Homogenization of an incompressible non-Newtonian flow through a thin porous medium

María Anguiano and Francisco Javier Suárez-Grau

Abstract. In this paper, we consider a non-Newtonian flow in a thin porous medium Ω_ε of thickness ε which is perforated by periodically solid cylinders of size a_ε. The flow is described by the 3D incompressible Stokes system with a nonlinear viscosity, being a power of the shear rate (power law) of flow index $1 < p < +\infty$. We consider the limit when domain thickness tends to zero and we obtain different models depending on the magnitude a_ε with respect to ε.

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1. Introduction

In this paper, we consider an incompressible viscous non-Newtonian flow through a thin porous medium. The study of partial differential equations (PDE's) in thin structures is a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. For example, in fluid mechanics the analysis of fluid-film bearings in thin domains gives rise to different types of nonlinear Reynolds equations, see for instance [13, 22, 29]. There are many other works in the recent literature about PDE's and thin structures, for instance [4, 5, 14, 20, 25, 26, 28].

In the case of Newtonian flow in a porous medium such geometry leads to Darcy's law as an averaged momentum equation, connecting the velocity to the gradient of the pressure. A number of papers address the derivation of Darcy's law from the fundamental hydrodynamical equations (see Lions [18] and Sanchez-Palencia [27] for derivation using homogenization). A good reference for physical aspects of this problem, as well as mathematical ones, is the book [17]. However, for the non-Newtonian flow the situation is completely different. The main reason is that the viscosity is a nonlinear function of the symmetrized gradient of the velocity. For some flow regimes linear models, analogous to Darcy's law are used, but in other situations measurements indicate that the filtration laws linking the velocity and gradient of the pressure are nonlinear. Bourgeat and Mikelić in [8] consider the stationary incompressible purely viscous non-Newtonian flow through a porous medium and use the homogenization technique called the two-scale convergence (see [2, 24]). Further extensions are to be found in [9, 15].

Thin porous media are common and of great importance for various industries and products. These include papers and cartons, filters and filtration cakes, porous coatings, fuel cells, textiles, and diapers and wipes, to name only a few. A thin porous medium is characterized by lateral dimensions

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much greater than its thickness. The above results about non-Newtonian flows relate to a fixed height domain. Our aim in the present paper is to extend it to the case of a domain of small height ε.

The thin porous medium considered involves two small parameters: the thickness of the domain ε and the interspatial distance between obstacles a_ε. We consider a fluid flow through periodic vertical cylinders confined between two parallel plates (see Figures 1 and 2). A representative elementary volume for the thin porous medium is a cube of lateral length a_ε and vertical length ε. The cube is repeated periodically in the space between the plates. Each cube can be divided into fluid part and a solid part, where the solid part has the shape of a vertical cylinder of length ε (see Figure 3).

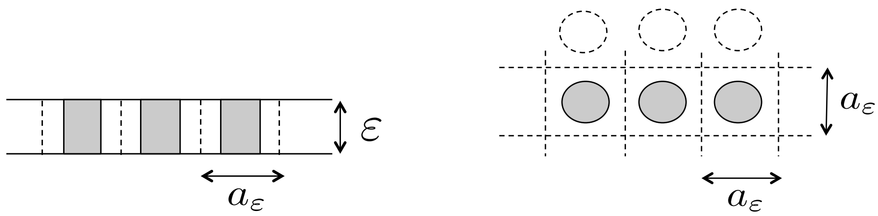

Figure 1. Views from lateral (left) and from above (right)

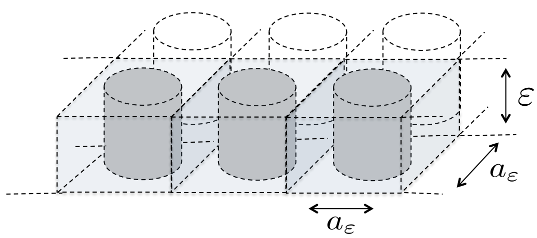

Figure 2. View of the domain

We consider that the viscosity satisfies the non linear power law. If u is the velocity and Du the gradient velocity tensor, denoting the shear rate by $\mathbb{D}\left[u\right] = \frac{1}{2}(Du + D^t u)$, the viscosity as a function of the shear rate is given by

$$\eta_p\left(\mathbb{D}\left[u\right]\right) = \mu \left|\mathbb{D}\left[u\right]\right|^{p-2}, \quad 1 < p < +\infty,$$

where the matrix norm $|\cdot|$ is defined by $|\xi|^2 = Tr(\xi\xi^t)$ with $\xi \in \mathbb{R}^3$, the two material parameters $\mu > 0$ and p are called the consistency and the flow index, respectively. Recall that $p = 2$ yields the Newtonian fluid. For $1 < p < 2$ the fluid is pseudoplastic (shear thinning), which is the characteristic of high polymers, polymer solutions, and many suspensions, whereas for $p > 2$ the fluid is dilatant (shear thickening), whose behavior is reported for certain slurries, like mud, clay, or cement, and implies an increased resistance to flow with intensified shearing.

We consider fluids satisfying the non-Newtonian Stokes system, $1 < p < +\infty$, in the thin porous medium described above. Our purpose is to study the asymptotic behavior of this system when ε tends to zero. The proof of our results is based on an adaptation of the unfolding method [3, 10, 11], which is strongly related to the two-scale convergence method, but here it is necessary to combine it with a rescaling in the height variable, in order to work with a domain of height one. The unfolding method is a very efficient tool to study periodic homogenization problems where the size of the periodic cell tends to zero. The idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable). Recently, in [4], a generalization of the unfolding method to the case of locally periodic media has been introduced. In particular, the behavior of the solutions of the Neumann problem for the Laplace operator in a thin domain with a highly oscillatory boundary is studied.

We show that the asymptotic behavior of this system depends on the parameter a_ε with respect to ε:

- If $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, i.e. when the cylinder height is proportional to the interspatial distance, with λ the proportionality constant, we obtained a 2D Darcy's law as a homogenized model with a permeability function which depends on the parameter λ and is obtained through local Stokes problem in 3D.
- If $a_\varepsilon \gg \varepsilon$, i.e. when the cylinder height is much smaller than the interspatial distance, we obtain a 2D Darcy's law, with the permeability function obtained by means of local Reynolds problem, which is a considerable simplification.
- If $a_\varepsilon \ll \varepsilon$, i.e. when the cylinder height is much larger than the interspatial distance, we obtain a 2D Darcy's law as a homogenized model with a permeability function which is obtained through local Stokes problem in 2D.

The paper is organized as follows. In Section 2, the domain and some the notations are introduced. In Section 3, we formulate the problem and state our main result, which is proved in Section 6 by means of an adaptation of the unfolding method. To apply this method, a priori estimates are stablished in Section 4 and some compactness results are proved in Section 5.

2. The domain and some notations

A periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = [-1/2, 1/2]^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \bigcup Y'_s = Y'$ and $Y'_f \bigcap Y'_s = \emptyset$). More precisely, we assume that ω is a smooth, bounded, connected set in $\mathbb{R}^2$, and that Y'_s is a smooth and connected set strictly included in Y'.

The microscale of a porous medium is a small positive number a_ε. The domain ω is covered by a regular mesh of size a_ε: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',a_\varepsilon} = a_\varepsilon k' + a_\varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},a_\varepsilon}$ and a solid part $Y'_{s_{k'},a_\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size a_ε. We also define $Y = Y' \times (0,1) \in \mathbb{R}^3$, and is divided in a fluid part Y_f and a solid part Y_s, and consequently $Y_{k',a_\varepsilon} = Y'_{k',a_\varepsilon} \times (0,1) \in \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},a_\varepsilon}$ and a solid part $Y_{s_{k'},a_\varepsilon}$.

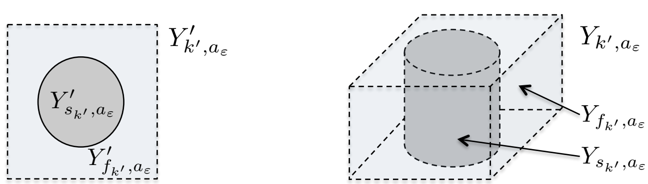

Figure 3. Views of a periodic cell in 2D (left) and 3D (right)

The fluid part ω_ε of a porous medium is defined by

$$\omega_\varepsilon = \omega \backslash \bigcup_{k' \in T_\varepsilon} Y'_{s_{k'},a_\varepsilon},$$

where $T_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',a_\varepsilon} \cap \omega \neq \emptyset\}$.

In order to apply the unfolding method, we will need the following notation. For $k' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \iff x' \in Y'_{k',1}. \tag{1}$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial Y'_{k',1}$). Moreover, for every $a_\varepsilon > 0$, we have

$$\kappa\left(\frac{x'}{a_\varepsilon}\right) = k' \iff x' \in Y'_{k',a_\varepsilon}.$$

We will consider the open set $\Omega_\varepsilon \subset \mathbb{R}^3$ given by

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \tag{2}$$

Then Ω_ε denotes the whole fluid part in the thin film.

We define $\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1)$, $\Omega = \omega \times (0,1)$ and $Q_\varepsilon = \omega \times (0, \varepsilon)$. We have that

$$\widetilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k' \in T_\varepsilon} Y_{s_{k'}, a_\varepsilon} = \Omega \cap \bigcup_{k' \in T_\varepsilon} Y_{f_{k'}, a_\varepsilon}.$$

We denote by $L^p_\sharp(Y)$, $W^{1,p}_\sharp(Y)$, the functional spaces

$$L^p_\sharp(Y) = \left\{ v \in L^p_{loc}(Y) \, : \, \int_Y |v|^p dy < +\infty, v(y' + k', y_3) = v(y) \ \ \forall k' \in \mathbb{Z}^2, \text{ a.e. } y \in Y \right\},$$

and

$$W^{1,p}_\sharp(Y) = \left\{ v \in W^{1,p}_{loc}(Y) \cap L^p_\sharp(Y) : \int_Y |\nabla_y v|^p dy < +\infty \right\}.$$

We denote by : the full contraction of two matrices: for $A = (a_{i,j})_{1 \le i,j \le 2}$ and $B = (b_{i,j})_{1 \le i,j \le 2}$, we have $A : B = \sum_{i,j=1}^2 a_{ij} b_{ij}$.

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line, and by C a generic positive constant which can change from line to line.

3. Setting and main result

Along this section, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

In this section we describe the asymptotic behavior of an incompressible viscous non-Newtonian fluid in the geometry Ω_ε described in Section 2. The proof of the corresponding results will be given in the next sections.

Our results are referred to the non-Newtonian Stokes system. Namely, let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in W^{1,p}_0(\Omega_\varepsilon)^3 \times L^{p'}(\Omega_\varepsilon)$, $1 < p < +\infty$, which satisfies

$$\left\{ \begin{array}{r} -\text{div } \left(\eta_p \left(\mathbb{D} \left[u_\varepsilon \right] \right) \mathbb{D} \left[u_\varepsilon \right] \right) + \nabla p_\varepsilon = f \ \text{ in } \ \Omega_\varepsilon, \\ \text{div}\, u_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon, \end{array} \right. \tag{3}$$

where Ω_ε is defined by (2) and $p' = p/(p-1)$ is the conjugate exponent of p. The right-hand side f is of the form

$$f(x) = (f'(x'), 0), \ \text{ a.e. } \ x \in \Omega,$$

where f is assumed in $L^{p'}(\omega)^2$. This choice of f is usual when we deal with thin domains. Since the thickness of the domain, ε, is small then the vertical component of the force can be neglected and, moreover the force can be considered independent of the vertical variable.

We deal the problem with Dirichlet boundary condition, i.e.

$$u_\varepsilon = 0 \text{ on } \ \partial\Omega_\varepsilon. \tag{4}$$

It is well known that (3)-(4) has a unique solution $(u_\varepsilon, p_\varepsilon) \in W^{1,p}_0(\Omega_\varepsilon)^3 \times L^{p'}(\Omega_\varepsilon)$ (see the classical theory [16, 18, 31]). This solution is unique up to an additive constant for p_ε, i.e. it is unique if we consider the corresponding equivalence class: $p_\varepsilon \in L^{p'}(\Omega_\varepsilon)/\mathbb{R}$.

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3

$$y_3 = \frac{x_3}{\varepsilon}, \tag{5}$$

in order to have the functions defined in an open set with fixed height.

Namely, we define $\tilde{u}_\varepsilon \in W^{1,p}_0(\widetilde{\Omega}_\varepsilon)^3$, $\tilde{p}_\varepsilon \in L^{p'}(\widetilde{\Omega}_\varepsilon)/\mathbb{R}$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \ \ \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \ \ a.e. \ (x', y_3) \in \widetilde{\Omega}_\varepsilon.$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v=(v',v_3)$ and a scalar function w, we will denote $\mathbb{D}_{x'}\left[v\right]=\frac{1}{2}(D_{x'}v+D^t_{x'}v)$ and $\partial_{y_3}\left[v\right]=\frac{1}{2}(\partial_{y_3}v+\partial^t_{y_3}v)$, where we denote $\partial_{y_3}=(0,0,\frac{\partial}{\partial y_3})^t$. Moreover, associated to the change of variables (5), we introduce the operators: $\mathbb{D}_\varepsilon$, D_ε, div_ε and ∇_ε, by

$$\mathbb{D}_\varepsilon\left[v\right]=\frac{1}{2}\left(D_\varepsilon v+D^t_\varepsilon v\right),$$

$$(D_\varepsilon v)_{i,j}=\partial_{x_j}v_i \text{ for } \ i=1,2,3,\ j=1,2,$$

$$(D_\varepsilon v)_{i,3}=\frac{1}{\varepsilon}\partial_{y_3}v_i \text{ for } \ i=1,2,3,$$

$$\mathrm{div}_\varepsilon v=\mathrm{div}_{x'}v'+\frac{1}{\varepsilon}\partial_{y_3}v_3,$$

$$\nabla_\varepsilon w=(\nabla_{x'}w,\frac{1}{\varepsilon}\partial_{y_3}w)^t.$$

Using the transformation (5), the system (3) can be rewritten as

$$\left\{\begin{array}{r}-\mathrm{div}_\varepsilon\left(\mu\left|\mathbb{D}_\varepsilon\left[\tilde u_\varepsilon\right]\right|^{p-2}\mathbb{D}_\varepsilon\left[\tilde u_\varepsilon\right]\right)+\nabla_\varepsilon\tilde p_\varepsilon=f \text{ in } \ \widetilde\Omega_\varepsilon,\\ \mathrm{div}_\varepsilon\tilde u_\varepsilon=0 \text{ in } \ \widetilde\Omega_\varepsilon,\end{array}\right. \tag{6}$$

with Dirichlet boundary condition, i.e.

$$\tilde u_\varepsilon=0 \text{ on } \ \partial\widetilde\Omega_\varepsilon.$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$.

The sequence of solutions $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)\in W^{1,p}_0(\widetilde\Omega_\varepsilon)^3\times L^{p'}(\widetilde\Omega_\varepsilon)/\mathbb{R}$ is not defined in a fixed domain independent of ε but rather in a varying set $\widetilde\Omega_\varepsilon$. In order to pass the limit if ε tends to zero, convergences in fixed Sobolev spaces (defined in Ω) are used which requires first that $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ be extended to the whole domain Ω. Then, by definition, an extension $(\tilde u_\varepsilon,\,\tilde P_\varepsilon)\in W^{1,p}_0(\Omega)^3\times L^{p'}(\Omega)/\mathbb{R}$ of $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ is defined on Ω and coincides with $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ on $\widetilde\Omega_\varepsilon$ (we will use the same notation, $\tilde u_\varepsilon$, for the velocity in $\widetilde\Omega_\varepsilon$ and its continuation in Ω).

In order to simplify the notation, we define S as the p-Laplace operator

$$S(\xi)=|\xi|^{p-2}\,\xi,\quad \forall\xi\in\mathbb{R}^{3\times 3}_{\mathrm{sym}},\quad 1<p<+\infty.$$

Our main result referred to the asymptotic behavior of the solution of (6) is given by the following theorem.

Theorem 3.1. *We distingue three cases depending on the relation between the parameter a_ε with respect to ε:*

i) *if $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, then the extension $(a_\varepsilon^{-\frac{p}{p-1}}\tilde u_\varepsilon,\tilde P_\varepsilon)$ of the solution of (6) converges weakly to $(\tilde u,\tilde P)$ in $W^{1,p}(0,1;L^p(\omega)^3)\times L^{p'}(\omega)/\mathbb{R}$. Moreover, it holds that $(\tilde U,\tilde P)$, with $\tilde U_3=0$, is the unique solution of Darcy's law*

$$\left\{\begin{array}{l}\tilde U'(x')=\dfrac{1}{\mu}A^\lambda\left(f'(x')-\nabla_{x'}\tilde P(x')\right) \ \textit{in} \ \ \omega,\\ \mathrm{div}_{x'}\tilde U'(x')=0 \ \textit{in} \ \ \omega,\\ \tilde U'(x')\cdot n=0 \ \textit{in} \ \ \partial\omega,\end{array}\right. \tag{7}$$

where $\tilde U(x')=\int_0^1\tilde u(x',y_3)\,dy_3$ and the permeability function $A^\lambda:\mathbb{R}^2\to\mathbb{R}^2$ is monotone and coercive, defined by

$$A^\lambda(\xi')=\int_{Y_f}w^{\xi'}(y)\,dy,\quad \forall\,\xi'\in\mathbb{R}^2, \tag{8}$$

where $w^{\xi'}(y)$, for every $\xi'\in\mathbb{R}^2$ with $\int_Y w_3^{\xi'}(y)dy=0$, denote the unique solution in $W^{1,p}_\sharp(Y_f)^3$ of the local Stokes problem in 3D

$$\left\{\begin{array}{rcll}-\mathrm{div}_\lambda S\left(\mathbb{D}_\lambda[w^{\xi'}]\right)+\nabla_\lambda\pi^{\xi'} &=& \xi' & \text{in } Y_f\,,\\ \mathrm{div}_\lambda w^{\xi'} &=& 0 & \text{in } Y_f\,,\\ w^{\xi'} &=& 0 & \text{in } \partial Y_s\,,\\ w^{\xi'},\pi^{\xi'}\ Y'-\text{periodic}, & & & \end{array}\right. \tag{9}$$

where $\mathbb{D}_\lambda[\cdot]=\mathbb{D}_{y'}[\cdot]+\lambda\partial_{y_3}[\cdot]$, $\nabla_\lambda=\nabla_{y'}+\lambda\partial_{y_3}$ and $\mathrm{div}_\lambda=\nabla_{x'}+\lambda\partial_{y_3}$.

ii) *if $a_\varepsilon\gg\varepsilon$, then the extension $(\varepsilon^{-\frac{p}{p-1}}\tilde u_\varepsilon,\tilde P_\varepsilon)$ of the solution of (6) converges weakly to $(\tilde u,\tilde P)$ in $W^{1,p}(0,1;L^p(\omega)^3)\times L^{p'}(\omega)/\mathbb{R}$. Moreover, it holds that $(\tilde U,\tilde P)$, with $\tilde U_3=0$, is the unique solution of Darcy's law*

$$\left\{\begin{array}{l}\tilde U'(x')=\dfrac{1}{\mu\,2^{\frac{p'}{2}}(p'+1)}A^\infty\left(f'(x')-\nabla_{x'}\tilde P(x')\right)\ \text{in}\ \omega,\\ \mathrm{div}_{x'}\tilde U'(x')=0\ \text{in}\ \omega,\\ \tilde U'(x')\cdot n=0\ \text{in}\ \partial\omega,\end{array}\right. \tag{10}$$

where $\tilde U(x')=\int_0^1\tilde u(x',y_3)\,dy_3$ and the permeability function $A^\infty:\mathbb{R}^2\to\mathbb{R}^2$ is monotone and coercive, defined by

$$A^\infty(\xi')=\int_{Y_f'}\left|\xi'+\nabla_{y'}\pi^{\xi'}\right|^{p'-2}\left(\xi'+\nabla_{y'}\pi^{\xi'}\right)dy',\quad\forall\,\xi'\in\mathbb{R}^2, \tag{11}$$

where, $\pi^{\xi'}(y')$, for every $\xi'\in\mathbb{R}^2$, denote the unique solution in $W^{1,p'}_\sharp(Y_f')\cap L^{p'}(Y_f')/\mathbb{R}$ of the local Reynolds problem

$$\left\{\begin{array}{rcll}\mathrm{div}_{y'}\left(\left|\xi'+\nabla_{y'}\pi^{\xi'}\right|^{p'-2}\left(\xi'+\nabla_{y'}\pi^{\xi'}\right)\right) &=& 0 & \text{in } Y_f',\\ \left(\left|\xi'+\nabla_{y'}\pi^{\xi'}\right|^{p'-2}\left(\xi'+\nabla_{y'}\pi^{\xi'}\right)\right)\cdot n &=& 0 & \text{in } \partial Y_s'.\end{array}\right. \tag{12}$$

iii) *if $a_\varepsilon\ll\varepsilon$, then the extension $(a_\varepsilon^{-\frac{p}{p-1}}\tilde u_\varepsilon,\tilde P_\varepsilon)$ of the solution of (6) converges weakly to $(\tilde u,\tilde P)$ in $L^p(\Omega)^3\times L^{p'}(\omega)/\mathbb{R}$. Moreover, it holds that $(\tilde U,\tilde P)$, with $\tilde U_3=0$, is the unique solution of Darcy's law*

$$\left\{\begin{array}{l}\tilde U'(x')=\dfrac{1}{\mu}A^0\left(f'(x')-\nabla_{x'}\tilde P(x')\right)\ \text{in}\ \omega,\\ \mathrm{div}_{x'}\tilde U'(x')=0\ \text{in}\ \omega,\\ \tilde U'(x')\cdot n=0\ \text{in}\ \partial\omega,\end{array}\right. \tag{13}$$

where $\tilde U(x')=\int_0^1\tilde u(x',y_3)\,dy_3$ and the permeability function $A^0:\mathbb{R}^2\to\mathbb{R}^2$ is monotone and coercive, defined by

$$A^0(\xi')=\int_{Y_f}w^{\xi'}(y)\,dy,\quad\forall\,\xi'\in\mathbb{R}^2, \tag{14}$$

where $w^{\xi'}(y)$, for every $\xi'\in\mathbb{R}^2$, denote the unique solution in $W^{1,p}_\sharp(Y_f)^2$ of the local Stokes problem in 2D

$$\left\{\begin{array}{rcll}-\mathrm{div}_{y'}S\left(\mathbb{D}_{y'}[w^{\xi'}]\right)+\nabla_{y'}\pi^{\xi'} &=& \xi' & \text{in } Y_f\,,\\ \mathrm{div}_{y'}w^{\xi'} &=& 0 & \text{in } Y_f\,,\\ w^{\xi'} &=& 0 & \text{in } \partial Y_s\,,\\ w^{\xi'},\pi^{\xi'}\ Y'-\text{periodic}. & & & \end{array}\right. \tag{15}$$

Remark 3.2. *The monotonicity and coerciveness properties of the permeability functions A^λ, A^∞ and A^0 given by (8), (11) and (14), respectively, can be found in* [9].

4. A Priori Estimates

Let us begin with a variant of the Korn's inequality in the thin domain Ω_ε.

Lemma 4.1. *There exists a constant C independent of ε, such that, for any function $v \in W_0^{1,p}(\Omega_\varepsilon)^3$ with $1 < p < +\infty$, one has*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then*

$$\|v\|_{L^p(\Omega_\varepsilon)^3} \leq C(a_\varepsilon^p + \varepsilon^p)^{\frac{1}{p}} \|\mathbb{D}[v]\|_{L^p(\Omega_\varepsilon)^{3\times 3}},$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\|v\|_{L^p(\Omega_\varepsilon)^3} \leq C\varepsilon \|\mathbb{D}[v]\|_{L^p(\Omega_\varepsilon)^{3\times 3}},$$

iii) *if $a_\varepsilon \ll \varepsilon$, then*

$$\|v\|_{L^p(\Omega_\varepsilon)^3} \leq Ca_\varepsilon \|\mathbb{D}[v]\|_{L^p(\Omega_\varepsilon)^{3\times 3}}.$$

Proof. Let us start with the proof of *i*). We observe that the microscale of the porous medium a_ε is similar or larger than the thickness of the domain ε, which lead us to divide the domain Ω_ε in small cubes of lateral length a_ε and vertical length ε.

For any function $v(z) \in W^{1,p}(Y_f)^3$ with $v = 0$ in ∂Y_s, using the results from Mjasnikov and Mosolov [23], we have

$$\int_{Y_f} |v|^p\, dz \leq C \int_{Y_f} |\mathbb{D}_z[v]|^p\, dz, \quad 1 < p < +\infty, \tag{16}$$

where the constant C depends only on Y_f.

For every $k' \in \mathbb{Z}^2$, by the change of variable

$$k' + z' = \frac{x'}{a_\varepsilon}, \quad z_3 = \frac{x_3}{\varepsilon}, \quad dz = \frac{dx}{\varepsilon\, a_\varepsilon^2}, \quad \partial_{z'} = a_\varepsilon \partial_{x'}, \quad \partial_{z_3} = \varepsilon \partial_{x_3}, \tag{17}$$

we rescale (16) from Y_f to $Q_{f_{k'},a_\varepsilon} = Y'_{f_{k'},a_\varepsilon} \times (0,\varepsilon)$. This yields that, for any function $v(x) \in W^{1,p}(Q_{f_{k'},a_\varepsilon})^3$ with $v = 0$ in $\partial Q_{s_{k'},a_\varepsilon}$ where $Q_{s_{k'},a_\varepsilon} = Y'_{s_{k'},a_\varepsilon} \times (0,\varepsilon)$, one has

$$\begin{aligned} \int_{Q_{f_{k'},a_\varepsilon}} |v|^p\, dx &\leq a_\varepsilon^p C \int_{Q_{f_{k'},a_\varepsilon}} |\mathbb{D}_{x'}[v]|^p\, dx + \varepsilon^p C \int_{Q_{f_{k'},a_\varepsilon}} |\partial_{x_3}[v]|^p\, dx \\ &\leq C(a_\varepsilon^p + \varepsilon^p) \int_{Q_{f_{k'},a_\varepsilon}} |\mathbb{D}_x[v]|^p\, dx, \end{aligned} \tag{18}$$

with the same constant C as in (16). Summing the inequalities (18) for all the periods $Q_{f_{k'},a_\varepsilon}$, gives

$$\int_{\Omega_\varepsilon} |v|^p\, dx \leq C(a_\varepsilon^p + \varepsilon^p) \int_{\Omega_\varepsilon} |\mathbb{D}_x[v]|^p\, dx.$$

In fact, we must consider separately the periods containing a portion of $\partial\omega$, but they yield at a distance $O(a_\varepsilon)$ of $\partial\omega$, where v is zero, and then the corresponding inequality is immediately obtained.

For the case *ii*), since the thickness of the domain ε is much smaller that the microscale of the porous medium a_ε, if we proceed as the previous case, we would obtain

$$\|v\|_{L^p(\Omega_\varepsilon)^3} \leq Ca_\varepsilon \|\mathbb{D}[v]\|_{L^p(\Omega_\varepsilon)^{3\times 3}}, \quad \forall v \in W_0^{1,p}(\Omega_\varepsilon)^3.$$

However, we are able to obtain a more optimal estimate. Thus, for any function $v(z', x_3) \in W^{1,p}(Y'_f \times (0,\varepsilon))^3$, $1 < p < +\infty$, with $v = 0$ in $\partial(Y'_s \times (0,\varepsilon))$, the Poincare inequality in $Y'_f \times (0,\varepsilon)$ states that

$$\int_{Y'_f \times (0,\varepsilon)} |v|^p\, dz' dx_3 \leq C\varepsilon^p \int_{Y'_f \times (0,\varepsilon)} |\partial_{x_3} v|^p\, dz' dx_3, \tag{19}$$

where the constant C is independent of ε. For every $k' \in \mathbb{Z}^2$, by the change of variable

$$k' + z' = \frac{x'}{a_\varepsilon}, \quad dz' = \frac{dx'}{a_\varepsilon^2}, \tag{20}$$

we rescale (19) from Y'_f to $Y'_{f_{k'},a_\varepsilon}$. This yields that, for any function $v(x)\in W^{1,p}(Q_{f_{k'},a_\varepsilon})^3$ with $v=0$ in $\partial Q_{s_{k'},a_\varepsilon}$, one has

$$\int_{Q_{f_{k'},a_\varepsilon}} |v|^p\,dx \le C\varepsilon^p \int_{Q_{f_{k'},a_\varepsilon}} |\partial_{x_3} v|^p\,dx \le C\varepsilon^p \int_{Q_{f_{k'},a_\varepsilon}} |D_x v|^p\,dx, \tag{21}$$

with the same constant C as in (19). Summing the inequalities (21), gives

$$\int_{\Omega_\varepsilon} |v|^p\,dx \le C\varepsilon^p \int_{\Omega_\varepsilon} |D_x v|^p\,dx,$$

and by the classical Korn inequality, we have the desired result.

On the other hand, for the case *iii*), we observe that the microscale of the porous medium a_ε is much smaller than the thickness of the domain ε. Thus, if we proceed as the first case, we would obtain

$$\|v\|_{L^p(\Omega_\varepsilon)^3} \le C\varepsilon \|\mathbb{D}[v]\|_{L^p(\Omega_\varepsilon)^{3\times 3}}, \quad \forall v\in W^{1,p}_0(\Omega_\varepsilon)^3.$$

Since $a_\varepsilon \ll \varepsilon$, we are able to reproduce the original proof of Tartar [30] dividing the domain Ω_ε in small cubes of lateral length a_ε and vertical length a_ε. This will lead to a more optimal estimate with a constant Ca_ε.

To do this, for every $k\in\mathbb{Z}^3$, by the change of variable

$$k+z=\frac{x}{a_\varepsilon},\quad dz=\frac{dx}{a_\varepsilon^3},\quad \partial_z = a_\varepsilon \partial_x, \tag{22}$$

we rescale (16) from Y_f to $Q_{f_k,a_\varepsilon}=a_\varepsilon\,k+a_\varepsilon Y_f$. This yields that, for any function $v(x)\in W^{1,p}(Q_{f_k,a_\varepsilon})^3$ with $v=0$ in $\partial Q_{s_k,a_\varepsilon}$ where $Q_{s_k,a_\varepsilon}=a_\varepsilon\,k+a_\varepsilon Y_s$, one has

$$\int_{Q_{f_k,a_\varepsilon}} |v|^p\,dx \quad\le\quad Ca_\varepsilon^p \int_{Q_{f_k,a_\varepsilon}} |\mathbb{D}_x[v]|^p\,dx,$$

with the same constant C as in (16). Summing for all the periods $Q_{k,a_\varepsilon}=a_\varepsilon\,k+a_\varepsilon Y$, gives

$$\int_{\Omega_\varepsilon} |v|^p\,dx \le Ca_\varepsilon^p \int_{\Omega_\varepsilon} |\mathbb{D}_x[v]|^p\,dx.$$

□

Considering the change of variables given in (5), we obtain the following result for the domain $\widetilde\Omega_\varepsilon$.

Lemma 4.2. *There exists a constant C independent of ε, such that, for any function $\tilde v\in W^{1,p}_0(\widetilde\Omega_\varepsilon)^3$, with $1<p<+\infty$, one has*

i) *if $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, then*

$$\|\tilde v\|_{L^p(\widetilde\Omega_\varepsilon)^3} \le C(a_\varepsilon^p+\varepsilon^p)^{\frac1p} \|\mathbb{D}_\varepsilon[\tilde v]\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}},$$

ii) *if $a_\varepsilon\gg\varepsilon$, then*

$$\|\tilde v\|_{L^p(\widetilde\Omega_\varepsilon)^3} \le C\varepsilon \|\mathbb{D}_\varepsilon[\tilde v]\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}},$$

iii) *if $a_\varepsilon\ll\varepsilon$, then*

$$\|\tilde v\|_{L^p(\widetilde\Omega_\varepsilon)^3} \le Ca_\varepsilon \|\mathbb{D}_\varepsilon[\tilde v]\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}}.$$

Remark 4.3. *Observe that, taking into account the relation between a_ε and ε, the estimates given in Lemma 4.2 can be expressed as follows*

i) *if $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, or $a_\varepsilon\ll\varepsilon$, then*

$$\|\tilde v\|_{L^p(\widetilde\Omega_\varepsilon)^3} \le Ca_\varepsilon \|\mathbb{D}_\varepsilon[\tilde v]\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}}, \quad \forall\tilde v\in W^{1,p}_0(\widetilde\Omega_\varepsilon)^3,\quad 1<p<+\infty, \tag{23}$$

ii) *if $a_\varepsilon\gg\varepsilon$, then*

$$\|\tilde v\|_{L^p(\widetilde\Omega_\varepsilon)^3} \le C\varepsilon \|\mathbb{D}_\varepsilon[\tilde v]\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}}, \quad \forall\tilde v\in W^{1,p}_0(\widetilde\Omega_\varepsilon)^3,\quad 1<p<+\infty. \tag{24}$$

Let us obtain some a priori estimates for $\tilde{u}_\varepsilon$.

Lemma 4.4. *There exists a constant C independent of ε, such that if $\tilde{u}_\varepsilon \in W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3$, with $1 < p < +\infty$, is the solution of the problem (6), one has*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$\|\tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3} \le C a_\varepsilon^{\frac{p}{p-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C a_\varepsilon^{\frac{1}{p-1}}, \tag{25}$$

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C a_\varepsilon^{\frac{1}{p-1}}, \tag{26}$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\|\tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3} \le C \varepsilon^{\frac{p}{p-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C \varepsilon^{\frac{1}{p-1}}, \tag{27}$$

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C \varepsilon^{\frac{1}{p-1}}. \tag{28}$$

Proof. Multiplying by $\tilde{u}_\varepsilon$ in the first equation of (6) and integrating over $\widetilde{\Omega}_\varepsilon$, we have

$$\mu \int_{\widetilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^{p-2}\, \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\, dx'dy_3 = \int_{\widetilde{\Omega}_\varepsilon} f \cdot \tilde{u}_\varepsilon\, dx'dy_3. \tag{29}$$

Using Hölder's inequality and the assumption of f, we obtain that

$$\int_{\widetilde{\Omega}_\varepsilon} f \cdot \tilde{u}_\varepsilon\, dx'dy_3 \le C \|\tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3},$$

and by (29), we have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^p_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C \|\tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3}. \tag{30}$$

For the cases $a_\varepsilon \approx \varepsilon$ or $a_\varepsilon \ll \varepsilon$, taking into account (23), we obtain

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times3}} \le C a_\varepsilon^{\frac{1}{p-1}},$$

and, consequently, from classical Korn's inequality we obtain (26). Now, from the previous estimate and (23), we deduce

$$\|\tilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3} \le C a_\varepsilon^{\frac{p}{p-1}}.$$

For the case $a_\varepsilon \gg \varepsilon$, proceeding similarly with (24), we obtain the desired result. □

4.1. The Extension of $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain Ω

In this section, we will extend the solution $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain Ω. It is easy to extend the velocity by zero in $\Omega \setminus \widetilde{\Omega}_\varepsilon$ (this is compatible with its Dirichlet boundary condition on $\partial\widetilde{\Omega}_\varepsilon$). We will use the same notation, $\tilde{u}_\varepsilon$, for the velocity in $\widetilde{\Omega}_\varepsilon$ and its continuation in Ω. It is well known that extension by zero preserves L^p and $W_0^{1,p}$ norms for $1 < p < +\infty$. We note that the extension $\tilde{u}_\varepsilon$ belongs to $W_0^{1,p}(\Omega)^3$.

Now, we give some properties of the restricted operator from $W_0^{1,p}(\Omega)^3$ into $W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3$ preserving divergence-free vectors. First, we extend the technique of Tartar [30], Allaire [1] and Mikelić [21] (see, for instance, [8] for more details) to the case of a thin domain.

Lemma 4.5. *There exists a (restriction) operator R_p^ε acting from $W_0^{1,p}(Q_\varepsilon)^3$ into $W_0^{1,p}(\Omega_\varepsilon)^3$, $1 < p < +\infty$, such that*

1. $R_p^\varepsilon v = v$, *if $v \in W_0^{1,p}(\Omega_\varepsilon)^3$ (elements of $W_0^{1,p}(\Omega_\varepsilon)^3$ are continuated by 0 to Q_ε)*
2. $\operatorname{div}(R_p^\varepsilon v) = 0$ *in Ω_ε, if $\operatorname{div} v = 0$ in Q_ε*
3. *For any $v \in W_0^{1,p}(Q_\varepsilon)^3$ (the constant C is independent of v and ε),*
   i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$\begin{aligned} \left\|R_p^\varepsilon v\right\|_{L^p(\Omega_\varepsilon)^3} &\le C\|v\|_{L^p(Q_\varepsilon)^3} + C a_\varepsilon \|Dv\|_{L^p(Q_\varepsilon)^{3\times3}}, \\ \left\|D R_p^\varepsilon v\right\|_{L^p(\Omega_\varepsilon)^{3\times3}} &\le \frac{C}{a_\varepsilon}\|v\|_{L^p(Q_\varepsilon)^3} + C\|Dv\|_{L^p(Q_\varepsilon)^{3\times3}}, \end{aligned}$$

ii) *if* $a_\varepsilon \gg \varepsilon$, *then*

$$\begin{aligned}\left\|R_p^\varepsilon v\right\|_{L^p(\Omega_\varepsilon)^3} &\leq C\left\|v\right\|_{L^p(Q_\varepsilon)^3} + Ca_\varepsilon\left\|Dv\right\|_{L^p(Q_\varepsilon)^{3\times 3}},\\ \left\|DR_p^\varepsilon v\right\|_{L^p(\Omega_\varepsilon)^{3\times 3}} &\leq \frac{C}{\varepsilon}\left\|v\right\|_{L^p(Q_\varepsilon)^3} + C\frac{a_\varepsilon}{\varepsilon}\left\|Dv\right\|_{L^p(Q_\varepsilon)^{3\times 3}}.\end{aligned}$$

Proof. Let us consider the linear map R_p constructed in Lemma 1.1 in [8] from $W_0^{1,p}(Y)^3$ to $W_0^{1,p}(Y)^3$, $1 < p < +\infty$, such that

$$\left\|R_p v\right\|_{W^{1,p}(Y)^3} \leq C\left\|v\right\|_{W^{1,p}(Y)^3}, \tag{31}$$

and $R_p v$ coincides with v if v is zero on Y_s (i.e., if $v \in W_0^{1,p}(Y_f)^3$) and $\operatorname{div} v = 0$ implies $\operatorname{div} R_p v = 0$. Then, R_p^ε is defined by applying R_p to each $Q_{k',a_\varepsilon} = Y'_{k',a_\varepsilon} \times (0,\varepsilon)$. Consequently, the two first items are satisfied. Finally, we will prove the third item. From (31), by the change of variable (17), as in Lemma 4.1, we have

$$\int_{\Omega_\varepsilon} |R_\varepsilon v|^p\,dx + a_\varepsilon^p\int_{\Omega_\varepsilon} |D_{x'}R_\varepsilon v|^p\,dx + \varepsilon^p\int_{\Omega_\varepsilon} |\partial_{x_3}R_\varepsilon v|^p\,dx \leq C\left(\int_{Q_\varepsilon}|v|^p\,dx + (a_\varepsilon^p+\varepsilon^p)\int_{Q_\varepsilon}|Dv|^p\,dx\right),$$

and 3.-i) for $a_\varepsilon \approx \varepsilon$ and 3.-ii) follow. Similarly, from (31), by the change of variable (22), as in Lemma 4.1, we have

$$\int_{\Omega_\varepsilon} |R_\varepsilon v|^p\,dx + a_\varepsilon^p\int_{\Omega_\varepsilon}|DR_\varepsilon v|^p\,dx \leq C\left(\int_{Q_\varepsilon}|v|^p\,dx + a_\varepsilon^p\int_{Q_\varepsilon}|Dv|^p\,dx\right),$$

and 3.-i) follows for $a_\varepsilon \ll \varepsilon$. □

Then, the following estimates in the fixed domain Ω are available.

Lemma 4.6. *Let us define* $\tilde R_p^\varepsilon(\tilde v) = R_p^\varepsilon(v)$ *for any* $\tilde v \in W_0^{1,p}(\Omega)^3$, $1 < p < +\infty$, *where* $\tilde v(x', y_3) = v(x', \varepsilon y_3)$ *and* R_p^ε *is defined in Lemma* 4.5. *Then, there exists a constant* C, *independent of* $\tilde v$ *and* ε, *such that*

1. $\tilde R_p^\varepsilon \tilde v = \tilde v$, *if* $\tilde v \in W_0^{1,p}(\widetilde\Omega_\varepsilon)^3$ *(elements of* $W_0^{1,p}(\widetilde\Omega_\varepsilon)^3$ *are continuated by* 0 *to* Ω*)*
2. $\operatorname{div}_\varepsilon(\tilde R_p^\varepsilon \tilde v) = 0$ *in* $\widetilde\Omega_\varepsilon$, *if* $\operatorname{div}_\varepsilon \tilde v = 0$ *in* Ω
3. *For any* $\tilde v \in W_0^{1,p}(\Omega)^3$, $1 < p < +\infty$, *(the constant* C *is independent of* $\tilde v$ *and* ε*),*
   i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, *or* $a_\varepsilon \ll \varepsilon$, *then*

$$\begin{aligned}\left\|\tilde R_p^\varepsilon \tilde v\right\|_{L^p(\widetilde\Omega_\varepsilon)^3} &\leq C\left\|\tilde v\right\|_{L^p(\Omega)^3} + Ca_\varepsilon\left\|D_\varepsilon \tilde v\right\|_{L^p(\Omega)^{3\times 3}},\\ \left\|D_\varepsilon \tilde R_p^\varepsilon \tilde v\right\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}} &\leq \frac{C}{a_\varepsilon}\left\|\tilde v\right\|_{L^p(\Omega)^3} + C\left\|D_\varepsilon \tilde v\right\|_{L^p(\Omega)^{3\times 3}},\end{aligned}$$

   ii) *if* $a_\varepsilon \gg \varepsilon$, *then*

$$\begin{aligned}\left\|\tilde R_p^\varepsilon \tilde v\right\|_{L^p(\widetilde\Omega_\varepsilon)^3} &\leq C\left\|\tilde v\right\|_{L^p(\Omega)^3} + Ca_\varepsilon\left\|D_\varepsilon \tilde v\right\|_{L^p(\Omega)^{3\times 3}},\\ \left\|D_\varepsilon \tilde R_p^\varepsilon \tilde v\right\|_{L^p(\widetilde\Omega_\varepsilon)^{3\times 3}} &\leq \frac{C}{\varepsilon}\left\|\tilde v\right\|_{L^p(\Omega)^3} + C\frac{a_\varepsilon}{\varepsilon}\left\|D_\varepsilon \tilde v\right\|_{L^p(\Omega)^{3\times 3}}.\end{aligned}$$

Proof. Considering the change of variables given in (5) and the estimates given in Lemma 4.5, we obtain the desired result. □

In order to extend the pressure to the whole domain Ω, we define a function $F_\varepsilon \in W^{-1,p'}(\Omega)^3$ by the following formula (brackets are for the duality products between $W^{-1,p'}$ and $W_0^{1,p}$):

$$\langle F_\varepsilon, \tilde v\rangle_\Omega = \left\langle \nabla_\varepsilon \tilde p_\varepsilon, \tilde R_p^\varepsilon \tilde v\right\rangle_{\widetilde\Omega_\varepsilon}, \text{ for any } \tilde v \in W_0^{1,p}(\Omega)^3, \tag{32}$$

where $\tilde R_p^\varepsilon$ is defined in Lemma 4.6. We calcul the right hand side of (32) by using (6) and we have

$$\langle F_\varepsilon, \tilde v\rangle_\Omega = \left\langle \operatorname{div}_\varepsilon\left(\mu\left|\mathbb{D}_\varepsilon\left[\tilde u_\varepsilon\right]\right|^{p-2}\mathbb{D}_\varepsilon\left[\tilde u_\varepsilon\right]\right), \tilde R_p^\varepsilon \tilde v\right\rangle_{\widetilde\Omega_\varepsilon} + \left\langle f, \tilde R_p^\varepsilon \tilde v\right\rangle_{\widetilde\Omega_\varepsilon}, \tag{33}$$

and by using Lemma 4.6, for fixed ε we see that it is a bounded functional on $W_0^{1,p}(\Omega)^3$ (see the proof of Lemma 4.7 below), and in fact $F_\varepsilon \in W^{-1,p'}(\Omega)^3$.

Moreover, if $\tilde{v} \in W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3$ and we continue it by zero out of $\widetilde{\Omega}_\varepsilon$, we see from (32) and the first point in Lemma 4.6 that $F_\varepsilon|_{\widetilde{\Omega}_\varepsilon} = \nabla_\varepsilon \tilde{p}_\varepsilon$.

Moreover, if $\text{div}_\varepsilon v = 0$ by the second point in Lemma 4.6 and (32), $\langle F_\varepsilon, \tilde{v}\rangle_\Omega = 0$ and this implies that F_ε is the gradient of some function $\tilde{P}_\varepsilon$ in $L^{p'}(\Omega)$. This means that F_ε is a continuation of $\nabla_\varepsilon \tilde{p}_\varepsilon$ to Ω, and that this continuation is a gradient. We also may say that $\tilde{p}_\varepsilon$ has been continuated to Ω and

$$F_\varepsilon \equiv \nabla_\varepsilon \tilde{P}_\varepsilon, \qquad \tilde{P}_\varepsilon \in L^{p'}(\Omega)/\mathbb{R}.$$

Lemma 4.7. *There exists a constant C independent of ε, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W_0^{1,p}(\Omega)^3 \times L^{p'}(\Omega)/\mathbb{R}$, $1 < p < +\infty$, of the solution $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ satisfies*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \le C a_\varepsilon^{\frac{p}{p-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^p(\Omega)^{3\times 3}} \le C a_\varepsilon^{\frac{1}{p-1}}, \tag{34}$$

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^p(\Omega)^{3\times 3}} \le C a_\varepsilon^{\frac{1}{p-1}}, \tag{35}$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \le C \varepsilon^{\frac{p}{p-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^p(\Omega)^{3\times 3}} \le C \varepsilon^{\frac{1}{p-1}}, \tag{36}$$

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^p(\Omega)^{3\times 3}} \le C \varepsilon^{\frac{1}{p-1}}, \tag{37}$$

and, moreover, in every cases,

$$\left\|\tilde{P}_\varepsilon\right\|_{L^{p'}(\Omega)/\mathbb{R}} \le C. \tag{38}$$

Proof. Taking into account Lemma 4.4, we have, after extension, the estimates of the velocity.

Let us estimate $\nabla_\varepsilon \tilde{P}_\varepsilon$ in the cases $a_\varepsilon \approx \varepsilon$ or $a_\varepsilon \ll \varepsilon$. We estimate the right side of (33). Using Hölder's inequality and from (25) we have

$$\begin{aligned}\left|\left\langle \text{div}_\varepsilon\left(\mu\,|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^{p-2}\,\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\right), \tilde{R}_p^\varepsilon \tilde{v}\right\rangle_{\widetilde{\Omega}_\varepsilon}\right| &\le \mu\,\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^{p-1}_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \left\|D_\varepsilon \tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \\ &\le C a_\varepsilon \left\|D_\varepsilon \tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}}.\end{aligned}$$

Using the assumption of f, we obtain

$$\left|\left\langle f, \tilde{R}_p^\varepsilon \tilde{v}\right\rangle_{\widetilde{\Omega}_\varepsilon}\right| \le C \left\|\tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^3}.$$

Then, from (33), we deduce

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \le C a_\varepsilon \left\|D_\varepsilon \tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C \left\|\tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^3}.$$

Taking into account 3.-i) in Lemma 4.6, we have

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \le C a_\varepsilon \left(\frac{1}{a_\varepsilon}\|\tilde{v}\|_{L^p(\Omega)^3} + \|D_\varepsilon \tilde{v}\|_{L^p(\Omega)^{3\times 3}}\right) + C\left(\|\tilde{v}\|_{L^p(\Omega)^3} + a_\varepsilon \|D_\varepsilon \tilde{v}\|_{L^p(\Omega)^{3\times 3}}\right).$$

Then, as $a_\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \le C\,\|\tilde{v}\|_{W_0^{1,p}(\Omega)^3}, \tag{39}$$

for any $\tilde{v} \in W_0^{1,p}(\Omega)^3$.

Now, we consider

$$\tilde{g} = \left|\tilde{P}_\varepsilon\right|^{p'-2} \tilde{P}_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \left|\tilde{P}_\varepsilon\right|^{p'-2} \tilde{P}_\varepsilon\, dx' dy_3,$$

where $\tilde{g} \in L^p(\Omega)$, due to $\tilde{P}_\varepsilon \in L^{p'}(\Omega)$, and $\int_\Omega \tilde{g}\, dx'dy_3 = 0$. We define $\tilde{v} = \mathcal{B}[\tilde{g}]$, where $\mathcal{B}$ is the Bogovskii operator associated to Ω. By Theorem 3.1 in Chapter III.3 in [16], we obtain that $\mathrm{div}_\varepsilon \tilde{v} = \tilde{g}$, $\tilde{v} \in W_0^{1,p}(\Omega)^3$ and there exists a positive constant C such that

$$\|\tilde{v}\|_{W_0^{1,p}(\Omega)^3} \le C\, \|\tilde{g}\|_{L^p(\Omega)}\,. \tag{40}$$

It is easy to prove that there exists a positive constant C such that

$$\|\tilde{g}\|_{L^p(\Omega)} \le C \left\|\tilde{P}_\varepsilon\right\|_{L^{p'}(\Omega)}^{p'-1}, \tag{41}$$

and

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| = \left\|\tilde{P}_\varepsilon\right\|_{L^{p'}(\Omega)}^{p'},$$

which, together with (39) and (40), give the estimate (38).

Finally, let us estimate $\nabla_\varepsilon \tilde{P}_\varepsilon$ in the case $a_\varepsilon \gg \varepsilon$. Similarly to the previous case, we estimate the right side of (33) by using Hölder's inequality and from (27), and we have

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \le C\varepsilon \left\|D_\varepsilon \tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C\left\|\tilde{R}_p^\varepsilon \tilde{v}\right\|_{L^p(\widetilde{\Omega}_\varepsilon)^3}.$$

Taking into account 3.-ii) in Lemma 4.6, we have

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \;\le\; C\varepsilon\left(\frac{1}{\varepsilon}\,\|\tilde{v}\|_{L^p(\Omega)^3} + \frac{a_\varepsilon}{\varepsilon}\,\|D_\varepsilon \tilde{v}\|_{L^p(\Omega)^{3\times 3}}\right) + C\left(\|\tilde{v}\|_{L^p(\Omega)^3} + a_\varepsilon\,\|D_\varepsilon \tilde{v}\|_{L^p(\Omega)^{3\times 3}}\right).$$

Then, as $a_\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\right\rangle_\Omega\right| \;\le\; C\,\|\tilde{v}\|_{W_0^{1,p}(\Omega)^3},$$

for any $\tilde{v} \in W_0^{1,p}(\Omega)^3$. Using the Bogovskii operator as above, we have the estimate (38). □

4.2. Adaptation of the Unfolding Method

The change of variable (5) does not provide the information we need about the behavior of $\tilde{u}_\varepsilon$ in the microstructure associated to $\widetilde{\Omega}_\varepsilon$. To solve this difficulty, we introduce an adaptation of the unfolding method (see [3, 10, 11]), which is strongly related to the two-scale convergence method (see [2, 24]). Let us introduce the adaption of the unfolding method in which we divide the domain Ω in cubes of lateral length a_ε and vertical length 1. For this purpose, given $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W_0^{1,p}(\Omega)^3 \times L^{p'}(\Omega)/\mathbb{R}$, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by

$$\hat{u}_\varepsilon(x', y) = \tilde{u}_\varepsilon\left(a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e. } (x', y) \in \omega \times Y, \tag{42}$$

$$\hat{P}_\varepsilon(x', y) = \tilde{P}_\varepsilon\left(a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e. } (x', y) \in \omega \times Y, \tag{43}$$

where the function κ is defined in (1).

Remark 4.8. *For $k' \in T_\varepsilon$, the restriction of $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ to $Y'_{k',a_\varepsilon} \times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ by using the change of variables*

$$y' = \frac{x' - a_\varepsilon k'}{a_\varepsilon}, \tag{44}$$

which transforms Y_{k',a_ε} into Y.

Let us obtain some estimates for the sequences $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$.

Lemma 4.9. *There exists a constant C independent of ε, such that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (42)-(43) satisfies*

i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, *or* $a_\varepsilon \ll \varepsilon$,

$$\|\mathbb{D}_{y'}[\hat{u}_\varepsilon]\|_{L^p(\omega\times Y)^{3\times 2}} \le C a_\varepsilon^{\frac{p}{p-1}}, \quad \|\partial_{y_3}[\hat{u}_\varepsilon]\|_{L^p(\omega\times Y)^3} \le C\varepsilon\, a_\varepsilon^{\frac{1}{p-1}}, \tag{45}$$

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^{3\times 2}} \le C a_\varepsilon^{\frac{p}{p-1}}, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \le C\varepsilon\, a_\varepsilon^{\frac{1}{p-1}}, \tag{46}$$

$$\|\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \le C a_\varepsilon^{\frac{p}{p-1}}, \tag{47}$$

ii) *if* $a_\varepsilon \gg \varepsilon$,

$$\|\mathbb{D}_{y'}[\hat{u}_\varepsilon]\|_{L^p(\omega\times Y)^{3\times 2}} \le C a_\varepsilon\, \varepsilon^{\frac{1}{p-1}}, \quad \|\partial_{y_3}[\hat{u}_\varepsilon]\|_{L^p(\omega\times Y)^3} \le C\varepsilon^{\frac{p}{p-1}}, \tag{48}$$

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^{3\times 2}} \le C a_\varepsilon\, \varepsilon^{\frac{1}{p-1}}, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \le C\varepsilon^{\frac{p}{p-1}}, \tag{49}$$

$$\|\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \le C\varepsilon^{\frac{p}{p-1}}, \tag{50}$$

and, moreover, in every cases,

$$\left\|\hat{P}_\varepsilon\right\|_{L^{p'}(\omega\times Y)/\mathbb{R}} \le C. \tag{51}$$

Proof. Let us obtain some estimates for the sequence $\hat{u}_\varepsilon$ defined by (42). Taking into account the definition (42) of $\hat{u}_\varepsilon$, we obtain

$$\begin{aligned}\int_{\omega\times Y} |\mathbb{D}_{y'}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy &\le \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y |\mathbb{D}_{y'}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy\\ &= \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y |\mathbb{D}_{y'}[\tilde{u}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)]|^p\,dx'dy.\end{aligned}$$

We observe that $\tilde{u}_\varepsilon$ does not depend on x', then we can deduce

$$\int_{\omega\times Y} |\mathbb{D}_{y'}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy \le a_\varepsilon^p\sum_{k'\in T_\varepsilon}\int_Y |\mathbb{D}_{y'}[\tilde{u}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)]|^p\,dy.$$

By the change of variables (44), we obtain

$$\begin{aligned}\int_{\omega\times Y} |\mathbb{D}_{y'}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy &\le a_\varepsilon^p\sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_0^1 |\mathbb{D}_{x'}[\tilde{u}_\varepsilon(x',y_3)]|^p\,dx'dy_3\\ &\le a_\varepsilon^p\int_{\omega\times(0,1)} |\mathbb{D}_{x'}[\tilde{u}_\varepsilon(x',y_3)]|^p\,dx'dy_3.\end{aligned}$$

Taking into account the second estimates in (34) and (36), we get the first estimates in (45) and (48) respectively.

Similarly, using Remark 4.8 and definition (42), we have

$$\int_{\omega\times Y} |\partial_{y_3}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy \le a_\varepsilon^2\sum_{k'\in T_\varepsilon}\int_Y |\partial_{y_3}[\tilde{u}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)]|^p\,dy.$$

By the change of variables (44) and the second estimate in (34), in the cases $a_\varepsilon\approx\varepsilon$ or $a_\varepsilon\ll\varepsilon$, we obtain

$$\int_{\omega\times Y} |\partial_{y_3}[\hat{u}_\varepsilon(x',y)]|^p\,dx'dy \le \int_{\omega\times(0,1)} |\partial_{y_3}[\tilde{u}_\varepsilon(x',y_3)]|^p\,dx'dy_3 \le C\varepsilon^p a_\varepsilon^{\frac{p}{p-1}},$$

so (45) is proved. Consequently, from classical Korn's inequality, we also have (46). Analogously, using the second estimate in (36), we get the second estimate in (48) and, from classical Korn's inequality, we have (49).

Similarly, using the definition (42), the change of variables (44) and the first estimate in (34), in the cases $a_\varepsilon\approx\varepsilon$ or $a_\varepsilon\ll\varepsilon$, we have

$$\int_{\omega\times Y} |\hat{u}_\varepsilon(x',y)|^p\,dx'dy \le C a_\varepsilon^{\frac{p^2}{p-1}},$$

and (47) holds. Analogously, using the first estimate in (36), we get (50).

Finally, let us obtain some estimates for the sequence $\hat{P}_\varepsilon$ defined by (43). We observe that using the definition (43) of $\hat{P}_\varepsilon$, we obtain

$$\int_{\omega\times Y}\left|\hat{P}_\varepsilon(x',y)\right|^{p'}dx'dy\leq\sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y\left|\tilde{P}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)\right|^{p'}dx'dy.$$

We observe that $\tilde{P}_\varepsilon$ does not depend on x', then we can deduce

$$\int_{\omega\times Y}\left|\hat{P}_\varepsilon(x',y)\right|^{p'}dx'dy\leq a_\varepsilon^2\sum_{k'\in T_\varepsilon}\int_Y\left|\tilde{P}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)\right|^{p'}dy.$$

By the change of variables (44), we obtain

$$\int_{\omega\times Y}\left|\hat{P}_\varepsilon(x',y)\right|^{p'}dx'dy\leq\int_{\omega\times(0,1)}\left|\tilde{P}_\varepsilon(x',y_3)\right|^{p'}dx'dy_3.$$

Taking into account (38), we have (51). □

5. Some compactness results

In this section we obtain some compactness results about the behavior of the sequences $(\tilde{u}_\varepsilon,\tilde{P}_\varepsilon)$ and $(\hat{u}_\varepsilon,\hat{P}_\varepsilon)$ satisfying a priori estimates given in Lemma 4.7 and Lemma 4.9 respectively. We obtain different behaviors depending on the magnitude a_ε with respect to ε.

Let us start giving a convergence result for the pressure $\tilde{P}_\varepsilon$.

Lemma 5.1. *For a subsequence of ε still denote by ε there exists $\tilde{P}\in L^{p'}(\Omega)/\mathbb{R}$ such that*

$$\tilde{P}_\varepsilon\rightharpoonup\tilde{P}\ \text{ in }\ L^{p'}(\Omega)/\mathbb{R}. \tag{52}$$

Proof. Observe that estimate (38) implies, up to a subsequence, the existence of $\tilde{P}\in L^{p'}(\Omega)/\mathbb{R}$ such that (52) holds. □

We will give a convergence result for $\tilde{u}_\varepsilon$.

Lemma 5.2. *For a subsequence of ε still denote by ε,*

i) *if $a_\varepsilon\approx\varepsilon$ with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, then there exists $\tilde{u}\in W^{1,p}(0,1;L^p(\omega)^3)$ where $\tilde{u}_3=0$, and $\tilde{u}=0$ on $\partial\Omega$, such that*

$$a_\varepsilon^{-\frac{p}{p-1}}\tilde{u}_\varepsilon\rightharpoonup(\tilde{u}',0)\ \text{ in }\ W^{1,p}(0,1;L^p(\omega)^3), \tag{53}$$

ii) *if $a_\varepsilon\gg\varepsilon$, then there exists $\tilde{u}\in W^{1,p}(0,1;L^p(\omega)^3)$ where $\tilde{u}_3=0$, and $\tilde{u}=0$ on $\partial\Omega$, such that*

$$\varepsilon^{-\frac{p}{p-1}}\tilde{u}_\varepsilon\rightharpoonup(\tilde{u}',0)\ \text{ in }\ W^{1,p}(0,1;L^p(\omega)^3), \tag{54}$$

iii) *if $a_\varepsilon\ll\varepsilon$, then there exists $\tilde{u}\in L^p(\Omega)^3$ where $\tilde{u}_3=0$, and $\tilde{u}=0$ on $\partial\Omega$, such that*

$$a_\varepsilon^{-\frac{p}{p-1}}\tilde{u}_\varepsilon\rightharpoonup(\tilde{u}',0)\ \text{ in }\ L^p(\Omega)^3. \tag{55}$$

Moreover, in every cases

$$\mathrm{div}_{x'}\left(\int_0^1\tilde{u}'(x',y_3)dy_3\right)=0\ \text{ in }\ \omega,\quad\left(\int_0^1\tilde{u}'(x',y_3)dy_3\right)\cdot n=0\ \text{ on }\ \partial\omega. \tag{56}$$

Proof. We proceed in four steps.

Step 1. Critical case $a_\varepsilon\approx\varepsilon$. In this case, the estimates (34)-(35) read

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)^3}\leq Ca_\varepsilon^{\frac{p}{p-1}},\quad\|D_{x'}\tilde{u}_\varepsilon\|_{L^p(\Omega)^{3\times 2}}\leq Ca_\varepsilon^{\frac{1}{p-1}},\quad\|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^p(\Omega)^3}\leq Ca_\varepsilon^{\frac{p}{p-1}}. \tag{57}$$

The above estimates imply the existence $\tilde{u} \in W^{1,p}(0,1;L^p(\omega)^3)$, such that, up to a subsequence, we have

$$a_\varepsilon^{-\frac{p}{p-1}}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \text{ in } W^{1,p}(0,1;L^p(\omega)^3), \tag{58}$$

which implies

$$a_\varepsilon^{-\frac{p}{p-1}}\mathrm{div}_{x'}\tilde{u}'_\varepsilon \rightharpoonup \mathrm{div}_{x'}\tilde{u}' \text{ in } W^{1,p}(0,1;W^{-1,p'}(\omega)). \tag{59}$$

Since $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in Ω, multiplying by $a_\varepsilon^{-\frac{p}{p-1}}$ we obtain

$$a_\varepsilon^{-\frac{p}{p-1}}\mathrm{div}_{x'}\tilde{u}'_\varepsilon + \frac{a_\varepsilon}{\varepsilon}a_\varepsilon^{-\frac{2p-1}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3} = 0, \quad \text{ in } \Omega, \tag{60}$$

which, combined with (59) and $a_\varepsilon/\varepsilon \to \lambda$, implies that $a_\varepsilon^{-\frac{2p-1}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ is bounded in $W^{1,p}(0,1;W^{-1,p'}(\omega))$. This implies that $a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ tends to zero in $W^{1,p}(0,1;W^{-1,p'}(\omega))$. Also, from (58), we have that $a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ tends to $\partial_{y_3}\tilde{u}_3$ in $L^p(\Omega)$. From the uniqueness of the limit, we have that $\partial_{y_3}\tilde{u}_3 = 0$, which implies that $\tilde{u}_3$ does not depend on y_3.

It remains to prove that $\tilde{u}_3 = 0$. In order to do that, let us first show that $\tilde{P}$ only depends on x'. As usual, we take a test function $v = (0, \varepsilon v_3)$ in the momentum equation in (6). From convergences (52) and (53), we deduce that $\partial_{y_3}\tilde{P} = 0$, which implies that $\tilde{P}$ only depends on x'. Next, as $\tilde{u}_3$ does not depend on y_3, we take a test function $v = (0, a_\varepsilon^{-\frac{p}{p-1}}v_3(x'))$ in (6), and passing to the limit, using monotonicity arguments, we can deduce that $\tilde{u}_3 = 0$.

Step 2. Supercritical case $a_\varepsilon \gg \varepsilon$. In this case, the estimates (34)-(35) read

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \leq C\varepsilon^{\frac{p}{p-1}}, \quad \|D_{x'}\tilde{u}_\varepsilon\|_{L^p(\Omega)^{3\times 2}} \leq C\varepsilon^{\frac{1}{p-1}}, \quad \|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \leq C\varepsilon^{\frac{p}{p-1}}. \tag{61}$$

The proof is similar to the Step 1 by taking ε instead of a_ε, so we omit it.

Step 3. Subcritical case $a_\varepsilon \ll \varepsilon$. In this case, the estimates (34)-(35) read

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \leq Ca_\varepsilon^{\frac{p}{p-1}}, \quad \|D_{x'}\tilde{u}_\varepsilon\|_{L^p(\Omega)^{3\times 2}} \leq Ca_\varepsilon^{\frac{1}{p-1}}, \quad \|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^p(\Omega)^3} \leq C\varepsilon\, a_\varepsilon^{\frac{1}{p-1}}. \tag{62}$$

The first estimate of (62) implies the existence $\tilde{u} \in L^p(\Omega)^3$, such that, up to a subsequence, convergence (55) holds. On the other hand, (60) combined with (55) implies that $\varepsilon^{-1}a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ is bounded in $L^p(0,1;W^{-1,p'}(\omega))$. This implies that $a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ tends to zero in $L^p(0,1;W^{-1,p'}(\omega))$. Also, from (55), we have that $a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\tilde{u}_{\varepsilon,3}$ tends to $\partial_{y_3}\tilde{u}_3$ in $W^{-1,p'}(0,1;L^p(\omega))$. From the uniqueness of the limit, we have that $\partial_{y_3}\tilde{u}_3 = 0$, which implies that $\tilde{u}_3$ does not depend on y_3. Finally, reasoning similarly as the step 1, we deduce that $\tilde{u}_3 = 0$.

Step 4. In this step we prove (56). To do this, we consider $v \in C^1_c(\omega)$ as test function in $\mathrm{div}_\varepsilon\tilde{u}_\varepsilon = 0$ in Ω, which gives

$$\int_\Omega \mathrm{div}_{x'}\tilde{u}'_\varepsilon\, v(x')\, dx'dy_3 = 0.$$

In the cases $a_\varepsilon \approx \varepsilon$ and $a_\varepsilon \ll \varepsilon$, multiplying by $a_\varepsilon^{-\frac{p}{p-1}}$ and from convergences (53) and (55) we get (56). Finally, in the case $a_\varepsilon \gg \varepsilon$, multiplying by $\varepsilon^{-\frac{p}{p-1}}$ and from convergence (54), we get (56). □

Now, we give a convergence result for the pressure $\hat{P}_\varepsilon$.

Lemma 5.3. *For a subsequence of ε still denote by ε there exists $\hat{P} \in L^{p'}(\omega \times Y)/\mathbb{R}$ such that*

$$\hat{P}_\varepsilon \rightharpoonup \hat{P} \text{ in } L^{p'}(\omega \times Y)/\mathbb{R}. \tag{63}$$

Proof. The estimate (51) implies the existence $\hat{P}:\omega\times Y\to\mathbb{R}$ such that (63) holds. By semicontinuity and the previous estimate of $\hat{P}_\varepsilon$, we have

$$\int_{\omega\times Y}\left|\hat{P}(x',y)\right|^{p'}dx'dy\le C,$$

which shows that $\hat{P}$ belongs to $L^{p'}(\omega\times Y)$. □

Next, we give a convergence result for $\hat{u}_\varepsilon$.

Lemma 5.4. *For a subsequence of ε still denote by ε,*

i) *if $a_\varepsilon\approx\varepsilon$ with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, then there exists $\hat{u}\in L^p(\omega;W^{1,p}_\sharp(Y)^3)$, with $\hat{u}=0$ on $\omega\times Y_s$, such that*

$$a_\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon\rightharpoonup\hat{u}\ \text{ in }\ L^p(\omega;W^{1,p}(Y)^3),\tag{64}$$

$$\operatorname{div}_\lambda\hat{u}=0\ \text{ in }\ \omega\times Y,\tag{65}$$

where $\operatorname{div}_\lambda=\operatorname{div}_{y'}+\lambda\partial_{y_3}$,

ii) *if $a_\varepsilon\gg\varepsilon$, then there exists $\hat{u}\in W^{1,p}(0,1;L^p_\sharp(\omega\times Y')^3)$, with $\hat{u}=0$ on $\omega\times Y_s$ and $\hat{u}_3$ independent of y_3, such that*

$$\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon\rightharpoonup\hat{u}\ \text{ in }\ W^{1,p}(0,1;L^p(\omega\times Y')^3)\,,\tag{66}$$

$$\operatorname{div}_{y'}\left(\int_0^1\hat{u}'(x',y)dy_3\right)=0\ \text{ in }\ \omega\times Y'\quad\text{ and }\quad\operatorname{div}_{y'}\hat{u}'=0\ \text{ in }\ \omega\times Y,\tag{67}$$

iii) *if $a_\varepsilon\ll\varepsilon$, then there exists $\hat{u}\in L^p(\Omega;W^{1,p}_\sharp(Y')^3)$, with $\hat{u}=0$ on $\omega\times Y_s$ and $\hat{u}_3$ independent of y_3, such that*

$$a_\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon\rightharpoonup\hat{u}\ \text{ in }\ L^p(\Omega;W^{1,p}(Y')^3),\tag{68}$$

$$\operatorname{div}_{y'}\hat{u}'=0\ \text{ in }\ \omega\times Y.\tag{69}$$

Moreover, in every cases

$$\operatorname{div}_{x'}\left(\int_Y\hat{u}'(x',y)dy\right)=0\ \text{ in }\ \omega,\quad\left(\int_Y\hat{u}'(x',y)dy\right)\cdot n=0\ \text{ on }\ \partial\omega.\tag{70}$$

Proof. We proceed in four steps.
Step 1. Critical case $a_\varepsilon\approx\varepsilon$. In this case, the estimates (46)-(47) read

$$\|\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3}\le Ca_\varepsilon^{\frac{p}{p-1}},\quad\|D_y\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^{3\times3}}\le Ca_\varepsilon^{\frac{p}{p-1}}.\tag{71}$$

Taking into account the Dirichlet condition, the above estimates imply the existence $\hat{u}:\omega\times Y\to\mathbb{R}^3$, such that, up to a subsequence, convergences (64) holds. By semicontinuity and the estimates given in (71), we have

$$\int_{\omega\times Y}|\hat{u}|^p\,dx'dy\le C,\quad\int_{\omega\times Y}|D_y\hat{u}|^p\,dx'dy\le C,$$

which shows that $\hat{u}\in L^p(\omega;W^{1,p}(Y)^3)$.

It remains to prove the Y'-periodicity of $\hat{u}$ in y'. To do this, we observe that by definition of $\hat{u}_\varepsilon$ given by (42), we have

$$\hat{u}_\varepsilon(x_1+\varepsilon,x_2,-1/2,y_2,y_3)=\hat{u}_\varepsilon(x',1/2,y_2,y_3)\text{ a.e. }(x',y_2,y_3)\in\omega\times(-1/2,1/2)\times(0,1)\,,$$

which, dividing by $\varepsilon^{\frac{p}{p-1}}$ and taking into account convergence (64), gives

$$\hat{u}(x',-1/2,y_2,y_3)=\hat{u}(x',1/2,y_2,y_3)\text{ a.e. }(x',y_2,y_3)\in\omega\times(-1/2,1/2)\times(0,1)\,.$$

Analogously, we can prove

$$\hat{u}(x',y_1,-1/2,y_3)=\hat{u}(x',y_1,1/2,y_3)\text{ a.e. }(x',y_1,y_3)\in\omega\times(-1/2,1/2)\times(0,1)\,,$$

These equalities prove the periodicity of $\hat{u}$.

Since $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in Ω, then by definition of $\hat{u}_\varepsilon$ we have $a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$. Multiplying by $a_\varepsilon^{-\frac{1}{p-1}}$ we obtain

$$a_\varepsilon^{-\frac{p}{p-1}}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \frac{a_\varepsilon}{\varepsilon} a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0, \quad \text{ in } \omega\times Y, \tag{72}$$

which, combined with (64) and $a_\varepsilon/\varepsilon \to \lambda$, proves (65).

Step 2. Supercritical case $a_\varepsilon \gg \varepsilon$. In this case, the estimates (49)-(50) read

$$\|\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \leq C\varepsilon^{\frac{p}{p-1}}, \quad \|D_{y'}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^{3\times 2}} \leq C\varepsilon^{\frac{1}{p-1}} a_\varepsilon, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \leq C\varepsilon^{\frac{p}{p-1}}. \tag{73}$$

Therefore from the first and third estimates, up to a subsequence and using a semicontinuity argument, there exists $\hat{u}\in W^{1,p}(0,1;L^p(\omega\times Y')^3)$ such that

$$\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } \ W^{1,p}(0,1;L^p(\omega\times Y')^3)\,. \tag{74}$$

And we can deduce that $\varepsilon^{-\frac{1}{p-1}}a_\varepsilon^{-1}\hat{u}_\varepsilon$ tends to zero in $W^{1,p}(0,1;L^p(\omega\times Y')^3)$, which implies that $\varepsilon^{-\frac{1}{p-1}}a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}_\varepsilon$ tends to zero. Since $\mathrm{div}_\varepsilon\tilde{u}_\varepsilon = 0$ in Ω, then by definition of $\hat{u}_\varepsilon$ we have $a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$. Multiplying by $\varepsilon^{-\frac{1}{p-1}}$ we obtain

$$\varepsilon^{-\frac{1}{p-1}}a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0, \quad \text{ in } \omega\times Y,$$

which, combined with (74), proves that $\partial_{y_3}\hat{u}_3 = 0$.

Now, we prove (67). To do this, we consider $v\in C_c^1(\omega\times Y')$ as test function in $a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$, which gives

$$\int_{\omega\times Y}\mathrm{div}_{y'}\hat{u}'_\varepsilon\, v(x',y')\,dx'dy = 0.$$

Multiplying by $\varepsilon^{-\frac{p}{p-1}}$ and from convergences (74) we get (67).

In order to proof the Y'-periodicity of $\hat{u}$ in y', we proceed similarly to the step 1.

Step 3. Subcritical case $a_\varepsilon \ll \varepsilon$. In this case, the estimates (46)-(47) read

$$\|\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \leq Ca_\varepsilon^{\frac{p}{p-1}}, \quad \|D_{y'}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^{3\times 2}} \leq Ca_\varepsilon^{\frac{p}{p-1}}, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^p(\omega\times Y)^3} \leq C\varepsilon a_\varepsilon^{\frac{1}{p-1}}. \tag{75}$$

Therefore from the two first estimates in (75), up to a subsequence and using a semicontinuity argument, there exists $\hat{u}\in L^p(\Omega;W^{1,p}(Y')^3)$ satisfying (68). And we can deduce that $\varepsilon^{-1}a_\varepsilon^{-\frac{1}{p-1}}\hat{u}_\varepsilon$ is bounded in $L^p(\Omega;W^{1,p}(Y')^3)$ and tends to zero. This together the third estimate in (75) implies that $\varepsilon^{-1}a_\varepsilon^{-\frac{1}{p-1}}\hat{u}_\varepsilon$ is bounded in $L^p(\omega;W^{1,p}(Y)^3)$ and tends to zero. Passing to the limit in (72), using (68), we obtain (69).

Now, we prove that $\hat{u}_3$ is independent of y_3. To do this, we consider $v\in C_c^1(\Omega)$ as test function in $a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$, which gives

$$\int_{\omega\times Y}\partial_{y_3}\hat{u}_{\varepsilon,3}\, v(x',y_3)\,dx'dy = 0.$$

Multiplying by $a_\varepsilon^{-\frac{p}{p-1}}$ and from convergence (68), we get $\hat{u}_3$ is independent of y_3.

In order to proof the Y'-periodicity of $\hat{u}$ in y', we proceed similarly to the step 1.

Step 4. In order to prove (70), let us first prove the following relation between $\tilde{u}$ and $\hat{u}$,

$$\int_Y \hat{u}(x',y)dy = \int_0^1 \tilde{u}(x',y_3)dy_3. \tag{76}$$

For this, let us consider $v\in C_c^1(\omega)$. We observe that using the definition (42) of $\hat{u}_\varepsilon$, we obtain

$$\int_\omega\int_Y \hat{u}_\varepsilon(x',y)v(x')dydx' = \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y \tilde{u}_\varepsilon(a_\varepsilon k' + a_\varepsilon y', y_3)\, v(a_\varepsilon k' + a_\varepsilon y')dydx' + O_\varepsilon.$$

We observe that $\tilde{u}_\varepsilon$ and v do not depend on x', then we can deduce

$$\int_\omega\int_Y \hat{u}_\varepsilon(x',y)v(x')dydx' = a_\varepsilon^2\sum_{k'\in T_\varepsilon}\int_{Y'}\int_0^1 \tilde{u}_\varepsilon(a_\varepsilon k'+a_\varepsilon y',y_3)\,v(a_\varepsilon k'+a_\varepsilon y')dy_3dy' + O_\varepsilon.$$

By the change of variables (44), we obtain

$$\begin{aligned}\int_\omega\int_Y \hat{u}_\varepsilon(x',y)v(x')dydx' &= \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_0^1 \tilde{u}_\varepsilon(x',y_3)\,v(x')dy_3dx'\\ &= \int_\omega\int_0^1 \tilde{u}_\varepsilon(x',y_3)\,v(x')dy_3dx' + O_\varepsilon.\end{aligned}$$

Multiplying by $a_\varepsilon^{-\frac{p}{p-1}}$ and taking into account the convergences (53) and (64) for the critical case, (55) and (68) for the subcritical case, we obtain (70) thanks to (56). Finally, multiplying by $\varepsilon^{-\frac{p}{p-1}}$ and taking into account the convergences (54) and (66) for the supercritical case, we obtain (70) thanks to (56). □

6. Homogenized models

In this section, we will multiply system (6) by a test function having the form of the limit $\hat{u}$ (as explicated in Lemma 5.4), and we will use the convergences given in the previous section in order to identify the homogenized model in every cases.

Theorem 6.1. *We distingue three cases:*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, then $(a_\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}(x',y), \tilde{P}(x'))$ in $L^p(\omega;W^{1,p}(Y)^3)\times L^{p'}(\omega)/\mathbb{R}$, with $\int_Y \hat{u}_3\,dy = 0$, of the homogenized problem*

$$\left\{\begin{array}{rcl} -\mu\,\mathrm{div}_\lambda\left(S\left(\mathbb{D}_\lambda\left[\hat{u}\right]\right)\right)+\nabla_\lambda\hat{q} &=& f'-\nabla_{x'}\tilde{P}\quad in\ \ \omega\times Y_f,\\ \mathrm{div}_\lambda\hat{u} &=& 0\quad in\ \ \omega\times Y_f,\\ \hat{u} &=& 0\ in\ \ \omega\times Y_s\\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x',y)dy\right) &=& 0\ in\ \ \omega,\\ \left(\displaystyle\int_Y \hat{u}'(x',y)dy\right)\cdot n &=& 0\ on\ \ \partial\omega,\\ y'\to\hat{u}(x',y),\hat{q}(x',y)\quad Y'-periodic, && \end{array}\right. \tag{77}$$

where $\mathbb{D}_\lambda\left[\cdot\right] = \mathbb{D}_{y'}\left[\cdot\right]+\lambda\partial_{y_3}\left[\cdot\right]$, $\nabla_\lambda = \nabla_{y'}+\lambda\partial_{y_3}$ and $\mathrm{div}_\lambda = \nabla_{x'}+\lambda\partial_{y_3}$.

ii) *if $a_\varepsilon \gg \varepsilon$, then $(\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}(x',y), \tilde{P}(x'))$ in $W^{1,p}(0,1;L^p(\omega\times Y')^3)\times L^{p'}(\omega)/\mathbb{R}$, with $\int_Y \hat{u}_3\,dy = 0$ and $\hat{u}_3$ independent of y_3, of the homogenized problem*

$$\left\{\begin{array}{rcl} -\mu\,\partial_{y_3}\left(S\left(\partial_{y_3}\left[\hat{u}'\right]\right)\right)+\nabla_{y'}\hat{q} &=& f'-\nabla_{x'}\tilde{P}\quad in\ \ \omega\times Y_f,\\ \mathrm{div}_{y'}\left(\displaystyle\int_0^1 \hat{u}'dy_3\right) &=& 0\quad in\ \ \omega\times Y'_f,\\ \hat{u}' &=& 0\ in\ \ \omega\times Y_s\\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x',y)dy\right) &=& 0\ in\ \ \omega,\\ \left(\displaystyle\int_Y \hat{u}'(x',y)dy\right)\cdot n &=& 0\ on\ \ \partial\omega,\\ y'\to\hat{u}'(x',y),\hat{q}(x',y')\quad Y'-periodic. && \end{array}\right. \tag{78}$$

iii) *if* $a_\varepsilon \ll \varepsilon$*, then* $(a_\varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ *converges to the unique solution* $(\hat{u}(x',y), \tilde{P}(x'))$ *in* $L^p(\Omega; W^{1,p}_\sharp(Y')^3)\times L^{p'}(\omega)/\mathbb{R}$*, with* $\hat{u}_3 = 0$*, of the homogenized problem*

$$\left\{\begin{array}{rcl} -\mu \mathrm{div}_{y'} S\left(\mathbb{D}_{y'}\left[\hat{u}'\right]\right) + \nabla_{y'}\hat{q} & = & f' - \nabla_{x'}\tilde{P} \quad in \;\; \omega\times Y_f, \\ \mathrm{div}_{y'}\hat{u}' & = & 0 \quad in \;\; \omega\times Y_f, \\ \hat{u}' & = & 0 \;\; in \;\; \omega\times Y_s \\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x',y)dy\right) & = & 0 \;\; in \;\; \omega, \\ \left(\displaystyle\int_Y \hat{u}'(x',y)dy\right)\cdot n & = & 0 \;\; on \;\; \partial\omega, \\ y' \to \hat{u}'(x',y), \hat{q}(x',y) \quad Y'-periodic. & & \end{array}\right. \tag{79}$$

Proof. First of all, we choose a test function $v(x',y)\in\mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with $v(x',y) = 0 \in \omega\times Y_s$ (thus, $v(x', x'/a_\varepsilon, y_3)\in W^{1,p}_0(\widetilde{\Omega}_\varepsilon)^3$). Multiplying (6) by $v(x', x'/a_\varepsilon, y_3)$ and integrating by parts, we have

$$\begin{array}{l} \mu\displaystyle\int_\Omega S\left(\mathbb{D}_\varepsilon\left[\tilde{u}_\varepsilon\right]\right) : \left(\mathbb{D}_{x'}\left[v\right] + \frac{1}{a_\varepsilon}\mathbb{D}_{y'}\left[v\right] + \frac{1}{\varepsilon}\partial_{y_3}\left[v\right]\right) dx'dy_3 \\ -\displaystyle\int_\Omega \tilde{P}_\varepsilon \,\mathrm{div}_{x'}v'\,dx'dy_3 - \frac{1}{a_\varepsilon}\int_\Omega \tilde{P}_\varepsilon\,\mathrm{div}_{y'}v'\,dx'dy_3 - \frac{1}{\varepsilon}\int_\Omega \tilde{P}_\varepsilon\,\partial_{y_3}v_3\,dx'dy_3 \\ = \displaystyle\int_\Omega f'\cdot v'\,dx'dy_3\,. \end{array}$$

By the change of variables given in Remark 4.8, we obtain

$$\begin{array}{l} \mu\displaystyle\int_{\omega\times Y} S\left(\frac{1}{a_\varepsilon}\mathbb{D}_{y'}\left[\hat{u}_\varepsilon\right] + \frac{1}{\varepsilon}\partial_{y_3}\left[\hat{u}_\varepsilon\right]\right) : \left(\frac{1}{a_\varepsilon}\mathbb{D}_{y'}\left[v\right] + \frac{1}{\varepsilon}\partial_{y_3}\left[v\right]\right) dx'dy \\ -\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_{x'}v'\,dx'dy - \frac{1}{a_\varepsilon}\int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_{y'}v'\,dx'dy - \frac{1}{\varepsilon}\int_{\omega\times Y} \hat{P}_\varepsilon\,\partial_{y_3}v_3\,dx'dy \\ = \displaystyle\int_{\omega\times Y} f'\cdot v'\,dx'dy + O_\varepsilon\,. \end{array} \tag{80}$$

This variational formulation will be useful in the following steps.

We proceed in three steps.

Step 1. Critical case $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$.

First, we prove that $\hat{P}$ does not depend on the microscopic variable y. To do this, we consider as test function $a_\varepsilon v(x', x'/a_\varepsilon, y_3)$ in (80), which gives

$$\begin{array}{l} \mu a_\varepsilon\displaystyle\int_{\omega\times Y} S\left(a_\varepsilon^{-\frac{p}{p-1}}\mathbb{D}_{y'}\left[\hat{u}_\varepsilon\right] + \frac{a_\varepsilon}{\varepsilon}a_\varepsilon^{-\frac{p}{p-1}}\partial_{y_3}\left[\hat{u}_\varepsilon\right]\right) : \left(\mathbb{D}_{y'}\left[v\right] + \frac{a_\varepsilon}{\varepsilon}\partial_{y_3}\left[v\right]\right) dx'dy \\ -a_\varepsilon\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_{x'}v'\,dx'dy - \int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_{y'}v'\,dx'dy - \frac{a_\varepsilon}{\varepsilon}\int_{\omega\times Y} \hat{P}_\varepsilon\,\partial_{y_3}v_3\,dx'dy \\ = a_\varepsilon\displaystyle\int_{\omega\times Y} f'\cdot v'\,dx'dy + O_\varepsilon\,. \end{array} \tag{81}$$

Thus, passing to the limit when ε tends to zero by using convergences (63) and (64), we have

$$\int_{\omega\times Y} \hat{P}\,\mathrm{div}_\lambda v\,dx'dy = 0,$$

which shows that $\hat{P}$ does not depend on y.

For all $\varphi\in\mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with $\varphi = 0$ in $\omega\times Y_s$, $\mathrm{div}_\lambda\varphi = 0$ in $\omega\times Y$ and $\mathrm{div}_{x'}(\int_Y \varphi'\,dy) = 0$ in ω, we choose $v_\varepsilon = (v'_\varepsilon, v_{\varepsilon,3})$ defined by

$$v'_\varepsilon = \varphi' - a_\varepsilon^{-\frac{p}{p-1}}\hat{u}'_\varepsilon, \quad v_{\varepsilon,3} = \lambda\,\frac{\varepsilon}{a_\varepsilon}\varphi_3 - a_\varepsilon^{-\frac{p}{p-1}}\hat{u}_{\varepsilon,3},$$

as a test function in (80). Due to monotonicity, we have

$$\mu\int_{\omega\times Y} S\left(\mathbb{D}_{y'}\left[\varphi\right]+\frac{a_\varepsilon}{\varepsilon}\partial_{y_3}\left[\varphi\right]\right):\left(\mathbb{D}_{y'}\left[v_\varepsilon\right]+\frac{a_\varepsilon}{\varepsilon}\partial_{y_3}\left[v_\varepsilon\right]\right)dx'dy$$
$$-\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_{x'}v'_\varepsilon\,dx'dy\geq\int_{\omega\times Y} f'\cdot v'_\varepsilon\,dx'dy+O_\varepsilon\,.$$

Thus, we can use the convergences (63) and (64). If we argue similarly as in [7], we have that the convergence of the pressure $\tilde{P}_\varepsilon$ is in fact strong. This implies that the convergence of the pressure $\hat{P}_\varepsilon$ is also in fact strong (see Proposition 2.9 in [12]). Then, when passing to the limit, the second term contributes nothing because the limit of $\hat{P}_\varepsilon$ does not depend on y and $\hat{u}'$ satisfies (70). Taking into account that $\lambda\,\varepsilon/a_\varepsilon\to 1$, we obtain

$$\mu\int_{\omega\times Y} S\left(\mathbb{D}_{y'}\left[\varphi\right]+\lambda\partial_{y_3}\left[\varphi\right]\right):\left(\mathbb{D}_{y'}\left[\varphi-\hat{u}\right]+\lambda\partial_{y_3}\left[\varphi-\hat{u}\right]\right)dx'dy$$
$$\geq\int_{\omega\times Y} f'\cdot(\varphi'-\hat{u}')\,dx'dy+O_\varepsilon\,,$$

which, due to Minty Lemma [18], is equivalent to

$$-\mu\,\mathrm{div}_\lambda\left(S\left(\mathbb{D}_\lambda\left[\hat{u}\right]\right)\right)=f'\quad\text{in}\quad\omega\times Y.$$

By density

$$\mu\int_{\omega\times Y} S\left(\mathbb{D}_\lambda\left[\hat{u}\right]\right)\mathbb{D}_\lambda\left[v\right]dx'dy=\int_{\omega\times Y} f'\,v'\,dx'dy \tag{82}$$

holds for every function v in the Hilbert space V defined by

$$V=\left\{\begin{array}{l} v(x',y)\in L^p(\omega;W^{1,p}_\sharp(Y)^3),\quad\text{such that}\\ \mathrm{div}_\lambda v(x',y)=0\ \text{ in }\omega\times Y,\quad \mathrm{div}_{x'}\left(\displaystyle\int_Y v(x',y)\,dy\right)=0\ \text{ in }\omega,\\ v(x',y)=0\ \text{ in }\omega\times Y_s,\quad\left(\displaystyle\int_Y v(x',y)\,dy\right)\cdot n=0\ \text{ on }\omega\end{array}\right\}.$$

By Lax-Milgram lemma, the variational formulation (82) in the Hilbert space V admits a unique solution $\hat{u}$ in V. Reasoning as in [1], the orthogonal of V with respect to the usual scalar product in $L^p(\omega\times Y)$ is made of gradients of the form $\nabla_{x'}q(x')+\nabla_\lambda\hat{q}(x',y)$, with $q(x')\in L^{p'}(\omega)/\mathbb{R}$ and $\hat{q}(x',y)\in L^{p'}(\omega;W^{1,p}_\sharp(Y))$. Therefore, by integration by parts, the variational formulation (82) is equivalent to the effective system (77). It remains to prove that the pressure $\tilde{P}(x')$ arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div}_{x'}(\int_Y\hat{u}(x',y)dy)=0$ is the same as the limit of the pressure $\tilde{P}_\varepsilon$. This can be easily done by multiplying equation (6) by a test function with div_λ equal to zero, and identifying limits. Since (77) admits a unique solution, then the complete sequence $(a_\varepsilon^{-p/(p-1)}\hat{u}_\varepsilon,\hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}(x',y),\tilde{P}(x'))$. This gives the desired result. Finally, observe that from (76) and $\tilde{u}_3=0$, we have that $\int_Y\hat{u}_3dy=0$.

Step 2. Supercritical case $a_\varepsilon\gg\varepsilon$.

First, we show that $\hat{P}$ does not depend on the vertical variable y_3. To do this, we consider as test function $v=(0,\varepsilon v_3(x',x'/a_\varepsilon,y_3))$ in (80), and passing to the limit by using convergences (63) and (66), we get

$$\int_{\omega\times Y}\hat{P}\,\partial_{y_3}v_3\,dx'dy=0.$$

This shows that $\hat{P}$ does not depend on y_3.

Let us now prove that $\hat{P}$ does not depend on the microscopic variable y'. For this, we take now as test function $v=(a_\varepsilon v'(x',x'/a_\varepsilon,y_3),0)$ in (80). By using convergences (63) and (66), we get

$$\int_{\omega\times Y'}\hat{P}\,\mathrm{div}_{y'}v'\,dx'dy=0,$$

which implies that $\hat{P}$ does not depend on y'. Thus, we conclude that $\hat{P}$ does not depend on the entire variable y.

For all $\varphi \in \mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with φ_3 independent of y_3, $\varphi = 0$ in $\omega \times Y_s$, satisfying (67) and (70), we choose $v_\varepsilon = \varphi - \varepsilon^{-\frac{p}{p-1}}\hat{u}_\varepsilon$, as a test function in (80). Using monotonicity, we have

$$\mu \int_{\omega\times Y} S\left(\frac{\varepsilon}{a_\varepsilon}\mathbb{D}_{y'}[\varphi] + \partial_{y_3}[\varphi]\right) : \left(\frac{\varepsilon}{a_\varepsilon}\mathbb{D}_{y'}[v_\varepsilon] + \partial_{y_3}[v_\varepsilon]\right) dx'dy$$
$$- \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon \, .$$

Thus, we can use the convergences (63) and (66). If we argue similarly as the step 1, we have that the convergence of the pressure $\hat{P}_\varepsilon$ is strong. Then, when passing to the limit, the second term contributes nothing because the limit of $\hat{P}_\varepsilon$ does not depend on y and $\hat{u}'$ satisfies (70). We obtain

$$\mu \int_{\omega\times Y} S\left(\partial_{y_3}[\varphi']\right) : \partial_{y_3}[\varphi' - \hat{u}'] \, dx'dy$$
$$\geq \int_{\omega\times Y} f' \cdot (\varphi' - \hat{u}') \, dx'dy + O_\varepsilon \, ,$$

which, due to Minty Lemma [18], is equivalent to

$$-\mu \, \partial_{y_3}\left(S\left(\partial_{y_3}[\hat{u}']\right)\right) = f' \quad \text{in} \quad \omega \times Y.$$

By density, and reasoning as in Step 1, this problem is equivalent to the effective system (78). Observe that the first condition in (67) implies that $\hat{q}$ does not depend on y_3. Finally, observe that from (76) and $\tilde{u}_3 = 0$, we have that $\int_Y \hat{u}_3 dy = 0$.

Step 3. Subcritical case $a_\varepsilon \ll \varepsilon$.

First, we show that $\hat{P}$ does not depend on the vertical variable y_3. To do this, we consider as test function $v = (0, \varepsilon v_3(x', x'/a_\varepsilon, y_3))$ in (80), and passing to the limit by using convergences (63) and (68), we get

$$\int_{\omega\times Y} \hat{P} \, \partial_{y_3} v_3 \, dx'dy = 0.$$

This shows that $\hat{P}$ does not depend on y_3.

Now, we consider as test function $v = (a_\varepsilon v'(x', x'/a_\varepsilon, y_3), 0)$ in (80). Passing to the limit, we have

$$\int_{\omega\times Y} \hat{P} \, \mathrm{div}_{y'} v' \, dx'dy = 0,$$

which shows that $\hat{P}$ does not depend on y', and so $\hat{P}$ only depends on x'.

For all $\varphi \in \mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with φ_3 independent of y_3, $\varphi = 0$ in $\omega \times Y_s$, satisfying (69) and (70), we choose

$$v_\varepsilon = \varphi - a_\varepsilon^{-\frac{p}{p-1}} \hat{u}_\varepsilon,$$

as a test function in (80). Using monotonicity, we have

$$\mu \int_{\omega\times Y} S\left(\mathbb{D}_{y'}[\varphi] + \frac{a_\varepsilon}{\varepsilon}\partial_{y_3}[\varphi]\right) : \left(\mathbb{D}_{y'}[v_\varepsilon] + \frac{a_\varepsilon}{\varepsilon}\partial_{y_3}[v_\varepsilon]\right) dx'dy$$
$$- \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon \, .$$

Thus, we can use the convergences (63) and (68). If we argue similarly as the step 1, we have that the convergence of the pressure $\hat{P}_\varepsilon$ is strong. Then, when passing to the limit, the second term contributes nothing because the limit of $\hat{P}_\varepsilon$ does not depend on y and $\hat{u}'$ satisfies (70). We obtain

$$\begin{aligned}&\mu\int_{\omega\times Y} S\left(\mathbb{D}_{y'}\left[\varphi\right]\right):\mathbb{D}_{y'}\left[\varphi-\hat{u}\right]dx'dy\\ &\geq\int_{\omega\times Y} f'\cdot\left(\varphi'-\hat{u}'\right)dx'dy+O_\varepsilon\,,\end{aligned}$$

which, due to Minty Lemma [18], is equivalent to

$$-\mu\,\mathrm{div}_{y'}\left(S\left(\mathbb{D}_{y'}\left[\hat{u}\right]\right)\right)=f'\quad\text{in}\quad\omega\times Y.$$

By density, and reasoning as in Step 1, this problem is equivalent to the homogenized system (79). It is straightforward to obtain that $\hat{u}_3=0$.

□

In the final step, we will eliminate the microscopic variable y in the homogenized problem. This is the focus of the Theorem 3.1.

Proof of Theorem 3.1. In the cases $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$ or $a_\varepsilon\ll\varepsilon$, the derivation of (7) and (13) from the homogenized problems (77) and (79) respectively, is straightforward by using the local problems (9) and (15), and definitions of the permeability functions (8) and (14) respectively.

In the case $a_\varepsilon\gg\varepsilon$, we proceed as the previous cases. We deduce that

$$\left\{\begin{array}{l}\tilde{U}'(x')=-\dfrac{1}{\mu}A^\infty\left(f'(x')-\nabla_{x'}\tilde{P}(x')\right)\ \text{in}\ \ \omega,\\ \mathrm{div}_{x'}\,\tilde{U}'(x')=0\ \text{in}\ \ \omega,\\ \tilde{U}'(x')\cdot n=0\ \text{in}\ \ \partial\omega,\end{array}\right.\tag{83}$$

where $\tilde{U}(x')=\int_0^1\tilde{u}(x',y_3)\,dy_3$ and the permeability function $A^\infty:\mathbb{R}^2\to\mathbb{R}^2$ is monotone and coercive, defined by

$$A^\infty(\xi')=\int_{Y_f}w^{\xi'}(y)\,dy,\quad\forall\,\xi'\in\mathbb{R}^2,\tag{84}$$

where, $w^{\xi'}(y')$, for every $\xi'\in\mathbb{R}^2$, denotes the unique solution in $W^{1,p}_\sharp(Y'_f)^2$ of the local Stokes problem in 2D

$$\left\{\begin{array}{rcll}-\partial_{y_3}S\left(\partial_{y_3}[w^{\xi'}]\right)+\nabla_{y'}\pi^{\xi'}&=&-\xi'&\text{in}\ \ Y_f,\\ \mathrm{div}_{y'}\left(\displaystyle\int_0^1 w^{\xi'}dy_3\right)&=&0&\text{in}\ \ Y'_f,\\ w^{\xi'}&=&0&\text{in}\ \ \partial Y_s,\\ \multicolumn{4}{l}{w^{\xi'}(x',y),\pi^{\xi'}(x',y')\ Y'-\text{periodic}.}\end{array}\right.\tag{85}$$

We observe that (85) can be solved, and we can give a Reynolds type equation.

Take into account that

$$\left|\partial_{y_3}\left[w^{\xi'}\right]\right|^{p-2}=\left|Tr\left(\partial_{y_3}\left[w^{\xi'}\right],\partial^t_{y_3}\left[w^{\xi'}\right]\right)\right|^{\frac{p}{2}-1},$$

implies

$$S(\partial_{y_3}[w^{\xi'}])=2^{-\frac{p}{2}}S(\partial_{y_3}w^{\xi'}),$$

from Proposition 3.4 in [22], we deduce that

$$w^{\xi'}(y)=-\frac{2^{\frac{p'}{2}}}{p'}\left(\frac{1}{2^{p'}}-\left|\frac{1}{2}-y_3\right|^{p'}\right)\left|\xi'+\nabla_{y'}\pi^{\xi'}\right|^{p'-2}\left(\xi'+\nabla_{y'}\pi^{\xi'}\right).$$

From the expression of the Darcy velocity (1.14) in [22], we have

$$\int_0^1 w^{\xi'}(y)\,dy_3=-\frac{1}{2^{\frac{p'}{2}}(p'+1)}\left|\xi'+\nabla_{y'}\pi^{\xi'}\right|^{p'-2}\left(\xi'+\nabla_{y'}\pi^{\xi'}\right).$$

Then, from (83)-(84) we have (10) and (11), and from the second equation in (85) we have (12).

□

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