Attractors for a non-autonomous Liénard equation

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Abstract

In this paper we prove the existence of pullback and uniform attractors for a non-autonomous Liénard equation. The relation among these attractors is also discussed. After that, we consider that the Liénard equation includes forcing terms which belong to a class of functions extending periodic and almost periodic functions recently introduced by Kloeden and Rodrigues in [14]. Finally, we estimate the Hausforff dimension of the pullback attractor. We illustrate these results with a numerical simulation: we present a simulation showing the pullback attractor for the non-autonomous Van der Pol equation, an important special case of the non-autonomous Liénard equation.

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1 Introduction and setting of the problem

The Liénard equation was introduced in 1928 by the French physicist A. Liénard in the paper [25]. The Liénard equation is a particularly interesting equation as many questions arising in the physical sciences are concerned with it (see for instance [29], [33], [34]). The Liénard equation, which is often taken as the typical example of nonlinear self-excited vibration problem, can be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. During the development of radio and vacuum tubes, Liénard equations were intensely studied as they can be used to model oscillating circuits. It can also be used to model certain mechanical systems which contain the nonlinear damping coefficients and the restoring force or stiffness. The Liénard equation has been studied for a long time, and there are many results. The general autonomous equation of Liénard type was first studied by Levinson and Smith in the classical paper [24], and later on several authors have contributed to the theory of this equation with respect to existence and uniqueness of nontrivial periodic solutions.

The non-autonomous Liénard equation has been well-studied (see for instance [10, 17, 18, 19, 27] as well as their lists of references) and there are many results on the dissipative nature of the non-autonomous Liénard equation (see [4], [20], [28]). Leonov proposed in [21] a method for localizing the attractors of the non-autonomous Liénard equation based on the construction of special piecewise-linear discontinuous comparison systems. Later, the universality of the construction of the comparison systems examined in [22] by Leonov enables to introduce different variable parameters which improve the localization theorem of [21].

There are basically two ways to define attraction of a compact and invariant non–autonomous set for a process on a metric space. The first, and perhaps more obvious, corresponds to the attraction in the sense of Lyapunov stability, which is called *forward attraction*, and involves a moving target, while the second, called *pullback attraction*, involves a fixed target set with progressively earlier starting time. In general, these two types of attraction are independent concepts, while for the autonomous case, they are equivalent. Physically, the pullback attractor provides a way to assess an asymptotic regime at time t (the time at which we observe the system) for a system starting to evolve from the remote past. The pullback dynamics contains interesting dynamical properties, which allow us to understand the forward attraction (see [3], [13] for more details). To our knowledge, there does not seem to be in the literature any study of the existence of pullback attractors for non-autonomous Liénard equations.

Let us introduce the model we will be involved with in this paper. We consider the following problem for a non-autonomous Liénard equation,

$$x'' + f(x)x' + a(t)x = E(t). \tag{1}$$

Denote by $y = x'$, then equation (1) can be reduced to the following equivalent first order system

$$\left.\begin{aligned} x' &= y, \\ y' &= -a(t)x - f(x)y + E(t), \end{aligned}\right\} \tag{2}$$

with initial condition

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{3}$$

where $t_0 \in \mathbb{R}$.

We assume that the following conditions are fulfilled:

(H1) f is a locally Lipschitz continuous function with respect to x and there exists $f_0 > 0$ such that $f(x) \geq f_0$ for all x.

(H2) $E : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function such that satisfies $\int_{-\infty}^{t} e^{f_0 s} E^2(s)ds < +\infty$ for all $t \in \mathbb{R}$.

(H3) $a : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function such that $0 < a_0 \le a(t) \le a_1$ for all $t \in \mathbb{R}$ and $(a(t)e^{f_0 t})' \le 0$ on $[t_0, \infty)$.

Van der Pol equation is an important special case of the Liénard equation. This equation was introduced in 1920 by the Dutch physicist Balthasar Van der Pol and is an example of the long-standing interaction between differential equations and the physical and biological sciences. A few years after, in [32] Van der Pol and Van der Mark modeled the electric activity of the heart rate. Later, Fitzhugh [9] and Nagumo [26] extended the Van der Pol equation in a planar field as a model for action potentials of neurons. Recently, this equation has also been utilized in seismology to model the two plates in a geological fault.

The non-autonomous Van der Pol equation

$$x'' - \mu(1 - x^2)x' + a(t)x = E(t), \tag{4}$$

where μ is a positive constant, describes the behavior of the Van der Pol oscillator when acted upon by a external disturbance in the presence of a linear restoring force and non-linear damping. When $|x|$ is small, the quadratic term x^2 is negligible and the system becomes a linear differential equation with a negative damping $-\mu$, i.e.,

$$x'' - \mu x' + a(t)x = E(t).$$

On the other hand, when $|x|$ is large, the term x^2 becomes dominant and the damping becomes positive and dissipation occurs. Therefore, an important special case of (1) is (4) for large $|x|$.

The first aim of this paper is to show the existence of a pullback and a uniform attractor for the process associated to (2)-(3). The fact that a and E are non-autonomous is the main novelties of our problem.

A temporally global solution, if it exists, of a non-autonomous ordinary differential equation need not be periodic, almost periodic or almost automorphic when the forcing term is periodic, almost periodic or almost automorphic, respectively. An alternative class of functions extending periodic and almost periodic functions, which has the property that a bounded temporally global solution of a non-autonomous ordinary differential equation belongs to this class when the forcing term does, is introduced by Kloeden and Rodrigues in [14]. Specifically, the class of functions consists of uniformly continuous functions, defined on the real line and taking values in a Banach space, which have pre-compact ranges. Besides periodic and almost periodic functions, this class also includes many nonrecurrent functions. The second aim of this paper is to consider that (2) includes forcing terms which belong to this class of functions introduced by Kloeden and Rodrigues.

On the other hand, the theory of topological dimension [12], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor. For investigating attractors, the Hausdorff dimension of a set is much better. In [23] Lyapunov-type functions are introduced into upper estimates for the Hausdorff dimension of negatively invariant sets of cocycles. In this sense, the third aim of this paper is to estimate the Hausdorff dimension of the pullback attractor of (2)-(3) using the recent method proposed by Leonov *et al.* in [23].

The structure of the paper is as follows. A brief recall on abstract results about the existence of pullback and uniform attractors is given in Section 2. In Section 3, the main goals of proving the existence of pullback and uniform attractors of (2)-(3) and the relation among them under certain suitable assumption, are established. In Section 4, we consider that (1) includes forcing terms which belong to a class of functions introduced by Kloeden and Rodrigues [14]. We illustrate these results with a numerical simulation: we present a simulation showing the pullback attractor for the non-autonomous Van der Pol equation (4) for large $|x|$. Finally, in Section 5 we estimate the Hausdorff dimension of the pullback attractor associated to (2)-(3).

2 Abstract results on Pullback and Uniform Attractors

In this section we recall some abstract results on the theory of pullback attractors (see [1, 2, 3]) and we establish some results on the theory of uniform attractors (see [3, 7]).

Let (X, d_X) be a metric space, and let us denote $\mathbb{R}^2_d = \{(t, t_0) \in \mathbb{R}^2 : t_0 \leq t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t, t_0, x) \mapsto U(t, t_0)x \in X$ with $U(t_0, t_0)x = x$ for any $(t_0, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, t_0)x) = U(t, t_0)x$ for any $t_0 \leq r \leq t$ and all $x \in X$.

Definition 1 *Let U be a process on X. U is said to be continuous if for any pair $t_0 \leq t$, the mapping $U(t, t_0) : X \to X$ is continuous.*

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D} = \{D(t) : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$. Let $\mathcal{D}$ be a nonempty set of parameterized families of nonempty bounded sets $\widehat{D} = \{D(t) = D \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $D \subset X$ is a bounded set.

In what follows, we will consider a fixed universe of attraction $\mathcal{D}$ and throughout our analysis the concepts of absorption and attraction will be referred to this fixed universe.

Definition 2 *It is said that $\widehat{D}_0 \subset \mathcal{P}(X)$ is pullback absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\widehat{t}_0(t, \widehat{D}) \leq t$ such that*

$$U(t, t_0)D(t_0) \subset D_0(t) \quad \text{for all } t_0 \leq \widehat{t}_0(t, \widehat{D}).$$

We denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x\in\mathcal{O}_1} \inf_{y\in\mathcal{O}_2} d_X(x, y) \quad \text{for } \mathcal{O}_1,\, \mathcal{O}_2 \subset X.$$

Definition 3 *It is said that $\widehat{D}_0 \subset \mathcal{P}(X)$ is pullback attracting if*

$$\lim_{t_0\to-\infty} \mathrm{dist}_X(U(t, t_0)D(t_0), D_0(t)) = 0 \quad \textit{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R}.$$

There exists now a wide literature on pullback attractors (see, e.g., [15, 16, 30]), but we would like to emphasize that these notions take the final time as fixed and moves the initial time backwards towards $-\infty$. Note that this does not mean that we are moving backwards in time, but we consider the state of the system at time t that had begun in earlier and earlier initial instants t_0, i.e., $t_0 \to -\infty$.

Definition 4 *Let $\widehat{D}_0(t) \subset \mathcal{P}(X)$. This family is said to be invariant with respect to the process U if*

$$U(t, t_0)D_0(t_0) = D_0(t) \quad \textit{for all } t_0 \le t.$$

Denote the omega-limit set of $\widehat{D}$ by

$$\Lambda(\widehat{D}, t) := \bigcap_{s\le t} \overline{\bigcup_{t_0\le s} U(t, t_0)D(t_0)}^X \quad \text{for all } t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Definition 5 *The family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor associated to the continuous process U if is invariant, attracts every $\{D(t)\} \in \mathcal{D}$ and minimal in the sense that if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

The general result on the existence of pullback attractor is a generalization of the abstract theory for autonomous dynamical systems [31]:

Theorem 6 *[Crauel et al. [8], Schmalfuss [30]] Assume that there exists a family of compact pullback absorbing sets $\{B(t)\}_{t\in\mathbb{R}}$. Then, the family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ defined by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D}, t\right)}^X,$$

is the pullback attractor, where $\Lambda\left(\widehat{D}, t\right)$ is the omega-limit set at time t of $\widehat{D} \in \mathcal{D}$, where $\mathcal{D}$ is the universe of fixed nonempty bounded subsets of X.

Another approach to the asymptotic dynamics of non-autonomous equations, the uniform attractor, has been developed by Chepyzhov and Vishik [7]. The theory of uniform attractors can be developed for a single non-autonomous process (see [6, 7]).

Definition 7 *A set $K \subseteq X$ is said to be uniformly (with respect to $t_0 \in \mathbb{R}$) attracting for the process $\{U(t,t_0)\}$ on X if for all $t_0 \in \mathbb{R}$ and for any bounded set $B \subset X$,*

$$\lim_{T \to +\infty} \left(\sup_{t_0 \in \mathbb{R}} \operatorname{dist}_X(U(T+t_0,t_0)B, K) \right) = 0. \tag{5}$$

Respectively, the process $\{U(t,t_0)\}$ is said to be uniformly asymptotically compact (with respect to $t_0 \in \mathbb{R}$) if there exists a compact uniformly (with respect to $t_0 \in \mathbb{R}$) attracting set of $\{U(t,t_0)\}$.

Definition 8 *A closed set $\mathcal{A}_1 \subseteq X$ is said to be a uniform (with respect to $t_0 \in \mathbb{R}$) attractor for a process $\{U(t,t_0)\}$ if it is the minimal closed uniformly (with respect to $t_0 \in \mathbb{R}$) attracting set for this process. Minimality is meant in the sense that any closed attracting set is contained in $\mathcal{A}_1$.*

Theorem 9 *[Chepyzhov and Vishik [5, 7], Haraux [11]] If a process $\{U(t,t_0)\}$ is uniformly asymptotically (with respect to $t_0 \in \mathbb{R}$) compact, then it has the uniform (with respect to $t_0 \in \mathbb{R}$) attractor $\mathcal{A}_1$. The set $\mathcal{A}_1$ is compact in X.*

To describe the structures of uniform attractors and to perform a comparison with the pullback attractor we introduce the notions of complete trajectory of a process, kernel of a process and cross-section of the kernel (the terminology is due to Chepyzhov and Vishik [6, 7]).

Definition 10 *A map $u : \mathbb{R} \to X$ is called a complete trajectory of a process $U(t,t_0)$ if*

$$U(t,t_0)u(t_0) = u(t) \quad \text{for all } t \geq t_0, \quad t, t_0 \in \mathbb{R}.$$

Definition 11 *The kernel $\mathbb{K}$ of a process $U(t,t_0)$ consists of all of its bounded complete trajectories of the process $U(t,t_0)$.*

Definition 12 *The set*

$$\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathbb{K}\}$$

is said to be the kernel section at a time moment $t = s$, $s \in \mathbb{R}$.

These kernel sections are, essentially, the fibres of the pullback attractor: if $U(\cdot,\cdot)$ is a process that has a pullback attractor $\mathcal{A}$, then any backwards bounded trajectory is contained in $\mathcal{A}(t)$, and we can deduce that if $\mathcal{A}(\cdot)$ is bounded then $\mathcal{A}(t) = \mathcal{K}(t)$ (see for instance [3]). Observe that Theorem 9 implies the existence of a (fixed) compact attracting set K for $U(\cdot,\cdot)$, so that, from (5) and Theorem 3.11 in [3] it also implies the existence of a pullback attractor, which is then uniformly included in $\mathcal{K}$. Just as $\mathcal{A}(t)$ must contain $\mathcal{K}(t)$ for each t, the uniform attractor must contain the union of all the kernel sections (see [3]).

Lemma 13 *If $U(\cdot,\cdot)$ has a uniform attractor $\mathcal{A}_1$, then*

$$\bigcup_{t\in\mathbb{R}} \mathcal{K}(t) \subseteq \mathcal{A}_1.$$

3 Pullback and uniform attractors

Since the functions on the right hand side of (2) are locally Lipschitz with respect to x and y, then for any $t_0 \in \mathbb{R}$ and any $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique local solution of the model (2)-(3), denoted by $u(t;t_0,u_0) := (x(t;t_0,(x_0,y_0)), y(t;t_0,(x_0,y_0)))$, and this solution is a global solution one (7) is proved.

3.1 Pullback Attractor

In this section, we will show the existence of a pullback attractor in $\mathbb{R}^2$ of our problem (2)-(3). First, thanks to the uniqueness of solution of (2)-(3), we can define a process $\{U(t,t_0), t_0 \le t\}$ in $\mathbb{R}^2$, by

$$U(t,t_0)u_0 = u(t;t_0,u_0) \quad \forall u_0 \in \mathbb{R}^2. \tag{6}$$

The process defined by (6) is continuous in $\mathbb{R}^2$.

Proposition 14 *Assume (H1)-(H3). Then, for any initial condition $u_0 \in \mathbb{R}^2$, the solution u of (2)-(3) satisfies*

$$|u(t;t_0,u_0)|^2 \le \frac{l_1}{l_0} e^{-f_0(t-t_0)} |u_0|^2 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{-\infty}^{t} e^{f_0 s} E^2(s) ds, \tag{7}$$

for all $t \ge t_0$, where $l_0 := min\{1, a_0\}$ and $l_1 := max\{1, a_1\}$.

Proof. We deduce that

$$\frac{d}{dt} y^2(t) = -2a(t)xy - 2f(x)y^2 + 2E(t)y.$$

We have

$$2f(x)y^2 \ge 2f_0 y^2,$$

and

$$2E(t)y \le f_0 y^2 + \frac{1}{f_0} E^2(t).$$

Then, we can deduce

$$\frac{d}{dt}y^2(t)+f_0y^2+2a(t)xy\le\frac{1}{f_0}E^2(t). \tag{8}$$

Multiplying (8) by e^{f_0t}, we obtain that

$$\frac{d}{dt}\left(e^{f_0t}y^2(t)\right)+2e^{f_0t}a(t)xy\le\frac{1}{f_0}e^{f_0t}E^2(t).$$

Integrating between t_0 and t

$$e^{f_0t}y^2(t)+2\int_{t_0}^{t}e^{f_0s}a(s)x(s)y(s)ds\le e^{f_0t_0}y_0^2+\frac{1}{f_0}\int_{t_0}^{t}e^{f_0s}E^2(s)ds. \tag{9}$$

Taking into account that $y=x'$, integrating by parts and using (H3), we have

$$\begin{aligned}2\int_{t_0}^{t}e^{f_0s}a(s)x(s)y(s)ds&=e^{f_0t}a(t)x^2(t)-e^{f_0t_0}a(t_0)x_0^2-\int_{t_0}^{t}x^2(s)(a(s)e^{f_0s})'ds\\&\ge e^{f_0t}a_0x^2(t)-e^{f_0t_0}a_1x_0^2,\end{aligned}$$

so that (9) becomes,

$$\begin{aligned}e^{f_0t}\left|u(t)\right|^2&\le\frac{l_1}{l_0}e^{f_0t_0}\left|u_0\right|^2+\frac{1}{l_0f_0}\int_{t_0}^{t}e^{f_0s}E^2(s)ds \\&\le\frac{l_1}{l_0}e^{f_0t_0}\left|u_0\right|^2+\frac{1}{l_0f_0}\int_{-\infty}^{t}e^{f_0s}E^2(s)ds,\end{aligned} \tag{10}$$

whence (7) follows. ■

We consider the universe of fixed nonempty bounded subsets of $\mathbb{R}^2$. Now, we prove that there exists a pullback absorbing family for the process $U(t,t_0)$ defined by (6).

Proposition 15 *Under the assumptions in Proposition 14, the family $\widehat{D}_0=\{D_0(t):t\in\mathbb{R}\}$ defined by $D_0(t)=\overline{B}_{\mathbb{R}^2}\left(0,\rho_0(t)\right)$, where $\rho_0(t)$ is the nonnegative number given by*

$$\rho_0^2(t)=1+\frac{1}{l_0f_0}e^{-f_0t}\int_{-\infty}^{t}e^{f_0s}E^2(s)ds,\ \forall t\in\mathbb{R},$$

is pullback absorbing family for the process U defined by (6).

Proof. Let $D\subset\mathbb{R}^2$ be bounded. Then, there exists $d>0$ such that $|u_0|\le d$ for all $u_0\in D$. Thanks to Proposition 14, we deduce that for every $t_0\le t$ and any $u_0\in D$,

$$|U(t,t_0)u_0|^2 \le \frac{l_1}{l_0} e^{-f_0 t} e^{f_0 t_0} |u_0|^2 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{-\infty}^{t} e^{f_0 s} E^2(s) ds$$
$$\le \frac{l_1}{l_0} e^{-f_0 t} e^{f_0 t_0} d^2 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{-\infty}^{t} e^{f_0 s} E^2(s) ds.$$

If we consider $T(t,D) := f_0^{-1} \log(\frac{l_0}{l_1} e^{f_0 t} d^{-2})$, we have

$$|U(t,t_0)u_0|^2 \le 1 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{-\infty}^{t} e^{f_0 s} E^2(s) ds,$$

for all $t_0 \le T(t,D)$ and for all $u_0 \in D$.

Consequently the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{\mathbb{R}^2}(0, \rho_0(t))$ is pullback absorbing for the process U defined by (6). ■

Now, as a direct consequence of the preceding results and Theorem 6, we have the existence of the pullback attractor for the process U defined by (6).

Theorem 16 *Under the assumptions in Proposition 14, the process U defined by (6) possesses a pullback attractor $\mathcal{A}$, which is given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda\left(\widehat{D}, t\right)}. \tag{11}$$

3.2 Uniform Attractor

Now, we suppose that E is translation bounded in $L^2_{loc}(\mathbb{R};\mathbb{R})$, i.e.,

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} E^2(s) ds < \infty. \tag{12}$$

In this subsection, using Theorem 9, we will prove that, under the assumption (12), the process $\{U(t,t_0)\}$ has a uniform (with respect to $t_0 \in \mathbb{R}$) attractor.

Remark 17 *Observe that assumption (12) implies (H2).*

Proposition 18 *Assume (H1) and (H3). Let E satisfies (12). Then, the process U defined by (6) is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact.*

Proof. Let $D \subset \mathbb{R}^2$ be bounded, and as in the proof of Proposition 15, let $d > 0$ such that $|u_0| \le d$ for all $u_0 \in D$. Using (10), we have for any $u_0 \in D$

$$|u(t;t_0,u_0)|^2 \le \frac{l_1}{l_0} e^{-f_0 t} e^{f_0 t_0} |u_0|^2 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{t_0}^{t} e^{f_0 s} E^2(s) ds$$
$$\le \frac{l_1}{l_0} e^{-f_0 t} e^{f_0 t_0} d^2 + \frac{1}{l_0 f_0} e^{-f_0 t} \int_{t_0}^{t} e^{f_0 s} E^2(s) ds, \tag{13}$$

for all $t \geq t_0$. We estimate the integral on the right-hand side of (13), taking into account (12),

$$\int_{t_0}^{t} e^{-f_0(t-s)}E^2(s)ds \leq \int_{-\infty}^{t} e^{-f_0(t-s)}E^2(s)ds \leq \sum_{n\geq 0}\int_{t-(n+1)}^{t-n} e^{-f_0(t-s)}E^2(s)ds$$
$$\leq \sum_{n\geq 0} e^{-f_0 n}\int_{t-(n+1)}^{t-n} E^2(s)ds = C_1(1-e^{-f_0})^{-1},$$

where $C_1 := \sup_{t\in\mathbb{R}} \int_t^{t+1} E^2(s)ds < \infty$. Then, we can deduce that there exists a positive constant C_α such that

$$|u(t;t_0,u_0)|^2 \leq \frac{l_1}{l_0}e^{-f_0(t-t_0)}d^2 + C_\alpha.$$

Replacing t by $t+t_0$, we have

$$|u(t+t_0;t_0,u_0)|^2 \leq \frac{l_1}{l_0}e^{-f_0 t}d^2 + C_\alpha,$$

and if we consider $t \geq T(D) := \frac{\log(\frac{l_1}{l_0}d^2)}{f_0}$, we obtain

$$|u(t+t_0;t_0,u_0)|^2 \leq 1 + C_\alpha,$$

for all t_0 and for all $u_0 \in D$.

Then, the set $B_0 := \overline{B}_{\mathbb{R}^2}(0, 1+C_\alpha)$ is compact and uniformly (with respect to $t_0 \in \mathbb{R}$) attracting for the process U defined by (6). Therefore, the process U is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact. ∎

We can now state a theorem about the existence of a uniform attractor of (2)-(3). Taking into account Theorem 9 and Lemma 13, we can deduce the following result.

Theorem 19 *Under the assumptions in Proposition 18, the process U defined by (6) has a uniform attractor $\mathcal{A}_1$, which is compact in $\mathbb{R}^2$. Moreover, we have the following relation:*

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subseteq \mathcal{A}_1, \tag{14}$$

where $\mathcal{A}(t)$ is given by (11).

4 Pullback attractors for a class of ODEs more general than almost periodic

In this section we use recent results due to Kloeden and Rodrigues [14], where the authors introduced a class of functions which has the property that a bounded temporally global solution of a nonautonomous ordinary differential equation belongs to this class when the forcing terms do.

Let $BUC(\mathbb{R},\mathbb{R}^2)$ denotes the space of bounded and uniformly continuous functions $g:\mathbb{R}\to\mathbb{R}^2$, with the supremum norm. We consider as in [14] the following class of functions,

$$\mathcal{F} := \{g \in BUC(\mathbb{R},\mathbb{R}^2) : g \text{ has precompact range } \mathcal{R}(g)\}.$$

The class $\mathcal{F}$ includes periodic functions. We now consider the class $\mathcal{F}_{ODE}$ defined by

$$\mathcal{F}_{ODE} := \{g : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2; \text{ is uniformly continuous in } t\in\mathbb{R}, \text{ uniformly in } (x,y) \\ \text{in compact subsets } C\subset\mathbb{R}^2, \text{with precompact range } \mathcal{R}_C(g)\},$$

where

$$\mathcal{R}_C(g) := \bigcup_{(x,y)\in C} \{g(t,x,y), t\in\mathbb{R}\}.$$

Functions in the class $\mathcal{F}$ belong trivially to the class $\mathcal{F}_{ODE}$. For our problem, we consider

$$g_1(t,x,y) := y, \tag{15}$$

$$g_2(t,x,y) := -a(t)x - f(x)y + E(t), \tag{16}$$

and we suppose that

$$a, E \in BUC(\mathbb{R},\mathbb{R}). \tag{17}$$

Proposition 20 *Under assumption (17), g_1 and g_2 defined by (15)-(16) belong to the class $\mathcal{F}_{ODE}$.*

Proof. We prove that $g_2 \in \mathcal{F}_{ODE}$. First, we have to prove that g_2 is uniformly continuous in $t\in\mathbb{R}$, uniformly in (x,y) in compact subsets $C\subset\mathbb{R}^2$, i.e., we have to prove that there is a function $\alpha_0(\theta,C)$, $\alpha_0(\theta,C)\mapsto 0_+$ $(\theta\mapsto 0+)$ such that

$$|g_2(t_1,x_1,y_1) - g_2(t_2,x_2,y_2)| \le \alpha_0(|t_1-t_2| + |x_1-x_2| + |y_1-y_2|, C), \tag{18}$$

for all $(x_1,y_1),(x_2,y_2)\in C$, where $C\subset\mathbb{R}^2$ is a compact subset, and $t_1,t_2\in\mathbb{R}$.

We deduce that there exists $R(C)>0$, depending on the compact set C, such that

$$|g_2(t_1,x_1,y_1) - g_2(t_2,x_2,y_2)| \le R\,|a(t_1)-a(t_2)| + R\,|y_1-y_2| + |E(t_1)-E(t_2)|,$$

and using (17), we have (18) with $\alpha_0(|t_1-t_2| + |y_1-y_2|, C) \mapsto 0_+$ $(|t_1-t_2| + |y_1-y_2| \mapsto 0+)$, so g_2 is uniformly continuous in $t\in\mathbb{R}$, uniformly in (x,y) in compact subsets $C\subset\mathbb{R}^2$.

Finally, thanks to (17), in particular we have that a and E are bounded functions in $t\in\mathbb{R}$, and we can deduce that $\mathcal{R}_C(g_2)$ is precompact, where $C\subset\mathbb{R}^2$ is a compact subset. Therefore, $g_2\in\mathcal{F}_{ODE}$. On the other hand, g_1 trivially belongs to $\mathcal{F}_{ODE}$. ■

Then, we can write (2) as

$$\frac{du}{dt} = g(t,u), \ t \in \mathbb{R}, \tag{19}$$

with initial condition

$$u(t_0) = u_0, \tag{20}$$

where $u(t;t_0,u_0) := (x(t;t_0,(x_0,y_0)), y(t;t_0,(x_0,y_0)))$, $t_0 \in \mathbb{R}$ and $g(t,u) := (g_1(t,x,y), g_2(t,x,y))$ belongs to the class $\mathcal{F}_{ODE}$.

Thanks to Lemma 8 in [14], on account of the following Proposition, the components sets of the pullback attractor and its entire solutions are in fact uniformly continuous.

Proposition 21 *Assume (H1) and (H3). Under assumption (17), problem (19)-(20) generates a process which possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ such that $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is precompact.*

Proof. Taking into account (17) we deduce that E satisfies (H2) and (12). Then, thanks to Theorem 16, there exists the pullback attractor for the process defined by (6). On the other hand, using Theorem 19, we have (14). Therefore, $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is bounded and therefore $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is precompact. ■

Lemma 22 *Under the assumptions in Proposition 21 we have that (ϕ_1,ϕ_2) belongs to the class $\mathcal{F}$ for every entire solution (ϕ_1,ϕ_2) of the problem (19)-(20) taking values in the pullback attractor.*

Now, we present a simulation showing the pullback attractor for the non-autonomous Van der Pol equation (4) for large $|x|$.

We consider the following parameters: $\mu = 2$, $a(t) = e^{-|t+2000|}$, $E(t) = cos(t)$, and the following initial conditions values: $x(-2000) = 1.6$ and $y(-2000) = 1$.

Notice that E satisfies (17) and a satisfies (H3) and (17). Also, we observe that $f(x) = -2(1-x^2) \geq 1$ for $|x| \geq \sqrt{\frac{3}{2}}$ and satisfies (H1) for large $|x|$.

Therefore, thanks to Proposition 21 we can deduce that the non-autonomous Van der Pol equation

$$x'' - 2(1-x^2)x' + e^{-|t+2000|}x = cos(t), \tag{21}$$

with initial condition

$$x(-2000) = 1.6, \quad y(-2000) = 1, \tag{22}$$

generates a process which possesses a pullback attractor.

In Figure 1 we present a simulation showing the pullback attractor for the non-autonomous Van der Pol equation (21)-(22).

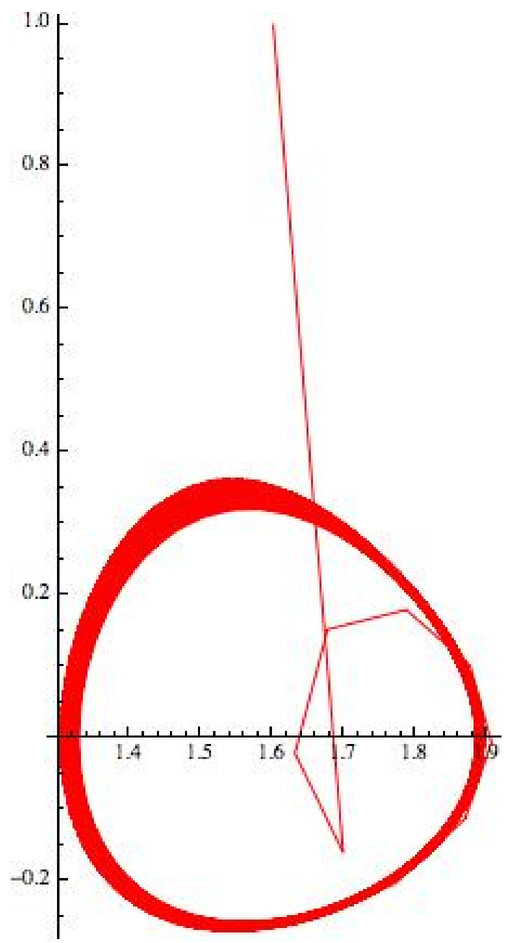

Figure 1: Numerical solution $(x(t), y(t))$

5 Upper Estimates for the Hausdorff Dimension of the Pullback Attractor

In this section we obtain an upper bound for the Hausdorff dimension of the pullback attractor of the process defined by (6). For this purpose, we use a method proposed by Leonov *et al.* in [23] in the framework of cocycle dynamical systems.

Assume that $a, E \in BUC(\mathbb{R}, \mathbb{R})$ and satisfy the following additional conditions:

(H4) Boundedness in time, i.e., there exists a nonnegative constant E_0 such that

$$|E(t)| \leq E_0, \quad \text{for all} \quad t \in \mathbb{R}.$$

(H5) $a \in C^1(\mathbb{R}, \mathbb{R})$ and there exists a nonnegative constant $\tilde{a}_1$ such that

$$\left|a'(t)\right| \leq \tilde{a}_1, \quad \text{for all} \quad t \in \mathbb{R}.$$

(H6) The hull of the function g denoting the right-hand side of (2), is a compact metric space, i.e., $\mathcal{H}(g) = \overline{\{g(t + \cdot, \cdot) : t \in \mathbb{R}\}}$ is a compact metric space.

Notice that if a and E are bounded and uniformly continuous functions, then the hull $\mathcal{H}(g)$ is a compact metric space where the closure is taken in the local uniform convergence topology (see Proposition 2.5, Chapter V in [7] for more details).

In Section 3 we have proved that the solution mapping of (2)-(3) defines a process given by (6) which has a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}} \subset \mathbb{R}^2$ defined by (11).

Also we can obtain a cocycle by considering

$$\left.\begin{aligned} v' &= \mathbb{F}(\sigma_t p, v), \\ v(0) &= v_0 \in \mathbb{R}^2, \end{aligned}\right\} \tag{23}$$

where $p \in \mathcal{H}(g)$, $\mathbb{F}(p, v) := p(0, v)$ and σ is defined as the shift mapping $\sigma_t : \mathcal{H}(g) \mapsto \mathcal{H}(g)$ given by

$$\sigma_t(\widetilde{g}) := \widetilde{g}(\cdot + t, \cdot),$$

for $t \in \mathbb{R}$ and $\widetilde{g} \in \mathcal{H}(g)$.

Then, the cocycle generated by (23) is given by

$$\varphi(t, p)v_0 = v(t; p, v_0),$$

where $v(t; p, v_0)$ denotes the solution of (23) with initial value v_0 at $t = 0$. If we take $p = g \in \mathcal{H}(g)$, then

$$\varphi(t, g)v_0 = v(t; g, v_0),$$

and (23) becomes

$$\left.\begin{aligned} v' &= \sigma_t g(0, v), \\ v(0) &= v_0 \in \mathbb{R}^2, \end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned} v' &= g(t, v), \\ v(0) &= v_0 \in \mathbb{R}^2, \end{aligned}\right\}$$

and we have

$$\varphi(t, g)v_0 = U(t, 0)v_0.$$

Then, our problem (2)-(3) generates a cocycle $(\{\varphi(t,p)\cdot\}_{p\in\mathcal{H}(g),t\in\mathbb{R}}, \mathbb{R}^2)$ over the base flow $(\{\sigma_t\}_{t\in\mathbb{R}}, \mathcal{H}(g))$, where

$$\varphi(t, \sigma_s g)v_0 = U(t + s, s)v_0. \tag{24}$$

If one introduces a new variable $y = x' + F(x)$, where $F(x) = \int_0^x f(s)ds$, then (1) goes into the system

$$\left.\begin{aligned} x' &= y - F(x), \\ y' &= -a(t)x + E(t), \end{aligned}\right\} \tag{25}$$

with initial condition

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{26}$$

where $t_0 \in \mathbb{R}$.

Now, to estimate the Hausdorff dimension of the pullback attractor associated to the process defined by (6), we will use Theorem 2 in [23], which is stated in the framework of cocycle dynamical systems. Then, for the cocycle generated by our system, we need to verify:

i) There exists a family of compact sets $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(g)}$ which is negatively invariant for the cocycle defined by (24), i.e.

$$\widetilde{\mathcal{A}}(\sigma_t p) \subset \varphi(t,p)\widetilde{\mathcal{A}}(p), \text{ for all } p \in \mathcal{H}(g), t \geq 0.$$

ii) There exists a compact set $\widetilde{K} \subset \mathbb{R}^2$ such that

$$\overline{\bigcup_{p\in\mathcal{H}(g)} \widetilde{\mathcal{A}}(p)} \subset \widetilde{K}.$$

iii) There exists a continuous function $V : \mathcal{H}(g) \times \mathbb{R}^2 \to \mathbb{R}$ with derivatives $\frac{d}{dt}V(\sigma_t p, \varphi(t,p)u_0)$ along a given trajectory such that

$$\lambda_1(\sigma_t p, \varphi(t,p)u_0) + s\lambda_2(\sigma_t p, \varphi(t,p)u_0) + \frac{d}{dt}V(\sigma_t p, \varphi(t,p)u_0) < 0, \tag{27}$$

for all $t \in \mathbb{R}$, $u_0 \in \widetilde{K}$, $p \in \mathcal{H}(g)$ and $s \in (0,1]$, where λ_i with $i = 1,2$ are the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (25) arranged in nonincreasing order $\lambda_1 \geq \lambda_2$.

Using the pullback attractor, $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, associated to the process defined by (6), we define the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(g)}$ by

$$\widetilde{\mathcal{A}}(p) = \begin{cases} \mathcal{A}(s) & \text{if } p = \sigma_s g, \\ \{x \in \mathbb{R}^2 : x = \lim_{t_n\to+\infty} x_{t_n},\ x_{t_n} \in \mathcal{A}(t_n)\} & \text{if } p \neq \sigma_s g, \end{cases} \tag{28}$$

where $s \in \mathbb{R}$ and $p \in \mathcal{H}(g)$.

The set $\widetilde{\mathcal{A}}(p)$ is compact for any $p \in \mathcal{H}(g)$. Moreover, the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(g)}$ is negatively invariant. Indeed, if $p = \sigma_s g$, taking into account (24) and the fact that $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is invariant for the process U defined by (6), we obtain that $\varphi(t,p)\widetilde{\mathcal{A}}(p) =$

$\widetilde{\mathcal{A}}(\sigma_t p)$ for all $t \geq 0$. If $p \neq \sigma_s g$, then $p = \lim_{t_n \to +\infty} \sigma_{t_n} g$ and it is easy to see that $\varphi(t,p)\widetilde{\mathcal{A}}(p) \supseteq \widetilde{\mathcal{A}}(\sigma_t p)$.

On the other hand, we can consider the following compact set

$$\widetilde{K} := \overline{\bigcup_{t \in \mathbb{R}} \mathcal{A}(t)} \subset \mathbb{R}^2,$$

and we have that

$$\overline{\bigcup_{p \in \mathcal{H}(g)} \widetilde{\mathcal{A}}(p)} \subset \widetilde{K},$$

and, consequently, condition i)-ii) hold.

We can now establish our result on the estimate of the Haussdorff dimension of the pullback attractor for our model. We denote by $dim_H K$ the Hausdorff dimension of K.

Theorem 23 *Assume (H1), (H3)-(H5), and that $a, E \in BUC(\mathbb{R}, \mathbb{R})$ satisfy (H6). Then the pullback attractor of the process U defined by (6) satisfies*

$$dim_H \mathcal{A}(t) \leq 2 - \frac{2f_0}{m + f_0}, \tag{29}$$

for all $t \in \mathbb{R}$, where m is a positive number given by $m := \frac{5}{4} + a_1^2 + k(1 + \frac{1}{2}\tilde{a}_1 + E_0)$, where k is a positive number depending on the compact set $\widetilde{K}$.

Proof. We need to verify iii).

It is easy to see that the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (25) are

$$\frac{1}{2}\left\{-f(x) \pm \sqrt{f^2(x) + (1 - a(t))^2}\right\}.$$

Hence, condition (27) can be written in the form

$$-f(x)(1+s) + (1-s)\sqrt{f^2(x) + (1-a(t))^2} + 2\frac{d}{dt}V_p(t,x,y) < 0, \tag{30}$$

for all $t \in \mathbb{R}$, $(x,y) \in \widetilde{K}$ and $p \in \mathcal{H}(g)$. Here,

$$V_p(t,x,y) \equiv V(\sigma_t p, \varphi(t,p)(x,y))$$

is a function defined for $(x,y) \in \widetilde{K}$, $p \in \mathcal{H}(g)$, and $t \in \mathbb{R}$ by the relation

$$V(\sigma_t p, x, y) := \frac{1}{4}(1-s)(a(t)x^2 + y^2).$$

Then

$$\frac{d}{dt}V_p = -\frac{1}{2}(1-s)a(t)F(x)x + \frac{1}{2}(1-s)E(t)y + \frac{1}{4}(1-s)a'(t)x^2,$$

and inequality (30) is equivalent to the following

$$-f_0(1+s) + (1-s)\vartheta(t,x,y) < 0, \tag{31}$$

where

$$\vartheta(t,x,y) := \sqrt{f^2(x) + (1-a(t))^2} - a(t)F(x)x + E(t)y + \frac{1}{2}a'(t)x^2.$$

Let us denote

$$m := \max_{t,x,y} \vartheta(t,x,y).$$

We have iii) from (31), and due to Theorem 2 in [23] we obtain

$$dim_H \widetilde{\mathcal{A}}(p) \leq 1 + \frac{-f_0 + m}{m + f_0} = 2 - \frac{2f_0}{m + f_0}, \tag{32}$$

for all $p \in \mathcal{H}(g)$.

We have

$$\begin{aligned}\vartheta(t,x,y) = &-\left(\gamma\sqrt{f^2(x) + (1-a(t))^2} - \frac{1}{2\gamma}\right)^2 + \gamma^2\left[f^2(x) + (1-a(t))^2\right] + \frac{1}{4\gamma^2} \\ &- a(t)F(x)x + E(t)y + \frac{1}{2}a'(t)x^2,\end{aligned}$$

where $\gamma \neq 0$ is a varied parameter. Further,

$$\vartheta(t,x,y) \leq \gamma^2\left[f^2(x) + (1-a(t))^2\right] + \frac{1}{4\gamma^2} - a(t)F(x)x + E(t)y + \frac{1}{2}a'(t)x^2.$$

Applying the mean value theorem for integrals, we observe that

$$-a(t)F(x)x = -a(t)f(x^*)x^2 \leq 0.$$

where $0 < x^* < x$. Then, we have that

$$\vartheta(t,x,y) \leq \gamma^2\left[f^2(x) + (1-a(t))^2\right] + \frac{1}{4\gamma^2} + E(t)y + \frac{1}{2}a'(t)x^2.$$

If we take the varied parameter $\gamma^2 = 1$ and taking into account (H3), (H4) and (H5), we deduce that there exists a positive constant k depending on the compact set $\widetilde{K}$ such that

$$\vartheta(t, x, y) \leq \frac{5}{4} + a_1^2 + k(1 + E_0 + \frac{1}{2}\tilde{a}_1),$$

and (28) and (32) imply (29). ■

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