Derivation of a quasi-stationary coupled Darcy-Reynolds equation for incompressible viscous fluid flow through a thin porous medium with a fissure

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Abstract

We consider a non-stationary Stokes system in a thin porous medium of thickness ε which is perforated by periodically distributed solid cylinders of size ε, and containing a fissure of width η_ε. Passing to the limit when ε goes to zero, we find a critical size $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ in which the flow is described by a 2D quasi-stationary Darcy law coupled with a 1D quasi-stationary Reynolds problem.

AMS classification numbers: 75A05, 76A20, 76M50, 35B27.

Keywords: Stokes equation; Darcy's law; Reynolds equation; thin porous medium; fissure.

1 Introduction

The aim of this work is to prove the convergence of the homogenization process for the non-stationary Stokes system in a thin porous medium $D_{\varepsilon\eta_\varepsilon}$ of thickness ε which is perforated by periodically distributed solid cylinders of size ε and contains a fissure $\{0 \leq x_2 \leq \eta_\varepsilon\}$ of width η_ε.

We consider the fluid flow through a periodic distribution of vertical cylinders and a fissure. The periodic distribution of vertical cylinders and the fissure are confined between two parallel plates (see Figure 1). A representative elementary volume for the thin porous medium is a cube of lateral length ε and vertical lentgth ε. The cube is repeated periodically in the space between the plates. Each cube can be divided into fluid part and a solid part, where the solid part has the shape of a vertical cylinder of height ε.

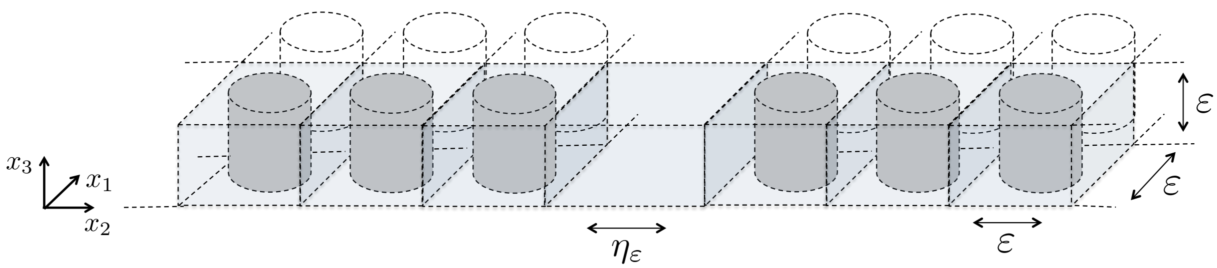

Figure 1: View of the domain $D_{\varepsilon\eta_\varepsilon}$

The question of a medium containing a fissure with properties different from those of the rest of the material has been the subject of many studies previously, see Ciarlet *et al* [1], Panasenko [2] and Chapter 13 of Sanchez-Palencia [3] among others. A similar problem of the one considered in this paper with a fixed height domain, but for the Laplace's equation, was studied in Bourgeat and Tapiero [4]. The peculiar behavior observed for the Laplace's equation when $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ has motivated the analogous study for the Stokes system in Bourgeat *et al* [5] (see [6] for the Navier-Stokes system and [7] for a non-stationary Stokes system).

In Anguiano [8], we consider a non-stationary Stokes system in a thin porous medium of thickness ε which is perforated by periodically distributed solid cylinders of size a_ε. We apply an adaptation of the unfolding method in order to obtain rigorously quasi-stationary Darcy's laws. The behavior observed when $a_\varepsilon \approx \varepsilon$ has motivated the fact of considering a thin porous medium containing a fissure. In this sense, our aim in the present paper is to extend the study of Bourgeat *et al* [5] to the case of a non-stationary Stokes system in a domain of small height ε, perforated by periodically distributed solid cylinders of size ε, containing a fissure of width η_ε, which makes necessary to rescale in the height variable in order to work with a domain of height one. We find the same critical size as in Bourgeat *et al* [5], what means that the evolutive model and the thin thickness of the domain do not modify the critical size. However, the thin thickness of the domain leads us to use techniques of reduction of the dimension together with homogenization in order to obtain more simplified effective models than those obtained in Bourgeat *et al* [5]. More precisely, we obtain the following results corresponding to three characteristic situations depending on the parameter η_ε with respect to ε:

- If $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ the fissure is not giving any contribution. In this case, in order to find the limit, we use the results developed in Anguiano [8] and we obtain a 2D quasi-stationary Darcy's law.

- If $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ the fissure is dominant. We introduce a rescaling of the fissure in order to work with a domain with size one, and then we prove that the limit of the velocity is a Dirac measure concentrated on the line $\{x_2 = 0\} \cap \{x_3 = 0\}$ representing the corresponding tangential line flow. Meanwhile in the porous medium the effective velocity is equal to zero.

- If $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$ with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, it appears a coupling effect and the effective flow behaves as 2D quasi-stationary Darcy flow in the porous medium coupled with the tangential flow of the line $\{x_2 = 0\} \cap \{x_3 = 0\}$. Compared to the first case $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, the effective velocity has now an additional tangential component concentrated on $\{x_2 = 0\} \cap \{x_3 = 0\}$. Moreover, the limit problem is now given by a new variational equation, in which appears the parameter λ, and consists of a 2D quasi-stationary Darcy law in the porous medium coupled with a 1D quasi-stationary Reynolds problem on the line $\{x_2 = 0\} \cap \{x_3 = 0\}$.

2 The domain and some notations

2.1 The domain

Let $\omega \subset \mathbb{R}^2$ be smooth bounded connected open set and $\Omega = \omega \times (0,1) \subset \mathbb{R}^3$. We define

$$\Omega_+ = \Omega \cap \{x_2 > 0\}, \quad \Omega_- = \Omega \cap \{x_2 < 0\}, \quad \Sigma = \Omega \cap \{x_2 = 0\}, \quad \Sigma_1 = \Sigma \cap \{x_3 = 0\}.$$

For some $\eta_0 > 0$ we define the domains

$$D = \Omega_- \cup (\eta_0 e_2 + \Omega_+) \cup (\Sigma \times [0, \eta_0] e_2), \quad D' = D \cap \{x_3 = 0\},$$

with $e_2 = (0, 1, 0)$.

Let $\varepsilon > 0$ be a small parameter devoted to tend to zero and $0 < \eta_\varepsilon < \eta_0$ be a small parameter devoted to tend to zero with ε.

A periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = [0,1]^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \bigcup Y'_s = Y'$ and $Y'_f \bigcap Y'_s = \emptyset$). More precisely, we assume that Y'_s is a smooth and connected set strictly included in Y'. For $k' = (k_1, k_2) \in \mathbb{Z}^2$, each cell $Y'_{k'} = k' + Y'$ is divided in a fluid part $Y'_{f_{k'}}$ and a solid part $Y'_{s_{k'}}$. We define $Y = Y' \times (0,1) \subset \mathbb{R}^3$, and is divided in a fluid part Y_f and a solid part Y_s.

We also denote

$$Y_s^- = \bigcup_{k' \in \mathbb{Z}^2_-} Y_{s_{k'}}, \quad Y_s^+ = \bigcup_{k' \in \mathbb{Z}^2_+} Y_{s_{k'}},$$

all the solid parts in $\mathbb{R}^2 \times (0,1)$, where $\mathbb{Z}^2_- = \{k' \in \mathbb{Z}^2,\ k_2 < 0\}$ and $\mathbb{Z}^2_+ = \{k' \in \mathbb{Z}^2,\ k_2 > 0\}$. It is obvious that $E_f = \big((\mathbb{R}^2 \times (0,1)) \setminus (Y_s^- \cup Y_s^+)\big) \cap \Omega$ is the fluid part in Ω.

Following [9], we make the following assumptions on Y_f, E_f, Y_s and $Y_s^* = Y_s^+ \cup Y_s^-$:

i) Y_f is an open connected set of strictly positive measure, with a locally Lipschitz boundary.

ii) Y_s has strictly positive measure in $\overline{Y}$.

iii) E_f and the interior of Y_s^* are open sets with boundaries of class $C^{0,1}$ and are locally located on one side of their boundaries. Moreover E_f is connected.

We also define

$$Y_{s,\varepsilon}^- = \varepsilon Y_s'^- \times (0,1), \quad Y_{s,\varepsilon\eta_\varepsilon}^+ = (\eta_\varepsilon e_2 + \varepsilon Y_s'^+) \times (0,1), \quad \widetilde{S}_{\varepsilon\eta_\varepsilon} = \partial(Y_{s,\varepsilon}^- \cup Y_{s,\varepsilon\eta_\varepsilon}^+).$$

We denote by

$\widetilde{A}_{\varepsilon\eta_\varepsilon} = (Y_{s,\varepsilon}^- \cup Y_{s,\varepsilon\eta_\varepsilon}^+) \cap D$ - the solid part of the domain D,
$\widetilde{D}_{\varepsilon\eta_\varepsilon} = D \setminus \widetilde{A}_{\varepsilon\eta_\varepsilon}$ - the fluid part of the domain D (including the fissure),
$\widetilde{I}_{\eta_\varepsilon} = \Sigma \times (0, \eta_\varepsilon)e_2$ - the fissure in D,
$\widetilde{\Omega}_{\varepsilon\eta_\varepsilon} = \widetilde{D}_{\varepsilon\eta_\varepsilon} \setminus \widetilde{I}_{\eta_\varepsilon}$ - the fluid part of the porous medium in D.

Let us define a domain with thickness ε, given by $\Omega^\varepsilon = \Omega \cap \{0 < x_3 < \varepsilon\} \subset \mathbb{R}^3$. We also define

$$\Omega_+^\varepsilon = \Omega_+ \cap \{0 < x_3 < \varepsilon\}, \quad \Omega_-^\varepsilon = \Omega_- \cap \{0 < x_3 < \varepsilon\}, \quad \Sigma^\varepsilon = \Omega^\varepsilon \cap \{x_2 = 0\},$$

and

$$D^\varepsilon = \Omega_-^\varepsilon \cup \left(\eta_0 e_2 + \Omega_+^\varepsilon\right) \cup (\Sigma^\varepsilon \times [0, \eta_0]e_2)\,.$$

The microscale of a porous medium is the small positive number ε. The domain ω is covered by a regular mesh of size ε: for $k' = (k_1, k_2) \in \mathbb{Z}^2$, each cell $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},\varepsilon}$ and a solid part $Y'_{s_{k'},\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size ε. We define $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0,1) \subset \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},\varepsilon}$ and a solid part $Y_{s_{k'},\varepsilon}$.

Now, we denote by $A_{\varepsilon\eta_\varepsilon}$, $D_{\varepsilon\eta_\varepsilon}$, I_{η_ε} and $\Omega_{\varepsilon\eta_\varepsilon}$ the sets $\widetilde{A}_{\varepsilon\eta_\varepsilon}$, $\widetilde{D}_{\varepsilon\eta_\varepsilon}$, $\widetilde{I}_{\eta_\varepsilon}$ and $\widetilde{\Omega}_{\varepsilon\eta_\varepsilon}$, respectively, with thickness ε, i.e.,

$A_{\varepsilon\eta_\varepsilon} = \widetilde{A}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the solid part of the domain D^ε,
$D_{\varepsilon\eta_\varepsilon} = \widetilde{D}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fluid part of the domain D^ε (including the fissure),
$I_{\eta_\varepsilon} = \widetilde{I}_{\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fissure in D^ε,
$\Omega_{\varepsilon\eta_\varepsilon} = \widetilde{\Omega}_{\varepsilon\eta_\varepsilon} \cap \{0 < x_3 < \varepsilon\}$ - the fluid part of the porous medium in D^ε.

Finally we define

$$\Omega_{\varepsilon\eta_\varepsilon}^+ = D_{\varepsilon\eta_\varepsilon} \cap \{x_2 > \eta_\varepsilon\}, \quad \Omega_{\varepsilon\eta_\varepsilon}^- = D_{\varepsilon\eta_\varepsilon} \cap \{x_2 < 0\}, \quad \Gamma_{\eta_\varepsilon} = \partial\Sigma^\varepsilon \times (0, \eta_\varepsilon)e_2,$$

and

$$D^+ = D \cap \{x_2 > 0\}, \quad D^- = \Omega_-\,.$$

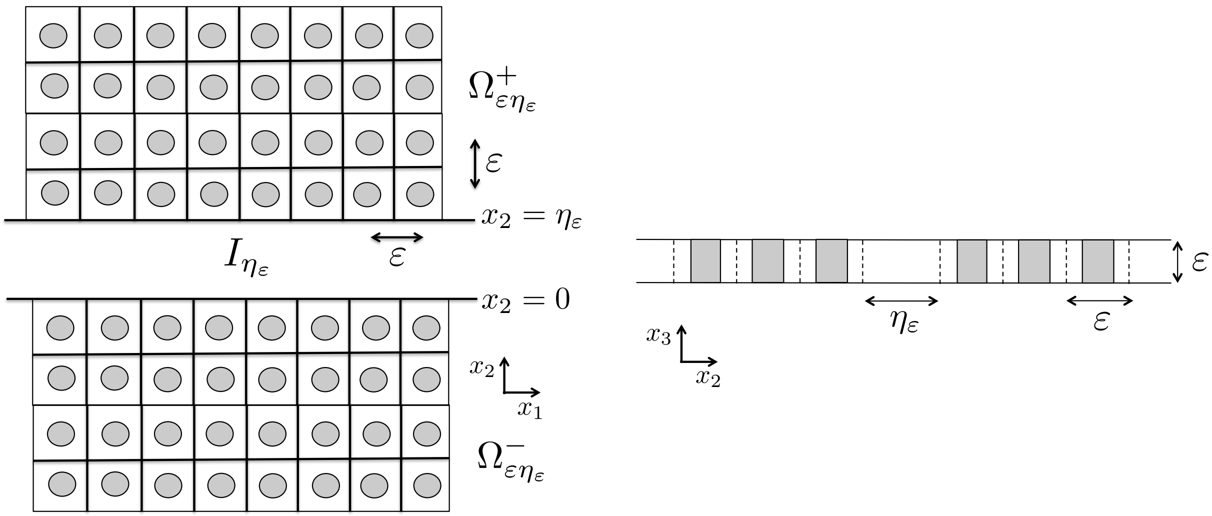

Figure 2: View of the domain $D_{\varepsilon\eta_\varepsilon}$ from above (left) and lateral (right)

2.2 Some notations

Let us introduce some notations which will be useful in the following. For a vectorial function $v = (v_1, v_2, v_3)$ and a scalar function w, we introduce the operators: D_ε, ∇_ε and div_ε by

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } \; i = 1,2,3, \; j = 1,2,$$

$$(D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } \; i = 1,2,3,$$

$$\nabla_\varepsilon w = (\nabla_{x'} w, \frac{1}{\varepsilon}\partial_{y_3} w)^t,$$

$$\text{div}_\varepsilon v = \text{div}_{x'} v' + \frac{1}{\varepsilon}\partial_{y_3} v_3,$$

and moreover the operators D_{η_ε}, $\nabla_{\eta_\varepsilon}$ and $\text{div}_{\eta_\varepsilon}$ by

$$(D_{\eta_\varepsilon} v)_{i,1} = \partial_{x_1} v_i \text{ for } \; i = 1,2,3,$$

$$(D_{\eta_\varepsilon} v)_{i,2} = \frac{1}{\eta_\varepsilon}\partial_{y_2} v_i \text{ for } \; i = 1,2,3,$$

$$(D_{\eta_\varepsilon} v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } \; i = 1,2,3,$$

$$\nabla_{\eta_\varepsilon} w = (\partial_{x_1} w, \frac{1}{\eta_\varepsilon}\partial_{y_2} w, \frac{1}{\varepsilon}\partial_{y_3} w)^t,$$

$$\text{div}_{\eta_\varepsilon} v = \partial_{x_1} v_1 + \frac{1}{\eta_\varepsilon}\partial_{y_2} v_2 + \frac{1}{\varepsilon}\partial_{y_3} v_3.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line. We denote by C a generic positive constant which can change from line to line.

3 Setting and main results

Hereinafter, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

In this section, we describe the asymptotic behavior of an incompressible viscous fluid in a thin porous medium with a fissure. The proof of the corresponding results will be given in the next sections.

Our results are referred to the non-stationary Stokes system. Namely, for $f \in C([0,T] \times \overline{D})^3$ let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in L^2(0,T; H_0^1(D_{\varepsilon\eta_\varepsilon}))^3 \times L^2(0,T; L^2(D_{\varepsilon\eta_\varepsilon}))$, which satisfies

$$\left\{\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \mu\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } & (0,T) \times D_{\varepsilon\eta_\varepsilon}, \\ \operatorname{div} u_\varepsilon = 0 \text{ in } & (0,T) \times D_{\varepsilon\eta_\varepsilon}, \\ u_\varepsilon(0,x) = 0, \quad & x \in D_{\varepsilon\eta_\varepsilon}, \end{aligned}\right. \tag{3.1}$$

where $T > 0$, $\mu > 0$ is the viscosity and $D_{\varepsilon\eta_\varepsilon}$ is defined in Section 2. The right-hand side f is of the form

$$f(t,x) = (f'(t,x'), 0), \text{ a.e. } x \in D, \tag{3.2}$$

where

$$f' \in C([0,T] \times \overline{D})^2. \tag{3.3}$$

This choice of f is usual when we deal with thin domains. Since the thickness of the domain ε is small then the vertical component of the force can be neglected and, moreover the force can be considered independent of the vertical variable.

Finally, we may consider Dirichlet boundary conditions without altering the generality of the problem under consideration,

$$u_\varepsilon = 0 \text{ on } (0,T) \times \partial D_{\varepsilon\eta_\varepsilon}. \tag{3.4}$$

For any fixed ε, under the assumptions of f and u_ε^0, a classical result (see Temam [10]) shows that (3.1)-(3.4) has at least one weak solution $(u_\varepsilon, p_\varepsilon) \in L^2(0,T; H_0^1(D_{\varepsilon\eta_\varepsilon}))^3 \times L^2(0,T; L^2(D_{\varepsilon\eta_\varepsilon}))$, where p_ε is uniquely defined up to an additive constant, that is, it is uniquely defined if we consider the corresponding equivalence class: $p_\varepsilon \in L^2(0,T; L^2(D_{\varepsilon\eta_\varepsilon})/\mathbb{R})$.

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3

$$y_3 = \frac{x_3}{\varepsilon}, \tag{3.5}$$

in order to have the functions defined in an open set with fixed height $\widetilde{D}_{\varepsilon\eta_\varepsilon}$ given in Section 2.

Namely, we define $\tilde{u}_\varepsilon \in L^2(0,T; H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon}))^3$, $\tilde{p}_\varepsilon \in L^2(0,T; L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})/\mathbb{R})$ by

$$\tilde{u}_\varepsilon(t,x',y_3) = u_\varepsilon(t,x',\varepsilon y_3), \quad \tilde{p}_\varepsilon(t,x',y_3) = p_\varepsilon(t,x',\varepsilon y_3), \quad a.e. \ (t,x',y_3) \in (0,T) \times \widetilde{D}_{\varepsilon\eta_\varepsilon}.$$

Using the transformation (3.5), the system (3.1) can be rewritten as

$$\left\{\begin{aligned} \frac{\partial \tilde{u}_\varepsilon}{\partial t} - \mu\Delta_\varepsilon \tilde{u}_\varepsilon + \nabla_\varepsilon \tilde{p}_\varepsilon = f \text{ in } & (0,T) \times \widetilde{D}_{\varepsilon\eta_\varepsilon}, \\ \operatorname{div}_\varepsilon \tilde{u}_\varepsilon = 0 \text{ in } & (0,T) \times \widetilde{D}_{\varepsilon\eta_\varepsilon}, \\ \tilde{u}_\varepsilon(0,x',y_3) = 0, \quad & (x',y_3) \in \widetilde{D}_{\varepsilon\eta_\varepsilon}, \end{aligned}\right. \tag{3.6}$$

with Dirichlet boundary conditions

$$\tilde{u}_\varepsilon = 0 \text{ on } \ (0,T) \times \partial \widetilde{D}_{\varepsilon\eta_\varepsilon}, \tag{3.7}$$

where we set $\Delta_\varepsilon w = \Delta_{x'} w + \varepsilon^{-2}\partial^2_{y_3} w$ and $\widetilde{D}_{\varepsilon\eta_\varepsilon}$ is defined in Section 2.

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon)$.

Moreover, in order to study the behavior of $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ in the fissure we rewrite our equations in the unit cylinder $\widetilde{I}_1 = \Sigma \times (0,1)e_2$ by introducing the change of variable

$$y_2 = \frac{x_2}{\eta_\varepsilon}\,, \tag{3.8}$$

which transform $\widetilde{I}_{\eta_\varepsilon}$ in a fixed domain $\widetilde{I}_1$. We define the new functions

$$\tilde{\mathcal{U}}^\varepsilon(t,x_1,y_2,y_3) = \tilde{u}_\varepsilon(t,x_1,\eta_\varepsilon y_2,y_3), \quad \tilde{P}^\varepsilon(t,x_1,y_2,y_3) = \tilde{p}_\varepsilon(t,x_1,\eta_\varepsilon y_2,y_3) - c_{\varepsilon\eta_\varepsilon}, \tag{3.9}$$

with

$$c_{\varepsilon\eta_\varepsilon} = \frac{1}{|\widetilde{I}_{\eta_\varepsilon}|}\int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon(t,x',y_3)\,dx'dy_3. \tag{3.10}$$

Using the transformation (3.8), the system (3.6) can be rewritten as

$$\left\{\begin{array}{c} \dfrac{\partial \tilde{\mathcal{U}}^\varepsilon}{\partial t} - \mu\Delta_{\eta_\varepsilon}\tilde{\mathcal{U}}^\varepsilon + \nabla_{\eta_\varepsilon}\tilde{P}^\varepsilon = f(t,x_1,\eta_\varepsilon y_2) \text{ in } \ (0,T)\times\widetilde{I}_1, \\ \operatorname{div}_{\eta_\varepsilon}\tilde{\mathcal{U}}^\varepsilon = 0 \text{ in } \ (0,T)\times\widetilde{I}_1, \\ \tilde{\mathcal{U}}^\varepsilon(0,x_1,\eta_\varepsilon y_2,y_3) = 0, \quad (x_1,\eta_\varepsilon y_2,y_3)\in\widetilde{I}_1, \end{array}\right. \tag{3.11}$$

with Dirichlet boundary conditions

$$\tilde{\mathcal{U}}^\varepsilon = 0 \text{ on } \ (0,T)\times\partial\widetilde{I}_1, \tag{3.12}$$

where we set $\Delta_{\eta_\varepsilon} w = \partial^2_{x_1} w + \eta_\varepsilon^{-2}\partial^2_{y_2} w + \varepsilon^{-2}\partial^2_{y_3} w$.

Our main result referred to the asymptotic behavior of the solution of (3.6) is given by the following theorem.

Theorem 3.1. *We distingue three cases depending on the relation between the parameter η_ε with respect to ε:*

i) if $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, then there exists $(\tilde{v},\tilde{p}) \in L^2((0,T)\times D)^3 \times L^2(0,T;L^2(D)/\mathbb{R})$, with $\tilde{v}_3 = 0$ and $\tilde{p}$ independent of y_3, such that the solution $(\varepsilon^{-2}\tilde{u}_\varepsilon,\tilde{p}_\varepsilon)$ of problem (3.6)-(3.7) satisfies

$$\varepsilon^{-2}\tilde{u}_\varepsilon \rightharpoonup \tilde{v} \quad \text{in } L^2((0,T)\times D)^3, \quad \tilde{p}_\varepsilon \to \tilde{p} \quad \text{in } L^2(0,T;L^2(D)/\mathbb{R}). \tag{3.13}$$

Moreover, $\tilde{p} \in L^2(0,T;H^1(D)/\mathbb{R})$ and $(\tilde{V},\tilde{p})$ is the unique solution of the 2D quasi-stationary Darcy law (where t is only a parameter)

$$\left\{\begin{array}{rcl} \tilde{V}'(t,x') &=& \dfrac{1}{\mu}K\left(f'(t,x') - \nabla_{x'}\tilde{p}(t,x')\right) \quad \text{in } (0,T)\times D', \\ \operatorname{div}_{x'}\tilde{V}(t,x') &=& 0 \quad \text{in } (0,T)\times D', \\ \tilde{V}(t,x')\cdot n &=& 0 \quad \text{in } (0,T)\times\partial D', \end{array}\right. \tag{3.14}$$

where $\tilde V(t,x') = \int_0^1 \tilde v(t,x',y_3)dy_3$ *and* $K \in \mathbb{R}^{2\times 2}$ *is a symmetric, positive, tensor defined by its entries*

$$K_{ij} = \int_{Y_f} D_y w^i(y) : D_y w^j(y)\, dy, \quad i,j = 1,2, \tag{3.15}$$

where $w^i(y)$, $i = 1,2$, *with* $\int_{Y_f} w_3^i dy = 0$, *denotes the unique solution in* $H^1_\#(Y_f)^3$ *of the local stationary Stokes problems in 3D*

$$\left\{\begin{array}{rcl} -\Delta_y w^i + \nabla_y q^i &=& e_i \quad \text{in } Y_f,\\ \operatorname{div}_y w^i &=& 0 \quad \text{in } Y_f,\\ w^i &=& 0 \quad \text{in } \partial(Y\setminus Y_f),\\ w^i, q^i \quad Y' - \text{periodic}. \end{array}\right. \tag{3.16}$$

ii) if $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ *and let* $(\tilde{\mathcal{U}}^\varepsilon, \tilde P^\varepsilon)$ *be a solution of (3.11)-(3.12). Then there exist* $\tilde{\mathcal{U}} \in L^2((0,T)\times \widetilde I_1)^3$, *independent of* y_3, *with* $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, *and* $\tilde P \in L^2(0,T;L^2(\widetilde I_1)/\mathbb{R})$ *only depending on* t *and* x_1, *such that for a subsequence,*

$$\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \tilde{\mathcal{U}} \quad \text{in } L^2((0,T)\times\widetilde I_1)^3, \qquad \tilde P^\varepsilon \rightharpoonup \tilde P \quad \text{in } L^2(0,T;L^2(\widetilde I_1)/\mathbb{R}),$$

where

$$\tilde{\mathcal{U}}_1(t,x_1,y_2) = \frac{y_2(1-y_2)}{2}\Big(f_1(t,x_1,0) - \partial_{x_1}\tilde P(t,x_1)\Big). \tag{3.17}$$

Moreover, it holds that

$$\eta_\varepsilon{}^{-3}\tilde u_\varepsilon \overset{\star}{\rightharpoonup} \tilde{\mathcal{V}}\delta_{\Sigma_1} \quad \text{in } L^2(0,T;\mathcal{M}(D))^3, \tag{3.18}$$

where $\tilde{\mathcal{V}} \in L^2((0,T)\times\Sigma_1)^3$, *with* $\tilde{\mathcal{V}}_2 = \tilde{\mathcal{V}}_3 = 0$, *such that*

$$\tilde{\mathcal{V}}_1(t,x_1) = \int_0^1 \tilde{\mathcal{U}}_1(t,x_1,y_2)\,dy_2 = \frac{1}{12}\Big(f_1(t,x_1,0) - \partial_{x_1}\tilde P(t,x_1)\Big), \tag{3.19}$$

and, in fact $\tilde P \in L^2(0,T;H^1(\Sigma_1)/\mathbb{R})$ *is the unique solution of the 1D quasi-stationary Reynolds problem on* Σ_1 *(where* t *is only a parameter)*

$$\left\{\begin{array}{l} \partial_{x_1}\Big(f_1(t,x_1,0) - \partial_{x_1}\tilde P(t,x_1))\Big) = 0 \quad \text{in } (0,T)\times\Sigma_1,\\ \Big(f_1(t,x_1,0) - \partial_{x_1}\tilde P(t,x_1)\Big)\cdot n = 0 \quad \text{on } (0,T)\times\partial\Sigma_1. \end{array}\right. \tag{3.20}$$

iii) if $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, *with* $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, *then there exist a Darcy velocity* $\tilde v$, *a Reynolds velocity* $\tilde{\mathcal{V}}$ *and a pressure field* $\tilde p$ *such that*

$$\begin{array}{ll} \varepsilon^{-2}\tilde u_\varepsilon \overset{\star}{\rightharpoonup} \tilde v + \lambda^3\tilde{\mathcal{V}}\delta_{\Sigma_1} & \text{in } L^2(0,T;\mathcal{M}(D))^3,\\ \tilde p_\varepsilon \to \tilde p & \text{in } L^2(0,T;L^2(D)/\mathbb{R}), \end{array} \tag{3.21}$$

where δ_{Σ_1} *is the Dirac measure concentrated on* Σ_1, *and* $\mathcal{M}(D)^3$ *is the space of Radon meaures on* D. *The velocities* $\tilde v$ *and* $\tilde{\mathcal{V}}$ *are linked with the pressure* $\tilde p$ *through the 2D Darcy law (3.14) in* $(0,T)\times D'$ *and the 1D Reynolds problem (3.20) on* $(0,T)\times\Sigma_1$. *The pressure field* $\tilde p \in L^2(0,T;H^1(D')/\mathbb{R})$ *with* $\tilde p(\cdot,0) \in L^2(0,T;H^1(\Sigma_1)/\mathbb{R})$, *is the unique solution of the variational problem*

$$\int_0^T\int_{D'}\frac{1}{\mu}K\left(f'(t,x') - \nabla_{x'}\tilde p(t,x')\right)\cdot\nabla_{x'}\varphi(t,x')\,dx'dt + \frac{\lambda^3}{12}\int_0^T\int_{\Sigma_1}\left(f_1(t,x_1,0) - \partial_{x_1}\tilde p(t,x_1)\right)\partial_{x_1}\varphi(t,x_1,0)\,dx_1 d \tag{3.22}$$

for every $\varphi \in L^2(0,T;H^1(D'))$ *with* $\varphi(\cdot,0) \in L^2(0,T;H^1(\Sigma_1))$.

Remark 3.2. *The coupled problem (3.22) corresponding to the critical case $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, can be considered as the general one. In fact, if λ tends to infinity in (3.22) we recover the 1D quasi-stationary Reynolds problem (3.20), meanwhile if λ tends to zero we recover the 2D quasi-stationary Darcy law (3.14).*

4 A Priori Estimates

Let us begin with a lemma on Poincaré inequality in the porous medium $\widetilde{\Omega}_{\varepsilon\eta_\varepsilon}$, which will be very useful (see for example Lemma 4.1 in [8]).

Lemma 4.1. *There exists a constant C independent of ε, such that, for any function $v \in H^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $\widetilde{S}_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3} \leq C\varepsilon \|D_\varepsilon v\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^{3\times 3}} . \tag{4.23}$$

Next, we give an useful estimate in the fissure $\widetilde{I}_{\eta_\varepsilon}$.

Lemma 4.2. *There exists a constant C independent of ε, such that, for any function $v \in H^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and $v = 0$ on $\widetilde{S}_{\varepsilon\eta_\varepsilon}$, one has*

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|D_\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} . \tag{4.24}$$

Proof. For any function $w(y) \in H^1(\widetilde{I}_1)^3$ with $w = 0$ in $\partial\widetilde{I}_1$, the Poincaré inequality in $\widetilde{I}_1$ states that

$$\int_{\widetilde{I}_1} |w|^2 \, dz \leq C \int_{\widetilde{I}_1} |\partial_{z_2} w|^2 \, dz, \tag{4.25}$$

where the constant C depends only on $\widetilde{I}_1$.

For every $k' \in \mathbb{Z}^2$, by the change of variable

$$z_1 = x_1, \quad z_2 = \frac{x_2}{\eta_\varepsilon}, \quad z_3 = \frac{x_3}{\varepsilon}, \quad dz = \frac{dx}{\varepsilon\eta_\varepsilon}, \qquad \partial_{z_2} = \eta_\varepsilon \partial_{x_2}, \tag{4.26}$$

we rescale (4.25) from $\widetilde{I}_1$ to I_{η_ε}. This yields that, for any function $w(x) \in H^1(I_{\eta_\varepsilon})^3$ with $w = 0$ in $\partial I_{\eta_\varepsilon}$, one has

$$\int_{I_{\eta_\varepsilon}} |w|^2 \, dx \leq C\eta_\varepsilon^2 \int_{I_{\eta_\varepsilon}} |\partial_{x_2} w|^2 \, dx \leq C\eta_\varepsilon^2 \int_{I_{\eta_\varepsilon}} |D_x w|^2 \, dx, \tag{4.27}$$

with the same constant C as in (4.25). Finally, applying the dilatation (3.5) in (4.27), we obtain

$$\int_{\tilde{I}_{\eta_\varepsilon}} |w|^2 \, dx' dy_3 \leq C\eta_\varepsilon^2 \int_{\tilde{I}_{\eta_\varepsilon}} |D_\varepsilon w|^2 \, dx' dy_3,$$

which gives

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon \|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}. \tag{4.28}$$

Next, if we choose a point $y \in A_{\varepsilon\eta_\varepsilon}$, which is close to the point $x \in I_{\eta_\varepsilon}$, then we have

$$v(x) - v(y) = Dv(\xi)(x - y) \leq (\varepsilon + \eta_\varepsilon)|Dv|.$$

Since $v(y) = 0$ because $y \in A_{\varepsilon\eta_\varepsilon}$, we have

$$\|v(x)\|_{L^2(I_{\eta_\varepsilon})^3} \leq C(\varepsilon + \eta_\varepsilon)\|Dv\|_{L^2(I_{\eta_\varepsilon})^{3\times 3}},$$

and applying the dilatation (3.5) gives

$$\|v\|_{L^2(\tilde{I}_{\eta_\varepsilon})^3} \leq C(\varepsilon + \eta_\varepsilon)\|D_\varepsilon v\|_{L^2(\tilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Finally, multiplying the above inequality with (4.28) we obtain

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}}\,\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}} \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}}\,\|D_\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}\,, \tag{4.29}$$

which is the desired estimate (4.24).

□

Let us give the classical estimate, [11], for a function in L^2 when we deal with a thin fissure.

Lemma 4.3. *Let $v \in L^2(\widetilde{I}_{\eta_\varepsilon})$ be such that $\int_{\widetilde{I}_{\eta_\varepsilon}} v\,dx'dy_3 = 0$. Then*

$$\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})} \leq \frac{C}{\eta_\varepsilon}\|\nabla_\varepsilon v\|_{H^{-1}(\widetilde{I}_{\eta_\varepsilon})^3}.$$

Now, we are in position to obtain some a priori estimates for $\tilde{u}_\varepsilon$.

Lemma 4.4. *There exists a constant C independent of ε, such that the solution $\tilde{u}_\varepsilon \in H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ of the problem (3.6) satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^2((0,T)\times\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3} \leq C(\eta_\varepsilon^{\frac{3}{2}}\varepsilon + \varepsilon^2), \tag{4.30}$$

$$\|\tilde{u}_\varepsilon\|_{L^2((0,T)\times\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\eta_\varepsilon^{\frac{5}{2}} + \varepsilon\eta_\varepsilon + \eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\right), \tag{4.31}$$

$$\|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2((0,T)\times\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq C(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon), \tag{4.32}$$

$$\|\tilde{u}_\varepsilon\|_{L^\infty(0,T;L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon}))^3} \leq C(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon), \tag{4.33}$$

$$\left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} \leq C, \quad \left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\widetilde{I}_{\eta_\varepsilon})^3} \leq C\eta_\varepsilon^{\frac{1}{2}}. \tag{4.34}$$

Proof. Multiplying by $\tilde{u}_\varepsilon$ in the first equation of (3.6), integrating over $\widetilde{D}_{\varepsilon\eta_\varepsilon}$ and using the energy equality, we have

$$\frac{1}{2}\frac{d}{dt}\,\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \mu\|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} = \int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f(t)\cdot\tilde{u}_\varepsilon(t)\,dx'dy_3\,. \tag{4.35}$$

Using Cauchy-Schwarz's inequality, we obtain that

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f(t)\cdot\tilde{u}_\varepsilon(t)\,dx'dy_3 \leq C\eta_\varepsilon^{\frac{1}{2}}\|f(t)\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3}\|\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} + \|f(t)\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}\|\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3},$$

and by inequalities (4.23) and (4.24), we have

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f(t)\cdot\tilde{u}_\varepsilon(t)\,dx'dy_3 \leq C\left(\eta_\varepsilon^{\frac{1}{2}}\eta_\varepsilon^{\frac{1}{2}}(\varepsilon + \eta_\varepsilon)^{\frac{1}{2}}\|f(t)\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3} + \varepsilon\|f(t)\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}\right)\|D_\varepsilon\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}.$$

Using Young's inequality, we obtain that

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f(t)\cdot\tilde{u}_\varepsilon(t)\,dx'dy_3 \le \frac{\mu}{2}\|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} + C\left(\eta_\varepsilon(\varepsilon+\eta_\varepsilon)^{\frac{1}{2}}\|f(t)\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3} + \varepsilon\|f(t)\|_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}\right)^2.$$

Therefore, from (4.35) we get

$$\frac{d}{dt}\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \mu\|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \le C\Big(\eta_\varepsilon^2(\varepsilon+\eta_\varepsilon)\|f(t)\|^2_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3} + \varepsilon^2\|f(t)\|^2_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}\Big), \quad (4.36)$$

and integrating between 0 and T and taking into account the assumption of f (3.2)-(3.3), in particular, we have

$$\|\tilde{u}_\varepsilon(T)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \int_0^T \|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}dt \le C\left(\eta_\varepsilon^2\varepsilon + \eta_\varepsilon^3 + \varepsilon^2\right).$$

Since $\eta_\varepsilon^2\varepsilon < \eta_\varepsilon^3$ if $\varepsilon < \eta_\varepsilon$ and $\eta_\varepsilon^2\varepsilon \le \eta_\varepsilon\varepsilon^2 < \varepsilon^2$ if $\eta_\varepsilon < \varepsilon$, the term $\eta_\varepsilon^2\varepsilon$ can be dropped. This gives (4.32) and (4.33).

On the other hand, applying (4.23) in (4.36), we have

$$\frac{d}{dt}\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + C\varepsilon^{-2}\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3} \le C\left(\eta_\varepsilon^2(\varepsilon+\eta_\varepsilon)\|f(t)\|^2_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3} + \varepsilon^2\|f(t)\|^2_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}\right),$$

and integrating between 0 and T and taking into account the assumption on f (3.2)-(3.3), in particular, we have

$$\int_0^T \|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_{\varepsilon\eta_\varepsilon})^3}dt \le C\varepsilon^2\left(\eta_\varepsilon^2\varepsilon + \eta_\varepsilon^3 + \varepsilon^2\right).$$

Reasoning as before, the term $\eta_\varepsilon^2\varepsilon$ can be dropped. This gives (4.30). Finally, applying (4.24) and (4.32) we get

$$\|\tilde{u}_\varepsilon\|_{L^2((0,T)\times\widetilde{I}_{\eta_\varepsilon})^3} \le C(\eta_\varepsilon + {\eta_\varepsilon}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})({\eta_\varepsilon}^{\frac{3}{2}} + \varepsilon) \le C\left({\eta_\varepsilon}^{\frac{5}{2}} + \varepsilon\eta_\varepsilon + \eta_\varepsilon^2\varepsilon^{\frac{1}{2}} + {\eta_\varepsilon}^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\right).$$

Since ${\eta_\varepsilon}^2\varepsilon^{\frac{1}{2}} < {\eta_\varepsilon}^{\frac{5}{2}}$ if $\eta_\varepsilon > \varepsilon$ and ${\eta_\varepsilon}^2\varepsilon^{\frac{1}{2}} < {\eta_\varepsilon}^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}$ if $\eta_\varepsilon < \varepsilon$, the term ${\eta_\varepsilon}^2\varepsilon^{\frac{1}{2}}$ can be dropped, and (4.31) holds.

Finally, we will prove (4.34). Now, we proceed formally. The rigorous proof schould be made using the Galerkin approximations. First, multiplying by $\frac{\partial\tilde{u}_\varepsilon}{\partial t}$ in the first equation of (3.6), integrating over $\widetilde{D}_{\varepsilon\eta_\varepsilon}$ and using the energy equality, we have

$$\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \mu\frac{1}{2}\frac{d}{dt}\|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} = \int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f\cdot\frac{\partial\tilde{u}_\varepsilon}{\partial t}\,dx'dy_3. \quad (4.37)$$

Using Cauchy-Schwarz's inequality and Young's inequality, we obtain that

$$\int_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} f\cdot\frac{\partial\tilde{u}_\varepsilon}{\partial t}\,dx'dy_3 \quad\le\quad \frac{1}{2}\|f(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^2} + \frac{1}{2}\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}.$$

Then, we deduce

$$\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \mu\frac{d}{dt}\|D_\varepsilon\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \le \|f(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^2},$$

and integrating between 0 and T

$$\int_0^T \left\| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t} \right\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} dt + \mu \left\| D_\varepsilon \tilde{u}_\varepsilon(T) \right\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} \leq \int_0^T \|f(t)\|^2_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^2} dt.$$

Taking into account the assumption of f (3.2)-(3.3), we obtain the first estimate in (4.34). Now, multiplying by $\frac{\partial \tilde{u}_\varepsilon}{\partial t}$ in the first equation of (3.6) and integrating over $\widetilde{I}_{\eta_\varepsilon}$, we have (4.37) in $\widetilde{I}_{\eta_\varepsilon}$. Taking into account that using Cauchy-Schwarz's inequality and Young's inequality, we have

$$\int_{\widetilde{I}_{\eta_\varepsilon}} f \cdot \frac{\partial \tilde{u}_\varepsilon}{\partial t}\, dx' dy_3 \quad \leq \quad \frac{1}{2}\eta_\varepsilon \|f(t)\|^2_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^2} + \frac{1}{2} \left\| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t} \right\|^2_{L^2(\widetilde{I}_{\eta_\varepsilon})^3},$$

we deduce, in particular, that

$$\int_0^T \left\| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t} \right\|^2_{L^2(\widetilde{I}_{\eta_\varepsilon})^3} dt \leq C\eta_\varepsilon,$$

and we have proved the second estimate in (4.34). □

In the next step we will estimate the pressure to the whole domain D. We give some properties of the restricted operator, R^ε, from $H_0^1(D)^3$ into $H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ preserving divergence-free vectors, which was introduced by Tartar [12]. Since the construction of the operator is local, having no obstacles in $\widetilde{I}_{\eta_\varepsilon}$ means that we do not have to use the extension in that part. Next, we give the properties of the operator R^ε.

Lemma 4.5. *There exists a linear continuous operator R^ε acting from $H_0^1(D)^3$ into $H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ such that*

1. $R^\varepsilon v = v$, *if* $v \in H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$
2. $\mathrm{div}_\varepsilon(R^\varepsilon v) = 0$, *if* $\mathrm{div}\, v = 0$
3. *For any $v \in H_0^1(D)^3$ (the constant $\widetilde{C}$ is independent of v and ε),*

$$\begin{aligned} \|R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} &\leq \widetilde{C}\, \|v\|_{L^2(D)^3} + \widetilde{C}\varepsilon \|D_\varepsilon v\|_{L^2(D)^{3\times 3}}, \\ \|D_\varepsilon R^\varepsilon v\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} &\leq \frac{\widetilde{C}}{\varepsilon} \|v\|_{L^2(D)^3} + \widetilde{C}\, \|D_\varepsilon v\|_{L^2(D)^{3\times 3}}. \end{aligned}$$

In order to extend the pressure to the whole domain D, we define, for all $T > 0$, a function $F_\varepsilon \in L^2(0,T; H^{-1}(D))^3$ by the following formula (brackets are for the duality products between H^{-1} and H_0^1):

$$\langle F_\varepsilon(t), v \rangle_D = \langle \nabla_\varepsilon \tilde{p}_\varepsilon(t), R^\varepsilon v \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}, \text{ for any } \; v \in H_0^1(D)^3, \quad \forall t \in (0,T), \tag{4.38}$$

where R^ε is defined in Lemma 4.5. We calcule the right hand side of (4.38) by using (3.6) and we have

$$\langle F_\varepsilon(t), v \rangle_D = \langle \mu \Delta_\varepsilon \tilde{u}_\varepsilon(t), R^\varepsilon v \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} + \langle f(t), R^\varepsilon v \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} - \left\langle \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}, R^\varepsilon v \right\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}, \tag{4.39}$$

and by using the third point in Lemma 4.5, for fixed ε, we deduce that $F_\varepsilon \in L^2(0,T; H^{-1}(D))^3$.

Moreover, if $v \in H_0^1(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3$ and we continue it by zero out of $\widetilde{D}_{\varepsilon\eta_\varepsilon}$, we see from (4.38) and the first point in Lemma 4.5 that $F_\varepsilon|_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}(t) = \nabla_\varepsilon \tilde{p}_\varepsilon(t)$, for all $t \in (0,T)$.

Moreover, if $\operatorname{div} v = 0$ by the second point in Lemma 4.5 and (4.38), $\langle F_\varepsilon(t), v\rangle_D = 0$, for all $t \in (0,T)$, and this implies (by the orthogonality property) that $F_\varepsilon(t)$ is the gradient of some function in $L^2(D)$, for all $t \in (0,T)$. This means that F_ε is a continuation of $\nabla_\varepsilon \tilde{p}_\varepsilon$ to $(0,T)\times D$, and that this continuation is a gradient. We also may say that $\tilde{p}_\varepsilon$ has been continuated to $(0,T)\times D$ and we denote the extended pressure again by $\tilde{p}_\varepsilon$ and

$$F_\varepsilon \equiv \nabla_\varepsilon \tilde{p}_\varepsilon, \qquad \tilde{p}_\varepsilon \in L^2(0,T;L^2(D)/\mathbb{R}).$$

Lemma 4.6. *Let $\tilde{p}_\varepsilon$ be the extension of the pressure defined as above. Then, there exists a constant C independent of ε such that*

$$\|\tilde{p}_\varepsilon\|_{L^2(0,T;L^2(D)/\mathbb{R})} \leq C\left(\frac{\eta_\varepsilon^{\frac{3}{2}}}{\varepsilon}+1\right), \tag{4.40}$$

$$\|\tilde{p}_\varepsilon - c_{\varepsilon\eta_\varepsilon}\|_{L^2((0,T)\times\widetilde{I}_{\eta_\varepsilon})} \leq C\left(\eta_\varepsilon^{\frac{1}{2}}+\frac{\varepsilon}{\eta_\varepsilon}\right), \tag{4.41}$$

where $c_{\varepsilon\eta_\varepsilon}$ is given by (3.10).

Proof. Let us first estimate $\nabla_\varepsilon \tilde{p}_\varepsilon$. To do this we estimate the right side of (4.39). Using Cauchy-Schwarz's inequality and the third point in Lemma 4.5, we have

$$\begin{aligned}\left|\langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon(t), R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| &\leq \mu\left\|D_\varepsilon \tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times3}}\left\|D_\varepsilon R^\varepsilon v\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times3}}\\ &\leq C\left\|D_\varepsilon \tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times3}}\left(\frac{1}{\varepsilon}\left\|v\right\|_{L^2(D)^3}+\left\|D_\varepsilon v\right\|_{L^2(D)^{3\times3}}\right),\end{aligned}$$

using the assumption of f, we obtain

$$\left|\langle f(t), R^\varepsilon v\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| \leq C\left\|f(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}\left(\left\|v\right\|_{L^2(D)^3}+\varepsilon\left\|D_\varepsilon v\right\|_{L^2(D)^{3\times3}}\right),$$

and

$$\left|\left\langle \frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}, R^\varepsilon v\right\rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}}\right| \leq C\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}\left(\left\|v\right\|_{L^2(D)^3}+\varepsilon\left\|D_\varepsilon v\right\|_{L^2(D)^{3\times3}}\right).$$

Then, from (4.39), we deduce

$$\begin{aligned}\left|\langle\nabla_\varepsilon \tilde{p}_\varepsilon(t), v\rangle_D\right| &\leq C\left\|D_\varepsilon \tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times3}}\left(\frac{1}{\varepsilon}\left\|v\right\|_{L^2(D)^3}+\left\|D_\varepsilon v\right\|_{L^2(D)^{3\times3}}\right)\\ &+ C\left(\left\|f(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}+\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}\right)\left(\left\|v\right\|_{L^2(D)^3}+\varepsilon\left\|D_\varepsilon v\right\|_{L^2(D)^{3\times3}}\right).\end{aligned}$$

Then, as $\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$\left|\langle\nabla_\varepsilon \tilde{p}_\varepsilon(t), v\rangle_D\right| \leq C\left(\frac{1}{\varepsilon}\left\|D_\varepsilon \tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times3}}+\left\|f(t)\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}+\left\|\frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}\right)\left\|v\right\|_{H_0^1(D)^3},$$

for any $v \in H_0^1(D)^3$. Consequently, we obtain

$$\|\nabla_\varepsilon \tilde{p}_\varepsilon(t)\|_{H^{-1}(D)^3} \leq C\left(\frac{1}{\varepsilon}\|D_\varepsilon \tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}} + \|f(t)\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3} + \left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{D}_{\varepsilon\eta_\varepsilon})^3}\right),$$

and from the Nečas inequality in D , integrating between 0 and T, and from (4.32), the first estimate in (4.34) and the assumption of f, we have the estimate (4.40).

Now, we prove the estimate (4.41). Let $v \in H_0^1(\widetilde{I}_{\eta_\varepsilon})^3$, then

$$\langle \nabla_\varepsilon \tilde{p}_\varepsilon(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}} = \langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}} + \langle f(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}} - \left\langle \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}, v\right\rangle_{\widetilde{I}_{\eta_\varepsilon}}.$$

We estimate the right hand side. Using Cauchy-Schwarz's inequality, we have

$$\left|\langle \mu\Delta_\varepsilon \tilde{u}_\varepsilon(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq \mu\|D_\varepsilon \tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}},$$

and

$$\left|\langle f(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C\eta_\varepsilon^{\frac{1}{2}}\|f(t)\|_{L^\infty(\widetilde{I}_{\eta_\varepsilon})^3}\|v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3},$$

and by estimate (4.29), we have

$$\left|\langle f(t), v\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C(\eta_\varepsilon^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}})\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Using again Cauchy-Schwarz's inequality and estimate (4.29), we obtain

$$\left|\left\langle \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}, v\right\rangle_{\widetilde{I}_{\eta_\varepsilon}}\right| \leq C\eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}}\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3}\|D_\varepsilon v\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}}.$$

Then, we have

$$\|\nabla_\varepsilon \tilde{p}_\varepsilon(t)\|_{H^{-1}(\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\|D_\varepsilon \tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}} + \eta_\varepsilon^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}} + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}}\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3}\right),$$

and taking into account that $\int_{\widetilde{I}_{\eta_\varepsilon}}(\tilde{p}_\varepsilon - c_{\varepsilon\eta_\varepsilon})dx'dy_3 = 0$, we use Lemma 4.3 and we can deduce

$$\|\tilde{p}_\varepsilon(t) - c_{\varepsilon\eta_\varepsilon}(t)\|_{L^2(\widetilde{I}_{\eta_\varepsilon})} \leq \frac{C}{\eta_\varepsilon}\left(\|D_\varepsilon \tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^{3\times 3}} + \eta_\varepsilon^{\frac{3}{2}} + \eta_\varepsilon\varepsilon^{\frac{1}{2}} + \eta_\varepsilon^{\frac{1}{2}}(\eta_\varepsilon + \varepsilon)^{\frac{1}{2}}\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{I}_{\eta_\varepsilon})^3}\right).$$

Integrating between 0 and T, and from the estimate (4.32), and the second estimate in (4.34), we have

$$\|\tilde{p}_\varepsilon - c_{\varepsilon\eta_\varepsilon}\|_{L^2((0,T)\times\widetilde{I}_{\eta_\varepsilon})} \leq \frac{C}{\eta_\varepsilon}\left(\eta_\varepsilon^{\frac{3}{2}} + \varepsilon + \eta_\varepsilon\varepsilon^{\frac{1}{2}}\right).$$

Reasoning as in the proof of Lemma 4.4, we observe that $\eta_\varepsilon\varepsilon^{\frac{1}{2}}$ can be dropped and so we obtain (4.41). □

5 Proof of the main result

In view of estimates (4.30), (4.32) of the velocity and (4.40) of the pressure, the proof of Theorem 3.1 will be divided in three characteristic cases: $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$.

5.1 Problem in the porous part $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$

The proof of Theorem 3.1-i) will be developed in different lemmas.

In this subsection, we need to extend the velocity $\tilde{u}_\varepsilon$ by zero in the fissure $\widetilde{I}_{\eta_\varepsilon}$, and we will denote the extended velocity by $\tilde{v}_\varepsilon$, i.e.

$$\tilde{v}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{in } \widetilde{\Omega}_{\varepsilon\eta_\varepsilon}, \\ 0 & \text{in } \widetilde{I}_{\eta_\varepsilon}. \end{cases} \tag{5.42}$$

Lemma 5.1. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and let $(\tilde{v}_\varepsilon, \tilde{p}_\varepsilon)$ be the extended solution of (3.6)-(3.7). Then there exist subsequences of $\tilde{v}_\varepsilon$ and $\tilde{p}_\varepsilon$ still denoted by the same, and functions $\tilde{v} \in L^2(0,T;H^1(0,1;L^2(\omega)^3))$ with $\tilde{v}_3 = 0$, $\tilde{p} \in L^2(0,T;L^2(D)/\mathbb{R})$ independent of y_3, such that*

$$\varepsilon^{-2}\tilde{v}_\varepsilon \rightharpoonup (\tilde{v}', 0) \quad \text{in } L^2(0,T;H^1(0,1;L^2(\omega)^3)), \quad \tilde{p}_\varepsilon \to \tilde{p} \quad \text{in } L^2(0,T;L^2(D)/\mathbb{R}). \tag{5.43}$$

Moreover, $\tilde{v}$ satisfies

$$\operatorname{div}_{x'}\left(\int_0^1 \tilde{v}'(t,x',y_3)dy_3\right) = 0 \quad \text{in } (0,T)\times\omega, \quad \left(\int_0^1 \tilde{v}'(t,x',y_3)dy_3\right)\cdot n = 0 \quad \text{on } (0,T)\times\partial w. \tag{5.44}$$

Proof. From estimates (4.30), (4.32) and (4.40), taking into account the extension of the velocity by zero to D and $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$, we have the following estimates

$$\|\tilde{v}_\varepsilon\|_{L^2((0,T)\times D)^3} \le C\varepsilon^2, \quad \|\tilde{p}_\varepsilon\|_{L^2(0,T;L^2(D)/\mathbb{R})} \le C,$$

$$\|D_{x'}\tilde{v}_\varepsilon\|_{L^2((0,T)\times D)^{3\times 2}} \le C\varepsilon, \quad \|\partial_{y_3}\tilde{v}_\varepsilon\|_{L^2((0,T)\times D)^3} \le C\varepsilon^2.$$

Now, we can use Lemma 5.1-(i) and Lemma 5.3-(i) in [8], because in the present paper $a_\varepsilon \approx \varepsilon$ in the porous part, in order to obtain (5.43), with the weak convergence of the pressure, and (5.44).

Finally, we prove that the convergence of the pressure is in fact strong. As $\tilde{v}_3 = 0$, let $w_\varepsilon = (w'_\varepsilon, 0) \in H_0^1(D)^3$ be such that

$$w_\varepsilon \rightharpoonup w \quad \text{in } H_0^1(D)^3. \tag{5.45}$$

We consider $\varphi \in C_c^1(0,T)$. Then (brackets are for the duality products between H^{-1} and H_0^1):

$$\begin{aligned} &\int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t), \varphi(t) w_\varepsilon >_D - < \nabla_{x'}\tilde{p}, \varphi(t) w >_D| \, dt \\ &\le \int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t), \varphi(t)(w_\varepsilon - w) >_D| \, dt + \int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t) - \nabla_{x'}\tilde{p}(t), \varphi(t) w >_D| \, dt. \end{aligned}$$

On the one hand, using the weak convergence of the pressure, we have

$$\int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t) - \nabla_{x'}\tilde{p}(t), \varphi(t) w >_D| \, dt = \int_0^T \int_D (\tilde{p}_\varepsilon(t) - \tilde{p}(t)) \operatorname{div}_{x'} \varphi(t) w' \, dx' dy_3 dt \to 0, \quad \text{as } \varepsilon \to 0.$$

On the other hand, we have

$$\int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t), \varphi(t)(w_\varepsilon - w) >_D| \, dt = \int_0^T \left| < \nabla_{x'} \tilde{p}_\varepsilon(t), \varphi(t) R^\varepsilon(w'_\varepsilon - w') >_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} \right| dt$$

$$= \int_0^T \left| \langle \mu \Delta_{x'} \tilde{v}'_\varepsilon(t), \varphi(t) R^\varepsilon(w'_\varepsilon - w') \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} + \langle f'(t), \varphi(t) R^\varepsilon(w'_\varepsilon - w') \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} - \langle \frac{\partial \tilde{v}'_\varepsilon(t)}{\partial t}, \varphi(t) R^\varepsilon(w'_\varepsilon - w') \rangle_{\widetilde{D}_{\varepsilon\eta_\varepsilon}} \right| dt,$$

and using Cauchy-Schwarz's inequality, estimate (4.32), the first estimate in (4.34), the estimates of the restricted operator R^ε applied to $D_{x'}$ instead of D_ε, and taking into account that $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and $\varepsilon \ll 1$, we get

$$\int_0^T |< \nabla_\varepsilon \tilde{p}_\varepsilon(t), \varphi(t)(w_\varepsilon - w) >_D| \, dt$$

$$\leq C \left(\left(\int_0^T \varphi(t)^2 \|w'_\varepsilon - w'\|^2_{L^2(D)^2} dt \right)^{1/2} + \varepsilon \left(\int_0^T \varphi(t)^2 \|D_{x'} w'_\varepsilon - D_{x'} w'\|^2_{L^2(D)^{2\times 2}} dt \right)^{1/2} \right) \to 0 \quad \text{as } \varepsilon \to 0,$$

by virtue of (5.45) and the Rellich Theorem. This implies that $\nabla_\varepsilon \tilde{p}_\varepsilon \to \nabla_{x'} \tilde{p}$ strongly in $L^2(0,T; H^{-1}(D))^3$, which implies the strong convergence of the pressure given in (5.43). □

Lemma 5.2. *Let $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and let $(\tilde{v}_\varepsilon, \tilde{p}_\varepsilon)$ be the extended solution of (3.6)-(3.7). Let $(\tilde{v}, \tilde{p}) \in L^2((0,T) \times D)^3 \times L^2(0,T; L^2(D)/\mathbb{R})$ be given by Lemma 5.1. Then, $\tilde{p} \in L^2(0,T; H^1(D)/\mathbb{R})$ and $(\tilde{v}, \tilde{p})$ is the unique solution of Darcy's law (3.14).*

Proof. We apply Theorem 3.1-(i) in [8], because in the present paper $a_\varepsilon \approx \varepsilon$ in the porous part, in order to obtain that $(\tilde{v}, \tilde{p})$ is the unique solution of Darcy's law (3.14).

Finally, the classical theory of the elliptic equation implies existence of the unique solution $\tilde{p}$ belongs to $L^2(0,T; H^1(D)/\mathbb{R})$.

□

Proof of Theorem 3.1-i). It remains to prove convergence (3.13) of the whole velocity $\tilde{u}_\varepsilon$, i.e. to prove

$$\varepsilon^{-2} \|\tilde{u}_\varepsilon\|_{L^2((0,T) \times \widetilde{I}_{\eta_\varepsilon})^3} \to 0. \tag{5.46}$$

For this, it is sufficient to prove that

$$\varepsilon^{-2} \|\tilde{u}_\varepsilon\|_{L^2((0,T) \times \widetilde{I}_{\eta_\varepsilon})^3} \to 0 \quad \text{for } \eta_\varepsilon \ll \varepsilon, \tag{5.47}$$

and

$$\varepsilon^{-2} \|\tilde{u}_\varepsilon\|_{L^q((0,T) \times \widetilde{I}_{\eta_\varepsilon})^3} \to 0 \quad \text{for } \varepsilon \ll \eta_\varepsilon \ll \varepsilon^{\frac{1}{\alpha}} \quad, 1 < \alpha < \frac{3}{2}, \tag{5.48}$$

for a q which will be defined below.

Using (4.31) and using $\eta_\varepsilon \ll \varepsilon$, we have

$$\varepsilon^{-2} \|\tilde{u}_\varepsilon\|_{L^2((0,T) \times \widetilde{I}_{\eta_\varepsilon})^3} \leq C \left(\frac{\eta_\varepsilon^{\frac{5}{2}}}{\varepsilon^2} + \frac{\eta_\varepsilon}{\varepsilon} + \left(\frac{\eta_\varepsilon}{\varepsilon} \right)^{\frac{1}{2}} \right),$$

so that (5.47) easily holds. Using Hölder's inequality with the conjugate exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ we obtain

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^q((0,T)\times\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\frac{\eta_\varepsilon^{\frac{1}{q}+2}}{\varepsilon^2} + \frac{\eta_\varepsilon^{\frac{1}{q}+\frac{1}{2}}}{\varepsilon} + \frac{\eta_\varepsilon^{\frac{1}{q}}}{\varepsilon^{\frac{1}{2}}}\right).$$

Now we take $\eta_\varepsilon = \varepsilon^{\frac{1}{\alpha}}$. Then we find that

$$\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^q((0,T)\times\widetilde{I}_{\eta_\varepsilon})^3} \leq C\left(\varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}+2\right)-2} + \varepsilon^{\frac{1}{\alpha}\left(\frac{1}{q}+\frac{1}{2}\right)-1} + \varepsilon^{\frac{1}{q\alpha}-\frac{1}{2}}\right). \tag{5.49}$$

We seek an optimal q such that the right hand side in (5.49) tends to zero. It is easy to prove that we have a convergence to zero for any $q \in \left(1, \frac{2}{2(\alpha-1)+1}\right)$. Therefore, (5.48) holds and so we have (5.46).

□

5.2 Problem in the fissure part $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$

The proof of Theorem 3.1-*ii*) will be developed in different lemmas.

Lemma 5.3. *Let $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ and let $(\tilde{\mathcal{U}}^\varepsilon, \tilde{P}^\varepsilon)$ be the solution of (3.11)-(3.12). Then there exist subsequences of $\tilde{\mathcal{U}}^\varepsilon$ and $\tilde{P}^\varepsilon$ still denoted by the same, and functions $\tilde{\mathcal{U}} \in L^2((0,T)\times\widetilde{I}_1)^3$, independent of y_3, with $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, $\tilde{P} \in L^2(0,T;L^2(\widetilde{I}_1)/\mathbb{R})$ such that*

$$\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \tilde{\mathcal{U}} \quad \text{in } L^2((0,T)\times\widetilde{I}_1)^3, \quad \tilde{P}^\varepsilon \rightharpoonup \tilde{P} \quad \text{in } L^2(0,T;L^2(\widetilde{I}_1)/\mathbb{R}). \tag{5.50}$$

Moreover, $\tilde{P} = \tilde{P}(x_1)$ and $\tilde{\mathcal{U}}_1$ is given by expression (3.17).

Proof. Taking into account $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ and estimates (4.31), (4.32), (4.33), (4.41) with the change of variable (3.8), we have

$$\|\tilde{\mathcal{U}}^\varepsilon\|_{L^2((0,T)\times\widetilde{I}_1)^3} \leq C\eta_\varepsilon{}^2, \tag{5.51}$$

$$\|\partial_{x_1}\tilde{\mathcal{U}}^\varepsilon\|_{L^2((0,T)\times\widetilde{I}_1)^3} \leq C\eta_\varepsilon, \quad \|\partial_{y_2}\tilde{\mathcal{U}}^\varepsilon\|_{L^2((0,T)\times\widetilde{I}_1)^3} \leq C\eta_\varepsilon{}^2, \tag{5.52}$$

$$\|\partial_{y_3}\tilde{\mathcal{U}}^\varepsilon\|_{L^2((0,T)\times\widetilde{I}_1)^3} \leq C\varepsilon\,\eta_\varepsilon, \tag{5.53}$$

$$\|\tilde{\mathcal{U}}^\varepsilon\|_{L^\infty(0,T;L^2(\widetilde{I}_1))^3} \leq C\eta_\varepsilon, \tag{5.54}$$

$$\|\tilde{P}^\varepsilon\|_{L^2(0,T;L^2(\widetilde{I}_1)/\mathbb{R})} \leq C. \tag{5.55}$$

From the estimates (5.51) and (5.55), there exist $\tilde{\mathcal{U}} \in L^2((0,T)\times\widetilde{I}_1)^3$, $\tilde{P} \in L^2(0,T;L^2(\widetilde{I}_1)/\mathbb{R})$ such that convergence (5.50) holds. Moreover

$$\eta_\varepsilon{}^{-2}\partial_{y_2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup \partial_{y_2}\tilde{\mathcal{U}} \quad \text{in } L^2((0,T)\times\widetilde{I}_1)^3, \tag{5.56}$$

and from (5.54), there exists $\tilde{\mathcal{W}} \in L^\infty(0,T;L^2(\widetilde{I}_1))^3$ such that

$$\eta_\varepsilon{}^{-1}\tilde{\mathcal{U}}^\varepsilon \overset{*}{\rightharpoonup} \tilde{\mathcal{W}} \quad \text{in } L^\infty(0,T;L^2(\widetilde{I}_1))^3. \tag{5.57}$$

The estimate (5.53) implies that $\varepsilon^{-1}\eta_\varepsilon^{-1}\partial_{y_3}\tilde{\mathcal{U}}^\varepsilon$ is bounded in $L^2((0,T)\times\widetilde{I}_1)^3$. This together with $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$ implies that $\eta_\varepsilon^{-2}\partial_{y_3}\tilde{\mathcal{U}}^\varepsilon$ tends to $\partial_{y_3}\tilde{\mathcal{U}} = 0$. This implies that $\tilde{\mathcal{U}}$ does not depend on y_3.

As $\tilde{\mathcal{U}}$ does not depend on y_3, let $\varphi \in C_0^\infty((0,T)\times \widetilde{I}_1)^3$ independent of y_3. Taking into account that $\operatorname{div}_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon = 0$ in $(0,T)\times \widetilde{I}_1$, we have

$$\begin{aligned}&{\eta_\varepsilon}^{-1}\int_0^T\int_{\widetilde{I}_1}\left(\partial_{x_1}\tilde{\mathcal{U}}_1^\varepsilon + {\eta_\varepsilon}^{-1}\partial_{y_2}\tilde{\mathcal{U}}_2^\varepsilon + \varepsilon^{-1}\partial_{y_3}\tilde{\mathcal{U}}_3^\varepsilon\right)\varphi\, dx_1dy_2dy_3dt\\ =\;& -{\eta_\varepsilon}^{-1}\int_0^T\int_{\widetilde{I}_1}\tilde{\mathcal{U}}_1^\varepsilon\partial_{x_1}\varphi\, dx_1dy_2dy_3dt - {\eta_\varepsilon}^{-2}\int_0^T\int_{\widetilde{I}_1}\tilde{\mathcal{U}}_2^\varepsilon\cdot\partial_{y_2}\varphi\, dx_1dy_2dy_3dt = 0.\end{aligned}$$

Taking the limit $\varepsilon\to 0$ we obtain

$$\int_0^T\int_{\widetilde{I}_1}\tilde{\mathcal{U}}_2\partial_{y_2}\varphi\, dx_1dy_2dy_3dt = 0,$$

so that $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_2(t,x_1)$.

Since $\tilde{\mathcal{U}}$, $\partial_{y_2}\tilde{\mathcal{U}} \in L^2((0,T)\times\widetilde{I}_1)^3$ the traces $\tilde{\mathcal{U}}(t,x_1,0), \tilde{\mathcal{U}}(t,x_1,1)$ are well defined in $L^2((0,T)\times\Sigma)^3$. Analogously to the proof of Lemma 4.2 we choose a point $\beta_{(x_1,y_3)}\in \widetilde{A}_{\varepsilon\eta_\varepsilon}$, which is close to the point $\alpha_{(x_1,y_3)}\in\Sigma$, then we have

$$\begin{aligned}\int_0^T\int_\Sigma |\tilde{\mathcal{U}}^\varepsilon(t,x',0,y_3)|^2dx_1dy_3dt &= \int_0^T\int_\Sigma |\tilde{u}_\varepsilon(t,x_1,0,y_3)|^2dx_1dy_3dt\\ &\le C\int_0^T\int_\Sigma\left(\int_{(\beta_{(x_1,y_3)},\alpha_{(x_1,y_3)})} D_\varepsilon\tilde{u}_\varepsilon\cdot(\alpha_{(x_1,y_3)}-\beta_{(x_1,y_3)})d\ell\right)^2 dx_1dy_3dt,\end{aligned}$$

so that, by Cauchy-Schwarz's inequality,

$$\|\tilde{\mathcal{U}}^\varepsilon(t,x_1,0,y_3)\|^2_{L^2((0,T)\times\Sigma)^3} \le C\varepsilon\|D_\varepsilon\tilde{u}_\varepsilon\|^2_{L^2((0,T)\times\widetilde{D}_{\varepsilon\eta_\varepsilon})^{3\times 3}}.$$

Taking into account estimate (4.32) and $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, we have

$${\eta_\varepsilon}^{-2}\|\tilde{\mathcal{U}}^\varepsilon(t,x_1,0,y_3)\|^2_{L^2((0,T)\times\Sigma)^3} \le C\varepsilon\eta_\varepsilon \to 0 \quad \text{as } \varepsilon\to 0,$$

which implies that

$$\tilde{\mathcal{U}}(t,x_1,0) = 0\,,$$

and analogously

$$\tilde{\mathcal{U}}(t,x_1,1) = 0\,.$$

Consequently

$$\tilde{\mathcal{U}}_2 = 0\,.$$

It remains to prove that $\tilde{\mathcal{U}}_3 = 0$. In order to do that, as $\tilde{\mathcal{U}}$ does not depend on y_3, we take a test function $v = (0,0,v_3(x_1,y_2))$ in (3.11),

$$\frac{d}{dt}\left(\int_{\widetilde{I}_1}\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3\right) + \int_{\widetilde{I}_1}\partial_{x_1}^2\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3 + \frac{1}{\eta_\varepsilon^2}\int_{\widetilde{I}_1}\partial_{y_2}^2\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3 = 0,$$

in $\mathcal{D}'(0,T)$. We consider $\varphi\in C_c^1([0,T])$ such that $\varphi(T)=0$ and $\varphi(0)\neq 0$. Multiplying by φ and integrating between 0 and T, we have

$$\begin{aligned}&-\int_0^T\frac{d}{dt}\varphi(t)\int_{\widetilde{I}_1}\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3dt + \int_0^T\varphi(t)\int_{\widetilde{I}_1}\partial_{x_1}^2\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3dt\\ &+\frac{1}{\eta_\varepsilon^2}\int_0^T\varphi(t)\int_{\widetilde{I}_1}\partial_{y_2}^2\tilde{\mathcal{U}}_3^\varepsilon(t)v_3\, dx_1dy_2dy_3dt = 0.\end{aligned}$$

We pass to the limit when ε tends to zero, and using the convergences (5.56) and (5.57) with

$$v_3\varphi(t) \in L^2((0,T)\times \widetilde{I}_1), \quad v_3\frac{d}{dt}\varphi(t) \in L^1(0,T;L^2(\widetilde{I}_1)),$$

we can deduce that $\tilde{\mathcal{U}}_3 = 0$.

Finally, we compute the expression of $\tilde{\mathcal{U}}$ given in (3.17). First, we take a test function $v = (0,0,\varepsilon v_3)$ in (3.11), and we obtain

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\left(\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3\right) + \varepsilon\int_{\widetilde{I}_1} \partial^2_{x_1}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3 + \frac{\varepsilon}{\eta_\varepsilon^2}\int_{\widetilde{I}_1} \partial^2_{y_2}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3\\
&+\frac{1}{\varepsilon}\int_{\widetilde{I}_1} \partial^2_{y_3}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3 - \int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{y_3} v_3\, dx_1dy_2dy_3 = 0,
\end{aligned}$$

in $\mathcal{D}'(0,T)$. Multiplying by φ and integrating between 0 and T, we have

$$\begin{aligned}
&-\varepsilon\int_0^T \frac{d}{dt}\varphi(t)\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3dt + \varepsilon\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial^2_{x_1}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3dt\\
&+\frac{\varepsilon}{\eta_\varepsilon^2}\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial^2_{y_2}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3dt + \frac{1}{\varepsilon}\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial^2_{y_3}\tilde{\mathcal{U}}_3^\varepsilon(t) v_3\, dx_1dy_2dy_3dt\\
&-\int_0^T \varphi(t)\int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{y_3} v_3\, dx_1dy_2dy_3dt = 0.
\end{aligned}$$

We pass to the limit when ε tends to zero, and using the estimate (5.53), the convergences (5.50) and (5.57) with

$$v_3\varphi(t) \in L^2((0,T)\times \widetilde{I}_1), \quad v_3\frac{d}{dt}\varphi(t) \in L^1(0,T;L^2(\widetilde{I}_1)),$$

we can deduce that $\tilde{P}$ does not depend on y_3.

We take a test function $v = (0,\eta_\varepsilon v_2, 0)$, independent of y_3, in (3.11), and we obtain

$$\begin{aligned}
&\eta_\varepsilon\frac{d}{dt}\left(\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3\right) + \eta_\varepsilon\int_{\widetilde{I}_1} \partial^2_{x_1}\tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3 + \frac{1}{\eta_\varepsilon}\int_{\widetilde{I}_1} \partial^2_{y_2}\tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3\\
&-\int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{y_2} v_2\, dx_1dy_2dy_3 = \eta_\varepsilon\int_{\widetilde{I}_1} f_2 v_2\, dx_1dy_2dy_3,
\end{aligned}$$

in $\mathcal{D}'(0,T)$. Multiplying by φ and integrating between 0 and T, we have

$$\begin{aligned}
&-\eta_\varepsilon\int_0^T \frac{d}{dt}\varphi(t)\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3dt + \eta_\varepsilon\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial^2_{x_1}\tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3dt\\
&+\frac{1}{\eta_\varepsilon}\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial^2_{y_2}\tilde{\mathcal{U}}_2^\varepsilon(t) v_2\, dx_1dy_2dy_3dt - \int_0^T \varphi(t)\int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{y_2} v_2\, dx_1dy_2dy_3dt\\
&= \eta_\varepsilon\int_0^T \varphi(t)\int_{\widetilde{I}_1} f_2 v_2\, dx_1dy_2dy_3dt.
\end{aligned}$$

We pass to the limit when ε tends to zero, and using the convergences (5.50) and (5.57) with

$$v_2\varphi(t) \in L^2((0,T)\times \widetilde{I}_1), \quad v_2\frac{d}{dt}\varphi(t) \in L^1(0,T;L^2(\widetilde{I}_1)),$$

we can deduce that $\tilde{P} = \tilde{P}(t, x_1)$. Now, taking into account that $\tilde{\mathcal{U}}$ does not depend on y_3 and $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, we take a test function $v = (v_1(x_1, y_2), 0, 0)$ in (3.11),

$$\frac{d}{dt}\left(\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3\right) + \int_{\widetilde{I}_1} \partial_{x_1}^2 \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3 + \frac{1}{\eta_\varepsilon^2}\int_{\widetilde{I}_1} \partial_{y_2}^2 \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3$$
$$-\int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{x_1} v_1\, dx_1 dy_2 dy_3 = \int_{\widetilde{I}_1} f_1(t, x_1, \eta_\varepsilon y_2) v_1\, dx_1 dy_2 dy_3,$$

in $\mathcal{D}'(0,T)$. Multiplying by φ and integrating between 0 and T, we have

$$-\int_0^T \frac{d}{dt}\varphi(t) \int_{\widetilde{I}_1} \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3 dt + \int_0^T \varphi(t) \int_{\widetilde{I}_1} \partial_{x_1}^2 \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3 dt$$
$$+\frac{1}{\eta_\varepsilon^2}\int_0^T \varphi(t)\int_{\widetilde{I}_1} \partial_{y_2}^2 \tilde{\mathcal{U}}_1^\varepsilon(t) v_1\, dx_1 dy_2 dy_3 dt - \int_0^T \varphi(t) \int_{\widetilde{I}_1} \tilde{P}^\varepsilon \partial_{x_1} v_1\, dx_1 dy_2 dy_3 dt$$
$$= \int_0^T \varphi(t) \int_{\widetilde{I}_1} f_1(t, x_1, \eta_\varepsilon y_2) v_1\, dx_1 dy_2 dy_3 dt.$$

We pass to the limit when ε tends to zero, and using the convergences (5.50) and (5.57) with

$$v_1\varphi(t) \in L^2((0,T)\times \widetilde{I}_1), \quad v_1 \frac{d}{dt}\varphi(t) \in L^1(0,T; L^2(\widetilde{I}_1)),$$

we obtain the ODE

$$\begin{cases} -\partial_{y_2}^2 \tilde{\mathcal{U}}_1(t, x_1, y_2) = f_1(t, x_1, 0) - \partial_{x_1}\tilde{P}(t, x_1), \\ \tilde{\mathcal{U}}_1(t, x_1, 0) = \tilde{\mathcal{U}}_1(t, x_1, 1) = 0, \end{cases}$$

□

which gives the expression (3.17) for $\tilde{\mathcal{U}}_1$.

Proof of Theorem 3.1-ii). It remains to prove the convergence (3.18) of the whole velocity to the function $\mathcal{V}$ given by (3.19), and also prove that $\tilde{P} \in L^2(0,T; H^1(\Sigma)/\mathbb{R})$ is the unique solution of the Reynolds problem (3.20).

Taking as test function $\varphi \in C^\infty((0,T)\times D)$, independent of y_3, in the equation $\mathrm{div}_\varepsilon\, \tilde{u}_\varepsilon = 0$ in $(0,T)\times D$, we obtain

$$\int_0^T\int_D \mathrm{div}_\varepsilon\, \tilde{u}_\varepsilon \varphi\, dx' dy_3 dt = -\int_0^T \int_D \tilde{v}'_\varepsilon \cdot \nabla_{x'}\varphi\, dx' dy_3 dt - \eta_\varepsilon \int_0^T \int_{\widetilde{I}_1} (\tilde{\mathcal{U}}^\varepsilon)' \cdot \nabla_{x'}\varphi(t, x_1, \eta_\varepsilon y_2)\, dx_1 dy_2 dy_3 dt = 0,$$

so that multiplying by ${\eta_\varepsilon}^{-3}$,

$$\int_0^T \int_{\widetilde{I}_1} {\eta_\varepsilon}^{-2} \tilde{\mathcal{U}}_1^\varepsilon \partial_{x_1}\varphi(t, x_1, \eta_\varepsilon y_2)\, dx_1 dy_2 dy_3 dt \tag{5.58}$$
$$= -\int_0^T \int_D {\eta_\varepsilon}^{-3} \tilde{v}_\varepsilon \cdot \nabla_{x'}\varphi\, dx' dy_3 dt - \int_0^T \int_{\widetilde{I}_1} {\eta_\varepsilon}^{-2} \tilde{\mathcal{U}}_2^\varepsilon \partial_{x_2}\varphi(t, x_1, \eta_\varepsilon y_2)\, dx_1 dy_2 dy_3 dt.$$

Using (4.30) and taking into account $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, we obtain

$$\eta_\varepsilon{}^{-3}\|\tilde{v}_\varepsilon\|_{L^2((0,T)\times D)^3} \le C\left(\frac{\varepsilon}{\eta_\varepsilon{}^{\frac{3}{2}}} + \frac{\varepsilon^2}{\eta_\varepsilon{}^3}\right) \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.59}$$

Taking the limit in (5.58) as $\varepsilon \to 0$, using convergence (5.50), $\tilde{\mathcal{U}}_2 = 0$ and $\tilde{\mathcal{U}}_1$ independent of y_3, we have

$$\int_0^T \int_\Sigma \tilde{\mathcal{U}}_1 \partial_{x_1}\varphi(x_1, 0)\, dx_1 dy_2 dt = 0,$$

and by definition (3.19), we get

$$\int_0^T \int_{\Sigma_1} \Big(f_1(t, x_1, 0) - \partial_{x_1}\tilde{P}(t, x_1)\Big)\, \partial_{x_1}\varphi(t, x_1, 0)\, dx_1 dt = 0.$$

Consequently, $\tilde{P} \in L^2(0, T; H^1(\Sigma_1)/\mathbb{R})$ and is the unique solution of (3.20). Finally, we consider $\varphi \in C_0((0, T) \times D)^3$, independent of y_3, and so we have

$$\int_0^T \int_D \eta_\varepsilon{}^{-3}\tilde{u}_\varepsilon \cdot \varphi\, dx' dy_3 dt = \int_0^T \int_D \eta_\varepsilon{}^{-3}\tilde{v}_\varepsilon \cdot \varphi\, dx' dy_3 dt + \int_0^T \int_{\widetilde{I}_1} \eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \cdot \varphi(t, x_1, \eta_\varepsilon y_2)\, dx_1 dy_2 dy_3 dt.$$

Using (5.59), convergence (5.50) and $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, we obtain

$$\begin{aligned}\int_0^T \int_D \eta_\varepsilon{}^{-3}\tilde{u}_\varepsilon \cdot \varphi\, dx' dy_3 dt \quad &\to \quad \int_0^T \int_\Sigma \tilde{\mathcal{U}}_1(t, x_1, y_2)\varphi_1(t, x_1, 0)\, dx_1 dy_2 dt \\ &= \int_0^T \int_{\Sigma_1} \tilde{\mathcal{V}}_1(t, x_1)\varphi_1(t, x_1, 0)\, dx_1 = \int_0^T \langle \tilde{\mathcal{V}}_1(t, x_1)\delta_{\Sigma_1}, \varphi\rangle_{\mathcal{M}(D)^3, C_0(D)^3} dt,\end{aligned}$$

which implies (3.18). □

5.3 Effects of coupling $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$

The conclusion of the previous two subsections is that for any sequence of solutions $(\tilde{v}_\varepsilon, \tilde{p}_\varepsilon)$ with $\eta_\varepsilon \ll \varepsilon^{\frac{2}{3}}$ and $(\tilde{\mathcal{U}}^\varepsilon, \tilde{P}^\varepsilon)$ with $\eta_\varepsilon \gg \varepsilon^{\frac{2}{3}}$, and letting $\varepsilon \to 0$, we can extract subsequences still denoted by $\tilde{v}_\varepsilon, \tilde{p}_\varepsilon, \tilde{\mathcal{U}}^\varepsilon, \tilde{P}^\varepsilon$ and find functions $\tilde{v} \in L^2(0, T; H^1(0, 1; L^2(\omega)^3))$ with $\tilde{v}_3 = 0$, $\tilde{p} \in L^2(0, T; H^1(D)/\mathbb{R})$, $\tilde{\mathcal{U}} \in L^2((0, T) \times \widetilde{I}_1)^3$, independent of y_3, with $\tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, $\tilde{P} \in L^2(0, T; H^1(\Sigma)/\mathbb{R})$ such that

$$\begin{aligned}&\varepsilon^{-2}\tilde{v}_\varepsilon \rightharpoonup (\tilde{v}', 0) \quad \text{in } L^2(0, T; H^1(0, 1; L^2(\omega)^3)), \quad \tilde{p}_\varepsilon \to \tilde{p} \quad \text{in } L^2(0, T; L^2(D)/\mathbb{R}), \\ &\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \rightharpoonup (\tilde{\mathcal{U}}_1, 0, 0) \quad \text{in } L^2((0, T) \times \widetilde{I}_1)^3, \quad \tilde{P}^\varepsilon \rightharpoonup \tilde{P} \quad \text{in } L^2(0, T; L^2(\widetilde{I}_1)/\mathbb{R}).\end{aligned} \tag{5.60}$$

Moreover such limit functions $\tilde{v}, \tilde{p}, \tilde{\mathcal{U}}, \tilde{P}$ necessarily satisfy the equations

$$\begin{aligned}&\tilde{V}'(t, x') = \frac{1}{\mu}K\left(f'(t, x') - \nabla_{x'}\tilde{p}(t, x')\right) \quad \text{in } (0, T) \times D', \\ &\tilde{\mathcal{U}}_1(t, x_1, y_2) = \frac{y_2(1 - y_2)}{2}\Big(f_1(t, x_1, 0) - \partial_{x_1}\tilde{P}(t, x_1)\Big) \quad \text{in } (0, T) \times \widetilde{I}_1,\end{aligned} \tag{5.61}$$

where $\tilde{V}'(t, x') = \int_0^1 \tilde{v}'(t, x', y_3) dy_3$.

We are going to find the connection between the functions $\tilde{p}$ and $\tilde{P}$, i.e. to find the coupling effects between the solution in the porous part and in the fissure.

Lemma 5.4. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, with $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and let $\tilde{p}_\varepsilon \in L^2(0,T;L^2(D)/\mathbb{R})$, $\tilde{p} \in L^2(0,T;H^1(D)/\mathbb{R})$, $\tilde{P} \in L^2(0,T;H^1(\Sigma)/\mathbb{R})$ be such that (5.60) and (5.61) hold. Then,*

$$\int_0^T\int_{D'}\frac{1}{\mu}K\left(f'(t,x')-\nabla_{x'}\tilde{p}(t,x')\right)\cdot\nabla_{x'}\varphi(t,x')\,dx'dt+\frac{\lambda^3}{12}\int_0^T\int_{\Sigma_1}\left(f_1(t,x_1,0)-\partial_{x_1}\tilde{P}(t,x_1)\right)\partial_{x_1}\varphi(t,x_1,0)\,dx_1dt= \tag{5.62}$$

for every $\varphi \in L^2(0,T;H^1(D'))$ with $\varphi(t,\cdot,0) \in L^2(0,T;H^1(\Sigma_1))$.

Proof. Let $\varphi_\varepsilon(t,x',y_3) = \varphi(t,x',\varepsilon y_3) \in L^2(0,T;H^1(D))$ with $\varphi \in L^2(0,T;H^1(\overline{D}))$ and $\varphi(t,\cdot,0) \in L^2(0,T;H^1(\Sigma))$. Taking into account the definitions (5.42) of $\tilde{v}_\varepsilon$ and (3.9) of $\tilde{\mathcal{U}}^\varepsilon$, and from $\mathrm{div}_\varepsilon\,\tilde{u}_\varepsilon = 0$ in $(0,T)\times D$ we have

$$\int_0^T\int_D\varepsilon^{-2}\tilde{u}_\varepsilon\cdot\nabla_\varepsilon\varphi_\varepsilon\,dx'dy_3dt=\int_0^T\int_D\varepsilon^{-2}\tilde{v}_\varepsilon\cdot\nabla_\varepsilon\varphi_\varepsilon\,dx'dy_3dt+\left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3\int_0^T\int_{\widetilde{I}_1}\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon\cdot\nabla_\varepsilon\varphi_\varepsilon(t,x_1,\eta_\varepsilon y_2,y_3)\,dx_1dy_2dy_3$$

and by the definition of φ_ε, we can deduce

$$\int_0^T\int_D\varepsilon^{-2}\tilde{v}_\varepsilon\cdot\nabla\varphi(t,x',\varepsilon y_3)\,dx'dy_3dt+\left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3\int_0^T\int_{\widetilde{I}_1}\eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon\cdot\nabla\varphi(t,x_1,\eta_\varepsilon y_2,\varepsilon y_3)\,dx_1dy_2dy_3dt=0.$$

Taking the limit as $\varepsilon \to 0$, using (5.60), $\tilde{v}_3 = \tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$, $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, and taking into account that $\tilde{\mathcal{U}}_1$ does not depend on y_3, we obtain

$$\int_0^T\int_D\tilde{v}'(t,x',y_3)\cdot\nabla_{x'}\varphi(t,x',0)\,dx'dy_3dt+\lambda^3\int_0^T\int_\Sigma\tilde{\mathcal{U}}_1(t,x_1,y_2)\partial_{x_1}\varphi(t,x_1,0,0)\,dx_1dy_2dt=0,$$

and taking into account expressions (5.61) and (3.19), we get (5.62). □

We are going to prove the relation $\tilde{p}(t,x_1,0) = \tilde{P}(t,x_1) + C$, with $C \in \mathbb{R}$. Then (3.22) follows from (5.62).

Lemma 5.5. *Let $\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}$, $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, $0 < \lambda < +\infty$, and let $\tilde{p}$, $\tilde{P}$ be the limit pressures from (5.60). Then, there exists $C \in \mathbb{R}$ such that*

$$\tilde{p}(t,x_1,0) = \tilde{P}(t,x_1) + C, \tag{5.63}$$

and $\tilde{p} \in L^2(0,T;H^1(D')/\mathbb{R})$ with $\tilde{p}(t,\cdot,0) \in L^2(0,T;H^1(\Sigma_1)/\mathbb{R})$ is the unique solution of the variational problem (3.22).

Proof. We need to extend the test functions considered in the proof of Lemma 5.2 to the fissure $\widetilde{I}_{\eta_\varepsilon}$. To do this, we define $I'_{\eta_\varepsilon} = \widetilde{I}_{\eta_\varepsilon} \cap \{x_3 = 0\}$, $B_{\eta_\varepsilon} = D'_- \cup \Sigma_1 \cup I'_{\eta_\varepsilon}$ and $Y_1 = \overline{Y}_f \cap \{x_2 = 0\}$, and we consider $\phi(y') \in C^\infty_\#(B_{\eta_\varepsilon})^3$ be such that $\phi(y') = 0$ in $Y' \setminus Y'_f$. We define

$$\phi_\varepsilon(x') = \begin{cases} \phi\left(\frac{x'}{\varepsilon}\right) & \text{in } D'_-\,, \\ K_2\,e_2 & \text{in } I'_{\eta_\varepsilon}, \quad \text{where } K_2 = \displaystyle\int_{Y_1}\phi_2(y_1,0)dy_1\,. \end{cases}$$

Let $\varphi \in C_0^\infty(B_1)$, with $B_1 = D_- \cup \Sigma \cup \widetilde{I}_1$ be such that

$$\int_\Sigma \varphi(x_1, 0, y_3)\, dx_1 dy_3 = 0. \tag{5.64}$$

Taking in (3.6) as test function

$$w_\varepsilon(x', y_3) = \begin{cases} \varphi(x', y_3)\phi\left(\frac{x'}{\varepsilon}\right) & \text{in } D_-, \\ \varphi\left(x_1, \frac{x_2}{\eta_\varepsilon}, y_3\right) K_2\, e_2 & \text{in } \widetilde{I}_{\eta_\varepsilon}, \end{cases}$$

we obtain

$$\frac{d}{dt}\left(\int_{B_{\eta_\varepsilon}} \tilde{u}_\varepsilon(t) \cdot w_\varepsilon\, dx' dy_3\right) + \mu \int_{B_{\eta_\varepsilon}} D_\varepsilon \tilde{u}_\varepsilon(t) : D_\varepsilon w_\varepsilon\, dx' dy_3 = \int_{B_{\eta_\varepsilon}} f'(t) \cdot w'_\varepsilon\, dx' dy_3 + \int_{B_{\eta_\varepsilon}} \tilde{p}_\varepsilon(t)\, \mathrm{div}_\varepsilon\, w_\varepsilon\, dx' dy_3.$$

We consider $\psi \in C_c^1([0,T])$ such that $\psi(T) = 0$ and $\psi(0) \neq 0$. Multiplying by ψ and integrating between 0 and T, we have

$$\begin{aligned} &-\int_0^T \frac{d}{dt}\psi(t) \int_{B_{\eta_\varepsilon}} \tilde{u}_\varepsilon(t) \cdot w_\varepsilon\, dx' dy_3 dt + \mu \int_0^T \psi(t) \int_{B_{\eta_\varepsilon}} D_\varepsilon \tilde{u}_\varepsilon(t) : D_\varepsilon w_\varepsilon\, dx' dy_3 dt \\ &= \int_0^T \psi(t) \int_{B_{\eta_\varepsilon}} f'(t) \cdot w'_\varepsilon\, dx' dy_3 dt + \int_0^T \psi(t) \int_{B_{\eta_\varepsilon}} \tilde{p}_\varepsilon(t)\, \mathrm{div}_\varepsilon\, w_\varepsilon\, dx' dy_3 dt. \end{aligned} \tag{5.65}$$

Using (5.51), we have

$$\begin{aligned} &\left| K_2 \int_0^T \frac{d}{dt}\psi(t) \int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{\mathcal{U}}_2^\varepsilon(t) \cdot \varphi\left(x_1, \frac{x_2}{\eta_\varepsilon}, y_3\right)\, dx' dy_3 dt \right| \\ &= \left| K_2 \eta_\varepsilon \int_0^T \frac{d}{dt}\psi(t) \int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{\mathcal{U}}_2^\varepsilon(t) \cdot \varphi\left(x_1, y_2, y_3\right)\, dx_1 dy_2 dy_3 dt \right| \leq C\eta_\varepsilon^3 \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

We observe that

$$\begin{aligned} &K_2 \int_0^T \psi(t) \int_{\widetilde{I}_{\eta_\varepsilon}} f'(t) \cdot \varphi'\left(x_1, \frac{x_2}{\eta_\varepsilon}, y_3\right) e_2\, dx' dy_3 dt \\ &= \eta_\varepsilon K_2 \int_0^T \psi(t) \int_{\widetilde{I}_1} f'(t) \cdot \varphi'(x_1, y_2, y_3) e_2\, dx_1 dy_2 dy_3 dt \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

and by the definition of w_ε in $\widetilde{I}_{\eta_\varepsilon}$ and using estimates (5.52), (5.53), we deduce

$$\begin{aligned} &\left| K_2 \int_0^T \psi(t) \int_{\widetilde{I}_{\eta_\varepsilon}} D_\varepsilon \tilde{\mathcal{U}}^\varepsilon(t) \partial_{x_2} \varphi(x_1, \frac{x_2}{\eta_\varepsilon}, y_3)\, dx' dy_3 dt \right| \\ &= \left| K_2 \int_0^T \psi(t) \int_{\widetilde{I}_1} D_{\eta_\varepsilon} \tilde{\mathcal{U}}^\varepsilon(t) \partial_{y_2} \varphi(x_1, y_2, y_3)\, dx_1 dy_2 dy_3 dt \right| \leq C\eta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

Then, from (5.65), we can deduce that

$$-\int_0^T \frac{d}{dt}\psi(t)\int_{D_-} \tilde{u}_\varepsilon(t)\cdot w_\varepsilon\, dx'dy_3dt + \int_0^T \psi(t)\int_{D_-} D_\varepsilon \tilde{v}_\varepsilon(t) : D_\varepsilon w_\varepsilon\, dx'dy_3dt \tag{5.66}$$
$$= \int_0^T \psi(t)\int_{D_-} f'(t)\cdot w'_\varepsilon\, dx'dy_3dt + \int_0^T \psi(t)\int_{D_-} \tilde{p}_\varepsilon(t)\mathrm{div}_\varepsilon\, w_\varepsilon\, dx'dy_3dt$$
$$+K_2\int_0^T \psi(t)\int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon(t)\partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt + O_\varepsilon.$$

For the last term on the right hand side, we have

$$K_2\int_0^T \psi(t)\int_{\widetilde{I}_{\eta_\varepsilon}} \tilde{p}_\varepsilon(t)\partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt = K_2\int_0^T \psi(t)\int_{\widetilde{I}_{\eta_\varepsilon}} c_{\varepsilon\eta_\varepsilon}(t)\partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt$$
$$+K_2\int_0^T \psi(t)\int_{\widetilde{I}_{\eta_\varepsilon}} (\tilde{p}_\varepsilon(t)-c_{\varepsilon\eta_\varepsilon}(t))\partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt,$$

where $c_{\varepsilon\eta_\varepsilon}$ is defined in (3.10).

Using (5.60), we obtain

$$K_2\int_0^T \psi(t)\int_{\widetilde{I}_{\eta_\varepsilon}} (\tilde{p}_\varepsilon(t)-c_{\varepsilon\eta_\varepsilon}(t))\partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt$$
$$= K_2\int_0^T \psi(t)\int_{\widetilde{I}_1} \tilde{P}^\varepsilon(t)\partial_{y_2}\varphi(x_1,y_2,y_3)\, dx_1dy_2dy_3dt \tag{5.67}$$
$$\to K_2\int_0^T \psi(t)\int_{\widetilde{I}_1} \tilde{P}(t,x_1)\partial_{y_2}\varphi(x_1,y_2,y_3)\, dx_1dy_2dy_3dt = -K_2\int_0^T \psi(t)\int_\Sigma \tilde{P}(t,x_1)\varphi(x_1,0,y_3)\, dx_1dy_3dt,$$

as $\varepsilon\to 0$, where $\tilde{P}^\varepsilon$ is given by (3.9), and using (5.64), we have

$$K_2\int_0^T \psi(t)c_{\varepsilon\eta_\varepsilon}(t)\int_{\widetilde{I}_{\eta_\varepsilon}} \partial_{x_2}\varphi(x_1,\frac{x_2}{\eta_\varepsilon},y_3)\, dx'dy_3dt = K_2\int_0^T \psi(t)c_{\varepsilon\eta_\varepsilon}(t)\int_{\widetilde{I}_1} \partial_{y_2}\varphi(x_1,y_2,y_3)\, dx_1dy_2dy_3dt = 0.$$

Passing to the limit in (5.66) similarly as in the proof of Theorem 6.1-(i) in [8] by using an adaptation of the unfolding method, and taking into account (5.67) and

$$\int_0^T \psi(t)\int_{D'_-\times Y} \tilde{p}(t,x')\,\mathrm{div}_{x'}(\varphi(x',y_3)\phi(y'))\, dx'dydt$$
$$= -\int_0^T \psi(t)\int_{D'_-\times Y} \nabla_{x'}\tilde{p}(t,x')\varphi(x',y_3)\phi(y')\, dx'dydt$$
$$+\int_0^T \psi(t)\int_{\Sigma\times Y_1} \tilde{p}(t,x_1,0)\varphi(x_1,0,y_3)\phi_2(y_1,0)\, dx_1dy_1dy_3dt$$
$$= -\int_0^T \psi(t)\int_{D'_-\times Y} \nabla_{x'}\tilde{p}(t,x')\varphi(x',y_3)\phi(y')\, dx'dydt + K_2\int_0^T \psi(t)\int_\Sigma \tilde{p}(t,x_1,0)\varphi(x_1,0,y_3)\, dx_1dy_3dt,$$

then we have

$$\int_0^T \psi(t)\int_\Sigma \left(\tilde{p}(t,x_1,0)-\tilde{P}(t,x_1)\right)\varphi(x_1,0,y_3)\, dx_1dy_3dt = 0,$$

so that

$$\int_{(0,T)\times\Sigma_1} \left(\tilde{p}(t,x_1,0) - \tilde{P}(t,x_1)\right) \vartheta(t,x_1)\, dx_1 dt = 0,$$

for every $\vartheta \in C_0^\infty((0,T)\times\Sigma_1)$ such that $\int_\Sigma \vartheta\, dx_1 = 0$, a.e. $t \in (0,T)$. Finally we conclude that there exists a constant $C \in \mathbb{R}$ such that (5.63) holds and $\tilde{p}(t,x_1,0) \in L^2(0,T;H^1(\Sigma_1)/\mathbb{R})$.

Using (5.63) into (5.62), we obtain the variational formulation (3.22) for the limit pressure $\tilde{p}$ in the Banach space of functions $v \in L^2(0,T;H^1(D'))$ such that $v(t,x_1,0) \in L^2(0,T;H^1(\Sigma_1))$. Since $K \in \mathbb{R}^{2\times 2}$ is a symmetric, positive, tensor given by (3.15), it can be proved that (3.22) has a unique solution in that Banach space with the norm $|v|_{L^2(0,T;H^1(D'))} + |v(x_1,0)|_{L^2(0,T;H^1(\Sigma_1))}$.

□

Proof of Theorem 3.1-iii). It remains to prove the convergence (3.21) of the whole velocity.

Let $\varphi \in C_0((0,T)\times D)^3$. Then

$$\begin{aligned}\int_0^T\int_D \varepsilon^{-2}\tilde{u}_\varepsilon \cdot \varphi\, dx' dy_3 dt = &\int_0^T\int_D \varepsilon^{-2}\tilde{v}_\varepsilon \cdot \varphi\, dx' dy_3 dt \\ &+ \left(\frac{\eta_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right)^3 \int_0^T\int_{\widetilde{I}_1} \eta_\varepsilon{}^{-2}\tilde{\mathcal{U}}^\varepsilon \cdot \varphi(t,x_1,\eta_\varepsilon y_2,y_3)\, dx_1 dy_2 dy_3 dt = 0.\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, using (5.60), $\tilde{v}_3 = \tilde{\mathcal{U}}_2 = \tilde{\mathcal{U}}_3 = 0$ and $\eta_\varepsilon/\varepsilon^{\frac{2}{3}} \to \lambda$, we obtain

$$\int_0^T\int_D \varepsilon^{-2}\tilde{u}_\varepsilon \cdot \varphi\, dx' dy_3 dt \to \int_0^T\int_D \tilde{v}' \cdot \varphi'\, dx' dy_3 dt + \lambda^3 \int_0^T\int_{\widetilde{I}_1} \tilde{\mathcal{U}}_1(t,x_1,y_2)\varphi(t,x_1,0,y_3)\, dx_1 dy_2 dy_3 dt.$$

Taking into account that

$$\begin{aligned}&\int_0^T\int_{\widetilde{I}_1} \tilde{\mathcal{U}}(t,x_1,y_2)\varphi(t,x_1,0,y_3)\, dx_1 dy_2 dy_3 dt \\ &= \int_0^T\int_{\Sigma_1} \mathcal{V}(t,x_1)\left(\int_0^1 \varphi(t,x_1,0,y_3) dy_3\right) dx_1 dt = \int_0^T \langle \mathcal{V}\delta_{\Sigma_1}, \varphi\rangle_{\mathcal{M}(D)^3, C_0(D)^3} dt,\end{aligned}$$

where $\mathcal{V}(t,x_1)$ is given by (3.19), we get (3.21). □

Acknowledgments

The author would like to thank the referees for the detailed remarks which allowed to improve this paper. The author has been supported by Junta de Andalucía (Spain), Proyecto de Excelencia P12-FQM-2466, and in part by European Commission, Excellent Science-European Research Council (ERC) H2020-EU.1.1.-639227.

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