Derivation of a coupled Darcy-Reynolds equation for a fluid flow in a thin porous medium including a fissure

María Anguiano and Francisco Javier Suárez-Grau

Abstract. We study the asymptotic behavior of a fluid flow in a thin porous medium of thickness ε, which characteristic size of the pores ε, and containing a fissure of width η_ε. We consider the limit when the size of the pores tends to zero and we find a critical size $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ in which the flow is described by a 2D Darcy law coupled with a 1D Reynolds problem. We also discuss the other cases.

Mathematics Subject Classification (2010). 75A05, 76A20, 76M50, 35B27.

Keywords. Stokes equation, Darcy's law, Reynolds equation, thin porous medium, fissure.

1. Introduction

The aim of this work is to apply the two-scale convergence method (see Allaire [2] and Nghetseng [9]) to the homogenized of a Stokes system in a thin porous medium $D_{\varepsilon\eta_\varepsilon}$ of thickness ε which is perforated by periodically distributed solid cylinders of size ε and contains a fissure $\{0 \leq x_2 \leq \eta_\varepsilon\}$ of width η_ε. But here, it is necessary to combine the two-scale convergence method in the horizontal variables with a rescaling in the height variable in order to work with a domain of height one.

We consider the fluid flow through a periodic distribution of vertical cylinders and a fissure. The periodic distribution of vertical cylinders and the fissure are confined between two parallel plates (see Fig. 1). A representative elementary volume for the thin porous medium is a cube of lateral length ε and vertical lentgth ε. The cube is repeated periodically in the space between the plates. Each cube can be divided into fluid part and a solid part, where the solid part has the shape of a vertical cylinder of height ε.

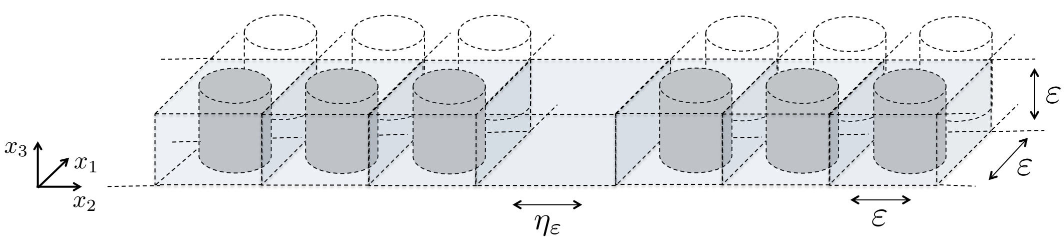

FIGURE 1. View of the domain $D_{\varepsilon\eta_\varepsilon}$

María Anguiano has been supported by Junta de Andalucía (Spain), Proyecto de Excelencia P12-FQM-2466, and in part by European Commission, Excellent Science-European Research Council (ERC) H2020-EU.1.1.-639227. Francisco Javier Suárez-Grau has been supported by Ministerio de Economía y Competitividad (Spain), Proyecto Excelencia MTM2014-53309-P.

Homogenization has been applied to the study of perforated materials for a long time. The question of a medium containing a fissure with properties different from those of the rest of the material has been the subject of many studies previously, see Ciarlet *et al* [8], Panasenko [10] and Chapter 13 of Sanchez-Palencia [11] among others. A similar problem of the one considered in this paper with a fixed height domain, but for the Laplace's equation, was studied in Bourgeat and Tapiero [5]. The peculiar behavior observed for the Laplace's equation when $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ has motivated the analogous study for the Stokes system in Bourgeat *et al* [6] (see Zhao and Yao [14] for the Navier-Stokes system).

All the above results relate to a fixed height domain. Our aim in the present paper is to extend the study of Bourgeat *et al* [6] to the case of a porous medium of small height ε. We find the same critical size as in Bourgeat *et al* [6], what means that the thickness of the domain does not have any influence in the relation between the fissure parameter η_ε with respect to porosity parameter ε. However, the thickness of the domain leads us to use techniques of reduction of the dimension together with homogenization in order to obtain more simplified effective models than those obtained in Bourgeat *et al* [6]. More precisely, we obtain the following results corresponding to three characteristic situations depending on the parameter η_ε with respect to ε:

- If $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ the fissure is not giving any contribution. In this case, in order to find the limit, we use the theory developed by Allaire [2] and Nguesteng [9] of two-scale convergence only in the horizontal variables and we obtain a 2D Darcy's law.
- If $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ the fissure is dominant. We introduce a rescaling in the fissure in order to work with a domain with size one, and then we prove that the limit of the velocity is a Dirac measure concentrated on the line $\{x_2 = 0\} \cap \{x_3 = 0\}$ representing the corresponding tangential line flow. Meanwhile in the porous medium the effective velocity is equal to zero.
- If $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, it appears a coupling effect and the effective flow behaves as 2D Darcy flow in the porous medium coupled with the tangential flow of the line $\{x_2 = 0\} \cap \{x_3 = 0\}$. Compared to the first case $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, the effective velocity has now an additional tangential component concentrated on $\{x_2 = 0\} \cap \{x_3 = 0\}$. Moreover, the limit problem is now given by a new variational equation, in which appears the parameter λ, and consists of a 2D Darcy law in the porous medium coupled with a 1D Reynolds problem on the line $\{x_2 = 0\} \cap \{x_3 = 0\}$.

2. The domain and some notations

Let $\omega \subset \mathbb{R}^2$ be smooth bounded connected open set and $\Omega = \omega \times (0,1) \subset \mathbb{R}^3$. We define

$$\Omega_+ = \Omega \cap \{x_2 > 0\}, \quad \Omega_- = \Omega \cap \{x_2 < 0\}, \quad \Sigma = \Omega \cap \{x_2 = 0\}, \quad \Sigma_1 = \Sigma \cap \{x_3 = 0\}.$$

For some $\eta_0 > 0$ we define the domains

$$D = \Omega_- \cup (\eta_0 e_2 + \Omega_+) \cup (\Sigma \times [0, \eta_0] e_2), \quad D' = D \cap \{x_3 = 0\},$$

with $e_2 = (0,1,0)$.

Let $\varepsilon > 0$ be a small parameter devoted to tend to zero and $0 < \eta_\varepsilon < \eta_0$ be a small parameter devoted to tend to zero with ε.

A periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = [0,1]^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \bigcup Y'_s = Y'$ and $Y'_f \bigcap Y'_s = \emptyset$). More precisely, we assume that Y'_s is a smooth, closed and connected set strictly included in Y'. For $k' = (k_1, k_2) \in \mathbb{Z}^2$, each cell $Y'_{k'} = k' + Y'$ is divided in a fluid part $Y'_{f_{k'}}$ and a solid part $Y'_{s_{k'}}$. We define $Y = Y' \times (0,1) \subset \mathbb{R}^3$, and is divided in a fluid part Y_f and a solid part Y_s.

We also denote

$$Y_s^- = \bigcup_{k' \in \mathbb{Z}^2_-} Y_{s_{k'}}, \quad Y_s^+ = \bigcup_{k' \in \mathbb{Z}^2_+} Y_{s_{k'}},$$

all the solid parts in $\mathbb{R}^2 \times (0,1)$, which are closed subsets of $\mathbb{R}^3$, where $\mathbb{Z}^2_- = \{k' \in \mathbb{Z}^2,\ k_2 < 0\}$ and $\mathbb{Z}^2_+ = \{k' \in \mathbb{Z}^2,\ k_2 > 0\}$. It is obvious that $E_f = (\mathbb{R}^2 \times (0,1)) \setminus (Y_s^- \cup Y_s^+)$ is an open subset of $\mathbb{R}^3$.

Following [1], we make the following assumptions on Y_f, E_f, Y_s and $Y_s^* = Y_s^+ \cup Y_s^-$:

i) Y_f is an open connected set of strictly positive measure, with a locally Lipschitz boundary.
ii) Y_s has strictly positive measure in $\overline{Y}$.
iii) E_f and the interior of Y_s^* are open sets with boundaries of class $C^{0,1}$ and are locally located on one side of their boundaries. Moreover E_f is connected.

The microscale of a porous medium is the small positive number ε. The domain ω is covered by a regular mesh of size ε: for $k' = (k_1, k_2) \in \mathbb{Z}^2$, each cell $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},\varepsilon}$ and a solid part $Y'_{s_{k'},\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size ε. We define $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0,1) \subset \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},\varepsilon}$ and a solid part $Y_{s_{k'},\varepsilon}$.

We also define

$$Y^-_{s,\varepsilon} = \bigcup_{k' \in \mathbb{Z}^2_-} Y_{s_{k'},\varepsilon}, \quad Y^+_{s,\varepsilon\eta_\varepsilon} = \eta_\varepsilon e_2 + \bigcup_{k' \in \mathbb{Z}^2_+} Y_{s_{k'},\varepsilon}, \quad \widetilde{S}_{\varepsilon\eta_\varepsilon} = \partial(Y^-_{s,\varepsilon} \cup Y^+_{s,\varepsilon\eta_\varepsilon}).$$

We denote by

$\widetilde{A}_{\varepsilon\eta_\varepsilon} = (Y^-_{s,\varepsilon} \cup Y^+_{s,\varepsilon\eta_\varepsilon}) \cap D$ - the solid part of the domain D,
$\widetilde{D}_{\varepsilon\eta_\varepsilon} = D \setminus \widetilde{A}_{\varepsilon\eta_\varepsilon}$ - the fluid part of the domain D (including the fissure),
$\widetilde{I}_{\eta_\varepsilon} = \Sigma \times (0, \eta_\varepsilon) e_2$ - the fissure in D,
$\widetilde{\Omega}_{\varepsilon\eta_\varepsilon} = \widetilde{D}_{\varepsilon\eta_\varepsilon} \setminus \widetilde{I}_{\eta_\varepsilon}$ - the fluid part of the porous medium in D.

Let us define a domain with thickness ε, given by $\Omega^\varepsilon = \Omega \cap \{0 < x_3 < \varepsilon\} \subset \mathbb{R}^3$. We also define

$$\Omega^\varepsilon_+ = \Omega_+ \cap \{0 < x_3 < \varepsilon\}, \quad \Omega^\varepsilon_- = \Omega_- \cap \{0 < x_3 < \varepsilon\}, \quad \Sigma^\varepsilon = \Omega^\varepsilon \cap \{x_2 = 0\},$$

and

$$D^\varepsilon = \Omega^\varepsilon_- \cup \left(\eta_0 e_2 + \Omega^\varepsilon_+\right) \cup (\Sigma^\varepsilon \times [0, \eta_0] e_2)\,.$$

Now, we denote by $A_{\varepsilon\eta_\varepsilon}$, $D_{\varepsilon\eta_\varepsilon}$, I_{η_ε} and $\Omega_{\varepsilon\eta_\varepsilon}$ the sets $\widetilde{A}_{\varepsilon\eta_\varepsilon}$, $\widetilde{D}_{\varepsilon\eta_\varepsilon}$, $\widetilde{I}_{\eta_\varepsilon}$ and $\widetilde{\Omega}_{\varepsilon\eta_\varepsilon}$, respectively, with thickness ε, i.e.,

$A_{\varepsilon\eta_\varepsilon} = \widetilde{A}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the solid part of the domain D^ε,
$D_{\varepsilon\eta_\varepsilon} = \widetilde{D}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fluid part of the domain D^ε (including the fissure),
$I_{\eta_\varepsilon} = \widetilde{I}_{\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fissure in D^ε,
$\Omega_{\varepsilon\eta_\varepsilon} = \widetilde{\Omega}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fluid part of the porous medium in D^ε.

Finally we define (see Fig. 2)

$$\Omega^+_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \cap \{x_2 > \eta_\varepsilon\}, \quad \Omega^-_{\varepsilon\eta_\varepsilon} = D_{\varepsilon\eta_\varepsilon} \cap \{x_2 < 0\}, \quad \Gamma_{\eta_\varepsilon} = \partial\Sigma^\varepsilon \times (0, \eta_\varepsilon) e_2,$$

and

$$D^+ = D \cap \{x_2 > 0\}, \quad D^- = \Omega_-\,.$$

Let us introduce some notations which will be useful in the following. For a vectorial function $v = (v_1, v_2, v_3)$ and a scalar function w, we introduce the operators: D_ε, ∇_ε and div_ε by

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } i = 1,2,3,\ j = 1,2, \quad (D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } i = 1,2,3,$$

$$\nabla_\varepsilon w = (\nabla_{x'} w, \frac{1}{\varepsilon}\partial_{y_3} w)^t, \quad \mathrm{div}_\varepsilon v = \mathrm{div}_{x'} v' + \frac{1}{\varepsilon}\partial_{y_3} v_3,$$

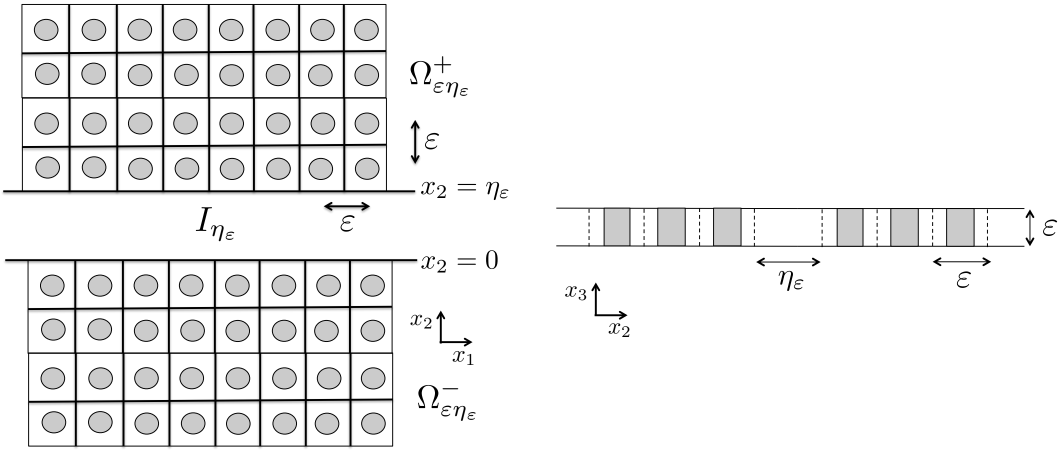

FIGURE 2. View of the domain $D_{\varepsilon\eta_\varepsilon}$ from above (left) and lateral (right)

and moreover the operators D_{η_ε}, $\nabla_{\eta_\varepsilon}$ and $\mathrm{div}_{\eta_\varepsilon}$ by

$$(D_{\eta_\varepsilon} v)_{i,1} = \partial_{x_1} v_i, \quad (D_{\eta_\varepsilon} v)_{i,2} = \frac{1}{\eta_\varepsilon}\partial_{y_2} v_i, \quad (D_{\eta_\varepsilon} v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } \ i = 1, 2, 3,$$

$$\nabla_{\eta_\varepsilon} w = (\partial_{x_1} w, \frac{1}{\eta_\varepsilon}\partial_{y_2} w, \frac{1}{\varepsilon}\partial_{y_3} w)^t, \quad \mathrm{div}_{\eta_\varepsilon} v = \partial_{x_1} v_1 + \frac{1}{\eta_\varepsilon}\partial_{y_2} v_2 + \frac{1}{\varepsilon}\partial_{y_3} v_3.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line. We denote by C a generic positive constant which can change from line to line.

3. Setting and main results

Hereinafter, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

In this section we describe the asymptotic behavior of an incompressible viscous Newtonian fluid in the thin porous medium with a fissure. The proof of the corresponding results will be given in the next sections.

Our results are referred to the Stokes system. Namely, for $f \in C(\overline{D})^3$ let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in H^1_0(D_{\varepsilon\eta_\varepsilon})^3 \times L^2_0(D_{\varepsilon\eta_\varepsilon})$, which satisfies

$$\left\{ \begin{array}{r} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \ D_{\varepsilon\eta_\varepsilon}, \\ \mathrm{div}\, u_\varepsilon = 0 \text{ in } \ D_{\varepsilon\eta_\varepsilon}, \end{array} \right. \tag{1}$$

where $\mu > 0$ is the viscosity and $L^2_0(D_{\varepsilon\eta_\varepsilon})$ is the space of functions of $L^2(D_{\varepsilon\eta_\varepsilon})$ with null integral. The right-hand side f is of the form

$$f(x) = (f'(x'), 0), \text{ a.e. } \ x \in D_{\varepsilon\eta_\varepsilon},$$

where f' is assumed in $C^1(\overline{D})^2$. This choice of f is usual when we deal with thin domains. Since the thickness of the domain ε is small then the vertical component of the force can be neglected and, moreover the force can be considered independent of the vertical variable.

Finally, we may consider Dirichlet boundary conditions without altering the generality of the problem under consideration,

$$u_\varepsilon = 0 \text{ on } \ \partial D_{\varepsilon\eta_\varepsilon}. \tag{2}$$

It is well known that (1)-(2) has a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1_0(D_{\varepsilon\eta_\varepsilon})^3 \times L^2_0(D_{\varepsilon\eta_\varepsilon})$ (see [12] for more details).

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3

$$y_3 = \frac{x_3}{\varepsilon}, \tag{3}$$

in order to have the functions defined in an open set with fixed height $\widetilde{D}_{\varepsilon\eta_\varepsilon}$ given in Section 2.

Namely, we define $\tilde{u}_\varepsilon \in H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$, $\tilde{p}_\varepsilon \in L_0^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \quad \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e.\ (x', y_3) \in \widetilde{D}_{\varepsilon\eta_\varepsilon}.$$

Using the transformation (3), the system (1) can be rewritten as

$$\begin{cases} -\mu\Delta_\varepsilon \tilde{u}_\varepsilon + \nabla_\varepsilon \tilde{p}_\varepsilon = f \text{ in } \widetilde{D}_{\varepsilon\eta_\varepsilon}, \\ \operatorname{div}_\varepsilon \tilde{u}_\varepsilon = 0 \text{ in } \widetilde{D}_{\varepsilon\eta_\varepsilon}, \end{cases} \tag{4}$$

with Dirichlet boundary condition, i.e.

$$\tilde{u}_\varepsilon = 0 \text{ on } \partial\widetilde{D}_{\varepsilon\eta_\varepsilon}. \tag{5}$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$.

Moreover, in order to study the behavior of $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ in the fissure we rewrite our equations in the unit cylinder $\widetilde{I}_1 = \Sigma \times (0,1)e_2$ by introducing the change of variable

$$y_2 = \frac{x_2}{\eta_\varepsilon}, \tag{6}$$

which transform $\widetilde{I}_{\eta_\varepsilon}$ in a fixed domain $\widetilde{I}_1$. We define the new functions

$$\tilde{\mathcal{U}}^\varepsilon(x_1, y_2, y_3) = \tilde{u}_\varepsilon(x_1, \eta_\varepsilon y_2, y_3), \quad \tilde{P}^\varepsilon(x_1, y_2, y_3) = \tilde{p}_\varepsilon(x_1, \eta_\varepsilon y_2, y_3) - c_{\varepsilon\eta_\varepsilon}, \tag{7}$$

with

$$c_{\varepsilon\eta_\varepsilon} = \frac{1}{|\widetilde{I}_{\eta_\varepsilon}|}\int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon \, dx' dy_3. \tag{8}$$

Using the transformation (6), the system (4) can be rewritten as

$$\begin{cases} -\mu\Delta_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon + \nabla_{\eta_\varepsilon} \tilde{P}^\varepsilon = f(x_1, \eta_\varepsilon y_2) \text{ in } \widetilde{I}_1, \\ \operatorname{div}_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon = 0 \text{ in } \widetilde{I}_1, \end{cases} \tag{9}$$

with Dirichlet boundary condition, i.e.

$$\tilde{\mathcal{U}}^\varepsilon = 0 \text{ on } \partial\widetilde{I}_1. \tag{10}$$

Our main result referred to the asymptotic behavior of the solution of (4) is given by the following theorem.

Theorem 3.1. *We distingue three cases depending on the relation between the parameter η_ε with respect to ε:*

i) *if $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, then there exists $(\tilde{v}, \tilde{p}) \in L^2(D)^3 \times L_0^2(D)$, with $\tilde{v}_3 = 0$ and $\tilde{p}$ independent of y_3, such that the solution $(\varepsilon^{-2}\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ of problem (4)-(5) satisfies*

$$\varepsilon^{-2}\tilde{u}_\varepsilon \rightharpoonup \tilde{v} \quad in\ L^2(D)^3, \quad \tilde{p}_\varepsilon \to \tilde{p} \quad in\ L_0^2(D). \tag{11}$$

Moreover, $\tilde{p} \in H^1(D) \cap L_0^2(D)$ and $(\tilde{V}, \tilde{p})$ is the unique solution of the 2D Darcy law

$$\begin{cases} \tilde{V}'(x') &= \dfrac{1}{\mu}K\left(f'(x') - \nabla_{x'}\tilde{p}(x')\right) \quad in\ D', \\ \operatorname{div}_{x'} \tilde{V}(x') &= 0 \quad in\ D', \\ \tilde{V}(x') \cdot n &= 0 \quad in\ \partial D', \end{cases} \tag{12}$$

where $\tilde{V}(x') = \int_0^1 \tilde{v}(x', y_3)dy_3$ and $K \in \mathbb{R}^{2\times 2}$ is a symmetric, positive, tensor defined by its entries

$$K_{ij} = \int_{Y_f} D_y w^i(y) : D_y w^j(y)\, dy, \quad i, j = 1, 2, \tag{13}$$

where $w^i(y)$, $i = 1, 2$, with $\int_{Y_f} w_3^i dy = 0$, denotes the unique solution in $H^1_\#(Y_f)^3$ of the local problem in 3D

$$\begin{cases} -\Delta_y w^i + \nabla_y q^i &= e_i \quad \text{in } Y_f, \\ \operatorname{div}_y w^i &= 0 \quad \text{in } Y_f, \\ w^i &= 0 \quad \text{in } \partial(Y \setminus Y_f), \\ w^i, q^i Y' - \text{periodic}. \end{cases} \tag{14}$$

ii) *if $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ and let $(\tilde{\mathcal{U}}^\varepsilon, \tilde{P}^\varepsilon)$ be a solution of (9)-(10). Then there exist $\tilde{\mathcal{U}} \in L^2(\widetilde{I}_1)^3$, independent of y_3, with $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, and $\tilde{P} \in L^2_0(\widetilde{I}_1)$ only depending on x_1, such that for a subsequence,*

$$\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \tilde{\mathcal{U}} \quad \text{in } L^2(\widetilde{I}_1)^3, \quad \tilde{P}^\varepsilon \rightharpoonup \tilde{P} \quad \text{in } L^2(\widetilde{I}_1),$$

where

$$\tilde{\mathcal{U}}_1(x_1, y_2) = \frac{y_2(1-y_2)}{2\mu}\left(f_1(x_1, 0) - \partial_{x_1}\tilde{P}(x_1)\right). \tag{15}$$

Moreover, it holds that

$$\eta_\varepsilon{}^{-3}\tilde{u}_\varepsilon \overset{\star}{\rightharpoonup} \tilde{\mathcal{V}}\delta_{\Sigma_1} \quad \text{in } \mathcal{M}(D)^3, \tag{16}$$

where $\tilde{\mathcal{V}} \in L^2(\Sigma_1)^3$, with $\tilde{\mathcal{V}}_2 = \tilde{\mathcal{V}}_3 = 0$, such that

$$\tilde{\mathcal{V}}_1(x_1) = \int_0^1 \tilde{\mathcal{U}}_1(x_1, y_2)\, dy_2 = \frac{1}{12\mu}\left(f_1(x_1, 0) - \partial_{x_1}\tilde{P}(x_1)\right), \tag{17}$$

and, in fact $\tilde{P} \in H^1(\Sigma_1) \cap L^2_0(\Sigma_1)$ is the unique solution of the 1D Reynolds problem on Σ_1

$$\begin{cases} \partial_{x_1}\left(f_1(x_1, 0) - \partial_{x_1}\tilde{P}(x_1))\right) = 0 \quad \text{in } \Sigma_1, \\ \left(f_1(x_1, 0) - \partial_{x_1}\tilde{P}(x_1)\right) \cdot n = 0 \quad \text{on } \partial\Sigma_1. \end{cases} \tag{18}$$

iii) *if $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, then there exist a Darcy velocity $\tilde{v}$, a Reynolds velocity $\tilde{\mathcal{V}}$ and a pressure field $\tilde{p}$ such that*

$$\begin{aligned} &\varepsilon^{-2}\tilde{u}_\varepsilon \overset{\star}{\rightharpoonup} \tilde{v} + \lambda^3\tilde{\mathcal{V}}\delta_{\Sigma_1} \quad \text{in } \mathcal{M}(D)^3, \\ &\tilde{p}_\varepsilon \to \tilde{p} \quad \text{in } L^2(D), \end{aligned} \tag{19}$$

where δ_{Σ_1} is the Dirac measure concentrated on Σ_1, and $\mathcal{M}(D)^3$ is the space of Radon meaures on D. The velocities $\tilde{v}$ and $\tilde{\mathcal{V}}$ are linked with the pressure $\tilde{p}$ through the 2D Darcy law (12) in D' and the 1D Reynolds problem (18) on Σ_1. The pressure field $\tilde{p} \in H^1(D') \cap L^2_0(D')$ with $\tilde{p}(\cdot, 0) \in H^1(\Sigma_1) \cap L^2_0(\Sigma_1)$, is the unique solution of the variational problem

$$\int_{D'} \frac{1}{\mu} K\left(f'(x') - \nabla_{x'}\tilde{p}(x')\right) \cdot \nabla_{x'}\varphi(x')\, dx' + \frac{\lambda^3}{12\mu}\int_{\Sigma_1} \left(f_1(x_1, 0) - \partial_{x_1}\tilde{p}(x_1)\right)\partial_{x_1}\varphi(x_1, 0)\, dx_1 = 0, \tag{20}$$

for every $\varphi \in H^1(D')$ with $\varphi(\cdot, 0) \in H^1(\Sigma_1)$.

Remark 3.2. *The coupled problem (20) corresponding to the critical case $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, can be considered as the general one. In fact, if λ tends to infinity in (20) we recover the 1D Reynolds problem (18), meanwhile if λ tends to zero we recover the 2D Darcy law (12).*

4. A Priori Estimates

Let us begin with a lemma on Poincaré inequality in the porous medium $\widetilde{\Omega}_{\varepsilon\eta_\varepsilon}$, which will be very useful (see for example Lemma 4.1 in [3]).

Lemma 4.1. *There exists a constant C independent of ε, such that, for any function $v \in H^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $\widetilde{S}_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3} \leq C\varepsilon \|D_\varepsilon v\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^{3\times 3}}. \tag{21}$$

Next, we give an useful estimate in the fissure $\widetilde{I}_{\eta_\varepsilon}$.

Lemma 4.2. *There exists a constant C independent of ε, such that, for any function $v \in H^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $\widetilde{S}_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|D_\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} . \tag{22}$$

Proof. For any function $w(y) \in H^1(\widetilde{I}_1)^3$ with $w = 0$ in $\partial\widetilde{I}_1$, the Poincaré inequality in $\widetilde{I}_1$ states that

$$\int_{\widetilde{I}_1} |w|^2 \, dz \leq C \int_{\widetilde{I}_1} |\partial_{z_2} w|^2 \, dz, \tag{23}$$

where the constant C depends only on $\widetilde{I}_1$.

For every $k' \in \mathbb{Z}^2$, by the change of variable

$$z_1 = x_1, \quad z_2 = \frac{x_2}{\eta_\varepsilon}, \quad z_3 = \frac{x_3}{\varepsilon}, \quad dz = \frac{dx}{\varepsilon\eta_\varepsilon}, \qquad \partial_{z_2} = \eta_\varepsilon \partial_{x_2},$$

we rescale (23) from $\widetilde{I}_1$ to I_{η_ε}. This yields that, for any function $w(x) \in H^1(I_{\eta_\varepsilon})^3$ with $w = 0$ in $\partial I_{\eta_\varepsilon}$, one has

$$\int_{I_{\eta_\varepsilon}} |w|^2 \, dx \leq C\eta_\varepsilon^2 \int_{I_{\eta_\varepsilon}} |\partial_{x_2} w|^2 \, dx \leq C\eta_\varepsilon^2 \int_{I_{\eta_\varepsilon}} |D_x w|^2 \, dx, \tag{24}$$

with the same constant C as in (23). Finally, applying the dilatation (3) in (24), we obtain

$$\int_{\widetilde{I}_{\eta_\varepsilon}} |w|^2 \, dx' dy_3 \leq C\eta_\varepsilon^2 \int_{\widetilde{I}_{\eta_\varepsilon}} |D_\varepsilon w|^2 \, dx' dy_3,$$

which gives

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon \|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}. \tag{25}$$

Next, if we choose a point $y \in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $x \in I_{\eta_\varepsilon}$, then we have

$$v(x) - v(y) = Dv(\xi)(x - y) \leq (\varepsilon + \eta_\varepsilon)|Dv|.$$

Since $v(y) = 0$ because $y \in A_{\varepsilon\eta_\varepsilon}$, we have

$$\|v(x)\|_{L^2(I_{\eta_\varepsilon})^3} \leq C(\varepsilon + \eta_\varepsilon)\|Dv\|_{L^2(I_{\eta_\varepsilon})^{3\times 3}},$$

and applying the dilatation (3) gives

$$\|v\|_{L^2(\tilde{I}_{\eta_\varepsilon})^3} \leq C(\varepsilon + \eta_\varepsilon)\|D_\varepsilon v\|_{L^2(\tilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Finally, multiplying the above inequality with (25) we obtain

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|D_\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} , \tag{26}$$

which is the desired estimate (22).

□

Let us give the classical estimate, [4], for the a function in L^2 when we deal with a thin fissure.

Lemma 4.3. *Let $v \in L^2(\widetilde{I}_{\eta_\varepsilon})$ be such that $\int_{\widetilde{I}_{\eta_\varepsilon}} v \, dx' dy_3 = 0$. Then*

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})} \leq \frac{C}{\eta_\varepsilon} \|\nabla_\varepsilon v\|_{H^{-1}(\widetilde{I}_{\eta_\varepsilon})^3}.$$

Now, we are in position to obtain some a priori estimates for $\tilde{u}_{\varepsilon\eta_\varepsilon}$.

Lemma 4.4. *There exists a constant C independent of ε, such that the solution $\tilde{u}_\varepsilon \in H^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ of the problem (4) satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3} \leq C(\eta_\varepsilon{}^{\frac{3}{2}}\varepsilon + \varepsilon^2), \tag{27}$$

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\eta_\varepsilon{}^{\frac{5}{2}} + \varepsilon\eta_\varepsilon + \eta_\varepsilon{}^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\right), \tag{28}$$

$$\|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq C(\eta_\varepsilon{}^{\frac{3}{2}} + \varepsilon). \tag{29}$$

Proof. Multiplying by $\tilde{u}_\varepsilon$ in the first equation of (4) and integrating over $\widetilde{D}_{\varepsilon\eta_\varepsilon}$, we have

$$\mu\|D_\varepsilon\tilde{u}_\varepsilon\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} = \int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f'\cdot\tilde{u}'_\varepsilon\,dx'\,. \tag{30}$$

Using Hölder's inequality and the assumption of f, we obtain that

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f'\cdot\tilde{u}'_\varepsilon\,dx' \leq C\eta_\varepsilon{}^{\frac{1}{2}}\|f'\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^2}\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} + \|f'\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^2}\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3},$$

and by inequalities (21) and (22), we have

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f'\cdot\tilde{u}'_\varepsilon\,dx' \leq C\left(\eta_\varepsilon(\varepsilon+\eta_\varepsilon)^{\frac{1}{2}} + \varepsilon\right)\|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq C\left(\eta_\varepsilon{}^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}} + \varepsilon\right)\|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}.$$

Therefore, from (30) we get

$$\|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq C\left(\eta_\varepsilon{}^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}} + \varepsilon\right).$$

Since $\eta_\varepsilon\varepsilon^{\frac{1}{2}} < \eta_\varepsilon{}^{\frac{3}{2}}$ if $\varepsilon < \eta_\varepsilon$ and $\eta_\varepsilon\varepsilon^{\frac{1}{2}} \leq \eta_\varepsilon^{\frac{1}{2}}\varepsilon < \varepsilon$ if $\eta_\varepsilon < \varepsilon$, the term $\eta_\varepsilon\varepsilon^{\frac{1}{2}}$ can be dropped. This gives (29).

Applying (21) together with (29) we obtain (27). Finally, applying (22) and (29) we get

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C(\eta_\varepsilon + \eta_\varepsilon{}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})(\eta_\varepsilon{}^{\frac{3}{2}} + \varepsilon) \leq C\left(\eta_\varepsilon{}^{\frac{5}{2}} + \varepsilon\eta_\varepsilon + \eta_\varepsilon{}^2\varepsilon^{\frac{1}{2}} + \eta_\varepsilon{}^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\right).$$

Since $\eta_\varepsilon{}^2\varepsilon^{\frac{1}{2}} < \eta_\varepsilon{}^{\frac{5}{2}}$ if $\eta_\varepsilon > \varepsilon$ and $\eta_\varepsilon{}^2\varepsilon^{\frac{1}{2}} < \eta_\varepsilon{}^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}$ if $\eta_\varepsilon < \varepsilon$, the term $\eta_\varepsilon{}^2\varepsilon^{\frac{1}{2}}$ can be dropped, and (28) holds.

□

In the next step we will estimate the pressure to the whole domain D. We give some properties of the restricted operator, R^ε, from $H^1_0(D)^3$ into $H^1_0(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ preserving divergence-free vectors, which was introduced by Tartar [12]. Since the construction of the operator is local, having no obstacles in $\widetilde{I}_{\eta_\varepsilon}$ means that we do not have to use the extension in that part. Next, we give the properties of the operator R^ε.

Lemma 4.5. *There exists a linear continuous (restriction) operator R^ε acting from $H^1_0(D)^3$ into $H^1_0(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ such that*

1. $R^\varepsilon v = v$, *if* $v \in H^1_0(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ *(elements of* $H^1_0(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ *are continuated by* 0 *to* D*)*
2. $\mathrm{div}_\varepsilon(R^\varepsilon v) = 0$, *if* $\mathrm{div}\, v = 0$
3. *For any* $v \in H^1_0(D)^3$ *(the constant* $\widetilde{C}$ *is independent of* v *and* ε*),*

$$\|R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} \leq \widetilde{C}\,\|v\|_{L^2(D)^3} + \widetilde{C}\varepsilon\,\|D_\varepsilon v\|_{L^2(D)^{3\times 3}}\,,$$

$$\|D_\varepsilon R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq \frac{\widetilde{C}}{\varepsilon}\,\|v\|_{L^2(D)^3} + \widetilde{C}\,\|D_\varepsilon v\|_{L^2(D)^{3\times 3}}\,.$$

In order to extend the pressure to the whole domain D, we define a function $F_\varepsilon \in H^{-1}(D)^3$ by the following formula (brackets are for the duality products between H^{-1} and H^1_0):

$$\langle F_\varepsilon, v\rangle_D = \langle\nabla_\varepsilon\tilde{p}_\varepsilon, R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\,,\ \text{for any}\ \ v \in H^1_0(D)^3, \tag{31}$$

where R^ε is defined in Lemma 4.5. We calcule the right hand side of (31) by using (4) and we have

$$\langle F_\varepsilon, v\rangle_D = \langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon, R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} + \langle f, R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}, \tag{32}$$

and by using the third point in Lemma 4.5, for fixed ε, we see that it is a bounded functional on $H_0^1(D)^3$, and in fact $F_\varepsilon \in H^{-1}(D)^3$.

Moreover, if $v \in H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and we continue it by zero out of $\widetilde{D}_{\varepsilon\eta_\varepsilon}$, we see from (31) and the first point in Lemma 4.5 that $F_\varepsilon|_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} = \nabla_\varepsilon \tilde{p}_\varepsilon$.

Moreover, if $\operatorname{div} v = 0$ by the second point in Lemma 4.5 and (31), $\langle F_\varepsilon, v\rangle_D = 0$ and this implies (by the orthogonality property) that F_ε is the gradient of some function in $L^2(D)$. This means that F_ε is a continuation of $\nabla_\varepsilon \tilde{p}_\varepsilon$ to D, and that this continuation is a gradient. We also may say that $\tilde{p}_\varepsilon$ has been continuated to D and we denote the extended pressure again by $\tilde{p}_\varepsilon$ and

$$F_\varepsilon \equiv \nabla_\varepsilon \tilde{p}_\varepsilon, \qquad \tilde{p}_\varepsilon \in L_0^2(D).$$

Lemma 4.6. *Let $\tilde{p}_\varepsilon$ be the extension of the pressure defined as above. Then*

$$\|\tilde{p}_\varepsilon\|_{L^2(D)} \leq C\left(\frac{\eta_\varepsilon^{\frac{3}{2}}}{\varepsilon} + 1\right), \tag{33}$$

$$\|\tilde{p}_\varepsilon - c_{\varepsilon\eta_\varepsilon}\|_{L^2(\widetilde{I}_{\eta_\varepsilon})} \leq C\left(\eta_\varepsilon^{\frac{1}{2}} + \frac{\varepsilon}{\eta_\varepsilon}\right), \tag{34}$$

where $c_{\varepsilon\eta_\varepsilon}$ is given by (8).

Proof. Let us first estimate $\nabla_\varepsilon \tilde{p}_\varepsilon$. To do this we estimate the right side of (32). Using Hölder's inequality and from (29) we have

$$\begin{aligned}\left|\langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon, R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| &\leq \mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \|D_\varepsilon R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}\\ &\leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right) \|D_\varepsilon R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}.\end{aligned}$$

Using the assumption of f, we obtain

$$\left|\langle f, R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| \leq C \|R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}.$$

Then, from (32), we deduce

$$|\langle \nabla_\varepsilon \tilde{p}_\varepsilon, v\rangle_D| \leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right) \|D_\varepsilon R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} + C\|R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}.$$

Taking into account the third point in Lemma 4.5, we have

$$|\langle \nabla_\varepsilon \tilde{p}_\varepsilon, v\rangle_D| \leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right)\left(\frac{1}{\varepsilon}\|v\|_{L^2(D)^3} + \|D_\varepsilon v\|_{L^2(D)^{3\times 3}}\right) + C\left(\|v\|_{L^2(D)^3} + \varepsilon\|D_\varepsilon v\|_{L^2(D)^{3\times 3}}\right).$$

Then, as $\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$|\langle \nabla_\varepsilon \tilde{p}_\varepsilon, v\rangle_D| \leq C\left(\frac{\eta_\varepsilon^{\frac{3}{2}}}{\varepsilon} + 1\right)\|v\|_{H_0^1(D)^3},$$

for any $v \in H_0^1(D)^3$. Therefore, we obtain

$$\|\nabla_\varepsilon \tilde{p}_\varepsilon\|_{H^{-1}(D)^3} \leq C\left(\frac{\eta_\varepsilon^{\frac{3}{2}}}{\varepsilon} + 1\right),$$

and the estimate (33) follows by using the Nečas inequality in D.

Now, we prove the estimate (34). Let $v \in H_0^1(\widetilde{I}_{\eta_\varepsilon})^3$, then

$$\langle \nabla_\varepsilon \tilde{p}_\varepsilon, v\rangle_{\widetilde{I}_{\eta_\varepsilon}} = \langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon, v\rangle_{\widetilde{I}_{\eta_\varepsilon}} + \langle f, v\rangle_{\widetilde{I}_{\eta_\varepsilon}}.$$

We estimate the right hand side. Using Hölder's inequality and (29) we have

$$\left|\langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon, v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right)\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Using again Hölder's inequality and assumption of f, we obtain that

$$\left|\langle f, v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C\eta_\varepsilon^{\frac{1}{2}}\|f\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3}\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3},$$

and by estimate (26), we have

$$\left|\langle f, v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C(\eta_\varepsilon^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}})\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Then, we have

$$\|\nabla_\varepsilon\tilde{p}_\varepsilon\|_{H^{-1}(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}} + \varepsilon\right).$$

Reasoning as in the proof of Lemma 4.4, we observe that $\eta_\varepsilon\varepsilon^{\frac{1}{2}}$ can be dropped and so we obtain

$$\|\nabla_\varepsilon\tilde{p}_\varepsilon\|_{H^{-1}(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right).$$

Finally, taking into account that $\int_{\widetilde{I}_{\eta_\varepsilon}}(\tilde{p}_\varepsilon - c_{\varepsilon\eta_\varepsilon})\,dx'dy_3 = 0$, we use Lemma 4.3 and we obtain the estimate (34). □

5. Proof of the main result

In view of estimates (27), (29) of the velocity and (33) of the pressure, the proof of Theorem 3.1 will be divided in three characteristic cases: $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$.

5.1. Problem in the porous part $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$

The proof of Theorem 3.1-*i*) will be developed in different lemmas.

In this subsection, we need to extend the velocity $\tilde{u}_\varepsilon$ by zero in the fissure $\widetilde{I}_{\eta_\varepsilon}$, and we will denote the extended veolcity by $\tilde{v}_\varepsilon$, i.e.

$$\tilde{v}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{in } \widetilde{\Omega}_{\varepsilon\eta_\varepsilon}, \\ 0 & \text{in } \widetilde{I}_{\eta_\varepsilon}. \end{cases} \tag{35}$$

Lemma 5.1. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and let $(\tilde{v}_\varepsilon, \tilde{p}_\varepsilon)$ be the extended solution of (4)-(5). Then there exist subsequences of $\tilde{v}_\varepsilon$ and $\tilde{p}_\varepsilon$ still denoted by the same, and functions $\tilde{v} \in H^1(0,1;L^2(\omega)^3)$ with $\tilde{v}_3 = 0$, $\tilde{p} \in L^2_0(D)$, which does not depend on y_3, such that*

$$\varepsilon^{-2}\tilde{v}_\varepsilon \rightharpoonup (\tilde{v}', 0) \quad \text{in } H^1(0,1;L^2(\omega)^3), \quad \tilde{p}_\varepsilon \to \tilde{p} \quad \text{in } L^2(D). \tag{36}$$

Moreover, $\tilde{v}$ satisfies

$$\text{div}_{x'}\left(\int_0^1 v'(x', y_3)dy_3\right) = 0 \quad \text{in } \omega, \quad \left(\int_0^1 v'(x', y_3)dy_3\right)\cdot n = 0 \quad \text{on } \partial w. \tag{37}$$

Proof. From estimates (27), (29) and (33), taking into account the extension of the velocity by zero to D and $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, we have the following estimates

$$\|\tilde{v}_\varepsilon\|_{L^2(D)^3} \leq C\varepsilon^2, \quad \|\tilde{p}_\varepsilon\|_{L^2(D)} \leq C,$$
$$\|D_{x'}\tilde{v}_\varepsilon\|_{L^2(D)^{3\times 2}} \leq C\varepsilon, \quad \|\partial_{y_3}\tilde{v}_\varepsilon\|_{L^2(D)^3} \leq C\varepsilon^2. \tag{38}$$

Then there exist $\tilde{v} \in H^1(0,1;L^2(\omega)^3)$ and $\tilde{p} \in L^2_0(D)$ such that, for a subsequence still denoted by $\tilde{v}_\varepsilon$, $\tilde{p}_\varepsilon$, it holds

$$\varepsilon^{-2}\tilde{v}_\varepsilon \rightharpoonup \tilde{v} \quad \text{in } H^1(0,1;L^2(\omega)^3), \quad \tilde{p}_\varepsilon \rightharpoonup \tilde{p} \quad \text{in } L^2(D), \tag{39}$$

which implies

$$\frac{1}{\varepsilon^2}\text{div}_{x'}\tilde{v}'_\varepsilon \rightharpoonup \text{div}_{x'}\tilde{v}' \text{ in } H^1(0,1;H^{-1}(\omega)). \tag{40}$$

Since $\text{div}_\varepsilon\tilde{v}_\varepsilon = 0$ in D, multiplying by ε^{-2} we obtain

$$\frac{1}{\varepsilon^2}\text{div}_{x'}\tilde{v}'_\varepsilon + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{v}_{\varepsilon,3} = 0, \quad \text{in } D,$$

which, combined with (40), implies that $\partial_{y_3}\tilde{v}_{\varepsilon,3}/\varepsilon^3$ is bounded in $H^1(0,1;H^{-1}(\omega))$. This implies that $\partial_{y_3}\tilde{v}_{\varepsilon,3}/\varepsilon^2$ tends to zero in $H^1(0,1;H^{-1}(\omega))$. Also, from the second estimate in (38), we have that $\partial_{y_3}\tilde{v}_{\varepsilon,3}/\varepsilon^2$ tends to $\partial_{y_3}\tilde{v}_3$ in $L^2(D)^3$. From the uniqueness of the limit, we have that $\partial_{y_3}\tilde{v}_3 = 0$, which implies that $\tilde{v}_3$ does not depend on y_3.

It remains to prove that $\tilde{v}_3 = 0$. In order to do that, let us first show that $\tilde{p}$ only depends on x'. As usual, we take a test function $\phi = (0, \varepsilon\phi_3)$ in the momentum equation in (4). From convergences (39), we deduce that $\partial_{y_3}\tilde{p} = 0$, which implies that $\tilde{p}$ only depends on x'. Next, as $\tilde{v}_3$ does not depend on y_3, we take a test function $\phi = (0, \varepsilon^{-2}\phi_3(x'))$ in (4), and passing to the limit we can deduce that $\tilde{v}_3 = 0$.

Next, we prove that the convergence of the pressure is in fact strong. As $\tilde{v}_3 = 0$, let $w_\varepsilon = (w'_\varepsilon, 0) \in H^1_0(D)^3$ be such that

$$w_\varepsilon \rightharpoonup w \quad \text{in } H^1_0(D)^3. \tag{41}$$

Then (brackets are for the duality products between H^{-1} and H^1_0):

$$|< \nabla_\varepsilon \tilde{p}_\varepsilon, w_\varepsilon >_D - < \nabla_{x'}\tilde{p}, w >_D| \le |< \nabla_\varepsilon \tilde{p}_\varepsilon, w_\varepsilon - w >_D| + |< \nabla_\varepsilon \tilde{p}_\varepsilon - \nabla_{x'}\tilde{p}, w >_D| .$$

On the one hand, using the second convergence in (39), we have

$$|< \nabla_\varepsilon \tilde{p}_\varepsilon - \nabla_{x'}\tilde{p}, w >_D| = \int_D (\tilde{p}_\varepsilon - \tilde{p})\,\mathrm{div}_{x'}\, w'\, dx \to 0, \quad \text{as } \varepsilon \to 0.$$

On the other hand, we have

$$\begin{aligned}|< \nabla_\varepsilon \tilde{p}_\varepsilon, w_\varepsilon - w >_D| &= \left|< \nabla_{x'}\tilde{p}_\varepsilon, R^\varepsilon(w'_\varepsilon - w') >_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| \\ &= \left|\langle \mu\Delta_{x'}\tilde{v}'_\varepsilon, R^\varepsilon(w'_\varepsilon - w')\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} - \langle f', R^\varepsilon(w'_\varepsilon - w')\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right|,\end{aligned}$$

and using Hölder's inequality, estimate (29), the estimates of the restricted operator R^ε applied to $D_{x'}$ instead of D_ε, and taking into account that $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and $\varepsilon \ll 1$, we get

$$\begin{aligned}&|< \nabla_\varepsilon \tilde{p}_\varepsilon, w_\varepsilon - w >_D| \le C\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon\right)\left(\frac{1}{\varepsilon}\|w'_\varepsilon - w'\|_{L^2(D)^2} + \|D_{x'}w'_\varepsilon - D_{x'}w'\|_{L^2(D)^{2\times 2}}\right) \\ &+C\left(\|w'_\varepsilon - w'\|_{L^2(D)^2} + \varepsilon\|D_{x'}w'_\varepsilon - D_{x'}w'\|_{L^2(D)^{2\times 2}}\right) \\ &\le C\left(\|w'_\varepsilon - w'\|_{L^2(D)^2} + \varepsilon\|D_{x'}w'_\varepsilon - D_{x'}w'\|_{L^2(D)^{2\times 2}}\right) \to 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

by virtue (41) and the Rellich Theorem. This implies that $\nabla_\varepsilon \tilde{p}_\varepsilon \to \nabla_{x'}\tilde{p}$ strongly in $H^{-1}(D)^3$, which implies the strong convergence of the pressure given in (36).

Finally, we prove (37). To do this, we consider $w \in C^1_c(\omega)$ as test function in $\mathrm{div}_\varepsilon \tilde{v}_\varepsilon = 0$ in D, which gives

$$\frac{1}{\varepsilon^2}\int_D \mathrm{div}_{x'}\tilde{v}_\varepsilon\, w(x')\, dx' dy_3 = 0.$$

From convergences (36), we get (37).

□

The proof of the following result will be showed by using the two-scale convergence introduced by Nguesteng [9] in the L^2-setting and developed by Allaire [2]. In this case, we use the two-scale convergence in the horizontal variables. By $\overset{2}{\rightharpoonup}$ we denote the limit in the two-scale sense and by $\sharp$ we denote Y'-periodicity.

Lemma 5.2. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and let $\tilde{v}_\varepsilon$ be the extended solution of (4)-(5). Then there exist subsequences of $\tilde{v}_\varepsilon$ still denoted by the same, and $\hat{v}(x', y', y_3) \in L^2(D'; H^1_\sharp(Y)^3)$ such that*

$$\begin{gathered}\varepsilon^{-2}\tilde{v}_\varepsilon \overset{2}{\rightharpoonup} \hat{v}(x', y', y_3) \quad \text{in } L^2(D' \times Y)^3, \qquad \varepsilon^{-1}D_{x'}\tilde{v}_\varepsilon \overset{2}{\rightharpoonup} D_{y'}\hat{v}(x', y', y_3) \quad \text{in } L^2(D' \times Y)^{3\times 3}, \\ \varepsilon^{-2}\partial_{y_3}\tilde{v}_\varepsilon \overset{2}{\rightharpoonup} \partial_{y_3}\hat{v}(x', y', y_3) \quad \text{in } L^2(D' \times Y)^3,\end{gathered} \tag{42}$$

The weak limit $v(x)$ and the two-scale limit $\hat{v}(x,y)$ are related as follows

$$\tilde{v}(x',y_3)=\int_{Y'}\hat{v}(x',y',y_3)\,dy'. \tag{43}$$

Moreover, $\hat{v}$ satisfies

$$\mathrm{div}_y\hat{v}(x',y',y_3)=0 \quad \text{in } D'\times Y, \quad \hat{v}=0 \quad \text{in } Y\setminus Y_s, \tag{44}$$

$$\mathrm{div}_{x'}\left(\int_Y \hat{v}(x',y)\,dy\right)=0 \ \text{ in } D', \quad \left(\int_Y \hat{v}(x',y)\,dy\right)\cdot n=0 \ \text{ on } \partial D'. \tag{45}$$

Proof. From estimates (27) and (29) and taking into account that $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, we get

$$\|\tilde{v}_\varepsilon\|_{L^2(D)^3}\leq C\varepsilon^2, \quad \|D_\varepsilon\tilde{v}_\varepsilon\|_{L^2(D)^{3\times 3}}\leq C\varepsilon.$$

Thus, from Lemma 1.5 in [7], there exist subsequences of $\tilde{v}_\varepsilon$, still denoted by $\tilde{v}_\varepsilon$, and function $\hat{v}\in L^2(D_1;H^1_\#(Y)^3)$ such that the convergences given in (42) hold.

Relation (43) is a classical property relating weak convergence and two-scale convergence, see Allaire [2] and Bourgeat and Mikelic [7] for more details. From $\mathrm{div}_\varepsilon\,\tilde{v}_\varepsilon=0$ in D and the convergences (42), then (44) straightforward. Finally, (37) and (43) imply (45).

□

Lemma 5.3. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and let $(\tilde{v}_\varepsilon,\tilde{p}_\varepsilon)$ be the extended solution of (4)-(5). Let $(\tilde{v},\tilde{p})\in L^2(D)^3\times L^2_0(D)$ be given by Lemma 5.1. Then, $\tilde{p}\in H^1(D)\cap L^2_0(D)$ and $(\tilde{v},\tilde{p})$ is the unique solution of Darcy's law (12).*

Proof. We choose $\phi_{+-}(x',y',y_3)\in\mathcal{D}(D'_{+-};C^\infty_\sharp(Y)^3)$ with $\phi_{+-}=0$ in $D'_{+-}\times Y_s$ and satisfying incompressibility condition (44). There exists $\eta_1>0$ such that *supp* $\phi_+(x',y',y_3)\subset D\setminus\widetilde{I}_{\eta_\varepsilon}$ for every $\eta_\varepsilon\in(0,\eta_1)$. Let $\eta_\varepsilon<\eta_1$. We define a test function $\phi^{+-}_\varepsilon(x',y_3)=\phi_{+-}(x',x'/\varepsilon,y_3)$ in (4)-(5). In the sequel, we use the elementary properties of the two-scale convergence. Using the two-scale convergence of $\varepsilon^{-2}\tilde{v}_\varepsilon$ given in (42), we have

$$\int_{D_{+-}} f\cdot\phi^{+-}_\varepsilon\,dx\to\int_{D'_{+-}}\int_Y f'\cdot\phi'_{+-}\,dx'dy,$$

and using that $\mathrm{div}_y\phi_{+-}=0$ in $D'_{+-}\times Y$ and the strong convergence of the pressure (36), we have

$$\langle\nabla_\varepsilon\tilde{p}_\varepsilon,\phi^{+-}_\varepsilon\rangle_{D_{+-}}=-\int_{D_{+-}}\tilde{p}_\varepsilon\,\mathrm{div}_{x'}\,(\phi'_\varepsilon)^{+-}\,dx'dy_3\to-\int_{D'_{+-}}\int_Y\tilde{p}\,\mathrm{div}_{x'}\phi'_{+-}(x',y)\,dx'dy, \quad \text{as } \varepsilon\to 0.$$

Therefore, passing to the limit, we obtain

$$\begin{aligned}&\mu\int_{D'_{+-}\times Y}D_{y'}\hat{v}:D_{y'}\phi_{+-}\,dx'dy+\mu\int_{D'_{+-}\times Y}\partial_{y_3}\hat{v}:\partial_{y_3}\phi_{+-}\,dx'dy\\ &=\int_{D'_{+-}\times Y}f'\cdot\phi'_{+-}\,dx'dy-\int_{D'_{+-}\times Y}\nabla_{x'}\tilde{p}\cdot\phi'_{+-}\,dx'dy\,.\end{aligned}$$

Consequently, there exists $\hat{\pi}\in L^2(D';L^2_0(Y))$ such that $(\hat{v},\hat{\pi})$ satisfies the homogenized problem

$$-\mu\Delta_y\hat{v}+\nabla_y\hat{\pi}=f'(x')-\nabla_{x'}\tilde{p}(x') \quad \text{in } Y_f, \tag{46}$$

$$\mathrm{div}_y\hat{v}(x',y)=0 \quad \text{in } Y_f, \tag{47}$$

$$(\hat{v},\hat{\pi}) \text{ is } Y'-\text{periodic}, \quad \hat{v}=0 \quad \text{in } Y\setminus Y_f, \tag{48}$$

a.e. $x'\in D'$, by using the variant of de Rham's formula in a periodic setting (see Nguetseng [9] and Temam [13]).

The derivation of (12) from the effective problems (46)-(48) is straightforward by using the local problems (14) and definitions of the permeability functions (13).

Since $\tilde{V}'\in L^2(D')^2$, thanks to (12), we get that $\tilde{p}$ belongs to $H^1(D')\cap L^2_0(D')$.

□

Proof of Theorem 3.1-*i*). It remains to prove convergence (11) of the whole velocity $\tilde{u}_\varepsilon$, i.e. to prove

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \to 0. \tag{49}$$

For this, it is sufficient to prove that

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \to 0 \quad \text{for } \eta_\varepsilon \ll \varepsilon, \tag{50}$$

and

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^q(\widetilde{I}_{\eta_\varepsilon})^3} \to 0 \quad \text{for } \varepsilon \ll \eta_\varepsilon \ll \varepsilon^{\frac{1}{\alpha}} \quad, 1 < \alpha < \frac{3}{2}, \tag{51}$$

for a q which will be defined below.

Using (28) and using $\eta_\varepsilon \ll \varepsilon$, we have

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\frac{\eta_\varepsilon^{\frac{5}{2}}}{\varepsilon^2} + \frac{\eta_\varepsilon}{\varepsilon} + \left(\frac{\eta_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\right),$$

so that (50) easily holds. Using Hölder's inequality with the conjugate exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ we obtain

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^q(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\frac{\eta_\varepsilon^{\frac{1}{q}+2}}{\varepsilon^2} + \frac{\eta_\varepsilon^{\frac{1}{q}+\frac{1}{2}}}{\varepsilon} + \frac{\eta_\varepsilon^{\frac{1}{q}}}{\varepsilon^{\frac{1}{2}}}\right).$$

Now we take $\eta_\varepsilon = \varepsilon^{\frac{1}{\alpha}}$. Then we find that

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^q(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}+2\right)-2} + \varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}+\frac{1}{2}\right)-1} + \varepsilon^{\frac{1}{q\alpha}-\frac{1}{2}}\right). \tag{52}$$

We seek an optimal q such that the right hand side in (52) tends to zero. It is easy to prove that we have a convergence to zero for any $q \in \left(1, \frac{2}{2(\alpha-1)+1}\right)$. Therefore, (51) holds and so we have (49).

□

5.2. Problem in the fissure part $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$

The proof of Theorem 3.1-*ii*) will be developed in different lemmas.

Lemma 5.4. *Let $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ and let $(\tilde{\mathcal{U}}^\varepsilon, \tilde{P}^\varepsilon)$ be the solution of* (9)*-*(10)*. Then there exist subsequences of $\tilde{\mathcal{U}}^\varepsilon$ and $\tilde{P}^\varepsilon$ still denoted by the same, and functions $\tilde{\mathcal{U}} \in L^2(\widetilde{I}_1)^3$, independent of y_3, with $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, $\tilde{P} \in L^2_0(\widetilde{I}_1)$ such that*

$$\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \tilde{\mathcal{U}} \quad \text{in } L^2(\widetilde{I}_1)^3, \quad \tilde{P}^\varepsilon \rightharpoonup \tilde{P} \quad \text{in } L^2(\widetilde{I}_1). \tag{53}$$

Moreover, $\tilde{P} = \tilde{P}(x_1)$ and $\tilde{\mathcal{U}}_1$ is given by expression (15)*.*

Proof. Taking into account $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ and estimates (28), (29), (34) with the change of variable (6), we have

$$\|\tilde{\mathcal{U}}^\varepsilon\|_{L^2(\widetilde{I}_1)^3} \leq C\eta_\varepsilon{}^2, \tag{54}$$

$$\|\partial_{x_1}\tilde{\mathcal{U}}^\varepsilon\|_{L^2(\widetilde{I}_1)^3} \leq C\eta_\varepsilon, \quad \|\partial_{y_2}\tilde{\mathcal{U}}^\varepsilon\|_{L^2(\widetilde{I}_1)^3} \leq C\eta_\varepsilon{}^2, \tag{55}$$

$$\|\partial_{y_3}\tilde{\mathcal{U}}^\varepsilon\|_{L^2(\widetilde{I}_1)^3} \leq C\varepsilon\,\eta_\varepsilon, \tag{56}$$

$$\|\tilde{P}^\varepsilon\|_{L^2(\widetilde{I}_1)} \leq C. \tag{57}$$

From these estimates (54) and (57), there exist $\tilde{\mathcal{U}} \in L^2(\widetilde{I}_1)^3$, $\tilde{P} \in L^2_0(\widetilde{I}_1)$ such that convergence (53) holds. Moreover

$$\eta_\varepsilon{}^{-2}\partial_{y_2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \partial_{y_2}\tilde{\mathcal{U}} \quad \text{in } L^2(\widetilde{I}_1)^3. \tag{58}$$

The estimate (56) implies that $\varepsilon^{-1}\eta_\varepsilon^{-1}\partial_{y_3}\mathcal{U}^\varepsilon$ is bounded in $L^2(\widetilde{I}_1)^3$. This together with $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ implies that $\eta_\varepsilon^{-2}\partial_{y_3}\mathcal{U}^\varepsilon$ tends to $\partial_{y_3}\tilde{\mathcal{U}} = 0$. This implies that $\tilde{\mathcal{U}}$ does not depend on y_3.

As $\tilde{\mathcal{U}}$ does not depend on y_3, let $\varphi \in C_0^\infty(\widetilde{I}_1)^3$ independent of y_3. Taking into account that $\mathrm{div}_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon = 0$ in $\widetilde{I}_1$, we have

$$\begin{aligned} & {\eta_\varepsilon}^{-1} \int_{\widetilde{I}_1} \left(\partial_{x_1} \tilde{\mathcal{U}}_1^\varepsilon + {\eta_\varepsilon}^{-1} \partial_{y_2} \tilde{\mathcal{U}}_2^\varepsilon + \varepsilon^{-1} \partial_{y_3} \tilde{\mathcal{U}}_3^\varepsilon \right) \varphi \, dx_1 dy_2 dy_3 \\ = \; & -{\eta_\varepsilon}^{-1} \int_{\widetilde{I}_1} \tilde{\mathcal{U}}_1^\varepsilon \partial_{x_1} \varphi \, dx_1 dy_2 dy_3 - {\eta_\varepsilon}^{-2} \int_{\widetilde{I}_1} \tilde{\mathcal{U}}_2^\varepsilon \cdot \partial_{y_2} \varphi \, dx_1 dy_2 dy_3 = 0. \end{aligned}$$

Taking the limit $\varepsilon \to 0$ we obtain

$$\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_2 \partial_{y_2} \varphi \, dx_1 dy_2 dy_3 = 0,$$

so that $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_2(x_1)$.

Since $\tilde{\mathcal{U}}$, $\partial_{y_2} \tilde{\mathcal{U}} \in L^2(\widetilde{I}_1)^3$ the traces $\tilde{\mathcal{U}}(x_1, 0)$, $\tilde{\mathcal{U}}(x_1, 1)$ are well defined in $L^2(\Sigma)^3$. Analogously to the proof of Lemma 4.2 we choose a point $\beta_{(x_1,y_3)} \in \widetilde{A}_{\varepsilon\eta_\varepsilon}$, which is close to the point $\alpha_{(x_1,y_3)} \in \Sigma$, then we have

$$\begin{aligned} \int_\Sigma |\tilde{\mathcal{U}}^\varepsilon(x', 0, y_3)|^2 dx_1 dy_3 &= \int_\Sigma |\tilde{u}_\varepsilon(x_1, 0, y_3)|^2 dx_1 dy_3 \\ &\leq C \int_\Sigma \left(\int_{(\beta_{(x_1,y_3)}, \alpha_{(x_1,y_3)})} D_\varepsilon \tilde{u}_\varepsilon \cdot (\alpha_{(x_1,y_3)} - \beta_{(x_1,y_3)}) d\ell \right)^2 dx_1 dy_3, \end{aligned}$$

so that, by Hölder's inequality,

$$\|\tilde{\mathcal{U}}^\varepsilon(x_1, 0, y_3)\|^2_{L^2(\Sigma)^3} \leq C\varepsilon \|D_\varepsilon \tilde{u}_\varepsilon\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}.$$

Taking into account estimate (29) and $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, we have

$${\eta_\varepsilon}^{-2} \|\tilde{\mathcal{U}}^\varepsilon(x_1, 0, y_3)\|^2_{L^2(\Sigma)^3} \leq C\varepsilon\eta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0,$$

which implies that

$$\tilde{\mathcal{U}}(x_1, 0) = 0\,,$$

and analogously

$$\tilde{\mathcal{U}}(x_1, 1) = 0\,.$$

Consequently

$$\tilde{\mathcal{U}}_2 = 0\,.$$

It remains to prove that $\tilde{\mathcal{U}}_3 = 0$. In order to do that, as $\tilde{\mathcal{U}}$ does not depend on y_3, we take a test function $v = (0, 0, v_3(x_1, y_2))$ in (9), and passing to the limit, with the convergence (58), we can deduce that $\tilde{\mathcal{U}}_3 = 0$.

Finally, we compute the expression of $\tilde{\mathcal{U}}$ given in (15). First, we take a test function $v = (0, \eta_\varepsilon v_2, \varepsilon v_3)$ in (9), and passing to the limit, with the convergences (53), we can deduce that $\tilde{P}$ only depends on x_1. Now, taking into account that $\tilde{\mathcal{U}}$ does not depend on y_3 and $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, we take a test function $v = (v_1(x_1, y_2), 0, 0)$ in (9), and passing to the limit, we obtain the ODE

$$\begin{cases} -\mu \partial^2_{y_2} \tilde{\mathcal{U}}_1(x_1, y_2) = f_1(x_1, 0) - \partial_{x_1} \tilde{P}(x_1), \\ \\ \tilde{\mathcal{U}}_1(x_1, 0) = \tilde{\mathcal{U}}_1(x_1, 1) = 0, \end{cases}$$

which gives the expression (15) for $\tilde{\mathcal{U}}_1$.

□

Proof of Theorem 3.1-*ii*). It remains to prove the convergence (16) of the whole velocity to the function $\mathcal{V}$ given by (17), and also prove that $\tilde{P} \in H^1(\Sigma) \cap L^2_0(\Sigma)$ is the unique solution of the Reynolds problem (18).

Taking as test function $\varphi \in C^\infty(D)$, independent of y_3, in the equation $\mathrm{div}_\varepsilon\, \tilde u_\varepsilon = 0$ in D, we obtain

$$\int_D \mathrm{div}_\varepsilon\, \tilde u_\varepsilon \varphi\, dx'dy_3 = -\int_D \tilde v_\varepsilon' \cdot \nabla_{x'}\varphi\, dx'dy_3 - \eta_\varepsilon \int_{\widetilde I_1} (\tilde{\mathcal U}^\varepsilon)' \cdot \nabla_{x'}\varphi(x_1,\eta_\varepsilon y_2)\, dx_1dy_2dy_3 = 0,$$

so that multiplying by ${\eta_\varepsilon}^{-3}$,

$$\begin{aligned} &\int_{\widetilde I_1} {\eta_\varepsilon}^{-2}\tilde{\mathcal U}_1^\varepsilon \partial_{x_1}\varphi(x_1,\eta_\varepsilon y_2)\, dx_1dy_2dy_3 \\ &= -\int_D {\eta_\varepsilon}^{-3}\tilde v_\varepsilon \cdot \nabla_{x'}\varphi\, dx'dy_3 - \int_{\widetilde I_1} {\eta_\varepsilon}^{-2}\tilde{\mathcal U}_2^\varepsilon \partial_{x_2}\varphi(x_1,\eta_\varepsilon y_2)\, dx_1dy_2dy_3. \end{aligned} \tag{59}$$

Using (27) and taking into account $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, we obtain

$${\eta_\varepsilon}^{-3}\|\tilde v_\varepsilon\|_{L^2(D)^3} \le C\left(\frac{\varepsilon}{{\eta_\varepsilon}^{\frac{3}{2}}} + \frac{\varepsilon^2}{{\eta_\varepsilon}^3}\right) \to 0 \quad \text{as } \varepsilon \to 0. \tag{60}$$

Taking the limit in (59) as $\varepsilon \to 0$, using convergence (53), $\tilde{\mathcal U}_2 = 0$ and $\tilde{\mathcal U}_1$ independent of y_3, we have

$$\int_\Sigma \tilde{\mathcal U}_1 \partial_{x_1}\varphi(x_1,0)\, dx_1dy_2 = 0,$$

and by definition (17), we get

$$\int_{\Sigma_1} \left(f_1(x_1,0) - \partial_{x_1}\tilde P(x_1)\right) \partial_{x_1}\varphi(x_1,0)\, dx_1 = 0.$$

Consequently, $\tilde P \in H^1(\Sigma_1) \cap L^2_0(\Sigma_1)$ and is the unique solution of (18). Finally, we consider $\varphi \in C_0(D)^3$, independent of y_3, and so we have

$$\int_D {\eta_\varepsilon}^{-3}\tilde u_\varepsilon \varphi\, dx'dy_3 = \int_D {\eta_\varepsilon}^{-3}\tilde v_\varepsilon \varphi\, dx'dy_3 + \int_{\widetilde I_1} {\eta_\varepsilon}^{-2}\tilde{\mathcal U}^\varepsilon \varphi(x_1,\eta_\varepsilon y_2)\, dx_1dy_2dy_3.$$

Using (60), convergence (53) and $\tilde{\mathcal U}_2 = \tilde{\mathcal U}_3 = 0$, we obtain

$$\begin{aligned} \int_D {\eta_\varepsilon}^{-3}\tilde u_\varepsilon \varphi\, dx'dy_3 \quad \to \quad & \int_\Sigma \tilde{\mathcal U}_1(x_1,y_2)\varphi_1(x_1,0)\, dx_1dy_2 \\ &= \int_{\Sigma_1} \tilde{\mathcal V}_1(x_1)\varphi_1(x_1,0)\, dx_1 = \langle \tilde{\mathcal V}_1(x_1)\delta_{\Sigma_1}, \varphi\rangle_{\mathcal M(D)^3, C_0(D)^3}, \end{aligned}$$

which implies (16). □

5.3. Effects of coupling $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$

The conclusion of the previous two subsections is that for any sequence of solutions $(\tilde v_\varepsilon, \tilde p_\varepsilon)$ with $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and $(\tilde{\mathcal U}^\varepsilon, \tilde P^\varepsilon)$ with $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, and letting $\varepsilon \to 0$, we can extract subsequences still denoted by $\tilde v_\varepsilon, \tilde p_\varepsilon, \tilde{\mathcal U}^\varepsilon, \tilde P^\varepsilon$ and find functions $\tilde v \in H^1(0,1;L^2(\omega)^3)$ with $\tilde v_3 = 0$, $\tilde p \in H^1(D) \cap L^2_0(D)$, $\tilde{\mathcal U} \in L^2(\widetilde I_1)^3$, independent of y_3, with $\tilde{\mathcal U}_2 = \tilde{\mathcal U}_3 = 0$, $\tilde P \in H^1(\Sigma) \cap L^2_0(\Sigma)$ such that

$$\begin{aligned} &\varepsilon^{-2}\tilde v_\varepsilon \rightharpoonup (\tilde v', 0) \quad \text{in } H^1(0,1;L^2(\omega)^3), \quad \tilde p_\varepsilon \to \tilde p \quad \text{in } L^2(D), \\ &{\eta_\varepsilon}^{-2}\tilde{\mathcal U}^\varepsilon \rightharpoonup (\tilde{\mathcal U}_1, 0, 0) \quad \text{in } L^2(\widetilde I_1)^3, \quad \tilde P^\varepsilon \rightharpoonup \tilde P \quad \text{in } L^2(\widetilde I_1). \end{aligned} \tag{61}$$

Moreover such limit functions $\tilde v, \tilde p, \tilde{\mathcal U}, \tilde P$ necessarily satisfy the equations

$$\begin{aligned} &\tilde V'(x') = \frac{1}{\mu}K\left(f'(x') - \nabla_{x'}\tilde p(x')\right) \quad \text{in } D', \\ &\tilde{\mathcal U}_1(x_1,y_2) = \frac{y_2(1-y_2)}{2\mu}\left(f_1(x_1,0) - \partial_{x_1}\tilde P(x_1)\right) \quad \text{in } \widetilde I_1, \end{aligned} \tag{62}$$

where $\tilde V'(x') = \int_0^1 \tilde v'(x',y_3)dy_3$.

We are going to find the connection between the functions $\tilde{p}$ and $\tilde{P}$, i.e. to find the coupling effects between the solution in the porous part and in the fissure.

Lemma 5.5. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and let $\tilde{p}_\varepsilon \in L^2_0(D)$, $\tilde{p} \in H^1(D) \cap L^2_0(D)$, $\tilde{P} \in H^1(\Sigma) \cap L^2_0(\Sigma)$ be such that (61) and (62) hold. Then,*

$$\int_{D'} \frac{1}{\mu} K\left(f'(x') - \nabla_{x'}\tilde{p}(x')\right) \cdot \nabla_{x'}\varphi(x')\, dx' + \frac{\lambda^3}{12\mu}\int_{\Sigma_1}\left(f_1(x_1,0) - \partial_{x_1}\tilde{P}(x_1)\right)\partial_{x_1}\varphi(x_1,0)\, dx_1 = 0, \tag{63}$$

for every $\varphi \in H^1(D')$ with $\varphi(\cdot,0) \in H^1(\Sigma_1)$.

Proof. Let $\varphi_\varepsilon(x', y_3) = \varphi(x', \varepsilon y_3) \in H^1(D)$ with $\varphi \in H^1(\overline{D})$ and $\varphi(\cdot, 0) \in H^1(\Sigma)$. Taking into account the definitions (35) of $\tilde{v}_\varepsilon$ and (7) of $\tilde{\mathcal{U}}^\varepsilon$, and from $\mathrm{div}_\varepsilon\, \tilde{u}_\varepsilon = 0$ in D we have

$$\int_D \varepsilon^{-2}\tilde{u}_\varepsilon \cdot \nabla_\varepsilon \varphi_\varepsilon\, dx' dy_3 = \int_D \varepsilon^{-2}\tilde{v}_\varepsilon \cdot \nabla_\varepsilon\varphi_\varepsilon\, dx'dy_3 + \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3 \int_{\widetilde{I}_1} \eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \cdot \nabla_\varepsilon \varphi_\varepsilon(x_1, \eta_\varepsilon y_2, y_3)\, dx_1 dy_2 dy_3 = 0,$$

and by the definition of φ_ε, we can deduce

$$\int_D \varepsilon^{-2}\tilde{v}_\varepsilon \cdot \nabla\varphi(x', \varepsilon y_3)\, dx'dy_3 + \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3 \int_{\widetilde{I}_1} \eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \cdot \nabla\varphi(x_1, \eta_\varepsilon y_2, \varepsilon y_3)\, dx_1 dy_2 dy_3 = 0.$$

Taking the limit as $\varepsilon \to 0$, using (61), $\tilde{v}_3 = \tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, and taking into account that $\tilde{\mathcal{U}}_1$ does not depend on y_3, we obtain

$$\int_D \tilde{v}'(x', y_3) \cdot \nabla_{x'}\varphi(x', 0)\, dx'dy_3 + \lambda^3 \int_\Sigma \tilde{\mathcal{U}}_1(x_1, y_2)\partial_{x_1}\varphi(x_1, 0, 0)\, dx_1 dy_2 = 0,$$

and taking into account expressions (62) and (17), we get (63). □

We are going to prove the relation $\tilde{p}(x_1, 0) = \tilde{P}(x_1) + C$, with $C \in \mathbb{R}$. Then (20) follows from (63).

Lemma 5.6. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and let $\tilde{p}$, $\tilde{P}$ be the limit pressures from (61). Then, there exists $C \in \mathbb{R}$ such that*

$$\tilde{p}(x_1, 0) = \tilde{P}(x_1) + C, \tag{64}$$

and $\tilde{p} \in H^1(D') \cap L^2_0(D')$ with $\tilde{p}(\cdot, 0) \in H^1(\Sigma_1) \cap L^2_0(\Sigma_1)$ is the unique solution of the variational problem (20).

Proof. We need to extend the test functions considered in the proof of Lemma 5.3 to the fissure $\widetilde{I}_{\eta_\varepsilon}$. To do this, we define $I'_{\eta_\varepsilon} = \widetilde{I}_{\eta_\varepsilon} \cap \{x_3 = 0\}$, $B_{\eta_\varepsilon} = D'_- \cup \Sigma_1 \cup I'_{\eta_\varepsilon}$ and $Y_1 = \overline{Y}_f \cap \{x_2 = 0\}$, and we consider $\phi(y') \in C^\infty_\#(B_{\eta_\varepsilon})^3$ be such that $\phi(y') = 0$ in $Y' \setminus Y'_f$. We define

$$\phi_\varepsilon(x') = \begin{cases} \phi\left(\frac{x'}{\varepsilon}\right) & \text{in } D'_-, \\ K_2\, e_2 & \text{in } I'_{\eta_\varepsilon}, \quad \text{where } K_2 = \displaystyle\int_{Y_1} \phi_2(y_1, 0) dy_1 . \end{cases}$$

Let $\varphi \in C^\infty_0(B_1)$, with $B_1 = D_- \cup \Sigma \cup \widetilde{I}_1$ be such that

$$\int_\Sigma \varphi(x_1, 0, y_3)\, dx_1 dy_3 = 0, \tag{65}$$

and $\mathrm{div}_y\left(\varphi(x', y_3)\phi(y')\right) = 0$ in Y_f.

Taking in (4) as test function

$$w_\varepsilon(x', y_3) = \begin{cases} \varphi(x', y_3)\phi\left(\frac{x'}{\varepsilon}\right) & \text{in } D_-, \\ \varphi\left(x_1, \frac{x_2}{\eta_\varepsilon}, y_3\right) K_2\, e_2 & \text{in } \widetilde{I}_{\eta_\varepsilon}, \end{cases}$$

we obtain

$$\mu \int_{B_{\eta_\varepsilon}} D_\varepsilon \tilde{u}_\varepsilon : D_\varepsilon w_\varepsilon \, dx' dy_3 = \int_{B_{\eta_\varepsilon}} f' \cdot w'_\varepsilon \, dx' dy_3 - \int_{B_{\eta_\varepsilon}} \tilde{p}_\varepsilon \, \mathrm{div}_\varepsilon \, w_\varepsilon \, dx' dy_3. \tag{66}$$

Taking into account that

$$K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} f' \cdot \varphi' \left(x_1, \frac{x_2}{\eta_\varepsilon}, y_3 \right) e_2 \, dx' dy_3 = \eta_\varepsilon K_2 \int_{\widetilde{I}_1} f' \cdot \varphi'(x_1, y_2, y_3) e_2 \, dx_1 dy_2 dy_3 \to 0 \quad \text{as } \varepsilon \to 0,$$

and by using estimates (55), (56), that

$$\left| K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} D_\varepsilon \tilde{\mathcal{U}}^\varepsilon \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 \right| = \left| K_2 \int_{\widetilde{I}_1} D_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon \partial_{y_2} \varphi(x_1, y_2, y_3) \, dx_1 dy_2 dy_3 \right| \leq C \eta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0,$$

from (66), we obtain

$$\mu \int_{D_-} D_\varepsilon \tilde{v}_\varepsilon : D_\varepsilon w_\varepsilon \, dx' dy_3 = \int_{D_-} f' \cdot w'_\varepsilon \, dx' dy_3 + \int_{D_-} \tilde{p}_\varepsilon \mathrm{div}_\varepsilon \, w_\varepsilon \, dx' dy_3 + K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 + O_\varepsilon. \tag{67}$$

For the last term on the right hand side, we have

$$\begin{aligned} K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 \quad = \quad & K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} (\tilde{p}_\varepsilon - c_{\varepsilon \eta_\varepsilon}) \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 \\ & + K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} c_{\varepsilon \eta_\varepsilon} \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3, \end{aligned}$$

where $c_{\varepsilon \eta_\varepsilon}$ is defined in (8). Using (61), we obtain

$$\begin{aligned} & K_2 \int_{\widetilde{I}_{\eta_\varepsilon}} (\tilde{p}_\varepsilon - c_{\varepsilon \eta_\varepsilon}) \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 = K_2 \int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{y_2} \varphi(x_1, y_2, y_3) \, dx_1 dy_2 dy_3 \\ & \to K_2 \int_{\widetilde{I}_1} \tilde{P}(x_1) \partial_{y_2} \varphi(x_1, y_2, y_3) \, dx_1 dy_2 dy_3 = -K_2 \int_{\Sigma} \tilde{P}(x_1) \varphi(x_1, 0, y_3) \, dx_1 dy_3, \qquad \text{as } \varepsilon \to 0, \end{aligned} \tag{68}$$

where $\tilde{P}^\varepsilon$ is given by (7), and using (65), we have

$$K_2 c_{\varepsilon \eta_\varepsilon} \int_{\widetilde{I}_{\eta_\varepsilon}} \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3) \, dx' dy_3 = K_2 c_{\varepsilon \eta_\varepsilon} \int_{\widetilde{I}_1} \partial_{y_2} \varphi(x_1, y_2, y_3)) \, dx_1 dy_2 dy_3 = 0.$$

Passing to the limit in (67) similarly as in the proof of Lemma 5.3, we know that $\hat{v}$ and $\tilde{p}$ are related by the variational formulation of problem (46)-(48), and taking into account (68) and

$$\begin{aligned} & \int_{D'_- \times Y} \tilde{p}(x') \, \mathrm{div}_{x'} (\varphi(x', y_3) \phi(y')) \, dx' dy \\ & = - \int_{D'_- \times Y} \nabla_{x'} \tilde{p}(x') \varphi(x', y_3) \phi(y') \, dx' dy + \int_{\Sigma \times Y_1} \tilde{p}(x_1, 0) \varphi(x_1, 0, y_3) \phi_2(y_1, 0) \, dx_1 dy_1 dy_3 \\ & = - \int_{D'_- \times Y} \nabla_{x'} \tilde{p}(x') \varphi(x', y_3) \phi(y') \, dx' dy + K_2 \int_{\Sigma} \tilde{p}(x_1, 0) \varphi(x_1, 0, y_3) \, dx_1 dy_3, \end{aligned}$$

then we have

$$\int_{\Sigma} \left(\tilde{p}(x_1, 0) - \tilde{P}(x_1) \right) \varphi(x_1, 0, y_3) \, dx_1 dy_3 = 0,$$

so that

$$\int_{\Sigma_1} \left(\tilde{p}(x_1, 0) - \tilde{P}(x_1) \right) \psi(x_1) \, dx_1 = 0,$$

for every $\psi \in C_0^\infty(\Sigma_1)$ such that $\int_\Sigma \psi \, dx_1 = 0$. Finally we conclude that there exists a constant $C \in \mathbb{R}$ such that (64) holds and $\tilde{p}(x_1, 0) \in H^1(\Sigma_1)$.

Using (64) into (63), we obtain the variational formulation (20) for the limit pressure $\tilde{p}$ in the Banach space of functions $v \in H^1(D')$ such that $v(x_1, 0) \in H^1(\Sigma_1)$. Since $K \in \mathbb{R}^{2 \times 2}$ is a symmetric, positive, tensor given by (13), it can be proved that (20) has a unique solution in that Banach space with the norm $|v|_{H^1(D')} + |v(x_1,0)|_{H^1(\Sigma_1)}$.

□

Proof of Theorem 3.1-*iii*). It remains to prove the convergence (19) of the whole velocity.

Let $\varphi \in C_0(D)^3$. Then

$$\int_D \varepsilon^{-2}\tilde{u}_\varepsilon \cdot \varphi \, dx' dy_3 = \int_D \varepsilon^{-2}\tilde{v}_\varepsilon \cdot \varphi \, dx' dy_3 + \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3 \int_{\widetilde{I}_1} \eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \cdot \varphi(x_1, \eta_\varepsilon y_2, y_3)\, dx_1 dy_2 dy_3 = 0.$$

Taking the limit as $\varepsilon \to 0$, using (61), $\tilde{v}_3 = \tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$ and $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, we obtain

$$\int_D \varepsilon^{-2}\tilde{u}_\varepsilon \cdot \varphi \, dx' dy_3 \to \int_D \tilde{v}' \cdot \varphi' \, dx' dy_3 + \lambda^3 \int_{\widetilde{I}_1} \tilde{\mathcal{U}}_1(x_1, y_2)\varphi(x_1, 0, y_3)\, dx_1 dy_2 dy_3.$$

Taking into account that

$$\int_{\widetilde{I}_1} \tilde{\mathcal{U}}(x_1, y_2)\varphi(x_1, 0, y_3)\, dx_1 dy_2 dy_3 = \int_{\Sigma_1} \mathcal{V}(x_1) \left(\int_0^1 \varphi(x_1, 0, y_3) dy_3\right) dx_1 = \langle \mathcal{V}\delta_{\Sigma_1}, \varphi\rangle_{\mathcal{M}(D)^3, C_0(D)^3},$$

where $\mathcal{V}(x_1)$ is given by (17), we get (19). □

References

[1] Allaire, G: Homogenization of the Stokes flow in a connected porous medium. Asymptot. Anal. **2**, 203-222 (1989).

[2] Allaire, G: Homogenization and two-scale convergence. SIAM J. Math. Anal. **23**, 1482-1518 (1992).

[3] Anguiano, M: Darcy's laws for non-stationary viscous fluid flow in a thin porous medium. Math. Meth. Appl. Sci. (2016) DOI: 10.1002/mma.4204.

[4] Bourgeat, A., ElAmri, H., Tapiero, R.: Existence d'une taille critique pour une fissure dans un milieu poreux. Second Colloque Franco Chilien de Mathematiques Appliquées, Cepadués Edts, Tolouse, 67-80 (1991).

[5] Bourgeat, A., Tapiero, R.: Homogenization in a perforated domain including a thin full interlayer. Int. Ser. Num. Math. **114**, 25-36 (1993).

[6] Bourgeat, A., Marušic-Paloka, E., Mikelić, A.: Effective fluid flow in a porous medium containing a thin fissure. Asymptot. Anal. **11**, 241-262 (1995).

[7] Bourgeat, A., Mikelić, A.: Homogenization of a polymer flow through a porous medium. Nonlin. Anal. **26**, 1221-1253 (1996).

[8] Ciarlet, P.G., Ledret, H., Nzwenga, R.: Modélisation de la jonction entre un corps élastique tridimensionnel et une plaque. C. R. Acad. Sci. Paris Ser. I **305**, 55-58 (1987).

[9] Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, 608-623 (1989).

[10] Panasenko, G.P.: Higher order asymptotics of solutions of problems on the contact of periodic structures. Math. U.S.S.R. Sbornik. **38**, 465-494 (1981).

[11] Sanchez-Palencia, E.: Non-Homogeneous Media and Vibration Theory. Springer Lecture Notes in Physics, vol. 127 (1980).

[12] Tartar, L.: Incompressible fluid flow in a porous medium convergence of the homogenization process. In: Appendix to Lecture Notes in Physics, vol. 127. Springer, Berlin (1980).

[13] Temam, R.: Navier-Stokes equations and nonlinear functional analysis. In: CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 41. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1983).

[14] Zhao, H., Yao, Z.: Effective models of the Navier-Stokes flow in porous media with a thin fissure. J. Math. Anal. Appl. **387**, 542-555 (2012).

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