Homogenization of a non-stationary non-Newtonian flow in a porous medium containing a thin fissure

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Abstract

We consider a non-stationary incompressible non-Newtonian Stokes system in a porous medium with characteristic size of the pores ε and containing a thin fissure of width η_ε. The viscosity is supposed to obey the power law with flow index $\frac{5}{3} \leq q \leq 2$. The limit when size of the pores tends to zero gives the homogenized behavior of the flow. We obtain three different models depending on the magnitude η_ε with respect to ε: if $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ the homogenized fluid flow is governed by a time-dependent nonlinear Darcy law, while if $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$ is governed by a time-dependent nonlinear Reynolds problem. In the critical case, $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$, the flow is described by a time-dependent nonlinear Darcy law coupled with a time-dependent nonlinear Reynolds problem.

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1 Introduction

In this paper we consider a non-stationary incompressible viscous non-Newtonian flow in a periodic porous medium with characteristic size of the pores ε and containing a thin fissure $\{0 \leq x_3 \leq \eta_\varepsilon\}$ of width η_ε with $\varepsilon, \eta_\varepsilon$ two small parameters that tend to zero (see Figure 1). Drilling and hydraulic fracturing fluids used in the oil industry are usually non-Newtonian liquids. Therefore during well drilling or hydraulic fracturing operations, the non-Newtonian drilling muds or hydraulic fluids will infiltrate into permeable formations surrounding the wellbore, which may seriously damage the formation. The rheological behavior of drilling muds, cement slurries and hydraulic fracturing fluids is often described by a power-law model (see Cloud and Clark [9], Shah [16]). The importance of modeling flow of non-Newtonian fluids from the wellbore into the surrounding formations has been recognized in the industry.

One way to study this problem is to use the homogenization theory, which has been applied to the study of perforated materials for a long time. The question of a medium containing a thin fissure with properties different from those of the rest of the material has been the subject of many studies previously. For instance, in Panasenko [14], a methodology is proposed for averaging in boundary value problems with plane bondary, and also problems on the contact of several microstructures with plane contact surface. The author considers both direct contact of two structures and contact of two media separated by a thin inhomogeneous layer having periodic structure. In [15], Pham Huy and Sanchez-Palencia study some properties of limit behavior of partial differential equations of the second order which model thermal conductivity problems.

Our goal in this paper is to find the homogenized system corresponding to the limit when the size of the pores, and so the width of the fissure, tends to zero. A similar problem of the one considered in this paper, but for the Laplace's equation, was studied in Bourgeat and Tapiero [4]. The peculiar behavior observed for the Laplace's equation when $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ has motivated the analogous study for the stationary Newtonian Stokes system in Bourgeat *et al.* [5] (see Zhao and Yao [20] for the stationary Newtonian Navier-Stokes system). Another work on this problem, for the non-stationary case, can be found in Zhao and Yao [19], where a non-stationary Newtonian Stokes flow is considered. But to our knowledge, there does not seem to be in the literature any study on the homogenization analysis of a non-stationary non-Newtonian Stokes system in a porous medium with a thin fissure.

In this paper, we consider that the viscosity is a nonlinear function of the symmetrized gradient of the velocity. The viscosity satisfies the nonlinear power law, which is widely used for melted polymers, oil, mud, etc. If u is the velocity and Du the gradient velocity tensor, denoting the shear rate by $\mathbb{D}[u] = \frac{1}{2}(Du + D^t u)$, the viscosity as a function of the shear rate is given by

$$\eta_q(\mathbb{D}[u]) = \mu\, |\mathbb{D}[u]|^{q-2}, \quad 1 < q < +\infty,$$

where the two material parameters $\mu > 0$ and q are called the consistency and the flow index, respectively. Recall that $q = 2$ yields the Newtonian fluid. For $1 < q < 2$ the fluid is pseudoplastic (shear thinning), which is the characteristic of high polymers, polymer solutions, and many suspensions, whereas for $q > 2$ the fluid is dilatant (shear thickening), whose behavior is reported for certain slurries, like mud, clay, or cement.

We consider fluids satisfying the non-stationary non-Newtonian power-law Stokes system, in the domain described above, and our goal is to generalize the study of Bourgeat *et al.* [5] to the non-stationary non-Newtonian case. We find new technical difficulties that needed to be overcome in comparison to [5]. Let us introduce a brief summary of the mathematical innovation in this paper as compared to that in [5]. By the classical theory (see, for instance, Lions [11]), we have the existence and uniqueness of solution for $1 < q < +\infty$. Some *a priori* estimates for velocity in the framework of Sobolev spaces and variational formulations are established for $1 < q < +\infty$ (see Lemma 4.4). To find these estimates and then the order of the limits, we use a variant of the Korn's inequality for this type of domain. Moreover, due to the non-stationary case, some more uniform *a priori* estimates in time for the velocity are established for $1 < q \leq 2$ (see Lemma 4.5), which are needed for estimating the pressure. To estimate the pressure first we need to extend the pressure to the whole domain using an operator in $W_0^{1,q}$, due to the non-Newtonian case, which is given in Bourgeat and Mikelić [6] and generalizes the results for $q = 2$. Moreover, due to the non-stationary case, we can not hope to get the estimation for pressure for $1 < q \leq 2$. First, we assume that $\frac{3}{2} \leq q$ in order to prove the estimation of the pressure in the whole domain. To do this, we use interpolation, with the interpolation parameter $\theta = \frac{4q-6}{q}$, and the Bogovskii operator (see Lemma 4.7). Secondly, we have to assume that $\frac{5}{3} \leq q$ in order to get the estimation of the pressure in the fissure contained in the domain (see Lemma 4.8). Therefore, we have to assume that $\frac{5}{3} \leq q \leq 2$ to overcome this difficulty in the present paper, so we consider pseudoplastic fluids and Newtonian fluids. To find the limit equations, we use the theory developed by Allaire [2] and Nguesteng [13] of *two-scale convergence*, which has proved to be very useful in homogenization theory.

The results obtained here correspond to three characteristic situations depending on the parameter η_ε with respect to ε:

- If $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ the fissure is not giving any contribution. In this case, in order to find the limit, we use two-scale convergence and we obtain a time-dependent nonlinear Darcy's law.

- If $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$ the fissure is dominant. We introduce a rescaling in the thin fissure in order to work with a domain with height one, and then we prove that the limit of the velocity is a Dirac measure concentrated on $\{x_3 = 0\}$ representing the corresponding tangential surface flow. Meanwhile in the porous medium the effective velocity is equal to zero. We obtain a time-dependent nonlinear Reynolds problem.

- If $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$ with $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}} \to \lambda$, $0 < \lambda < +\infty$, it appears a coupling effect and the effective flow behaves as Darcy flow in the porous medium coupled with the tangential flow of the surface $\{x_3 = 0\}$. Compared to the first case $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, the effective velocity has now an additional tangential component concentrated on $\{x_3 = 0\}$. Moreover, the limit problem is now given by a new variational equation, in which appears the parameter λ, and consists of a time-dependent nonlinear Darcy law in the porous medium and an additional time-dependent nonlinear Reynolds problem on the surface $\{x_3 = 0\}$.

  The paper is organized as follows. In Section 2, the domain and some notations are introduced. In Section 3, we formulate the problem and state the main result (Theorem 3.1), which is proved in Section 5. To prove the main result, *a priori* estimates are established in Section 4. The proof of main result in Section 5 is divided in three subsections corresponding to the three characteristic situations depending on the parameter η_ε with respect to ε. A conclusion section is established in Section 6.

2 The domain and some notations

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and

$$\Omega_+ = \Omega \cap \{x_3 > 0\}, \quad \Omega_- = \Omega \cap \{x_3 < 0\}, \quad \Sigma = \Omega \cap \{x_3 = 0\}.$$

For some $\eta_0 > 0$ we define the domain

$$D = \Omega_- \cup (\eta_0 e_3 + \Omega_+) \cup (\Sigma \times [0, \eta_0]) \,,$$

with $e_3 = (0, 0, 1)$.

Let $\varepsilon > 0$ be a small parameter that tends to zero and $0 < \eta_\varepsilon < \eta_0$ be a small parameter that tends to zero with ε.

With Ω we associate a microstructure through the periodic cell $Y = (0,1)^3$ made of two complementary parts: the solid part A, which is closed and strictly contained in Y with a smooth boundary ∂A, and the fluid part $Y^* = Y \setminus A$. Defining $Y^k = k + Y$, $k \in \mathbb{Z}^3$, we set A^k and $Y^{*k} = Y^k \setminus A^k$ as the solid and fluid part in Y^k respectively.

We also denote

$$A^- = \bigcup_{k \in Z^3_-} A^k, \quad A^+ = \bigcup_{k \in Z^3_+} A^k,$$

all the solid parts in $\mathbb{R}^3$, where $Z^k_- = \{k \,:\, k \in \mathbb{Z}^3,\ k_3 < 0\}$ and $Z^k_+ = \{k \,:\, k \in \mathbb{Z}^3,\ k_3 > 0\}$. It is obvious that $E^* = \mathbb{R}^3 \setminus (A^- \cup A^+)$ is an open subset in $\mathbb{R}^3$.

Following Allaire [1], we make the following assumptions on Y^*, E^*, A and $A^* = A^+ \cup A^-$:

i) Y^* is an open connected set of strictly positive measure, with a locally Lipschitz boundary.

ii) A has strictly positive measure in $\overline{Y}$.

iii) E^* and the interior of A^* are open sets with boundaries of class $C^{0,1}$ and are locally located on one side of their boundaries. Moreover E^* is connected.

We also define

$$Y^{*k}_\varepsilon = \varepsilon Y^{*k}, \quad k \in \mathbb{Z}^3,$$
$$A^-_\varepsilon = \varepsilon A^-, \quad A^+_{\varepsilon\eta_\varepsilon} = \eta_\varepsilon e_3 + \varepsilon A^+, \quad S_{\varepsilon\eta_\varepsilon} = \partial \left(A^-_\varepsilon \cup A^+_{\varepsilon\eta_\varepsilon} \right).$$

We denote by

$A_{\varepsilon\eta_\varepsilon} = A^-_\varepsilon \cup A^+_{\varepsilon\eta_\varepsilon}$	- the solid part of the domain D,
$D_{\varepsilon\eta_\varepsilon} = D \setminus A_{\varepsilon\eta_\varepsilon}$	- the fluid part of the domain D (including the fissure),
$I_{\eta_\varepsilon} = \Sigma \times (0, \eta_\varepsilon)$	- the fissure in D,
$\Omega_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \setminus I_{\eta_\varepsilon}$	- the fluid part of the porous medium,

and

$$\Omega^+_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \cap \{x_3 > \eta_\varepsilon\}, \quad \Omega^-_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \cap \{x_3 < 0\}, \quad \Gamma_{\eta_\varepsilon} = \partial\Sigma \times (0, \eta_\varepsilon).$$

Finally we define

$$D_+ = D \cap \{x_3 > 0\}, \quad D_- = \Omega_- \,.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line. We denote by C a generic positive constant which can change from line to line.

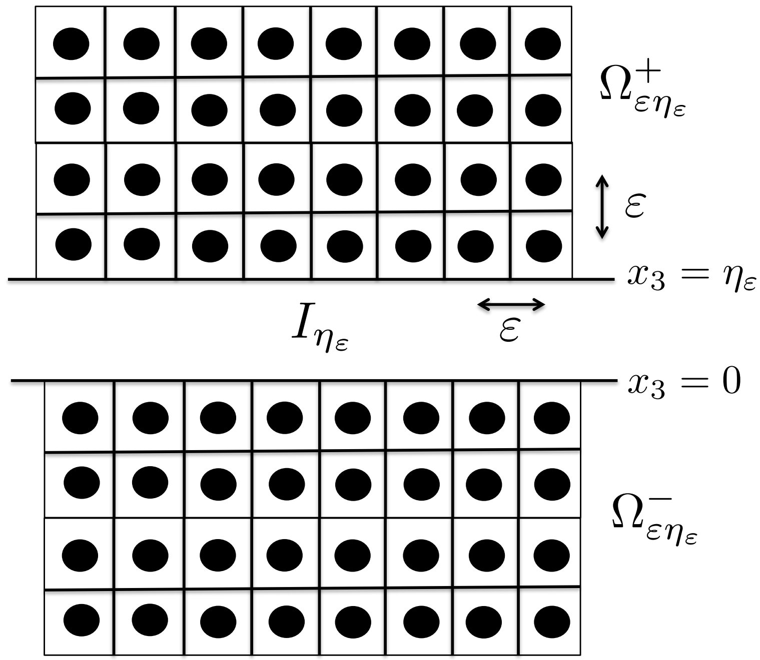

Figure 1: View of the domain $D_{\varepsilon\eta_\varepsilon}$

3 Setting and main result

In the following, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We use the notation $\tilde{\cdot}$ to denote a generic function of $\mathbb{R}^2$.

In this section we describe the asymptotic behavior of a non-stationary incompressible viscous non-Newtonian fluid in the porous medium with a thin fissure. The proof of the corresponding results will be given in the next sections.

Our results are referred to the non-stationary incompressible non-Newtonian power-law Stokes system. Namely, for $f \in C^1([0,T]\times\overline{D})^3$, let us consider a sequence $(u_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon}) \in L^q(0,T; W_0^{1,q}(D_{\varepsilon\eta_\varepsilon}))^3 \times L^{q'}(0,T; L_0^{q'}(D_{\varepsilon\eta_\varepsilon}))$, $1 < q < +\infty$, which satisfies

$$\left\{ \begin{array}{rl} \dfrac{\partial u_{\varepsilon\eta_\varepsilon}}{\partial t} - \operatorname{div}\left(\mu \left|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}\right]\right|^{q-2} \mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}\right]\right) + \nabla p_{\varepsilon\eta_\varepsilon} = f & \text{in } \ (0,T) \times D_{\varepsilon\eta_\varepsilon}, \\ \operatorname{div} u_{\varepsilon\eta_\varepsilon} = 0 & \text{in } \ (0,T) \times D_{\varepsilon\eta_\varepsilon}, \\ u_{\varepsilon\eta_\varepsilon}(0,x) = 0, & \ x \in D_{\varepsilon\eta_\varepsilon}, \end{array} \right. \tag{3.1}$$

where $T > 0$, $\mu > 0$ is the consistency, $q' = q/(q-1)$ is the conjugate exponent of q and $L_0^{q'}(D_{\varepsilon\eta_\varepsilon})$ is the space of functions of $L^{q'}(D_{\varepsilon\eta_\varepsilon})$ with null integral. We consider the problem with Dirichlet boundary condition, i.e.

$$u_{\varepsilon\eta_\varepsilon} = 0 \text{ on } \ (0,T) \times \partial D_{\varepsilon\eta_\varepsilon}. \tag{3.2}$$

For every $\varepsilon, \eta_\varepsilon > 0$, with $1 < q < +\infty$, it is well known that (3.1)-(3.2) has a unique solution $(u_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon}) \in L^q(0,T; W_0^{1,q}(D_{\varepsilon\eta_\varepsilon}))^3 \times L^{q'}(0,T; L_0^{q'}(D_{\varepsilon\eta_\varepsilon}))$ (see the classical theory [11] for more details).

Our aim is to study the asymptotic behavior of $u_{\varepsilon\eta_\varepsilon}$ and $p_{\varepsilon\eta_\varepsilon}$ when ε tends to zero.

As usual, in order to study the behavior of $u_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$ in the fissure we rewrite our equations in the unit cylinder $I_1 = \Sigma \times (0,1)$ by introducing the change of variable

$$z = \frac{x_3}{\eta_\varepsilon}\,, \tag{3.3}$$

which transforms I_{η_ε} in a fixed domain I_1. We define the new functions

$$\mathcal{U}^{\varepsilon\eta_\varepsilon}(t,x',z) = u_{\varepsilon\eta_\varepsilon}(t,x',\eta_\varepsilon z), \quad P^{\varepsilon\eta_\varepsilon}(t,x',z) = p_{\varepsilon\eta_\varepsilon}(t,x',\eta_\varepsilon z) - c_{\varepsilon\eta_\varepsilon}, \tag{3.4}$$

and

$$\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon} = (\mathcal{U}_1^{\varepsilon\eta_\varepsilon}, \mathcal{U}_2^{\varepsilon\eta_\varepsilon})\,,$$

with

$$c_{\varepsilon\eta_\varepsilon}(t) = \frac{1}{|I_{\eta_\varepsilon}|}\int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}(t,x)\,dx. \tag{3.5}$$

Let us introduce some notation which will be useful in the analysis below. We will denote $\mathbb{D}_{x'}[u_{\varepsilon\eta_\varepsilon}] = \frac{1}{2}(D_{x'}u_{\varepsilon\eta_\varepsilon} + D^t_{x'}u_{\varepsilon\eta_\varepsilon})$ and $\partial_z[u_{\varepsilon\eta_\varepsilon}] = \frac{1}{2}(\partial_z u_{\varepsilon\eta_\varepsilon} + \partial^t_z u_{\varepsilon\eta_\varepsilon})$, where we denote $\partial_z = (0,0,\frac{\partial}{\partial z})^t$, and associated to the change of variables (3.3), we introduce the operators: $\mathbb{D}_{\eta_\varepsilon}$, D_{η_ε} and $\mathrm{div}_{\eta_\varepsilon}$, by

$$\mathbb{D}_{\eta_\varepsilon}[v] = \frac{1}{2}\left(D_{\eta_\varepsilon}v + D^t_{\eta_\varepsilon}v\right), \quad \mathrm{div}_{\eta_\varepsilon}v = \mathrm{div}_{x'}\tilde{v} + \frac{1}{\eta_\varepsilon}\partial_z v_3,$$

$$(D_{\eta_\varepsilon}v)_{i,j} = \partial_{x_j}v_i \text{ for } i=1,2,3,\ j=1,2, \quad (D_{\eta_\varepsilon}v)_{i,3} = \frac{1}{\eta_\varepsilon}\partial_z v_i \text{ for } i=1,2,3.$$

Using the transformation (3.3), the system (3.1) can be rewritten as

$$\left\{\begin{array}{r} \dfrac{\partial \mathcal{U}^{\varepsilon\eta_\varepsilon}}{\partial t} - \mathrm{div}_{\eta_\varepsilon}\left(\mu\,|\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]|^{q-2}\,\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]\right) + \nabla_{\eta_\varepsilon}P^{\varepsilon\eta_\varepsilon} = f(t,x',\eta_\varepsilon z) \text{ in } (0,T)\times I_1, \\ \mathrm{div}_{\eta_\varepsilon}\mathcal{U}^{\varepsilon\eta_\varepsilon} = 0 \text{ in } (0,T)\times I_1, \\ \mathcal{U}^{\varepsilon\eta_\varepsilon}(0,x',\eta_\varepsilon z) = 0 \quad (x',\eta_\varepsilon z)\in I_1, \end{array}\right. \tag{3.6}$$

with Dirichlet boundary condition,

$$\mathcal{U}^{\varepsilon\eta_\varepsilon} = 0 \text{ on } \partial\Sigma\times(0,1), \text{ for all } t\in(0,T). \tag{3.7}$$

In order to simplify the notation, we define S_q as the q-Laplace operator

$$S_q(\xi) = |\xi|^{q-2}\,\xi, \quad \forall \xi \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \quad 1<q<+\infty.$$

Our main result for the asymptotic behavior of the solution of (3.1)-(3.2) is given by the following theorem. Observe that we need further assumptions on q and f to carry out this study.

Theorem 3.1. *Assume that $\frac{5}{3}\le q\le 2$ and $f(0)=0$. We distinguish three cases depending on the relation between the parameter η_ε with respect to ε:*

i) if $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, then there exists $(v,p)\in L^q((0,T)\times D)^3\times L^{q'}(0,T;L^{q'}(D))$ such that the solution $(u_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ of the problem (3.1)-(3.2) satisfies

$$\varepsilon^{-\frac{q}{q-1}}u_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \textit{in } L^q((0,T)\times D)^3, \quad p_{\varepsilon\eta_\varepsilon}\to p \quad \textit{in } L^{q'}(0,T;L^{q'}(D)). \tag{3.8}$$

Moreover, $p \in L^{q'}(0,T;W^{1,q'}(D))$ and (v,p) is the unique solution of time-dependent nonlinear Darcy's law

$$\left\{\begin{array}{rcl} v(t,x) &=& \dfrac{1}{\mu}K\left(f(t,x)-\nabla p(t,x)\right) \quad \text{in } (0,T)\times D,\\ \operatorname{div} v(t,x) &=& 0 \quad \text{in } (0,T)\times D,\\ v(t,x)\cdot n &=& 0 \quad \text{on } (0,T)\times \partial D,\end{array}\right. \tag{3.9}$$

where n is the outward normal to ∂D and the permeability function $K:\mathbb{R}^3\to\mathbb{R}^3$ is monotone and coercive, and is defined by

$$K(\xi)=\int_{Y^*} w^\xi(y)\,dy, \quad \forall\,\xi\in\mathbb{R}^3, \tag{3.10}$$

where $w^\xi(y)$, for every $\xi\in\mathbb{R}^3$, denotes the unique solution in $W^{1,q}_{\#}(Y^*)^3$ ($\#$ denotes Y-periodicity) of the local problem

$$\left\{\begin{array}{rcl} -\operatorname{div}_y S_q\left(\mathbb{D}[w^\xi]\right)+\nabla_y \pi^\xi &=& \xi \quad \text{in } Y^*,\\ \operatorname{div}_y w^\xi &=& 0 \quad \text{in } Y^*,\\ w^\xi &=& 0 \quad \text{in } \partial A,\\ w^\xi,\pi^\xi\ Y-\text{periodic}, && \end{array}\right. \tag{3.11}$$

ii) if $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$ and let $(\mathcal{U}^{\varepsilon\eta_\varepsilon},P^{\varepsilon\eta_\varepsilon})$ be a solution of (3.6)-(3.7). Then there exist $\mathcal{U}\in L^q((0,T)\times I_1)^3$ and $P\in L^{q'}(0,T;L^{q'}(I_1))$, such that for a subsequence,

$$\eta_\varepsilon^{-\frac{q}{q-1}}\mathcal{U}^{\varepsilon\eta_\varepsilon}\rightharpoonup \mathcal{U} \quad \text{in } L^q((0,T)\times I_1)^3, \quad P^{\varepsilon\eta_\varepsilon}\rightharpoonup P \quad \text{in } L^{q'}(0,T;L^{q'}(I_1)),$$

where

$$\tilde{\mathcal{U}}(t,x',z)=\frac{2^{\frac{q'}{2}}}{q'\mu^{q'-1}}\left(\left(\frac{1}{2}\right)^{q'}-\left|\frac{1}{2}-z\right|^{q'}\right)S_{q'}\left(\tilde{f}(t,x',0)-\nabla_{x'}P(t,x')\right), \quad \mathcal{U}=(\tilde{\mathcal{U}},0). \tag{3.12}$$

Moreover, it holds that

$$\eta_\varepsilon^{-\frac{2q-1}{q-1}}u_{\varepsilon\eta_\varepsilon}\overset{\star}{\rightharpoonup}\mathcal{V}\delta_\Sigma \quad \text{in } L^q(0,T;\mathcal{M}(D))^3, \tag{3.13}$$

where $\mathcal{V}\in L^q((0,T)\times\Sigma)^3$ such that

$$\tilde{\mathcal{V}}(t,x')=\int_0^1\tilde{\mathcal{U}}(t,x',z)\,dz=\frac{1}{2^{\frac{q'}{2}}(q+1)\mu^{q'-1}}S_{q'}\left(\tilde{f}(t,x',0)-\nabla_{x'}P(t,x')\right), \quad \mathcal{V}=(\tilde{\mathcal{V}},0), \tag{3.14}$$

and, in fact $P\in L^{q'}(0,T;W^{1,q'}(\Sigma))$ is the unique solution of the time-dependent nonlinear Reynolds problem on Σ

$$\left\{\begin{array}{l} -\operatorname{div}_{x'}\tilde{\mathcal{V}}(t,x')=0 \quad \text{in } (0,T)\times\Sigma,\\ \tilde{\mathcal{V}}(t,x')\cdot\tilde{n}=0 \quad \text{on } (0,T)\times\partial\Sigma,\end{array}\right. \tag{3.15}$$

where $\tilde{n}$ is the outward normal to $\partial\Sigma$.

iii) if $\eta_\varepsilon\approx\varepsilon^{\frac{q}{2q-1}}$, with $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}}\to\lambda$, $0<\lambda<+\infty$ and let $(u_{\varepsilon\eta_\varepsilon},p_{\varepsilon\eta_\varepsilon})$ be the solution of the problem (3.1)-(3.2), then there exist a Darcy's velocity $v\in L^q((0,T)\times D)^3$, a Reynolds velocity $\mathcal{V}\in L^q((0,T)\times\Sigma)^3$, with $\mathcal{V}_3=0$, and a pressure field $p\in L^{q'}(0,T;W^{1,q'}(D))$ such that

$$\begin{array}{l} \varepsilon^{-\frac{q}{q-1}}u_{\varepsilon\eta_\varepsilon}\overset{\star}{\rightharpoonup} v+\lambda^{\frac{2q-1}{q-1}}\mathcal{V}\delta_\Sigma \quad \text{in } L^q(0,T;\mathcal{M}(D))^3,\\ p_{\varepsilon\eta_\varepsilon}\to p \quad \text{in } L^{q'}(0,T;L^{q'}(D)),\end{array} \tag{3.16}$$

where δ_Σ is the Dirac measure concentrated on Σ, and $\mathcal{M}(D)^3$ is the space of Radon measures on D. The velocities v and $\mathcal{V}$ are given by the first equation in (3.9) and (3.14), respectively, where the pressure $P \in L^{q'}(0,T;W^{1,q'}(\Sigma))$ is connected with the pressure p by the relation

$$p(t,x',0) = P(t,x') + \widetilde{C}, \quad \widetilde{C} \in \mathbb{R}.$$

Moreover, the pressure $p \in V_\Sigma = \{\varphi \in L^{q'}(0,T;W^{1,q'}(D)) : \varphi(\cdot,0) \in L^{q'}(0,T;W^{1,q'}(\Sigma))\}$ is the unique solution of the variational problem

$$\int_0^T \int_D v(t,x)\cdot\nabla\varphi(t,x)\,dxdt + \lambda^{\frac{2q-1}{q-1}} \int_0^T\int_\Sigma \tilde{\mathcal{V}}(t,x')\cdot\nabla_{x'}\varphi(t,x',0)\,dx'dt = 0, \quad \forall\varphi \in V_\Sigma. \tag{3.17}$$

Remark 3.2. *The monotonicity and coerciveness properties of the permeability function K given by (3.10) can be found in sections 2 and 4 in [7], which implies that (3.9) is well posed. On the other hand, the q'-Laplace operator is well know that is monotone and coercive (see [12] for more details), which implies that (3.15) is well posed. Therefore, we can deduce that the problem (3.17) is also well posed.*

4 A Priori estimates

Let us begin with the following variant of the Korn's inequality in the porous medium $\Omega_{\varepsilon\eta_\varepsilon}$, which will be very useful (see for example Bourgeat and Mikelic in [6]).

Lemma 4.1. *There exists a constant C independent of ε, such that, for any function $v \in W^{1,q}(D_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $S_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^q(\Omega_{\varepsilon\eta_\varepsilon})^3} \le C\varepsilon \|\mathbb{D}[v]\|_{L^q(\Omega_{\varepsilon\eta_\varepsilon})^{3\times3}}, \quad 1<q<+\infty. \tag{4.18}$$

Next, we give an useful estimate in the fissure I_{η_ε}.

Lemma 4.2. *There exists a constant C independent of ε, such that, for any function $v \in W^{1,q}(D_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $S_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^q(I_{\eta_\varepsilon})^3} \le C\eta_\varepsilon^{\frac12}(\eta_\varepsilon+\varepsilon)^{\frac12}\|\mathbb{D}[v]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}, \quad 1<q<+\infty. \tag{4.19}$$

Proof. Because the thickness of I_{η_ε} is η_ε, we have, by the classical Poincaré inequality,

$$\|v\|_{L^q(I_{\eta_\varepsilon})^3} \le C\eta_\varepsilon\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}. \tag{4.20}$$

Next, if we choose a point $x_1 \in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $x \in I_{\eta_\varepsilon}$, then we have

$$v(x) - v(x_1) = Dv(\xi)(x-x_1) \le (\varepsilon+\eta_\varepsilon)|Dv|.$$

Since $v(x_1) = 0$ because $x_1 \in A_{\varepsilon\eta_\varepsilon}$, we have

$$\|v(x)\|_{L^q(I_{\eta_\varepsilon})^3} \le C(\varepsilon+\eta_\varepsilon)\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}.$$

Multiplying the above inequality with (4.20) we obtain

$$\|v\|_{L^q(I_{\eta_\varepsilon})^3} \le C\eta_\varepsilon^{\frac12}(\eta_\varepsilon+\varepsilon)^{\frac12}\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}} \le C\eta_\varepsilon^{\frac12}(\eta_\varepsilon+\varepsilon)^{\frac12}\|Dv\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}, \tag{4.21}$$

and from the classical Korn inequality we obtain (4.19).

□

Let us give the classical estimate for a function in L^q when we deal with a thin fissure (see Bourgeat *et al.* [3] for more details).

Lemma 4.3. *There exists a constant C independent of ε, such that, for any function $v \in L^q(I_{\eta_\varepsilon})$ with $\int_{I_{\eta_\varepsilon}} v\,dx = 0$, one has*

$$\|v\|_{L^q(I_{\eta_\varepsilon})} \le \frac{C}{\eta_\varepsilon}\|\nabla v\|_{W^{-1,q}(I_{\eta_\varepsilon})^3}, \quad 1 < q < +\infty.$$

Now, we are in position to obtain some *a priori* estimates for $u_{\varepsilon\eta_\varepsilon}$.

Lemma 4.4. *Assume that $1 < q < +\infty$. Then, there exists a constant C independent of ε, such that the solution $u_{\varepsilon\eta_\varepsilon} \in L^q(0,T;W^{1,q}(D_{\varepsilon\eta_\varepsilon}))^3$ of the problem (3.1)-(3.2) satisfies*

$$\|u_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times\Omega_{\varepsilon\eta_\varepsilon})^3} \le C\left(\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}\varepsilon + \varepsilon^{\frac{q}{q-1}}\right), \tag{4.22}$$

$$\|u_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_{\eta_\varepsilon})^3} \le C\left(\eta_\varepsilon\eta_\varepsilon^{\frac{2q-1}{q(q-1)}} + \varepsilon^{\frac{1}{q-1}}\eta_\varepsilon + \eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{q+1}{2(q-1)}}\right), \tag{4.23}$$

$$\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}\right]\|_{L^q((0,T)\times D_{\varepsilon\eta_\varepsilon})^{3\times3}} \le C\left(\eta_\varepsilon^{\frac{2q-1}{q(q-1)}} + \varepsilon^{\frac{1}{q-1}}\right), \tag{4.24}$$

$$\|Du_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times D_{\varepsilon\eta_\varepsilon})^{3\times3}} \le C\left(\eta_\varepsilon^{\frac{2q-1}{q(q-1)}} + \varepsilon^{\frac{1}{q-1}}\right). \tag{4.25}$$

Proof. Multiplying by $u_{\varepsilon\eta_\varepsilon}$ in the first equation of (3.1), integrating over $D_{\varepsilon\eta_\varepsilon}$ and using the energy equality, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{\varepsilon\eta_\varepsilon}(t)\|^2_{L^2(D_{\varepsilon\eta_\varepsilon})^3} + \mu\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]\|^q_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}} = \int_{D_{\varepsilon\eta_\varepsilon}} f(t)\cdot u_{\varepsilon\eta_\varepsilon}(t)\,dx\,. \tag{4.26}$$

Using Hölder's inequality, we obtain that

$$\int_{D_{\varepsilon\eta_\varepsilon}} f(t)\cdot u_{\varepsilon\eta_\varepsilon}(t)\,dx \le C\eta_\varepsilon^{\frac{1}{q'}}\|f(t)\|_{L^\infty(I_{\eta_\varepsilon})^3}\|u_{\varepsilon\eta_\varepsilon}(t)\|_{L^q(I_{\eta_\varepsilon})^3} + \|f(t)\|_{L^{q'}(\Omega_{\varepsilon\eta_\varepsilon})^3}\|u_{\varepsilon\eta_\varepsilon}(t)\|_{L^q(\Omega_{\varepsilon\eta_\varepsilon})^3},$$

and by inequalities (4.18) and (4.19), we have

$$\int_{D_{\varepsilon\eta_\varepsilon}} f(t)\cdot u_{\varepsilon\eta_\varepsilon}(t)\,dx \le C\left(\eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}(\varepsilon+\eta_\varepsilon)^{\frac{1}{2}}\|f(t)\|_{L^\infty(I_{\eta_\varepsilon})^3} + \varepsilon\|f(t)\|_{L^{q'}(\Omega_{\varepsilon\eta_\varepsilon})^3}\right)\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}.$$

Using Young's inequality with the conjugate exponent q and $q' = q/(q-1)$, we obtain that

$$\int_{D_{\varepsilon\eta_\varepsilon}} f(t)\cdot u_{\varepsilon\eta_\varepsilon}(t)\,dx \le \frac{\mu}{2}\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]\|^q_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}} + C\left(\eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}(\varepsilon+\eta_\varepsilon)^{\frac{1}{2}}\|f(t)\|_{L^\infty(I_{\eta_\varepsilon})^3} + \varepsilon\|f(t)\|_{L^{q'}(\Omega_{\varepsilon\eta_\varepsilon})^3}\right)^{q'}.$$

Therefore, from (4.26) we get

$$\begin{aligned}&\frac{d}{dt}\|u_{\varepsilon\eta_\varepsilon}(t)\|^2_{L^2(D_{\varepsilon\eta_\varepsilon})^3} + \mu\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]\|^q_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}} \\ &\le C\Big(\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}(\varepsilon+\eta_\varepsilon)^{\frac{q'}{2}}\|f(t)\|^{q'}_{L^\infty(I_{\eta_\varepsilon})^3} + \varepsilon^{q'}\|f(t)\|^{q'}_{L^{q'}(\Omega_{\varepsilon\eta_\varepsilon})^3}\Big),\end{aligned} \tag{4.27}$$

and integrating between 0 and T and taking into account the assumption of f, in particular, we have

$$\int_0^T \|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]\|^q_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}dt \le C\left(\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}} + \eta_\varepsilon^{q'+1} + \varepsilon^{q'}\right).$$

Since $\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}} < \eta_\varepsilon{}^{q'+1}$ if $\varepsilon < \eta_\varepsilon$ and $\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}} \le \eta_\varepsilon\varepsilon^{q'} < \varepsilon^{q'}$ if $\eta_\varepsilon < \varepsilon$, the term $\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}}$ can be dropped. Taking into account that $q'+1 = (2q-1)/(q-1)$, this gives (4.24) and from the classical Korn inequality we have (4.25).

On the other hand, applying (4.18) in (4.27), we have

$$\frac{d}{dt}\|u_{\varepsilon\eta_\varepsilon}(t)\|^2_{L^2(D_{\varepsilon\eta_\varepsilon})^3} + C\varepsilon^{-q}\|u_{\varepsilon\eta_\varepsilon}(t)\|^q_{L^q(\Omega_{\varepsilon\eta_\varepsilon})^3} \le C\left(\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}(\varepsilon+\eta_\varepsilon)^{\frac{q'}{2}}\|f(t)\|^{q'}_{L^\infty(I_{\eta_\varepsilon})^3} + \varepsilon^{q'}\|f(t)\|^{q'}_{L^{q'}(\Omega_{\varepsilon\eta_\varepsilon})^3}\right),$$

and integrating between 0 and T and taking into account the assumption of f, in particular, we have

$$\int_0^T \|u_{\varepsilon\eta_\varepsilon}(t)\|^q_{L^q(\Omega_{\varepsilon\eta_\varepsilon})^3}dt \le C\varepsilon^q\left(\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}} + \eta_\varepsilon{}^{q'+1} + \varepsilon^{q'}\right).$$

Reasoning as before, the term $\eta_\varepsilon\eta_\varepsilon^{\frac{q'}{2}}\varepsilon^{\frac{q'}{2}}$ can be dropped. Taking into account that $q'/q = 1/(q-1)$, this gives (4.22).

Finally, applying (4.19) and (4.24), we get

$$\|u_{\varepsilon\eta_\varepsilon}\|^q_{L^q((0,T)\times I_{\eta_\varepsilon})^3} \le C\left(\eta_\varepsilon\eta_\varepsilon^{\frac{2q-1}{q(q-1)}} + \eta_\varepsilon^{\frac{1}{2}}\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}\varepsilon^{\frac{1}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{q-1}} + \eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{q-1}}\varepsilon^{\frac{1}{2}}\right).$$

Since $\eta_\varepsilon^{\frac{1}{2}}\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}\varepsilon^{\frac{1}{2}} < \eta_\varepsilon\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}$ if $\varepsilon < \eta_\varepsilon$ and $\eta_\varepsilon^{\frac{1}{2}}\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}\varepsilon^{\frac{1}{2}} < \eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{q-1}}\varepsilon^{\frac{1}{2}}$ if $\eta_\varepsilon < \varepsilon$, the term $\eta_\varepsilon^{\frac{1}{2}}\eta_\varepsilon^{\frac{2q-1}{q(q-1)}}\varepsilon^{\frac{1}{2}}$ can be dropped, and (4.23) holds.

□

For estimating the pressure in the next step, we need more uniform estimates in time.

Lemma 4.5. *Assume that $1 < q \le 2$ and $f(0) = 0$. Then, there exists a constant C independent of ε, such that the solution $u_{\varepsilon\eta_\varepsilon} \in L^q(0,T;W^{1,q}(D_{\varepsilon\eta_\varepsilon}))^3$ of the problem (3.1)-(3.2) satisfies*

$$\|\partial_t u_{\varepsilon\eta_\varepsilon}\|_{L^\infty(0,T;L^2(D_{\varepsilon\eta_\varepsilon}))^3} \le C\left(\varepsilon + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right), \tag{4.28}$$

$$\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}\right]\|_{L^2(0,T;L^q(D_{\varepsilon\eta_\varepsilon}))^{3\times 3}} \le C\left(\varepsilon + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right), \tag{4.29}$$

where $\partial_t u_{\varepsilon\eta_\varepsilon} := \dfrac{\partial u_{\varepsilon\eta_\varepsilon}}{\partial t}$.

Proof. Taking into account (4.18)-(4.19) and arguing similarly to Proposition 2.2. in Clopeau and Mikelić [8], we have the desired result. □

In order to investigate the behavior of solutions to (3.1)-(3.2), as $\varepsilon \to 0$, we need to extend the pressure to the whole domain D. The extension is closely related to the construction of a restriction operator. Such extension for the case of porous medium without fissure is given in Tartar [17] for the case $q = 2$. We need an operator, R^ε_q, between $W^{1,q}_0(D)^3$ into $W^{1,q}_0(D_{\varepsilon\eta_\varepsilon})^3$ with similar properties, which is given in Bourgeat and Mikelić [6]. Since the construction of the operator is local, having no obstacles in I_{η_ε} means that we do not have to use the extension in that part. Next, we give the properties of the operator R^ε_q (see Lemma 1.2. in [6] for more details).

Lemma 4.6. *There exists a linear continuous operator R_q^ε acting from $W_0^{1,q}(D)^3$ into $W_0^{1,q}(D_{\varepsilon\eta_\varepsilon})^3$, $1 < q < +\infty$, such that*

1. $R_q^\varepsilon v = v$, *if* $v \in W_0^{1,q}(D_{\varepsilon\eta_\varepsilon})^3$
2. $\operatorname{div}(R_q^\varepsilon v) = 0$, *if* $\operatorname{div} v = 0$
3. *For any* $v \in W_0^{1,q}(D)^3$ *(the constant C is independent of v and ε),*

$$\begin{aligned} \left\| R_q^\varepsilon v \right\|_{L^q(D_{\varepsilon\eta_\varepsilon})^3} &\leq C \left\| v \right\|_{L^q(D)^3} + C\varepsilon \left\| Dv \right\|_{L^q(D)^{3\times 3}}, \\ \left\| DR_q^\varepsilon v \right\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}} &\leq \frac{C}{\varepsilon} \left\| v \right\|_{L^q(D)^3} + C \left\| Dv \right\|_{L^q(D)^{3\times 3}}. \end{aligned}$$

In order to extend the pressure to the whole domain D, we define, for all $T > 0$, a function $F_{\varepsilon\eta_\varepsilon} \in L^{q'}(0,T; W^{-1,q'}(D))^3$ by the following formula (brackets are for the duality products between $W^{-1,q'}$ and $W_0^{1,q}$):

$$\left\langle F_{\varepsilon\eta_\varepsilon}(t), v \right\rangle_D = \left\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), R_q^\varepsilon v \right\rangle_{D_{\varepsilon\eta_\varepsilon}} \; a.e. \; t \in (0,T), \text{ for any } \; v \in W_0^{1,q}(D)^3, \tag{4.30}$$

where R_q^ε is defined in Lemma 4.6. We calcule the right hand side of (4.30) by using (3.1) and we have

$$\langle F_{\varepsilon\eta_\varepsilon}(t), v\rangle_D = \left\langle \operatorname{div}\left(\mu \left|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right|^{q-2} \mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right), R_q^\varepsilon v \right\rangle_{D_{\varepsilon\eta_\varepsilon}} + \left\langle f(t), R_q^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}} - \left\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), R_q^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}}, \tag{4.31}$$

and by using the third point in Lemma 4.6, for fixed ε, η_ε we deduce that $F_{\varepsilon\eta_\varepsilon} \in L^{q'}(0,T; W^{-1,q'}(D))^3$.

Moreover, if $v \in W_0^{1,q}(D_{\varepsilon\eta_\varepsilon})^3$ and we continue it by zero out of $D_{\varepsilon\eta_\varepsilon}$, we see from (4.30) and the first point in Lemma 4.6 that $F_{\varepsilon\eta_\varepsilon}|_{D_{\varepsilon\eta_\varepsilon}}(t) = \nabla p_{\varepsilon\eta_\varepsilon}(t)$, *a.e.* $t \in (0,T)$.

Moreover, if $\operatorname{div} v = 0$ by the second point in Lemma 4.6 and (4.30), $\langle F_{\varepsilon\eta_\varepsilon}(t), v\rangle_D = 0$, *a.e.* $t \in (0,T)$, and this implies that $F_{\varepsilon\eta_\varepsilon}(t)$ is the gradient of some function in $L^{q'}(D)$, *a.e.* $t \in (0,T)$. This means that $F_{\varepsilon\eta_\varepsilon}$ is a continuation of $\nabla p_{\varepsilon\eta_\varepsilon}$ to $(0,T) \times D$, and that this continuation is a gradient. We also may say that $p_{\varepsilon\eta_\varepsilon}$ has been continuated to $(0,T) \times D$. We denote the extended pressure again by $p_{\varepsilon\eta_\varepsilon}$ and since it is defined up to a constant we take $p_{\varepsilon\eta_\varepsilon}$ such that $\int_D p_{\varepsilon\eta_\varepsilon} dx = 0$. Moreover, we have

$$F_{\varepsilon\eta_\varepsilon} \equiv \nabla p_{\varepsilon\eta_\varepsilon}.$$

For such extended pressure we obtain the following result.

Lemma 4.7. *Assume that $\frac{3}{2} \leq q \leq 2$ and $f(0) = 0$. Then, there exists a constant C independent of ε such that if $p_{\varepsilon\eta_\varepsilon} \in L^{q'}(0,T; L_0^{q'}(D))$ is the extended pressure to the whole domain D, one has*

$$\| p_{\varepsilon\eta_\varepsilon} \|_{L^{q'}(0,T;L^{q'}(D))} \leq C \left(\frac{\eta_\varepsilon^{\frac{q'+1}{q'}}}{\varepsilon} + 1 \right). \tag{4.32}$$

Proof. Let us first estimate $\nabla p_{\varepsilon\eta_\varepsilon}$. To do this we estimate the right side of (4.31). Using Hölder's inequality and the third point in Lemma 4.6, we have

$$\begin{aligned} &\left| \left\langle \operatorname{div}\left(\mu \left|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right|^{q-2} \mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right), R_q^\varepsilon v \right\rangle_{D_{\varepsilon\eta_\varepsilon}} \right| \leq \mu \left\| \mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right] \right\|^{q-1}_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}} \left\| DR_q^\varepsilon v \right\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}} \\ &\qquad \leq \mu \left\| \mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right] \right\|^{q-1}_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}} \left(\frac{C}{\varepsilon} \left\| v \right\|_{L^q(D)^3} + C \left\| Dv \right\|_{L^q(D)^{3\times 3}} \right), \end{aligned}$$

and

$$\left|\left\langle f(t), R_q^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}}\right| \leq \|f(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3}\left(C\|v\|_{L^q(D)^3} + C\varepsilon\|Dv\|_{L^q(D)^{3\times 3}}\right).$$

We also deduce

$$\left|\left\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), R_q^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}}\right| \leq \|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3}\left\|R_q^\varepsilon v\right\|_{L^q(D_{\varepsilon\eta_\varepsilon})^3}. \tag{4.33}$$

Now, we introduce the interpolation parameter $\theta = \frac{4q-6}{q}$. Since $\frac{3}{2} \leq q \leq 2$, we have that $0 \leq \theta \leq 1$ such that

$$\frac{1}{q'} = \frac{\theta}{q} + \frac{1-\theta}{q^\star},$$

where $q^\star = \frac{3\,q}{3-q}$. We have the interpolation

$$\|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3} \leq \|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|^{\theta}_{L^q(D_{\varepsilon\eta_\varepsilon})^3}\|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|^{1-\theta}_{L^{q^\star}(D_{\varepsilon\eta_\varepsilon})^3},$$

and by the Sobolev embedding, $W_0^{1,q} \hookrightarrow L^{q^\star}$, the classical Korn's inequality and (4.18)-(4.19), we obtain

$$\begin{aligned}\|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3} &\leq \|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|^{\theta}_{L^q(D_{\varepsilon\eta_\varepsilon})^3}\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{1-\theta}_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}} \\ &\leq C\left(\varepsilon + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right)^{\theta}\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}.\end{aligned} \tag{4.34}$$

Taking into account (4.34), with $\eta_\varepsilon \ll 1$ and $\varepsilon \ll 1$, and the third point in Lemma 4.6, from (4.33) we can deduce

$$\left|\left\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), R_q^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}}\right| \leq C\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}\left(C\|v\|_{L^q(D)^3} + C\varepsilon\|Dv\|_{L^q(D)^{3\times 3}}\right).$$

Then, from (4.31), we deduce

$$\begin{aligned}\left|\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), v\rangle_D\right| &\leq \mu\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{q-1}_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}\left(\frac{C}{\varepsilon}\|v\|_{L^q(D)^3} + C\|Dv\|_{L^q(D)^{3\times 3}}\right) \\ &+ \left(\|f(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3} + C\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}\right)\left(C\|v\|_{L^q(D)^3} + C\varepsilon\|Dv\|_{L^q(D)^{3\times 3}}\right).\end{aligned}$$

Then, as $\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$\begin{aligned}\left|\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), v\rangle_D\right| &\leq C\frac{1}{\varepsilon}\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{q-1}_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}\|v\|_{W_0^{1,q}(D)^3} \\ &+ C\left(\|f(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3} + \|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times 3}}\right)\|v\|_{W_0^{1,q}(D)^3},\end{aligned} \tag{4.35}$$

a.e. $t \in (0,T)$ and for any $v \in W_0^{1,q}(D)^3$.

Now, we consider

$$g(t) = |p_{\varepsilon\eta_\varepsilon}(t)|^{q'-2}\,p_{\varepsilon\eta_\varepsilon}(t) - \frac{1}{|D|}\int_D |p_{\varepsilon\eta_\varepsilon}(t)|^{q'-2}\,p_{\varepsilon\eta_\varepsilon}(t)dx, \quad a.e.\ t \in (0,T), \tag{4.36}$$

where $g(t) \in L^q(D)$, due to $p_{\varepsilon\eta_\varepsilon}(t) \in L^{q'}(D)$ *a.e.* $t \in (0,T)$, and $\int_D g(t)dx = 0$.

We define $v(t) = \mathcal{B}[g(t)]$ *a.e.* $t \in (0,T)$, where $\mathcal{B}$ is the Bogovskii operator associated with D. By Theorem 3.1 in Chapter III.3 in Galdi [10], we obtain that $\operatorname{div} v(t) = g(t)$, $v(t) \in W_0^{1,q}(D)^3$ *a.e.* $t \in (0,T)$ and there exists a positive constant C such that

$$\|v(t)\|_{W_0^{1,q}(D)^3} \leq C\,\|g(t)\|_{L^q(D)}\,, \quad a.e.\ t \in (0,T). \tag{4.37}$$

Using (4.36), it is easy to prove that there exists a positive constant C such that

$$\|g(t)\|_{L^q(D)} \leq C\|p_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D)}^{q'-1},$$

and by (4.37) we can deduce

$$\|v(t)\|_{W_0^{1,q}(D)^3} \leq C\|p_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D)}^{q'-1}, \quad a.e.\ t \in (0,T). \tag{4.38}$$

We observe that

$$\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), v(t)\rangle_D = -\langle p_{\varepsilon\eta_\varepsilon}(t), g(t)\rangle_D = -\|p_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D)}^{q'},$$

which, together with (4.35) and (4.38), we have

$$\|p_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(D)} \leq C\left(\frac{1}{\varepsilon}\,\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}^{q-1} + \|f(t)\|_{L^{q'}(D_{\varepsilon\eta_\varepsilon})^3} + \|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\right).$$

Integrating between 0 and T, and from the estimate (4.24), (4.29) with $\eta_\varepsilon \ll 1$ and $\varepsilon \ll 1$, the assumption of f and taking into account that $(2q-1)/q = (q'+1)/q'$, we have the estimate (4.32). $\square$

Lemma 4.8. *Assume that $\frac{5}{3} \leq q \leq 2$ and $f(0) = 0$. Then, there exists a constant C independent of ε such that if $p_{\varepsilon\eta_\varepsilon} \in L^{q'}(0,T;L_0^{q'}(D))$ is the extended pressure to the whole domain D, one has*

$$\|p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}\|_{L^{q'}((0,T)\times I_{\eta_\varepsilon})} \leq C\left(\eta_\varepsilon^{\frac{1}{q'}} + \frac{\varepsilon}{\eta_\varepsilon}\right), \tag{4.39}$$

where $c_{\varepsilon\eta_\varepsilon}$ is given by (3.5).

Proof. Let $v \in W_0^{1,q}(I_{\eta_\varepsilon})^3$, then

$$\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), v\rangle_{I_{\eta_\varepsilon}} = \left\langle \operatorname{div}\left(\mu\,|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]|^{q-2}\,\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right), v\right\rangle_{I_{\eta_\varepsilon}} + \langle f(t), v\rangle_{I_{\eta_\varepsilon}} - \langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), v\rangle_{I_{\eta_\varepsilon}}. \tag{4.40}$$

We estimate the right hand side. Using Hölder's inequality, we have

$$\left|\left\langle \operatorname{div}\left(\mu\,|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]|^{q-2}\,\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\right), v\right\rangle_{I_{\eta_\varepsilon}}\right| \leq \mu\,\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}^{q-1}\,\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}. \tag{4.41}$$

Using again Hölder's inequality and assumption of f, we obtain that

$$\left|\langle f(t), v\rangle_{I_{\eta_\varepsilon}}\right| \leq C\eta_\varepsilon^{\frac{1}{q'}}\,\|f\|_{L^\infty(I_{\eta_\varepsilon})^3}\|v\|_{L^q(I_{\eta_\varepsilon})^3},$$

and by estimate (4.21), we have

$$\left|\langle f(t), v\rangle_{I_{\eta_\varepsilon}}\right| \leq C(\eta_\varepsilon^{\frac{2q-1}{q}} + \eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}. \tag{4.42}$$

We also deduce

$$\left|\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), v\rangle_{I_{\eta_\varepsilon}}\right| \le \|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(I_{\eta_\varepsilon})^3}\,\|v\|_{L^q(I_{\eta_\varepsilon})^3}\,.$$

We introduce the interpolation parameter $\theta = \frac{4q-6}{q}$. Since $\frac{5}{3} \le q \le 2$, we have that $0 \le \theta \le 1$ and reasoning as in the proof of the Lemma 4.7 together with (4.19), we obtain

$$\begin{aligned}
\|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(I_{\eta_\varepsilon})^3} &\le \|\partial_t u_{\varepsilon\eta_\varepsilon}(t)\|^{\theta}_{L^q(I_{\eta_\varepsilon})^3}\,\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{1-\theta}_{L^q(I_{\eta_\varepsilon})^{3\times3}}\\
&\le C\left(\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right)^{\theta}\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,.
\end{aligned}$$

Then, we can deduce that

$$\left|\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), v\rangle_{I_{\eta_\varepsilon}}\right| \le C\left(\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right)^{\theta}\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,\|v\|_{L^q(I_{\eta_\varepsilon})^3}\,,$$

and by estimate (4.21), we have

$$\left|\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), v\rangle_{I_{\eta_\varepsilon}}\right| \le C\left({\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}\right)\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}\,. \tag{4.43}$$

Using (4.41)-(4.43) in (4.40), we obtain

$$\begin{aligned}
|\langle \nabla p_{\varepsilon\eta_\varepsilon}(t), v\rangle|_{I_{\eta_\varepsilon}} &\le C\left(\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{q-1}_{L^q(I_{\eta_\varepsilon})^{3\times3}} + \eta_\varepsilon^{\frac{2q-1}{q}} + \eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\right)\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}\\
&+ C\left({\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}\right)\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,\|Dv\|_{L^q(I_{\eta_\varepsilon})^{3\times3}}\,.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|\nabla p_{\varepsilon\eta_\varepsilon}(t)\|_{W^{-1,q'}(I_{\eta_\varepsilon})^3} &\le C\left(\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{q-1}_{L^q(I_{\eta_\varepsilon})^{3\times3}} + \eta_\varepsilon^{\frac{2q-1}{q}} + \eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\right)\\
&+ C\left({\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}\right)\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,,
\end{aligned}$$

and taking into account that $\int_{I_{\eta_\varepsilon}}(p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon})dx = 0$, we use Lemma 4.3 and we can deduce

$$\begin{aligned}
\|p_{\varepsilon\eta_\varepsilon}(t) - c_{\varepsilon\eta_\varepsilon}(t)\|_{L^{q'}(I_{\eta_\varepsilon})} &\le \frac{C}{\eta_\varepsilon}\left(\|\mathbb{D}\left[u_{\varepsilon\eta_\varepsilon}(t)\right]\|^{q-1}_{L^q(I_{\eta_\varepsilon})^{3\times3}} + \eta_\varepsilon^{\frac{2q-1}{q}} + \eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\right)\\
&+ \frac{C}{\eta_\varepsilon}\left({\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}\right)\|\mathbb{D}\left[\partial_t u_{\varepsilon\eta_\varepsilon}(t)\right]\|_{L^q(D_{\varepsilon\eta_\varepsilon})^{3\times3}}\,.
\end{aligned}$$

Integrating between 0 and T, and from the estimate (4.24), and (4.29) with $\eta_\varepsilon \ll 1$ and $\varepsilon \ll 1$, we have

$$\|p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}\|_{L^{q'}((0,T)\times I_{\eta_\varepsilon})} \le \frac{C}{\eta_\varepsilon}\left(\eta_\varepsilon^{\frac{2q-1}{q}} + \varepsilon + \eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} + {\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}\right).$$

Reasoning as in the proof of Lemma 4.4, we observe that $\eta_\varepsilon^{\frac{1}{q'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ can be dropped.

Using that $\frac{5}{3} \le q$, we can prove that ${\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}} < {\eta_\varepsilon}^{\theta+1} < \eta_\varepsilon^{\frac{2q-1}{q}}$ if $\varepsilon < \eta_\varepsilon$. Moreover, ${\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}} < \varepsilon^{\theta+1} < \varepsilon$ if $\eta_\varepsilon < \varepsilon$. Therefore, the term ${\eta_\varepsilon}^{\theta+1}+{\eta_\varepsilon}^{\frac{\theta+1}{2}}\varepsilon^{\frac{\theta+1}{2}}$ can be dropped, and so we obtain

$$\|p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}\|_{L^{q'}((0,T)\times I_{\eta_\varepsilon})} \le \frac{C}{\eta_\varepsilon}\left(\eta_\varepsilon^{\frac{2q-1}{q}} + \varepsilon\right).$$

□

5 Proof of the main result

In view of estimates (4.22), (4.25) of the velocity and (4.32) of the pressure, the proof of Theorem 3.1 will be divided in three characteristic cases: $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$ and $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$, with $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}} \to \lambda$, $0<\lambda<+\infty$.

5.1 Problem in the porous part $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$

The proof of Theorem 3.1-*i*) will be developed in different lemmas.

In this subsection, we need to extend the velocity $u_{\varepsilon\eta_\varepsilon}$ by zero in the fissure I_{η_ε}, and we will denote the extended veolcity by $v_{\varepsilon\eta_\varepsilon}$, i.e.

$$v_{\varepsilon\eta_\varepsilon} = \begin{cases} u_{\varepsilon\eta_\varepsilon} & \text{in } (0,T)\times\Omega_{\varepsilon\eta_\varepsilon}, \\ 0 & \text{in } (0,T)\times I_{\eta_\varepsilon}. \end{cases} \tag{5.44}$$

Lemma 5.1. *Assume the assumptions in Lemma 4.7. Let* $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ *and let* $(v_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ *be the extended solution of (3.1)-(3.2). Then there exist subsequences of* $v_{\varepsilon\eta_\varepsilon}$ *and* $p_{\varepsilon\eta_\varepsilon}$ *still denoted by the same, and functions* $v\in L^q((0,T)\times D)^3$, $p\in L^{q'}(0,T;L^{q'}(D))$ *such that*

$$\varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \textit{in } L^q((0,T)\times D)^3, \quad p_{\varepsilon\eta_\varepsilon} \to p \quad \textit{in } L^{q'}(0,T;L^{q'}(D)). \tag{5.45}$$

Moreover, v *satisfies*

$$\operatorname{div} v = 0 \quad \textit{in } (0,T)\times D, \qquad v\cdot n = 0 \quad \textit{on } (0,T)\times\partial D. \tag{5.46}$$

Proof. From estimates (4.22) and (4.32), taking into account the extension of the velocity by zero to D and $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, we have the following estimates

$$\|v_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times D)^3} \le C\varepsilon^{\frac{q}{q-1}}, \quad \|p_{\varepsilon\eta_\varepsilon}\|_{L^{q'}(0,T;L^{q'}(D))} \le C. \tag{5.47}$$

Then there exist $v\in L^q((0,T)\times D)^3$ and $p\in L^{q'}(0,T;L^{q'}(D))$ such that, for a subsequence still denoted by $v_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$, it holds

$$\varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \text{in } L^q((0,T)\times D)^3, \quad p_{\varepsilon\eta_\varepsilon} \rightharpoonup p \quad \text{in } L^{q'}(0,T;L^{q'}(D)). \tag{5.48}$$

Next, we prove that the convergence of the pressure is in fact strong. Let w_ε be a sequence of elements of $W_0^{1,q}(D)^3$ such that

$$w_\varepsilon \rightharpoonup w \quad \text{in } W_0^{1,q}(D)^3. \tag{5.49}$$

We consider $\varphi\in C_c^1(0,T)$. We have, (brackets are for the duality products between $W^{-1,q'}$ and $W_0^{1,q}$):

$$\begin{aligned}&\int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t), \varphi(t) w_\varepsilon >_D - < \nabla p(t), \varphi(t) w >_D|\, dt \\ &\le \int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t), \varphi(t)(w_\varepsilon - w) >_D|\, dt + \int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t) - \nabla p(t), \varphi(t) w >_D|\, dt.\end{aligned}$$

On the one hand, taking into account the second convergence in (5.48), we have

$$\int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t) - \nabla p(t), \varphi(t) w >_D| \, dt = \int_0^T \int_D (p_{\varepsilon\eta_\varepsilon}(t) - p(t)) \operatorname{div} \varphi(t) w \, dx dt \to 0, \quad \text{as } \varepsilon \to 0.$$

On the other hand, we have

$$\begin{aligned}
&\int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t), \varphi(t)(w_\varepsilon - w) >_D| \, dt = \int_0^T \left|< \nabla p_{\varepsilon\eta_\varepsilon}(t), \varphi(t) R_q^\varepsilon(w_\varepsilon - w) >_{D_{\varepsilon\eta_\varepsilon}}\right| dt \\
&\leq \int_0^T \left| \langle \operatorname{div} \left(\mu \, |\mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)]|^{q-2} \, \mathbb{D}[u_{\varepsilon\eta_\varepsilon}(t)] \right), \varphi(t) R_q^\varepsilon(w_\varepsilon - w) \rangle_{D_{\varepsilon\eta_\varepsilon}} \right| dt \\
&+ \int_0^T \left| \langle f(t), \varphi(t) R_q^\varepsilon(w_\varepsilon - w) \rangle_{D_{\varepsilon\eta_\varepsilon}} - \left\langle \partial_t u_{\varepsilon\eta_\varepsilon}(t), \varphi(t) R_q^\varepsilon(w_\varepsilon - w) \right\rangle_{D_{\varepsilon\eta_\varepsilon}} \right| dt.
\end{aligned}$$

Using Hölder's inequality, estimate (4.24), $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ and the estimates of the restricted operator R_q^ε given in Lemma 4.6 for the first term, and using Hölder's inequality, the assumption of f, (4.34) and (4.29) with $\eta_\varepsilon \ll 1$ and $\varepsilon \ll 1$, and the estimates of the restricted operator R_q^ε for the second term, we get

$$\begin{aligned}
&\int_0^T |< \nabla p_{\varepsilon\eta_\varepsilon}(t), \varphi(t)(w_\varepsilon - w) >_D| \, dt \\
&\leq C \left(\left(\int_0^T \varphi(t)^q \|w_\varepsilon - w\|^q_{L^q(D)^3} dt \right)^{1/q} + \varepsilon \left(\int_0^T \varphi(t)^q \|Dw_\varepsilon - Dw\|^q_{L^q(D)^{3\times 3}} dt \right)^{1/q} \right),
\end{aligned}$$

which tends to zero (as $\varepsilon \to 0$) by virtue of (5.49) and the Rellich Theorem. This implies that $\nabla p_{\varepsilon\eta_\varepsilon} \to \nabla p$ strongly in $L^{q'}(0,T; W^{-1,q'}(D))^3$, and we have the strong convergence of the pressure given in (5.45).

Finally, from $\operatorname{div} v_{\varepsilon\eta_\varepsilon} = 0$ in $(0,T) \times D$ and the weak convergence of the velocity given in (5.45), we easily obtain (5.46).

□

The proof of the following result will be showed by using the two-scale convergence introduced by Nguesteng [13] in the L^2-setting and developed by Allaire [2], who also introduced the L^q-setting. By $\overset{2}{\rightharpoonup}$ we denote the limit in the two-scale sense.

Lemma 5.2. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ with $1 < q < +\infty$ and let $v_{\varepsilon\eta_\varepsilon}$ be the extended solution of (3.1)-(3.2). Then there exist subsequences of $v_{\varepsilon\eta_\varepsilon}$ still denoted by the same, and $\hat{v}(t,x,y) \in L^q(0,T; L^q(D; W^{1,q}_\#(Y^*)^3))$ such that*

$$\varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon} \overset{2}{\rightharpoonup} \hat{v}(t,x,y) \text{ in } L^q((0,T) \times D \times Y^*)^3, \tag{5.50}$$

$$\varepsilon^{-\frac{1}{q-1}} D v_{\varepsilon\eta_\varepsilon} \overset{2}{\rightharpoonup} D_y \hat{v}(t,x,y) \text{ in } L^q((0,T) \times D \times Y^*)^{3\times 3}. \tag{5.51}$$

The weak limit $v(t,x)$ and the two-scale limit $\hat{v}(t,x,y)$ are related as follows

$$v(t,x) = \int_{Y^*} \hat{v}(t,x,y) \, dy. \tag{5.52}$$

Moreover, $\hat{v}$ satisfies

$$\operatorname{div}_y \hat{v}(t,x,y) = 0 \quad \text{in } (0,T) \times Y^*, \quad \hat{v} = 0 \quad \text{in } (0,T) \times Y \setminus Y^*, \tag{5.53}$$

$$\operatorname{div}_x \left(\int_{Y^*} \hat{v}(t,x,y) \, dy \right) = 0 \text{ in } (0,T) \times D, \quad \left(\int_{Y^*} \hat{v}(t,x,y) \, dy \right) \cdot n = 0 \text{ on } (0,T) \times \partial D. \tag{5.54}$$

Proof. From estimates (4.22) and (4.25) and taking into account that $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, we get

$$\|v_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times D)^3} \leq C\varepsilon^{\frac{q}{q-1}}, \quad \|Dv_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times D)^{3\times 3}} \leq C\varepsilon^{\frac{1}{q-1}}.$$

Thus, from Lemma 1.5 in [6] (the proof can easily be carried over to the time dependent case), there exists subsequences of $v_{\varepsilon\eta_\varepsilon}$, still denoted by $v_{\varepsilon\eta_\varepsilon}$, and function $\hat{v} \in L^q(0,T;L^q(D;W^{1,q}_{\#}(Y^*)^3))$ such that the convergences given in (5.50) hold.

Relation (5.52) is a classical property relating weak convergence and two-scale convergence, see Allaire [2] and Bourgeat and Mikelic [6] for more details. From $\operatorname{div} v_{\varepsilon\eta_\varepsilon} = 0$ in $(0,T)\times D$, then (5.53) straightforward. Finally, (5.46) and (5.52) imply (5.54).

□

Lemma 5.3. *Assume the assumptions in Lemma 4.5. Then, there exists a constant C independent of ε, such that the extension $v_{\varepsilon\eta_\varepsilon} \in L^q(0,T;W^{1,q}(D))^3$ satisfies*

$$\|\partial_t v_{\varepsilon\eta_\varepsilon}\|_{L^\infty(0,T;L^2(D))^3} \leq C\left(\varepsilon + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}}\right). \tag{5.55}$$

Proof. Taking into account Lemma 4.5, it is clear that, after extension, (5.55) holds. □

Lemma 5.4. *Assume the assumptions in Lemma 4.7. Let $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ and let $(v_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ be the extended solution of (3.1)-(3.2). Let $(v,p) \in L^q((0,T)\times D)^3 \times L^{q'}(0,T;L^{q'}(D))$ be given by Lemma 5.1. Then, $p \in L^{q'}(0,T;W^{1,q'}(D))$ and (v,p) is the unique solution of Darcy's law (3.9).*

Proof. First of all, we choose a test function $w(x) \in C_0^\infty(D)^3$ with $w(x) = 0$ on ∂D. Multiplying (3.1)-(3.2) by $w(x)$ and integrating by parts, we have

$$\int_D \partial_t v_{\varepsilon\eta_\varepsilon}(t)\cdot w\,dx + \int_D \mu S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}(t)]) : \mathbb{D}[w]\,dx = \langle f(t) - \nabla p_{\varepsilon\eta_\varepsilon}(t), w\rangle_D,$$

in $\mathcal{D}'(0,T)$. We observe that using (5.55), the first term contributes nothing. Therefore, we consider $\phi \in C_c^1(0,T)$, multiplying by ϕ and integrating between 0 and T, we have

$$\mu\int_0^T \phi(t)\int_D S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}(t)]) : \mathbb{D}[w]\,dxdt = \int_0^T \phi(t)\langle f(t) - \nabla p_{\varepsilon\eta_\varepsilon}(t), w\rangle_D\,dt + O_\varepsilon. \tag{5.56}$$

Considering $\varphi \in L^q(0,T;W_0^{1,q}(D))^3$, we define $w_\varepsilon(t,x) = \varphi(t,x) - \varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}(t,x)$ as test function in the variational formulation (5.56) and we have

$$\mu\int_0^T\int_D S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]) : \mathbb{D}[w_\varepsilon]\,dxdt = \int_0^T \langle f - \nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon\rangle_D\,dt + O_\varepsilon.$$

Observe that

$$S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]) = \varepsilon^q S_q(\mathbb{D}[\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}]).$$

Therefore,

$$\int_0^T\int_D S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]) : \mathbb{D}[w_\varepsilon]\,dxdt = \int_0^T\int_D \varepsilon^q S_q(\mathbb{D}[\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}]) : \mathbb{D}[\varphi]\,dxdt - \int_0^T\int_D |\varepsilon\mathbb{D}[\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}]|^q dxdt.$$

Using Hölder and Young inequalities in the first term of the right hand side, we have

$$\int_0^T\int_D S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]):\mathbb{D}[w_\varepsilon]\,dxdt \leq \frac{1}{q}\int_0^T\int_D |\varepsilon\mathbb{D}[\varphi]|^q dxdt - \frac{1}{q}\int_0^T\int_D |\varepsilon\mathbb{D}[\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}]|^q dxdt,$$

and so the variational formulation of problem (3.1)-(3.2) is equivalent to

$$\begin{aligned}\frac{\mu}{q}\int_0^T\int_D |\varepsilon\mathbb{D}[\varphi]|^q dxdt - \frac{\mu}{q}\int_0^T\int_D |\varepsilon\mathbb{D}[\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}]|^q dxdt \\ \geq \int_0^T\int_D f\cdot w_\varepsilon\,dxdt - \int_0^T\langle\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon\rangle_D\,dt + O_\varepsilon.\end{aligned}\tag{5.57}$$

Let $\psi^{+-}(t,x,y)\in\mathcal{D}((0,T)\times D_{+-};C^\infty_\#(Y^*)^3)$. There exists $\eta_1>0$ such that $supp\ \psi^{+-}(t,x,y)\subset D\setminus I_{\eta_\varepsilon}$ for every $\eta_\varepsilon\in(0,\eta_1)$. Let $\eta_\varepsilon<\eta_1$. We define $\psi^{+-}_\varepsilon(t,x)=\psi^{+-}(t,x,x/\varepsilon)$, and we insert $\varphi=\psi^{+-}_\varepsilon$ in (5.57). In the sequel, we use the elementary properties of the two scale convergence (for the time independent case we refer e.g. to [2]. This elementary properties can easily be carried over to the time dependent case). Using the two-scale convergence of $\varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon}$ given in (5.50), we have

$$\int_0^T\int_{D_{+-}} f\cdot w_\varepsilon\,dxdt \to \int_0^T\int_{D_{+-}}\int_Y f\cdot(\psi-\hat{v})\,dxdydt,$$

and using $\operatorname{div} v_{\varepsilon\eta_\varepsilon}=0$ in $(0,T)\times D$ and the strong convergence of the pressure (5.45), we have

$$\int_0^T\langle\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon\rangle_{D_{+-}}\,dt = \int_0^T\int_{D_{+-}} p_{\varepsilon\eta_\varepsilon}\operatorname{div}\psi^{+-}_\varepsilon\,dxdt \to \int_0^T\int_{D_{+-}}\int_Y p\,\operatorname{div}_x\psi(t,x,y)\,dxdydt, \quad \text{as } \varepsilon\to 0.$$

Therefore, passing to the limit in the variational formulation (5.57) and taking into account (5.50)-(5.51) and (5.54), we get

$$\begin{aligned}\frac{\mu}{q}\int_0^T\int_{D_{+-}}\int_Y |\mathbb{D}_y[\psi]|^q\,dxdydt - \frac{\mu}{q}\int_0^T\int_{D_{+-}}\int_Y |\mathbb{D}_y[\hat{v}]|^q dxdydt \\ \geq \int_0^T\langle f(t,x)-\nabla p(t,x), \int_Y(\psi-\hat{v})dy\rangle_{D_{+-}}\,dt.\end{aligned}\tag{5.58}$$

Consequently, there exists $\hat{\pi}\in L^{q'}(0,T;L^{q'}(D;L^{q'}_0(Y^*)))$ such that $(\hat{v},\hat{\pi})$ satisfies the homogenized problem

$$-\operatorname{div}_y\left(\mu|\mathbb{D}_y[\hat{v}]|^{q-2}\mathbb{D}_y[\hat{v}]\right)+\nabla_y\hat{\pi} = f(t,x)-\nabla p(t,x) \quad \text{in } (0,T)\times Y^*, \tag{5.59}$$

$$\operatorname{div}_y\hat{v}(t,x,y)=0 \quad \text{in } (0,T)\times Y^*, \tag{5.60}$$

$$(\hat{v},\hat{\pi}) \text{ is } Y-\text{periodic}, \quad \hat{v}=0 \quad \text{in } (0,T)\times Y\setminus Y^*, \tag{5.61}$$

by using the variant of de Rham's formula in a periodic setting (see Nguetseng [13] and Temam [18]).

The derivation of (3.9) from the effective problems (5.59)-(5.61) is straightforward by using the local problems (3.11) and definitions of the permeability functions (3.10).

Finally, reasoining as in Theorem 4 in [7], we get that the pressure p belongs to $L^{q'}(0,T;W^{1,q'}(D))$. □

Proof of Theorem 3.1-i). It remains to prove convergence (3.8) of the whole velocity $u_{\varepsilon\eta_\varepsilon}$, i.e. to prove

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_{\eta_\varepsilon})^3}\to 0. \tag{5.62}$$

For this, it is sufficient to prove that

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_{\eta_\varepsilon})^3}\to 0\quad \text{for } \eta_\varepsilon\ll\varepsilon, \tag{5.63}$$

and

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r((0,T)\times I_{\eta_\varepsilon})^3}\to 0\quad \text{for } \varepsilon\ll\eta_\varepsilon\ll\varepsilon^{\frac{1}{\alpha}}\quad ,1<\alpha<\frac{2q-1}{q}, \tag{5.64}$$

for a r which will be defined below.

Using (4.23) and using $\eta_\varepsilon\ll\varepsilon$, we have

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_{\eta_\varepsilon})^3}\le C\left(\frac{\eta_\varepsilon^{1+\frac{2q-1}{q(q-1)}}}{\varepsilon^{\frac{q}{q-1}}}+\frac{\eta_\varepsilon}{\varepsilon}+\left(\frac{\eta_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\right),$$

so that (5.63) easily holds. Using Hölder's inequality with the conjugate exponents $\frac{q}{r}$ and $\frac{q}{q-r}$ we obtain

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r((0,T)\times I_{\eta_\varepsilon})^3}\le C\left(\frac{\eta_\varepsilon^{\frac{1}{r}+\frac{q}{q-1}}}{\varepsilon^{\frac{q}{q-1}}}+\frac{\eta_\varepsilon^{\frac{1}{r}-\frac{1}{q}+1}}{\varepsilon}+\frac{\eta_\varepsilon^{\frac{1}{r}-\frac{1}{q}+\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right).$$

Now we take $\eta_\varepsilon=\varepsilon^{\frac{1}{\alpha}}$. Then we find that

$$\varepsilon^{-\frac{q}{q-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r((0,T)\times I_{\eta_\varepsilon})^3}\le C\left(\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{r}+\frac{q}{q-1}\right)-\frac{q}{q-1}}+\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{r}-\frac{1}{q}+1\right)-1}+\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{r}-\frac{1}{q}+\frac{1}{2}\right)-\frac{1}{2}}\right). \tag{5.65}$$

We seek an optimal r such that the right hand side in (5.65) tends to zero. It is easy to prove that we have a convergence to zero for any $r\in\left(1,\frac{q}{q(\alpha-1)+1}\right)$. Therefore, (5.64) holds and so we have (5.62). □

5.2 Problem in the fissure part $\eta_\varepsilon\gg\varepsilon^{\frac{q}{2q-1}}$

The proof of Theorem 3.1-*ii*) will be developed in different lemmas.

Lemma 5.5. *Assume the assumptions in Lemma 4.8. Let $\eta_\varepsilon\gg\varepsilon^{\frac{q}{2q-1}}$ and let $(\mathcal{U}^{\varepsilon\eta_\varepsilon},P^{\varepsilon\eta_\varepsilon})$ be the solution of (3.6)-(3.7). Then there exist subsequences of $\mathcal{U}^{\varepsilon\eta_\varepsilon}$ and $P^{\varepsilon\eta_\varepsilon}$ still denoted by the same, and functions $\mathcal{U}\in L^q((0,T)\times I_1)^3$, with $\mathcal{U}_3=0$, $P\in L^{q'}(0,T;L^{q'}(I_1))$ such that*

$$\eta_\varepsilon^{-\frac{q}{q-1}}\mathcal{U}^{\varepsilon\eta_\varepsilon}\rightharpoonup\mathcal{U}\quad \text{in } L^q((0,T)\times I_1)^3,\quad P^{\varepsilon\eta_\varepsilon}\rightharpoonup P\quad \text{in } L^{q'}(0,T;L^{q'}(I_1)). \tag{5.66}$$

Moreover, $P=P(t,x')\in L^{q'}(0,T;W^{1,q'}(\Sigma))$ and $\tilde{\mathcal{U}}$ is given by expression (3.12).

Proof. Taking into account $\eta_\varepsilon\gg\varepsilon^{\frac{q}{2q-1}}$, estimates (4.23), (4.25), (4.39) with the change of variable (3.3), we have

$$\|\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_1)^3}\le C\eta_\varepsilon^{\frac{q}{q-1}}, \tag{5.67}$$

$$\|\nabla_{x'}\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_1)^{3\times 2}} \leq C\eta_\varepsilon^{\frac{1}{q-1}}, \tag{5.68}$$

$$\|\partial_z\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times I_1)^3} \leq C\eta_\varepsilon^{\frac{q}{q-1}}, \tag{5.69}$$

$$\|P^{\varepsilon\eta_\varepsilon}\|_{L^{q'}(0,T;L^{q'}(I_1)/\mathbb{R})} \leq C. \tag{5.70}$$

From these estimates (5.67) and (5.70), there exist $\mathcal{U} \in L^q((0,T)\times I_1)^3$, $P \in L^{q'}(0,T;L^{q'}(I_1))$ such that convergence (5.66) holds. Moreover

$$\eta_\varepsilon^{-\frac{q}{q-1}}\partial_z\mathcal{U}^{\varepsilon\eta_\varepsilon} \rightharpoonup \partial_z\mathcal{U} \quad \text{ in } L^q((0,T)\times I_1)^3.$$

Let $\varphi \in C_0^\infty((0,T)\times I_1)^3$, then

$$\begin{aligned}
&\eta_\varepsilon^{-\frac{1}{q-1}}\int_0^T\int_{I_1}\left(\mathrm{div}_{x'}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon} + \eta_\varepsilon^{-1}\partial_z\mathcal{U}_3^{\varepsilon\eta_\varepsilon}\right)\varphi\, dxdt \\
&= -\eta_\varepsilon^{-\frac{1}{q-1}}\int_0^T\int_{I_1}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon}D_{x'}\varphi\, dxdt - \eta_\varepsilon^{-\frac{q}{q-1}}\int_0^T\int_{I_1}\mathcal{U}_3^{\varepsilon\eta_\varepsilon}\cdot\partial_z\varphi\, dxdt = 0.
\end{aligned}$$

Taking the limit $\varepsilon \to 0$ we obtain

$$\int_0^T\int_{I_1}\mathcal{U}_3\partial_z\varphi\, dxdt = 0,$$

so that $\mathcal{U}_3 = \mathcal{U}_3(t,x')$.

Since $\mathcal{U}$, $\partial_z\mathcal{U} \in L^q((0,T)\times I_1)^3$ the traces $\mathcal{U}(t,x',0)$, $\mathcal{U}(t,x',1)$ are well defined in $L^q((0,T)\times\Sigma)^3$. Analogously to the proof of Lemma 4.2 we choose a point $\beta_{x'} \in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $\alpha_{x'} \in \Sigma$, then we have

$$\begin{aligned}
\int_0^T\int_\Sigma|\mathcal{U}^{\varepsilon\eta_\varepsilon}(t,x',0)|^q dx'dt &= \int_0^T\int_\Sigma|u_{\varepsilon\eta_\varepsilon}(t,x',0)|^q dx'dt \\
&\leq C\int_0^T\int_\Sigma\left(\int_{(\beta_{x'},\alpha_{x'})}Du_{\varepsilon\eta_\varepsilon}\cdot(\alpha_{x'}-\beta_{x'})d\ell\right)^q dx'dt,
\end{aligned}$$

so that, by Hölder's inequality,

$$\|\mathcal{U}^{\varepsilon\eta_\varepsilon}(t,x',0)\|^q_{L^q((0,T)\times\Sigma)^3} \leq C\varepsilon\|Du_{\varepsilon\eta_\varepsilon}\|^q_{L^q((0,T)\times D_{\varepsilon\eta_\varepsilon})^{3\times 3}}.$$

Taking into account estimate (4.25) and $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$, we have

$$\eta_\varepsilon^{-\frac{q}{q-1}}\|\mathcal{U}^{\varepsilon\eta_\varepsilon}(t,x',0)\|^q_{L^q((0,T)\times\Sigma)^3} \leq C\varepsilon\eta_\varepsilon \to 0 \quad \text{ as } \varepsilon \to 0,$$

which implies that

$$\mathcal{U}(t,x',0) = 0\,,$$

and analogously

$$\mathcal{U}(t,x',1) = 0\,.$$

Consequently

$$\mathcal{U}_3(t,x',z) = 0\,,$$

which finishes (5.66).

Finally, we choose a test function $w(x',z)\in C_0^\infty(I_1)^3$ with $w(x',z)=0$ on $\partial\Sigma$. Multiplying (3.6)-(3.7) by $w(x',z)$ and integrating by parts, we have

$$\int_{I_1}\partial_t\mathcal{U}^{\varepsilon\eta_\varepsilon}(t)\cdot w\,dx'dz+\int_{I_1}\mu S_q(\mathbb{D}[\mathcal{U}^{\varepsilon\eta_\varepsilon}(t)]):\mathbb{D}[w]\,dx'dz=\langle f(t)-\nabla P^{\varepsilon\eta_\varepsilon}(t),w\rangle_{I_1},$$

in $\mathcal{D}'(0,T)$. We observe that using (4.28) with the change of variable (3.3), the first term contributes nothing. Therefore, we consider $\phi\in C_c^1(0,T)$, multiplying by ϕ and integrating between 0 and T, we have

$$\mu\int_0^T\phi(t)\int_{I_1}S_q(\mathbb{D}[\mathcal{U}^{\varepsilon\eta_\varepsilon}(t)]):\mathbb{D}[w]\,dx'dzdt=\int_0^T\phi(t)\langle f(t)-\nabla P^{\varepsilon\eta_\varepsilon}(t),w\rangle_{I_1}\,dt+O_\varepsilon.$$

Finally, we need to indentify the limit problem for $(\tilde{\mathcal{U}},P)$ and compute the expression of the solution. For this, thanks to Propositions 3.1 and 3.2 in Mikelić and Tapiero [12] and using (5.66), we have that the limit problem is given by

$$-\partial_z\left(|\partial_z\tilde{\mathcal{U}}|^{q-2}\partial_z\tilde{\mathcal{U}}\right)=\frac{2^{\frac{q}{2}}}{\mu}\left(\tilde f(t,x',0)-\nabla_{x'}P(t,x')\right),\quad\text{in }(0,T)\times I_1,$$

$$\operatorname{div}_{x'}\left(\int_0^1\tilde{\mathcal{U}}(t,x',z)\,dz\right)=0\ \text{ in }(0,T)\times\Sigma,\quad\left(\int_0^1\tilde{\mathcal{U}}(t,x',z)\,dz\right)\cdot\tilde\nu=0\ \text{ on }(0,T)\times\partial\Sigma,$$

and taking into account Proposition 3.3. in [12] we have that $P=P(t,x')\in L^{q'}(0,T;W^{1,q'}(\Sigma))$. Then, in order to compute the expression of $\tilde{\mathcal{U}}$ given in (3.12), we refer to Proposition 3.4 in Mikelić and Tapiero [12].

□

Proof of Theorem 3.1-ii). It remains to prove the convergence (3.13) of the whole velocity to the function $\mathcal{V}$ given by (3.14), and also prove that $P\in L^{q'}(0,T;W^{1,q'}(\Sigma))$ is the unique solution of the Reynolds problem (3.15).

Taking as test function $\varphi\in C^\infty((0,T)\times D)$ in $\operatorname{div}u_{\varepsilon\eta_\varepsilon}=0$ in $(0,T)\times D$, we obtain

$$\int_0^T\int_D\operatorname{div}u_{\varepsilon\eta_\varepsilon}\varphi\,dxdt=-\int_0^T\int_D v_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dxdt-\eta_\varepsilon\int_0^T\int_{I_1}\mathcal{U}^{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi(t,x',\eta_\varepsilon z)\,dx'dzdt=0,$$

so that multiplying by $\eta_\varepsilon^{-\frac{2q-1}{q-1}}$,

$$\begin{aligned}&\int_0^T\int_{I_1}\eta_\varepsilon^{-\frac{q}{q-1}}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon}\cdot\nabla_{x'}\varphi(t,x',\eta_\varepsilon z)\,dx'dzdt\\&=-\int_0^T\int_D\eta_\varepsilon^{-\frac{2q-1}{q-1}}v_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dxdt-\int_0^T\int_{I_1}\eta_\varepsilon^{-\frac{q}{q-1}}\mathcal{U}_3^{\varepsilon\eta_\varepsilon}\partial_3\varphi(t,x',\eta_\varepsilon z)\,dx'dzdt.\end{aligned}\tag{5.71}$$

Using (4.22) and taking into account $\eta_\varepsilon\gg\varepsilon^{\frac{q}{2q-1}}$, we obtain

$$\eta_\varepsilon^{-\frac{2q-1}{q-1}}\|v_{\varepsilon\eta_\varepsilon}\|_{L^q((0,T)\times D)^3}\le C\left(\frac{\varepsilon}{\eta_\varepsilon^{\frac{2q-1}{q}}}+\frac{\varepsilon^{\frac{q}{q-1}}}{\eta_\varepsilon^{\frac{2q-1}{q-1}}}\right)\to0\quad\text{ as }\varepsilon\to0.\tag{5.72}$$

Taking the limit in (5.71) as $\varepsilon \to 0$, using (5.72), the convergence (5.66) and $\mathcal{U}_3 = 0$, we have

$$\int_0^T \int_{I_1} \tilde{\mathcal{U}} \cdot \nabla_{x'} \varphi(t, x', 0)\, dx'dzdt = 0,$$

and by definition (3.14), we get the Reynolds problem (3.15). Consequently, P and is the unique solution of (3.15) (see Proposition 3.4 in Mikelic and Tapiero [12] for more details).

Finally, we consider $\varphi \in C_0((0,T) \times D)^3$ and so we have

$$\int_0^T \int_D \eta_\varepsilon^{-\frac{2q-1}{q-1}} u_{\varepsilon\eta_\varepsilon} \varphi\, dxdt = \int_0^T \int_D \eta_\varepsilon^{-\frac{2q-1}{q-1}} v_{\varepsilon\eta_\varepsilon} \varphi\, dxdt + \int_0^T \int_{I_1} \eta_\varepsilon^{-\frac{q}{q-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon} \varphi(t, x', \eta_\varepsilon z)\, dx'dzdt.$$

Using (5.72) and convergence (5.66) and $\mathcal{U}_3 = 0$, we obtain

$$\begin{aligned}\int_0^T \int_D \eta_\varepsilon^{-\frac{2q-1}{q-1}} u_{\varepsilon\eta_\varepsilon} \varphi\, dxdt \to \quad & \int_0^T \int_{I_1} \tilde{\mathcal{U}}(t, x', z)\tilde{\varphi}(t, x', 0)\, dx'dzdt \\ = & \int_0^T \int_\Sigma \tilde{\mathcal{V}}(t, x')\tilde{\varphi}(t, x', 0)\, dx'dt = \int_0^T \langle \mathcal{V}(t, x')\delta_\Sigma, \varphi\rangle_{\mathcal{M}(D)^3, C_0(D)^3} dt,\end{aligned}$$

which implies (3.13). □

5.3 Effects of coupling $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$

The conclusion of the previous two subsections is that for any sequence of solutions $(v_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ with $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$ and $(\mathcal{U}^{\varepsilon\eta_\varepsilon}, P^{\varepsilon\eta_\varepsilon})$ with $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$, and letting $\varepsilon \to 0$, we can extract subsequences still denoted by $v_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$, $\mathcal{U}^{\varepsilon\eta_\varepsilon}$, $P^{\varepsilon\eta_\varepsilon}$ and find functions $v \in L^q((0,T) \times D)^3$, $p \in L^{q'}(0,T; W^{1,q'}(D))$, $\tilde{\mathcal{U}} \in L^q((0,T) \times I_1)^2$, $P \in L^{q'}(0,T; W^{1,q'}(\Sigma))$ such that

$$\begin{aligned}&\varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \text{in } L^q((0,T) \times D)^3, \quad p_{\varepsilon\eta_\varepsilon} \to p \quad \text{in } L^{q'}(0,T; L^{q'}(D)), \\ &\eta_\varepsilon^{-\frac{q}{q-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon} \rightharpoonup \mathcal{U} \quad \text{in } L^q((0,T) \times I_1)^3 \text{ with } \mathcal{U} = (\tilde{\mathcal{U}}, 0), \quad P^{\varepsilon\eta_\varepsilon} \rightharpoonup P \quad \text{in } L^{q'}(0,T; L^{q'}(I_1)).\end{aligned} \tag{5.73}$$

Moreover such limit functions v, p, $\mathcal{U}$, P necessarily satisfy the equations

$$\begin{aligned}&v = \frac{1}{\mu} K\left(f(t,x) - \nabla p(t,x)\right) \quad \text{in } (0,T) \times D, \quad v \cdot n = 0 \;\text{ on } (0,T) \times \partial D, \\ &\tilde{\mathcal{U}} = \frac{2^{\frac{q'}{2}}}{q'\mu^{q'-1}} \left(\left(\frac{1}{2}\right)^{q'} - \left|\frac{1}{2} - z\right|^{q'} \right) S_{q'}\left(\tilde{f}(t, x', 0) - \nabla_{x'} P(t, x')\right) \quad \text{in } (0,T) \times I_1, \\ &\tilde{\mathcal{U}} \cdot \tilde{n} = 0 \;\text{ on } (0,T) \times \partial\Sigma.\end{aligned} \tag{5.74}$$

We are going to find the connection between the functions p and P, i.e. to find the coupling effects between the solution in the porous part and in the fissure part.

Lemma 5.6. *Assume the assumptions in Lemma 4.8. Let $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$, with $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}} \to \lambda$, $0 < \lambda < +\infty$, and let $\{p_{\varepsilon\eta_\varepsilon}\} \in L^{q'}(0,T; L^{q'}(D))$, $p \in L^{q'}(0,T; W^{1,q'}(D))$, $P \in L^{q'}(0,T; W^{1,q'}(\Sigma))$ be such that (5.73) and (5.74) hold. Then,*

$$\begin{aligned}&\int_0^T \int_D \frac{1}{\mu} K\left(f(t,x) - \nabla p(t,x)\right) \cdot \nabla\varphi(t,x)\, dxdt \\ &+ \lambda^{\frac{2q-1}{q-1}} \int_0^T \int_\Sigma \frac{1}{2^{\frac{q'}{2}}(q+1)\mu^{q'-1}} S_{q'}\left(\tilde{f}(t, x', 0) - \nabla_{x'} P(t, x')\right) \cdot \nabla_{x'}\varphi(t, x', 0)\, dx'dt = 0,\end{aligned} \tag{5.75}$$

for every $\varphi \in V_\Sigma$.

Proof. Let $\varphi \in V_\Sigma$. Taking into account the definitions (5.44) of $v_{\varepsilon\eta_\varepsilon}$ and (3.4) of $\mathcal{U}^{\varepsilon\eta_\varepsilon}$, and from $\operatorname{div} u_{\varepsilon\eta_\varepsilon} = 0$ in $(0,T)\times D$ we have

$$\begin{aligned}&\int_0^T\int_D \varepsilon^{-\frac{q}{q-1}} u_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dxdt\\ &=\int_0^T\int_D \varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dxdt + \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{q}{2q-1}}}\right)^{\frac{2q-1}{q-1}}\int_0^T\int_{I_1} \eta_\varepsilon{}^{-\frac{q}{q-1}}\mathcal{U}^{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi(t,x',\eta_\varepsilon z)\,dx'dzdt = 0.\end{aligned}$$

Taking the limit as $\varepsilon\to 0$, using (5.73), $\mathcal{U}_3 = 0$ and $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}}\to\lambda$, we obtain

$$\int_0^T\int_D v(t,x)\cdot\nabla\varphi(t,x)\,dxdt + \lambda^{\frac{2q-1}{q-1}}\int_0^T\int_{I_1}\tilde{\mathcal{U}}(t,x',z)\cdot\nabla_{x'}\varphi(t,x',0)\,dx'dzdt = 0,$$

and taking into account expressions (3.14) and (5.74), we get (5.75). □

In the following result, we are going to prove the relation between the pressure p and P.

Lemma 5.7. *Assume the assumptions in Lemma 4.8. Let $\eta_\varepsilon \approx \varepsilon^{\frac{q}{2q-1}}$, $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}}\to\lambda$, $0<\lambda<+\infty$, and let p, P be the limit pressures from (5.73). Then, there exists $\widetilde{C}\in\mathbb{R}$ such that*

$$p(t,x',0) = P(t,x') + \widetilde{C}, \tag{5.76}$$

and $p\in V_\Sigma$ is the unique solution of the variational problem (3.17).

Proof. We need to extend the test functions considered in the proof of Lemma 5.4 to the fissure I_{η_ε}. To do this, we define $B_{\eta_\varepsilon} = D_-\cup\Sigma\cup I_{\eta_\varepsilon}$ and $Y' = \overline{Y}^*\cap\{x_3=0\}$, and we consider $\phi(y)\in C^\infty_\#(B_{\eta_\varepsilon})^3$ be such that $\phi(y)=0$ in $Y\setminus Y^*$ and $\operatorname{div}_y\phi(y)=0$ in Y^*. We define

$$\phi_\varepsilon(x) = \begin{cases}\phi\left(\frac{x}{\varepsilon}\right) & \text{in } D_-\,,\\ K_3\,e_3 & \text{in } I_{\eta_\varepsilon},\quad \text{where } K_3 = \displaystyle\int_{Y'}\phi_3(y',0)dy'\,.\end{cases}$$

Let $\varphi\in C_0^\infty((0,T)\times B_1)$, with $B_1 = D_-\cup\Sigma\cup I_1$ be such that

$$\int_\Sigma \varphi(t,x',0)\,dx' = 0 \quad \text{a.e. } t\in(0,T). \tag{5.77}$$

We define

$$\varphi_{\eta_\varepsilon}(t,x) = \begin{cases}\varphi(t,x) & \text{in } (0,T)\times D_-\\ \varphi\left(t,x',\frac{x_3}{\eta_\varepsilon}\right) & \text{in } (0,T)\times I_{\eta_\varepsilon}.\end{cases}$$

Reasoning as in the first part of Lemma 5.4, we take in (3.1)-(3.2) as test function

$$w_\varepsilon(t,x) = \begin{cases}\varphi(t,x)\phi\left(\frac{x}{\varepsilon}\right) - \varepsilon^{-\frac{q}{q-1}}v_{\varepsilon\eta_\varepsilon} & \text{in } (0,T)\times D_-,\\ \varphi\left(t,x',\frac{x_3}{\eta_\varepsilon}\right)K_3\,e_3 & \text{in } (0,T)\times I_{\eta_\varepsilon},\end{cases}$$

and we obtain

$$\mu\int_0^T\int_{B_{\eta_\varepsilon}} S_q(\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]) : Dw_\varepsilon\,dxdt = \int_0^T\int_{B_{\eta_\varepsilon}} f\cdot w_\varepsilon\,dxdt - \int_0^T\int_{B_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}\operatorname{div} w_\varepsilon\,dxdt + O_\varepsilon. \tag{5.78}$$

Taking into account that

$$K_3 \int_0^T \int_{I_{\eta_\varepsilon}} f \cdot \varphi \left(t, x', \frac{x_3}{\eta_\varepsilon}\right) e_3 \, dxdt = \eta_\varepsilon K_3 \int_0^T \int_{I_1} f \cdot \varphi(t, x', z) e_3 \, dx'dzdt \to 0 \quad \text{as } \varepsilon \to 0,$$

and by using estimates (5.68), (5.69), that

$$\begin{aligned}
&\left| K_3 \int_0^T \int_{I_{\eta_\varepsilon}} S_q(\mathbb{D}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]) \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt \right| = \left| K_3 \int_0^T \int_{I_1} S_q(\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]) \partial_z \varphi(t, x', z) \, dx'dzdt \right| \\
&\leq C \eta_\varepsilon^{\frac{1}{q-1}} \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

from (5.78), we obtain

$$\begin{aligned}
\mu \int_0^T \int_{D_-} S_q(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]) : Dw_\varepsilon \, dxdt &= \int_0^T \int_{D_-} f \cdot w_\varepsilon \, dxdt + \int_0^T \int_{D_-} p_{\varepsilon\eta_\varepsilon} \operatorname{div} w_\varepsilon \, dxdt \\
&+ K_3 \int_0^T \int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon} \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt + O_\varepsilon.
\end{aligned} \tag{5.79}$$

For the last term on the right hand side, we have

$$\begin{aligned}
K_3 \int_0^T \int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon} \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt &= K_3 \int_0^T \int_{I_{\eta_\varepsilon}} (p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}) \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt \\
&+ K_3 \int_0^T \int_{I_{\eta_\varepsilon}} c_{\varepsilon\eta_\varepsilon} \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt,
\end{aligned}$$

where $c_{\varepsilon\eta_\varepsilon}$ is defined in (3.5). Using (5.73), we obtain

$$\begin{aligned}
&K_3 \int_0^T \int_{I_{\eta_\varepsilon}} (p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}) \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt = K_3 \int_0^T \int_{I_1} P^{\varepsilon\eta_\varepsilon} \partial_z \varphi(t, x', z) \, dx'dzdt \\
&\to K_3 \int_0^T \int_{I_1} P(t, x') \partial_z \varphi(t, x', z) \, dx'dzdt = -K_3 \int_0^T \int_\Sigma P(t, x') \varphi(t, x', 0) \, dx'dt, \qquad \text{as } \varepsilon \to 0,
\end{aligned} \tag{5.80}$$

where $P^{\varepsilon\eta_\varepsilon}$ is given by (3.4), and using (5.77), we obtain

$$K_3 \int_0^T c_{\varepsilon\eta_\varepsilon} \int_{I_{\eta_\varepsilon}} \partial_{x_3} \varphi(t, x', \frac{x_3}{\eta_\varepsilon}) \, dxdt = K_3 \int_0^T c_{\varepsilon\eta_\varepsilon} \int_{I_1} \partial_z \varphi(t, x', z) \, dx'dzdt = 0.$$

Passing to the limit in (5.79) similarly as in the proof of Lemma 5.4, we know that $\hat{v}$ and p are related by the variational formulation of problem (5.59)-(5.61), and taking into account (5.80) and

$$\begin{aligned}
&\int_0^T \int_{D \times Y} p(t, x) \operatorname{div}_x(\varphi(t, x) \phi(t, y)) \, dxdydt \\
&= -\int_0^T \int_{D \times Y} \nabla_x p(t, x) \varphi(t, x) \phi(t, y) \, dxdydt + \int_0^T \int_{\Sigma \times Y'} p(t, x', 0) \varphi(t, x', 0) \phi_3(t, y', 0) \, dx'dy'dt \\
&= -\int_0^T \int_{D \times Y} \nabla_x p(t, x) \varphi(t, x) \phi(t, y) \, dxdydt + K_3 \int_0^T \int_\Sigma p(t, x', 0) \varphi(t, x', 0) \, dx'dt,
\end{aligned}$$

then we have

$$\int_0^T \int_\Sigma \big(p(t,x',0) - P(t,x')\big)\, \varphi(t,x',0)\, dx'dt = 0,$$

so that

$$\int_0^T \int_\Sigma \big(p(t,x',0) - P(t,x')\big)\, \psi(t,x')\, dx'dt = 0,$$

for every $\psi \in C_0^\infty((0,T)\times\Sigma)$ such that $\int_\Sigma \psi\, dx' = 0$ a.e. $t \in (0,T)$. Finally we conclude that there exists a constant $\widetilde{C} \in \mathbb{R}$ such that (5.76) holds and $p(t,x',0) \in L^{q'}(0,T;W^{1,q'}(\Sigma))$, i.e. $p \in V_\Sigma$.

Using (5.75) and (5.76), we obtain the variational formulation (3.17) for the limit pressure p in the space V_Σ. Since K and $S_{q'}$ are coercive and monotone (see Remark 3.2 for more details), it can be proved that (3.17) has a unique solution in that Banach space $V_\Sigma/\mathbb{R}$ equipped with the norm $|v|_{V_\Sigma} = |v|_{L^{q'}(0,T;W^{1,q'}(D))} + |v(\cdot,0)|_{L^{q'}(0,T;W^{1,q'}(\Sigma))}$, by direct application of Lax-Milgram Theorem. Therefore, the whole sequence converges to p, the unique solution of the problem (3.17). $\square$

Proof of Theorem 3.1-iii). It remains to prove the convergence (3.16) of the whole velocity.

Let $\varphi \in C_0((0,T)\times D)^3$. Then

$$\begin{aligned}\int_0^T \int_D \varepsilon^{-\frac{q}{q-1}} u_{\varepsilon\eta_\varepsilon}\cdot\varphi\, dxdt &= \int_0^T \int_D \varepsilon^{-\frac{q}{q-1}} v_{\varepsilon\eta_\varepsilon}\cdot\varphi\, dxdt \\ &+ \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{q}{2q-1}}}\right)^{\frac{2q-1}{q-1}} \int_0^T \int_{I_1} \eta_\varepsilon^{-\frac{q}{q-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon}\cdot\varphi(t,x',\eta_\varepsilon z)\, dx'dzdt = 0.\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, using (5.73), $\mathcal{U}_3 = 0$ and $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}} \to \lambda$, we obtain

$$\int_0^T \int_D \varepsilon^{-\frac{q}{q-1}} u_{\varepsilon\eta_\varepsilon}\cdot\varphi\, dxdt \to \int_0^T \int_D v\cdot\varphi\, dxdt + \lambda^{\frac{2q-1}{q-1}} \int_0^T \int_{I_1} \tilde{\mathcal{U}}(t,x',z)\varphi(t,x',0)\, dx'dzdt.$$

Taking into account that

$$\int_0^T \int_{I_1} \tilde{\mathcal{U}}(t,x',z)\varphi(t,x',0)\, dx'dzdt = \int_0^T \int_\Sigma \mathcal{V}(t,x')\varphi(t,x',0)\, dx'dt = \int_0^T \langle \mathcal{V}\delta_\Sigma, \varphi\rangle_{\mathcal{M}(D)^3, C_0(D)^3} dt,$$

where $\mathcal{V}(t,x')$ is given by (3.14), we get (3.16). $\square$

6 Conclusions

In this paper, we consider a fluid flow in a porous media (with periodically distributed obstacles of size ε) when a thin fissure of width η_ε is present in the media. If the model is a Stokes flow, Bourgeat *et al.* [5] proved that there is a critical size, namely $\eta_\varepsilon \approx \varepsilon^{3/2}$, below which the fissure does not play a role and above which it is dominant. At the critical size, a coupled problem appears.

The reason for the interest in such models comes from Hydrocarbon exploration, and the need to model cracks in geological strata. Regular oil is often simplified as Newtonian, but it is better modeled with a shear thinning ($1 < q < 2$) law. In this sense, we consider a non-stationary non-Newtonian power-law fluid with $\frac{5}{3} \leq q \leq 2$ (the linear case appears here also). The main result in this paper (Theorem 3.1) could be summarized by the following expansion for the velocity field

$$\varepsilon^{-\frac{q}{q-1}} u_{\varepsilon\eta_\varepsilon} \sim v + \lambda^{\frac{2q-1}{q-1}} \mathcal{V}\delta_\Sigma,$$

where v is a Darcy flow coming the homogenization in the porous media and $\mathcal{V}$ is a Reynolds flow tangent to the fissure.

We see that depending on λ, where $\eta_\varepsilon/\varepsilon^{\frac{q}{2q-1}} \to \lambda$, $0 \leq \lambda \leq +\infty$, one or both flow will be dominant in the limit. In Theorem 3.1, we investigate all three cases $\lambda = 0$, $\lambda = +\infty$ and $\lambda \in (0, +\infty)$. This last case (case iii) in Theorem 3.1 is the most interesting since both flows have the same order, leading to the variational problem (3.17) for the pressure.

Observe that, formally, (3.17) is the weak formulation of the following boundary value problem

$$\left\{ \begin{array}{rcl} -\mathrm{div}\, v(t,x) - \lambda^{\frac{2q-1}{q-1}} \mathrm{div}_{x'} \left(\tilde{\mathcal{V}}(t,x')\delta_\Sigma \right) & = & 0 \ \text{ in } (0,T) \times D, \\ v(t,x) \cdot n + \lambda^{\frac{2q-1}{q-1}} \tilde{\mathcal{V}}(t,x')\delta_{\partial\Sigma} \cdot \tilde{n} & = & 0 \ \text{ on } (0,T) \times \partial D. \end{array} \right. \tag{6.81}$$

In the case $\lambda = 0$, i.e. $\eta_\varepsilon \ll \varepsilon^{\frac{q}{2q-1}}$, then the fissure is not giving any contribution. In fact, if λ tends to zero in (6.81) we obtain the Darcy's law (3.9).

On the other hand, in the case $\lambda = +\infty$, i.e. $\eta_\varepsilon \gg \varepsilon^{\frac{q}{2q-1}}$, then the fissure is dominant. In fact, multiplying (6.81) by $\lambda^{-\frac{2q-1}{q-1}}$ and tending λ to $+\infty$, we obtain the Reynolds problem (3.15).

Using the present study as a starting point, various improvements can be proposed. The first one is the generalization of the asymptotic study, which leads to the coupled Darcy Reynolds equation, to a truly non-stationary nonlinear Navier-Stokes system (and not only Stokes system). Another possible way is to study by means of homogenization techniques the modeling of two fluid flows through fractured porous media. Finally, another problem could be introducing micro-roughness for the fissure. Mathematical models of such domain include several small parameters, one is connected to the fissure thickness and the others to the microstructure. This approach could be very interesting, as it combines the effect of surface roughness in full film lubrication with the behavior of the flow in the porous media.

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