ANALYSIS OF THE EFFECTS OF A FISSURE FOR A NON-NEWTONIAN FLUID FLOW IN A POROUS MEDIUM

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Abstract. We study the solution of a non-Newtonian flow in a porous medium which characteristic size of the pores ε and containing a fissure of width η_ε. The flow is described by the incompressible Stokes system with a nonlinear viscosity, being a power of the shear rate (power law) of flow index $1<r<+\infty$. We consider the limit when size of the pores tends to zero and we obtain different models depending on the magnitude η_ε with respect to ε.

Key words. Non-Newtonian flow; Stokes equation; Darcy's law; porous medium; fissure

AMS subject classifications. 76M50, 35B27

1. Introduction

In this paper we consider an incompressible viscous non-Newtonian flow in a periodic porous medium with characteristic size of the pores ε and containing a fissure $\{0\leq x_n\leq \eta_\varepsilon\}$ of width η_ε with $\varepsilon,\eta_\varepsilon$ two small parameters devoted to tend to zero (see Figure 2.1). Modeling of non-Newtonian flow in fractured medium has encountered a renewed interest because it is essential in hydraulic fracturing operations, largely used for optimal exploitation of oil, gas and thermal reservoirs. Complex fluids interact with pre-existing rock fractures also during drilling operations, enhanced oil recovery, environmental remediation, and other natural phenomena such as magma and sand intrusions, and mud volcanoes.

The aim of this work is to find the effective system corresponding to the limit when the size of the pores, and so the width of the fissure, tends to zero. Homogenization has been applied to the study of perforated materials for a long time. The question of a medium containing a fissure with properties different from those of the rest of the material has been the subject of many studies previously, see Ciarlet *et al.* [8], Panasenko [11] and Chapter 13 of Sanchez-Palencia [12] among others. A similar problem of the one considered in this paper, but for the Laplace's equation, was studied in Bourgeat and Tapiero [4]. The peculiar behavior observed for the Laplace's equation when $\eta_\varepsilon\approx\varepsilon^{\frac{2}{3}}$ has motivated the analogous study for the Newtonian Stokes system in Bourgeat *et al.* [5] (see Zhao and Yao [15] for the Newtonian Navier-Stokes system).

A lot of fluid used in industrial practice are modeled with a shear thinning law. For this reason, in this paper we extend the previous studies obtained for Newtonian fluids to the case of power law fluid, whose situation is completely different. The main reason is that the viscosity is a nonlinear function of the symmetrized gradient of the velocity. In this sense, we consider that the viscosity satisfies the non linear power law, which is widely used for melted polymers, oil, mud, etc. If u is the velocity and Du the gradient velocity tensor, denoting the shear rate by $\mathbb{D}[u]=\frac{1}{2}(Du+D^tu)$, the viscosity

*María Anguiano has been supported by Junta de Andalucía (Spain), Proyecto de Excelencia P12-FQM-2466. Francisco Javier Suárez-Grau has been supported by Ministerio de Economía y Competitividad (Spain), Proyecto Excelencia MTM2014-53309-P.

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as a function of the shear rate is given by

$$\eta_r(\mathbb{D}[u]) = \mu|\mathbb{D}[u]|^{r-2}, \quad 1 < r < +\infty,$$

where the two material parameters $\mu > 0$ and r are called the consistency and the flow index, respectively.

Recall that $r = 2$ yields the Newtonian fluid. For $1 < r < 2$ the fluid is pseudoplastic (shear thinning), which is the characteristic of high polymers, polymer solutions, and many suspensions, whereas for $r > 2$ the fluid is dilatant (shear thickening), whose behavior is reported for certain slurries, like mud, clay, or cement, and implies an increased resistance to flow with intensified shearing.

We consider fluids satisfying the non-Newtonian Stokes system in the domain described above, and our goal is to generalize the study of Bourgeat *et al.* [5] to the non-Newtonian case. We first establish *a priori* estimates in the framework of Sobolev spaces and variational formulations. To find these estimates and then the order of the limits, we use a variant of the Korn's inequality for this type of domain. The results obtained here correspond to three characteristic situations depending on the parameter η_ε with respect to ε:

- If $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ the fissure is not giving any contribution. In this case, in order to find the limit, we use the theory developed by Allaire [2] and Nguesteng [10] of two-scale convergence and we obtain a nonlinear Darcy's law.
- If $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$ the fissure is dominant. We introduce a rescaling in the fissure in order to work with a domain with height one, and then we prove that the limit of the velocity is a Dirac measure concentrated on $\{x_n = 0\}$ representing the corresponding tangential surface flow. Meanwhile in the porous medium the effective velocity is equal to zero.
- If $\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}$ with $\eta_\varepsilon/\varepsilon^{\frac{r}{2r-1}} \to \lambda$, $0 < \lambda < +\infty$, it appears a coupling effect and the effective flow behaves as Darcy flow in the porous medium coupled with the tangential flow of the surface $\{x_n = 0\}$. Compared to the first case $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$, the effective velocity has now an additional tangential component concentrated on $\{x_n = 0\}$. Moreover, the limit problem is now given by a new variational equation, in which appears the parameter λ, and consists of a nonlinear Darcy law in the porous medium and an additional Reynolds problem on the surface $\{x_n = 0\}$.

2. The domain and some notations

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded open domain and

$$\Omega_+ = \Omega \cap \{x_n > 0\}, \quad \Omega_- = \Omega \cap \{x_n < 0\}, \quad \Sigma = \Omega \cap \{x_n = 0\}.$$

For some $\eta_0 > 0$ we define the domain

$$D = \Omega_- \cup (\eta_0 e_n + \Omega_+) \cup (\Sigma \times [0, \eta_0]).$$

Let $\varepsilon > 0$ and $0 < \eta_\varepsilon < \eta_0$ be two small parameters devoted to tend to zero. With Ω we associate a microstructure through the periodic cell $Y = (0,1)^n$ made of two complementary parts: the solid part A, which is a closed subset of $\overline{Y}$, and the fluid part $Y^* = Y \setminus A$. Defining $Y^k = k + Y$, $k \in \mathbb{Z}^n$, we set A^k and $Y^{*k} = Y^k \setminus A^k$ as the solid and fluid part in Y^k respectively.

We also denote

$$A^- = \bigcup_{k \in Z^n_-} A^k, \quad A^+ = \bigcup_{k \in Z^n_+} A^k,$$

all the solid parts in $\mathbb{R}^n$, where $Z_-^k = \{k : k \in \mathbb{Z}^n,\ k_n < 0\}$ and $Z_+^k = \{k : k \in \mathbb{Z}^n,\ k_n > 0\}$. It is obvious that $E^* = \mathbb{R}^n \setminus (A^- \cup A^+)$ is an open subset in $\mathbb{R}^n$.

Following Allaire [1], we make the following assumptions on Y^*, E^*, A and $A^* = A^+ \cup A^-$:

i) Y^* is an open connected set of strictly positive measure, with a locally Lipschitz boundary.

ii) A has strictly positive measure in $\overline{Y}$.

iii) E^* and the interior of A^* are open sets with boundaries of class $C^{0,1}$ and are locally located on one side of their boundaries. Moreover E^* is connected.

We also define

$$Y_\varepsilon^{*k} = \varepsilon Y^{*k}, \quad k \in \mathbb{Z}^n,$$

$$A_\varepsilon^- = \varepsilon A^-, \quad A_{\varepsilon\eta_\varepsilon}^+ = \eta_\varepsilon e_n + \varepsilon A^+, \quad S_{\varepsilon\eta_\varepsilon} = \partial\left(A_\varepsilon^- \cup A_{\varepsilon\eta_\varepsilon}^+\right).$$

We denote by

$A_{\varepsilon\eta_\varepsilon} = A_\varepsilon^- \cup A_{\varepsilon\eta_\varepsilon}^+$ - the solid part of the domain D,
$D_{\varepsilon\eta_\varepsilon} = D \setminus A_{\varepsilon\eta_\varepsilon}$ - the fluid part of the domain D (including the fissure),
$I_{\eta_\varepsilon} = \Sigma \times (0, \eta_\varepsilon)$ - the fissure in D,
$\Omega_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \setminus I_{\eta_\varepsilon}$ - the fluid part of the porous medium,

and

$$\Omega_{\varepsilon\eta_\varepsilon}^+ = D_{\varepsilon\eta_\varepsilon} \cap \{x_n > \eta_\varepsilon\}, \quad \Omega_{\varepsilon\eta_\varepsilon}^- = D_{\varepsilon\eta_\varepsilon} \cap \{x_n < 0\}, \quad \Gamma_{\eta_\varepsilon} = \partial\Sigma \times (0, \eta_\varepsilon).$$

Finally we define

$$D_+ = D \cap \{x_n > 0\}, \quad D_- = \Omega_-.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line. We denote by C a generic positive constant which can change from line to line.

3. Setting and main results

In the following, the points $x \in \mathbb{R}^n$ will be decomposed as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. We use the notation $\tilde{\cdot}$ to denote a generic function of $\mathbb{R}^{n-1}$.

In this section we describe the asymptotic behavior of an incompressible viscous non-Newtonian fluid in the porous medium with a fissure. The proof of the corresponding results will be given in the next sections.

Our results are referred to the non-Newtonian Stokes system. Namely, for $f \in C(\overline{D})^n$ let us consider a sequence $(u_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon}) \in W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n \times L_0^{r'}(D_{\varepsilon\eta_\varepsilon})$, $1 < r < +\infty$, which satisfies

$$\begin{cases} -\mathrm{div}\left(\mu |\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2} \mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right) + \nabla p_{\varepsilon\eta_\varepsilon} = f \text{ in } D_{\varepsilon\eta_\varepsilon}, \\ \mathrm{div}\, u_{\varepsilon\eta_\varepsilon} = 0 \text{ in } D_{\varepsilon\eta_\varepsilon}, \end{cases} \tag{3.1}$$

where $\mu > 0$ is the consistency, $r' = r/(r-1)$ is the conjugate exponent of r and $L_0^{r'}(D_{\varepsilon\eta_\varepsilon})$ is the space of functions of $L^{r'}(D_{\varepsilon\eta_\varepsilon})$ with null integral. We may consider Dirichlet boundary conditions without altering the generality of the problem under consideration,

$$u_{\varepsilon\eta_\varepsilon} = 0 \text{ on } \partial D_{\varepsilon\eta_\varepsilon}. \tag{3.2}$$

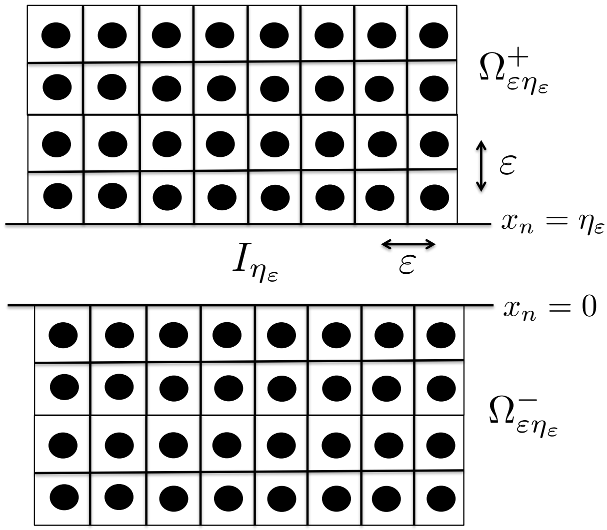

FIGURE 2.1. *View of the domain* $D_{\varepsilon\eta_\varepsilon}$

It is well known that (3.1)-(3.2) has a unique solution $(u_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon}) \in W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n \times L_0^{r'}(D_{\varepsilon\eta_\varepsilon})$ for every $\varepsilon, \eta_\varepsilon > 0$ (see the classical theory [14] for more details).

Our aim is to study the asymptotic behavior of $u_{\varepsilon\eta_\varepsilon}$ and $p_{\varepsilon\eta_\varepsilon}$ when ε tends to zero.

As usual, in order to study the behavior of $u_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$ in the fissure we rewrite our equations in the unit cylinder $I_1 = \Sigma \times (0,1)$ by introducing the change of variable

$$z = \frac{x_n}{\eta_\varepsilon}, \tag{3.3}$$

which transform I_{η_ε} in a fixed domain I_1. We define the new functions

$$\mathcal{U}^{\varepsilon\eta_\varepsilon}(x', z) = u_{\varepsilon\eta_\varepsilon}(x', \eta_\varepsilon z), \quad P^{\varepsilon\eta_\varepsilon}(x', z) = p_{\varepsilon\eta_\varepsilon}(x', \eta_\varepsilon z) - c_{\varepsilon\eta_\varepsilon}, \tag{3.4}$$

and

$$\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon} = (\mathcal{U}_1^{\varepsilon\eta_\varepsilon}, \mathcal{U}_2^{\varepsilon\eta_\varepsilon}, \ldots, \mathcal{U}_{n-1}^{\varepsilon\eta_\varepsilon}),$$

with

$$c_{\varepsilon\eta_\varepsilon} = \frac{1}{|I_{\eta_\varepsilon}|} \int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}\, dx. \tag{3.5}$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v = (\tilde{v}, v_n)$, we will denote $\mathbb{D}_{x'}[v] = \frac{1}{2}(D_{x'}v + D^t_{x'}v)$ and $\partial_z[v] = \frac{1}{2}(\partial_z v + \partial^t_z v)$, where we denote $\partial_z = (0, 0, \frac{\partial}{\partial z})^t$, and associated to the change of variables (3.3), we introduce the operators: $\mathbb{D}_{\eta_\varepsilon}$, D_{η_ε} and $\mathrm{div}_{\eta_\varepsilon}$, by

$$\mathbb{D}_{\eta_\varepsilon}[v] = \frac{1}{2}\left(D_{\eta_\varepsilon}v + D^t_{\eta_\varepsilon}v\right), \quad \mathrm{div}_{\eta_\varepsilon} v = \mathrm{div}_{x'}\tilde{v} + \frac{1}{\eta_\varepsilon}\partial_z v_n,$$

$$(D_{\eta_\varepsilon}v)_{i,j} = \partial_{x_j} v_i \text{ for } i = 1, \ldots, n, \; j = 1, \ldots, n-1, \quad (D_{\eta_\varepsilon}v)_{i,n} = \frac{1}{\eta_\varepsilon}\partial_z v_i \text{ for } i = 1, \ldots, n.$$

Using the transformation (3.3), the system (3.1) can be rewritten as

$$\begin{cases} -\mathrm{div}_{\eta_\varepsilon}\left(\mu|\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}]\right)+\nabla_{\eta_\varepsilon}P^{\varepsilon\eta_\varepsilon}=f(x',\eta_\varepsilon z) \text{ in } I_1, \\ \mathrm{div}_{\eta_\varepsilon}\mathcal{U}^{\varepsilon\eta_\varepsilon}=0 \text{ in } I_1, \end{cases} \tag{3.6}$$

with Dirichlet boundary condition,

$$\mathcal{U}^{\varepsilon\eta_\varepsilon}=0 \text{ on } \partial\Sigma\times(0,1). \tag{3.7}$$

In order to simplify the notation, we define S_r as the r-Laplace operator

$$S_r(\xi)=|\xi|^{r-2}\xi, \quad \forall\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}, \quad 1<r<+\infty,$$

and $K:\mathbb{R}^n\to\mathbb{R}^n$ as a function defined by

$$K(\xi)=\int_{Y^*}w^\xi(y)\,dy, \quad \forall\xi\in\mathbb{R}^n, \tag{3.8}$$

where $w^\xi(y)$, for every $\xi\in\mathbb{R}^n$, denote the unique solution in $W^{1,r}_{\#}(Y^*)^n$ (# denotes Y-periodicity) of the local problem

$$\begin{cases} -\mathrm{div}_y S\left(\mathbb{D}[w^\xi]\right)+\nabla_y\pi^\xi=\xi & \text{in } Y^*, \\ \mathrm{div}_y w^\xi=0 & \text{in } Y^*, \\ w^\xi=0 & \text{in } \partial A, \\ w^\xi,\pi^\xi \ Y-\text{periodic}. \end{cases} \tag{3.9}$$

Our main result referred to the asymptotic behavior of the solution of (3.1)-(3.2) is given by the following theorem.

Theorem 3.1. *Let $\eta_\varepsilon\approx\varepsilon^{\frac{r}{2r-1}}$, with $\eta_\varepsilon/\varepsilon^{\frac{r}{2r-1}}\to\lambda$, $0<\lambda<+\infty$, $1<r<+\infty$ and let $(u_{\varepsilon\eta_\varepsilon},p_{\varepsilon\eta_\varepsilon})$ be the solution of problem (3.1)-(3.2). Then there exist a Darcy velocity $v\in L^r(D)^n$, a Reynolds velocity $\mathcal{V}\in L^r(\Sigma)^n$, with $\mathcal{V}_n=0$, and a pressure $p\in W^{1,r'}(D)$ such that*

$$\begin{aligned} &\varepsilon^{-\frac{r}{r-1}}u_{\varepsilon\eta_\varepsilon}\overset{\star}{\rightharpoonup}v+\lambda^{\frac{2r-1}{r-1}}\mathcal{V}\delta_\Sigma \quad \text{in } \mathcal{M}(D), \\ &p_{\varepsilon\eta_\varepsilon}\to p \quad \text{in } L^{r'}(D), \end{aligned} \tag{3.10}$$

where δ_Σ is the Dirac measure concentrated on Σ, and $\mathcal{M}(D)^n$ is the space of Radon measures on D. The velocities v and $\mathcal{V}$ are given by

$$v(x)=\frac{1}{\mu}K\left(f(x)-\nabla p(x)\right), \quad \text{in } D, \tag{3.11}$$

$$\tilde{\mathcal{V}}(x')=\frac{1}{2^{\frac{r'}{2}}(r+1)\mu^{r'-1}}S_{r'}\left(\tilde{f}(x',0)-\nabla_{x'}P(x')\right), \quad \mathcal{V}=(\tilde{\mathcal{V}},0), \quad \text{in } \Sigma, \tag{3.12}$$

where the pressure $P\in W^{1,r'}(\Sigma)$ is connected with the pressure p by the relation

$$p(x',0)=P(x')+\widetilde{C}, \quad \widetilde{C}\in\mathbb{R}.$$

Moreover, the pressure $p\in V_\Sigma=\{\varphi\in W^{1,r'}(D):\varphi(\cdot,0)\in W^{1,r'}(\Sigma)\}$ is the unique solution of the variational problem

$$\int_D v(x)\cdot\nabla\varphi(x)\,dx+\lambda^{\frac{2r-1}{r-1}}\int_\Sigma\tilde{\mathcal{V}}(x')\cdot\nabla_{x'}\varphi(x',0)\,dx'=0, \quad \forall\varphi\in V_\Sigma. \tag{3.13}$$

Remark 3.2. *Formally, (3.13) is the weak formulation of the following boundary value problem*

$$\begin{cases} -\mathrm{div}\, v(x) - \lambda^{\frac{2r-1}{r-1}} \mathrm{div}_{x'}\left(\tilde{\mathcal{V}}(x')\delta_\Sigma\right) = 0 & \text{in } D, \\ v(x)\cdot\nu + \lambda^{\frac{2r-1}{r-1}} \tilde{\mathcal{V}}(x')\delta_{\partial\Sigma}\cdot\tilde{\nu} = 0 & \text{on } \partial D, \end{cases} \tag{3.14}$$

where ν is the outward normal to ∂D and $\tilde{\nu}$ is the outward normal to $\partial\Sigma$.

In the case $\lambda=0$, i.e. $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$, then the fissure is not giving any contribution. In fact, if λ tends to zero in (3.14) we obtain the following Darcy's law on D

$$\begin{cases} -\mathrm{div}\, v(x) = 0 & \text{in } D, \\ v(x)\cdot\nu = 0 & \text{on } \partial D, \end{cases} \tag{3.15}$$

where v is given by (3.11).

On the other hand, in the case $\lambda=+\infty$, i.e. $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$, then the fissure is dominant. In fact, multiplying (3.14) by $\lambda^{-\frac{2r-1}{r-1}}$ and tending λ to $+\infty$, we obtain the following Reynolds problem on Σ

$$\begin{cases} -\mathrm{div}_{x'}\tilde{\mathcal{V}}(x') = 0 & \text{in } \Sigma, \\ \tilde{\mathcal{V}}(x')\cdot\tilde{\nu} = 0 & \text{on } \partial\Sigma, \end{cases} \tag{3.16}$$

where $\tilde{\mathcal{V}}$ is given by (3.12).

Remark 3.3. *The monotonicity and coerciveness properties of the permeability function K given by (3.8) can be found in sections 2 and 4 in [7], which implies that (3.15) is well posed. On the other hand, the r'-Laplace operator is well know that is monotone and coercive (see [9] for more details), which implies that (3.16) is well posed. Therefore, the problem (3.13) is also well posed.*

In Section 4 we establish *a priori* estimates of the velocity and the pressure. Section 5 is devoted to prove Theorem 3.1, whose proof is divided in three subsections. In Subsection 5.1 we analyze the problem in the porous part ($\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$) while in Subsection 5.2 the problem in the fissure part ($\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$) is analyzed, which give the rigorously proof of (3.15) and (3.16), respectively. Finally, in Subsection 5.3 we prove that there is a balanced interaction between the fissure and the porous medium giving Theorem 3.1.

4. A Priori Estimates

Let us begin with the following variant of the Korn's inequality in the porous medium $\Omega_{\varepsilon\eta_\varepsilon}$, which will be very useful (see for example Bourgeat and Mikelic in [6]).

Lemma 4.1. *There exists a constant C independent of ε, such that, for any function $v\in W^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$ and $v=0$ on $S_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^r(\Omega_{\varepsilon\eta_\varepsilon})^n} \le C\varepsilon \|\mathbb{D}[v]\|_{L^r(\Omega_{\varepsilon\eta_\varepsilon})^{n\times n}}, \quad 1<r<+\infty. \tag{4.1}$$

Next, we give an useful estimate in the fissure I_{η_ε}.

Lemma 4.2. *There exists a constant C independent of ε, such that, for any function $v\in W^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$ and $v=0$ on $S_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^r(I_{\eta_\varepsilon})^n} \le C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon+\varepsilon)^{\frac{1}{2}} \|\mathbb{D}[v]\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}, \quad 1<r<+\infty. \tag{4.2}$$

Proof. Because the thickness of I_{η_ε} is η_ε, we have, by the classical Poincaré inequality,

$$\|v\|_{L^r(I_{\eta_\varepsilon})^n} \leq C\eta_\varepsilon \|Dv\|_{L^r(I_{\eta_\varepsilon})^{n\times n}}. \tag{4.3}$$

Next, if we choose a point $x_1 \in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $x \in I_{\eta_\varepsilon}$, then we have

$$v(x) - v(x_1) = Dv(\xi)(x - x_1) \leq (\varepsilon + \eta_\varepsilon)|Dv|.$$

Since $v(x_1) = 0$ because $x_1 \in A_{\varepsilon\eta_\varepsilon}$, we have

$$\|v(x)\|_{L^r(I_{\eta_\varepsilon})^n} \leq C(\varepsilon + \eta_\varepsilon)\|Dv\|_{L^r(I_{\eta_\varepsilon})^{n\times n}}.$$

Multiplying the above inequality with (4.3) we obtain

$$\|v\|_{L^r(I_{\eta_\varepsilon})^n} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|Dv\|_{L^r(I_{\eta_\varepsilon})^{n\times n}} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|Dv\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}, \tag{4.4}$$

and from the classical Korn inequality we obtain (4.2).
□

Let us give the classical estimate [3], for the a function in L^r when we deal with a fissure.
Lemma 4.3. *There exists a constant C independent of ε, such that, for any function $v \in L^r(I_{\eta_\varepsilon})$ with $\int_{I_{\eta_\varepsilon}} v\,dx = 0$, one has*

$$\|v\|_{L^r(I_{\eta_\varepsilon})} \leq \frac{C}{\eta_\varepsilon}\|\nabla v\|_{W^{-1,r}(I_{\eta_\varepsilon})^n}, \quad 1 < r < +\infty.$$

Now, we are in position to obtain some *a priori* estimates for $u_{\varepsilon\eta_\varepsilon}$.
Lemma 4.4. *There exists a constant C independent of ε, such that if $u_{\varepsilon\eta_\varepsilon} \in W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$, with $1 < r < +\infty$, is the solution of the problem (3.1)-(3.2), one has*

$$\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(\Omega_{\varepsilon\eta_\varepsilon})^n} \leq C(\eta_\varepsilon^{\frac{2r-1}{r}}\varepsilon^{r-1} + \varepsilon^r)^{\frac{1}{r-1}}, \tag{4.5}$$

$$\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n} \leq C\left(\eta_\varepsilon^{r-1}\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\eta_\varepsilon^{r-1}\right)^{\frac{1}{r-1}} + \eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{r+1}{2(r-1)}}, \tag{4.6}$$

$$\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} \leq C(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon)^{\frac{1}{r-1}}, \tag{4.7}$$

$$\|Du_{\varepsilon\eta_\varepsilon}\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} \leq C(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon)^{\frac{1}{r-1}}. \tag{4.8}$$

Proof. Multiplying by $u_{\varepsilon\eta_\varepsilon}$ in the first equation of (3.1) and integrating over $D_{\varepsilon\eta_\varepsilon}$, we have

$$\mu\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|^r_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} = \int_{D_{\varepsilon\eta_\varepsilon}} f\cdot u_{\varepsilon\eta_\varepsilon}\,dx. \tag{4.9}$$

Using Hölder's inequality and the assumption of f, we obtain that there exists a constant C such that

$$\int_{D_{\varepsilon\eta_\varepsilon}} f\cdot u_{\varepsilon\eta_\varepsilon}\,dx \leq C\eta_\varepsilon^{\frac{1}{r'}}\|f\|_{L^\infty(I_{\eta_\varepsilon})^n}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n} + \|f\|_{L^{r'}(\Omega_{\varepsilon\eta_\varepsilon})^n}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(\Omega_{\varepsilon\eta_\varepsilon})^n},$$

and by inequalities (4.1) and (4.2), we have

$$\int_{D_{\varepsilon\eta_\varepsilon}} f\cdot u_{\varepsilon\eta_\varepsilon}\,dx \le C\left(\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}(\varepsilon+\eta_\varepsilon)^{\frac{1}{2}}+\varepsilon\right)\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}$$
$$\le C\left(\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon+\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}+\varepsilon\right)\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}.$$

Therefore, from (4.9) we get

$$\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} \le C\left(\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon+\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}+\varepsilon\right)^{\frac{1}{r-1}}.$$

Since $\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}<\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon$ if $\varepsilon<\eta_\varepsilon$ and $\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\le\eta_\varepsilon^{\frac{1}{r'}}\varepsilon<\varepsilon$ if $\eta_\varepsilon<\varepsilon$, the term $\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ can be dropped. Taking into account that $1/r'+1=(2r-1)/r$, this gives (4.7) and from the classical Korn inequality we have (4.8).

Applying (4.1) together with (4.7) we obtain (4.5). Finally, applying (4.2) and (4.7) we get

$$\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n} \le C(\eta_\varepsilon+\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})(\eta_\varepsilon^{\frac{2r-1}{r}}+\varepsilon)^{\frac{1}{r-1}}$$
$$\le C\left(\eta_\varepsilon^{r-1}\eta_\varepsilon^{\frac{2r-1}{r}}+\varepsilon\eta_\varepsilon^{r-1}\right)^{\frac{1}{r-1}}+\left(\eta_\varepsilon^{\frac{r-1}{2}}\eta_\varepsilon^{\frac{2r-1}{r}}\varepsilon^{\frac{r-1}{2}}+\eta_\varepsilon^{\frac{r-1}{2}}\varepsilon^{\frac{r+1}{2}}\right)^{\frac{1}{r-1}}.$$

Since $\eta_\varepsilon^{\frac{r-1}{2}}\eta_\varepsilon^{\frac{2r-1}{r}}\varepsilon^{\frac{r-1}{2}}<\eta_\varepsilon^{r-1}\eta_\varepsilon^{\frac{2r-1}{r}}$ if $\eta_\varepsilon>\varepsilon$ and $\eta_\varepsilon^{\frac{r-1}{2}}\eta_\varepsilon^{\frac{2r-1}{r}}\varepsilon^{\frac{r-1}{2}}<\eta_\varepsilon^{\frac{r-1}{2}}\varepsilon^{\frac{r+1}{2}}$ if $\eta_\varepsilon<\varepsilon$, the term $\eta_\varepsilon^{\frac{r-1}{2}}\eta_\varepsilon^{\frac{2r-1}{r}}\varepsilon^{\frac{r-1}{2}}$ can be dropped, and (4.6) holds.

□

In order to investigate the behavior of solutions to (3.1)-(3.2), as $\varepsilon\to 0$, we need to extend the pressure $p_{\varepsilon\eta_\varepsilon}$ to the whole of D. Extending the pressure is a difficult task. The extension is closely related to the construction of a restriction operator. Such extension for the case of a porous medium without fissure is given in Tartar [13] for the case $r=2$. We need a restriction operator, R_r^ε, between $W_0^{1,r}(D)^n$ into $W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$ with similar properties, which is given in [6]. Since the construction of the operator is local, having no obstacles in I_{η_ε} means that we do not have to use the extension in that part.

Next, we give the properties of the restriction operator R_r^ε (see Lemma 1.2. in [6] for more details).

Lemma 4.5. *There exists a linear continuous operator R_r^ε acting from $W_0^{1,r}(D)^n$ into $W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$, $1<r<+\infty$, such that*

1. $R_r^\varepsilon v=v$, *if* $v\in W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$
2. $\operatorname{div}(R_r^\varepsilon v)=0$, *if* $\operatorname{div} v=0$
3. *For any* $v\in W_0^{1,r}(D)^n$ *(the constant C is independent of v and ε),*

$$\|R_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^n} \le C\|v\|_{L^r(D)^n}+C\varepsilon\|Dv\|_{L^r(D)^{n\times n}},$$
$$\|DR_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} \le \frac{C}{\varepsilon}\|v\|_{L^r(D)^n}+C\|Dv\|_{L^r(D)^{n\times n}}.$$

In order to extend the pressure to the whole domain D, we define a function $F_{\varepsilon\eta_\varepsilon}\in W^{-1,r'}(D)^n$ by the following formula (brackets are for the duality products between $W^{-1,r'}$ and $W_0^{1,r}$):

$$\langle F_{\varepsilon\eta_\varepsilon},v\rangle_D=\langle\nabla p_{\varepsilon\eta_\varepsilon},R_r^\varepsilon v\rangle_{D_{\varepsilon\eta_\varepsilon}},\ \text{for any}\ \ v\in W_0^{1,r}(D)^n, \tag{4.10}$$

where R_r^ε is the operator defined in Lemma 4.5. We calcule the right hand side of (4.10) by using (3.1) and we have

$$\langle F_{\varepsilon\eta_\varepsilon}, v\rangle_D = \left\langle \operatorname{div}\left(\mu|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right), R_r^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}} + \langle f, R_r^\varepsilon v\rangle_{D_{\varepsilon\eta_\varepsilon}}, \tag{4.11}$$

and by using the third point in Lemma 4.5, for fixed ε, η_ε we see that it is a bounded functional on $W_0^{1,r}(D)^3$, and in fact $F_{\varepsilon\eta_\varepsilon}\in W^{-1,r'}(D)^n$.

Moreover, if $v\in W_0^{1,r}(D_{\varepsilon\eta_\varepsilon})^n$ and we continue it by zero out of $D_{\varepsilon\eta_\varepsilon}$, we see from (4.10) and the first point in Lemma 4.5 that $F_{\varepsilon\eta_\varepsilon}|_{D_{\varepsilon\eta_\varepsilon}} = \nabla p_{\varepsilon\eta_\varepsilon}$.

On the other hand, if $\operatorname{div} v = 0$ by the second point in Lemma 4.5 and (4.10), we have that $\langle F_{\varepsilon\eta_\varepsilon}, v\rangle_\Omega = 0$ and this implies that $F_{\varepsilon\eta_\varepsilon}$ is the gradient of some function in $L^{r'}(D)$. This means that $F_{\varepsilon\eta_\varepsilon}$ is a continuation of $\nabla p_{\varepsilon\eta_\varepsilon}$ to D, and that this continuation is a gradient. We also may say that $p_{\varepsilon\eta_\varepsilon}$ has been continuated to D. We denote the extended pressure again by $p_{\varepsilon\eta_\varepsilon}$ and since it is defined up to a constant we take $p_{\varepsilon\eta_\varepsilon}$ such that $\int_D p_{\varepsilon\eta_\varepsilon}\,dx = 0$. Moreover, we have

$$F_{\varepsilon\eta_\varepsilon} \equiv \nabla p_{\varepsilon\eta_\varepsilon}.$$

For such extended pressure we obtain the following result.

LEMMA 4.6. *There exists a constant C independent of ε, such that if $p_{\varepsilon\eta_\varepsilon}\in L_0^{r'}(D)$, with r' the conjugate exponent of r and $1<r<+\infty$, is the extended pressure to the whole domain D, one has*

$$\|p_{\varepsilon\eta_\varepsilon}\|_{L^{r'}(D)} \le C\left(\frac{\eta_\varepsilon^{\frac{r'+1}{r'}}}{\varepsilon} + 1\right), \tag{4.12}$$

$$\|p_{\varepsilon\eta_\varepsilon} - c_{\varepsilon\eta_\varepsilon}\|_{L^{r'}(I_{\eta_\varepsilon})} \le C\left(\eta_\varepsilon^{\frac{1}{r'}} + \frac{\varepsilon}{\eta_\varepsilon}\right), \tag{4.13}$$

where $c_{\varepsilon\eta_\varepsilon}$ is given by (3.5).

Proof. Let us first estimate $\nabla p_{\varepsilon\eta_\varepsilon}$. To do this we estimate the right side of (4.11). Using Hölder's inequality and from (4.7) we have

$$\begin{aligned}\left|\left\langle \operatorname{div}\left(\mu|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right), R_r^\varepsilon v\right\rangle_{D_{\varepsilon\eta_\varepsilon}}\right| &\le \mu\|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\|^{r-1}_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}\|DR_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}\\ &\le C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\right)\|DR_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}.\end{aligned}$$

Using the assumption of f, we obtain that there exists a constant C such that

$$\left|\langle f, R_r^\varepsilon v\rangle_{D_{\varepsilon\eta_\varepsilon}}\right| \le C\|R_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^n}.$$

Then, from (4.11), we deduce

$$\left|\langle\nabla p_{\varepsilon\eta_\varepsilon}, v\rangle_D\right| \le C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\right)\|DR_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}} + C\|R_r^\varepsilon v\|_{L^r(D_{\varepsilon\eta_\varepsilon})^n}.$$

Taking into account the third point in Lemma 4.5, we have

$$\begin{aligned}\left|\langle\nabla p_{\varepsilon\eta_\varepsilon}, v\rangle_D\right| \le\, & C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\right)\left(\frac{1}{\varepsilon}\|v\|_{L^r(D)^n} + \|Dv\|_{L^r(D)^{n\times n}}\right)\\ &+ C\left(\|v\|_{L^r(D)^n} + \varepsilon\|Dv\|_{L^r(D)^{n\times n}}\right).\end{aligned}$$

Then, as $\varepsilon \ll 1$, we can deduce that

$$\left|\langle \nabla p_{\varepsilon\eta_\varepsilon}, v\rangle_D\right| \leq C\left(\frac{\eta_\varepsilon^{\frac{2r-1}{r}}}{\varepsilon}+1\right)\|v\|_{W_0^{1,r}(D)^n},$$

for any $v \in W_0^{1,r}(D)^n$. Therefore, we obtain

$$\|\nabla p_{\varepsilon\eta_\varepsilon}\|_{W^{-1,r'}(D)^n} \leq C\left(\frac{\eta_\varepsilon^{\frac{2r-1}{r}}}{\varepsilon}+1\right),$$

and the estimate (4.12) follows by using the Nečas inequality in D.

Let $v \in W_0^{1,r}(I_{\eta_\varepsilon})^n$, then

$$\langle \nabla p_{\varepsilon\eta_\varepsilon}, v\rangle_{I_{\eta_\varepsilon}} = \left\langle \operatorname{div}\left(\mu|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right), v\right\rangle_{I_{\eta_\varepsilon}} + \langle f, v\rangle_{I_{\eta_\varepsilon}}.$$

We estimate the right hand side. Using Hölder's inequality and (4.7) we have

$$\left|\left\langle \operatorname{div}\left(\mu|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right), v\right\rangle_{I_{\eta_\varepsilon}}\right| \leq C\left(\eta_\varepsilon^{\frac{2r-1}{r}}+\varepsilon\right)\|Dv\|_{L^r(I_{\eta_\varepsilon})^{n\times n}}.$$

Using again Hölder's inequality and assumption of f, we obtain that there exists a constant C such that

$$\left|\langle f, v\rangle_{I_{\eta_\varepsilon}}\right| \leq C\eta_\varepsilon^{\frac{1}{r'}}\|f\|_{L^\infty(I_{\eta_\varepsilon})^n}\|v\|_{L^r(I_{\eta_\varepsilon})^n},$$

and by the estimate (4.4), we have

$$\left|\langle f, v\rangle_{I_{\eta_\varepsilon}}\right| \leq C(\eta_\varepsilon^{\frac{2r-1}{r}} + \eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})\|Dv\|_{L^r(I_{\eta_\varepsilon})^{n\times n}},$$

Then, we have

$$\|\nabla p_{\varepsilon\eta_\varepsilon}\|_{W^{-1,r'}(I_{\eta_\varepsilon})^n} \leq C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} + \varepsilon\right).$$

Reasoning as in the proof of Lemma 4.4, we observe that $\eta_\varepsilon^{\frac{1}{r'}}\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ can be dropped and so we obtain

$$\|\nabla p_{\varepsilon\eta_\varepsilon}\|_{W^{-1,r'}(I_{\eta_\varepsilon})^n} \leq C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\right).$$

Using Lemma 4.3 we obtain the estimate (4.13).

□

5. Proof of the main result

In view of estimates (4.5), (4.8) of the velocity and (4.12) of the pressure, the proof of Theorem 3.1 will be divided in three characteristic cases: $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$, $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$ and $\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}$, with $\eta_\varepsilon/\varepsilon^{\frac{r}{2r-1}} \to \lambda$, $0 < \lambda < +\infty$.

5.1. Problem in the porous part $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$

In this subsection, we need to extend the velocity $u_{\varepsilon\eta_\varepsilon}$ by zero in the fissure I_{η_ε}, and we will denote the extended velocity by $v_{\varepsilon\eta_\varepsilon}$, i.e.

$$v_{\varepsilon\eta_\varepsilon}=\begin{cases} u_{\varepsilon\eta_\varepsilon} & \text{in } \Omega_{\varepsilon\eta_\varepsilon}, \\ 0 & \text{in } I_{\eta_\varepsilon}. \end{cases} \tag{5.1}$$

LEMMA 5.1. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ with $1<r<+\infty$ and let $(v_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ be the extended solution of (3.1)-(3.2). Then there exist subsequences of $v_{\varepsilon\eta_\varepsilon}$ and $p_{\varepsilon\eta_\varepsilon}$ still denoted by the same, and functions $v\in L^r(D)^n$, $p\in L^{r'}(D)$ such that*

$$\varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \text{in } L^r(D)^n, \quad p_{\varepsilon\eta_\varepsilon} \to p \quad \text{in } L^{r'}(D). \tag{5.2}$$

Moreover, v satisfies

$$\operatorname{div} v = 0 \quad \text{in } D, \quad v\cdot\nu = 0 \quad \text{on } \partial D. \tag{5.3}$$

Proof. From estimates (4.5) and (4.12), taking into account the extension of the velocity by zero to D and $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$, we have the following estimates

$$\|v_{\varepsilon\eta_\varepsilon}\|_{L^r(D)^n} \le C\varepsilon^{\frac{r}{r-1}}, \quad \|p_{\varepsilon\eta_\varepsilon}\|_{L^{r'}(D)} \le C.$$

Then there exist $v\in L^r(D)^n$ and $p\in L^{r'}(D)$ such that, for a subsequence still denoted by $v_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$, it holds

$$\varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \text{in } L^r(D)^n, \quad p_{\varepsilon\eta_\varepsilon} \rightharpoonup p \quad \text{in } L^{r'}(D).$$

Next, we prove that the convergence of the pressure is in fact strong. Let $w_\varepsilon \in W_0^{1,r}(D)^n$ be such that $w_\varepsilon \rightharpoonup w$ in $W_0^{1,r}(D)^n$. Then (brackets are for the duality products between $W^{-1,r'}$ and $W_0^{1,r}$):

$$|<\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon>_D - <\nabla p, w>_D| \le |<\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon - w>_D| + |<\nabla p_{\varepsilon\eta_\varepsilon} - \nabla p, w>_D|.$$

On the one hand, we have

$$|<\nabla p_{\varepsilon\eta_\varepsilon} - \nabla p, w>_D| = \int_D (p_{\varepsilon\eta_\varepsilon} - p)\operatorname{div} w\, dx \to 0, \quad \text{as } \varepsilon \to 0.$$

On the other hand, we have

$$\begin{aligned} &|<\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon - w>_D| = \left|<\nabla p_{\varepsilon\eta_\varepsilon}, R_r^\varepsilon(w_\varepsilon - w)>_{D_{\varepsilon\eta_\varepsilon}}\right| \\ &= \left|\langle \operatorname{div}\left(\mu|\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]|^{r-2}\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]\right), R_r^\varepsilon(w_\varepsilon - w)\rangle_{D_{\varepsilon\eta_\varepsilon}} - \langle f, R_r^\varepsilon(w_\varepsilon - w)\rangle_{D_{\varepsilon\eta_\varepsilon}}\right|, \end{aligned}$$

and using Hölder's inequality, estimate (4.7), the estimates of the restricted operator R_r^ε given in Lemma 4.5, $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ and $\varepsilon \ll 1$, we get

$$\begin{aligned} &|<\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon - w>_D| \le C\left(\eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon\right)\left(\frac{1}{\varepsilon}\|w_\varepsilon - w\|_{L^r(D)^n} + \|Dw_\varepsilon - Dw\|_{L^r(D)^{n\times n}}\right) \\ &+ C\left(\|w_\varepsilon - w\|_{L^r(D)^n} + \varepsilon\|Dw_\varepsilon - Dw\|_{L^r(D)^{n\times n}}\right) \\ &\le C\left(\|w_\varepsilon - w\|_{L^r(D)^n} + \varepsilon\|Dw_\varepsilon - Dw\|_{L^r(D)^{n\times n}}\right) \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Therefore, we have that $\nabla p_{\varepsilon\eta_\varepsilon} \to \nabla p$ strongly in $W^{-1,r'}(D)^n$, which implies the strong convergence of the pressure given in (5.2).

Finally, from $\operatorname{div} v_{\varepsilon\eta_\varepsilon}=0$ in D and the weak convergence of the velocity given in (5.2), we easily obtain (5.3).

□

The proof of the following result will be showed by using the two-scale convergence introduced by Nguesteng [10] in the L^2-setting and developed by Allaire [2], who also introduced the L^r-setting. By $\stackrel{2}{\rightharpoonup}$ we denote the limit in the two-scale sense.

LEMMA 5.2. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ with $1<r<+\infty$ and let $v_{\varepsilon\eta_\varepsilon}$ be the extended solution of (3.1)-(3.2). Then there exist subsequences of $v_{\varepsilon\eta_\varepsilon}$ still denoted by the same, and $\hat{v}(x,y)\in L^r(D;W^{1,r}_{\#}(Y^*)^n)$ such that*

$$\varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \stackrel{2}{\rightharpoonup} \hat{v}(x,y) \quad \text{in } L^r(D\times Y^*)^n, \quad \varepsilon^{-\frac{1}{r-1}} D v_{\varepsilon\eta_\varepsilon} \stackrel{2}{\rightharpoonup} D_y\hat{v}(x,y) \quad \text{in } L^r(D\times Y^*)^{n\times n}. \tag{5.4}$$

The weak limit $v(x)$ and the two-scale limit $\hat{v}(x,y)$ are related as follows

$$v(x)=\int_{Y^*}\hat{v}(x,y)\,dy. \tag{5.5}$$

Moreover, $\hat{v}$ satisfies

$$\operatorname{div}_y \hat{v}(x,y)=0 \quad \text{in } Y^*, \quad \hat{v}=0 \quad \text{in } Y\setminus Y^*, \tag{5.6}$$

$$\operatorname{div}_x\left(\int_{Y^*}\hat{v}(x,y)\,dy\right)=0 \quad \text{in } D, \quad \left(\int_{Y^*}\hat{v}(x,y)\,dy\right)\cdot\nu=0 \quad \text{on } \partial D. \tag{5.7}$$

Proof. From estimates (4.5) and (4.8) and taking into account that $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$, we get

$$\|v_{\varepsilon\eta_\varepsilon}\|_{L^r(D)^n}\le C\varepsilon^{\frac{r}{r-1}}, \quad \|Dv_{\varepsilon\eta_\varepsilon}\|_{L^r(D)^{n\times n}}\le C\varepsilon^{\frac{1}{r-1}}.$$

Thus, from Lemma 1.5 in [6], there exist subsequences of $v_{\varepsilon\eta_\varepsilon}$, still denoted by $v_{\varepsilon\eta_\varepsilon}$, and a function $\hat{v}\in L^r(D;W^{1,r}_{\#}(Y^*)^n)$ such that the convergences given in (5.4) hold.

Relation (5.5) is a classical property relating weak convergence and two-scale convergence, see Allaire [2] and Bourgeat and Mikelic [6] for more details. From $\operatorname{div} v_{\varepsilon\eta_\varepsilon}=0$ in D, then (5.6) straightforward. Finally, (5.3) and (5.5) imply (5.7).

□

LEMMA 5.3. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ with $1<r<+\infty$ and let $(v_{\varepsilon\eta_\varepsilon},p_{\varepsilon\eta_\varepsilon})$ be the extended solution of (3.1)-(3.2). Let $(v,p)\in L^r(D)^n\times L^{r'}(D)$ be given by Lemma 5.1. Then, $p\in W^{1,r'}(D)$ and (v,p) is the unique solution of Darcy's law (3.15).*

Proof. Considering $\varphi\in W^{1,r}_0(D)^n$, we define $w_\varepsilon(x)=\varphi(x)-\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}(x)$ as test function in (3.1)-(3.2) and we have

$$\int_D \mu S_r(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]):\mathbb{D}[w_\varepsilon]\,dx=\langle f-\nabla p_{\varepsilon\eta_\varepsilon},w_\varepsilon\rangle_D.$$

Observe that

$$S_r(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}])=\varepsilon^r S_r(\mathbb{D}[\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}]).$$

Therefore,

$$\int_D \mu S_r(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]):\mathbb{D}[w_\varepsilon]\,dx = \int_D \mu\varepsilon^r S_r(\mathbb{D}[\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}]):\mathbb{D}[\varphi]\,dx - \int_D \mu|\varepsilon\mathbb{D}[\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}]|^r dx.$$

Using Hölder and Young inequalities in the first term of the right hand side, we can deduce

$$\int_D \mu S_r(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]):\mathbb{D}[w_\varepsilon]\,dx \le \int_D \frac{\mu}{r}|\varepsilon\mathbb{D}[\varphi]|^r - \int_D \frac{\mu}{r}|\varepsilon\mathbb{D}[\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}]|^r dx,$$

and so the variational formulation of problem (3.1)-(3.2) is equivalent to

$$\int_D \frac{\mu}{r}|\varepsilon\mathbb{D}[\varphi]|^r - \int_D \frac{\mu}{r}|\varepsilon\mathbb{D}[\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}]|^r dx \ge \int_D f\cdot w_\varepsilon\,dx - \langle\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon\rangle_D. \tag{5.8}$$

Now, we choose $\psi(x,y)^{+-}\in\mathcal{D}(D_{+-};C^\infty_\#(Y^*)^n)$. There exists $\eta_1>0$ such that *supp* $\psi(x,y)^{+-}\subset D/I_{\eta_\varepsilon}$ for every $\eta_\varepsilon\in(0,\eta_1)$. Let $\eta_\varepsilon<\eta_1$. We define $\psi_\varepsilon(x)^{+-}=\psi(x,x/\varepsilon)^{+-}$ and we insert $\varphi=\psi_\varepsilon^{+-}$ in (5.8). In the sequel, we use the elementary properties of the two scale convergence. Using the two-scale convergence of $\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon}$ given in (5.4), we have

$$\int_{D_{+-}} f\cdot w_\varepsilon\,dx \to \int_{D_{+-}}\int_Y f\cdot(\psi-\hat{v})\,dxdy,$$

and using $\operatorname{div} v_{\varepsilon\eta_\varepsilon}=0$ in D and the strong convergence of the pressure (5.2), we have

$$\langle\nabla p_{\varepsilon\eta_\varepsilon}, w_\varepsilon\rangle_{D_{+-}} = \int_{D_{+-}} p_{\varepsilon\eta_\varepsilon}\operatorname{div}\psi_\varepsilon\,dx \to \int_{D_{+-}}\int_Y p\operatorname{div}_x\psi(x,y)\,dxdy, \quad \text{as } \varepsilon\to 0.$$

Therefore, passing to the limit in the variational formulation (5.8) and taking into account (5.7), we get

$$\int_{D_{+-}}\int_Y \frac{\mu}{r}|\mathbb{D}[\psi]|^r dxdy - \int_{D_{+-}}\int_Y \frac{\mu}{r}|\mathbb{D}[\hat{v}]|^r dxdy \ge \langle f(x)-\nabla p(x), \int_Y(\psi-\hat{v})dy\rangle_{D_{+-}}.$$

Consequently, there exists $\hat{\pi}\in L^{r'}(D;L^{r'}(Y^*)/\mathbb{R})$ such that $(\hat{v},\hat{\pi})$ satisfies the homogenized problem

$$-\operatorname{div}_y\left(\mu|\mathbb{D}_y[\hat{v}]|^{r-2}\mathbb{D}_y[\hat{v}]\right)+\nabla_y\hat{\pi}=f(x)-\nabla p(x) \quad \text{in } Y^*, \tag{5.9}$$

$$\operatorname{div}_y\hat{v}(x,y)=0 \quad \text{in } Y^*, \tag{5.10}$$

$$(\hat{v},\hat{\pi}) \text{ is } Y-\text{periodic}, \quad \hat{v}=0 \quad \text{in } Y\setminus Y^*, \tag{5.11}$$

by using the variant of de Rham's formula in a periodic setting (see Nguetseng [10] and Temam [14]). Reasoning as in Theorem 8 in [6], we get that the pressure p belongs to $W^{1,r'}(D)$.

Finally, the derivation of (3.15) from the effective problems (5.9)-(5.11) is straightforward by using the local problems (3.9) and definitions of the permeability functions (3.8).

□

It remains to prove the convergence of the whole velocity

$$\varepsilon^{-\frac{r}{r-1}}u_{\varepsilon\eta_\varepsilon}\rightharpoonup v \quad \text{in } L^r(D)^n, \tag{5.12}$$

which is equivalent to prove that the velocity in the fissure tends to zero, i.e. to prove

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n}\to 0. \tag{5.13}$$

For this, it is sufficient to prove that

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n}\to 0 \quad \text{for } \eta_\varepsilon \ll \varepsilon, \tag{5.14}$$

and

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q(I_{\eta_\varepsilon})^n}\to 0 \quad \text{for } \varepsilon \ll \eta_\varepsilon \ll \varepsilon^{\frac{1}{\alpha}}, \quad 1<\alpha<\frac{2r-1}{r}, \tag{5.15}$$

for a q which will be defined below.

Using (4.6) and using $\eta_\varepsilon \ll \varepsilon$, we have

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^r(I_{\eta_\varepsilon})^n}\le C\left(\frac{\eta_\varepsilon^{1+\frac{2r-1}{r(r-1)}}}{\varepsilon^{\frac{r}{r-1}}}+\frac{\eta_\varepsilon}{\varepsilon}+\left(\frac{\eta_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\right),$$

so that (5.14) easily holds. Using Hölder's inequality with the conjugate exponents $\frac{r}{q}$ and $\frac{r}{r-q}$ we obtain

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q(I_{\eta_\varepsilon})^n}\le C\left(\frac{\eta_\varepsilon^{\frac{1}{q}+\frac{r}{r-1}}}{\varepsilon^{\frac{r}{r-1}}}+\frac{\eta_\varepsilon^{\frac{1}{q}-\frac{1}{r}+1}}{\varepsilon}+\frac{\eta_\varepsilon^{\frac{1}{q}-\frac{1}{r}+\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right).$$

Now we take $\eta_\varepsilon=\varepsilon^{\frac{1}{\alpha}}$. Then we find that

$$\varepsilon^{-\frac{r}{r-1}}\|u_{\varepsilon\eta_\varepsilon}\|_{L^q(I_{\eta_\varepsilon})^n}\le C\left(\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}+\frac{r}{r-1}\right)-\frac{r}{r-1}}+\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{r}+1\right)-1}+\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{r}+\frac{1}{2}\right)-\frac{1}{2}}\right).$$

We seek an optimal q such that the right hand side in (5.1) tends to zero. It is easy to prove that we have a convergence to zero for any $q\in\left(1,\frac{r}{r(\alpha-1)+1}\right)$. Therefore, (5.15) holds and so we have (5.13).

5.2. Problem in the fissure part $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$

Using estimates in Lemma 4.4, the functions (3.4) and compactness, we prove the following lemma.

LEMMA 5.4. *Let $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$ with $1<r<+\infty$ and let $(\mathcal{U}^{\varepsilon\eta_\varepsilon},P^{\varepsilon\eta_\varepsilon})$ be the solution of (3.6)-(3.7). Then there exist subsequences of $\mathcal{U}^{\varepsilon\eta_\varepsilon}$ and $P^{\varepsilon\eta_\varepsilon}$ still denoted by the same, and functions $\mathcal{U}\in L^r(I_1)^n$, with $\mathcal{U}_n=0$, $P\in L^{r'}(I_1)$ such that*

$$\eta_\varepsilon^{-\frac{r}{r-1}}\mathcal{U}^{\varepsilon\eta_\varepsilon}\rightharpoonup\mathcal{U} \quad \text{in } L^r(I_1)^n, \quad P^{\varepsilon\eta_\varepsilon}\rightharpoonup P \quad \text{in } L^{r'}(I_1). \tag{5.16}$$

Moreover, $P=P(x')\in W^{1,r'}(\Sigma)$ and $\tilde{\mathcal{U}}$ is given by

$$\tilde{\mathcal{U}}(x',z)=\frac{2^{\frac{r'}{2}}}{r'\mu^{r'-1}}\left(\left(\frac{1}{2}\right)^{r'}-\left|\frac{1}{2}-z\right|^{r'}\right)|\tilde f(x',0)-\nabla_{x'}P(x')|^{r'-2}(\tilde f(x',0)-\nabla_{x'}P(x')). \tag{5.17}$$

Proof. Taking into account $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$ and estimates (4.6), (4.8), (4.13) we have

$$\|\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^r(I_1)^n}\le C\eta_\varepsilon^{\frac{r}{r-1}}, \tag{5.18}$$

$$\|\nabla_{x'}\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^r(I_1)^{n\times(n-1)}}\leq C\eta_\varepsilon^{\frac{1}{r-1}}, \tag{5.19}$$

$$\|\partial_z\mathcal{U}^{\varepsilon\eta_\varepsilon}\|_{L^r(I_1)^{n-1}}\leq C\eta_\varepsilon^{\frac{r}{r-1}}, \tag{5.20}$$

$$\|P^{\varepsilon\eta_\varepsilon}\|_{L^{r'}(I_1)}\leq C. \tag{5.21}$$

From the estimates (5.18) and (5.21), there exist $\mathcal{U}\in L^r(I_1)^n$, $P\in L^{r'}(I_1)$ such that convergence (5.16) holds. Moreover

$$\eta_\varepsilon^{-\frac{r}{r-1}}\partial_z\mathcal{U}^{\varepsilon\eta_\varepsilon}\rightharpoonup\partial_z\mathcal{U}\quad\text{ in } L^r(I_1)^n.$$

Let $\varphi\in C_0^\infty(I_1)^n$, then

$$\begin{aligned}&\eta_\varepsilon^{-\frac{1}{r-1}}\int_{I_1}\left(\mathrm{div}_{x'}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon}+\eta_\varepsilon^{-1}\partial_z\mathcal{U}_n^{\varepsilon\eta_\varepsilon}\right)\varphi\,dx\\&=-\eta_\varepsilon^{-\frac{1}{r-1}}\int_{I_1}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon}D_{x'}\varphi\,dx-\eta_\varepsilon^{-\frac{r}{r-1}}\int_{I_1}\mathcal{U}_n^{\varepsilon\eta_\varepsilon}\cdot\partial_z\varphi\,dx=0.\end{aligned}$$

Taking the limit $\varepsilon\to 0$ we obtain

$$\int_{I_1}\mathcal{U}_n\partial_z\varphi\,dx=0,$$

so that $\mathcal{U}_n=\mathcal{U}_n(x')$.

Since $\mathcal{U}$, $\partial_z\mathcal{U}\in L^r(I_1)^n$ the traces $\mathcal{U}(x',0)$, $\mathcal{U}(x',1)$ are well defined in $L^r(\Sigma)^n$. Analogously to the proof of Lemma 4.2 we choose a point $\beta_{x'}\in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $\alpha_{x'}\in\Sigma$, then we have

$$\int_\Sigma|\mathcal{U}^{\varepsilon\eta_\varepsilon}(x',0)|^r dx'=\int_\Sigma|u_{\varepsilon\eta_\varepsilon}(x',0)|^r dx'\leq C\int_\Sigma\left(\int_{(\beta_{x'},\alpha_{x'})}Du_{\varepsilon\eta_\varepsilon}\cdot(\alpha_{x'}-\beta_{x'})d\ell\right)^r dx',$$

so that, by Hölder's inequality,

$$\|\mathcal{U}^{\varepsilon\eta_\varepsilon}(x',0)\|^r_{L^r(\Sigma)^n}\leq C\varepsilon\|Du_{\varepsilon\eta_\varepsilon}\|^r_{L^r(D_{\varepsilon\eta_\varepsilon})^{n\times n}}.$$

Taking into account estimate (4.8) and $\eta_\varepsilon\gg\varepsilon^{\frac{r}{2r-1}}$, we have

$$\eta_\varepsilon^{-\frac{r}{r-1}}\|\mathcal{U}^{\varepsilon\eta_\varepsilon}(x',0)\|^r_{L^r(\Sigma)^n}\leq C\varepsilon\eta_\varepsilon\to 0\quad\text{ as }\varepsilon\to 0,$$

which implies that

$$\mathcal{U}(x',0)=0,$$

and analogously

$$\mathcal{U}(x',1)=0.$$

Consequently

$$\mathcal{U}_n=0,$$

which finishes (5.16). Finally, we need to indentify the effective system and compute the expression of the solution. For this, thanks to Propositions 3.1 and 3.2 in Mikelić and Tapiero [9] we have that the effective system is given by

$$-\partial_z\left(|\partial_z\tilde{\mathcal{U}}|^{r-2}\partial_z\tilde{\mathcal{U}}\right)=2^{\frac{r}{2}}\mu\left(\tilde{f}(x',0)-\nabla_{x'}P(x')\right),\quad \text{in } I_1,$$

$$\mathrm{div}_{x'}\left(\int_0^1\tilde{\mathcal{U}}(x',z)\,dz\right)=0\ \text{ in } \Sigma,\quad \left(\int_0^1\tilde{\mathcal{U}}(x',z)\,dz\right)\cdot\tilde{\nu}=0\ \text{ on } \partial\Sigma,$$

taking into account Proposition 3.3. in [9] we have that $P=P(x')\in W^{1,r'}(\Sigma)$ and thanks to Proposition 3.4 in [9] we have that the expression of the solution is given by (5.17). ◻

It remains to prove the convergence of the whole velocity

$$\eta_\varepsilon^{-\frac{2r-1}{r-1}}u_{\varepsilon\eta_\varepsilon}\stackrel{\star}{\rightharpoonup}\mathcal{V}\delta_\Sigma\quad\text{in }\mathcal{M}(D)^n, \tag{5.22}$$

where $\mathcal{V}(x')=\int_0^1\mathcal{U}(x',z)dz$ is given by (3.12), and also prove that the pressure P is the unique solution of the Reynolds problem (3.16).

Taking as test function $\varphi\in C^\infty(D)$ in $\mathrm{div}\,u_{\varepsilon\eta_\varepsilon}=0$ in D, we obtain

$$\int_D \mathrm{div}\,u_{\varepsilon\eta_\varepsilon}\varphi\,dx=-\int_D v_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dx-\eta_\varepsilon\int_{I_1}\mathcal{U}^{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi(x',\eta_\varepsilon z)\,dx'dz=0,$$

so that multiplying by $\eta_\varepsilon^{-\frac{2r-1}{r-1}}$,

$$\begin{aligned}&\int_{I_1}\eta_\varepsilon^{-\frac{r}{r-1}}\tilde{\mathcal{U}}^{\varepsilon\eta_\varepsilon}\cdot\nabla_{x'}\varphi(x',\eta_\varepsilon z)\,dx'dz\\&=-\int_D\eta_\varepsilon^{-\frac{2r-1}{r-1}}v_{\varepsilon\eta_\varepsilon}\cdot\nabla\varphi\,dx-\int_{I_1}\eta_\varepsilon^{-\frac{r}{r-1}}\mathcal{U}_n^{\varepsilon\eta_\varepsilon}\partial_n\varphi(x',\eta_\varepsilon z)\,dx'dz.\end{aligned}\tag{5.23}$$

Using (4.5) and taking into account $\eta_\varepsilon\gg\varepsilon^{\frac{r}{2r-1}}$, we obtain

$$\eta_\varepsilon^{-\frac{2r-1}{r-1}}\|v_{\varepsilon\eta_\varepsilon}\|_{L^r(D)^n}\le C\left(\frac{\varepsilon}{\eta_\varepsilon^{\frac{2r-1}{r}}}+\frac{\varepsilon^{\frac{r}{r-1}}}{\eta_\varepsilon^{\frac{2r-1}{r-1}}}\right)\to0\quad\text{as }\varepsilon\to0. \tag{5.24}$$

Taking the limit in (5.23) as $\varepsilon\to0$, using convergence (5.16) and $\mathcal{U}_n=0$, we have

$$\int_{I_1}\tilde{\mathcal{U}}\cdot\nabla_{x'}\varphi(x',0)\,dx'dz=0,$$

and by definition (3.12), we get the Reynolds problem (3.16). Consequently, P is the unique solution of (3.16) (see Proposition 3.4 in Mikelic and Tapiero [9] for more details). Finally, we consider $\varphi\in C_0(D)^n$ and so we have

$$\int_D\eta_\varepsilon^{-\frac{2r-1}{r-1}}u_{\varepsilon\eta_\varepsilon}\varphi\,dx=\int_D\eta_\varepsilon^{-\frac{2r-1}{r-1}}v_{\varepsilon\eta_\varepsilon}\varphi\,dx+\int_{I_1}\eta_\varepsilon^{-\frac{r}{r-1}}\mathcal{U}^{\varepsilon\eta_\varepsilon}\varphi(x',\eta_\varepsilon z)\,dx'dz.$$

Using (5.24) and convergence (5.16) and $\mathcal{U}_n=0$, we obtain

$$\begin{aligned}\int_D\eta_\varepsilon^{-\frac{2r-1}{r-1}}u_{\varepsilon\eta_\varepsilon}\varphi\,dx\to&\int_{I_1}\tilde{\mathcal{U}}(x',z)\tilde{\varphi}(x',0)\,dx'dz\\&=\int_\Sigma\tilde{\mathcal{V}}(x')\tilde{\varphi}(x',0)\,dx'=\langle\mathcal{V}(x')\delta_\Sigma,\varphi\rangle_{\mathcal{M}(D)^n,C_0(D)^n},\end{aligned}$$

which implies (5.22).

5.3. Effects of coupling $\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}$

The conclusion of the previous two subsections is that for any sequence of solutions $(v_{\varepsilon\eta_\varepsilon}, p_{\varepsilon\eta_\varepsilon})$ with $\eta_\varepsilon \ll \varepsilon^{\frac{r}{2r-1}}$ and $(\mathcal{U}^{\varepsilon\eta_\varepsilon}, P^{\varepsilon\eta_\varepsilon})$ with $\eta_\varepsilon \gg \varepsilon^{\frac{r}{2r-1}}$, and letting $\varepsilon \to 0$, we can extract subsequences still denoted by $v_{\varepsilon\eta_\varepsilon}$, $p_{\varepsilon\eta_\varepsilon}$, $\mathcal{U}^{\varepsilon\eta_\varepsilon}$, $P^{\varepsilon\eta_\varepsilon}$ and find functions $v \in L^r(D)^n$, $p \in W^{1,r'}(D)$, $\tilde{\mathcal{U}} \in L^r(I_1)^{n-1}$, $P \in W^{1,r'}(\Sigma)$ such that

$$\begin{aligned} &\varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \rightharpoonup v \quad \text{in } L^r(D)^n, \quad p_{\varepsilon\eta_\varepsilon} \to p \quad \text{in } L^{r'}(D), \\ &\eta_\varepsilon^{-\frac{r}{r-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon} \rightharpoonup \mathcal{U} \quad \text{in } L^r(I_1)^n \text{ with } \mathcal{U} = (\tilde{\mathcal{U}}, 0), \quad P^{\varepsilon\eta_\varepsilon} \rightharpoonup P \quad \text{in } L^{r'}(I_1). \end{aligned} \tag{5.25}$$

Moreover such limit functions v, p, $\mathcal{U}$, P necessarily satisfy the equations

$$\begin{aligned} &v = \frac{1}{\mu} K(f(x) - \nabla p(x)) \quad \text{in } D, \quad v \cdot \nu = 0 \quad \text{on } \partial D, \\ &\tilde{\mathcal{U}} = \frac{2^{\frac{r'}{2}}}{r'} \left(\left(\tfrac{1}{2}\right)^{r'} - \left|\tfrac{1}{2} - z\right|^{r'} \right) |\tilde{f}(x',0) - \nabla_{x'} P(x')|^{r'-2} (\tilde{f}(x',0) - \nabla_{x'} P(x')) \quad \text{in } I_1. \end{aligned} \tag{5.26}$$

We are going to find the connection between the functions p and P, i.e. to find the coupling effects between the solution in the porous part and in the fissure.

Lemma 5.5. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}$, with $\eta_\varepsilon / \varepsilon^{\frac{r}{2r-1}} \to \lambda$, $0 < \lambda < +\infty$, $1 < r < +\infty$, and let $\{p_{\varepsilon\eta_\varepsilon}\} \subset L^{r'}(D)$, $p \in W^{1,r'}(D)$, $P \in W^{1,r'}(\Sigma)$ be such that (5.25) and (5.26) hold. Then,*

$$\begin{aligned} &\int_D \frac{1}{\mu} K(f(x) - \nabla p(x)) \cdot \nabla \varphi(x)\, dx \\ &+ \lambda^{\frac{2r-1}{r-1}} \int_\Sigma \frac{\left|\tilde{f}(x',0) - \nabla_{x'} P(x')\right|^{r'-2}}{2^{\frac{r'}{2}} (r+1) \mu^{r'-1}} (\tilde{f}(x',0) - \nabla_{x'} P(x')) \cdot \nabla_{x'} \varphi(x',0)\, dx' = 0, \end{aligned} \tag{5.27}$$

for every $\varphi \in V_\Sigma$.

Proof. Let $\varphi \in V_\Sigma$. Taking into account the definitions (5.1) of $v_{\varepsilon\eta_\varepsilon}$ and (3.4) of $\mathcal{U}^{\varepsilon\eta_\varepsilon}$, and from $\operatorname{div} u_{\varepsilon\eta_\varepsilon} = 0$ in D we have

$$\begin{aligned} \int_D \varepsilon^{-\frac{r}{r-1}} u_{\varepsilon\eta_\varepsilon} \cdot \nabla \varphi\, dx &= \int_D \varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \cdot \nabla \varphi\, dx \\ &+ \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{r}{2r-1}}} \right)^{\frac{2r-1}{r-1}} \int_{I_1} \eta_\varepsilon^{-\frac{r}{r-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon} \cdot \nabla \varphi(x', \eta_\varepsilon z)\, dx' dz = 0. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$, using (5.25), $\mathcal{U}_n = 0$ and $\eta_\varepsilon / \varepsilon^{\frac{r}{2r-1}} \to \lambda$, we obtain

$$\int_D v(x) \cdot \nabla \varphi(x)\, dx + \lambda^{\frac{2r-1}{r-1}} \int_{I_1} \tilde{\mathcal{U}}(x', z) \cdot \nabla_{x'} \varphi(x', 0)\, dx' dz = 0,$$

and taking into account expressions (5.26) and (3.12), we get (5.27). □

In the following result, we are going to prove the relation between the pressures p and P.

Lemma 5.6. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}$, $\eta_\varepsilon / \varepsilon^{\frac{r}{2r-1}} \to \lambda$, $0 < \lambda < +\infty$, $1 < r < +\infty$ and let p, P be the limit pressures from (5.25). Then, there exists $\widetilde{C} \in \mathbb{R}$ such that*

$$p(x', 0) = P(x') + \widetilde{C}, \tag{5.28}$$

and $p \in V_\Sigma$ is the unique solution of the variational problem (3.13).

Proof. We need to extend the test functions considered in the proof of Lemma 5.3 to the fissure I_{η_ε}. To do this, we define $B_{\eta_\varepsilon} = D_- \cup \Sigma \cup I_{\eta_\varepsilon}$ and $Y' = \overline{Y}^* \cap \{x_n = 0\}$, and we consider $\phi(y)\in C^\infty_\#(B_{\eta_\varepsilon})^n$ be such that $\phi(y)=0$ in $Y\setminus Y^*$ and $\mathrm{div}_y\phi(y)=0$ in Y^*. We define

$$\phi_\varepsilon(x)=\begin{cases}\phi\left(\frac{x}{\varepsilon}\right) & \text{in } D_-,\\ K_n e_n & \text{in } I_{\eta_\varepsilon}, \quad \text{where } K_n=\int_{Y'}\phi_n(y',0)dy'.\end{cases}$$

Let $\varphi\in C_0^\infty(B_1)$, with $B_1=D_-\cup\Sigma\cup I_1$ be such that

$$\int_\Sigma \varphi(x',0)\,dx'=0. \tag{5.29}$$

We define

$$\varphi_{\eta_\varepsilon}(x)=\begin{cases}\varphi(x) & \text{in } D_-\\ \varphi\left(x',\frac{x_n}{\eta_\varepsilon}\right) & \text{in } I_{\eta_\varepsilon}.\end{cases}$$

Taking in (3.1) as test function

$$w_\varepsilon(x)=\begin{cases}\varphi(x)\phi\left(\frac{x}{\varepsilon}\right)-\varepsilon^{-\frac{r}{r-1}}v_{\varepsilon\eta_\varepsilon} & \text{in } D_-,\\ \varphi\left(x',\frac{x_n}{\eta_\varepsilon}\right)K_n e_n & \text{in } I_{\eta_\varepsilon},\end{cases}$$

we obtain

$$\int_{B_{\eta_\varepsilon}} S_r(\mathbb{D}[u_{\varepsilon\eta_\varepsilon}]):Dw_\varepsilon\,dx=\int_{B_{\eta_\varepsilon}} f\cdot w_\varepsilon\,dx-\int_{B_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}\mathrm{div}\,w_\varepsilon\,dx. \tag{5.30}$$

Taking into account that

$$K_n\int_{I_{\eta_\varepsilon}} f\cdot\varphi\left(x',\frac{x_n}{\eta_\varepsilon}\right)e_n\,dx=\eta_\varepsilon K_n\int_{I_1} f\cdot\varphi(x',z)e_n\,dx'dz\to 0 \quad \text{as } \varepsilon\to 0,$$

and by using estimates (5.19), (5.20), that

$$\left|K_n\int_{I_{\eta_\varepsilon}} S_r(\mathbb{D}[\mathcal{U}^{\varepsilon\eta_\varepsilon}])\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx\right|=\left|K_n\int_{I_1} S_r(\mathbb{D}_{\eta_\varepsilon}[\mathcal{U}^{\varepsilon\eta_\varepsilon}])\partial_z\varphi(x',z)\,dx'dz\right|$$
$$\le C\eta_\varepsilon^{\frac{1}{r-1}}\to 0, \quad \text{as } \varepsilon\to 0,$$

from (5.30), we obtain

$$\int_{D_-} S_r(\mathbb{D}[v_{\varepsilon\eta_\varepsilon}]):Dw_\varepsilon\,dx \tag{5.31}$$
$$=\int_{D_-} f\cdot w_\varepsilon\,dx+\int_{D_-} p_{\varepsilon\eta_\varepsilon}\mathrm{div}\,w_\varepsilon\,dx+K_n\int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx+O_\varepsilon.$$

For the last term on the right hand side, we have

$$K_n\int_{I_{\eta_\varepsilon}} p_{\varepsilon\eta_\varepsilon}\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx$$
$$=K_n\int_{I_{\eta_\varepsilon}} (p_{\varepsilon\eta_\varepsilon}-c_{\varepsilon\eta_\varepsilon})\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx+K_n\int_{I_{\eta_\varepsilon}} c_{\varepsilon\eta_\varepsilon}\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx,$$

where $c_{\varepsilon\eta_\varepsilon}$ is defined in (3.5). Using (5.25) and (5.29) we obtain

$$\begin{aligned}&K_n\int_{I_{\eta_\varepsilon}}(p_{\varepsilon\eta_\varepsilon}-c_{\varepsilon\eta_\varepsilon})\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx=K_n\int_{I_1}P^{\varepsilon\eta_\varepsilon}\partial_z\varphi(x',z)\,dx'dz\\&\to K_n\int_{I_1}P(x')\partial_z\varphi(x',z)\,dx'dz=-K_n\int_\Sigma P(x')\varphi(x',0)\,dx',\qquad\text{as }\varepsilon\to 0,\end{aligned}\tag{5.32}$$

where $P^{\varepsilon\eta_\varepsilon}$ is given by (3.4), and

$$K_nc_{\varepsilon\eta_\varepsilon}\int_{I_{\eta_\varepsilon}}\partial_{x_n}\varphi(x',\frac{x_n}{\eta_\varepsilon})\,dx=K_nc_{\varepsilon\eta_\varepsilon}\int_{I_1}\partial_z\varphi(x',z))\,dx'dz=0.$$

Passing to the limit in (5.31) similarly as in the proof of Lemma 5.3, we know that $\hat{v}$ and p are related by the variational formulation of problem (5.9)-(5.11), and taking into account (5.32) and

$$\begin{aligned}&\int_{D\times Y}p(x)\,\mathrm{div}_x(\varphi(x)\phi(y))\,dxdy\\&=-\int_{D\times Y}\nabla_xp(x)\varphi(x)\phi(y)\,dxdy+\int_{\Sigma\times Y'}p(x',0)\varphi(x',0)\phi_n(y',0)\,dx'dy'\\&=-\int_{D\times Y}\nabla_xp(x)\varphi(x)\phi(y)\,dxdy+K_n\int_\Sigma p(x',0)\varphi(x',0)\,dx',\end{aligned}$$

then we have

$$\int_\Sigma(p(x',0)-P(x'))\,\varphi(x',0)\,dx'=0,$$

so that

$$\int_\Sigma(p(x',0)-P(x'))\,\psi(x')\,dx'=0,$$

for every $\psi\in C_0^\infty(\Sigma)$ such that $\int_\Sigma\psi\,dx'=0$. Finally we conclude that there exists a constant $\widetilde{C}\in\mathbb{R}$ such that (5.28) holds and $p(\cdot,0)\in W^{1,r'}(\Sigma)$, i.e. $p\in V_\Sigma$. Using (5.28) and (5.27), we obtain the variational formulation for the limit pressure p in the space V_Σ in the form

$$\begin{aligned}&\int_D\frac{1}{\mu}K\,(f(x)-\nabla p(x))\cdot\nabla\varphi(x)\,dx\\&+\lambda^{\frac{2r-1}{r-1}}\int_\Sigma\frac{\left|\tilde f(x',0)-\nabla_{x'}p(x',0)\right|^{r'-2}}{2^{\frac{r'}{2}}(r+1)\mu^{r'-1}}(\tilde f(x',0)-\nabla_{x'}p(x',0))\cdot\nabla_{x'}\varphi(x',0)\,dx'=0,\end{aligned}\tag{5.33}$$

for every $\varphi\in V_\Sigma$.

Since K and $S_{r'}$ are coercive and monotone (see Remark 3.3 for more details), it can be proved that (5.33) has a unique solution in the Banach space $V_\Sigma/\mathbb{R}$ equipped with the norm $|v|_{V_\Sigma}=|v|_{W^{1,r'}(D)}+|v(\cdot,0)|_{W^{1,r'}(\Sigma)}$, by direct application of Lax-Milgram Theorem. Therefore the whole sequence converges to p, the unique solution of the problem (3.13).
□

Proof. [Proof of Theorem 3.1] It remains to prove the convergence (3.10) of the whole velocity.

Let $\varphi \in C_0(D)^n$. Then

$$\int_D \varepsilon^{-\frac{r}{r-1}} u_{\varepsilon\eta_\varepsilon} \cdot \varphi \, dx$$
$$= \int_D \varepsilon^{-\frac{r}{r-1}} v_{\varepsilon\eta_\varepsilon} \cdot \varphi \, dx + \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{r}{2r-1}}} \right)^{\frac{2r-1}{r-1}} \int_{I_1} \eta_\varepsilon^{-\frac{r}{r-1}} \mathcal{U}^{\varepsilon\eta_\varepsilon} \cdot \varphi(x', \eta_\varepsilon z) \, dx' dz.$$

Taking the limit as $\varepsilon \to 0$, using (5.25), $\mathcal{U}_n = 0$ and $\eta_\varepsilon / \varepsilon^{\frac{r}{2r-1}} \to \lambda$, we obtain

$$\int_D \varepsilon^{-\frac{r}{r-1}} u_{\varepsilon\eta_\varepsilon} \cdot \varphi \, dx \to \int_D v \cdot \varphi \, dx + \lambda^{\frac{2r-1}{r-1}} \int_{I_1} \tilde{\mathcal{U}}(x', z) \varphi(x', 0) \, dx' dz.$$

Taking into account that

$$\int_{I_1} \tilde{\mathcal{U}}(x', z) \varphi(x', 0) \, dx' dz = \int_\Sigma \mathcal{V}(x') \varphi(x', 0) \, dx' = \langle \mathcal{V} \delta_\Sigma, \varphi \rangle_{\mathcal{M}(D)^n, C_0(D)^n},$$

where $\mathcal{V}(x')$ is given by (3.12), we get (3.10).

□

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