Asymptotic behaviour of the nonautonomous SIR equations with diffusion*

María Anguiano
Dpto. de Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla, Apdo. de Correos 1160
41080 Sevilla, Spain
E-mail: anguiano@us.es

P.E. Kloeden
Institut für Mathematik, Goethe Universität
D-60054 Frankfurt am Main, Germany
E-mail: kloeden@math.uni-frankfurt.de

Abstract

The existence and uniqueness of positive solutions of a nonautonomous system of SIR equations with diffusion are established as well as the continuous dependence of such solutions on initial data. The proofs are facilitated by the fact that the nonlinear coefficients satisfy a global Lipschitz property due to their special structure. An explicit disease-free nonautonomous equilibrium solution is determined and its stability investigated. Uniform weak disease persistence is also shown. The main aim of the paper is to establish the existence of a nonautonomous pullback attractor is established for the nonautonomous process generated by the equations on the positive cone of an appropriate function space. For this an energy method is used to determine a pullback absorbing set and then the flattening property is verified, thus giving the required asymptotic compactness of the process.

1 Introduction and setting of the problem

The SIR model for the transmission of infectious diseases, which was introduced by Kermack and McKendrick [18] in 1927, is one of the fundamental models of

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mathematical epidemiology [1, 4, 34]. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptibles S, infectives I and recovereds R, of a constant total population.

Many generalizations of this model have been proposed and studied, for instance, to include age structure, time delays, spatial diffusion and variable infectivity; see, for example, [4, 5, 9, 13, 16]. Most papers have focused on the persistence and extinction of the disease, the existence of the threshold value for which the infectious disease will grow or die out, the local and global stability of disease-free and endemic equilibria, and the existence of periodic solutions. Persistence is an important property of dynamical systems in epidemiology and ecology, which addresses the long-term survival of some or all components of a system. See Smith and Thieme [28] and Thieme [32, 33] for background information and references. Loosely speaking, a population is uniformly weakly persistent if its size, while it may come arbitrarily close to 0 every now and then, always climbs back to a level that eventually is independent of the initial data.

More recently, nonautonomous versions of the SIR model and related epidemic systems for which the total population may vary in time have been investigated [15, 19, 23, 26, 32, 33]. Webb [36] introduced a spatially inhomogeneous version of the SIR model in a bounded environment in terms of a system of parabolic partial differential equations in 1981. Many variations of this model have since been considered [12, 26]. Discrete analogs have been considered in which people move between cities (see [2, 3, 11] and Section 3.2. in [28]). These models allow different moving rate for susceptibles and infective individuals, but are time-autonomous.

In this paper we will analyse the asymptotic behavior of a temporally-forced SIR model with diffusion on a bounded domain $\Omega \subset \mathbb{R}^d$, where $d \geq 1$, with a smooth boundary $\partial\Omega$ from the perspective of the theory of nonautonomous dynamical systems [6, 21, 27]. Temporally varying forcing is typical of seasonal variation of a disease [17, 29]. In particular, we consider the system of parabolic partial differential equations

$$\left.\begin{aligned} \frac{\partial S}{\partial t} - \Delta S &= aq(t) - aS + bI - \gamma \tfrac{SI}{N}, \\ \frac{\partial I}{\partial t} - \Delta I &= -(a+b+c)I + \gamma \tfrac{SI}{N}, \\ \frac{\partial R}{\partial t} - \Delta R &= cI - aR, \end{aligned}\right\} \tag{1}$$

where

$$N(t) = S(t) + I(t) + R(t),$$

with the Dirichlet boundary condition

$$S(x,t) = I(x,t) = R(x,t) = 0 \text{ on } \quad \partial\Omega \times (t_0, +\infty) \tag{2}$$

and initial condition

$$S(x,t_0) = S_0(x), \quad I(x,t_0) = I_0(x), \quad R(x,t_0) = R_0(x) \text{ for } \ x \in \Omega, \tag{3}$$

where $t_0 \in \mathbb{R}$ and the parameters a, b, c and γ are positive constants. The temporal forcing term is given by a continuous function $q : \mathbb{R} \to \mathbb{R}$ taking positive bounded values, i.e., $q(t) \in [q^-, q^+]$ for all $t \in \mathbb{R}$, where $0 < q^- \leq q^+$.

The system (1) is chosen as a representative model amongst other possibilities with the main aim of paper being to establish the existence of a nonautonomous pullback attractor, which contains the counterparts of equilibria and limit cycles of autonomous systems. The particular choices of frequency-dependent incidence and nonautonomous terms give the nonautonomous model (1) a quasiautonomous feature as the sharp threshold conditions for disease extinction can be chosen to be time independent. Other choices like mass action (density-dependent) incidence or time-dependent infection rates would require considering time-averages to get sharp extinction conditions, if they could be found at all (see [32, 33], Chapters 13 and 15 in [28]).

The following functions spaces will be used. $L^2(\Omega)$ denotes the space of square integrable real valued functions defined on Ω with the norm $|\cdot|_{L^2(\Omega)}$ corresponding to the scalar product defined

$$(u, v)_{L^2(\Omega)} = \int_\Omega u \cdot v dx$$

for all u, $v \in L^2(\Omega)$, while $H_0^1(\Omega)$ denotes the space of such functions satisfying the Dirichlet boundary condition that have square integrable generalized derivatives with the scalar product

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$$

for all $u \in H_0^1(\Omega)$ and the norm $|u|_{H_0^1(\Omega)} := |\nabla u|_{L^2(\Omega)}$. In addition, X_3 denotes the space of functions $(u_1, u_2, u_3) \in L^2(\Omega)^3$ with the scalar product

$$((u_1, u_2, u_3), (v_1, v_2, v_3))_{L^2(\Omega)} = (u_1, v_1)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\Omega)},$$

and norm

$$|(u_1, u_2, u_3))|_{L^2(\Omega)} = |u_1|_{L^2(\Omega)} + |u_2|_{L^2(\Omega)} + |u_3|_{L^2(\Omega)}$$

for all (u_1, u_2, u_3),$(v_1, v_2, v_3) \in X_3$. Finally, let X_3^+ be the subspace of nonnegative functions in X_3.

Recent developments in the theory of nonautonomous dynamical systems [8, 21] offer new concepts, which allow the asymptotic dynamics to be characterized. The main aim of this paper is to show the existence of a pullback attractor for the process associated to (1)–(3) using the theory of nonautonomous dynamical systems formulated as a process, which is also called two-parameter semi-group. The pullback attractor is the nonautonomous counterpart of an autonomous attractor and similarly contains the limiting behaviour. The choice of Dirichlet boundary conditions and the space $L^2(\Omega)$ here are to facilitate the derivation of the required estimates. Solutions in the space $L^1(\Omega)$ are more typical in many biological situations, but due to the special structure of the system (and its possible variants) we note that the solutions have stronger regularity, in particular are also in the space $L^{\infty}(\Omega)$, and $L^{1}(\Omega) \cap L^{\infty}(\Omega)$ is a subspace of $L^{2}(\Omega)$.

The structure of the paper is as follows. In Section 2, partly following [28], we analyze the total population equation and its asymptotic behaviour. In Section 3 we establish the existence and uniqueness of a positive solution for our model and we prove a continuous dependence result with respect to initial data. (Due to their special structure, the nonlinear coefficients are, in fact, globally Lipschitz, which is also verified here). A result about the asymptotic stability (in the forward sense) for $\gamma < \lambda_1 + a + b + c$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ on the domain Ω with a Dirichlet boundary condition, of a nonautonomous equilibrium solution representing a disease-free solution and its loss of linear stability are addressed in Section 4. In Section 5 we establish uniform weak persistence of the disease if $\gamma > \lambda_1 + a + b + c$. In Section 6, the main goal of proving the existence of a family of a pullback attractor or the attracting universe of fixed bounded sets is established via the flattening property.

Remark 1 *The assumption that $q^{-} > 0$ is not essential for the existence of the pullback attractor, but without it the persistence result need not hold. For example, if $q(t) \to 0$ exponentially fast as $t \to 0$, then the total population $N(t) \to 0$ as $t \to 0$ and hence so do the component populations. See also Remark 11 in Section 5.*

2 Total population equation

Adding both sides of the above PDEs gives

$$\frac{\partial N}{\partial t} + [-\Delta + a]\, N = aq(t), \tag{4}$$

where $N(t) = S(t) + I(t) + R(t)$ is the total population. Standard existence and uniqueness theorems for scalar reaction-diffusion equations provide the existence and uniqueness of a solution in $L^{2}(\Omega)$ for the mild form of the PDE (4) with a Dirichlet boundary condition. By the variation of constants formula this solution is given explicitly by

$$N(t) = e^{-\widetilde{A}(t-t_0)} N(t_0) + a e^{-\widetilde{A}t} \int_{t_0}^{t} e^{\widetilde{A}r} q(r)\, dr,$$

where $\widetilde{A}$ is the linear operator associated with $-\Delta + a$ on the domain Ω with the Dirichlet boundary condition.

From (4) we obtain the energy equality

$$\frac{d}{dt} |N(t)|^2_{L^2(\Omega)} + 2 |\nabla N(t)|^2_{L^2(\Omega)} = -2a |N(t)|^2_{L^2(\Omega)} + 2a \left(q(t), N(t)\right)_{L^2(\Omega)},$$

and hence, by the Poincaré inequality,

$$\frac{d}{dt}\left|N(t)\right|_{L^2(\Omega)}^2 + 2(\lambda_1 + a)\left|N(t)\right|_{L^2(\Omega)}^2 \leq 2\left(aq(t), N(t)\right)_{L^2(\Omega)},$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ on the domain Ω with Dirichlet boundary condition. Since

$$2\left(aq(t), N(t)\right)_{L^2(\Omega)} \leq \frac{(aq^+)^2}{\lambda_1 + a}\left|\Omega\right| + (\lambda_1 + a)\left|N(t)\right|_{L^2(\Omega)}^2$$

we have the differential inequality

$$\frac{d}{dt}\left|N(t)\right|_{L^2(\Omega)}^2 + (\lambda_1 + a)\left|N(t)\right|_{L^2(\Omega)}^2 \leq \frac{(aq^+)^2}{\lambda_1 + a}\left|\Omega\right|.$$

Multiplying both sides by the integrating factor $e^{(\lambda_1+a)t}$ and integrating between t_0 and t, then simplifying, we obtain

$$\left|N(t)\right|_{L^2(\Omega)}^2 \leq \left(\frac{aq^+}{\lambda_1 + a}\right)^2\left|\Omega\right| + \left|N_0\right|_{L^2(\Omega)}^2 e^{-(\lambda_1+a)(t-t_0)}. \tag{5}$$

The PDE (4) has what Chueshov [10] called a *nonautonomous equilibrium solution*, which is found by taking the pullback limit (i.e., as $t_0 \to -\infty$ with t held fixed, see [21]), namely

$$\widehat{N}(x,t) = ae^{-\widetilde{A}t}\int_{-\infty}^{t} e^{\widetilde{A}r}q(r)dr. \tag{6}$$

This also forward attracts all other solution of the PDE (4). We observe that

$$\frac{\partial}{\partial t}(N - \widehat{N}) + \left[-\Delta + a\right](N - \widehat{N}) = 0$$

with Dirchlet boundary conditions. As above, using the Poincaré inequality, we obtain the differential inequality

$$\frac{d}{dt}\left|N(t) - \widehat{N}(t)\right|_{L^2(\Omega)}^2 + 2(\lambda_1 + a)\left|N(t) - \widehat{N}(t)\right|_{L^2(\Omega)}^2 \leq 0.$$

Hence

$$\left|N(t) - \widehat{N}(t)\right|_{L^2(\Omega)}^2 \leq e^{-2(\lambda_1+a)(t-t_0)}\left|N(t_0) - \widehat{N}(t_0)\right|_{L^2(\Omega)}^2 \to 0 \text{ as } t \to \infty. \tag{7}$$

Then, we can deduce

$$N(x,t) - \widehat{N}(x,t) \to 0 \text{ } a.e. \text{ in } \Omega \times (t_0, +\infty) \text{ as } t \to \infty. \tag{8}$$

Summarizing

Theorem 2 *The nonautonomous equilibrium solution $\widehat{N}(x,t)$ is globally asymptotically stable (in the forward and pullback senses) with respect to both L^2 and almost everywhere convergence.*

3 Existence and uniqueness of solutions

We state and sketch the proof of a result on the existence and uniqueness of positive solutions of (1)–(3) for initial data in X_3^+. Similar results can be found in [24, 25, 34].

Theorem 3 *For any initial value $(S_0, I_0, R_0) \in X_3^+$, there exists a unique positive solution*

$$(S(t), I(t), R(t)) = (S(t, t_0; S_0, I_0, R_0), I(t, t_0; S_0, I_0, R_0), R(t, t_0; S_0, I_0, R_0))$$

of the problem (1)–(3) on $t \geq t_0$.

Proof. Let $(S(t), I(t), R(t))$ be a solution of (1)–(3) with initial condition (S_0, I_0, R_0). If we denote

$$\begin{aligned} f_1(S, I, R, t) &:= aq(t) - aS + bI - \gamma \frac{SI}{N}, \\ f_2(S, I, R, t) &:= -(a + b + c)I + \gamma \frac{SI}{N}, \\ f_3(S, I, R, t) &:= cI - aR, \end{aligned}$$

it is easy to verify that non-negative initial data imply non-negative solutions using [12, Theorem 2.1], since

$$f_1(0, I, R, t) \geq aq^+ + bI > 0, \; f_2(S, 0, R, t) = 0, \; f_3(S, I, 0, t) = cI \geq 0.$$

On the other hand, there is a unique local solution of (1)–(3) since the coefficients of the equation are, in fact, globally Lipschitz for any given non-negative initial value (S_0, I_0, R_0) in X_3^+. (This is true also for the nonlinear terms due to their special structure, which will be shown in the proof of Theorem 5 below). Finally, thanks to the upper bounds (5), the positive solutions of (1)–(3) are always bounded. In particular, all solutions of (1)–(3) are globally defined. ∎

3.1 Solution mapping as a process

The globally defined nonnegative solutions of (1)–(3) generate a nonautonomous 2-parameter semigroup or process in the Banach space X_3^+, i.e., a family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$ with $t \geq t_0$ in $\mathbb{R}$ satisfying

$$U_{t,t_0} x = x, \quad U_{t,t_0} x = U_{t,r} \circ U_{r,t_0} x \tag{9}$$

for all $t_0 \leq r \leq t$ and $x \in X_3^+$.

From Theorem 3 and Theorem 5 we have.

Proposition 4 *The 2-parameter family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$, $t_0 \leq t$, given by*

$$U_{t,t_0}(S_0, I_0, R_0) = (S(t), I(t), R(t)), \tag{10}$$

where $(S(t), I(t), R(t))$ is the unique positive solution of (1)–(3) with the initial value (S_0, I_0, R_0) defines a continuous process on X_3^+.

3.2 Continuity in initial data

Theorem 5 *The process defined by (10) is continuous in* X_3^+.

Proof. We denote

$$(\overline{S}, \overline{I}, \overline{R}) := (S_1, I_1, R_1) - (S_2, I_2, R_2),$$

where (S_1, I_1, R_1) is the solution of (1)–(3) for the initial condition (S_0^1, I_0^1, R_0^1) and (S_2, I_2, R_2) is the solution for the initial condition (S_0^2, I_0^2, R_0^2). Then, $(\overline{S}, \overline{I}, \overline{R})$ is the solution for the following problem

$$\left.\begin{aligned}
\frac{\partial \overline{S}}{\partial t} - \Delta \overline{S} &= -a\overline{S} + b\overline{I} - \gamma F_{1,2}, \\
\frac{\partial \overline{I}}{\partial t} - \Delta \overline{I} &= -(a+b+c)\overline{I} + \gamma F_{1,2}, \\
\frac{\partial \overline{R}}{\partial t} - \Delta \overline{R} &= c\overline{I} - a\overline{R},
\end{aligned}\right\} \tag{11}$$

in $\Omega \times (t_0, +\infty)$ with Dirichlet boundary condition

$$\overline{S}(x,t) = \overline{I}(x,t) = \overline{R}(x,t) = 0 \text{ on } \quad \partial\Omega \times (t_0, +\infty) \tag{12}$$

and initial condition

$$\overline{S}(x,t_0) = S_0^1(x) - S_0^2(x),\ \overline{I}(x,t_0) = I_0^1(x) - I_0^2(x),\ \overline{R}(x,t_0) = R_0^1(x) - R_0^2(x) \tag{13}$$

for $x \in \Omega$, where

$$F_{1,2} := \frac{S_1 I_1}{N_1} - \frac{S_2 I_2}{N_2}. \tag{14}$$

Now

$$F_{1,2} := \frac{S_1 I_1}{N_1} - \frac{S_2 I_2}{N_2} = \frac{(S_1 I_1 - S_2 I_2)}{N_1} - \frac{\overline{N} S_2 I_2}{N_1 N_2},$$

so

$$\begin{aligned}
|F_{1,2}| &\leq \frac{|I_1|}{|N_1|}|S_1 - S_2| + \frac{|S_2|}{|N_1|}|I_1 - I_2| + \left|\frac{S_2 I_2}{N_2 N_2}\right| \left|\frac{N_2}{N_1}\right| |\overline{N}| \\
&\leq |\overline{S}| + \left|\frac{N_2}{N_1}\right| |\overline{I}| + \left|\frac{N_2}{N_1}\right| |\overline{N}|
\end{aligned} \tag{15}$$

since $\frac{I_1}{N_1}$, $\frac{I_2}{N_2}$ $\frac{S_2}{N_2}$ take values in $[0,1]$. Similarly, interchanging the indices in the above derivation, we also have

$$|F_{1,2}| \leq |\overline{S}| + \left|\frac{N_1}{N_2}\right| |\overline{I}| + \left|\frac{N_1}{N_2}\right| |\overline{N}|. \tag{16}$$

Fix $t \geq t_0$. Applying (15) at the interior points x of Ω where $N_2(x,t) \leq N_1(x,t)$ and (16) at the interior points x of Ω where $N_1(x,t) \leq N_2(x,t)$, we obtain

$$|F_{1,2}| \leq |\overline{S}| + |\overline{I}| + |\overline{N}| .$$

Hence taking the L^2 norms

$$|F_{1,2}(t)|^2_{L^2(\Omega)} \leq 3\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + 3\left|\overline{I}(t)\right|^2_{L^2(\Omega)} + 3\left|\overline{N}(t)\right|^2_{L^2(\Omega)} . \tag{17}$$

From the energy equality applied to each component of the system (11)-(13) and using the Poincaré inequality we obtain

$$\begin{aligned} \frac{d}{dt}\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + 2\lambda_1\left|\overline{S}(t)\right|^2_{L^2(\Omega)} \leq & -2a\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + 2b\left(\overline{S}(t),\overline{I}(t)\right)_{L^2(\Omega)} \\ & +2\gamma\left(F_{1,2}(t),\overline{S}(t)\right)_{L^2(\Omega)} , \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{d}{dt}\left|\overline{I}(t)\right|^2_{L^2(\Omega)} + 2\lambda_1\left|\overline{I}(t)\right|^2_{L^2(\Omega)} \leq & -2(a+b+c)\left|\overline{I}(t)\right|^2_{L^2(\Omega)} \\ & +2\gamma\left(F_{1,2}(t),\overline{I}(t)\right)_{L^2(\Omega)} , \end{aligned} \tag{19}$$

and

$$\frac{d}{dt}\left|\overline{R}(t)\right|^2_{L^2(\Omega)} + 2\lambda_1\left|\overline{R}(t)\right|^2_{L^2(\Omega)} \leq 2c\left(\overline{I}(t),\overline{R}(t)\right)_{L^2(\Omega)} - 2a\left|\overline{R}(t)\right|^2_{L^2(\Omega)} . \tag{20}$$

Defining

$$\Sigma(t) := \left|\overline{S}(t)\right|^2_{L^2(\Omega)} + \left|\overline{I}(t)\right|^2_{L^2(\Omega)} + \left|\overline{R}(t)\right|^2_{L^2(\Omega)}$$

and adding we obtain

$$\begin{aligned} \frac{d}{dt}\Sigma(t) + 2(\lambda_1+a)\Sigma(t) \leq & \; 2b\left(\overline{S}(t),\overline{I}(t)\right)_{L^2(\Omega)} + 2\gamma\left(F_{1,2}(t),\overline{S}(t)\right)_{L^2(\Omega)} \\ & -2(b+c)\left|\overline{I}(t)\right|^2_{L^2(\Omega)} + 2\gamma\left(F_{1,2}(t),\overline{I}(t)\right)_{L^2(\Omega)} \\ & +2c\left(\overline{I}(t),\overline{R}(t)\right)_{L^2(\Omega)} \\ \leq & \; \frac{1}{2}b\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + \frac{1}{2}c\left|\overline{R}(t)\right|^2_{L^2(\Omega)} \\ & +2\gamma\left(F_{1,2}(t),\overline{S}(t)\right)_{L^2(\Omega)} + 2\gamma\left(F_{1,2}(t),\overline{I}(t)\right)_{L^2(\Omega)} , \\ \leq & \left(\frac{1}{2}b+\gamma\right)\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + \gamma\left|\overline{I}(t)\right|^2_{L^2(\Omega)} + \frac{1}{2}c\left|\overline{R}(t)\right|^2_{L^2(\Omega)} \\ & +2\gamma\left|F_{1,2}(t)\right|^2_{L^2(\Omega)} , \end{aligned}$$

where we have used the Cauchy-Schwarz inequality four times. Now we use the inequality (17) to obtain

$$\begin{aligned} \frac{d}{dt}\Sigma(t) + 2(\lambda_1+a)\Sigma(t) \leq & \left(\frac{1}{2}b+7\gamma\right)\left|\overline{S}(t)\right|^2_{L^2(\Omega)} + 7\gamma\left|\overline{I}(t)\right|^2_{L^2(\Omega)} \\ & +\frac{1}{2}c\left|\overline{R}(t)\right|^2_{L^2(\Omega)} + 6\gamma\left|\overline{N}(t)\right|^2_{L^2(\Omega)} . \end{aligned}$$

This gives the differential inequality

$$\frac{d}{dt}\Sigma(t) + \rho\Sigma(t) \leq 6\gamma \left|\overline{N}(t)\right|^2_{L^2(\Omega)}, \tag{21}$$

for an appropriate nonzero constant ρ (which may be negative).

Now, we observe that $\overline{N} = \overline{S} + \overline{I} + \overline{R}$ satisfies

$$\frac{\partial \overline{N}}{\partial t} + [-\Delta + a]\,\overline{N} = 0,$$

so as in the derivation of the estimate (7), we have

$$\left|\overline{N}(t)\right|^2_{L^2(\Omega)} \quad \leq \quad e^{-2(\lambda_1+a)(t-t_0)} \left|\overline{N}_0\right|^2_{L^2(\Omega)} \leq \Sigma(t_0), \tag{22}$$

since $\left|\overline{N}_0\right|^2_{L^2(\Omega)} \leq \Sigma(t_0)$. Thus from the differential inequality (21) we obtain

$$\frac{d}{dt}\Sigma(t) + \rho\Sigma(t) \leq 6\gamma\Sigma(t_0),$$

which we integrate between t_0 and t, to obtain

$$\Sigma(t) \leq \left[\left(1 - \frac{6\gamma}{\rho}\right) e^{-\rho(t-t_0)} + \frac{6\gamma}{\rho}\right] \Sigma(t_0)$$

Hence $\Sigma(t) \to 0$ as $\Sigma(t_0) \to 0$ for each $t \geq t_0$. Hence we have shown that the process defined by (10) is continuous in X_3^+. ■

Remark 6 *The SIR model (1)–(3) is, strictly speaking, not defined at the origin of X_3^+, but we can extend it continuously to hold there by defining the nonlinear term IS/N to be zero there. The estimates in the preceeding proof remain valid.*

4 Nonautonomous equilibrium solutions

It is clear from the above considerations, in particular from the estimate (5) for $|N(t)|^2_{L^2}$ that the closed and bounded subset

$$\Sigma_3^+ := \left\{(S, I, R) \in X_3^+ : N = S + I + R, |N|^2_{L^2(\Omega)} \leq 1 + \left(\frac{aq^+}{\lambda_1 + a}\right)^2 |\Omega|\right\} \tag{23}$$

of X_3^+ attracts (in both the forward and pullback senses) all populations starting outside it and that populations starting within it remain there, see Proposition 16 below.

Hence we can restrict attention to the dynamics in Σ_3^+ and to the asymptotically stable (in the forward and pullback senses) limiting population $\widehat{N}$ given by (6),

$$\left.\begin{aligned}\frac{\partial S}{\partial t}-\Delta S&=aq(t)-aS+bI-\gamma\tfrac{SI}{\widehat{N}},\\ \frac{\partial I}{\partial t}-\Delta I&=-(a+b+c)I+\gamma\tfrac{SI}{\widehat{N}},\\ \frac{\partial R}{\partial t}-\Delta R&=cI-aR,\end{aligned}\right\}\tag{24}$$

with the Dirichlet boundary condition and initial condition $(S(t_0), I(t_0), R(t_0)) = (S_0, I_0, R_0)$.

4.1 Disease–free limiting solution

We eliminate the S variable and just consider the equations for the I and R variables. Specifically, we replace S by $S = \widehat{N} - I - R$ in (24) to obtain the IR system

$$\left.\begin{aligned}\frac{\partial I}{\partial t}-\Delta I&=(\gamma-a-b-c)I-\gamma\tfrac{I(I+R)}{\widehat{N}},\\ \frac{\partial R}{\partial t}-\Delta R&=cI-aR\end{aligned}\right\}$$

with Dirichlet boundary condition and non-negative initial condition $(I(t_0), R(t_0)) = (I_0, R_0)$ satisfying $0 \le I_0(x),\ R_0(x) \le \widehat{N}(x, t_0)$ for every $x \in \Omega$.

Remark 7 *The use of $\widehat{N}$ instead of N here is no restriction since it corresponds to the stable manifold of the system, which contains the limiting dynamics of the system. It will only be attained from outside asymptotically, but nevertheless provides important information about the long-term dynamics of the system.*

Note that $I(t) = R(t) = 0$ is a solution, in which case system (24) reduces to

$$\frac{\partial S}{\partial t} - \Delta S = aq(t) - aS \tag{25}$$

with Dirichlet boundary condition $S(x,t) = 0$ on $\partial\Omega \times (t_0, +\infty)$ and the initial condition $S(x, t_0) = S_0(x)$ for $x \in \Omega$. This is exactly the same as equation (4) for the total population, so it has the pullback limit

$$\widehat{S}_1(t) = \widehat{N}(t) = ae^{-\widetilde{A}t}\int_{-\infty}^{t} e^{\widetilde{A}r}q(r)dr. \tag{26}$$

Lemma 8 *The nonautonomous equilibrium solution $(\widehat{S}_1(x,t), 0, 0) \in X_3^+$ is globally asymptotically stable (in the forward sense) in Σ_3 provided $\gamma < \lambda_1 + a + b + c$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ on the domain Ω with Dirichlet boundary condition.*

Proof. Since I satisfies

$$\frac{\partial I}{\partial t} - \Delta I = -(a+b+c)I + \gamma\frac{SI}{\widehat{N}},$$

so from the energy equality and the Poincaré inequality, we have

$$\frac{d}{dt}\left|I(t)\right|^2_{L^2(\Omega)} + 2(\lambda_1 + a + b + c)\left|I(t)\right|^2_{L^2(\Omega)} \leq 2\gamma\left(\frac{SI}{\widehat{N}}, I\right)_{L^2(\Omega)}.$$

On the other hand, taking into account that $0 \leq \frac{S}{\widehat{N}} \leq 1$, we obtain

$$\frac{d}{dt}\left|I(t)\right|^2_{L^2(\Omega)} \leq -2(\lambda_1 + a + b + c - \gamma)\left|I(t)\right|^2_{L^2(\Omega)}$$

and hence

$$\left|I(t)\right|^2_{L^2(\Omega)} \leq \left|I_0\right|^2_{L^2(\Omega)} e^{-2(\lambda_1+a+b+c-\gamma)(t-t_0)}. \tag{27}$$

Taking t to infinite in (27), we see that

$$I(t) \to 0 \text{ in } L^2(\Omega) \text{ as } t \to \infty,$$

provided $\gamma < \lambda_1 + a + b + c$, from which we can deduce that

$$I(x,t) \to 0 \ a.e. \text{ in } \Omega \times (t_0, +\infty) \text{ as } t \to \infty. \tag{28}$$

Similarly, R satisfies

$$\frac{\partial R}{\partial t} - \Delta R = cI - aR,$$

from the energy equality and the Poincaré inequality we have

$$\frac{d}{dt}\left|R(t)\right|^2_{L^2(\Omega)} + 2(\lambda_1 + a)\left|R(t)\right|^2_{L^2(\Omega)} \leq 2c\left(I, R\right)_{L^2(\Omega)}.$$

Using

$$2c\left(I, R\right)_{L^2(\Omega)} \leq \frac{c^2}{\lambda_1 + a}\left|I(t)\right|^2_{L^2(\Omega)} + (\lambda_1 + a)\left|R(t)\right|^2_{L^2(\Omega)},$$

this gives

$$\frac{d}{dt}\left|R(t)\right|^2_{L^2(\Omega)} + (\lambda_1 + a)\left|R(t)\right|^2_{L^2(\Omega)} \leq \frac{c^2}{\lambda_1 + a}\left|I(t)\right|^2_{L^2(\Omega)}.$$

Taking into account (27), we have

$$\frac{d}{dt}\left|R(t)\right|^2_{L^2(\Omega)} + (\lambda_1 + a)\left|R(t)\right|^2_{L^2(\Omega)} \leq \frac{c^2}{\lambda_1 + a}\left|I_0\right|^2_{L^2(\Omega)} e^{-2(\lambda_1+a+b+c-\gamma)(t-t_0)}.$$

Integrating between t_0 and t, then simplifying, we obtain

$$\left|R(t)\right|^2_{L^2(\Omega)} \leq \left|R_0\right|^2_{L^2(\Omega)} e^{-(\lambda_1+a)(t-t_0)} + \frac{c^2\left|I_0\right|^2_{L^2(\Omega)} e(t,t_0)}{(\lambda_1 + a)(\lambda_1 + a + 2b + 2c - 2\gamma)}, \tag{29}$$

where

$$e(t,t_0) := e^{-(\lambda_1+a)(t-t_0)} - e^{-2(\lambda_1+a+b+c-\gamma)(t-t_0)}.$$

Taking t to infinite, since $\gamma < \lambda_1 + a + b + c$, we have that

$$R(t) \to 0 \text{ in } L^2(\Omega) \text{ as } t \to \infty,$$

then we can deduce that

$$R(x,t) \to 0 \; a.e. \text{ in } \Omega \times (t_0, +\infty) \text{ as } t \to \infty. \tag{30}$$

Finally, from

$$S(x,t) - \widehat{S}_1(x,t) = (N(x,t) - I(x,t) - R(x,t)) - \widehat{S}_1(x,t),$$

taking into account (8), (28) and (30), we obtain

$$S(x,t) - \widehat{S}_1(x,t) \to 0 \; a.e. \text{ in } \Omega \times (t_0, +\infty) \text{ as } t \to \infty.$$

Thus we have proved that the nonautonomous equilibrium solution $(\widehat{S}_1(x,t), 0, 0) \in X_3^+$ is globally asymptotically stable (in the forward sense) in Σ_3^+ provided $\gamma < \lambda_1 + a + b + c$. ■

Lemma 9 *The disease–free nonautonomous equilibrium solution* $(\widehat{S}_1(x,t), 0, 0) \in X_3^+$ *loses forward linear stability at* $\gamma = \lambda_1 + a + b + c$.

Proof. Since linearized equations about the disease-free nonautonomous equilibrium solution are given by the autonomous system

$$\left.\begin{aligned} \frac{\partial \bar{S}}{\partial t} - \Delta \bar{S} &= -a\bar{S} + (b-\gamma)\bar{I}, \\ \frac{\partial \bar{I}}{\partial t} - \Delta \bar{I} &= (\gamma - a - b - c)\bar{I}, \\ \frac{\partial \bar{R}}{\partial t} - \Delta \bar{R} &= c\bar{I} - a\bar{R} \end{aligned}\right\}$$

since $\widehat{S}_1(t) \equiv \widehat{N}(t)$. The largest eigenvalue of the matrix on the right hand side of this system (when written in matrix-vector form) is $\gamma - \lambda_1 - a - b - c$. ■

5 Weak uniform persistence of the disease

The loss of linearized stability in Lemma 9 is of limited use, but nevertheless indicates that a change of behaviour may occur. In fact, the disease is then weakly uniformly weakly persistent.

Theorem 10 *If* $\gamma > \lambda_1 + a + b + c$*, then the disease is weakly uniformly persistent in the sense that there exists an* $\varepsilon_0 > 0$ *such that*

$$\limsup_{t\to\infty} \int_\Omega I(t,x)dx \geq \varepsilon_0$$

for all nonnegative solutions with $I_0 \not\equiv 0$ *at the initial time* t_0.

Proof. We will use contradiction arguments as in [11, 28]. Suppose that the disease is not uniformly weakly persistent. Then, for any $\varepsilon > 0$, there exists a solution with $I_0 \not\equiv 0$ such that

$$\limsup_{t\to\infty} \int_\Omega I(t,x)dx < \varepsilon.$$

We will work with the equations

$$\frac{\partial I}{\partial t} - \Delta I = \alpha I - \gamma \frac{I(I+R)}{N}, \quad \frac{\partial R}{\partial t} - \Delta R = cI - aR,$$

with the Dirichlet boundary condition, where $\alpha = \gamma - a - b - c$. By assumption, $\alpha - \lambda_1 > 0$. Let G be the Green's function for $\frac{\partial}{\partial t} - \Delta$ on Ω with the Dirichlet boundary condition. There exists a $t_\varepsilon > 0$ such that

$$\int_\Omega I(t,x)dx < \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

We have

$$\frac{\partial I}{\partial t} - \Delta I \leq \alpha I.$$

Consider $e^{-\alpha t} I(t,x)$. Then for all $s \geq t_\varepsilon + 1$, there exists a positive constant M such that

$$\begin{aligned} I(s,x) &\leq e^\alpha \int_\Omega G(1,x,y) I(s-1,y)\, dy \\ &\leq M \int_\Omega I(s-1,y)dy \leq M\varepsilon. \end{aligned} \tag{31}$$

Now for all $t \geq 0$, $x \in \Omega$ and using (31) we can deduce that there exists $\tilde{M} > 0$ such that

$$I(t+2+t_\varepsilon, x) \leq e^\alpha \int_\Omega G(1,x,y) I(t+1+t_\varepsilon, y)\, dy \leq \tilde{M}\varepsilon u(x), \tag{32}$$

where

$$u(x) := \int_\Omega G(1,x,y)\, dy.$$

Further, taking into account (4), if we consider $e^{at}N(t,x)$ and use the variation of constants formula, we have

$$N(t,x) \geq \int_0^t \int_\Omega G(t-s,x,y) q(s)\, dsdy \geq q^- \int_0^t \int_\Omega G(s,x,y) dsdy.$$

For $t \geq 0$, using Lemma 6.4 in [30], we obtain that there exists a $\delta > 0$ such that

$$N(t+2+t_\varepsilon, x) \geq q^- \int_{1/2}^2 \int_\Omega G(s,x,y)\, dsdy \geq \delta u(x). \tag{33}$$

On the other hand, we have

$$\frac{\partial R}{\partial t} - \Delta R \leq cI + aR.$$

Consider $e^{-at}R(t,x)$. Using the variation of constants formula, we obtain for all $t \geq 0$

$$R(t+2+t_\varepsilon, x) \leq \int_{t+1+t_\varepsilon}^{t+2+t_\varepsilon} \int_\Omega I(s,y) G(t+2+t_\varepsilon - s, x, y)\, ds dy$$

Now, taking into account (31) we can deduce

$$R(t{+}2{+}t_\varepsilon, x) \leq M\varepsilon \int_{t+1+t_\varepsilon}^{t+2+t_\varepsilon} \int_\Omega G(t{+}2{+}t_\varepsilon{-}s, x, y)\, ds dy = M\varepsilon \int_0^1 \int_\Omega G(s,x,y) ds dy.$$

Using Lemma 6.4 in [30], we have

$$R(t+2+t_\varepsilon, x) \leq M\varepsilon u(x). \tag{34}$$

We combine (32) and (34) and we conclude that for $t \geq 0$ and $x \in \Omega$,

$$\frac{(I+R)}{N}(t+2+t_\varepsilon, x) \leq \frac{(\tilde{M}+M)}{\delta}\varepsilon = \widetilde{M}\varepsilon,$$

with $\widetilde{M}$ not depending on ε.

Set $u(t,x) := I(t+2+t_\varepsilon, x)$. Then

$$\frac{\partial u}{\partial t} - \Delta u \geq \alpha u - \gamma \widetilde{M}\varepsilon u.$$

Let v_1 be the eigenvector associated with λ_1. Without restriction of generality, $v_1 \leq 1$. Set

$$w(t) = \int_\Omega u(t,x) v_1(x)\, dx.$$

Recall that w depends on ε. Then

$$w' \geq (\alpha - \lambda_1 - \gamma\widetilde{M}\varepsilon) w.$$

Since we can choose $\varepsilon > 0$ as small as we want, we can arrange that $\alpha - \lambda_1 - \gamma\widetilde{M}\varepsilon > 0$. But $w(0) > 0$, so we have $w(t) \to \infty$ as $t \to \infty$, which contradicts

$$w(t) \leq \int_\Omega u(t,x) dx = \int_\Omega I(t+2+t_\varepsilon, x) dx \leq \varepsilon, \quad t \geq 0.$$

■

Remark 11 *The assumption that $q^- > 0$ is essential for the above proof. As remarked in the Introduction, the result will not hold in certain situations without it. Nevertheless, we expect that it may still hold, e.g., when $q(t)$ switches periodically between a zero and nonzero value or, more generally when its asymptotic average $\lim_{t\to\infty} \int_{t_0}^t q(s)\, ds > 0$. We will investigate this in a future paper, since the focus of this one is on the pullback attractor.*

6 Existence of a pullback attractor

A pullback attractor for the process U in the space X_3^+ is a family $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ of nonempty compact subsets of X_3^+, which is invariant in the sense that

$$U_{t,t_0} A(t_0) = A(t) \qquad \text{for all } t \geq t_0$$

and pullback attracts bounded subsets D of X_3^+, i.e.,

$$\mathrm{dist}_{X_3^+}\left(U_{t,t_0} D, A(t)\right) \to 0 \quad \text{as} \quad t_0 \to -\infty.$$

Since all bounded subsets D of X_3^+ are pullback absorbed into the positively invariant bounded subset Σ_3^+ in a finite time we can restrict attention to the set Σ_3^+.

The existence of the pullback attractor and following characterization of its components sets follows from [8, 21, 27]. An important requirement is that the process is an asymptotically compact operator [6, 7, 8], which is implied by the flattening property [8, 20, 21, 35].

Definition 12 *A process U_{t,t_0} on a Banach space $(X, \|\cdot\||)$ is said to be* pullback flattening *if for every bounded set B of X, $\epsilon > 0$ and $t_0 \in \mathbb{R}$, there exists a T_0 $:= T_0(B, t_0, \epsilon) > 0$ and a finite-dimensional subspace X_ϵ of X such that for each $t \in \mathbb{R}$*

$$\bigcup_{t_0 \leq t - T_0} P_\epsilon U_{t,t_0} B \text{ is bounded} \tag{35}$$

and

$$\left\| (I - P_\epsilon) \bigcup_{t_0 \leq t - T_0} U_{t,t_0} B \right\|_X < \epsilon \,, \tag{36}$$

where $P_\epsilon : X \to X_\epsilon$ is a bounded projection and (36) *is understood in the sense that $\|(I - P_\epsilon) U_{t,t_0} x_0\|_X < \epsilon$ for all $x_0 \in B$ and $t \geq T_0$.*

Theorem 13 *The closed and bounded subset Σ_3^+ of X_3^+ is positively invariant for the process U_{t,t_0} defined by (10) and pullback absorbs bounded subsets of X_3^+. Moreover, the process U_{t,t_0} satisfies the flattening property, so is asymptotically compact.*

Hence the process U_{t,t_0} has a unique pullback attractor $\mathcal{A}$ with component sets given by

$$\mathcal{A}(t) := \overline{\bigcap_{s \leq t} U_{t,s} \Sigma_3^+}^{L^2(\Omega)^3}$$

for all $t \in \mathbb{R}$.

The proof of the theorem will be given in the following subsections.

Remark 14 *The pullback attractor $\mathcal{A}$ is obtained by pullback convergence that uses information about the system in the past. It includes, and is perhaps most* *realistic, when the nonautonomity arises from asymptotic autonomity or some sort of temporal recurrence such as periodicity or almost periodicity. More complicated, even random, variation is possible. The pullback attractor consists of the bounded total solutions of the system. See [8, 21].*

Remark 15 *The pullback attractor $\mathcal{A}$ has component sets $\mathcal{A}(t) \subset \Sigma_3^+ \cap \widehat{N}^*(t)$ for each $t \in \mathbb{R}$. In the case that the disease–free nonautonomous equilibrium solution $(\widehat{S}_1(t), 0, 0)$ is asymptotically stable (in the forward sense), then $\mathcal{A}(t) = \{(\widehat{S}_1(t), 0, 0)\}$ for each $t \in \mathbb{R}$. Since this solution loses stability (in the forward sense) at $\gamma = \lambda_1 + a + b + c$ there must be other nontrivial total solutions inside the pullback attractor for $\gamma > \lambda_1 + a + b + c$, see [27]. It would be interesting to investigate the internal structure of the pullback attractor for persistent and non-persistent components as was done in [28] for an autonomous attractor.*

6.1 Proof of Theorem 13: absorbing set

Proposition 16 *For every bounded subset D of X_3^+ and $t \in \mathbb{R}$ there exists $T_0(D) \geq 0$ such that the solution $(S(t), I(t), R(t))$ of (1)–(3) with initial value $(S_0, I_0, R_0) \in X_3^+$ at time t_0 satisfies*

$$U_{t,t_0}(S_0, I_0, R_0) := (S(t), I(t), R(t)) \in \Sigma_3^+ \quad \textit{for all } t_0 \leq t - T_0(D).$$

Proof. The proof follows from the definition of the total population and the inequality (5) in section 2, i.e.,

$$|N(t)|^2_{L^2(\Omega)} \leq \left(\frac{aq^+}{\lambda_1 + a}\right)^2 |\Omega| + |N_0|^2_{L^2(\Omega)} e^{-(\lambda_1 + a)(t - t_0)}.$$

■

Denote the bound in the set Σ_3^+ by

$$B := 1 + \left(\frac{aq^+}{\lambda_1 + a}\right)^2 |\Omega|$$

and note that

$$|S|^2_{L^2(\Omega)} \leq B, \quad |I|^2_{L^2(\Omega)} \leq B, \quad |R|^2_{L^2(\Omega)} \leq B, \tag{37}$$

for $(S, I, R) \in \Sigma_3^+$.

6.2 Proof of Theorem 13: flattening property

Let $A : H_0^1(\Omega) \to (H_0^1(\Omega))'$ be the linear operator associated with the negative Laplacian. Since the space $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, the operator A is symmetric, coercive and continuous. In particular, there exists a non-decreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues of A with $\lim_{j\to\infty} \lambda_j = +\infty$ and corresponding eigenvalues

$$Av_j = \lambda_j v_j \text{ for all } j \geq 1.$$

Moreover, $\{v_j : j \geq 1\} \subset H_0^1(\Omega)$ is a Hilbert basis of $L^2(\Omega)$ and span $\{v_j : j \geq 1\}$ is densely embedded in $H_0^1(\Omega)$.

Let P_m be the projection of $L^2(\Omega)$ onto the finite-dimensional subspace spanned by $\{v_1, \ldots, v_m\}$ and let $Q_m = I - P_m$. Then any $u \in L^2(\Omega)$ has the unique orthogonal decomposition $u = u_m + q_m$, where $u_m = P_m u$ and $q_m = Q_m u$. Similarly, $\nabla u = \nabla u_m + \nabla q_m$ for $u \in H_0^1(\Omega)$.

We can restrict attention to the dynamics in Σ_3^+. A solution $(S(t), I(t), R(t))$ system (24) can be decomposed its components $P_m X_3^+$ and $Q_m X_3^+$, where the latter satisfy

$$\begin{aligned}
\frac{\partial Q_m S}{\partial t} - \Delta Q_m S &= -a Q_m S + b Q_m I - \gamma Q_m \left(\frac{SI}{N}\right), \\
\frac{\partial Q_m I}{\partial t} - \Delta Q_m I &= -(a+b+c) Q_m I + \gamma Q_m \left(\frac{SI}{N}\right), \\
\frac{\partial Q_m R}{\partial t} - \Delta Q_m R &= c Q_m I - a Q_m R
\end{aligned}$$

with initial condition $(Q_m S(t_0), Q_m I(t_0), Q_m R(t_0)) = (Q_m S_0, Q_m I_0, Q_m R_0)$ and Dirchlet boundary condition.

The energy inequalities as above give

$$\begin{aligned}
\frac{d}{dt} |Q_m S|^2_{L^2(\Omega)} + 2 |\nabla Q_m S|^2_{L^2(\Omega)} &\leq -2a |Q_m S|^2_{L^2(\Omega)} + 2b (Q_m I, Q_m S)_{L^2(\Omega)} \\
&\quad -2\gamma \left(Q_m \left(\frac{SI}{N}\right), Q_m S \right)_{L^2(\Omega)} \\
&\leq (b + 2\gamma) |Q_m S|^2_{L^2(\Omega)} + b |Q_m I|^2_{L^2(\Omega)},
\end{aligned}$$

where

$$\left(Q_m \left(\frac{IS}{N}\right), Q_m S \right)_{L^2(\Omega)} \leq \int_\Omega \left|\frac{I}{N}\right| |Q_m S|^2 \, dx \leq |Q_m S|^2_{L^2(\Omega)},$$

since $|I/N| \leq 1$. Hence by the positive invariance of the set Σ_3^+ and the bounds (37), for solutions starting in Σ_3^+ we have

$$\frac{d}{dt} |Q_m S|^2_{L^2(\Omega)} + 2 |\nabla Q_m S|^2_{L^2(\Omega)} \leq kB$$

for an appropriate constant positive constant k. But, by a generalization of the Poincaré inequality,

$$\lambda_{m+1} |Q_m S|^2_{L^2(\Omega)} \leq |\nabla Q_m S|^2_{L^2(\Omega)},$$

so

$$\frac{d}{dt} |Q_m S(t)|^2_{L^2(\Omega)} + 2\lambda_{m+1} |Q_m S(t)|^2_{L^2(\Omega)} \leq kB.$$

Integrating thus gives

$$\begin{aligned} |Q_m S(t)|^2_{L^2(\Omega)} &\leq |Q_m S(t_0)|^2_{L^2(\Omega)}\, e^{-2\lambda_{m+1}(t-t_0)} + \frac{kB}{2\lambda_{m+1}}\left(1 - e^{-2\lambda_{m+1}(t-t_0)}\right). \\ &\leq Be^{-2\lambda_{m+1}(t-t_0)} + \frac{kB}{2\lambda_{m+1}}. \end{aligned}$$

Given $\epsilon > 0$ pick m large enough so that

$$2Be^{-2\lambda_{m+1}} < \epsilon, \qquad kB < \lambda_{m+1}\epsilon.$$

Then

$$|Q_m S(t)|^2_{L^2(\Omega)} \leq \epsilon.$$

Similar estimates hold for $|Q_m I(t)|^2_{L^2(\Omega)}$ and $|Q_m R(t)|^2_{L^2(\Omega)}$. This verifies condition (36) of the flattening property with the bounded set Σ_3^+ and $T_0 = 1$. It thus holds for all other bounded subsets of X_3^+ since these are pullback absorbed into Σ_3^+ in a finite time. It is clear that condition (35) by the $L^2(\Omega)$-boundedness and positive invariance of Σ_3^+

6.3 Proof of Theorem 13: existence of the pullback attractor

By the flattening property the process U_{t,t_0} defined by (10) is asymptotically compact in $L^2(\Omega)$. Moreover the bounded subset Σ_3^+ is positive invariant and a pullback absorbing set for the process U_{t,t_0}. The existence of a unique pullback attractor and its representation then follows from theorems in the literature (see [8] and, specifically, Theorem 7 in [6]).

This completes the proof of Theorem 13.

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