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On the Kneser property for reaction-diffusion equations in some unbounded domains with an H^{-1}-valued non-autonomous forcing term

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Abstract

In this paper we prove the Kneser property for a reaction-diffusion equation on an unbounded domain satisfying the Poincaré inequality with an external force taking values in the space H^{-1}. Using this property of solutions we check also the connectedness of the associated global pullback attractor.

We study also similar properties for systems of reaction-diffusion equations in which the domain is the whole $\mathbb{R}^N$.

Finally, the results are applied to a generalized logistic equation.

AMS Subject Classification: 35B40, 35B41, 35K55, 35K57, 37B25, 58C06

Keywords: reaction-diffusion system, setvalued dynamical system, global attractor, Kneser property, unbounded domain

1 Introduction and setting of the problem

When we consider a partial differential equation with non-uniqueness of the Cauchy problem it is interesting to consider the Kneser property, that is, the connectedness and compactness of the set of values attained by the solutions at any moment of time. In particular, this problem has been studied for reaction-diffusion equations by several authors so far.

In this direction some results are known for scalar reaction-diffusion equations in bounded domains in the case where the nonlinearity has at most linear growth [12]. Such results were extended later on in [14, 15] for systems of reaction-diffusion equations with nonlinearities having more than linear growth (see also [13]), with applications to the complex Ginzburg-Landau equation and the Lotka-Volterra system with diffusion, among others. Also, the Kneser property for degenerate reaction-diffusion equations was considered in [3, 4].

When the domain is unbounded the problem has additional technical difficulties. A first result in this direction was given [16], in which it is studied a scalar reaction-diffusion on unbounded domains in which the nonlinear term is equal to $|u|^{\frac{1}{2}}$. In [22] the results of [15] were extended to unbounded domains. However, due to technical difficulties it was necessary to assume an additional condition concerning the derivatives of the nonlinear terms. In this paper we improve the method of the proof given in [22] in order to avoid such condition.

We consider first the following problem.

Let $\Omega \subset \mathbb{R}^N$ be a nonempty unbounded open set and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2 \, dx \leq \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2 \, dx, \quad \forall u \in H_0^1(\Omega). \tag{1}$$

This condition is satisfied when the set Ω is bounded in one direction. For example, in the two-dimensional case we can take the strip

$$\Omega = \{(x,y) : a \leq y \leq b, -\infty < x < \infty\}.$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(x,u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that $f(\cdot, u)$ is a measurable function for any $u \in \mathbb{R}$, $f(x, \cdot) \in C(\mathbb{R})$ for almost every $x \in \Omega$, and satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $p \geq 2$ and positive functions $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, $C_2(x) \in L^1(\Omega)$ such that

$$|f(x,s)|^{\frac{p}{p-1}} \leq \alpha_1 |s|^p + C_1(x) \quad \forall s \in \mathbb{R}, \, x \in \Omega, \tag{3}$$

$$f(x,s)s \leq -\alpha_2 |s|^p + C_2(x) \quad \forall s \in \mathbb{R}, \, x \in \Omega. \tag{4}$$

We observe that

$$|f(x,s)| \leq \alpha_1^{\frac{p-1}{p}} |s|^{p-1} + C_1(x)^{\frac{p-1}{p}} \qquad \forall s \in \mathbb{R}, \, x \in \Omega. \tag{5}$$

By $\|\cdot\|_{L^2(\Omega)}$ we denote the norm in $L^2(\Omega)$, by $\|\cdot\|_{H_0^1(\Omega)} = \|\nabla \cdot\|_{L^2(\Omega)}$ the norm in $H_0^1(\Omega)$ and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$. We will use $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$ or $[L^2(\Omega)]^N$, and $\langle \cdot, \cdot \rangle$ to denote either the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $L^{p'}(\Omega)$ and $L^p(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Note that the function f is just continuous on u, and that no condition on the derivative is imposed.

In the second section we prove that the Kneser property holds for this problem.

In the previous paper [2] concerning equation (2) we proved the existence of a global compact pullback attractor. It is interesting to prove that the global attractor is connected. Using the Kneser property we are able to obtain such result, which is given in the third section. We observe that in [2] the function $C_1(x)$ belongs just to $L^1(\Omega)$, but for the Kneser property we need the stronger condition $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$.

In the fourth section we consider a system of reaction-diffusion equations with $\Omega = \mathbb{R}^N$, which was studied before in [22]. Using a similar technique as for problem (2) we improve the results of that paper, proving the Kneser property and the connectedness of the global attractor without using an extra condition on the derivative of the nonlinear term of the equation. Instead, as in problem 2, we have to assume that the functions appearing in the growth condition of the nonlinear term belong to $L^\infty(\Omega)$.

In the last section we apply these results to a generalized logistic equation.

2 The Kneser property

In this section we shall prove that the set of values attained by the solutions of equation (2) at any moment of time is connected. For this aim for each $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$ let us denote by $S(\tau, u_\tau)$ the set of all weak solutions of (2) defined for all $t \geq \tau$. Such a set is non-empty as in [2] it is proved that at least one weak global solution exists for any $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$.

We define a multi-valued map $U : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ by

$$U(t, \tau, u_\tau) = \{u(t) : \ u \in S(\tau, u_\tau)\}, \quad \tau \leq t, \quad u_\tau \in L^2(\Omega), \tag{6}$$

where $\mathcal{P}(L^2(\Omega))$ is the set of all non-empty subsets of $L^2(\Omega)$.

In [2] it is shown that the multi-valued mapping U defined by (6) is a strict multi-valued non-autonomous dynamical system on $L^2(\Omega)$ (see Definition 9). Our aim is to prove the connectedness of the set $U(t,\tau,u_\tau) \subset L^2(\Omega)$ for any $t \geq \tau$. We note that the compactness of $U(t,\tau,u_\tau)$ in $L^2(\Omega)$ is a consequence of Proposition 16 in [2], as in that paper it is shown that $U(t,\tau,u_\tau)$ is precompact and closed for any u_τ.

We shall obtain now that $U(t,\tau,u_\tau)$ is conected in $L^2(\Omega)$ and for this aim we need some preliminary lemmas.

We take a sequence $0 < \epsilon_k < 1$ converging to 0 as $k \to \infty$ and define a sequence of smooth functions $\psi_k : \mathbb{R}^+ \longrightarrow [0,1]$ satisfying

$$\psi_k(s) := \begin{cases} 1, \text{ if } 0 \leq s \leq \sqrt{\epsilon_k}, \\ 0 \leq \psi_k \leq 1, \text{ if } \sqrt{\epsilon_k} \leq s \leq 2\sqrt{\epsilon_k}, \\ 0, \text{ if } 2\sqrt{\epsilon_k} \leq s \leq 1/\epsilon_k, \\ 0 \leq \psi_k \leq 1, \text{ if } 1/\epsilon_k \leq s \leq 1/\epsilon_k + 1, \\ 1, \text{ if } s \geq 1/\epsilon_k + 1. \end{cases}$$

Let $\rho_{\epsilon_k} : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a mollifier, that is, $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R};\mathbb{R})$, $\operatorname{supp}\rho_{\epsilon_k} \subset B_{\epsilon_k}$, $\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\,ds = 1$ and $\rho_{\epsilon_k}(s) \geq 0$ for all $s \in \mathbb{R}$, where $B_{\epsilon_k} = \{u \in \mathbb{R} : |u| \leq \epsilon_k\}$.

We define the following approximating function

$$f^k(x,u) := \psi_k(|u|)\left(C_0^1 |u|^{p-2} u + f(x,0)\right) + (1-\psi_k(|u|)) \int_{\mathbb{R}} \rho_{\epsilon_k}(s) f(x,u-s)\,ds,$$

where $k \geq 1$, $p \geq 2$, and C_0^1 is a negative constant. Then it is easy to check that for a.a. $x \in \Omega$,

$$\sup_{|u|\leq A} \left|f^k(x,u) - f(x,u)\right| \longrightarrow 0, \text{ as } k \longrightarrow \infty, \text{ for any } A > 0.$$

Lemma 1 *Assume (3)-(4). Then the function f^k satisfy conditions (3)-(4), i.e., there exist constants $\widehat{\alpha}_1$, $\widehat{\alpha}_2 > 0$, and positive functions $\widehat{C}_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\widehat{C}_2(x) \in L^1(\Omega)$, not depending on k, such that*

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \leq \widehat{\alpha}_1 |u|^p + \widehat{C}_1(x) \quad \forall u \in \mathbb{R},\ x \in \Omega, \tag{7}$$

$$f^k(x,u)u \leq -\widehat{\alpha}_2 |u|^p + \widehat{C}_2(x) \quad \forall u \in \mathbb{R},\ x \in \Omega, \tag{8}$$

for k great enough.

Proof. Indeed, for the first property we have the following cases.

1) If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, then we have

$$|f^k(x,u)| \leq \left|C_0^1\right| |u|^{p-1} + |f(x,0)|,$$

so (3) yields

$$\begin{aligned} |f^k(x,u)|^{\frac{p}{p-1}} &\leq 2^{\frac{1}{p-1}} \left|C_0^1\right|^{\frac{p}{p-1}} |u|^p + 2^{\frac{1}{p-1}} |f(x,0)|^{\frac{p}{p-1}} \\ &\leq 2^{\frac{1}{p-1}} \left|C_0^1\right|^{\frac{p}{p-1}} |u|^p + 2^{\frac{1}{p-1}} C_1(x). \end{aligned}$$

2) If $\sqrt{\epsilon_k} \leq |u| \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq |u| \leq 1/\epsilon_k + 1$, then using (5) we have

$$\begin{aligned} |f^k(x,u)| &\leq \left|C_0^1\right| |u|^{p-1} + |f(x,0)| \\ &\quad + \int_{\mathbb{R}} \rho_{\epsilon_k}(s) (\alpha_1^{\frac{p-1}{p}} |u-s|^{p-1} + C_1(x)^{\frac{p-1}{p}})\,ds \\ &\leq \left|C_0^1\right| |u|^{p-1} + 2C_1(x)^{\frac{p-1}{p}} \\ &\quad + \alpha_1^{\frac{p-1}{p}} 2^{p-2} \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(|u|^{p-1} + |s|^{p-1}\right) ds \\ &\leq 2C_1(x)^{\frac{p-1}{p}} + \left(\left|C_0^1\right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1}\right) |u|^{p-1}, \end{aligned}$$

and then

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \le 2^{\frac{p+1}{p-1}} C_1(x) + 2^{\frac{1}{p-1}} \left(\left|C_0^1\right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1} \right)^{\frac{p}{p-1}} |u|^p .$$

3) If $2\sqrt{\epsilon_k} \le |u| \le 1/\epsilon_k$, then arguing as in the previous case we have

$$\left|f^k(x,u)\right| \le C_1(x)^{\frac{p-1}{p}} + \left(\alpha_1^{\frac{p-1}{p}} 2^{p-1} \right) |u|^{p-1} ,$$

and then

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \le 2^{\frac{1}{p-1}} C_1(x) + 2^{\frac{1}{p-1}} 2^p \alpha_1 \, |u|^p .$$

Finally, we obtain

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \le \widehat{C}_1(x) + \widehat{\alpha}_1 \, |u|^p ,$$

where $\widehat{C}_1(x) := 2^{\frac{p+1}{p-1}} C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, and $\widehat{\alpha}_1 := 2^{\frac{1}{p-1}} \left(\left|C_0^1\right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1} \right)^{\frac{p}{p-1}} > 0$.

On the other hand, note that

$$f^k(x,u)u = \psi_k(|u|)\, C_0^1 \, |u|^p + \psi_k(|u|)\, f(x,0)u + (1-\psi_k(|u|)) \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds. \tag{9}$$

Hence, for the second property we have the following cases.

1) If $2\sqrt{\epsilon_k} \le |u| \le 1/\epsilon_k$, then using (4), the Young inequality $ab \le \frac{a^p}{\varepsilon^{p-1}p} + \frac{\varepsilon b^{p'}}{p'}$ and (3) we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds &= \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\,(u-s)\, ds + \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, s ds \\
&\le \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(-\alpha_2 |u-s|^p + C_2(x)\right) ds \\
&+ \frac{\alpha_2}{p'\alpha_1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |f(x,u-s)|^{p'} ds + \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds \\
&\le \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(-\alpha_2 |u-s|^p + C_2(x)\right) ds \\
&+ \frac{\alpha_2}{p'\alpha_1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(\alpha_1 |u-s|^p + C_1(x)\right) ds + \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds \\
&\le \left(-\alpha_2 + \frac{\alpha_2}{p'}\right) \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(\frac{|u|^p}{2^{p-1}} - |s|^p\right) ds \\
&+ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds + \overline{C}_2(x),
\end{aligned} \tag{10}$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p, and $\overline{C}_2(x) := C_2(x) + \frac{\alpha_2}{p'\alpha_1} C_1(x)$, and where in the last inequality we have used

$$|u|^p = |u-s+s|^p \le 2^{p-1}\left(|u-s|^p + |s|^p\right).$$

We observe that $|u| \ge \sqrt{\epsilon_k}$, so that for k large enough,

$$|s|^p \le \epsilon_k^p \le \frac{1}{2^p} \epsilon_k^{\frac{p}{2}} \le \frac{1}{2^p} |u|^p .$$

Then from (10) we obtain

$$\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds \le \left(-\alpha_2 + \frac{\alpha_2}{p'}\right) \frac{1}{2^p} |u|^p + \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds + \overline{C}_2(x). \tag{11}$$

Since for k large enough, we have

$$|s|^p \le \epsilon_k^p \le -p\left(\frac{\alpha_2}{\alpha_1}\right)^{p-1}\left(-\alpha_2+\frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}}\epsilon_k^{\frac{p}{2}} \le -p\left(\frac{\alpha_2}{\alpha_1}\right)^{p-1}\left(-\alpha_2+\frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}}|u|^p,$$

we obtain

$$\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds \le \beta |u|^p + \overline{C}_2(x), \tag{12}$$

where $\beta = \left(-\alpha_2+\frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}} < 0$ and $\overline{C}_2(x) \in L^1(\Omega)$ is a positive function.

2) If $0 \le |u| \le \sqrt{\epsilon_k}$ or $|u| \ge 1/\epsilon_k + 1$, then using the Young inequality $ab \le \frac{\varepsilon a^p}{p} + \frac{b^{p'}}{\varepsilon^{\frac{1}{p-1}} p'}$ and (3) we obtain

$$\begin{aligned} f^k(x,u)u = C_0^1 |u|^p + f(x,0)u &\le C_0^1\left(1-\frac{1}{p}\right)|u|^p + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x) \\ &\le \frac{C_0^1}{2}|u|^p + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x), \end{aligned}$$

where $\frac{C_0^1}{2}$ is a negative constant and $\frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x) \in L^1(\Omega)$ is a positive function.

3) If $\sqrt{\epsilon_k} \le |u| \le 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \le |u| \le 1/\epsilon_k + 1$, we argue as in the first case to obtain (12). From (9), using the Young inequality and (3), we have

$$\begin{aligned} f^k(x,u)u &\le \psi_k(|u|)\, C_0^1 |u|^p - \psi_k(|u|)\frac{C_0^1}{p}|u|^p + \psi_k(|u|)\frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x) \\ &\quad + (1-\psi_k(|u|))\left(\beta|u|^p + \overline{C}_2(x)\right) \\ &\le \psi_k(|u|)\frac{C_0^1}{2}|u|^p + (1-\psi_k(|u|))\,\beta|u|^p + \left(\overline{C}_2(x) + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x)\right) \\ &\le \widetilde{\beta}|u|^p + \widehat{C}_2(x), \end{aligned}$$

where $\widetilde{\beta} := \max\left\{\beta, \frac{C_0^1}{2}\right\} < 0$ and $\widehat{C}_2 \in L^1(\Omega)$ is a positive function.

Then, we have

$$f^k(x,u)u \le \widetilde{\beta}|u|^p + \widehat{C}_2(x),$$

where $\widetilde{\beta} < 0$ and $\widehat{C}_2(x) \in L^1(\Omega)$. ■

Lemma 2 *Assume (3)-(4). Then, there exist D_{ϵ_k} such that*

$$\frac{\partial f^k}{\partial u}(x,u) \le D_{\epsilon_k},\ \forall u \in \mathbb{R},\ \textit{for a.a. } x \in \Omega. \tag{13}$$

Proof. We note that $\frac{\partial}{\partial u}\psi_k(|u|)$ is uniformly bounded on $\mathbb{R}$. We have

$$\begin{aligned} \frac{\partial f^k}{\partial u}(x,u) &= C_0^1 \psi_k(|u|)(p-1)|u|^{p-2} + C_0^1 |u|^{p-2} u \frac{\partial}{\partial u}\psi_k(|u|) \\ &\quad + f(x,0)\frac{\partial}{\partial u}\psi_k(|u|) + (1-\psi_k(|u|))\int_{\mathbb{R}} \rho'_{\epsilon_k}(u-s)\, f(x,s)\, ds \\ &\quad - \frac{\partial}{\partial u}\psi_k(|u|)\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, ds. \end{aligned} \tag{14}$$

We consider each term in (14).

- As C_0^1 is a negative constant, for the first term, we have

$$C_0^1 \psi_k\left(|u|\right)(p-1)\left|u\right|^{p-2} \leq 0.$$

- For the second term, we get

$$\left| C_0^1 \left|u\right|^{p-2} u \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| \leq \left|C_0^1\right| \left(\frac{1}{\epsilon_k}+1\right)^{p-1} C_{\psi_k}.$$

- For the third term using (5), we obtain

$$\left| f(x,0) \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| \leq \left|f(x,0)\right| C_{\psi_k} \leq C_1(x)^{\frac{p-1}{p}} C_{\psi_k} \leq \left\|C_1\right\|_\infty^{\frac{p-1}{p}} C_{\psi_k}.$$

- For the fourth term, we have to consider several cases.
  If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, we obtain

$$\left(1-\psi_k\left(|u|\right)\right) \int_{\mathbb{R}} \rho'_{\epsilon_k}\left(u-s\right) f\left(x,s\right) ds = 0.$$

  If $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, then using (5) we have

$$\begin{aligned}
&\left| \left(1-\psi_k\left(|u|\right)\right) \int_{\mathbb{R}} \rho'_{\epsilon_k}\left(u-s\right) f\left(x,s\right) ds \right| \\
&\leq \int_{B_{\epsilon_k}} \left|\rho'_{\epsilon_k}\left(s\right)\right| \left(\alpha_1^{\frac{p-1}{p}} \left|u-s\right|^{p-1} + C_1(x)^{\frac{p-1}{p}}\right) ds \\
&\leq \left\|C_1\right\|_\infty^{\frac{p-1}{p}} \int_{\mathbb{R}} \left|\rho'_{\epsilon_k}\left(s\right)\right| ds + 2^{p-1} \alpha_1^{\frac{p-1}{p}} \left(1/\epsilon_k + 1\right)^{p-1} \int_{\mathbb{R}} \left|\rho'_{\epsilon_k}\left(s\right)\right| ds \leq D_{\epsilon_k},
\end{aligned}$$

  as $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty\left(\mathbb{R};\mathbb{R}\right)$.

- For the last term, if $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, using (5) we have

$$\begin{aligned}
&\left| -\frac{\partial}{\partial u} \psi_k\left(|u|\right) \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) f\left(x, u-s\right) ds \right| \\
&\leq \left| \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) \alpha_1^{\frac{p-1}{p}} \left|u-s\right|^{p-1} ds \\
&+ \left| \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) C_1(x)^{\frac{p-1}{p}} ds \\
&\leq \left| \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| \left\|C_1\right\|_\infty^{\frac{p-1}{p}} + \left| \frac{\partial}{\partial u} \psi_k\left(|u|\right) \right| 2^{p-2} \alpha_1^{\frac{p-1}{p}} \int_{B_{\epsilon_k}} \rho_{\epsilon_k}\left(s\right) \left(|u|^{p-1} + |s|^{p-1}\right) ds \\
&\leq C_{\psi_k} \left\|C_1\right\|_\infty^{\frac{p-1}{p}} + C_{\psi_k} 2^{p-1} \alpha_1^{\frac{p-1}{p}} \left(1/\epsilon_k + 1\right)^{p-1} = D_{\epsilon_k}.
\end{aligned}$$

  In other case $\psi_{ku}\left(|u|\right) = 0$. Then (13) holds.

■

Let $T > \tau$ be arbitrary. In order to prove the Kneser property let us consider the following auxiliary problem

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} - \triangle u = f^k(x,u) + h(t) \quad \text{in} \quad \Omega \times (\gamma, +\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\gamma, +\infty), \\ u(x,\gamma) = u^\gamma(x), \quad x \in \Omega, \end{array} \right. \tag{15}$$

where $\gamma \in [\tau, T]$. In view of Lemma 1 for all $k \geq 1$ the function f^k satisfies (3) and (4), so that by [2, Theorem 2] for any $u^\gamma \in L^2(\Omega)$ problem (15) has at least one weak solution $u_\gamma^k(\cdot)$ defined on $[\gamma, T]$. Using Lemma 2 it is standard to check that for the difference $w(t)$ of two solutions we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \|\nabla w(t)\|^2_{L^2(\Omega)} \leq D_{\epsilon_k}\|w(t)\|^2_{L^2(\Omega)}.$$

Hence, from Gronwall's lemma we obtain

$$\|w(t)\|_{L^2(\Omega)} \leq \|w(\gamma)\|_{L^2(\Omega)} e^{D_{\epsilon_k}(t-\gamma)}, \tag{16}$$

so that the solution is unique.

We need some preliminary estimates.

Lemma 3 *Suppose that $\Omega \subset \mathbb{R}^N$ is a non-empty open unbounded set which satisfies (1) and suppose that $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is such that $f(\cdot, u)$ is a measurable function for any $u \in \mathbb{R}$, $f(x, \cdot) \in C(\mathbb{R})$ for almost every $x \in \Omega$, and satisfies (3) and (4). Let $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, with $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ for all $1 \leq i \leq N$, such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s}\|h_i(s)\|^2_{L^2(\Omega)}\,ds < +\infty \quad \forall t \in \mathbb{R}. \tag{17}$$

Then there exists $R = R(B, T)$ (not depending neither on γ nor k), where B is a bounded set of $L^2(\Omega)$, such that

$$\left\|u_\gamma^k(t)\right\|_{L^2(\Omega)} \leq R, \ \ \forall t \in [\gamma, T], \tag{18}$$

and

$$\left\|u_\gamma^k(\cdot)\right\|_{L^p(\gamma,T;L^p(\Omega))} \leq R, \tag{19}$$

for any $u^\gamma \in B$, where $u_\gamma^k(\cdot)$ is the unique solution to (15) with $u_\gamma^k(\gamma) = u^\gamma$.

Proof. We note that using (7) and (8) for f^k one can easily obtain that the functions u_γ^k satisfy the estimate

$$\frac{1}{2}\frac{d}{dt}\left\|u_\gamma^k(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla u_\gamma^k(t)\right\|^2_{L^2(\Omega)} + \widehat{\alpha}_2\left\|u_\gamma^k(t)\right\|^p_{L^p(\Omega)} \leq \frac{1}{2}\|h(t)\|^2_* + \left\|\widehat{C}_2\right\|_{L^1(\Omega)}.$$

Integrating between γ to t, we have

$$\begin{aligned} &\left\|u_\gamma^k(t)\right\|^2_{L^2(\Omega)} + \int_\gamma^t \left\|\nabla u_\gamma^k(s)\right\|^2_{L^2(\Omega)} ds + 2\widehat{\alpha}_2\int_\gamma^t \left\|u_\gamma^k(s)\right\|^p_{L^p(\Omega)} ds \\ &\leq \|u^\gamma\|^2_{L^2(\Omega)} + \int_\gamma^T \|h(s)\|^2_* \,ds + 2\left\|\widehat{C}_2\right\|_{L^1(\Omega)}(T-\gamma). \end{aligned} \tag{20}$$

Hence, (18) and (19) follow. ■

Lemma 4 *We suppose the same assumptions for Ω, f and h as in Lemma 3. Let K be a relatively compact set in $L^2(\Omega)$. Then, for all $\tau \leq T$ and $\varepsilon > 0$ there exists $M = M(\gamma, T, \varepsilon, K)$ such that*

$$\int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}} \left|u_\gamma^k(x,t)\right|^2 dx \leq \varepsilon, \, \forall t \in [\gamma, T], \quad \forall \gamma \in [\tau, T], \ \ \forall m \geq M, \, \forall k,$$

for any $u^\gamma \in K$, where $u_\gamma^k(\cdot)$ is the unique solution to (15) with $u_\gamma^k(\gamma) = u^\gamma$.

Proof. Thanks to Lemma 1 we have that f^k satisfies (7) and (8). If we argue as in [2, Lemma 15] we obtain a similar estimate for u_γ^k. ■

Remark 5 *Condition (17) is necessary in order to prove Lemma 4, but not for Lemma 3. However, for the sake of clarity we use the same conditions in all lemmas.*

Theorem 6 *Under the assumptions for Ω, f and h as in Lemma 3, the set $U(t,\tau,u_\tau)$ is connected in $L^2(\Omega)$ for any $t\in[\tau,T]$.*

Proof. The case $t=\tau$ is obvious. Suppose that for some $t^*\in(\tau,T]$ the set $U(t^*,\tau,u_\tau)$ is not connected. Then there exist two compact sets $A_1,\ A_2\subset L^2(\Omega)$ such that $A_1\cup A_2=U(t^*,\tau,u_\tau)$, $A_1\cap A_2=\emptyset$. Let $u_1(\cdot),\ u_2(\cdot)\in S(\tau,u_\tau)$ be such that $u_1(t^*)\in E_1,\ u_2(t^*)\in E_2$, where $E_1,\ E_2$ are disjoint open neighborhoods of $A_1,\ A_2$, respectively.

Let $u_i^k(t,\gamma)$, $i=1,2$, be equal to $u_i(t)$, if $t\in[\tau,\gamma]$, and let $u_i^k(t,\gamma)$ be the unique solution of problem (15), if $t\in[\gamma,T]$.

The proof of the following lemma is the same as in [14, Theorem 5] or [22, Lemma 8], so that we omit it.

Lemma 7 *The maps $\gamma\longrightarrow u^k(t,\gamma)$ are continuous for each fixed $k\geq1$ and $t\in[\tau,T]$.*

Now we put

$$\gamma(\lambda)=\begin{cases}\tau-(T-\tau)\,\lambda, \text{ if } \lambda\in[-1,0],\\ \tau+(T-\tau)\,\lambda, \text{ if } \lambda\in[0,1],\end{cases}$$

and define the function

$$\varphi^k(\lambda)(t)=\begin{cases}u_1^k(t,\gamma(\lambda)) \text{ if } \lambda\in[-1,0],\\ u_2^k(t,\gamma(\lambda)) \text{ if } \lambda\in[0,1].\end{cases}$$

We have $\varphi^k(-1)(t)=u_1^k(t,T)=u_1(t)$, $\varphi^k(1)(t)=u_2^k(t,T)=u_2(t)$. The map $\lambda\mapsto\varphi^k(\lambda)(t)\in L^2(\Omega)$ is continuous for any fixed $k\geq1$, $t\in[\tau,T]$ (note that $u_1^k(t,\tau)=u_2^k(t,\tau)$) and $\varphi^k(-1)(t^*)\in E_1$, $\varphi^k(1)(t^*)\in E_2$, so that there exists $\lambda_k\in[-1,1]$ such that $\varphi^k(\lambda_k)(t^*)\notin E_1\cup E_2$. Denote $u^k(t)=\varphi^k(\lambda_k)(t)$. Note that for each $k\geq1$ either $u^k(t)=u_1^k(t,\gamma(\lambda_k))$ or $u^k(t)=u_2^k(t,\gamma(\lambda_k))$. For some subsequence it is equal to one of them, say $u_1^k(t,\gamma(\lambda_k))$. Now we shall consider the function $u_1^k(t,\gamma(\lambda_k))$, $t\in[\tau,T]$. We have

$$u^k(t)=\begin{cases}u_1(t), \text{ if } t\in[\tau,\gamma(\lambda_k)],\\ u_1^k(t,\gamma(\lambda_k)), \text{ if } t\in[\gamma(\lambda_k),T],\end{cases}$$

where $\gamma(\lambda_k)\longrightarrow\gamma_0\in[\tau,T]$. We define the functions

$$\widetilde{f}_k(t,x,v)=\begin{cases}f(x,v), \text{ if } t\in[\tau,\gamma(\lambda_k)],\\ f_k(x,v), \text{ if } t\in(\gamma(\lambda_k),T].\end{cases}$$

By continuity $u_1(\gamma(\lambda_k))\longrightarrow u_1(\gamma_0)$, as $k\longrightarrow\infty$.

Further, by (20),

$$\left\{u^k(\cdot)\right\} \text{ is bounded in } L^\infty\left(\tau,T;L^2(\Omega)\right)\cap L^2\left(\tau,T;H_0^1(\Omega)\right)\cap L^p\left(\tau,T;L^p(\Omega)\right). \tag{21}$$

It follows also that $\frac{du^k}{dt}$ is bounded in $L^{p'}\left(\tau,T;L^{p'}(\Omega)\right)+L^2\left(\tau,T;H^{-1}(\Omega)\right)$. Then for some function $u=u(x,t)$ we have

$$u^k\rightharpoonup u \text{ weakly in } L^2\left(\tau,T;H_0^1(\Omega)\right), \tag{22}$$

$$u^k\rightharpoonup u \text{ weakly in } L^p\left(\tau,T;L^p(\Omega)\right),$$

$$\frac{du^k}{dt}\rightharpoonup\frac{du}{dt} \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega)\right)+L^2\left(\tau,T;H^{-1}(\Omega)\right),$$

$$u^k\overset{*}{\rightharpoonup}u \text{ weakly star in } L^\infty\left(\tau,T;L^2(\Omega)\right).$$

Arguing in a similar way as in [24, p.75] we first deduce

$$\lim_{a\to0}\sup_k\int_\tau^{T-a}\left\|u^k(t+a)-u^k(t)\right\|_{L^2(\Omega)}^2dt=0, \tag{23}$$

for all $T>\tau$.

Now, for all $m \in \mathbb{Z}$, $m \geq 1$, we denote

$$\Omega_m = \Omega \cap \left\{ x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m \right\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$. Let $\phi \in C^1([0,+\infty))$ be a function such that

$$0 \leq \phi(s) \leq 1,$$

$$\phi(s) = 1 \quad \forall s \in [0,1],$$

$$\phi(s) = 0 \quad \forall s \geq 2.$$

For each k and $m \geq 1$, we define

$$u^{k,m}(x,t) = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u^k(x,t) \qquad \forall x \in \Omega_{2m}, \forall k, \ \forall m \geq 1. \tag{24}$$

We obtain from (21) that, for all $m \geq 1$, the sequence $\left\{u^{k,m}\right\}_{k\geq 1}$ is bounded in $L^\infty\left(\tau,T;L^2\left(\Omega_{2m}\right)\right) \cap L^p\left(\tau,T;L^p\left(\Omega_{2m}\right)\right) \cap L^2\left(\tau,T;H_0^1\left(\Omega_{2m}\right)\right)$, for all $T > \tau$.

In particular, it follows that

$$\lim_{a\to 0} \sup_k \left(\int_\tau^{\tau+a} \left\|u^{k,m}(t)\right\|^2_{L^2(\Omega_{2m})} dt + \int_{T-a}^{T} \left\|u^{k,m}(t)\right\|^2_{L^2(\Omega_{2m})} dt \right) = 0.$$

On the other hand, from (23) we deduce that for $m \geq 1$,

$$\lim_{a\to 0} \sup_k \left(\int_\tau^{T-a} \left\|u^{k,m}(t+a) - u^{k,m}(t)\right\|^2_{L^2(\Omega_{2m})} dt \right) = 0.$$

Moreover, as Ω_{2m} is a bounded set, $H_0^1\left(\Omega_{2m}\right)$ is included in $L^2\left(\Omega_{2m}\right)$ with compact injection.

Then, by the Compactness Theorem 13.3 and Remark 13.1 of [25] with $X = L^2\left(\Omega_{2m}\right)$, $Y = H_0^1\left(\Omega_{2m}\right)$, $p = 2$ and $\mathcal{G} = \left\{u^{k,m}\right\}_{k\geq 1}$, we obtain that

$$\left\{u^{k,m}\right\}_{k\geq 1} \text{ is relatively compact in } L^2\left(\tau,T;L^2\left(\Omega_{2m}\right)\right),$$

and thus, taking into account that $u^{k,m}(x,t) = u^k(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$

$$\left\{u^k_{|\Omega_m}\right\}_{k\geq 1} \text{ is precompact in } L^2\left(\tau,T;L^2\left(\Omega_m\right)\right). \tag{25}$$

It is not difficult to conclude from (25), (22), via a diagonal procedure, the existence of a subsequence of $\left\{u^k\right\}_{k\geq 1}$ (which we denote as the sequence) such that

$$u^k_{|\Omega_m}(x,t) \longrightarrow u_{|\Omega_m}(x,t) \text{ for a.a. } (x,t) \in \Omega_m \times (\tau,T), \text{ for all } m \geq 1. \tag{26}$$

It is easy to obtain (see [20, Chapter 3]) that $\frac{du^k_{|\Omega_m}}{dt}$ is bounded in the space $L^{p'}\left(\tau,T;L^{p'}\left(\Omega_m\right)\right) + L^2\left(\tau,T;H^{-1}\left(\Omega_m\right)\right)$, which is continuously embedded in $L^q\left(\tau,T;H^{-s}\left(\Omega_m\right)\right)$ for $s = \max\left\{1, N\left(\frac{1}{p'} - \frac{1}{2}\right)\right\}$.

From (26), $\widetilde{f}_k(x,u^k_{|\Omega_m}(x,t)) \longrightarrow f(x,u_{|\Omega_m}(x,t))$ for a.a. $(x,t) \in \Omega_m \times (\tau,T)$, and then the boundedness of $\widetilde{f}_k(x,u^k_{|\Omega_m})$ in $L^{p'}\left(\tau,T;L^{p'}\left(\Omega_m\right)\right)$ implies that $\widetilde{f}_k(x,u^k_{|\Omega_m})$ converges to $f(x,u_{|\Omega_m})$ weakly in $L^{p'}\left(\tau,T;L^{p'}\left(\Omega_m\right)\right)$ for any $m \geq 1$ (see [17]). Also, we note that (25), Lemma 4 and [2, Lemma 15] imply (up to a subsequence) that

$$u^k \longrightarrow u \text{ strongly in } L^2\left(\tau,T;L^2(\Omega)\right), \tag{27}$$

$$u^k(t) \longrightarrow u(t) \text{ in } L^2(\Omega) \text{ for a.a. } t \in (\tau,T).$$

Moreover, $u^k_{|\Omega_m}(t) \rightharpoonup u_{|\Omega_m}(t)$ weakly in $L^2(\Omega)$ for all $t \in [\tau, T]$ and $m \geq 1$. Indeed, as $\frac{du^k_{|\Omega_m}}{dt}$ is a bounded sequence of the space $L^q(\tau, T; H^{-s}(\Omega_m))$, we have that $u^k_{|\Omega_m}(t) : [\tau, T] \longrightarrow H^{-s}(\Omega_m)$ is an equicontinuous family of functions. By (18) for each fixed $r \in [\tau, T]$ the sequence $u^k_{|\Omega_m}(r)$ is bounded in $L^2(\Omega_m)$, so that the compact embedding $L^2(\Omega_m) \subset H^{-s}(\Omega_m)$, implies that it is precompact in $H^{-s}(\Omega_m)$. Applying the Ascoli-Arzelà Theorem we deduce that $\left\{u^k_{|\Omega_m}(t)\right\}$ is a precompact sequence in $\mathcal{C}([\tau, T], H^{-s}(\Omega_m))$. Hence, since $u^k_{|\Omega_m} \rightharpoonup u_{|\Omega_m}$ weakly in $L^2(\tau, T; H^{-s}(\Omega_m))$, we have $u^k_{|\Omega_m} \longrightarrow u_{|\Omega_m}$ in $\mathcal{C}([\tau, T], H^{-s}(\Omega_m))$. The boundedness of $u^k_{|\Omega_m}(r)$ in $L^2(\Omega_m)$ implies then by a standard argument that $u^k_{|\Omega_m}(r) \rightharpoonup u_{|\Omega_m}(r)$ weakly in $L^2(\Omega_m)$ for all r.

Then it follows easily that $u^k(t) \rightharpoonup u(t)$ weakly in $L^2(\Omega)$ for any $t \in [\tau, T]$.

Also, we deduce $u(\tau) = u_\tau$. As $u^k_{|\Omega_m}$ is a weak solution with f replaced by $\widetilde{f}_k$ and Ω by Ω_m, passing to the limit we obtain that u is a weak solution.

Finally, we shall prove the following:

Lemma 8 *We have*

$$u^k(t^*) \longrightarrow u(t^*) \text{ strongly in } L^2(\Omega).$$

Proof. We note that using (4) one can easily obtain that any solution v of (2) satisfies the estimate

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla v(t)\|^2_{L^2(\Omega)} + \alpha_2 \|v(t)\|^p_{L^p(\Omega)} \leq \frac{1}{2}\|h(t)\|^2_* + \|C_2\|_{L^1(\Omega)}.$$

By integration and using (20) we have

$$\left\|u^k(t)\right\|^2_{L^2(\Omega)} \leq \left\|u^k(s)\right\|^2_{L^2(\Omega)} + \int_s^t \|h(\xi)\|^2_* \, d\xi + 2M(t-s), \tag{28}$$

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(s)\|^2_{L^2(\Omega)} + \int_s^t \|h(\xi)\|^2_* \, d\xi + 2M(t-s), \tag{29}$$

for all $t \geq s$, t, $s \in [\tau, T]$, where the constant $M > 0$ does not depend on k. From (28) and (29) the functions $J_k(t) = \left\|u^k(t)\right\|^2_{L^2(\Omega)} - 2Mt - \int_\tau^t \|h(\xi)\|^2_* \, d\xi$, $J(t) = \|u(t)\|^2_{L^2(\Omega)} - 2Mt - \int_\tau^t \|h(\xi)\|^2_* \, d\xi$ are continuous and non-increasing on $[\tau, T]$. We state that $\limsup J_k(t^*) \leq J(t^*)$. We know that $J_k(t) \longrightarrow J(t)$, for a.a. $t \in (\tau, T)$. Let t_m be a sequence such that $\tau < t_m < t^*$, $t_m \longrightarrow t^*$, as $m \longrightarrow \infty$, and $J_k(t_m) \longrightarrow J(t_m)$, as $k \longrightarrow \infty$, for any fixed m. Hence, using the continuity of J and the monotonicity of J_k, J we have that for any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $K(\varepsilon, m)$ such that

$$\begin{aligned} J_k(t^*) - J(t^*) &= J_k(t^*) - J_k(t_m) + J_k(t_m) - J(t_m) + J(t_m) - J(t^*) \\ &\leq |J_k(t_m) - J(t_m)| + |J(t_m) - J(t^*)| \leq 2\varepsilon, \end{aligned}$$

if $k \geq K$. Hence,

$$\begin{aligned} \limsup J_k(t^*) &= \limsup \left\|u^k(t^*)\right\|^2_{L^2(\Omega)} - 2Mt^* - \int_\tau^{t^*} \|h(\xi)\|^2_* \, d\xi \\ &\leq \|u(t^*)\|^2_{L^2(\Omega)} - 2Mt^* - \int_\tau^{t^*} \|h(\xi)\|^2_* \, d\xi. \end{aligned}$$

Therefore, $\limsup \left\|u^k(t^*)\right\|_{L^2(\Omega)} \leq \|u(t^*)\|_{L^2(\Omega)}$. Since $u^k(t^*) \rightharpoonup u(t^*)$ weakly in $L^2(\Omega)$, we have $\liminf \left\|u^k(t^*)\right\|_{L^2(\Omega)} \geq \|u(t^*)\|_{L^2(\Omega)}$. Thus, $u^k(t^*) \longrightarrow u(t^*)$ in $L^2(\Omega)$. ■

From this we immediately obtain that $u(t^*) \notin E_1 \cup E_2$, which is a contradiction and we conclude the proof of the theorem. ■

3 Connectedness of the pullback attractor

In [2] the existence of a global pullback attractor for (2) was proved. Our aim now is to obtain that this attractor is also connected in $L^2(\Omega)$.

First we recall some basic definitions for set-valued non-autonomous dynamical systems and establish a sufficient condition for the existence of a pullback attractor for these systems (see [6], [10], [18] for more details and [8], [9] and [19] for related items).

Let $X = (X, d_X)$ be a metric space, and let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X, and let us denote $\mathbb{R}^2_d := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$.

Definition 9 *A multi-valued map $U : \mathbb{R}^2_d \times X \longrightarrow \mathcal{P}(X)$ is called a multi-valued non-autonomous dynamical system (**MNDS**) on X (also named a multi-valued process on X) if*

$$U(s,s,\cdot) = id_X(\cdot) \text{ for all } s \in \mathbb{R},$$

$$U(t,\tau,x) \subset U(t,s,U(s,\tau,x)) \text{ for all } \tau \leq s \leq t,\ x \in X,$$

where $U(t,\tau,V) := \bigcup_{x_0 \in V} U(t,\tau,x_0)$ for any non-empty set $V \subset X$.

*An **MNDS** is said to be strict if*

$$U(t,\tau,x) = U(t,s,U(s,\tau,x)) \text{ for all } \tau \leq s \leq t,\ x \in X.$$

Definition 10 *An MNDS U on X is said to be **upper-semicontinuous** if for all $t \geq \tau$ the mapping $U(t,\tau,\cdot)$ is upper-semicontinuous from X into $\mathcal{P}(X)$, i.e., for any $x_0 \in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau,x_0)$, there exists $\delta > 0$ such that $U(t,\tau,y) \subset \mathcal{N}$ whenever $d_X(x_0,y) < \delta$.*

Let $\mathcal{D}$ be a class of sets parameterized in time, $\widehat{D} = \{D(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$. We will say that the class $\mathcal{D}$ is inclusion-closed, if $\widehat{D} \in \mathcal{D}$ and $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, imply that $\widehat{D'} = \{D'(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$ belongs to $\mathcal{D}$.

Definition 11 *We say that a family $\widehat{B} = \{B(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$ is pullback $\mathcal{D}$-**absorbing** if for every $\widehat{D} \in \mathcal{D}$ and every $t \in \mathbb{R}$, there exists $\tau(t,\widehat{D}) \leq t$ such that*

$$U(t,\tau,D(\tau)) \subset B(t) \text{ for all } \tau \leq \tau(t,\widehat{D}).$$

Definition 12 *The MNDS U is **asymptotically compact** with respect to a family $\widehat{B} = \{B(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$ if for all $t \in \mathbb{R}$ and every sequence $\tau_n \leq t$ tending to $-\infty$, any sequence $y_n \in U(t,\tau_n,B(\tau_n))$ is precompact.*

Let $dist_X(\cdot,\cdot)$ denote the Hausdorff semidistance, defined by

$$dist_X(C_1,C_2) := \sup_{x \in C_1} \inf_{y \in C_2} d_X(x,y) \quad \text{for } C_1,\, C_2 \subset X.$$

Definition 13 *A family $\widehat{A} = \{A(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$ is said to be a **global pullback** $\mathcal{D}$-**attractor** for the MNDS U if it satisfies:*

1. *$A(t)$ is compact for any $t \in \mathbb{R}$,*
2. *$\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau,D(\tau)),A(t)) = 0 \quad \forall t \in \mathbb{R},$$

for all $\widehat{D} \in \mathcal{D}$,

3. *$\widehat{A}$ is negatively invariant, i.e.,*

$$A(t) \subset U(t,\tau,A(\tau)), \text{ for any } (t,\tau) \in \mathbb{R}^2_d.$$

*$\widehat{A}$ is said to be a **strict global pullback** $\mathcal{D}$-**attractor** if the invariance property in the third item is strict, i.e.,*

$$A(t) = U(t,\tau,A(\tau)), \text{ for } (t,\tau) \in \mathbb{R}^2_d.$$

Theorem 14 *Assume that the MNDS U is upper-semicontinuous with closed values, and let $\widehat{B} = \{B(t) \in \mathcal{P}(X) : t \in \mathbb{R}\}$ be pullback $\mathcal{D}$-absorbing and such that U is asymptotically compact with respect to $\widehat{B}$. Then, the following statements hold:*

1) The set $\widehat{A}$ given by

$$A(t) := \Lambda\left(\widehat{B}, t\right) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t, \tau, B(\tau))} \quad t \in \mathbb{R}, \tag{30}$$

is a pullback $\mathcal{D}$-attractor for the MNDS U.

Moreover, suppose that $\mathcal{D}$ is inclusion closed, $\hat{B} \in \mathcal{D}$, and $B(t)$ is closed in X for any $t \in \mathbb{R}$. Then the family $\widehat{A}$ defined by (30) belongs to $\mathcal{D}$, and is the unique pullback $\mathcal{D}$-attractor with this property. In addition, in this case, if U is a strict MNDS, then $\widehat{A}$ is strictly invariant.

2) If, in addition to the main assumptions, $U(t, \tau, \cdot)$ has connected values and $A(t) \subset C(t)$, where $\widehat{C} \in \mathcal{D}$ is connected, then $\widehat{A}$ is connected, which means that any $A(t)$ is connected for any $t \ge \tau$.

Proof. For the first statement, see [10] and [18].

For the second statement, suppose that $\widehat{A}$ is not connected. Then there exist $t \in \mathbb{R}$ and two open sets θ_1, θ_2 satisfying $A(t) \cap \theta_i \neq \emptyset$ for $i = 1, 2$, $A(t) \subset \theta_1 \cup \theta_2$ and $\theta_1 \cap \theta_2 = \emptyset$.

It is well known (see [5], [11] or also [14, Theorem 24]) that an upper semicontinuous map with connected values maps any connected set into a connected one. Since the set $C(\tau)$ is connected, $U(t, \tau, C(\tau))$ is connected.

As $\widehat{A}$ is negatively invariant, we have

$$A(t) \subset U(t, \tau, A(\tau)) \subset U(t, \tau, C(\tau)).$$

Hence, $U(t, \tau, C(\tau)) \cap \theta_i \neq \emptyset$ for $i = 1, 2$, and by the connectedness of $U(t, \tau, C(\tau))$ we obtain that $\theta_1 \cup \theta_2$ does not contain $U(t, \tau, C(\tau))$. Thus for any $\tau \le t$ there exists $\xi_\tau \in U(t, \tau, C(\tau))$ such that $\xi_\tau \notin \theta_1 \cup \theta_2$.

Now, as $\widehat{A}$ is pullback $\mathcal{D}$-attracting and $\widehat{C} \in \mathcal{D}$, for each $n \ge 1$ there exist $\tau_n \le t$ and $y_n \in A(t)$ such that

$$d_X(\xi_{\tau_n}, y_n) \le \frac{1}{n}. \tag{31}$$

As $A(t)$ is compact for any $t \in \mathbb{R}$, we can extract a converging subsequence

$$y_m \longrightarrow y \in A(t).$$

By (31), we obtain

$$\xi_{\tau_m} \longrightarrow y \in A(t) \subset \theta_1 \cup \theta_2.$$

But taking into account that $\theta_1 \cup \theta_2$ is an open set then there exists m_0 for which $\xi_{\tau_m} \in \theta_1 \cup \theta_2$, for all $m > m_0$, which is a contradiction. ■

We shall apply Theorem 14 to equation (2).

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D} = \left\{D(t) \in \mathcal{P}(L^2(\Omega)) : t \in \mathbb{R}\right\}$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$ for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$, where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$. Observe that the class $\mathcal{D}_{\lambda_1}$ is inclusion-closed.

Theorem 15 *Under the assumptions for Ω, f and h as in Lemma 3, the MNDS U defined by (6) has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ belonging to $\mathcal{D}_{\lambda_1}$, which is strictly invariant and connected.*

Proof. In [2, Proposition 16 and Lemma 17] it is proved that the MNDS U defined by (6) is upper semicontinuous and has closed values.

It follows also by [2, Lemma 12] that the family $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, where the family $\widehat{B}_{\lambda_1}$ is defined by $B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$, and $R_{\lambda_1}(t)$ is the nonnegative number given for each $t \in \mathbb{R}$ by

$$R_{\lambda_1}^2(t) = 2e^{-\lambda_1 t}\sum_{i=1}^{N} \int_{-\infty}^{t} e^{\lambda_1 s} \left\| h_i(s) \right\|_{L^2(\Omega)}^2 ds + 2\lambda_1^{-1} \left\| C_2 \right\|_{L^1(\Omega)} + 1.$$

In [2, Lemma 18] it is proved that the MNDS U defined by (6) is asymptotically compact with respect to the family $\widehat{B}_{\lambda_1}$.

Also, in [2, Theorem 19] we prove the existence of a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ for U which is strictly invariant and belongs to $\mathcal{D}_{\lambda_1}$.

Finally, we shall study the connectedness of the pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$.

By Theorem 6, $U(t, \tau, u_\tau)$ has connected values in $L^2(\Omega)$. On the other hand, as $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, in particular we have that there exists $\tau(t, \widehat{A}) \leq t$ such that

$$U(t, \tau, A(\tau)) \subset B_{\lambda_1}(t) \ \text{ for all } \tau \leq \tau(t, \widehat{A}).$$

Since $\widehat{A}$ is negatively semi-invariant, we have

$$A(t) \subset B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t)),$$

where $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ is connected.

Hence, all conditions of the second statement of Theorem 14 are satisfied. Then, we have that $\widehat{A}$ is connected. ■

4 The Kneser property for a system of reaction-diffusion equations

We shall extend now the results of the previous section to the following system of reaction-diffusion equations

$$u_t = a\Delta u - f(x, u),\ x \in \mathbb{R}^N, t > 0, \tag{32}$$

$$u(0) = u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d, \tag{33}$$

where u is an unknown vector function, that is, $u(x,t) = (u^1, .., u^d)$, $x \in \mathbb{R}^N$, $t > 0$, $f(x,u) = (f^1, ..., f^d)$, and $u_t = \dfrac{\partial u}{\partial t}$. We assume the next conditions:

$(H1)$ The real $d \times d$ matrix a has a positive symmetric part $\frac{1}{2}(a + a^*) \geq AI$, where $A > 0$.

$(H2)$ $f = f_0 + f_1$, $f_0(x,u) = (f_0^1, .., f_0^d)$, $f_1(x,u) = (f_1^1, .., f_1^d)$ and f_i are Caratheodory functions, that is, they are continuous on u and measurable on x.

$(H3)$ There exist positive functions $C_0(x)$, $C_1(x) \in L^1(\mathbb{R}^N)$ and constants $\alpha, \beta > 0$, $p_i \geq 2$ verifying

$$\left(f_0\left(x,u\right), u\right) \geq \alpha \left|u\right|^2 - C_0\left(x\right), \tag{34}$$

$$(f_1(x,u), u) \geq \beta \sum_{i=1}^{d} \left|u^i\right|^{p_i} - C_1(x). \tag{35}$$

$(H4)$ There exist positive functions $C_2(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), C_3(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and constants $\gamma, \eta > 0$ verifying

$$|f_0(x,u)| \leq C_2(x) + \eta |u|, \tag{36}$$

$$\sum_{i=1}^{d} |f_1^i(x,u)|^{\frac{p_i}{p_i - 1}} \leq C_3(x) + \gamma \sum_{i=1}^{d} |u^i|^{p_i}. \tag{37}$$

Here, $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^m$ for $m \geq 1$, $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^d$.

The Kneser property for this system was studied before in [22] but considering $C_2(x) \in L^2(\mathbb{R}^N)$, $C_3(x) \in L^1(\mathbb{R}^N)$ and assuming an additional condition on the derivatives of f_0, f_1. Our aim is to apply the technique of the previous section in order to avoid such condition. Instead, we have to assume that $C_2(x), C_3(x) \in L^\infty(\mathbb{R}^N)$.

First, we shall state the equivalent statements of Lemmas 1 and 2. As the proofs are essentialy rather similar, we shall omit them. A detailed proof can be found in [1].

We take a sequence $0 < \epsilon_k < 1$ converging to 0 as $k \to \infty$ and define a sequence of smooth functions $\psi_k : \mathbb{R}^+ \longrightarrow [0,1]$ satisfying

$$\psi_k(s) := \begin{cases} 1, \text{ if } 0 \leq s \leq \sqrt{\epsilon_k}, \\ 0 \leq \psi_k \leq 1, \text{ if } \sqrt{\epsilon_k} \leq s \leq 2\sqrt{\epsilon_k}, \\ 0, \text{ if } 2\sqrt{\epsilon_k} \leq s \leq 1/\epsilon_k, \\ 0 \leq \psi_k \leq 1, \text{ if } 1/\epsilon_k \leq s \leq 1/\epsilon_k + 1, \\ 1, \text{ if } s \geq 1/\epsilon_k + 1. \end{cases}$$

Let $\rho_{\epsilon_k} : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ be a mollifier, that is, $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R}^d;\mathbb{R})$, $\operatorname{supp}\rho_{\epsilon_k} \subset B_{\epsilon_k}$, $\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)ds = 1$ and $\rho_{\epsilon_k}(s) \geq 0$ for all $s \in \mathbb{R}^d$, where $B_{\epsilon_k} = \left\{u \in \mathbb{R}^d : |u| \leq \epsilon_k\right\}$.

We define the following approximating functions

$$f_{0k}^i(x,u) := \psi_k(|u|)\left(C_0^0 u^i + f_0^i(x,0)\right) + (1 - \psi_k(|u|)) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) f_0^i(x, u-s)\, ds,$$

$$f_{1k}^i(x,u) := \psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\left(C_0^1 \left|u^i\right|^{p_i-2} u^i + f_1^i(x,0)\right) + \left(1 - \psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) f_1^i(x, u-s)\, ds,$$

where $k \geq 1$, $p_i \geq 2$, and C_0^0, and C_0^1 are positive constants. Let $f^k = f_{0k} + f_{1k}$. Then for a.a. $x \in \mathbb{R}^N$ we have

$$\sup_{|u| \leq A} \left|f^k(x,u) - f(x,u)\right| \longrightarrow 0, \text{ as } k \longrightarrow \infty, \text{ for any } A > 0.$$

Lemma 16 *Assume (34)-(37). Then the functions f_{0k}, f_{1k} also satisfy conditions (34)-(37) with constants and functions not depending on k, i.e. there exist constants $\widehat{\alpha}, \widehat{\beta}, \widehat{\eta}, \widehat{\gamma} > 0$, and positive functions $\widehat{C}_0(x)$, $\widehat{C}_1(x) \in L^1(\mathbb{R}^N)$, $\widehat{C}_2(x) \in L^2(\Omega) \cap L^\infty(\Omega)$, $\widehat{C}_3(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ such that f_{0k}, f_{1k} satisfy (34)-(37), for k great enough.*

Lemma 17 *Assume (34)-(37). Then, f_{0k}, f_{1k} are continuously differentiable on u and there exist $D_{0\epsilon_k}$, $D_{1\epsilon_k}$ such that*

$$(f_{0ku}(x,u)w, w) \geq -D_{0\epsilon_k} |w|^2, \tag{38}$$

$$(f_{1ku}(x,u)w, w) \geq -D_{1\epsilon_k} |w|^2, \ \forall w, u \in \mathbb{R}^d, \text{ for a.a. } x \in \mathbb{R}^N, \tag{39}$$

where f_{0ku}, f_{1ku} denote the jacobian matrixes of f_{0k} and f_{1k}, respectively.

As in the previous sections for each $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$ let us denote by $S(u_0)$ the set of all weak solutions of (32)-(33) defined for all $t \geq 0$. Such a set is non-empty as in [21] it is proved that at least one weak global solution exists for any $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

We define a multi-valued map $G : \mathbb{R}^+ \times \left[L^2\left(\mathbb{R}^N\right)\right]^d \to \mathcal{P}(\left[L^2\left(\mathbb{R}^N\right)\right]^d)$ by

$$G(t,u_0) = \{u(t) : \ u \in S(u_0)\}, \quad t \geq 0, \quad u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d. \tag{40}$$

In [21] it is shown that the multi-valued mapping G defined by (40) is a strict multivalued semiflow on $L^2(\Omega)$, that is, $G(0,\cdot) = Id$ and $G(t+s, u_0) = G(t, G(s,u_0))$ for all $t, s \geq 0, u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

In [21] it is proved that the set $G(t,u_0)$ is compact. Our aim is to prove the connectedness of the set $G(t,u_0) \subset \left[L^2\left(\mathbb{R}^N\right)\right]^d$ for any $t \geq 0$, $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$. Then we obtain the Kneser property.

Using lemmas 16, 17 and the same proof of Theorem 7 in [22] we have the following result.

Theorem 18 *Assume (34)-(37). Then* $G(t,u_0)$ *is connected for any* $t \geq 0$, $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

Remark 19 *The same result is true, with slight changes in the proofs, for the following system*

$$\begin{aligned} u_t &= a\Delta u - f(x,u) + h(x),\ x \in \mathbb{R}^N, t > 0, \\ u(0) &= u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d, \end{aligned}$$

where $h \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$ *and* $(H1)-(H4)$ *hold.*

Finally, we observe that in [21] the existence of a global compact invariant attractor $\mathcal{A}$ for the multivalued semiflow G is proved. Using Theorem 18 and arguing as in [22, Section 4] we obtain the following.

Theorem 20 *The global attractor* $\mathcal{A}$ *of* G *is connected.*

5 A generalized logistic equation

Consider the following problem

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + c(x)\left|u\right|^r - u^{p-1} + h(t), \\ \quad u = 0 \text{ on } \partial\Omega \times (\tau, +\infty), \\ \quad u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{array} \right. \tag{41}$$

where $\Omega \subset \mathbb{R}^N$ satisfies the Poincaré inequality, p is and even natural number, $0 < r < p-1$, $c(x) \in L^{\frac{p}{p-r-1}}(\Omega) \cap L^\infty(\Omega), c(x) \geq 0$, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ satisfies (17). This kind of nonlinearities for the logistic equation (instead of the classical $(1-u)u$) has been considered in [23, Chapter 11].

We note that

$$f(x,u) = c(x)\left|u\right|^r - u^{p-1}$$

and

$$f(x,u)u = c(x)\left|u\right|^r u - u^p \leq -\frac{1}{2}\left|u\right|^p + K_1 c(x)^{\frac{p}{p-r-1}}, \tag{42}$$

$$\left|f(x,u)\right|^{\frac{p}{p-1}} \leq K_2\left(c(x)^{\frac{p}{p-1}}\left|u\right|^{\frac{pr}{p-1}} + \left|u\right|^p\right) \leq K_3\left(c(x)^{\frac{p}{p-r-1}} + \left|u\right|^p\right), \tag{43}$$

so that conditions (3)-(4) hold.

In view of Theorems 6, 15 we obtain the following result.

Theorem 21 *Problem (41) generates a MNDS* U *such that:*

1. $U(t,\tau,u_\tau)$ *is connected in* $L^2(\Omega)$ *for any* $t \geq \tau$ *and* $u_\tau \in L^2(\Omega)$.
2. *The MNDS* U *has a unique pullback* $\mathcal{D}_{\lambda_1}$*-attractor* $\widehat{A}$ *belonging to* $\mathcal{D}_{\lambda_1}$*, which is strictly invariant and connected, where* λ_1 *is the constant in (1).*

When $\Omega = \mathbb{R}^N$ (so that the Poincaré inequality is not satisfied) and $h \equiv 0$ we can obtain the following result.

Theorem 22 *If* $p = 2$ *and* $\Omega = \mathbb{R}^N$*, then problem (41) generates a multivalued semiflow* G *such that:*

1. $G(t,u_0)$ *is connected in* $L^2(\Omega)$ *for any* $t \geq 0$ *and* $u_0 \in L^2(\Omega)$.
2. G *possesses a global compact invariant connected attractor* $\mathcal{A}$.

Proof. We take $f_0(x,u) = f_1(x,u) = \frac{1}{2}(-c(x)\left|u\right|^r + u)$, $d = 1$, $p_1 = 2$. Then, in view of (42)-(43) conditions (34)-(37) hold. The results follow from Theorems 18 and 20. ■

Remark 23 *The results given in [22] are not applicable to problem (41), as the condition on the derivative used in that paper is not satisfied. On the other hand, the results of the previous section can be applied also to the complex Ginzburg-Landau equation (as done in [22]). More precisely, it follows from Theorems 18, 20 and Remark 19 that Theorem 14 in [22] is true.*

Acknowledgements

This work was partially supported by spanish Ministerio de Ciencia e Innovación, Project MTM2008-00088 and MTM2009-11820, the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía), Proyecto de Excelencia P07-FQM-02468, and the Consejería de Cultura y Educación (Comunidad Autónoma de Murcia), grant 08667/PI/08.

We thank the anonymous referees for several useful remarks and comments.

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