyapunov functions for hybrid systems have been proposed in [35]. In [34] a quadratic estimate of the domain of attraction is obtained. A polyhedric approach is presented in [36].

It is well-known (see again [36]) that the computation of the maximal robust control invariant set for a piecewise affine system requires a computational complexity that grows exponentially with the number of partitions of the state space. In this chapter an algorithm that circumvents the huge computational complexity associated to the obtainment of the maximal robust control invariant set is provided. A two-phase algorithm is proposed. In the first phase of the algorithm, a polyhedric outer bound of the maximal control invariant set for the piecewise affine system is obtained. This outer estimation is used, in the second phase of the algorithm, to obtain a robust control invariant set for the system (not necessarily the maximal one). The algorithm is based on inner and outer approximations of a given non-convex set.

8.2 Problem Statement

Let us suppose that X is a bounded convex polyhedron. Suppose also that the convex polyhedrons X_i, $i = 1, \ldots, r$ form a partition of X. That is,

$$X = \bigcup_{i=1}^{r} X_i, \quad X_i \bigcap X_j = \emptyset \text{ if } i \neq j$$

We consider the following piecewise affine system,

$$x^+ = f(x, u, w) = A_i x + B_i u + E_i w + q_i \text{ if } x \in X_i \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the state vector, x^+ denotes the successor state; $u \in U = \{ u \in \mathbb{R}^m : \|u\|_\infty \leq u_{max} \}$ is the control input; w denotes a bounded additive uncertainty: $w \in W = \{ w \in \mathbb{R}^{n_w} : \|w\|_\infty \leq \epsilon \}$.

The objective of this chapter consists in providing a procedure to obtain a convex outer approximation of the maximal robust control invariant set of the piecewise affine system (denoted C_∞). This outer bound has a number of practical and relevant applications,

(i) It captures the geometry of C_∞ and makes it easier the computation of a robust control invariant set for the system (this use is explored in section 8.5).

(ii) The constraints that define the outer bound can be included as hard constraints in an hybrid MPC scheme. Moreover, the inclusion of the afore mentioned constraints can be used to improve the convex relaxations of the nonlinear optimization problems that appear in the context of hybrid MPC.

(iii) The outer bound can be used as a measure of the controllable region of an hybrid system. This can be used in the design of the hybrid system itself.

(iv) The obtained convex region can be also used to induce a control Lyapunov function.

In order to present the results of this chapter it is important to refer to the notion of one step set [36] applied to this specific family of systems. This is an extension of the general definition of $Q(\cdot)$ given in definition 9 on chapter 2.

Definition 44 (one step set) *Given a region Ω, and the system $x^+ = f(x, u, w)$, the one step set is given by:*

$$Q(\Omega) = \{ x \in X \ : \ \text{there is } \ u \in U \text{ such that } f(x, u, w) \in \Omega, \forall w \in W \}$$

Based on this definition, the maximal robust control invariant set can be obtained by means of the following algorithm.

Algorithm 1

(i) Set the initial region C_0 equal to X, $k = 0$.

(ii) $C_{k+1} = Q(C_k)$.

(iii) If $C_{k+1} = C_k$ then C_k is the greatest control invariant set. Stop. Else, set $k = k+1$ and return to step (ii).

Note that the evolution of any initial condition belonging to set C_k can be robustly maintained in X at least k sample times. Therefore, $C_\infty = \lim_{k\to\infty} C_k$ constitutes the set of initial conditions where the system is robustly controllable in an admissible way, that is, the maximal invariant set. In order to apply previous algorithm to a piecewise affine system the operators $Q_i(\cdot)$, $i = 1, \ldots, r$ are defined:

Definition 45 *Given a region Ω, set $Q_i(\Omega)$ denotes the following set,*

$$Q_i(\Omega) = \{\ x \in X_i\ :\ \exists u \in U \text{ such that } A_i x + B_i u + E_i w + q_i \in \Omega, \forall w \in W\ \}.$$

The following well-known properties allow us to compute $Q(\Omega)$ for a piecewise affine system [36]:

Property 14 *Given a convex polyhedron Ω, set $Q_i(\Omega)$ is a convex polyhedron.*

PROOF :

Definition 45 can be written in this way,

$$Q_i(\Omega) = \{\ x \in X_i\ :\ \exists u \in U \text{ such that } A_i x + B_i u + q_i \in \Omega \sim E_i W\ \},$$

where $\sim$ operator is the Pointriagin difference defined in notation 7 of chapter 6.

Note that if Ω is a convex polihedron and W is also a convex polihedron, $\Omega \sim W$ is also a convex polihedron (see proof of property 12 item 2).

Note also that $\{\ x \in X_i, u \in U\ :\ A_i x + B_i u + q_i \in \Omega \sim E_i W\ \}$ is a convex polihedron in dimension $\mathbb{R}^{n+m}$, therefore the projection of this set in X_i is also a convex polihedron in dimension $\mathbb{R}^n$. ■

Property 15 *Given a convex polyhedron Ω: $Q(\Omega) = \bigcup_{i=1}^{r} Q_i(\Omega)$.*

PROOF :

This property stems from the fact that $X = \bigcup_{i=1}^{r} X_i$ and for all X_i there exist a $Q_i(\cdot)$ operator. ■

It is possible, in principle, to compute the maximal robust control invariant set C_∞ by means of algorithm 1. However, it is well known (see for example [36]) that the computational burden of algorithm 1 grows exponentially with the number of regions. Moreover, the complexity of the representation of each of the obtained sets C_k also grows exponentially with k. In this chapter we propose an algorithm (based on convex outer (and inner) approximations of the one step set) that can be used to compute a convex robust control invariant set for the piecewise affine system.

8.3 Complementary of $Q(\Omega)$

The outer and inner approximations presented in this chapter for $Q(\Omega)$ rely on the computation of the complementary of $Q(\Omega)$. In this section it will be shown how to compute the complementary of $Q(\Omega)$ in X. For that purpose, the following definitions are required:

Definition 46 *Given set Ω, $Q^c(\Omega)$ denotes the complementary of $Q(\Omega)$ in X. That is, $Q^c(\Omega) = \{\ x \in X\ :\ x \notin Q(\Omega)\ \}$.*

Definition 47 *Given set Ω, $Q_i^c(\Omega)$ denotes the complementary of $Q_i(\Omega)$ in X_i. That is, $Q_i^c(\Omega) = \{\ x \in X_i\ :\ x \notin Q_i(\Omega)\ \}$.*

The following property stems directly from previous definitions,

Property 16 *Given set Ω: $Q^c(\Omega) = \bigcup\limits_{i=1}^{r} Q_i^c(\Omega)$.*

Therefore, in order to compute $Q^c(\Omega)$ it suffices to compute $Q_i^c(\Omega)$, $i = 1, \ldots, r$. The following property shows how to compute $Q_i^c(\Omega)$.

Property 17 *Suppose that $Q_i(\Omega) = \{\ x \in X_i\ :\ G_i x \leq g_i\ \}$, where $G_i \in \mathbb{R}^{n_x \times L_i}$ and $g_i \in \mathbb{R}^{L_i}$. Then,*

$$Q_i^c(\Omega) = \bigcup_{j=1}^{L_i} S_{i,j}(G_i, g_i)$$

where $S_{i,j}(G_i, g_i) = \{\ x \in X_i\ :\ G_i(j)x > g_i(j), G_i(l)x \leq g_i(l)$ for $l = 1, \ldots, j-1\ \}$, and $G_i(j)$ is the j-eseme row of G_i and $g_i(j)$ is the j-eseme component of g_i..

PROOF :

Suppose that $x \in Q_i^c(\Omega)$. Then, by definition, $x \in X_i$ and $x \notin \{\ x\ :\ G_i x \leq g_i\ \}$. That is, from the constraints $G_i(k)x \leq g_i(k)$, $k = 1, \ldots, L^i$ there is at least one of them that is not satisfied. Denote j the index corresponding to the first inequality that is not satisfied. Then, $G_i(j)x > g_i(j)$ and $G_i(l)x \leq g_i(l)$ for $l = 1, \ldots, j-1$. That is, $x \in S_{i,j}(G_i, g_i)$. From this it is inferred that $Q_i^c(\Omega) \subseteq \bigcup\limits_{j=1}^{L_i} S_{i,j}(G_i, g_i)$. On the other hand, it is easy to see that $S_{i,j}(G_i, g_i) \subseteq Q_i^c(\Omega)$, $j = 1, \ldots, L^i$. That is, $Q_i^c(\Omega) \supseteq \bigcup\limits_{j=1}^{L_i} S_{i,j}(G_i, g_i)$. ■

8.4 Outer bound of the maximal robust control invariant set

A procedure to obtain an outer bound of the maximal robust control invariant set is proposed in this section. The outer bound is obtained in a recursive way. Suppose that Ω is an outer bound of C_∞, then a sharper outer bound is obtained intersecting Ω with a number of semi-planes of the form $\{\ x\ :\ c^\top x \leq 1\ \}$. These semi-planes are obtained in such a way that they do not exclude any point contained in C_∞. That is, $C_\infty \subseteq \Omega \bigcap \{\ x\ :\ c^\top x \leq 1\ \}$. The construction of such semi-planes relies on the notion of outer supporting constraint:

Definition 48 *Given sets S and R, we say that $\{\ x\ :\ c^\top x \leq 1\ \}$ is an outer supporting constraint of S over R if c is the solution of the following maximization problem*

$$\begin{aligned} \max_{c,\rho} \quad & \rho \\ s.t. \quad & c^\top x > 1, \forall x \in \frac{1}{\rho} S \\ & c^\top x \leq 1, \forall x \in R \end{aligned}$$

As it will be shown later, the algorithm that computes the outer bound of C_∞ relies on the computation of outer supporting constraints of each of the subsets of $Q^c(\Omega)$ over $Q(\Omega)$. Suppose that $Q^c(\Omega) = \bigcup\limits_{j=1}^{n_c} S_j$ and

$Q(\Omega) = \bigcup_{i=1}^{r} T_i$ where S_j, $j = 1, \ldots, n_c$ and T_i, $i = 1, \ldots, r$ are polyhedrons; then the following property allows us to compute an outer supporting constraint of each subset S_j over $Q(\Omega) = \bigcup_{i=1}^{r} T_i$ by means of the solution of a linear optimization problem. In this way n_c outer supporting constraints are obtained.

Property 18 *Consider the polyhedron $S = \{ x : Fx \leq f \}$ and the polyhedrons $T_l = \{ x : M_l x \leq m_l \}$, $l = 1, \ldots, r$. Suppose that the scalar ρ and the vectors c, λ, and β_l, $l = 1, \ldots, r$ satisfy the following constraints:*

$$\begin{aligned}
\rho &> 0 && (8.2)\\
\lambda &\geq 0 && (8.3)\\
\beta_l &\geq 0, \quad l = 1, \ldots, r && (8.4)\\
\rho + f^\top \lambda &< 0 && (8.5)\\
-1 + m_l^\top \beta_l &\leq 0, \quad l = 1, \ldots, r && (8.6)\\
c + F^\top \lambda &= 0 && (8.7)\\
c - M_l^\top \beta_l &= 0, \quad l = 1, \ldots, r && (8.8)
\end{aligned}$$

then

$$\begin{aligned}
c^\top x &> 1, \quad \forall x \in \frac{1}{\rho} S && (8.9)\\
c^\top x &\leq 1, \quad \forall x \in \bigcup_{l=1}^{r} T_l && (8.10)
\end{aligned}$$

PROOF :

First, inequality (8.9) will be proved. From constraints (8.2-8.3) it is inferred that $\frac{\lambda}{\rho} \geq 0$. Taking now into account that $\frac{1}{\rho} S = \{ x : \rho F x \leq f \}$ it results that

$$\left(\frac{\lambda}{\rho}\right)^\top (\rho F x - f) \leq 0, \quad \forall x \in \frac{1}{\rho} S$$

From this:

$$c^\top x - 1 \geq c^\top x - 1 + \lambda^\top (Fx - \frac{f}{\rho}) \tag{8.11}$$

$$= (c + F^\top \lambda)^\top x - 1 - \frac{f^\top \lambda}{\rho}, \quad \forall x \in \frac{1}{\rho} S \tag{8.12}$$

From constraints (8.7) and (8.5) it is inferred that $c + F^\top \lambda = 0$ *and* $\rho + f^\top \lambda < 0$. *Therefore, from equation (8.12) it is finally concluded that:*

$$c^\top x - 1 \geq -1 - \frac{f^\top \lambda}{\rho} = -\frac{1}{\rho}(\rho + f^\top \lambda) > 0, \quad \forall x \in \frac{1}{\rho} S$$
$$c^\top x > 1, \ \ \forall x \in \frac{1}{\rho} S$$

In order to prove inequality (8.10) it suffices to show that

$$c^\top x \leq 1, \ \ \forall x \in T_l, \ \ l = 1, \ldots, r$$

Note that $T_l = \{ \ x \ : \ M_l x \leq m_l \ \}$, $l = 1, \ldots, r$. *From this and constraint (8.4) it is inferred that* $\beta_l^\top (m_l - M_l x) \geq 0$, *for all* $x \in T_l$. *Therefore:*

$$c^\top x - 1 \leq c^\top x - 1 + \beta_l^\top (m_l - M_l x) \tag{8.13}$$

$$= (c - M_l^\top \beta_l)^\top x - 1 + m_l^\top \beta_l, \ \ \forall x \in T_l, \ l = 1, \ldots, r \tag{8.14}$$

From constraints (8.8) and (8.6) it is inferred that $c - M_l^\top \beta_l = 0$ *and* $-1 + m_l^\top \beta_l < 0$. *Therefore, from equation (8.14) it is finally concluded that:*

$$c^\top x - 1 \leq -1 + m_l^\top \beta_l \leq 0, \ \ \forall x \in T_l, \ l = 1, \ldots, r$$
$$c^\top x \leq 1, \ \ \forall x \in \bigcup_{l=1}^{r} T_l.$$

■

8.4.1 Outer bound of the maximal control invariant set: proposed algorithm

The following algorithm provides a convex polyhedron that serves as an outer bound of the maximal robust control invariant set of a piecewise affine system:

Algorithm 2

(i) $k = 0$, $\hat{C}_0 = X$.

(ii) Given $\hat{C}_k = \{ x : Hx \leq h \}$, obtain $T_i = Q_i(\hat{C}_k)$, $i = 1, \ldots, r$.

(iii) Obtain $Q^c(\hat{C}_k) = \bigcup_{i=1}^{r} Q_i^c(\hat{C}_k) = \bigcup_{j=1}^{n_c} S_j$ by means of property 17.

(iv) For every $j = 1, \ldots, n_c$ obtain $\{ x : c_j^\top x \leq 1 \}$, the outer supporting constraint of S_j over $Q(\hat{C}_k) = \bigcup_{i=1}^{r} T_i$. This can be achieved by means of property 18.

(v) $\hat{C}_{k+1} = \hat{C}_k \bigcap_{j=1}^{n_c} \{ x : c_j^\top x \leq 1 \}$.

(vi) Go to step (ii).

Note that the algorithm can be finished when there is no significant improvement of the outer bound. That is, when C_k is almost identical to C_{k-1}.

Property 19 *Each one of the polyhedrons $\hat{C}_k$ obtained by means of algorithm 2 constitutes an outer bound of the maximal robust control invariant set of the piecewise affine system. That is, $C_\infty \subseteq \hat{C}_k$, for all $k \geq 0$.*

PROOF :

Note that $C_\infty \subseteq X = \hat{C}_0$. It suffices to show that $C_\infty \subseteq \hat{C}_k$ implies $C_\infty \subseteq \hat{C}_{k+1}$. Assume that $C_\infty \subseteq \hat{C}_k$. Then:

$$C_\infty = \hat{C}_k \bigcap C_\infty = \hat{C}_k \bigcap Q(C_\infty) \subseteq \hat{C}_k \bigcap Q(\hat{C}_k) \tag{8.15}$$

By definition, the outer supporting constraints satisfy: $c_j^\top x \leq 1$, for all $x \in Q(\hat{C}_k)$ and for all $1 \leq j \leq n_c$. Therefore $Q(\hat{C}_k) \subseteq \bigcap_{j=1}^{n_c} \{ x : c_j^\top x \leq 1 \}$. Thus, from equation (8.15) it is finally inferred that:

$$C_\infty \subseteq \hat{C}_k \bigcap Q(\hat{C}_k) \subseteq \hat{C}_k \bigcap_{j=1}^{n_c} \{ x : c_j^\top x \leq 1 \} = \hat{C}_{k+1}.$$

■

8.5 Inner approximation of the maximal robust control invariant set

In this section a recursive procedure to obtain an inner bound of the maximal robust control invariant set is proposed. The procedure uses as initial approximation of C_∞ the outer bound obtained by means of algorithm 2. In this way, the geometry of C_∞ is captured in some sense. The construction of the inner approximation relies on the notion of inner supporting constraint:

Definition 49 *Given sets S and R, we say that $\{\ x\ :\ c^\top x \leq 1\ \}$ is an inner supporting constraint of S over R if c is the solution of the following minimization problem*

$$\begin{array}{ll} \min\limits_{c,\rho} & \rho \\ s.t. & c^\top x > 1, \forall x \in S \\ & c^\top x \leq 1, \forall x \in \dfrac{1}{\rho} R \end{array}$$

The following property (its proof is similar to the one of property 18) allows to compute an inner supporting constraint of a polyhedron S over $Q(\Omega) = \bigcup\limits_{i=1}^{r} T_i$ by means of the solution of a linear optimization problem.

Property 20 *Consider polyhedron $S = \{\ x\ :\ Fx \leq f\ \}$ and the polyhedrons $T_l = \{\ x\ :\ M_l x \leq m_l\ \}$, $l = 1, \ldots, r$. Suppose that the scalar ρ and the vectors c, λ, and β_l, $l = 1, \ldots, r$ satisfy the following constraints:*

$$\begin{aligned} \rho &> 0 && (8.16)\\ \lambda &\geq 0 && (8.17)\\ \beta_l &\geq 0, \ \ l = 1, \ldots, r && (8.18)\\ 1 + f^\top \lambda &< 0 && (8.19)\\ -\rho + m_l^\top \beta_l &\leq 0, \ \ l = 1, \ldots, r && (8.20)\\ c + F^\top \lambda &= 0 && (8.21)\\ c - M_l^\top \beta_l &= 0, \ \ l = 1, \ldots, r && (8.22) \end{aligned}$$

then

$$c^\top x \;>\; 1, \;\; \forall x \in S \tag{8.23}$$

$$c^\top x \;\leq\; 1, \;\; \forall x \in \bigcup_{l=1}^{r} \frac{1}{\rho} T_l \tag{8.24}$$

8.5.1 Robust control invariant set: proposed algorithm

The following algorithm serves to compute a robust control invariant set for a piecewise affine system:

Algorithm 3

(i) $k = 0$,

(ii) Choose a contracting factor $\tilde{\lambda} \in (0, 1)$.

(iii) Make $\tilde{C}_0$ equal to the outer approximation of C_∞ obtained by means of algorithm 2.

(iv) Given $\tilde{C}_k = \{ \; x \; : \; Hx \leq h \; \}$, obtain $T_i = Q_i(\tilde{\lambda}\tilde{C}_k)$, $i = 1, \ldots, r$.

(v) Obtain $Q^c(\tilde{\lambda}\tilde{C}_k) = \bigcup\limits_{i=1}^{r} Q_i^c(\tilde{\lambda}\tilde{C}_k) = \bigcup\limits_{j=1}^{n_c} S_j$ by means of property 17.

(vi) For every $j = 1, \ldots, n_c$ obtain $\{ \; x \; : \; c_j^\top x \leq 1 \; \}$, the inner supporting constraint of S_j over $Q(\tilde{\lambda}\tilde{C}_k) = \bigcup\limits_{i=1}^{r} T_i$. This can be achieved by means of property 20.

(vii) $\tilde{C}_{k+1} = \bigcap\limits_{j=1}^{n_c} \{ \; x \; : \; c_j^\top x \leq 1 \; \}$.

(viii) If $\tilde{C}_{k+1} \subseteq Q(\tilde{C}_{k+1})$ then $\tilde{C}_{k+1}$ is a robust control invariant set. Stop. Else, go to step (ii).

Note that algorithm 3 finishes only if $\tilde{C}_{k+1} \subseteq Q(\tilde{C}_{k+1})$. This is the geometrical condition of robust invariance [17]. That is, if algorithm 3 finishes then a robust control invariant set is obtained. This set serves as an inner approximation of C_∞. It is not guaranteed that algorithm 3 converges to a robust control invariant set. Note, however, that it can be shown that each one of the obtained sets $\tilde{C}_k$ constitutes an inner approximation of C_k. The proof of this statement is based on the fact that, by definition, $Q^c(\tilde{\lambda}\tilde{C}_k) \bigcap \{ \; x \; : \; c_j^\top x \leq 1 \; \} = \emptyset$. That is, $\bigcap\limits_{j=1}^{n_c} \{ \; x \; : \; c_j^\top x \leq 1 \; \} \subseteq Q(\tilde{\lambda}\tilde{C}_k)$.

PROOF :

First, inequality (8.23) will be proved. From the definition of S and constraint (8.17) it is inferred that $\lambda^\top(Fx - f) \leq 0$, for all $x \in S$.

From this,

$$\begin{aligned} c^\top x - 1 &\geq c^\top x - 1 + \lambda^\top(Fx - f) && (8.25)\\ &= (c + F^\top\lambda)^\top x - 1 - f^\top\lambda, \quad \forall x \in S. && (8.26) \end{aligned}$$

From constraints (8.21) and (8.19) it is inferred that $c + F^\top\lambda = 0$ and $1 + f^\top\lambda < 0$. Therefore, from equation (8.26) it is concluded that,

$$c^\top x - 1 \geq -1 - f^\top\lambda = -(1 + f^\top\lambda) > 0, \quad \forall x \in S,$$
$$c^\top x > 1, \ \ \forall x \in S.$$

In order to prove inequality (8.24) it suffices to show that

$$c^\top x \leq 1, \ \ \forall x \in \frac{1}{\rho}T_l, \ \ l = 1, \ldots, r.$$

From constraints (8.16) and (8.18) it is inferred that $\dfrac{\beta_l}{\rho} \geq 0, l = 1, \ldots, r$. Taking now into account that $\dfrac{1}{\rho}T_l = \{ \ x \ : \ \rho M_l x \leq m_l \ \}$ it results that

$$\left(\frac{\beta_l}{\rho}\right)^\top (m_l - \rho M_l x) \geq 0, \quad \forall x \in \frac{1}{\rho}T_l, \ l = 1, \ldots, r.$$

From this,

$$\begin{aligned} c^\top x - 1 &\leq c^\top x - 1 + \beta_l^\top(\frac{m_l}{\rho} - M_l x) && (8.27)\\ &= (c - M_l^\top\beta_l)^\top x - 1 + \frac{m_l^\top\beta_l}{\rho}, \quad \forall x \in \frac{1}{\rho}T_l, \ l = 1, \ldots, r. && (8.28) \end{aligned}$$

From constraints (8.22) and (8.20) it is inferred that $c - M_l^\top\beta_l = 0$ and $-\rho + m_l^\top\beta_l \leq 0$, $l = 1, \ldots, r$. Therefore, from equation (8.28) it is finally concluded that,

$$c^\top x - 1 \leq -1 + \frac{m_l^\top \beta_l}{\rho} = \frac{1}{\rho}(-\rho + m_l^\top \beta_l) \leq 0, \quad \forall x \in \frac{1}{\rho} T_l, \; l = 1, \ldots, r$$

$$c^\top x \leq 1, \;\; \forall x \in \frac{1}{\rho} T_l, \; l = 1, \ldots, r.$$

■

8.6 Numerical example

In this example, region $X = \{\, x \; : \; \|x\|_\infty \leq 15 \,\}$ is subdivided into the subregions X_1, X_2 and X_3. These subregions are defined as follows:

$$X_1 = \{\, x \in X \; : \; x_1 - x_2 \leq 0 \,\}$$

$$X_2 = \{\, x \in X \; : \; x_1 - x_2 > 0 \text{ and } x_1 + x_2 \geq 0 \,\}$$

$$X_2 = \{\, x \in X \; : \; x_1 - x_2 > 0 \text{ and } x_1 + x_2 < 0 \,\}$$

Consider the following piecewise affine system:

$$x^+ = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w & \text{if } \; x \in X_1 \\ \begin{bmatrix} 1 & 1 \\ 0.5 & 1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w & \text{if } \; x \in X_2 \\ \begin{bmatrix} 1 & -0.5 \\ 0 & 1.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w & \text{if } \; x \in X_3 \end{cases}$$

In this example it is assumed that $U = \{\, u \in \mathbb{R} \; : \; \|u\|_\infty \leq 2 \,\}$ and $W = \{\, w \in \mathbb{R} \; : \; \|w\|_\infty \leq 0.1 \,\}$. The contracting factor for algorithm 3 has been set equal to 0.95. In figure 8.1 the sequence of outer bounds $\hat{C}_k$ is displayed. The most inner polyhedron is used in algorithm 3 as initial guess for the obtainment of a robust control invariant set.

In figure 8.2 a sequence of sets $\tilde{C}_k$ leading to a robust control invariant set is displayed. The most inner polyhedron is a robust control invariant set for the piecewise affine system.

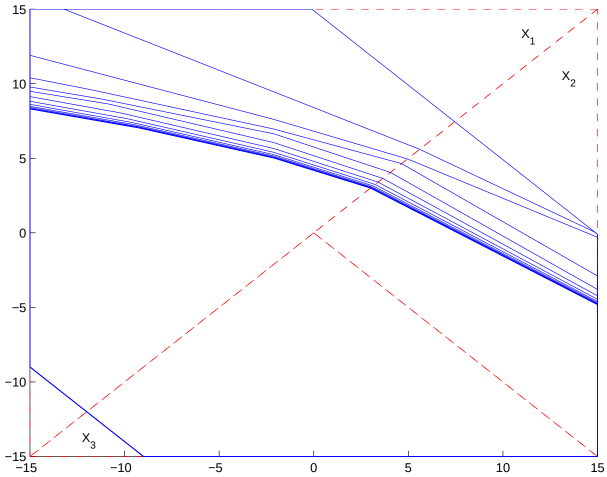

Figure 8.1: Sequence of outer bounds $\hat{C}_k$

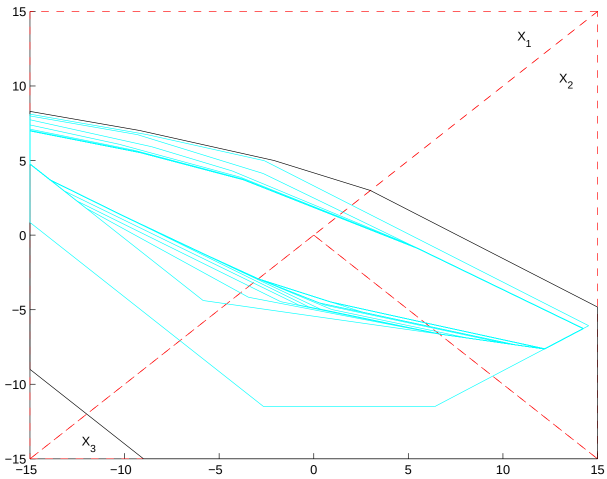

Figure 8.2: Sequence of sets $\tilde{C}_k$ leading to a robust control invariant set.

8.7 Conclusions

In this chapter two approximation operators are introduced. They provide an outer and inner approximation of the one step set. Based on them, two algorithms are presented. One of them obtains an outer approximation of the maximal robust control invariant set. The other one uses the outer approximation to compute a sequence of inner approximations of the maximal robust control invariant set. These sets are an extension of control invariant sets for saturated systems or lure systems applied to *PWA* systems.

9

Conclusion

9.1 Overview

Most of the work shown in this thesis is related to the saturation function. Saturation is the most commonly analyzed non-linearity in control engineering. All fisical controllers are saturated and many times this saturation should be taken into account if controllers are supposed to be working near the saturation. In the other hand, the domain of attraction of a system is the set of states of the system for which the controller leads the system to the origin (or in a generalized point of view to a set that includes the origin). This set is very useful because it gives an idea of the performance of the controller, the bigger domain of attraction is, the greater the fraction of the state space that can be controlled to the origin.

The exact domain of attraction of linear systems subject to control saturation is usually very difficult for systems in continuous time and very computational time demanding for discrete time systems. Even more, the domain of attraction of these systems is non-convex in general and the characterization of this set is complex. Due to these difficulties, an estimation of the domain of attraction set is usually obtained [29, 33].

For these reasons, the estimation of the domain of attraction of linear systems subject to control saturation has been studied in the last years, see for example, [23, 4, 29, 33] and references therein.

This work describes new methods to obtain estimations of the domain of attraction (and invariance sets) for linear systems subject to control saturation. It also extends this new methods to different family of systems, as Lur'e systems or piecewise affine systems. These methods are divided in methods to be applied on continuous time systems, and methods to be applied on discrete time systems. Note that the estimation of the domain of attraction for continuous time systems obtained in the literature are ellipsoids or based in ellipsoidal sets. However, estimations of the domain of attraction for discrete time systems can be ellipsoidal based sets or polihedrical sets. Ellipsoidal sets are usually easily characterized but polihedrical sets are larger sets (ellipsoidal sets are included in polihedrical sets if enough computation time is provided).

One of the most relevant approaches to the analysis of saturated systems is based on a linear difference inclusion (LDI) of the saturation nonlinearity. For example a linear difference inclusion is used to obtain an invariant ellipsoid in [24, 33] for this type of systems. In [41] this approach is also used to obtain a polyhedral invariant set for a saturated system.

The domain of attraction of a given saturated system can be approximated by means of an ellipsoid. In [33] and [29], the authors present a *Linear Difference Inclusion* for a linear saturated system. Based on that LDI, the authors also propose how to choose simultaneously both the matrix H, that characterizes the LDI, and the greatest ellipsoid that is invariant under the corresponding LDI.

This work presents an unified approach to the polyhedric estimation of the domain of attraction of a saturated linear system based in [29]. This generalization provides a polyhedric estimation that it is less conservative but at the expense of an increased representation complexity and the expense of computing time.

Moreover, given the obtained LDI, it is characterized the maximum domain of attraction provided by the LDI (denoted H-domain of attraction). This H-domain of attraction is an estimation of the domain of attraction of the nonlinear system. And under mild conditions, the proposed algorithm obtains the exact H-domain of attraction of the system.

This H-domain of attraction is analyzed in chapter 3. As continuous time systems are analyzed in [29] and discrete time systems are analyzed in this chapter, no more LDI method analysis are studied.

Estimations of the domain of attraction based in a LDI have an artificial limitation. This set must be included in $\mathcal{L}(H)$ set (see Notation 2). A new concept can be used to overpass this limitation. This concept is called SNS and it is presented in chapter 4 for discrete time systems and in chapter 5 for continuous time systems.

SNS concept are based only in geometrical properties, and in chapter 4 it is presented a simple algorithm to estimate the domain of attraction of a discrete time saturated linear system. Any domain of attraction obtained by means of an LDI representation of the system is included in the estimation provided by this proposed algorithm.

LDI methods to obtain an estimation of the domain of attraction for continuous time saturated systems shown in [29] are based in the solution of a optimization problem subject to *Linear Matrix Inequalities* ($LMIs$), constraints. In chapter 5 an optimization problem subject to $LMIs$ constraints with the SNS method is presented. Estimations of the domain of attraction based in a LDI are feasible sets in the optimization problem with the SNS method, and in concordance, the estimation of the domain of attraction is greater than the one obtained by means of an LDI approach

As far as SNS methods are presented for discrete time and continuous time saturated systems, chapters 4 and 5 fills an objective of this work, the design of a new concept to be applied in linear systems with saturated controller that yields better results than LDI methods.

In chapter 6, the SNS methods are extended to robust synthesis. It is provided an algorithm to determinate what K provides the best estimation of the domain of attraction in the controller

$$u = \sigma(Kx)$$

where $\sigma(\cdot)$ is the saturation function.

The algorithm presented provides a robust estimation of the domain of attraction, that is, the target system is the discrete time system

$$x^+ = Ax + Bu + E\theta$$

where x^+ is the successor of the state and θ are the constrained uncertain. This algorithm uses the SNS method shown in chapter 4 to get better estimations of the domain of attraction than provided by LDI methods.

Chapter 7 applies SNS methods to $L'ure$ systems. Therefore, it shows an extension to a different non-linearity of the controller. There exists in the literature a LDI based estimation of the domain of attraction. The estimation obtained by means of the algorithm shown in this chapter improves the obtained by means of the LDI.

This work finishes with an algorithm to obtain estimations of the domain of attraction to robust piecewise affine systems. This algorithm is presented in chapter 8.

9.2 Future lines of work

SNS methods has multiple possibilities that go out of the scope of this work. Some of them are:

- Systems analyzed in chapters 4 and 5 are not robust. Robustness can be included in a similar way that used in chapter 6.
- SNS methods are related to saturated linear controllers. This saturation function is an even function. Some modifications can be made to the SNS methods to manage non simetrical saturation functions and even different non-linearities than saturation. Union of this non-linearities can be addressed to obtain estimations of the domain of attraction for a different class of piecewise affine systems.
- SNS methods are only applied for obtaining estimations of the domain of attraction (and/or invariant sets). However this tool is even more powerful, and can also be applied in MPC controllers or non-linear controllers.

Apendix

A

Appendix

In this appendix it will be shown some lemmas that have not been included in previous chapter for clarity.

Lemma 10 *Given* $a \in \mathbb{R}$ *and* $y \in \mathbb{R}$*,*

$$a\sigma(y) \leq \max\{ay, -|a|\}$$

PROOF :

1. $|y| \leq 1$*:* $\max\{ay, -|a|\} = ay = a\sigma(y)$
2. $|y| > 1$ *and* $ay \geq 0$*:* $\max\{ay, -|a|\} = ay \geq a\,\mathrm{sign}\,(y) = a\sigma(y)$
3. $|y| > 1$ *and* $ay < 0$*:* $\max\{ay, -|a|\} = -|a| = a\,\mathrm{sign}\,(-a) = a\,\mathrm{sign}\,(y) = a\sigma(y)$

∎

Lemma 11 *Given two sets* $\{a_1, a_2, \ldots, a_m\}$*, and* $\{b_1, b_2, \ldots, b_m\}$*,* $\mathcal{V}$ *like definition 6, and* S^c *like in notation 1, both in chapter 2,*

$$\sum_{i=1}^{m} \max \{a_i, b_i\} = \max_{S \in \mathcal{V}} \{ \sum_{i \in S^c} a_i + \sum_{i \in S} b_i \}$$

Proof :

Let $\hat{S} \subseteq \mathcal{V}$ *the subset such that, for all* $i \in \hat{S}$*,* $b_i \geq a_i$*, and* $i \in \mathcal{V}$*,* $i \notin \hat{S}$*,* $a_i > b_i$*. Then*

$$\sum_{i=1}^{m} \max \{a_i, b_i\} = \sum_{i \in \hat{S}^c} a_i + \sum_{i \in \hat{S}} b_i.$$

It can be concluded that

$$\sum_{i \in \hat{S}^c} a_i + \sum_{i \in \hat{S}} b_i = \max_{S \in \mathcal{V}} \{ \sum_{i \in S^c} a_i + \sum_{i \in S} b_i \},$$

note that for all $S \in \mathcal{V}$*,*

$$\sum_{i \in S^c} a_i + \sum_{i \in S} b_i = \sum_{i \in \hat{S}^c} a_i + \sum_{i \in \hat{S}} b_i + \sum_{i \in S^c, i \notin \hat{S}^c} a_i - \sum_{i \in \hat{S}^c, i \notin S^c} a_i + \sum_{i \in S, i \notin \hat{S}} b_i - \sum_{i \in \hat{S}, i \notin S} b_i.$$

Subset $S^c \bigcap (\mathcal{V} - \hat{S}^c)$ *is equal to* $\hat{S} \bigcap (\mathcal{V} - S)$ *and* $\hat{S}^c \bigcap (\mathcal{V} - S^c)$ *is equal to* $S \bigcap (\mathcal{V} - \hat{S})$*, therefore*

$$\sum_{i \in S^c} a_i + \sum_{i \in S} b_i = \sum_{i \in \hat{S}^c} a_i + \sum_{i \in \hat{S}} b_i - \sum_{i \in \hat{S}^c, i \notin S^c} (a_i - b_i) - \sum_{i \in \hat{S}, i \notin S} (b_i - a_i),$$

and taking into account that for all $i \in \hat{S}$*,* $b_i \geq a_i$*, and for all* $i \in \mathcal{V}$*,* $i \notin \hat{S}$*,* $a_i > b_i$*, then*

$$\sum_{i \in S^c} a_i + \sum_{i \in S} b_i \leq \sum_{i \in \hat{S}^c} a_i + \sum_{i \in \hat{S}} b_i.$$

∎

A.1 Schur's complement

Definition 50 *Suppose matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$ defined as*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^m$. Suppose also that D is invertible. Then the Schur's complement is the expression

$$A - BD^{-1}C.$$

Lemma 12 *(Schur) Suppose matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$ defined as*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^m$ and D is invertible. Then M^{-1} can be expressed as

$$M^{-1} = \begin{bmatrix} I_n & 0 \\ -D^{-1}C & I_m \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix},$$

where I_p denotes a $p \times p$ unit matrix.

PROOF :

Let us define

$$T = \begin{bmatrix} I_n & 0 \\ -D^{-1}C & D^{-1} \end{bmatrix},$$

note that T is invertible because D is invertible.

Multplicating M times T,

$$MT = \begin{bmatrix} A - BD^{-1} & BD^{-1} \\ 0 & I_m \end{bmatrix}.$$

And the inverse is

$$(MT)^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ 0 & I_m \end{bmatrix}$$

$$(MT)^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix}.$$

In the other hand,

$$(MT)^{-1} = T^{-1}M^{-1}$$
$$M^{-1} = T(MT)^{-1}.$$

Therefore,

$$M^{-1} = \begin{bmatrix} I_n & 0 \\ -D^{-1}C & D^{-1} \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix},$$

and finally,

$$M^{-1} = \begin{bmatrix} I_n & 0 \\ -D^{-1}C & I_m \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix},$$

■

Last property can be applied in positive definite matrixes.

Lemma 13 *(Schur) Suppose matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$ defined as*

$$M = \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix}$$

where $A \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^m$ and $D > 0$. Then $M \geq 0$ is equivalent to

$$A - BD^{-1}B^\top > 0$$

PROOF :

Note that $M > 0$ is equivalent to $M^{-1} > 0$. This positive definite property of M^{-1} means that for all $x \in \mathbb{R}^{m+n}$, $x^\top M^{-1}x > 0$. As long as D is invertible,

$$x^\top M^{-1}x = x^\top \begin{bmatrix} I_n & 0 \\ -D^{-1}B^\top & I_m \end{bmatrix} \begin{bmatrix} (A - BD^{-1}B^\top)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix} x \geq 0,$$

and with the change of variable

$$y = \begin{bmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{bmatrix} x,$$

results that

$$\begin{bmatrix} (A - BD^{-1}B^\top)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \geq 0.$$

Note that $D^{-1} > 0$*, therefore* $A - BD^{-1}B^\top > 0$.

■

B

Acronyms

In this work the following notation and acronyms have been used

$\mathbb{R}$:= the set of real numbers,
$\mathbb{R}^+$:= the set of possitive real numbers, including zero,
$\mathbb{R}^n$:= the set of n-dimensional real vectors,
$\mathbb{R}^{n\times m}$:= the set of $n \times m$ real matrices,
$\mathbb{N}$:= the set of natural numbers,
$\mathbb{N}^+$:= the set of possitive natural numbers, excluding zero,
$[a, b]$:= the closed real interval,
(a, b) := the open real interval,
$|x|$:= the Euclidean norm, or 2-norm, of $x \in \mathbb{R}^n$,
$\|x\|_\infty$:= ∞-norm of $x \in \mathbb{R}^n$,
$|X|$:= a matrix with elements equal to the absolute values of the elements of matrix X,
I_k := an identity matrix of dimension $k \times k$,
$det(X)$:= the determinant of a square matrix X,
$X^\top$:=
$\mathcal{E}(P, \rho)$:= the ellipsoid $\{ x \ : \ x^\top P x \le \rho \}$,

$\mathcal{E}(P)$:= the ellipsoid $\mathcal{E}(P,1)$,

$\bar{\mathbf{1}}_n$:= a vector in $\mathbb{R}^n$ with all its components equal to 1,

$\Omega \sim \Theta$:= *Pointriagin difference of* Ω *and* Θ. *That is,* $\Omega \sim \Theta = \{\ x \in \Omega\ :\ x + \theta \in \Omega,\ \forall \theta \in \Theta\ \}$,

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