Asymptotic behaviour of nonlocal reaction-diffusion equations*

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Abstract

The existence of a global attractor in $L^2(\Omega)$ is established for a reaction-diffusion equation on a bounded domain Ω in $\mathbb{R}^d$ with Dirichlet boundary conditions, where the reaction term contains an operator $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ which is nonlocal and possibly nonlinear. Existence of weak solutions is established, but uniqueness is not required. Compactness of the multivalued flow is obtained via estimates obtained from limits of Galerkin approximations. In contrast with the usual situation, these limits apply for all and not just for almost all time instants.

1 Introduction

A simple population model with spatial dependence is given by the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + u\,(1-u)\,, \tag{1}$$

on a bounded domain Ω in $\mathbb{R}^d$ with Dirichlet boundary conditions. The long term dynamics of this model is well understood.

In the above model the population at a point $x \in \Omega$ depends only on its value at this point, apart from the diffusivity term. More realistically, it could

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depend on the population size at other points, in particularly at near by points. For example, it could depend on the average over a small neighbourhood

$$\overline{u}_\delta(t,x) = \int_{B(x;\delta)} u(t,y)\,dy \Bigg/ \int_{B(x;\delta)} dy \ ,$$

for some small $\delta > 0$ instead of on $u(t,x)$ itself. This leads to a nonlocal PDE

$$\frac{\partial u}{\partial t} = \Delta u + \overline{u}_\delta\left(1 - \overline{u}_\delta\right). \tag{2}$$

Alternatively, $\overline{u}_\delta(t,x)$ could be some other functional of the solution $u(\cdot,t)$ evaluated at the point x.

There are many applications of nonlocal effects in partial differential equations in the literature, e.g., in combustion theory [13] and the Navier-Stokes equations [5]. The nonlocal term is often an integral operator and the equations are then called "integro-differential" equations. Boltzmann equations are a very well known class of integro-differential, but are first order unlike those of interest here. However, the nonlocal term could be different, see, e.g., the review article by Bates [3]. There is a large literature on the existence, regularity and blow–up of solutions of nonlocal evolution equations, see for example [15, 16, 17] and the papers cited therein.

Global attractors for nonlocal evolution equations have been investigated recently for the globally modified Navier-Stokes equations by Caraballo *et al.* [5], for m-Laplacian parabolic equations with a nonlocal nonlinearity by Chen [6] and by Hilhorst *et al* [8] for a nonlocal Kuramoto–Sivashinsky equation. Several aspects of reaction-diffusion equations are being analyzed over the last years, particularly, their asymptotic behaviour, see for example [14,15] and [17]. In this paper we consider general nonlinear nonlocal terms in autonomous reaction-diffusion equations, which generate strict multivalued semiflows. In particular, we establish the existence of a global attractor after first proving weak solutions and the compactness of attainability sets of the multivalued semiflow. For this we use estimates obtained as limits of Galerkin approximations which hold for every time instant and not just for almost all time instants. The problem is formulated in the next section and dissipativity estimates are presented in Section 3 with some longer proofs given at the end of the paper in Section 7. The existence of weak solutions is established in Section 4, while the generation of a strict multivalued semiflow and the existence of a global attractor are shown in Section 5. Finally an explicit example is presented in Section 6.

2 Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, it satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_\Omega > 0$ such that

$$\int_\Omega u^2(x)dx \le \lambda_\Omega^{-1} \int_\Omega \left(\nabla u(x)\right)^2 dx, \ \ \forall u \in H_0^1(\Omega). \tag{3}$$

Let $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$ and $\|\cdot\|_{L^2(\Omega)}$ the corresponding norm in $L^2(\Omega)$. In addition, let $\langle\cdot,\cdot\rangle$ denote the duality product between spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Consider the following initial boundary value problem for a nonlocal reaction–diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} + A\,u = \mathcal{F}(u) & \text{in } \Omega \times (0,T)\,, \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \tag{4}$$

where A is a uniformly parabolic operator in divergence form with $a^{ij} = a^{ji} \in L^\infty(\Omega)$, $1 \le i,j \le N$, for which there exist constants λ_A, $\Lambda_A > 0$ such that

$$\lambda_A\,|\eta|^2 \;\le\; \sum_{i,j=1}^{N} a^{ij}(x)\,\eta_i\,\eta_j \;\le\; \Lambda_A\,|\eta|^2$$

and

$$Au(x) \;:=\; -\sum_{i,j=1}^{N} \partial_{x_j}\left(a^{ij}(x)\,\partial_{x_i} u(x)\right) \tag{5}$$

for all $x \in \Omega$ and $\eta \in \mathbb{R}^N$.

Remark 1 *The operator induced by A can be interpreted as*

$$A \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$$

and is symmetric with

$$\langle Av,\, v\rangle \ge \lambda_A\,\|\nabla v\|^2_{L^2(\Omega)} \;\text{ for all } v \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is included in $L^2(\Omega)$ with compact injection, as a consequence of the Hilbert-Schmidt Theorem there exists a nondecreasing sequence of positive real numbers,

$$0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le ......,$$

with $\lim_{n\to\infty}\lambda_n = +\infty$ and there exists an orthonormal basis $\{w_k : k \ge 1\}$ of $L^2(\Omega)$. Moreover, $\{w_k : k \ge 1\}$ is an orthogonal basis of $H_0^1(\Omega)$ with $A\,w_k = \lambda_k\,w_k$ for all $k \ge 1$, where

$$(u,v)_{H_0^1(\Omega)} \overset{\text{Def.}}{=} \langle Au, v\rangle\,.$$

The operator $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ fulfills the following assumptions:

a) $\mathcal{F}$ is continuous with respect to the L^2 norm.

b) there exist $\beta \in (0, \lambda_A \lambda_\Omega)$ and $C_\beta > 0$ with

$$(u,\ \mathcal{F}(u)) \ \leq \ \beta \, \|u\|^2_{L^2(\Omega)} + \ C_\beta \ \text{ for every } u \in L^2(\Omega), \tag{6}$$

c) there is some nondecreasing $\Psi : [0, \infty) \to \mathbb{R}$ such that for all $u \in L^2(\Omega)$,

$$\|\mathcal{F}(u)\|^2_{L^2(\Omega)} \ \leq \ \Psi\left(\|u\|_{L^2(\Omega)}\right) \tag{7}$$

In special cases $\mathcal{F}$ will also be assumed to satisfy a local Lipschitz condition:

d) For all $R > 0$ there exists L_R such that if $v, w \in L^2(\Omega)$ with $\|v\|_{L^2(\Omega)} \leq R$, $\|w\|_{L^2(\Omega)} \leq R$, then

$$\|\mathcal{F}(v) - \mathcal{F}(w)\|_{L^2(\Omega)} \leq L_R \, \|v - w\|_{L^2(\Omega)} \,. \tag{8}$$

Remark 2 *In the subsequent statements and proofs, condition (c) can be easily replaced by the slightly weaker assumption that there is a nondecreasing function* $\widetilde{\Psi} : [0, \infty) \to \mathbb{R}$ *such that for every* $u \in H^1_0(\Omega)$ *with* $Au \in H^1_0(\Omega)$*,*

$$|\langle \mathcal{F}(u),\ A\,u\rangle| \quad \leq \quad \widetilde{\Psi}\big(\|u\|_{L^2(\Omega)}\big). \tag{9}$$

$$\|\mathcal{F}(u)\|^2_{L^2(\Omega)} \quad \leq \quad \widetilde{\Psi}\big(\|u\|_{H^{1,2}(\Omega)}\big). \tag{10}$$

3 Dissipativity estimates

Both the existence of weak solutions and of bounded absorbing sets in $L^2(\Omega)$, $H^1_0(\Omega)$, respectively, are based on the following a priori estimates. Their proofs do not require the uniquenss of weak solutions for a given initial value. Some of the auxiliary results are formulated for Galerkin approximations, and their (quite technical) proofs are postponed to § 7.

Proposition 3 *If* $\mathcal{F}$ *satisfies hypothesis (b), then every weak solution* $u \in L^2\big(0, T;\, H^1_0(\Omega)\big)$ *of (4) with* $u' \in L^2\big(0, T;\, H^{-1}(\Omega)\big)$ *fulfills the estimates*

$$\begin{aligned} \big\|u(t)\big\|^2_{L^2(\Omega)} \quad &\leq \quad \frac{M}{L} + e^{-L\,t}\, \big\|u(0)\big\|^2_{L^2(\Omega)} \\ \lambda_A \int_0^t \|\nabla u\|^2_{L^2(\Omega)}\, ds \quad &\leq \quad \left(\frac{1}{2} + \frac{\beta}{L}\right)\, \big\|u(0)\big\|^2_{L^2(\Omega)} + \left(C_\beta + \frac{\beta M}{L}\right)\, t \end{aligned}$$

for every $t \in [0, T]$ *with the constants* $M := 2\, C_\beta > 0$ *and* $L := 2\, \lambda_\Omega\, \lambda_A - 2\beta > 0$.

The proof is given in § 7.

Proposition 4 *Suppose that conditions (a) – (c) hold for $\mathcal{F}$ and that $T < \infty$. Then there exists positive constants C_0, C_1 and C_2 depending only on β, C_β, Λ_A, λ_A and λ_Ω such that* every *weak solution $u \in L^2\big(0,T;\, H_0^1(\Omega)\big)$ of (4) with $u' \in L^2\big(0,T;\, H^{-1}(\Omega)\big)$ and $\|u_0\|_{L^2(\Omega)} \le \rho$ satisfies the a priori estimates*

$$\|u(s)\|^2_{L^2(\Omega)} \le C_1 + e^{-C_0\, s}\, \rho^2$$

$$\big\|\nabla u(t)\big\|^2_{L^2(\Omega)} \le C_1 \cdot \max\{1, \tfrac{1}{t}\} \cdot \big(1+e^{-C_2\, t}\, \rho^2 + \Psi\big(C_1 \cdot (1+e^{-C_2\, t}\, \rho^2)\big)\big)$$

for every $0 \le s < t \le T$.

The proof, given below, uses analogous estimates for the Galerkin approximations in the following lemma, which is proved in the § 7.

Lemma 5 *Let $\{w_k : k \ge 1\}$ be an orthogonal basis of $H_0^1(\Omega)$ as in Remark 1. For each $n \in \mathbb{N}$, suppose that $u_n(t) = \sum_{k=1}^{n} u_{nk}(t) \cdot w_k$ is a solution of*

$$\begin{cases} \dfrac{d}{dt}\,(u_n(t),\, w_k) + \langle A u_n(t),\, w_k\rangle = (\mathcal{F}(u_n(t)),\, w_k) \\ \qquad\qquad\qquad\qquad (u_n(0),\, w_k) = (u_0,\, w_k)\,, \quad k = 1 \ldots n. \end{cases} \tag{11}$$

If $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ satisfies the hypotheses (b) and (c), then there exist positive constants C_1, C_2 and C_3 depending only on β, C_β, Λ_A, λ_A and λ_Ω such that whenever $\|u_0\|_{L^2(\Omega)} \le \rho$, the following estimates holds for every s_0, s, $t \in (0,T]$ with $0 < s_0 \le s \le t$

$$\big\|\nabla u_n(t)\big\|^2_{L^2(\Omega)} \le C_1 \cdot \max\big\{1,\, \tfrac{1}{t}\big\} \cdot \big(1+e^{-C_2\, t}\, \rho^2 + \Psi\big(C_1\,(1+e^{-C_2\, t}\, \rho^2)\big)\big)$$

$$\int_s^t \|u_n'(\xi)\|_{L^2}\, d\xi \le C_3 \cdot \sqrt{t-s} \quad \cdot \mathrm{const}\,(s_0,\, T,\, \rho)\,.$$

$$\|u_n'(t)\|_{H^{-1}(\Omega)} \le \mathrm{const}\,(\beta,\, C_\beta,\, \Lambda_A,\, \lambda_A,\, \lambda_\Omega,\, \rho)\,.$$

Proof of Proposition 4. The inclusions $u \in L^2\big(0,T;\, H_0^1(\Omega)\big)$ and $u' \in L^2\big(0,T;\, H^{-1}(\Omega)\big)$ always imply that $u \in C^0\big([0,T];\, L^2(\Omega)\big)$, see [7, § 5.9].

Let $\{w_k : k \ge 1\}$ be an orthogonal basis of $H_0^1(\Omega)$ as in Remark 1. Then for each $n \in \mathbb{N}$,

$$u_n : [0,T] \to H_0^1(\Omega), \quad t \longmapsto \sum_{k=1}^{n} \big(u(t),\, w_k\big)\, w_k$$

is induced by the orthogonal projection of $u(t)$ on $\mathrm{span}\{w_1 \ldots w_n\}$ in $L^2(\Omega)$. It solves the perturbed nonlocal Galerkin problem

$$\begin{cases} \frac{d}{dt}\,(u_n(t),\, w_k) + \langle A u_n(t),\, w_k\rangle = (\mathcal{G}(t),\, w_k) \\ \qquad\qquad\qquad\qquad (u_n(0),\, w_k) = (u_0,\, w_k)\,, \quad k = 1 \ldots n, \end{cases} \tag{12}$$

with the map $\mathcal{G} : [0,T] \to L^2(\Omega)$ defined by $\mathcal{G}(t) := \mathcal{F}(u(t))$ for each $t \in [0,T]$, which depends only on time in combination with the weak solution $u(\cdot)$ (but not on u_n explicitly). In particular, $\mathcal{G}$ fulfills the conditions (a)–(c) (uniformly with respect to time). Hence, Lemma 5 provides a priori estimates for each Galerkin approximation $u_n(\cdot)$ depending essentially only on the L^2 norm of the initial value u_0.

Since $\{w_k, k \geq 1\}$ is an orthonormal basis of $L^2(\Omega)$, the sequence $(u_n(t))_{n\in\mathbb{N}}$ converges to $u(t)$ in $L^2(\Omega)$ at each time $t \in [0,T]$. The L^2 bound of the H_0^1 norm implies a weakly convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ in $L^2\left(0,T;\, H_0^1(\Omega)\right)$ and, its weak limit is $u \in L^2\left(0,T;\, H_0^1(\Omega)\right)$ (again). For each $s_0 \in (0,T)$, we even have a L^∞ bound of the H_0^1 norms of $(u_n)_{n\in\mathbb{N}}$ in $[s_0,T]$ and so Lemma 7 below guarantees the same estimates holds for $\left\|\nabla u(t)\right\|^2_{L^2(\Omega)}$ at every time $t \in [s_0,T]$, not just for Lebesgue-almost all such t. ■

Fixing $s_0 \in (0,T)$ arbitrarily, the Galerkin approximations of any weak solution $u(\cdot)$ with $\|u(0)\|_{L^2(\Omega)} \leq \rho$ are equi–continuous w.r.t. $L^2(\Omega)$ in $[s_0,T]$ due to the second estimate in Lemma 5. Now the pointwise convergence of the Galerkin approximations to $u(\cdot)$ implies the following statement directly:

Lemma 6 *Suppose that the conditions (a)–(c) hold for $\mathcal{F}$ and that $T < \infty$. For every $\rho > 0$, the subset of $C^0\big([0,T];L^2(\Omega)\big)$ consisting of all weak solutions $u \in L^2\big(0,T;\, H_0^1(\Omega)\big)$ of (4) with $u' \in L^2\big(0,T;\, H^{-1}(\Omega)\big)$ and $\|u(0)\|_{L^2(\Omega)} \leq \rho$ is equi-continuous in the subinterval $[s_0,T]$ for every $s_0 \in (0,T)$.*

Lemma 7 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(t_0,T;X)$ such that $u_n \rightharpoonup u$ weakly in $L^p(t_0,T;X)$ for some $p \in [1,+\infty)$ and $u \in C^0([t_0,T];Y)$.*

Then, for every $t \in [t_0,T]$, $u(t)$ belongs to X and satisfies

$$\|u(t)\|_X \;\leq\; \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)}\;.$$

Proof of Lemma 7. We denote

$$C \;:=\; \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)}\;.$$

As $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(t_0,T;X)$, there exist a subsequence (u_μ) and $v \in L^\infty(t_0,T;X)$ such that $u_\mu \overset{*}{\rightharpoonup} v$ in $L^\infty(t_0,T;X)$, i.e.,

$$\int_{t_0}^T \langle w^*(t), u_\mu(t)\rangle\, dt \longrightarrow \int_{t_0}^T \langle w^*(t), v(t)\rangle\, dt \quad \forall\;\, w^* \in L^1(t_0,T;X'),$$

where by $\langle\cdot,\cdot\rangle$ we denote the duality product between X' and X.
In particular, we have this convergence for all $w^* \in L^{p'}(t_0,T;X')$. Then, $u_\mu \rightharpoonup v$ weakly in $L^p(t_0,T;X)$, and as we also have $u_n \rightharpoonup u$ weakly in $L^p(t_0,T;X)$, then $v = u$.

Then, $u_\mu \overset{*}{\rightharpoonup} u$ in $L^\infty(t_0, T; X)$ and by the $*-$weak lower semicontinuity of the norm, we obtain

$$\|u\|_{L^\infty(t_0,T;X)} \;\leq\; \liminf_{\mu\to\infty} \|u_\mu\|_{L^\infty(t_0,T;X)} \;\leq\; C. \tag{13}$$

Now fix $t \in [t_0, T]$. By (13) there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, T]$ such that $t_n \to t$ and $u(t_n) \in X$ with $\|u(t_n)\|_X \leq C$ for all $n \in \mathbb{N}$.
As X is reflexive, there exist a subsequence (t_μ) and $x \in X$ such that $u(t_\mu) \rightharpoonup x$ weakly in X. The inclusion $X \subset Y$ is assumed to be continuous and so,

$$u(t_\mu) \rightharpoonup x \text{ weakly in } Y. \tag{14}$$

Due to $u \in C^0([t_0, T]; Y)$, we have in addition that $u(t_n) \to u(t)$ in Y. This implies $u(t) = x \in X$.
Finally the weak lower semi-continuity of the norm implies for every $t \in [t_0, T]$

$$\|u(t)\|_X \;=\; \|x\|_X \;\leq\; \liminf_{\mu\to\infty} \|u(t_\mu)\|_X \;\leq\; C$$

■

4 Existence of weak solutions

Proposition 8 *Suppose that hypotheses (a) – (c) hold for $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$. Then, for every $u_0 \in L^2(\Omega)$, there exists a weak solution $u \in L^2\big(0, T;\, H_0^1(\Omega)\big)$ of problem (4). Moreover, u belongs to $C^0\big([0, T];\, L^2(\Omega)\big)$ and, $u|_{(0,T]}$ is locally bounded in $H_0^1(\Omega)$.*
Furthermore, if $\mathcal{F}$ is locally Lipschitz as in hypothesis (d), then the weak solution is unique.

Proof. The proof is based on a sequence of Galerkin approximations and the a priori estimates in § 3. Let $\{w_k, k \geq 1\}$ be an orthogonal basis of $H_0^1(\Omega)$ as in Remark 1 and for each $n \in \mathbb{N}$, consider a solution $u_n(t) = \sum_{k=1}^{n} u_{nk}(t)\, w_k$ of (11).

Fix $s_0 \in (0, T)$ arbitrarily.
By Proposition 3, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2\big(0, T;\, H_0^1(\Omega)\big)$ Due to Lemma 5, $(u_n')_{n\in\mathbb{N}}$ is bounded both in $L^2\big(s_0, T;\, L^2(\Omega)\big)$ and $L^2(0, T; H^{-1}(\Omega))$. Hence Alaoglu's Theorem provides a subsequence (again denoted by) $(u_n)_{n\in\mathbb{N}}$ and functions $u \in L^2\big(0, T;\, H_0^1(\Omega)\big)$, $v \in L^2\big(s_0, T;\, L^2(\Omega)\big)$, $w \in L^2\big(0, T;\, H^{-1}(\Omega)\big)$ with

$$\begin{cases} u_n \longrightarrow u & \text{weakly in } L^2\big(0,\ T;\ \ H_0^1(\Omega)\big) \\ u_n' \longrightarrow v & \text{weakly in } L^2\big(s_0, T;\ \ \ L^2(\Omega)\big) \\ u_n' \longrightarrow w & \text{weakly in } L^2\big(0,\ T;\, H^{-1}(\Omega)\big). \end{cases}$$

In particular, $u' = v = w$ holds Lebesgue-almost everywhere, which a simple check of the distributional derivative property reveals. This implies that $u \in L^2(0,T;\, H^{-1}(\Omega)) \cap C^0\big([0,T]; L^2(\Omega)\big)$. Standard arguments conclude $u(0) = u_0$ from $u_n(0) \longrightarrow u_0$ in $L^2(\Omega)$ for $n \to \infty$ (see e.g. [7, § 7.1]).

The sequence $(u_n)_{n\in\mathbb{N}}$ is equi–continuous in $C^0\big([s_0,T];\, L^2(\Omega)\big)$ by Lemma 6. Moreover, the set $\big\{u_n(t) \;\big|\; t \in\; [s_0,T],\; n \in \mathbb{N}\big\}$ is relatively compact in $L^2(\Omega)$ as a consequence of Lemma 5 and the Sobolev Embedding Theorem. Hence the Arzelà–Ascoli Theorem provides a further subsequence (again denoted by) $(u_n)_{n\in\mathbb{N}}$ with

$$u_n(t) \longrightarrow u(t) \quad \text{in } L^2(\Omega) \text{ uniformly for } t \in [s_0,T].$$

Hypothesis (a) on the continuity of $\mathcal{F}$ implies that $\mathcal{F}(u_n(t)) \to \mathcal{F}(u(t))$ in $L^2(\Omega)$ for every $t \in [s_0,T]$.

Taking the limit as $n \to \infty$ gives that $u \in L^2\big(s_0,T;\, H_0^1(\Omega)\big)$ is a weak solution to the partial differential equation in problem (4) with $u' \in L^2\big(s_0,T;\, H^{-1}(\Omega)\big)$. Finally, $u|_{[s_0,T]} : [s_0,T] \to H_0^1(\Omega)$ is bounded due to Proposition 4.

If $\mathcal{F}$ satisfies the local Lipschitz condition (d) in addition the uniqueness of the weak solution follows by a standard argument. ■

5 The multivalued semiflow of weak solutions

Define $\Phi_{\mathcal{F}} : [0,\infty) \times L^2(\Omega) \to \mathcal{P}\big(L^2(\Omega)\big)$, where $\mathcal{P}\big(L^2(\Omega)\big)$ consists of all nonempty subsets of $L^2(\Omega)$, by

$$\Phi_{\mathcal{F}}(t,u_0) := \Big\{ u(t) \in L^2(\Omega) \;\big|\; \exists\, u(\cdot) \in L^2\big(0,t;\, H_0^1(\Omega)\big) : u' \in L^2\big(0,t;\, H^{-1}(\Omega)\big),\; u(0) = u_0 \text{ and } u \text{ is weak solution of (4) in } \Omega \times (0,t) \Big\}. \tag{15}$$

It is clear that this multivalued map $\Phi_{\mathcal{F}}$ forms a strict multivalued semiflow on $L^2(\Omega)$ as in the the following definition of Kapustyan *et al.* [9, Definition 2.1].

Definition 9 *Let (X,d) be a metric space and, $\mathcal{P}(X)$ consists of all its nonempty subsets.*
A map $\Phi : [0,\infty) \times X \to \mathcal{P}(X)$ is called strict multivalued semiflow (m–semiflow) *on X if it satisfies the following conditions:*

(A) $\Phi(0,x) = \{x\}$ *for all* $x \in X$
(B) $\Phi(t+s,x) = \Phi(t,\, \Phi(s,x))$ *for all* $s,t \geq 0,\; x \in X$.

Remark 10 *Kapustyan* et al. *[9, Definition 2.1] define, in fact, a more general m–semiflow $\Phi : [0,\infty) \times X \to \mathcal{P}(X)$, which satisfies condition (A), but* *instead of the equality condition (B), satisfies the following weaker inclusion condition*

$(C) \quad \Phi(t+s,x) \subset \Phi(t,\ \Phi(s,x)) \quad$ *for all* $s,t \geq 0,\ x \in X.$

The reason is that the counterpart of the setvalued mapping $\Phi_{\mathcal{F}}$ above for many systems, in particular the 3-dimensional Navier–Stokes equations, is restricted to weak solutions that satisfy an energy inequality. However, such energy inequalities only hold (or can only be proved to hold) for almost all time instants. Hence a concatentation of weak solutions satisfying the energy inequality on adjacent time intervals may not satisfy the energy inequality on the concatenated time interval, which means that the strict property (B) need not hold. see also Morillas & Valero [12]. This situation does not arise for the mapping $\Phi_{\mathcal{F}}$ defined above for the problem (4) *under consideration in this paper.*

Every strict multivalued semiflow $\Phi : [0,\infty) \times X \to \mathcal{P}(X)$ (in the sense of Definition 9) induces a map $\tilde{\Phi} : [0,\infty) \times \mathcal{P}(X) \to \mathcal{P}(X)$ in the following canonical way:

$$\tilde{\Phi}(t,M) \stackrel{\text{Def.}}{=} \bigcup_{x\in M} \Phi(t,x) \quad \text{for all nonempty } M \subset X.$$

Obviously, it satisfies the corresponding conditions

$(A^*) \quad \tilde{\Phi}(0,M) = M \quad$ for all $M \in \mathcal{P}(X)$

$(B^*) \quad \tilde{\Phi}(t+s,M) = \tilde{\Phi}(t,\ \tilde{\Phi}(s,M)) \quad$ for all $s,t \geq 0,\ M \in \mathcal{P}(X)$

or in the general nonstrict case

$(C^*) \quad \tilde{\Phi}(t+s,M) \subset \tilde{\Phi}(t,\ \tilde{\Phi}(s,M)) \quad$ for all $s,t \geq 0,\ M \in \mathcal{P}(X)$

Hence one usually does not distinguish between $\tilde{\Phi}$ and Φ.

5.1 Global attractors of multivalued semiflows: General results

Definition 11 *A* global attractor *of a strict multivalued semiflow Φ on a metric space (X,d) is a nonempty subset A of X which is Φ-invariant, i.e. $\Phi(t,A) = A$ for all $t \geq 0$, and attracts every bounded subset B of X, i.e.*

$$\text{dist}_X\left(\Phi(t,B),A\right) \to 0 \quad \textit{as } t \to \infty,$$

where dist_X *is the Hausdorrf semi–distance on $\mathcal{P}(X)$. For a general non-strict multivalued semiflow the attractor A is only required to be Φ-negativly invariant, i.e. $A \subset \Phi(t,A)$ for all $t \geq 0$* [11, Definition 6].

The existence of a global attractor is usually concluded from the compactness or asymptotic compactness of the multivalued mapping $\Phi(t,\cdot)$ for $t > 0$:

Definition 12 [11, Definition 4] *A multivalued semiflow* Φ *on a metric space* (X, d) *is called* asymptotically upper semicompact *if it satisfies the following condition:*
Let $B \subset X$ *be any bounded subset such that* $\bigcup_{t \geq t_B} \Phi(t, B)$ *is bounded in* (X, d) *for some* $t_B \geq 0$. *Then every sequence* $(\xi_n)_{n \in \mathbb{N}}$ *with* $\xi_n \in \Phi(t_n, B)$ *for some sequence* $t_n \to \infty$ *is precompact in* X.

Theorem 13 [11, Theorem 2 & Remark 2]
Let the multivalued semiflow $\Phi : [0, \infty) \times X \to \mathcal{P}(X)$ *be asymptotically upper semicompact with nonempty compact values. Suppose for any* $t > 0$ *that* $\Phi(t, \cdot) : X \to \mathcal{P}(X)$ *has closed graph. Furthermore assume that for every bounded subset* $B \subset X$, *there exists* $t_B > 0$ *such that* $\bigcup_{t \geq t_B} \Phi(t, B)$ *is bounded in* (X, d) .

Then Φ *has a global attractor* $\mathcal{A}$ *in* X *given by*

$$\mathcal{A} = \bigcup_{\substack{B \subset X \\ B \, bounded}} \bigcap_{t \geq 0} \overline{\Phi(t, B)}.$$

For the parabolic differential equations considered here, however, we can draw essentially the same conclusions without investigating asymptotic features of $\Phi(t, \cdot)$ (for $t \to \infty$) explicitly:

Theorem 14 [11, Theorem 4 & Remark 7] *Let* $\Phi : [0, \infty) \times X \to \mathcal{P}(X)$ *be a multivalued semiflow such that for every* $t > 0$, $\Phi(t, \cdot) : X \to \mathcal{P}(X)$ *has closed graph and compact values. Furthermore suppose the existence of a compact set* $K \subset X$ *such that for every bounded subset* $B \subset X$,

$$\text{dist}_X (\Phi(t, B), K) \to 0 \quad \text{as } t \to \infty.$$

Then Φ *has a compact global attractor and, it is the minimal closed set attracting every bounded subset of* X.

5.2 The main result about $\Phi_{\mathcal{F}}$: existence of a global attractor

The subsequent Propositions 16, 18 and 21 verify theta $\Phi_{\mathcal{F}}$ satisfies the assumptions of Theorem 14 above. Hence, the main result for the nonlocal reaction-diffusion problem (4) is the following statement:

Theorem 15 *Assume that* Ω *is a bounded open set and that* $\mathcal{F} : L^2(\Omega) \longrightarrow L^2(\Omega)$ *satisfies hypotheses (a)–(c).*

Then the strict multivalued semiflow $\Phi_{\mathcal{F}} : [0, \infty) \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ *defined in* (15) *has a compact global attractor in* $L^2(\Omega)$.

5.3 Topological properties of $\Phi_{\mathcal{F}}$

In combination with the Sobolev Embedding Theorem, the estimates in Proposition 4 and Proposition 8 (about existence of weak solutions) imply immediately:

Proposition 16 *If $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ satisfies hypotheses (a) – (c) then for every $t \in (0, \infty)$, the multivalued semiflow $\Phi_{\mathcal{F}}(t, \cdot)$ maps every nonempty bounded subset of $L^2(\Omega)$ into a nonempty and relatively compact subset of $L^2(\Omega)$.*

Lemma 17 *Assume that Ω is a bounded open set and $\mathcal{F} : L^2(\Omega) \longrightarrow L^2(\Omega)$ satisfies hypotheses (a)–(c).*
Then, the multivalued map $\Phi_{\mathcal{F}}$ defined in (15) is a strict multivalued semiflow on $L^2(\Omega)$.

Proof. It is easy to check that $\Phi_{\mathcal{F}}$ is well defined. Moreover, $\Phi_{\mathcal{F}}$ satisfies condition (A) in Definition 9.
Let us now prove that $\Phi_{\mathcal{F}}(t+s, u_0) \subset \Phi_{\mathcal{F}}(t, \Phi_{\mathcal{F}}(s, u_0))$ also holds for all $t, s \in \mathbb{R}^+$, $u_0 \in L^2(\Omega)$. Consider $y \in \Phi_{\mathcal{F}}(t+s, u_0)$. Then from the definition of $\Phi_{\mathcal{F}}$, there exists a solution $u \in \mathcal{D}(u_0)$ such that $u(t+s) = y$. As $s \in \mathbb{R}^+$, then $u(s) \in \Phi_{\mathcal{F}}(s, u_0)$, and the result follows if we prove $y \in \Phi_{\mathcal{F}}(t, u(s))$. Let $\overline{u}(\cdot) = u(\cdot + s)$. It is straightforward to prove by a change of variable that $\overline{u}$ is a weak solution and $\overline{u}(t) = u(t+s) = y$, $\overline{u}(0) = u(s)$. Then $y \in \Phi_{\mathcal{F}}(t, u(s)) \subset \Phi_{\mathcal{F}}(t, \Phi_{\mathcal{F}}(s, u_0))$.

Now we shall prove that $\Phi_{\mathcal{F}}(t, \Phi_{\mathcal{F}}(s, u_0)) \subset \Phi_{\mathcal{F}}(t+s, u_0)$.
Let $y \in \Phi_{\mathcal{F}}(t, \Phi_{\mathcal{F}}(s, u_0))$, then there exist z_1, $u_1(\cdot) \in \mathcal{D}(u_0)$, and $u_2(\cdot) \in \mathcal{D}(z_1)$, verifying

$$u_1(0) = u_0, \qquad u_1(s) = z_1,$$

$$u_2(0) = z_1, \qquad u_2(t) = y.$$

We shall check that there is $u(\cdot) \in \mathcal{D}(u_0)$ verifying $u(0) = u_0$, $u(t+s) = y$. As u_1, $u_2 \in C^0\big([0,T];\, L^2(\Omega)\big)$, we can define u as

$$u(r) = \begin{cases} u_1(r), & if \quad 0 \le r \le s, \\ u_2(r-s), & if \quad s \le r. \end{cases}$$

If we prove that u is a weak solution, then it is evident that $y \in \Phi_{\mathcal{F}}(t+s, u_0)$. For any $v \in \mathcal{C}_0^\infty([0,T] \times \Omega)$, using the change of variable $\tau = r - s$, and the definition of u_1 and u_2, we have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial}{\partial r} u, v \right\rangle dr + \int_0^T \left[\langle Au, v\rangle - (\mathcal{F}(u), v) \right] dr \\
&= \int_0^s \left\langle \frac{\partial}{\partial r} u_1, v \right\rangle dr + \int_0^s \left[\langle Au_1, v\rangle - (\mathcal{F}(u_1), v) \right] dr \\
&+ \int_s^T \left\langle \frac{\partial}{\partial r} u_2(r-s), v \right\rangle dr + \int_s^T \left[\langle Au_2(r-s), v\rangle - (\mathcal{F}(u_2(r-s)), v) \right] dr \\
&= \int_0^s \left\langle \frac{\partial}{\partial r} u_1, v \right\rangle dr + \int_0^s \left[\langle Au_1, v\rangle - (\mathcal{F}(u_1), v) \right] dr \\
&+ \int_0^{T-s} \left\langle \frac{\partial}{\partial r} u_2, v \right\rangle dr + \int_0^{T-s} \left[\langle Au_2, v\rangle - (\mathcal{F}(u_2), v) \right] dr.
\end{aligned}$$

Since u_1, u_2 are weak solutions, then the two last integrals are equal to zero. Hence, u is a weak solution. ■

Proposition 18 *Suppose hypotheses (a) – (c) for* $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$. *For every* $t > 0$, *the graph of the multivalued map* $\Phi_{\mathcal{F}}(t, \cdot) : L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ *is closed with respect to the norm topology in* $L^2(\Omega)$.

Proof.

Choose sequences $(u_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ in $L^2(\Omega)$ converging to u, ξ respectively with $\xi_n \in \Phi_{\mathcal{F}}(t, u_n)$ for every $n \in \mathbb{N}$.

By definition of $\Phi_{\mathcal{F}}$, there exists a weak solution $v_n(\cdot) \in L^2\left(0, t;\, H_0^1(\Omega)\right)$ of (4) for each $n \in \mathbb{N}$ such that $v_n(0) = u_n$, $v_n(t) = \xi_n$ and $v_n' \in L^2\left(0, t;\, H^{-1}(\Omega)\right)$. This implies $v_n(\cdot) \in C^0\left([0, t];\, L^2(\Omega)\right)$.

Due to the uniform bounds in Proposition 3, the Alaoglu Theorem provides a subsequence (again denoted by) $(v_n(\cdot))_{n\in\mathbb{N}}$ and some $w \in L^2\left(0, T;\, H_0^1(\Omega)\right)$ with $w' \in L^2\left(0, T;\, H^{-1}(\Omega)\right)$ and

$$\begin{cases} v_n \longrightarrow w & \text{weakly in } L^2\big(0, T;\ \ H_0^1(\Omega)\big) \\ v_n' \longrightarrow w' & \text{weakly in } L^2\big(0, T;\, H^{-1}(\Omega)\big). \end{cases}$$

In particular, $w \in C^0\left([0, t];\, L^2(\Omega)\right)$.

Fixing $s_0 \in (0, t)$ arbitrarily, the sequence of restrictions $v_n(\cdot)|_{[s_0,t]}$ is equicontinuous according to Lemma 6 and, all their values are in a relatively compact subset of $L^2(\Omega)$ due to Proposition 4 and the Sobolev Embedding Theorem. Hence the Arzelà–Ascoli Theorem ensures a subsequence of $\left(v_n(\cdot)|_{[s_0,t]}\right)_{n\in\mathbb{N}}$ which converges uniformly in $C^0\left([s_0, t];\, L^2(\Omega)\right)$.

By means of Cantor's diagonal construction, we obtain a subsequence (again denoted by) $(v_n(\cdot))_{n\in\mathbb{N}}$ and some $v \in C^0\left((0, t];\, L^2(\Omega)\right)$ such that for *every* $s_0 \in (0, t)$,

$$v_n|_{[s_0,t]} \longrightarrow v|_{[s_0,t]} \quad \text{uniformly w.r.t. } L^2(\Omega).$$

The Mazur Lemma implies $v(s) = w(s) \in H^1_0(\Omega)$ for Lebesgue-almost every $s \in [0,t]$. Finally, hypothesis (a) about the continuity of $\mathcal{F}$ and the weak convergences mentioned above lead to the conclusion that for every $\varphi \in H^1_0(\Omega)$, the limit of

$$(v_n',\ \varphi) \ + \ \langle A\, v_n,\ \varphi\rangle \ = \ (\mathcal{F}(v_n),\ \varphi)$$

for $n \to \infty$ is

$$(v',\ \varphi) \ + \ \langle A\, v,\ \varphi\rangle \ = \ (\mathcal{F}(v),\ \varphi)$$

Lebesgue-almost everywhere in $[0,t]$.

As regards the claim that $\xi \in \Phi_{\mathcal{F}}(t,u)$, we still have to check $v(0) = u$. Following the standard arguments (as in [7, § 7.1 c], for example), the weak solution property of $v_n, v \in L^2\big(0,t;\ H^1_0(\Omega)\big)$ guarantees

$$\begin{cases} -\displaystyle\int_0^t \big((v_n,\ \eta') + \langle A\, v_n,\ \eta\rangle\big)\ ds \ = \ \displaystyle\int_0^t (\mathcal{F}(v_n),\ \eta)\ ds + (u_n,\ \eta(0)) \\ -\displaystyle\int_0^t \big((v,\ \eta') \ + \langle A\, v,\ \eta\rangle\big)\ ds \ \ = \ \displaystyle\int_0^t (\mathcal{F}(v),\ \eta)\ ds \ \ + (v(0),\ \eta(0)) \end{cases}$$

for every $\eta \in C^1\big([0,t];\ H^1_0(\Omega)\big)$ with $\eta(t) = 0$. The convergence of $(v_n)_{n\in\mathbb{N}}$ mentioned above leads to $(u_n,\ \eta(0)) \longrightarrow (v(0),\ \eta(0))$ for every $\eta(0) \in H^1_0(\Omega)$, i.e.

$$v(0) \ = \ \lim_{n\to\infty} u_n \ = \ u \quad \text{in } L^2(\Omega). \qquad \blacksquare$$

5.4 Absorbing sets of $\Phi_{\mathcal{F}}$ in $L^2(\Omega)$ and $H^1_0(\Omega)$

Definition 19 *Let Φ be a multivalued semiflow on a metric space (X,d). A subset $M \subset X$ is called* bounded absorbing set *of Φ in X if it satisfies the following conditions that*

- *(i) M is bounded in (X,d), i.e. there exist $x_0 \in X$ and $\rho > 0$ with $d(x,x_0) \le \rho$ for all $x \in M$,*
- *(ii) for every bounded set $B \subset X$, there exists $t_B \ge 0$ such that $\Phi(t,B) \subset M$ holds for all $t \ge t_B$.*

Proposition 20 *If $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ satisfies hypothesis (b), then*

$$M_0 \ := \ \left\{ v \in L^2(\Omega) : \|v\|^2_{L^2(\Omega)} \le \frac{M}{L} + 1 \right\}.$$

is a bounded absorbing set of the multivalued semiflow $\Phi_{\mathcal{F}}$ in $L^2(\Omega)$.

Proof. According to Proposition 3, every weak solution $u \in L^2_{\text{loc}}\big(0,\infty;\ H^1_0(\Omega)\big)$ of (4) with $u' \in L^2_{\text{loc}}\big(0,\infty;\ H^{-1}(\Omega)\big)$ satisfies at every time $t \ge 0$

$$\|u(t)\|^2_{L^2(\Omega)} \le \frac{M}{L} + e^{-Lt}\, \|u(0)\|^2_{L^2(\Omega)}\,.$$

For every bounded set $B \subset L^2(\Omega)$, there exists some $t_B \geq 0$ such that all $u_0 \in B$ and $t \geq t_B$ fulfill

$$e^{-Lt} \ \|u_0\|^2_{L^2(\Omega)} \ \leq \ 1$$

and so, all weak solutions $u \in L^2_{\mathrm{loc}}\big(0,\infty;\, H^1_0(\Omega)\big)$ of (4) with $u(0) \in B$ satisfy

$$\|u(t)\|^2_{L^2(\Omega)} \ \leq \ \frac{M}{L} + 1 \quad \forall\, t \geq t_B.$$

■

Proposition 21 *If $\mathcal{F} : L^2(\Omega) \longrightarrow L^2(\Omega)$ satisfies hypothesis (a) – (c), then the multivalued semiflow $\Phi_{\mathcal{F}}$ of problem (4) has a bounded absorbing set in $H^1_0(\Omega)$ in the following sense:*
There exists a bounded set $M_1 \subset H^1_0(\Omega)$ such that for every bounded subset $B \subset L^2(\Omega)$, there exists some $t_B \geq 0$ with $\Phi_{\mathcal{F}}(t, B) \subset M_1$ for all $t \geq t_B$.

Proof. It results from Proposition 20 and the a priori estimates in Proposition 4. Consider

$$M_1 \ := \ \left\{ v \in H^1_0(\Omega) \ : \ \|v\|^2_{L^2(\Omega)} \leq \frac{M}{L} + 1, \ \ \|\nabla v\|^2_{L^2(\Omega)} \leq C_1 \ (2 + \Psi(2\, C_1)) \right\}$$

and the constants C_1, C_2 mentioned in Proposition 4 and depending only on β, C_β, Λ_A, λ_A and λ_Ω. For every bounded set $B \subset L^2(\Omega)$, there exists a finite time $t_B \geq 1$ such that all $u_0 \in B$ satisfy

$$e^{-L\, t_B} \ \|u_0\|_{L^2(\Omega)} \leq 1, \qquad e^{-C_2\, t_B} \ \|u_0\|_{L^2(\Omega)} \leq 1$$

Then every $v \in \Phi_{\mathcal{F}}(t, u_0)$ with $t \geq t_B$ and $u_0 \in B$ fulfills $v \in M_1$. ■

6 An example

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a smooth boundary. Let us consider the following problem for a nonlocal reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u)\overline{u}\,(1 - \overline{u})\,, \ \text{ in } \Omega \times (0, +\infty)\,, \\ u = 0, \ \text{ on } \partial\Omega \times (0, +\infty)\,, \\ u(x, 0) = u_0(x), \ \ x \in \Omega, \end{cases} \tag{16}$$

where we now define

$$\overline{u}(x) = \frac{1}{|B(x;\delta)|} \int_{B(x;\delta)\cap\Omega} u(y)dy, \tag{17}$$

for some small $\delta > 0$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$, f is continuous and there exists a constant $C > 0$ such that f satisfies

$$|f(r)| \leq C, \text{ for all } r \in \mathbb{R},$$

and

$$rf(r) \geq 0, \text{ for all } r \in \mathbb{R}.$$

Remark 22 *This structure is chosen to simplify the estimates presented below. In particular, $|B(x;\delta)|$ in the denominator of (17) could be replaced by $|B(x;\delta) \cap \Omega|$ if the domain Ω satisfies a uniform interior cone condition, since this would ensure that $|B(x;\delta) \cap \Omega|$ is uniformly bounded from below.*

We observe that this problem is a particular case. If we consider $a^{ij} = \delta_{ij}$, we have $A = -\Delta$, $\lambda_A = \Lambda_A = 1$ and $\mathcal{F}(u)(x) = f(u(x))\overline{u}\,(1 - \overline{u})$.

Now, we will prove the hypothesis (6) and (10).

We observe that thanks to the assumptions of f,

$$\begin{aligned}
\int_\Omega u\mathcal{F}(u)dx &= \int_\Omega uf(u)\overline{u}\,(1-\overline{u})\,dx \\
&= \int_\Omega uf(u)\overline{u}dx - \int_\Omega uf(u)\overline{u}^2dx \\
&\leq \frac{1}{4}\int_\Omega uf(u)dx + \int_\Omega uf(u)\overline{u}^2dx - \int_\Omega uf(u)\overline{u}^2dx \\
&= \frac{1}{4}\int_\Omega uf(u)dx \\
&\leq \frac{1}{4}\int_\Omega \left(\frac{\lambda_\Omega}{2}|u|^2 + \frac{|f(u)|^2}{2\lambda_\Omega}\right)dx \\
&= \frac{\lambda_\Omega}{8}\|u\|^2_{L^2(\Omega)} + \frac{C^2}{8\lambda_\Omega}|\Omega|,
\end{aligned}$$

and then we have (6) with $\beta = \frac{\lambda_\Omega}{8} \in (0, \lambda_\Omega)$ and $C_\beta = \frac{C^2}{8\lambda_\Omega}|\Omega| > 0$.

We also have

$$\int_\Omega |\mathcal{F}(u)(x)|^2\,dx \;\leq C\left(\int_\Omega \overline{u}^2dx + \int_\Omega \overline{u}^4dx - 2\int_\Omega \overline{u}^3dx\right).$$

Taking into account that

$$\begin{aligned}
|\overline{u}| &= \left|\frac{1}{B(x;\delta)}\int_{B(x;\delta)\cap\Omega} u(y)dy\right| \\
&\leq \frac{1}{|B(x;\delta)|}\left(\int_{B(x;\delta)\cap\Omega} u^2(y)dy\right)^{1/2}\left(\int_{B(x;\delta)\cap\Omega} 1dy\right)^{1/2} \\
&\leq \frac{1}{|B(0;\delta)|^{1/2}}\|u\|_{L^2(\Omega)},
\end{aligned}$$

we observe that if we consider $m \geq 1$, we have

$$\int_{\Omega} |\overline{u}|^m \, dx \leq \frac{|\Omega|}{|B(0;\delta)|^{m/2}} \|u\|^m_{L^2(\Omega)} . \tag{18}$$

We also obtain

$$\begin{aligned} \int_{\Omega} (\mathcal{F}(u))^2 dx \leq & \frac{C\,|\Omega|}{|B(0;\delta)|} \|u\|^2_{L^2(\Omega)} \\ & + \frac{C\,|\Omega|}{|B(0;\delta)|^2} \|u\|^4_{L^2(\Omega)} \\ & + \frac{2C\,|\Omega|}{|B(0;\delta)|^{3/2}} \|u\|^3_{L^2(\Omega)} , \end{aligned}$$

and then we have (7) with $\Phi\big(\|u\|_{L^2(\Omega)}\big) = \frac{C|\Omega|}{|B(0;\delta)|} \|u\|^2_{L^2(\Omega)} + \frac{C|\Omega|}{|B(0;\delta)|^2} \|u\|^4_{L^2(\Omega)} + \frac{2C|\Omega|}{|B(0;\delta)|^{3/2}} \|u\|^3_{L^2(\Omega)}$.

We observe that if $u \in L^2(\Omega)$, then $\mathcal{F}(u) \in L^\infty(\Omega) \subset L^2(\Omega)$.

Now, we prove that $\mathcal{F}$ is continuous. We suppose that $u_n \longrightarrow u$ strongly in $L^2(\Omega)$, then we have to prove that $\mathcal{F}(u_n) \longrightarrow \mathcal{F}(u)$ strongly in $L^2(\Omega)$. It is sufficient to prove that there exists $\{u_\mu\} \subset \{u_n\}$ such that $\mathcal{F}(u_\mu) \longrightarrow \mathcal{F}(u)$ strongly in $L^2(\Omega)$.

As $u_n \longrightarrow u$ strongly in $L^2(\Omega)$, then (see [4]) there exists $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu(x) \longrightarrow u(x) \text{ a.e. in } \Omega.$$

Then, as f is continuous we have

$$f(u_\mu(x)) \longrightarrow f(u(x)) \text{ a.e. in } \Omega. \tag{19}$$

We observe that arguing as before we obtain

$$\begin{aligned} |\overline{u}_\mu(x) - \overline{u}(x)| &= \frac{1}{|B(x;\delta)|} \left| \int_{B(x;\delta)\cap\Omega} (u_\mu(y) - u(y))\, dy \right| \\ &\leq \frac{1}{|B(0;\delta)|^{1/2}} \|u_\mu - u\|_{L^2(\Omega)} \longrightarrow 0, \end{aligned}$$

and then

$$\overline{u}_\mu(x) \longrightarrow \overline{u}(x) \text{ a.e. in } \Omega. \tag{20}$$

From (19) and (20) we have

$$\mathcal{F}(u_\mu)(x) \longrightarrow \mathcal{F}(u)(x) \text{ a.e. in } \Omega. \tag{21}$$

On the other hand, we observe that $\|u_\mu\|_{L^2(\Omega)}$ is bounded because Ω is a bounded domain and $u_\mu \longrightarrow u$ strongly in $L^2(\Omega)$. Then, we have

$$\begin{aligned} |\mathcal{F}(u_\mu(x))| &\leq C\,|\overline{u}_\mu(x)|\,|1-\overline{u}_\mu(x)| \qquad (22)\\ &\leq C\frac{1}{|B(0;\delta)|^{1/2}}\,\|u_\mu\|_{L^2(\Omega)}\left(1+\frac{1}{|B(0;\delta)|^{1/2}}\,\|u_\mu\|_{L^2(\Omega)}\right)\\ &\leq \widetilde{C}, \end{aligned}$$

where $\widetilde{C}$ is a positive constant.

Then, from (21), (22) and by the Dominated Convergence Theorem, we obatin

$$\mathcal{F}(u_\mu) \longrightarrow \mathcal{F}(u) \text{ strongly in } L^2(\Omega),$$

then $\mathcal{F}$ is continuous.

From Theorem 8 we have that there exists a weak solution $u \in L^2\big(0,T;\, H_0^1(\Omega)\big)$ of problem (16) with $u' \in L^2\big(0,T;\, H^{-1}(\Omega)\big)$. According to Theorem 15, the problem (16) defines a multivalued semiflow in $L^2(\Omega)$, which possesses a compact global attractor $\mathcal{A}$. Also, $\mathcal{A}$ is the minimal closed attracting set.

7 Proofs of the dissipativity estimates

Proof of Proposition 3. The uniform ellipticity condition of the linear operator A and assumption (b) on the nonlocal operator $\mathcal{F}$ used in the energy equality,

$$\frac{1}{2}\,\frac{d}{dt}\,\|u\|^2_{L^2(\Omega)} + \int_\Omega u\,(A\,u)\,dx \;=\; \int_\Omega u\,\mathcal{F}(u)\,dx \qquad (23)$$

give

$$\frac{1}{2}\,\frac{d}{dt}\,\|u\|^2_{L^2(\Omega)} + \lambda_A\,\|\nabla u\|^2_{L^2(\Omega)} \;\leq\; \beta\,\|u\|^2_{L^2(\Omega)} \;+\; C_\beta. \qquad (24)$$

Hence, by the Poincaré inequality (3),

$$\frac{1}{2}\,\frac{d}{dt}\,\|u\|^2_{L^2(\Omega)} \;+\; \lambda_\Omega\,\lambda_A\,\|u\|^2_{L^2(\Omega)} \;\leq\; \beta\,\|u\|^2_{L^2(\Omega)} +\; C_\beta.$$

i.e.,

$$\frac{d}{dt}\,\|u\|^2_{L^2(\Omega)} \leq M - L\,\|u\|^2_{L^2(\Omega)}\,, \qquad (25)$$

where $M := 2\,C_\beta > 0$ and $L := 2\,\lambda_\Omega\,\lambda_A - 2\beta > 0$.

Multiplying by e^{Lt}, and integrating between 0 and t, it follows

$$\begin{aligned} e^{Lt}\,\|u(t)\|^2_{L^2(\Omega)} &\leq \|u(0)\|^2_{L^2(\Omega)} + M\int_0^t e^{Ls}\,ds\\ &\leq \|u(0)\|^2_{L^2(\Omega)} + M\int_{-\infty}^t e^{Ls}\,ds, \end{aligned}$$

i.e., for every $t \in [0,T]$,

$$\|u(t)\|^2_{L^2(\Omega)} \;\leq\; \frac{M}{L} + e^{-Lt}\, \|u(0)\|^2_{L^2(\Omega)} \,. \tag{26}$$

Finally, integrating equation (24) with respect to time, we obtain

$$\begin{aligned}
&\lambda_A \int_0^t \|\nabla u\|^2_{L^2(\Omega)} \, ds \\
&\quad\leq\; \frac{1}{2} \left(\|u_0\|^2_{L^2(\Omega)} - \|u(t)\|^2_{L^2(\Omega)} \right) \;+\; \beta \int_0^t \|u\|^2_{L^2(\Omega)} \, ds \;+\; C_\beta \, t \\
&\quad\leq\; \|u_0\|^2_{L^2(\Omega)} \left(\frac{1}{2} \;+\; \beta \int_0^t e^{-L\, s} \, ds \right) \;+\; \left(C_\beta + \frac{\beta M}{L} \right) \, t.
\end{aligned}$$

■

Proof of Lemma 5. Let $\{w_k : k \geq 1\}$ denote an orthogonal basis of $H^1_0(\Omega)$ as mentioned in Remark 1. For each $n \geq 1$, let

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t) \, w_k,$$

be a Galerkin approximation of problem (4) satisfying the finite dimensional system of ordinary differential equations in the sense that

$$\left\{ \begin{array}{r} \frac{d}{dt} \, (u_n(t), w_k) + \langle A u_n(t), w_k \rangle = (\mathcal{F}(u_n(t)), w_k) \\ (u_n(0), w_k) = (u_0, w_k) \,, \quad k = 1, \ldots, n. \end{array} \right. \tag{27}$$

We observe that

$$A u_n(t) \;=\; \sum_{k=1}^{n} \; u_{nk}(t) \; A w_k \;=\; \sum_{k=1}^{n} \; u_{nk}(t) \; \lambda_k \, w_k \;\in\; \text{span}\, \{w_1, \ldots, w_n\} \,.$$

Then, multiplying equation (27) by $u_{nk}(t)\, \lambda_k$, summing from $k = 1$ to n, and using property (c) of $\mathcal{F}$, gives

$$\begin{aligned}
\langle u_n'(t),\, A u_n(t) \rangle \;+\; \|A u_n(t)\|^2_{L^2(\Omega)} \;&=\; (\mathcal{F}(u_n(t)),\, A u_n(t)) \\
&\leq\; \tfrac{1}{2} \left(\|\mathcal{F}(u_n(t))\|^2_{L^2(\Omega)} + \|A u_n(t)\|^2_{L^2(\Omega)} \right) \\
&\leq\; \Psi(u_n(t)) \;+\; \tfrac{1}{2} \, \|A u_n(t)\|^2_{L^2(\Omega)}
\end{aligned} \tag{28}$$

for Lebesgue-almost all $t \in [0,T]$. On the other hand,

$$\langle u_n'(t), A u_n(t)\rangle = -\int_\Omega \sum_{i,j=1}^N \partial_{x_j}\big(a^{ij}(x)\,\partial_{x_i} u_n\big)\, u_n'\, dx \tag{29}$$

$$= \sum_{i,j=1}^N \int_\Omega a^{ij}(x)\,\partial_{x_i} u_n\,(\partial_t\,\partial_{x_j} u_n)\, dx$$

$$= \frac{d}{dt}\left(\frac{1}{2}\cdot \mathcal{A}\,[u_n, u_n]\right) \tag{30}$$

for the symmetric bilinear form

$$\mathcal{A}\,[u,v] := \int_\Omega \sum_{i,j=1}^N a^{ij}(x)\,\partial_{x_i} u\,\partial_{x_j} v\, dx,\;\; u, v \in H_0^1(\Omega).$$

since it is assumed that $a^{ij} = a^{ji}$, with $i,j = 1,..,N$, and these coefficients do not depend on t.

Integrating from s and t, then gives

$$\frac{1}{2}\cdot \mathcal{A}\,[u_n(t), u_n(t)] \;\le\; \frac{1}{2}\cdot \mathcal{A}\,[u_n(s), u_n(s)] \;+\; \int_s^t \Psi\big(\|u_n(\xi)\|_{L^2(\Omega)}\big)\, d\xi. \tag{31}$$

Hence, it follows from the inequalities of coercivity and continuity

$$\lambda_A\,\|\nabla u_n(s)\|^2_{L^2(\Omega)} \;\le\; \mathcal{A}\,[u_n(s), u_n(s)] \;\le\; \Lambda_A\,\|\nabla u_n(s)\|^2_{L^2(\Omega)}$$

that

$$\frac{\lambda_A}{2}\cdot \|\nabla u_n(t)\|^2_{L^2(\Omega)} \;\le\; \frac{\Lambda_A}{2}\cdot \|\nabla u_n(s)\|^2_{L^2(\Omega)} \;+\; \int_s^t \Psi\big(\|u_n(\xi)\|_{L^2(\Omega)}\big)\, d\xi. \tag{32}$$

Integrating now the variable s between $t_0 := \max\{0, t-1\}$ and t, gives

$$\begin{aligned}\frac{\lambda_A}{2}\cdot \min\{t,1\}\cdot \|\nabla u_n(t)\|^2_{L^2(\Omega)} \quad &\le \quad \frac{\Lambda_A}{2}\int_{\max\{0,t-1\}}^t \|\nabla u_n(s)\|^2_{L^2(\Omega)}\, ds \;+ \\ &\qquad \int_{\max\{0,t-1\}}^t \Psi\big(\|u_n(s)\|_{L^2(\Omega)}\big)\, ds.\end{aligned}$$

On the other hand, integrating (24) with respect to time in $[t_0, t]$, it follows that

$$\begin{aligned}&\|u_n(t)\|^2_{L^2(\Omega)} - \|u_n(t_0)\|^2_{L^2(\Omega)} \;+\; 2\,\lambda_A \int_{t_0}^t \|\nabla u_n(s)\|^2_{L^2(\Omega)}\, ds \\ &\le\; 2\,\beta \int_{t_0}^t \|u_n(s)\|^2_{L^2(\Omega)}\, ds + 2\,C_\beta.\end{aligned}$$

Then, using Proposition 3,

$$\Lambda_A \int_{t_0}^{t} \|\nabla u_n(s)\|^2_{L^2(\Omega)}\, ds \;\le\; \Lambda_A\, \frac{2\,\beta+1}{2\,\lambda_A} \sup_{s\in[t_0,t]} \|u_n(s)\|^2_{L^2(\Omega)} \;+\; \frac{C_\beta\,\Lambda_A}{\lambda_A}$$

$$\le\; C_2\cdot\Big(1+e^{-L\,t}\,\|u_0\|^2_{L^2(\Omega)}\Big) \tag{33}$$

for all $t\ge 0$ with a constant $C_2 = C_2(\beta,\Lambda_A,\lambda_A,\lambda_\Omega,C_\beta) > 1$. Thus, whenever $\|u_0\|_{L^2(\Omega)} \le \rho$,

$$\|\nabla u_n(t)\|^2_{L^2(\Omega)} \;\le\; \frac{2\,C_2}{\lambda_A\ \min\{t,1\}}\ \left(1+e^{-L\,t}\,\rho^2 \;+\; \Psi\big(\frac{M}{L}+e^{-L\,t}\,\rho^2\big)\right). \tag{34}$$

Moreover, this upper bound holds for *every* $t\in(0,T]$ — in contrast to the immediate consequence of inequality (28) and the coercivity of $\mathcal{A}$.

The next step is to establish an estimate for $\|u_n'\|_{L^\infty(0,T;\,H^{-1}(\Omega))}$. This follows essentially the arguments for the standard energy estimate, taking the preceding a priori bounds into consideration.
For every $v\in H^1_0(\Omega)$, there is a unique representation $v=v^1+v^2$ with $v^1\in \mathrm{span}\{w_1,\ldots,w_n\}$ and $(v^2,w_k)=0$ for $k=1,\ldots,n$ since $\{w_k, k\ge 1\}$ is an orthonormal basis of $L^2(\Omega)$.

For Lebesgue-almost every $t\in[0,T]$ it follows that

$$\begin{aligned}
\big(u_n'(t),\,v^1\big)+\big\langle Au_n(t),\,v^1\big\rangle \;&=\; \big(\mathcal{F}(u_n(t)),\,v^1\big)\\
\langle u_n'(t),\,v\rangle \;=\; (u_n'(t),\,v) \;&=\; \big(u_n'(t),\,v^1\big)\\
&=\; \big(\mathcal{F}(u_n(t)),\,v^1\big)-\big\langle Au_n(t),\,v^1\big\rangle\\
&\le\; \big(\|\mathcal{F}(u_n(t))\|_{L^2}+\Lambda_A\,\|\nabla u_n(t)\|_{L^2}\big)\ \|v^1\|_{H^1_0(\Omega)}.
\end{aligned}$$

Then, inequality (7) in assumption (c) of $\mathcal{F}$ implies that

$$\langle u_n'(t),\,v\rangle \;\le\; \Big(\sqrt{\Psi\left(\|u_n(t)\|_{L^2}\right)} \;+\; \Lambda_A\,\|\nabla u_n(t)\|_{L^2}\Big)\ \|v^1\|_{H^1_0(\Omega)}.$$

Hence, in view of the estimates (26) and (34),

$$\begin{aligned}
\|u_n'(t)\|_{H^{-1}(\Omega)} \;&\le\; \sqrt{\Psi\left(\|u_n(t)\|_{L^2}\right)} \;+\; \Lambda_A\,\|\nabla u_n(t)\|_{L^2}\\
&\le\; \mathrm{const}\,\big(\beta,\,C_\beta,\,\Lambda_A,\,\lambda_A,\,\lambda_\Omega,\,\|u_0\|_{L^2(\Omega)}\big)\,.
\end{aligned}$$

Finally, multiplying (27) by $u_{nk}'(t)$ and summing from $k=1$ to n, it follows from inequality (10) in assumption (c) of $\mathcal{F}$ that

$$\begin{aligned} \langle u_n'(t),\, u_n'(t)\rangle + \langle Au_n(t),\, u_n'(t)\rangle &= (\mathcal{F}(u_n(t)),\, u_n'(t)) \\ &\leq \|\mathcal{F}(u_n(t))\|^2_{L^2(\Omega)} \;+\; \frac{1}{4}\,\|u_n'(t)\|^2_{L^2(\Omega)} \\ &\leq \Psi\big(\|u_n(t)\|_{L^2(\Omega)}\big) \;+\; \frac{1}{4}\,\|u_n'(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Using (30), this can be reformulated as

$$\|u_n'(t)\|^2_{L^2(\Omega)} \;+\; \frac{d}{dt}\left(\frac{1}{2}\,\mathcal{A}\,[u_n, u_n]\right) \;\leq\; \Psi\big(\|u_n(t)\|_{L^2(\Omega)}\big) \;+\; \frac{1}{4}\,\|u_n'(t)\|^2_{L^2(\Omega)}$$

i.e., for Lebesgue-almost every $t \in [0, T]$,

$$\|u_n'(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\,\mathcal{A}\,[u_n, u_n] \;\leq\; 2\;\Psi\big(\|u_n(t)\|_{L^2(\Omega)}\big).$$

Now fix $s_0 \in\;]0, T[$ arbitrarily and integrate between s_0 and T to obtain

$$\begin{aligned} \int_{s_0}^{T} \|u_n'(\xi)\|^2_{L^2(\Omega)}\, d\xi + A\,[u_n(T), u_n(T)] \;\leq\;& A\,[u_n(s_0), u_n(s_0)] \\ &+ 2\int_{s_0}^{T} \Psi\big(\|u_n(\xi)\|_{L^2(\Omega)}\big)\; d\xi. \end{aligned}$$

The general inequalities of coercivity and continuity

$$0 \;\leq\; \lambda_A\;\|\nabla u_n(t)\|^2_{L^2(\Omega)} \;\leq\; \mathcal{A}\,[u_n(t), u_n(t)] \;\leq\; \Lambda_A\;\|\nabla u_n(t)\|^2_{L^2(\Omega)}\,,$$

imply that

$$\int_{s_0}^{T} \|u_n'(\xi)\|^2_{L^2(\Omega)}\;\, d\xi \;\leq\; \Lambda_A\;\, \|\nabla u_n(s_0)\|^2_{L^2(\Omega)} \;+\; 2\,\int_{s_0}^{T} \Psi\big(\|u_n(\xi)\|_{L^2(\Omega)}\big)\; d\xi.$$

Hence, from the estimates (26), (34) and the monotonicity of Ψ one concludes that

$$\int_{s_0}^{T} \|u_n'(\xi)\|^2_{L^2(\Omega)}\;\, d\xi \;\leq\; \mathrm{const}(\beta, \Lambda_A, \lambda_A, \lambda_\Omega, C_\beta,\;\, s_0,\;\, T,\;\, \|u_0\|_{L^2(\Omega)}).$$

Finally, Hölder's inequality guarantees the claimed inequality for every $s, t \in [s_0, T]$ with $s < t$, i.e.,

$$\int_{s}^{t} \|u_n'(\xi)\|_{L^2(\Omega)}\;\, d\xi \;\leq\; \mathrm{const}\,\big(\beta, \Lambda_A, \lambda_A, \lambda_\Omega, C_\beta,\;\, s_0,\;\, T,\;\, \|u_0\|_{L^2(\Omega)}\big)\,\cdot\,\sqrt{t-s}\,.$$

■

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