Darcy's laws for non-stationary viscous fluid flow in a thin porous medium

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Abstract

We consider a non-stationary Stokes system in a thin porous medium Ω_ε of thickness ε which is perforated by periodically solid cylinders of size a_ε. We are interested here to give the limit behavior when ε goes to zero. To do so, we apply an adaptation of the unfolding method. Time-dependent Darcy's laws are rigorously derived from this model depending on the comparison between a_ε and ε.

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1 Introduction

The aim of this work is to apply an adaptation of the unfolding method (see Arbogast *et al.* [1], Casado-Díaz [2] and Cioranescu *et al.* [3]) to the homogenization of a non-stationary Stokes system in a thin porous medium Ω_ε of thickness ε which is perforated by periodically solid cylinders of size a_ε. The unfolding method is a very efficient tool to study periodic homogenization problems where the size of the periodic cell tends to zero. The idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable), but here it is necessary to combine it with a rescaling in the height variable, in order to work with a domain of height one.

We consider the fluid flow through periodic vertical cylinders confined between two parallel plates (see Figures 1 and 2). A representative elementary volume for the thin porous medium is a cube of lateral length a_ε and vertical length ε. The cube is repeated periodically in the space between the plates. Each cube can be divided into fluid part and a solid part, where the solid part has the shape of a vertical cylinder of length ε (see Figure 3).

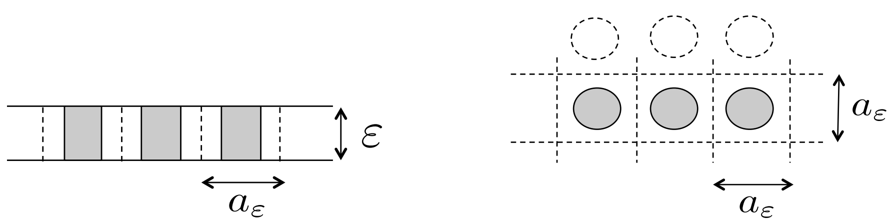

Figure 1: Views from lateral (left) and from above (right)

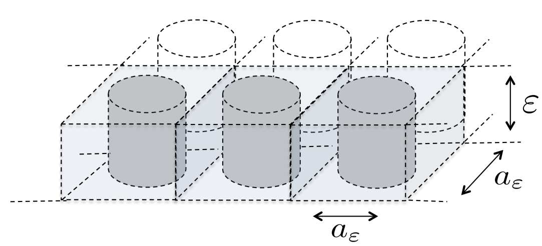

Figure 2: View of the domain

Thin porous media are common and of great importance for various industries and products. These include papers and cartons, filters and filtrateion cakes, porous coatings, fuel cells, textiles, and diapers and wipes, to name only a few. A thin porous medium is obviously characterized by lateral dimensions much greater than its thickness. As an example, the thickness of the so-called gas diffusion layers of proton exchange membrane fuel cells is typically on the order of 200μm, whereas its lateral dimension is on the order of 20cm, leading here to a ratio lateral dimension/thickness on the order of 10^3.

In 1856, H. Darcy [4] investigated water flow through a sand column and found that the driving force and fluid transport obeyed the relation

$$u = -\frac{K}{\mu}\nabla p,$$

where p is the pressure, μ is the viscosity of the fluid, K is the permeability tensor, and u is the flux. In 1949, H.C. Brinkman [5] introduced another extension to the traditional form of Darcy's law which is used to account for transitional flow between boundaries

$$\beta \Delta u + u = -\frac{K}{\mu} \nabla p,$$

where β is the effective viscosity.

The Stokes equations for a viscous fluid in a porous medium yield the Darcy's law as a homogenized model. Quite early, many papers have been devoted to the derivation of Darcy's law by means of homogenization, using formal asymptotic expansions (see for example Keller [6], Lions [7] and Sanchez-Palencia [8]). The first rigorous proof (including the difficult construction of a pressure extension) appeared in Tartar [9]. Further extensions are to be found in Allaire [10], Lipton and Avellaneda [11] and Mikelic [12].

The above results relate to a fixed height domain. Our aim in the present paper is to extend them to the case of a non-stationary Stokes system in a domain of small height ε. In particular, time-dependent Darcy's law and time-dependent Brinkman's law are rigorously derived from this model as the parameter ε tends to zero.

We show that the asymptotic behavior of this system depends on the parameter a_ε with respect to ε:

- If $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, i.e. when the cylinder height is proportional to the interspatial distance with λ the proportionality constant, we obtained a time-dependent Darcy's law as an homogenized model with a permeability tensor which depends on the parameter λ and is obtained through local stationary Stokes problems in 3D.
- If $a_\varepsilon \gg \varepsilon$, i.e. when the cylinder height is much smaller than the interspatial distance, we obtain a pure 2D time-dependent Darcy's law, with the permeability tensor obtained by means of local stationary Stokes problems in 2D, which is a considerable simplification.
- If $a_\varepsilon \ll \varepsilon$, i.e. when the cylinder height is much larger than the interspatial distance, a time-dependent Brinkman's law is derived as an homogenized model with local stationary Stokes problems in 2D.

The paper is organized as follows. In Section 2, the domain and some the notations are introduced. In Section 3, we formulate the problem and state our main result, which is proved in Section 6 by means of an adaptation of the unfolding method. To apply this method, a priori estimates are stablished in Section 4 and some compactness results are proved in Section 5.

2 The domain and some notations

A periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = [-1/2, 1/2]^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \bigcup Y'_s = Y'$ and $Y'_f \bigcap Y'_s = \varnothing$). More precisely, we assume that ω is a smooth, bounded, connected set in $\mathbb{R}^2$, and that Y'_s is a smooth and connected set strictly included in Y'.

The microscale of a porous medium is a small positive number a_ε. The domain ω is covered by a regular mesh of size a_ε: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',a_\varepsilon} = a_\varepsilon k' + a_\varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},a_\varepsilon}$ and a solid part $Y'_{s_{k'},a_\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size a_ε. We also define $Y = Y' \times (0,1) \in \mathbb{R}^3$, and is divided in a fluid part Y_f and a solid part Y_s, and consequently $Y_{k',a_\varepsilon} = Y'_{k',a_\varepsilon} \times (0,1) \in \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},a_\varepsilon}$ and a solid part $Y_{s_{k'},a_\varepsilon}$.

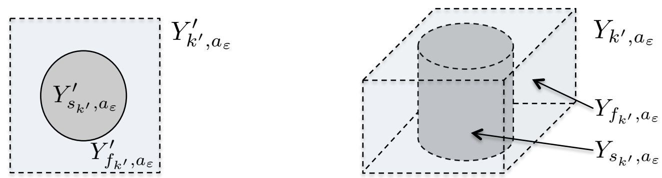

Figure 3: Views of a periodic cell in 2D (left) and 3D (right)

The fluid part ω_ε of a porous medium is defined by

$$\omega_\varepsilon = \omega \setminus \bigcup_{k' \in T_\varepsilon} Y'_{s_{k'},a_\varepsilon},$$

where $T_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',a_\varepsilon} \cap \omega \neq \emptyset\}$.

In order to apply the unfolding method, we will need the following notation. For $k' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \iff x' \in Y'_{k',1}. \tag{2.1}$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial Y'_{k',1}$). Moreover, for every $a_\varepsilon > 0$, we have

$$\kappa\left(\frac{x'}{a_\varepsilon}\right) = k' \iff x' \in Y'_{k',a_\varepsilon}.$$

We will consider the open set $\Omega_\varepsilon \subset \mathbb{R}^3$ given by

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \tag{2.2}$$

Then Ω_ε denotes the whole fluid part in the thin film.

We define $\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1)$ and $\Omega = \omega \times (0,1)$. We have that

$$\widetilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k' \in T_\varepsilon} Y_{s_{k'},a_\varepsilon} = \Omega \cap \bigcup_{k' \in T_\varepsilon} Y_{f_{k'},a_\varepsilon}.$$

We denote by $L^2_\sharp(Y)$, $H^1_\sharp(Y)$, the functional spaces

$$L^2_\sharp(Y) = \Big\{ v \in L^2_{loc}(Y) \,:\, \int_Y |v|^2 dy < +\infty, v(y' + k', y_3) = v(y) \;\; \forall k' \in \mathbb{Z}^2, \text{ a.e. } y \in Y \Big\},$$

and

$$H^1_\sharp(Y) = \Big\{ v \in H^1_{loc}(Y) \cap L^2_\sharp(Y) : \int_Y |\nabla_y v|^2 dy < +\infty \Big\}.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line and by C a generic positive constant which also can change from line to line.

3 Setting and main results

Along this paper, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

In this section we describe the asymptotic behavior of a viscous fluid in the geometry Ω_ε described in Section 2. The proof of the corresponding results will be given in the next sections.

Our results are referred to the non-stationary Stokes system. Namely, let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in L^2(0,T; H_0^1(\Omega_\varepsilon))^3 \times L^2(0,T; L^2(\Omega_\varepsilon))$, which satisfies

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \mu \Delta u_\varepsilon + \nabla p_\varepsilon = f \quad \text{in } (0,T)\times\Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \quad \text{in } (0,T)\times\Omega_\varepsilon, \\ u_\varepsilon(0,x) = u_\varepsilon^0(x), \quad x \in \Omega_\varepsilon, \end{cases} \tag{3.3}$$

where $T > 0$, Ω_ε is defined by (2.2) and $\mu > 0$ is the viscosity. The right-hand side f is of the form

$$f(t,x) = (f'(t,x'), 0), \text{ a.e. } x \in \Omega,$$

where f is assumed in $L^2((0,T)\times\omega)^2$. We deal the problem with Dirichlet boundary condition, i.e.

$$u_\varepsilon = 0 \text{ on } (0,T)\times\partial\Omega_\varepsilon. \tag{3.4}$$

In the sequel, we always assume that

$$\varepsilon^{-1/2}(a_\varepsilon^2+\varepsilon^2)^{-1/2}\left\|u_\varepsilon^0\right\|_{L^2(\Omega_\varepsilon)^3} + \varepsilon^{-1/2}\left\|Du_\varepsilon^0\right\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C. \tag{3.5}$$

For any fixed ε, under the assumptions on $f(t,x)$ and u_ε^0, a classical result (see Temam [13]) shows that (3.3)-(3.4) has at least one weak solution $(u_\varepsilon, p_\varepsilon) \in L^2(0,T; H_0^1(\Omega_\varepsilon))^3 \times L^2((0,T)\times\Omega_\varepsilon)$, where p_ε is uniquely defined up to an additive constant, i.e. it is uniquely defined if we consider the corresponding equivalence class: $p_\varepsilon \in L^2(0,T; L^2(\Omega_\varepsilon)/\mathbb{R})$.

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3

$$y_3 = \frac{x_3}{\varepsilon}, \tag{3.6}$$

in order to have the functions defined in an open set with fixed height.

Namely, we define $\tilde{u}_\varepsilon \in L^2(0,T; H_0^1(\widetilde{\Omega}_\varepsilon))^3$, $\tilde{p}_\varepsilon \in L^2(0,T; L^2(\widetilde{\Omega}_\varepsilon)/\mathbb{R})$ by

$$\tilde{u}_\varepsilon(t,x',y_3) = u_\varepsilon(t,x',\varepsilon y_3), \quad \tilde{p}_\varepsilon(t,x',y_3) = p_\varepsilon(t,x',\varepsilon y_3), \quad \textit{a.e. } (t,x',y_3) \in (0,T)\times\widetilde{\Omega}_\varepsilon.$$

Using the transformation (3.6), the system (3.3) can be rewritten as

$$\begin{cases} \dfrac{\partial \tilde{u}_\varepsilon}{\partial t} - \mu\Delta_{x'}\tilde{u}_\varepsilon - \varepsilon^{-2}\mu\partial_{y_3}^2\tilde{u}_\varepsilon + \nabla_{x'}\tilde{p}_\varepsilon + \varepsilon^{-1}\partial_{y_3}\tilde{p}_\varepsilon = f \quad \text{in } (0,T)\times\widetilde{\Omega}_\varepsilon, \\ \operatorname{div}_{x'}\tilde{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\tilde{u}_{\varepsilon,3} = 0 \quad \text{in } (0,T)\times\widetilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon(0,x',y_3) = \tilde{u}_\varepsilon^0(x',y_3), \quad (x',y_3) \in \widetilde{\Omega}_\varepsilon, \end{cases} \tag{3.7}$$

with Dirichlet boundary condition, i.e.

$$\tilde{u}_\varepsilon = 0 \text{ on } (0,T)\times\partial\widetilde{\Omega}_\varepsilon.$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon)$.

The sequence of solutions $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon) \in L^2(0,T;H_0^1(\widetilde{\Omega}_\varepsilon)^3) \times L^2(0,T;L^2(\widetilde{\Omega}_\varepsilon)/\mathbb{R})$ is not defined in a fixed domain independent of ε but rather in a varying set $\widetilde{\Omega}_\varepsilon$. In order to pass the limit if ε tends to zero, convergences in fixed Sobolev spaces (defined in Ω) are used which requires first that $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon)$ be extended to the whole domain Ω. Then, by definition, an extension $(\tilde{u}_\varepsilon,\, \tilde{P}_\varepsilon) \in L^2(0,T;H_0^1(\Omega))^3 \times L^2(0,T;L^2(\Omega)/\mathbb{R})$ of $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon)$ is defined on $(0,T)\times\Omega$ and coincides with $(\tilde{u}_\varepsilon,\, \tilde{p}_\varepsilon)$ on $(0,T)\times\widetilde{\Omega}_\varepsilon$ (we will use the same notation, $\tilde{u}_\varepsilon$, for the velocity in $(0,T)\times\widetilde{\Omega}_\varepsilon$ and its continuation in $(0,T)\times\Omega$).

Our main result referred to the asymptotic behavior of the solution of (3.7) is given by the following theorem.

Theorem 3.1. *We distingue three cases depending on the relation between the parameter a_ε with respect to ε:*

i) if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then the extension $(\tilde{u}_\varepsilon/a_\varepsilon^2, \tilde{P}_\varepsilon)$ of the solution of (3.7) converges weakly to $(\tilde{u}', \tilde{P})$ in $L^2(0,T;L^2(\Omega))^2 \times L^2(0,T;L^2(\Omega)/\mathbb{R})$. Moreover, it holds that $(\tilde{U}', \tilde{P})$ is the unique solution of Darcy's law

$$\begin{cases} \tilde{U}'(t,x') = \dfrac{A^\lambda}{\mu}\left(f'(t,x') - \nabla_{x'}\tilde{P}(t,x')\right) \text{ in } (0,T)\times\omega, \\ \operatorname{div}_{x'}\tilde{U}'(t,x') = 0 \text{ in } (0,T)\times\omega, \\ \tilde{U}'(t,x')\cdot n = 0 \text{ in } (0,T)\times\partial\omega, \end{cases} \tag{3.8}$$

where $\tilde{U}'(t,x') = \int_0^1 \tilde{u}'(t,x',y_3)\,dy_3$ and A^λ is a symmetric, positive, tensor defined by its entries

$$A_{ij}^\lambda = \int_{Y_f} D_\lambda w^i(y) : D_\lambda w^j(y)\,dy, \quad i,j = 1,2\,,$$

where $D_\lambda = D_{y'} + \lambda\partial_{y_3}$ and $w^i(y)$, for $i=1,2$, with $\int_Y w_3^i dy = 0$, denote the unique solutions in $H_\sharp^1(Y_f)^3$ of the local stationary Stokes problems in 3D

$$\begin{cases} -\Delta_\lambda w^i + \nabla_\lambda q^i = e_i \text{ in } Y_f\,, \\ \operatorname{div}_\lambda w^i = 0 \text{ in } Y_f\,, \\ w^i = 0 \text{ in } \partial Y_s\,, \\ w^i, q^i\ Y'-\text{periodic}, \end{cases}$$

with $\Delta_\lambda = \Delta_{y'} + \lambda^2\partial_{y_3}^2$, $\nabla_\lambda = \nabla_{y'} + \lambda\partial_{y_3}$ and $\operatorname{div}_\lambda = \nabla_{x'} + \lambda\partial_{y_3}$.

ii) if $a_\varepsilon \gg \varepsilon$, then the extension $(\tilde{u}_\varepsilon/a_\varepsilon^2, \tilde{P}_\varepsilon)$ of the solution of (3.7) converges, weakly in $L^2(0,T;L^2(\omega))^2 \times L^2(0,T;L^2(\omega)/\mathbb{R})$, to the unique solution $(\tilde{u}', \tilde{P})$ of Darcy's law

$$\begin{cases} \tilde{u}'(t,x') = \dfrac{A}{\mu}\left(f'(t,x') - \nabla_{x'}\tilde{P}(t,x')\right) \text{ in } (0,T)\times\omega, \\ \operatorname{div}_{x'}\tilde{u}'(t,x') = 0 \text{ in } (0,T)\times\omega, \\ \tilde{u}'(t,x')\cdot n = 0 \text{ in } (0,T)\times\partial\omega, \end{cases} \tag{3.9}$$

where A is a symmetric, positive, tensor defined by its entries

$$A_{ij} = \int_{Y_f'} D_{y'}w^i(y') : D_{y'}w^j(y')\,dy', \quad i = 1,2\,, \tag{3.10}$$

where, for $i = 1, 2$, $w^i(y')$ denote the unique solutions in $H^1_\sharp(Y'_f)^2$ of the local stationary Stokes problems in 2D

$$\begin{cases} -\Delta_{y'} w^i + \nabla_{y'} q^i &= e_i \quad \text{in } Y'_f \\ \mathrm{div}_{y'} w^i &= 0 \quad \text{in } Y'_f \\ w^i &= 0 \quad \text{in } \partial Y'_s \\ w^i, q^i \; Y' - \text{periodic.} \end{cases} \tag{3.11}$$

iii) if $a_\varepsilon \ll \varepsilon$, then the extension $(\tilde u_\varepsilon/\varepsilon^2, (a_\varepsilon/\varepsilon)\tilde P_\varepsilon)$ of the solution of (3.7) converges, weakly in $L^2(0,T;L^2(\Omega))^2 \times L^2(0,T;L^2(\omega)/\mathbb{R})$, to the unique solution $(\tilde u', \tilde P)$ of Brinkman's law

$$\begin{cases} -\mu \partial^2_{y_3} \tilde u'(t,x',y_3) + \mu A\, \tilde u'(t,x',y_3) &= f'(t,x') - \nabla_{x'} \tilde P(t,x') \quad \text{in } (0,T)\times\Omega, \\ \tilde u'(t,x',y_3) &= 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \mathrm{div}_{x'}\left(\displaystyle\int_0^1 \tilde u'(t,x',y_3)dy_3\right) &= 0 \quad \text{in } (0,T)\times\omega, \\ \left(\displaystyle\int_0^1 \tilde u'(t,x',y_3)dy_3\right)\cdot n &= 0 \quad \text{in } (0,T)\times\omega\,, \end{cases} \tag{3.12}$$

where A is given by (3.10).

4 A Priori Estimates

Let us begin with a lemma on Poincaré inequality in $\widetilde\Omega_\varepsilon$. We reproduce the original proof of Tartar [9].

Let us introduce some notation which will be useful in the following. We introduce the operators: D_ε and div_ε, by

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } \; i = 1,2,3, \; j = 1,2,$$

$$(D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } \; i = 1,2,3,$$

$$\mathrm{div}_\varepsilon v = \mathrm{div}_{x'} v' + \frac{1}{\varepsilon}\partial_{y_3} v_3.$$

Lemma 4.1. *There exists a constant $C > 0$ independent of ε, such that*

$$\|v\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\sqrt{a_\varepsilon^2 + \varepsilon^2}\, \|D_\varepsilon v\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}}, \quad \forall v \in H^1_0(\widetilde\Omega_\varepsilon)^3. \tag{4.13}$$

Proof. For any function $w(y) \in H^1_0(Y_f)^3$, the Poincaré inequality in Y_f states that

$$\int_{Y_f} |w|^2\, dy' dy_3 \le C \int_{Y_f} |D_y w|^2\, dy' dy_3, \tag{4.14}$$

where the constant C depends only on Y_f.

For every $k' \in \mathbb{Z}^2$, by the change of variable

$$k' + y' = \frac{x'}{a_\varepsilon}, \quad dy' = \frac{dx'}{a_\varepsilon^2} \quad \partial_{y'} = a_\varepsilon \partial_{x'}, \tag{4.15}$$

we rescale (4.14) from Y_f to $Y_{f_{k'},a_\varepsilon}$. This yields that, for any function $w(x',y_3)\in H_0^1(Y_{f_{k'},a_\varepsilon})^3$, one has

$$\begin{aligned}\int_{Y_{f_{k'},a_\varepsilon}} |w|^2\,dx'dy_3 &\leq a_\varepsilon^2 C\int_{Y_{f_{k'},a_\varepsilon}} |D_{x'}w|^2\,dx'dy_3+\varepsilon^2 C\int_{Y_{f_{k'},a_\varepsilon}}\left|\frac{1}{\varepsilon}\partial_{y_3}w\right|^2 dx'dy_3 \\ &\leq C(a_\varepsilon^2+\varepsilon^2)\left(\int_{Y_{f_{k'},a_\varepsilon}} |D_{x'}w|^2\,dx'dy_3+\int_{Y_{f_{k'},a_\varepsilon}}\left|\frac{1}{\varepsilon}\partial_{y_3}w\right|^2 dx'dy_3\right),\end{aligned} \tag{4.16}$$

with the same constant C as in (4.14). Summing the inequalities (4.16) for every $k'\in T_\varepsilon$, which cover the domain $\widetilde{\Omega}_\varepsilon$, gives the desired result (4.13).

□

We observe that using the hypothesis on the initial data (3.5), it is easy to deduce that

$$(a_\varepsilon^2+\varepsilon^2)^{-1/2}\left\|\tilde u_\varepsilon^0\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\left\|D_\varepsilon \tilde u_\varepsilon^0\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq C. \tag{4.17}$$

Let us obtain some priori estimates for $\tilde u_\varepsilon$.

Lemma 4.2. *Assume that $f\in L^2((0,T)\times\omega)^2$. Then, for any initial condition $\tilde u_\varepsilon^0\in H_0^1(\widetilde{\Omega}_\varepsilon)^3$ satisfying (4.17), there exists a constant C independent of ε, such that the solution $\tilde u_\varepsilon\in L^2(0,T;H_0^1(\widetilde{\Omega}_\varepsilon))^3$ of the problem (3.7) satisfies*

$$\|\tilde u_\varepsilon\|_{L^2((0,T)\times\widetilde{\Omega}_\varepsilon)^3} \leq C(a_\varepsilon^2+\varepsilon^2),\quad \|D_\varepsilon\tilde u_\varepsilon\|_{L^2((0,T)\times\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq C\sqrt{a_\varepsilon^2+\varepsilon^2}, \tag{4.18}$$

$$\|\tilde u_\varepsilon\|_{L^\infty(0,T;L^2(\widetilde{\Omega}_\varepsilon))^3} \leq C\sqrt{a_\varepsilon^2+\varepsilon^2},\quad \left\|\frac{\partial\tilde u_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\widetilde{\Omega}_\varepsilon)^3}\leq C. \tag{4.19}$$

Proof. Multiplying by $\tilde u_\varepsilon$ in the first equation of (3.7), integrating over $\widetilde{\Omega}_\varepsilon$ and using the energy equality, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\mu\|D_\varepsilon\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}=\int_{\widetilde{\Omega}_\varepsilon} f\cdot\tilde u_\varepsilon\,dx'dy_3. \tag{4.20}$$

Using Cauchy-Schwarz's inequality, Young's inequality and (4.13), we obtain that

$$\begin{aligned}\int_{\widetilde{\Omega}_\varepsilon} f\cdot\tilde u_\varepsilon\,dx'dy_3 &\leq \frac{1}{2}\frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}+\frac{1}{2}\frac{\mu}{C^2(a_\varepsilon^2+\varepsilon^2)}\|\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}\\ &\leq \frac{1}{2}\frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}+\frac{1}{2}\mu\|D_\varepsilon\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}.\end{aligned}$$

Thus, from (4.20), we deduce

$$\frac{d}{dt}\|\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\mu\|D_\varepsilon\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq\frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}, \tag{4.21}$$

and integrating between 0 and T

$$\|\tilde u_\varepsilon(T)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\mu\int_0^T\|D_\varepsilon\tilde u_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}dt\leq\left\|\tilde u_\varepsilon^0\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\int_0^T\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}dt.$$

Taking into account the assumption of f and (4.17), we obtain

$$\|\tilde{u}_\varepsilon(T)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu \int_0^T \|D_\varepsilon \tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\, dt \le C(a_\varepsilon^2 + \varepsilon^2),$$

and therefore, we deduce the second inequality in (4.18) and the first inequality in (4.19).

On the other hand, taking into account (4.13) in (4.21), we obtain

$$\frac{d}{dt}\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu\, C(a_\varepsilon^2+\varepsilon^2)^{-1}\|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le \frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\,\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)},$$

and integrating between 0 and T

$$\|\tilde{u}_\varepsilon(T)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu\, C(a_\varepsilon^2+\varepsilon^2)^{-1}\int_0^T \|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}\, dt \le \left\|\tilde{u}_\varepsilon^0\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \frac{C^2}{\mu}(a_\varepsilon^2+\varepsilon^2)\int_0^T \|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}\, dt.$$

Taking into account the assumption of f and (4.17), we obtain

$$\|\tilde{u}_\varepsilon(T)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu\, C(a_\varepsilon^2+\varepsilon^2)^{-1}\int_0^T \|\tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}\, dt \le C(a_\varepsilon^2+\varepsilon^2),$$

and therefore, we deduce the first inequality in (4.18).

Finally, we will prove the second estimate in (4.19). Now, we proceed formally. The rigorous proof schould be made using the Galerkin approximations. Multiplying by $\frac{\partial \tilde{u}_\varepsilon}{\partial t}$ in the first equation of (3.7), integrating over $\widetilde{\Omega}_\varepsilon$ and using the energy equality, we have

$$\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu\frac{1}{2}\frac{d}{dt}\|D_\varepsilon \tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} = \int_{\widetilde{\Omega}_\varepsilon} f\cdot\frac{\partial \tilde{u}_\varepsilon}{\partial t}\, dx'dy_3.$$

Using Cauchy-Schwarz's inequality and Young's inequality, we obtain that

$$\int_{\widetilde{\Omega}_\varepsilon} f\cdot\frac{\partial \tilde{u}_\varepsilon}{\partial t}\, dx'dy_3 \quad\le\quad \frac{1}{2}\|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)} + \frac{1}{2}\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}.$$

Then, we deduce

$$\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \mu\frac{d}{dt}\|D_\varepsilon \tilde{u}_\varepsilon(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le \|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)},$$

and integrating between 0 and T

$$\int_0^T \left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3} dt + \mu\,\|D_\varepsilon \tilde{u}_\varepsilon(T)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le \mu\left\|D_\varepsilon \tilde{u}_\varepsilon^0\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + \int_0^T \|f(t)\|^2_{L^2(\widetilde{\Omega}_\varepsilon)}\, dt.$$

Taking into account the assumption of f and (4.17), we obtain the second estimate in (4.19). □

4.1 The Extension of ($\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$) to the whole domain $(0,T)\times\Omega$

In this section, we will extend the solution $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain $(0,T)\times\Omega$. It is easy to extend the velocity by zero in $(0,T)\times\Omega\backslash\widetilde{\Omega}_\varepsilon$ (this is compatible with its Dirichlet boundary condition on $(0,T)\times\partial\widetilde{\Omega}_\varepsilon$). We will use the same notation, $\tilde{u}_\varepsilon$, for the velocity in $(0,T)\times\widetilde{\Omega}_\varepsilon$ and its continuation in $(0,T)\times\Omega$. We note that the extension $\tilde{u}_\varepsilon$ belongs to $L^2(0,T;H_0^1(\Omega))^3$.

Now, we give some properties of the restricted operator from $H_0^1(\Omega)^3$ into $H_0^1(\widetilde{\Omega}_\varepsilon)^3$ preserving divergence-free vectors, which was introduced by Tartar [9].

Lemma 4.3. *There exists a linear continuous operator R_ε acting from $H_0^1(\Omega)^3$ into $H_0^1(\widetilde{\Omega}_\varepsilon)^3$ such that*

1. $R_\varepsilon v = v$ *in* $\widetilde{\Omega}_\varepsilon$, *if* $v\in H_0^1(\widetilde{\Omega}_\varepsilon)^3$
2. $\mathrm{div}_\varepsilon\,(R_\varepsilon v) = 0$ *in* $\widetilde{\Omega}_\varepsilon$, *if* $\mathrm{div}_\varepsilon v = 0$ *in* Ω
3. *For any* $v\in H_0^1(\Omega)^3$ *(the constant* $\widetilde{C}$ *is independent of* v *and* ε*),*

$$\begin{aligned}\|R_\varepsilon v\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} &\leq \widetilde{C}\,\|v\|_{L^2(\Omega)^3} + \widetilde{C}a_\varepsilon\,\|D_\varepsilon v\|_{L^2(\Omega)^{3\times3}}\,,\\ \|D_\varepsilon R_\varepsilon v\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}} &\leq \frac{\widetilde{C}}{a_\varepsilon}\,\|v\|_{L^2(\Omega)^3} + \widetilde{C}\,\|D_\varepsilon v\|_{L^2(\Omega)^{3\times3}}\,.\end{aligned}$$

In order to extend the pressure to the whole domain Ω, we define a function $F_\varepsilon\in L^2(0,T;H^{-1}(\Omega))^3$, for all $T>0$, by the following formula (brackets are for the duality products between H^{-1} and H_0^1):

$$\langle F_\varepsilon(t), v\rangle_\Omega = \langle \nabla_\varepsilon\tilde{p}_\varepsilon(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon}\,,\ \text{for any}\ \ v\in H_0^1(\Omega)^3,\quad \forall t\in(0,T),\tag{4.22}$$

where R_ε is defined in Lemma 4.3. We calcule the right hand side of (4.22) by using (3.7) and we have

$$\langle F_\varepsilon(t), v\rangle_\Omega = \mu\,\langle \Delta_{x'}\tilde{u}_\varepsilon(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon} + \mu\,\langle \varepsilon^{-2}\partial^2_{y_3}\tilde{u}_\varepsilon(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon} + \langle f(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon} - \left\langle \frac{\partial\tilde{u}_\varepsilon(t)}{\partial t}, R_\varepsilon v\right\rangle_{\widetilde{\Omega}_\varepsilon}\,,\tag{4.23}$$

and by using the third point in Lemma 4.3 and (4.18)-(4.19), for fixed ε we can deduce that $F_\varepsilon\in L^2(0,T;H^{-1}(\Omega))^3$.

Moreover, if $v\in H_0^1(\widetilde{\Omega}_\varepsilon)^3$ and we continue it by zero out of $\widetilde{\Omega}_\varepsilon$, we see from (4.22) and the first point in Lemma 4.3 that $F_\varepsilon|_{\widetilde{\Omega}_\varepsilon}(t) = \nabla_\varepsilon\tilde{p}_\varepsilon(t)$, for all $t\in(0,T)$.

Moreover, if $\mathrm{div}_\varepsilon v = 0$ by the second point in Lemma 4.3 and (4.22), $\langle F_\varepsilon(t), v\rangle_\Omega = 0$, for all $t\in(0,T)$, and this implies that $F_\varepsilon(t)$ is the gradient of some function $\tilde{P}_\varepsilon(t)$ in $L^2(\Omega)$, for all $t\in(0,T)$. This means that F_ε is a continuation of $\nabla_\varepsilon\tilde{p}_\varepsilon$ to $(0,T)\times\Omega$, and that this continuation is a gradient. We also may say that $\tilde{p}_\varepsilon$ has been continuated to $(0,T)\times\Omega$ and

$$F_\varepsilon\equiv\nabla_\varepsilon\tilde{P}_\varepsilon,\qquad \tilde{P}_\varepsilon\in L^2(0,T;L^2(\Omega)/\mathbb{R}).$$

Lemma 4.4. *Assume that $f\in L^2((0,T)\times\omega)^2$. Then, for any initial condition $\tilde{u}_\varepsilon^0\in H_0^1(\widetilde{\Omega})^3$ satisfying (4.17) in Ω, there exists a constant C independent of ε, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)\in L^2(0,T;H_0^1(\Omega))^3\times L^2(0,T;L^2(\Omega)/\mathbb{R})$ of the solution $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ of the problem (3.7) satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^3} \leq C(a_\varepsilon^2+\varepsilon^2),\quad \|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^{3\times3}}\leq C\sqrt{a_\varepsilon^2+\varepsilon^2},\tag{4.24}$$

$$\|\tilde{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))^3} \leq C\sqrt{a_\varepsilon^2+\varepsilon^2},\quad \left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\Omega)^3}\leq C,\tag{4.25}$$

$$\left\|\tilde{P}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega)/\mathbb{R})} \leq C\left(\frac{\sqrt{a_\varepsilon^2+\varepsilon^2}}{a_\varepsilon}+1\right). \tag{4.26}$$

Proof. We first estimate the velocity. Taking into account Lemma 4.2, it is clear that, after extension, (4.24) and (4.25) hold.

Let us estimate $\nabla_\varepsilon \tilde{P}_\varepsilon$. Taking into account the third point in Lemma 4.3, we have

$$\begin{aligned}\left|\langle \Delta_{x'}\tilde{u}_\varepsilon(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon}\right| &\leq \|D_{x'}\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 2}}\|D_{x'}R_\varepsilon v\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 2}}\\ &\leq \|D_{x'}\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 2}}\left(\frac{\widetilde{C}}{a_\varepsilon}\|v\|_{L^2(\Omega)^3}+\widetilde{C}\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right),\end{aligned}$$

$$\begin{aligned}\left|\left\langle \varepsilon^{-2}\partial^2_{y_3}\tilde{u}_\varepsilon(t), R_\varepsilon v\right\rangle_{\widetilde{\Omega}_\varepsilon}\right| &\leq \left\|\frac{1}{\varepsilon}\partial_{y_3}\tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\left\|\frac{1}{\varepsilon}\partial_{y_3}R_\varepsilon v\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\\ &\leq \left\|\frac{1}{\varepsilon}\partial_{y_3}\tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\left(\frac{\widetilde{C}}{a_\varepsilon}\|v\|_{L^2(\Omega)^3}+\widetilde{C}\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right),\end{aligned}$$

$$\left|\langle f(t), R_\varepsilon v\rangle_{\widetilde{\Omega}_\varepsilon}\right| \leq \|f(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\left(\widetilde{C}\|v\|_{L^2(\Omega)^3}+\widetilde{C}a_\varepsilon\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right),$$

and

$$\left|\left\langle \frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}, R_\varepsilon v\right\rangle_{\widetilde{\Omega}_\varepsilon}\right| \leq \left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\left(\widetilde{C}\|v\|_{L^2(\Omega)^3}+\widetilde{C}a_\varepsilon\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right).$$

Then, from (4.23), we deduce

$$\begin{aligned}\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon(t), v\right\rangle_\Omega\right| &\leq \|D_\varepsilon\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\left(\frac{\widetilde{C}}{a_\varepsilon}\|v\|_{L^2(\Omega)^3}+\widetilde{C}\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right)\\ &\quad+\left(\|f(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\right)\left(\widetilde{C}\|v\|_{L^2(\Omega)^3}+\widetilde{C}a_\varepsilon\|D_\varepsilon v\|_{L^2(\Omega)^{3\times 3}}\right).\end{aligned}$$

Then, as $a_\varepsilon \ll 1$, we see that there exists a positive constant C such that

$$\begin{aligned}\left|\left\langle \nabla_\varepsilon \tilde{P}_\varepsilon(t), v\right\rangle_\Omega\right| &\leq \frac{C}{a_\varepsilon}\|D_\varepsilon\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\|v\|_{H^1_0(\Omega)^3}\\ &\quad+\left(\|f(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\right)\|v\|_{H^1_0(\Omega)^3},\end{aligned}$$

and consequently

$$\left\|\nabla_\varepsilon \tilde{P}_\varepsilon(t)\right\|_{H^{-1}(\Omega)^3} \leq \frac{C}{a_\varepsilon}\|D_\varepsilon\tilde{u}_\varepsilon(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}+\|f(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}.$$

From the classical inequality (see [9])

$$\left\|\tilde{P}_\varepsilon\right\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega)\left\|\nabla_\varepsilon \tilde{P}_\varepsilon\right\|_{H^{-1}(\Omega)^3},$$

we obtain

$$\left\|\tilde{P}_\varepsilon(t)\right\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\frac{1}{a_\varepsilon}\left\|D_\varepsilon \tilde{u}_\varepsilon(t)\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + \|f(t)\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} + \left\|\frac{\partial \tilde{u}_\varepsilon(t)}{\partial t}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\right).$$

Integrating between 0 and T, and from (4.18)-(4.19) and the assumption of f, we have (4.26). □

4.2 Adaptation of the Unfolding Method

The change of variable (3.6) does not provide the information we need about the behavior of $\tilde{u}_\varepsilon$ in the microstructure associated to $\widetilde{\Omega}_\varepsilon$. To solve this difficulty, we introduce an adaptation of the unfolding method (see [1, 2, 3]), which is strongly related to the two-scale convergence method (see Allaire [14] and Nghetseng [15]). For this purpose, given $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in L^2(0,T;H_0^1(\Omega))^3 \times L^2(0,T;L^2(\Omega)/\mathbb{R})$, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by

$$\hat{u}_\varepsilon(t,x',y) = \tilde{u}_\varepsilon\left(t, a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e.} \quad (t,x',y) \in (0,T)\times w \times Y, \tag{4.27}$$

$$\hat{P}_\varepsilon(t,x',y) = \tilde{P}_\varepsilon\left(t, a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e.} \quad (t,x',y) \in (0,T)\times w \times Y, \tag{4.28}$$

where the function κ is defined in (2.1).

Remark 4.5. *For $k' \in T_\varepsilon$ and for all $t \in (0,T)$, the restriction of $(\hat{u}_\varepsilon(t), \hat{P}_\varepsilon(t))$ to $Y'_{k',a_\varepsilon} \times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde{u}_\varepsilon(t), \tilde{P}_\varepsilon(t))$ by using the change of variables*

$$y' = \frac{x' - a_\varepsilon k'}{a_\varepsilon}, \tag{4.29}$$

which transforms Y_{k',a_ε} into Y.

Let us obtain some estimates for the sequences $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$.

Lemma 4.6. *Under the assumptions in Lemma 4.4, there exists a constant C independent of ε, such that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (4.27)-(4.28) satisfies*

$$\left\|D_{y'}\hat{u}_\varepsilon\right\|_{L^2((0,T)\times\omega\times Y)^{3\times 2}} \leq C a_\varepsilon \sqrt{a_\varepsilon^2 + \varepsilon^2}, \quad \left\|\partial_{y_3}\hat{u}_\varepsilon\right\|_{L^2((0,T)\times\omega\times Y)^3} \leq C\varepsilon\sqrt{a_\varepsilon^2+\varepsilon^2}, \tag{4.30}$$

$$\|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\omega\times Y))^3} \leq C\sqrt{a_\varepsilon^2+\varepsilon^2}, \quad \|\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \leq C\left(a_\varepsilon^2+\varepsilon^2\right), \tag{4.31}$$

$$\left\|\hat{P}_\varepsilon\right\|_{L^2(0,T;L^2(\omega\times Y)/\mathbb{R})} \leq C\left(\frac{\sqrt{a_\varepsilon^2+\varepsilon^2}}{a_\varepsilon} + 1\right). \tag{4.32}$$

Proof. Let us obtain some estimates for the sequence $\hat{u}_\varepsilon$ defined by (4.27). Taking into account the definition (4.27) of $\hat{u}_\varepsilon$, we obtain

$$\begin{aligned}\int_{\omega\times Y}\left|D_{y'}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy &\leq \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y \left|D_{y'}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy \\ &= \sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y \left|D_{y'}\tilde{u}_\varepsilon(t,a_\varepsilon k' + a_\varepsilon y', y_3)\right|^2 dx'dy.\end{aligned}$$

We observe that $\tilde{u}_\varepsilon$ does not depend on x', then we can deduce

$$\int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy \leq a_\varepsilon^2 \sum_{k'\in T_\varepsilon} \int_Y \left|D_{y'}\tilde{u}_\varepsilon(t,a_\varepsilon k'+a_\varepsilon y',y_3)\right|^2 dy.$$

By the change of variables (4.29), we obtain

$$\begin{aligned} \int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy &\leq a_\varepsilon^2 \sum_{k'\in T_\varepsilon} \int_{Y'_{k',a_\varepsilon}} \int_0^1 \left|D_{x'}\tilde{u}_\varepsilon(t,x',y_3)\right|^2 dx'dy_3 \\ &\leq a_\varepsilon^2 \int_{\omega\times(0,1)} \left|D_{x'}\tilde{u}_\varepsilon(t,x',y_3)\right|^2 dx'dy_3. \end{aligned}$$

Integrating between 0 and T, and taking into account (4.24), we have

$$\int_0^T \int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dydt \leq C a_\varepsilon^2 \left(a_\varepsilon^2+\varepsilon^2\right).$$

Similarly, using Remark 4.5 and definition (4.27), we have

$$\int_{\omega\times Y} \left|\partial_{y_3}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy \leq a_\varepsilon^2 \sum_{k'\in T_\varepsilon} \int_Y \left|\partial_{y_3}\tilde{u}_\varepsilon(t,a_\varepsilon k'+a_\varepsilon y',y_3)\right|^2 dy.$$

By the change of variables (4.29), we obtain

$$\int_{\omega\times Y} \left|\partial_{y_3}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy \leq \int_{\omega\times(0,1)} \left|\partial_{y_3}\tilde{u}_\varepsilon(t,x',y_3)\right|^2 dx'dy_3,$$

integrating between 0 and T and taking into account (4.24), we have

$$\int_0^T \int_{\omega\times Y} \left|\partial_{y_3}\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dydt \leq C\varepsilon^2(a_\varepsilon^2+\varepsilon^2),$$

so (4.30) is proved.

Similarly, using the definition (4.27) and the change of variables (4.29), we have

$$\int_{\omega\times Y} \left|\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dy \leq \int_\Omega \left|\tilde{u}_\varepsilon(t,x',y_3)\right|^2 dx'dy_3.$$

Taking into account the first estimate in (4.25), we obtain the fist estimate in (4.31). On the other hand, integrating between 0 and T and taking into account the fist estimate in (4.24), we have

$$\int_0^T \int_{\omega\times Y} \left|\hat{u}_\varepsilon(t,x',y)\right|^2 dx'dydt \leq C\left(a_\varepsilon^2+\varepsilon^2\right)^2,$$

and (4.31) holds.

Finally, let us obtain some estimates for the sequence $\hat{P}_\varepsilon$ defined by (4.28). We observe that using the definition (4.28) of $\hat{P}_\varepsilon$, we obtain

$$\int_{\omega\times Y} \left|\hat{P}_\varepsilon(t,x',y)\right|^2 dx'dy \leq \sum_{k'\in T_\varepsilon} \int_{Y'_{k',a_\varepsilon}} \int_Y \left|\tilde{P}_\varepsilon(t,a_\varepsilon k'+a_\varepsilon y',y_3)\right|^2 dx'dy.$$

We observe that $\tilde{P}_\varepsilon$ does not depend on x', then we can deduce

$$\int_{\omega\times Y}\left|\hat{P}_\varepsilon(t,x',y)\right|^2 dx'dy \leq a_\varepsilon^2 \sum_{k'\in T_\varepsilon}\int_Y \left|\tilde{P}_\varepsilon(t,a_\varepsilon k'+a_\varepsilon y',y_3)\right|^2 dy.$$

By the change of variables (4.29), we obtain

$$\int_{\omega\times Y}\left|\hat{P}_\varepsilon(t,x',y)\right|^2 dx'dy \leq \int_{\omega\times(0,1)}\left|\tilde{P}_\varepsilon(t,x',y_3)\right|^2 dx'dy_3.$$

Integrating between 0 and T and taking into account (4.26), we have

$$\int_0^T\int_{\omega\times Y}\left|\hat{P}_\varepsilon(t,x',y)\right|^2 dx'dydt \leq C\left(\frac{\sqrt{a_\varepsilon^2+\varepsilon^2}}{a_\varepsilon}+1\right)^2,$$

and (4.32) holds. □

Remark 4.7. *From (4.17) in Ω, it is easy to deduce*

$$\left(a_\varepsilon^2+\varepsilon^2\right)^{-1/2}\left\|\hat{u}_\varepsilon^0\right\|_{L^2(\omega\times Y)^3} + a_\varepsilon^{-1}\left\|D_{y'}\hat{u}_\varepsilon^0\right\|_{L^2(\omega\times Y)^{3\times 2}} + \varepsilon^{-1}\left\|\partial_{y_3}\hat{u}_\varepsilon^0\right\|_{L^2(\omega\times Y)^3} \leq C. \tag{4.33}$$

5 Some compactness results

In this section we obtain some compactness results about the behavior of the sequences $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ and $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ satisfying a priori estimates given in Lemma 4.4 and Lemma 4.6 respectively. We obtain different behaviors depending on the magnitude a_ε with respect to ε. Namely, we find a critical regime $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, and therefore we distingue three different regimes.

Let us start giving a convergence result for the pressure $\tilde{P}_\varepsilon$.

Lemma 5.1. *We distingues three regimes depending on the relation between the parameter a_ε with respect to ε:*

i) if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, or $a_\varepsilon \gg \varepsilon$, then for a subsequence of ε still denote by ε there exists $\tilde{P} \in L^2(0,T;L^2(\Omega)/\mathbb{R})$ such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ in } L^2((0,T)\times\Omega), \tag{5.34}$$

ii) if $a_\varepsilon \ll \varepsilon$, then for a subsequence of ε still denote by ε there exists $\tilde{P} \in L^2(0,T;L^2(\Omega)/\mathbb{R})$ such that

$$\frac{a_\varepsilon}{\varepsilon}\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ in } L^2((0,T)\times\Omega). \tag{5.35}$$

Proof. Taking into account the estimate of the pressure (4.26), we realize that we have to distingue three different cases. Namely, the critical case $a_\varepsilon \approx \varepsilon$, the supercritical case $a_\varepsilon \gg \varepsilon$ and the subcritical case $a_\varepsilon \ll \varepsilon$. Observe that in the critical and supercritical cases, the estimate (4.26) reads

$\left\|\tilde{P}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega)/\mathbb{R})} \le C$, which implies the existence $\tilde{P} : (0,T) \times \Omega \to \mathbb{R}$ such that (5.34) holds. By semicontinuity and the previous estimate of $\tilde{P}_\varepsilon$, we have

$$\int_0^T \int_\Omega \left|\tilde{P}(t)\right|^2 dx' dy_3 dt \le C,$$

which shows that $\tilde{P}$ belongs to $L^2(0,T;L^2(\Omega)/\mathbb{R})$. For the subcritical case, the estimate (4.26) reads $\left\|\tilde{P}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega)/\mathbb{R})} \le C\varepsilon/a_\varepsilon$. Reasoning similarly, we have (5.35). □

We will give a convergence result for $\tilde{u}_\varepsilon$.

Lemma 5.2. *For a subsequence of ε still denote by ε,*

i) if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then there exists $\tilde{u} \in L^2(0,T;H^1(0,1;L^2(\omega))^3)$ where $\tilde{u}_3 = 0$, and $\tilde{u} = 0$ on $(0,T) \times \partial\Omega$, such that

$$\frac{\tilde{u}_\varepsilon}{a_\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \ \ in \ \ L^2(0,T;H^1(0,1;L^2(\omega))^3), \tag{5.36}$$

ii) if $a_\varepsilon \gg \varepsilon$, then there exists $\tilde{u} \in L^2(0,T;L^2(\Omega)^3)$ where $\tilde{u}_3 = 0$ and $\tilde{u}'$ does not depend on y_3, with $\tilde{u} = 0$ on $(0,T) \times \partial\Omega$, such that

$$\frac{\tilde{u}_\varepsilon}{a_\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \ \ in \ \ L^2(0,T;L^2(\omega)^3), \tag{5.37}$$

iii) if $a_\varepsilon \ll \varepsilon$, then there exist $\tilde{u} \in L^2(0,T;H^1(0,1;L^2(\omega))^3)$ where $\tilde{u}_3 = 0$, $\tilde{w} \in L^2(0,T;H^2(0,1;H^{-1}(\omega)))$, with $\tilde{u} = \tilde{w} = 0$ on $(0,T) \times \partial\Omega$, such that

$$\frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \ \ in \ \ L^2(0,T;H^1(0,1;L^2(\omega))^3), \tag{5.38}$$

$$\frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup \tilde{w} \ \ in \ \ L^2(0,T;H^2(0,1;H^{-1}(\omega))), \tag{5.39}$$

$$\operatorname{div}_{x'}\tilde{u}' + \partial_{y_3}\tilde{w} = 0 \ \ in \ \ (0,T) \times \Omega\,. \tag{5.40}$$

Furthermore, it holds

$$\operatorname{div}_{x'}\left(\int_0^1 \tilde{u}'(t,x',y_3)dy_3\right) = 0 \ \ in \ \ (0,T) \times \omega, \quad \left(\int_0^1 \tilde{u}'(t,x',y_3)dy_3\right) \cdot n = 0 \ \ on \ \ (0,T) \times \partial\omega. \tag{5.41}$$

Proof. Taking into account the estimate of the velocity (4.24) for every case, we argue similarly to the proof of Lemma 5.2 in [16]. □

Now, we give a convergence result for the pressure $\hat{P}_\varepsilon$.

Lemma 5.3. *We distingues three regimes depending on the relation between the parameter a_ε with respect to ε:*

i) if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \gg \varepsilon$, then for a subsequence of ε still denote by ε there exists $\hat{P} \in L^2(0,T;L^2(\omega \times Y)/\mathbb{R})$ such that

$$\hat{P}_\varepsilon \rightharpoonup \hat{P} \ \ in \ \ L^2((0,T) \times \omega \times Y), \tag{5.42}$$

ii) if $a_\varepsilon \ll \varepsilon$, then for a subsequence of ε still denote by ε there exists $\hat{P} \in L^2(0,T;L^2(\omega \times Y)/\mathbb{R})$ such that

$$\frac{a_\varepsilon}{\varepsilon}\hat{P}_\varepsilon \rightharpoonup \hat{P} \ \ in \ \ L^2((0,T) \times \omega \times Y). \tag{5.43}$$

Proof. In *i)* the estimates (4.32) reads $\left\|\hat{P}_\varepsilon\right\|_{L^2(0,T;L^2(\omega\times Y)/\mathbb{R})} \leq C$, which implies the existence $\hat{P}$: $(0,T) \times \omega \times Y \to \mathbb{R}$ such that (5.42) holds. By semicontinuity and the previous estimate of $\hat{P}_\varepsilon$, we have

$$\int_0^T \int_{\omega\times Y} \left|\hat{P}(t)\right|^2 dx'dydt \leq C,$$

which shows that $\hat{P}$ belongs to $L^2(0,T;L^2(\omega \times Y)/\mathbb{R})$. In *ii)*, we have that the estimate (4.32) reads $\left\|\hat{P}_\varepsilon\right\|_{L^2(0,T;L^2(\omega\times Y)/\mathbb{R})} \leq C\varepsilon/a_\varepsilon$. Reasoning similarly, we have (5.43). □

Next, we give a convergence result for $\hat{u}_\varepsilon$.

Lemma 5.4. *For a subsequence of ε still denote by ε,*

i) if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then there exist $\hat{u} \in L^2(0,T;L^2(\omega;H^1_\sharp(Y)^3))$, $\hat{w} \in L^\infty(0,T;L^2(\omega;L^2_\sharp(Y)^3))$, with $\hat{u} = \hat{w} = 0$ on $(0,T) \times \omega \times Y_s$, such that

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon^2} \rightharpoonup \hat{u} \ \ in \ \ L^2(0,T;L^2(\omega;H^1(Y)^3)), \tag{5.44}$$

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \ \ in \ \ L^\infty(0,T;L^2(\omega \times Y))^3, \tag{5.45}$$

$$\mathrm{div}_\lambda \hat{u} = 0 \ \ in \ \ (0,T) \times \omega \times Y, \tag{5.46}$$

$$\mathrm{div}_{x'}\left(\int_Y \hat{u}'(t,x',y)dy\right) = 0 \ \ in \ \ (0,T) \times \omega, \quad \left(\int_Y \hat{u}'(t,x',y)dy\right) \cdot n = 0 \ \ on \ \ (0,T) \times \partial\omega, \tag{5.47}$$

where $\mathrm{div}_\lambda = \mathrm{div}_{y'} + \lambda\partial_{y_3}$,

ii) if $a_\varepsilon \gg \varepsilon$, then there exist $\hat{u} \in L^2(0,T;L^2(\omega;H^1_\sharp(Y')^3))$ independent of y_3, $\hat{w} \in L^\infty(0,T;L^2(\omega;L^2_\sharp(Y)^3))$, with $\hat{u} = \hat{w} = 0$ on $(0,T) \times \omega \times Y'_s$, such that

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon^2} \rightharpoonup \hat{u} \ \ in \ \ L^2(0,T;L^2(\omega;H^1(Y')^3)), \tag{5.48}$$

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \ \ in \ \ L^\infty(0,T;L^2(\omega \times Y))^3, \tag{5.49}$$

$$\mathrm{div}_{y'} \hat{u}' = 0 \ \ in \ \ (0,T) \times \omega \times Y'. \tag{5.50}$$

$$\mathrm{div}_{x'}\left(\int_{Y'} \hat{u}'(t,x',y')dy'\right) = 0 \ \ in \ \ (0,T) \times \omega, \quad \left(\int_Y' \hat{u}'(t,x',y')dy'\right) \cdot n = 0 \ \ on \ \ (0,T) \times \partial\omega, \tag{5.51}$$

iii) if $a_\varepsilon \ll \varepsilon$, then there exist $\hat{u} \in L^2(0,T;L^2(\Omega;H^1_\sharp(Y')^3))$, $\hat{w} \in L^\infty(0,T;L^2(\omega\times Y))^3$, with $\hat{u} = \hat{w} = 0$ on $\omega \times Y_s$, such that

$$\frac{\hat{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \ \text{in} \ \ L^2(0,T;H^1(0,1;L^2(\omega\times Y')^3)), \tag{5.52}$$

$$\frac{\hat{u}_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \ \text{in} \ \ L^\infty(0,T;L^2(\omega\times Y))^3, \tag{5.53}$$

$$a_\varepsilon^{-1}\varepsilon^{-1} D_{y'}\hat{u}_\varepsilon \rightharpoonup D_{y'}\hat{u} \ \text{in} \ \ L^2(0,T;L^2(\omega\times Y)^{3\times 2}), \tag{5.54}$$

$$\mathrm{div}_{y'}\hat{u}' = 0 \ \text{in} \ \ (0,T)\times\omega\times Y. \tag{5.55}$$

Proof. We proceed in four steps.
Step 1. Critical case $a_\varepsilon \approx \varepsilon$. In this case, the estimates (4.30)-(4.31) read

$$\|\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \le C a_\varepsilon^2, \quad \|D_y \hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^{3\times 3}} \le C a_\varepsilon^2, \quad \|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\omega\times Y))^3} \le C a_\varepsilon. \tag{5.56}$$

Taking into account the Dirichlet condition, the above estimates imply the existence $\hat{u}, \hat{w} : (0,T)\times\omega\times Y \to \mathbb{R}^3$, such that, up to a subsequence, convergences (5.44)-(5.45) hold. By semicontinuity and the estimates given in (5.56), we have

$$\int_0^T \int_{\omega\times Y} |\hat{u}|^2 \, dx'dydt \le C, \quad \int_0^T \int_{\omega\times Y} |D_y \hat{u}|^2 \, dx'dydt \le C, \quad \sup_{t\in(0,T)} \mathrm{ess} \, \|\hat{w}(t)\|_{L^2(\omega\times Y)^3} \le C,$$

which shows that $\hat{u} \in L^2(0,T;L^2(\omega;H^1(Y)^3))$, $\hat{w} \in L^\infty(0,T;L^2(\omega\times Y))^3$.

It remains to prove the Y'-periodicity of $\hat{u}$ in y'. To do this, we observe that by definition of $\hat{u}_\varepsilon$ given by (4.27), we have

$$\hat{u}_\varepsilon(t, x_1+\varepsilon, x_2, -1/2, y_2, y_3) = \hat{u}_\varepsilon(t, x', 1/2, y_2, y_3) \ \text{a.e.} \ (t, x', y_2, y_3) \in (0,T)\times\omega\times(-1/2,1/2)\times(0,1)\,,$$

which, dividing by a_ε^4 and taking into account convergence (5.44), gives

$$\hat{u}(t, x', -1/2, y_2, y_3) = \hat{u}(t, x', 1/2, y_2, y_3) \ \text{a.e.} \ (t, x', y_2, y_3) \in (0,T)\times\omega\times(-1/2,1/2)\times(0,1)\,.$$

Analogously, we can prove

$$\hat{u}(t, x', y_1, -1/2, y_3) = \hat{u}(t, x', y_1, 1/2, y_3) \ \text{a.e.} \ (t, x', y_1, y_3) \in (0,T)\times\omega\times(-1/2,1/2)\times(0,1)\,.$$

These equalities prove the periodicity of $\hat{u}$.

Since $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in $(0,T)\times\Omega$, then by definition of $\hat{u}_\varepsilon$ we have $a_\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$. Multipying by a_ε^{-1}, we obtain

$$a_\varepsilon^{-2}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \frac{a_\varepsilon}{\varepsilon} a_\varepsilon^{-2}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0, \quad \text{in } (0,T)\times\omega\times Y, \tag{5.57}$$

which combined with (5.44) and $a_\varepsilon/\varepsilon \to \lambda$, proves (5.46).

Step 2. Supercritical case $a_\varepsilon \gg \varepsilon$. In this case, the estimates (4.30)-(4.31) read

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^{3\times 2}} \le C a_\varepsilon^2, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \le C\varepsilon a_\varepsilon, \tag{5.58}$$

$$\|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\omega\times Y))^3} \le C a_\varepsilon \quad \|\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \le C a_\varepsilon^2. \tag{5.59}$$

Therefore from the first estimate in (5.58) and (5.59), up to a subsequence and using a semicontinuity argument, there exist $\hat{u} \in L^2(0,T;L^2(\Omega;H^1(Y')^3))$, $\hat{w} \in L^\infty(0,T;L^2(\omega\times Y))^3$ such that

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon^2} \rightharpoonup \hat{u} \text{ in } \ L^2(0,T;L^2(\Omega;H^1(Y')^3)), \tag{5.60}$$

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \text{ in } \ L^\infty(0,T;L^2(\omega\times Y))^3.$$

Since $\varepsilon^{-1}a_\varepsilon^{-1}\partial_{y_3}\hat{u}_\varepsilon$ is bounded in $L^2((0,T)\times\omega\times Y)^3$, we observe that $a_\varepsilon^{-2}\partial_{y_3}\hat{u}_\varepsilon$ is also bounded in $L^2((0,T)\times\omega\times Y)^3$ and tends to zero. This together (5.60) implies

$$a_\varepsilon^{-2}\partial_{y_3}\hat{u}_\varepsilon \to \partial_{y_3}\hat{u} \text{ in } \ L^2(0,T;L^2(\Omega;H^1(Y')^3)).$$

By the uniqueness of the limit, we deduce that $\partial_{y_3}\hat{u} = 0$ and so $\hat{u}$ does not depend on y_3.

In order to proof the Y'-periodicity of $\hat{u}$ in y', we proceed similarly to the step 1.

Integrating (5.57) in the variable y_3 between 0 and 1, and taking into account the convergence (5.60), we deduce $\operatorname{div}_{y'}\left(\int_0^1 \hat{u}'(t,x',y)dy_3\right) = 0$. Since $\hat{u}'$ does not depend on y_3, we get (5.50).

Step 3. In order to prove (5.47) and (5.51), let us first prove the following relation between $\tilde{u}$ and $\hat{u}$,

$$\int_Y \hat{u}(t,x',y)dy = \int_0^1 \tilde{u}(t,x',y_3)dy_3. \tag{5.61}$$

For this, let us consider $v \in C_c^1(\omega)$. We observe that using the definition (4.27) of $\hat{u}_\varepsilon$, we obtain

$$\frac{1}{a_\varepsilon^2}\int_\omega\int_Y \hat{u}_\varepsilon(t,x',y)v(x')dydx' = \frac{1}{a_\varepsilon^2}\sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_Y \tilde{u}_\varepsilon(t,a_\varepsilon k' + a_\varepsilon y', y_3)\, v(a_\varepsilon k' + a_\varepsilon y')dydx' + O_\varepsilon.$$

We observe that $\tilde{u}_\varepsilon$ and v do not depend on x', then we can deduce

$$\frac{1}{a_\varepsilon^2}\int_\omega\int_Y \hat{u}_\varepsilon(t,x',y)v(x')dydx' = \sum_{k'\in T_\varepsilon}\int_{Y'}\int_0^1 \tilde{u}_\varepsilon(t,a_\varepsilon k' + a_\varepsilon y', y_3)\, v(a_\varepsilon k' + a_\varepsilon y')dy_3dy' + O_\varepsilon.$$

By the change of variables (4.29), we obtain

$$\begin{aligned}\frac{1}{a_\varepsilon^2}\int_\omega\int_Y \hat{u}_\varepsilon(t,x',y)v(x')dydx' &= \frac{1}{a_\varepsilon^2}\sum_{k'\in T_\varepsilon}\int_{Y'_{k',a_\varepsilon}}\int_0^1 \tilde{u}_\varepsilon(t,x',y_3)\, v(x')dy_3dx' \\ &= \frac{1}{a_\varepsilon^2}\int_\omega\int_0^1 \tilde{u}_\varepsilon(t,x',y_3)\, v(x')dy_3dx' + O_\varepsilon.\end{aligned}$$

We consider $\varphi \in C_c^1([0,T])$ such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$. Multiplying by φ and integrating between 0 and T, we have

$$\frac{1}{a_\varepsilon^2}\int_0^T\int_\omega\int_Y \hat{u}_\varepsilon(t,x',y)v(x')\varphi(t)dydx'dt = \frac{1}{a_\varepsilon^2}\int_0^T\int_\omega\int_0^1 \tilde{u}_\varepsilon(t,x',y_3)\, v(x')\varphi(t)dy_3dx'dt + O_\varepsilon.$$

Taking into account the convergences (5.36) and (5.44) for the critical case, and (5.37) and (5.48) for the supercritical case, we obtain (5.61) for both cases. This together with (5.41) implies (5.47) and (5.51).

Step 4. Subcritical case $a_\varepsilon \ll \varepsilon$. In this case, the estimates (4.30)-(4.31) read

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^{3\times 2}} \leq C\varepsilon a_\varepsilon, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \leq C\varepsilon^2, \tag{5.62}$$

$$\|\hat{u}_\varepsilon\|_{L^\infty(0,T;L^2(\omega\times Y))^3} \leq C\varepsilon \quad \|\hat{u}_\varepsilon\|_{L^2((0,T)\times\omega\times Y)^3} \leq C\varepsilon^2. \tag{5.63}$$

Therefore from the second estimate in (5.62) and (5.63), up to a subsequence and using a semicontinuity argument, there exist $\hat{v} \in L^2(0,T;H^1(0,1;L^2(\omega\times Y')^3))$, $\hat{w} \in L^\infty(0,T;L^2(\omega\times Y))^3$ such that

$$\frac{\hat{u}_\varepsilon}{\varepsilon^2} \rightharpoonup \hat{v} \text{ in } \ L^2(0,T;H^1(0,1;L^2(\omega\times Y')^3)), \tag{5.64}$$

$$\frac{\hat{u}_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \text{ in } \ L^\infty(0,T;L^2(\omega\times Y))^3.$$

Since $\varepsilon^{-1}a_\varepsilon^{-1}D_{y'}\hat{u}_\varepsilon$ is bounded in $L^2((0,T)\times\omega\times Y)^{3\times 2}$, we observe that $\varepsilon^{-2}D_{y'}\hat{u}_\varepsilon$ is also bounded in $L^2((0,T)\times\omega\times Y)^{3\times 2}$ and tends to zero. This together (5.64) implies

$$\varepsilon^{-2}D_{y'}\hat{u}_\varepsilon \rightharpoonup D_{y'}\hat{v} \text{ in } \ L^2(0,T;H^1(0,1;L^2(\omega\times Y')^{3\times 2})).$$

By the uniqueness of the limit, we deduce that $D_{y'}\hat{v} = 0$ and so $\hat{v}$ does not depend on y'.

Taking into account (5.38) and (5.64), and proceeding as in (5.61), we obtain

$$\tilde{u}(t,x',y_3) = \int_{Y'} \hat{v}(t,x',y)dy'. \tag{5.65}$$

Since $\hat{v}$ does not depend on y', we have that $\hat{v} = (\tilde{u}', 0)$.

From the first estimate in (5.62), up to a subsequence and using a semicontinuity argument, there exists $\hat{u} \in L^2(0,T;L^2(\Omega;H^1(Y')^3))$ such that

$$a_\varepsilon^{-1}\varepsilon^{-1}D_{y'}\hat{u}_\varepsilon \rightharpoonup D_{y'}\hat{u} \text{ in } \ L^2(0,T;L^2(\omega\times Y)^{3\times 2}). \tag{5.66}$$

Since $\text{div}_\varepsilon\tilde{u}_\varepsilon = 0$ in Ω, then by definition of $\hat{u}_\varepsilon$ we have

$$a_\varepsilon^{-1}\varepsilon^{-1}\text{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-2}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0 \text{ in } \ (0,T)\times\omega\times Y,$$

which passing to the limit and taking into account that $\tilde{u}_3 = 0$, we obtain (5.55).

In order to proof the Y'-periodicity of $\hat{u}$ in y', we proceed similarly to the step 1.

□

Remark 5.5. *From (4.33), it is easy to deduce*

i) if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then there exists $\hat{u}^0 \in L^2(\omega\times Y)^3$, such that

$$\frac{\hat{u}^0_\varepsilon}{a_\varepsilon} \rightharpoonup \hat{u}^0 \ \textit{in} \ \ L^2(\omega\times Y)^3,$$

ii) if $a_\varepsilon \gg \varepsilon$ then there exists $\hat{u}^0 \in L^2(\omega \times Y')^3$, such that

$$\frac{\hat{u}^0_\varepsilon}{a_\varepsilon} \rightharpoonup \hat{u}^0 \ \ in \ \ L^2(\omega \times Y')^3,$$

iii) if $a_\varepsilon \ll \varepsilon$, then there exists $\hat{u}^0 \in L^2(\omega \times Y)^3$, such that

$$\frac{\hat{u}^0_\varepsilon}{\varepsilon} \rightharpoonup \hat{u}^0 \ \ in \ \ L^2(\omega \times Y)^3.$$

6 Homogenized models

In this section, we will multiply system (3.7) by a test function having the form of the limit $\hat{u}$ (see Lemma 5.4), and we will use the convergences given in the previous section in order to identify the homogenized model in every cases. This is the focus of the following theorem.

Theorem 6.1. *We distingue the three cases:*

i) if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then $(\hat{u}_\varepsilon/a_\varepsilon^2, \hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}(t,x',y), \tilde{P}(t,x'))$, with $\int_Y \hat{u}_3 dy = 0$, of the homogenized problem

$$\left\{\begin{array}{rcl} -\mu\Delta_\lambda \hat{u} + \nabla_\lambda \hat{q} &=& f' - \nabla_{x'}\tilde{P} \quad in \ \ (0,T)\times\omega\times Y_f, \\ \mathrm{div}_\lambda \hat{u} &=& 0 \quad in \ \ (0,T)\times\omega\times Y_f, \\ \hat{u} &=& 0 \ in \ \ (0,T)\times\omega\times Y_s \\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(t,x',y)dy\right) &=& 0 \ in \ \ (0,T)\times\omega, \\ \left(\displaystyle\int_Y \hat{u}'(t,x',y)dy\right)\cdot n &=& 0 \ on \ \ (0,T)\times\partial\omega, \\ y' \to \hat{u}, \hat{q} \quad Y'-periodic, && \end{array}\right. \tag{6.67}$$

where $\Delta_\lambda = \Delta_{y'} + \lambda^2\partial^2_{y_3}$, $\nabla_\lambda = \nabla_{y'} + \lambda\partial_{y_3}$ and $\mathrm{div}_\lambda = \nabla_{x'} + \lambda\partial_{y_3}$.

ii) if $a_\varepsilon \gg \varepsilon$, then $(\hat{u}_\varepsilon/a_\varepsilon^2, \hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}'(t,x',y'), \tilde{P}(t,x'))$, with $\hat{u}_3 = 0$, of the homogenized problem

$$\left\{\begin{array}{rcl} -\mu\Delta_{y'} \hat{u}' + \nabla_{y'} \hat{q} &=& f' - \nabla_{x'}\tilde{P} \quad in \ \ (0,T)\times\omega\times Y'_f, \\ \mathrm{div}_{y'} \hat{u}' &=& 0 \quad in \ \ (0,T)\times\omega\times Y'_f, \\ \hat{u}' &=& 0 \ in \ \ (0,T)\times\omega\times Y'_s \\ \mathrm{div}_{x'}\left(\displaystyle\int_{Y'} \hat{u}'(t,x',y')dy'\right) &=& 0 \ in \ \ (0,T)\times\omega, \\ \left(\displaystyle\int_{Y'} \hat{u}'(t,x',y')dy'\right)\cdot n &=& 0 \ on \ \ (0,T)\times\partial\omega, \\ y' \to \hat{u}', \hat{q} \quad Y'-periodic. && \end{array}\right. \tag{6.68}$$

iii) if $a_\varepsilon \ll \varepsilon$, then $(\tilde{u}_\varepsilon/\varepsilon^2, (a_\varepsilon/\varepsilon)\tilde{P}_\varepsilon)$ converges to the unique solution $(\tilde{u}'(t,x',y_3), \tilde{P}(t,x'))$ of the *homogenized problem*

$$\left\{\begin{array}{rcl} -\mu\partial_{y_3}^2 \tilde{u}' + \mu A\tilde{u}' & = & f' - \nabla_{x'}\tilde{P} \quad \text{in } (0,T)\times\Omega\,, \\ \tilde{u}' & = & 0 \quad \text{on } (0,T)\times\partial\Omega\,, \\ \operatorname{div}_{x'}\left(\int_0^1 \tilde{u}'(t,x',y_3)dy_3\right) & = & 0 \quad \text{in } (0,T)\times\omega\,, \\ \left(\int_0^1 \tilde{u}'(t,x',y_3)dy_3\right)\cdot n & = & 0 \quad \text{in } (0,T)\times\omega\,. \end{array}\right. \tag{6.69}$$

where A is a symmetric, positive definite, tensor defined in (3.10) through the solution $w^i(y')$, $i = 1, 2$, of local stationary Stokes problems in 2D (3.11).

Furthermore, $\hat{u}'_\varepsilon/(\varepsilon a_\varepsilon)$ converges weakly to $\hat{u}'$, which is given by the relation

$$\hat{u}'(t,x',y) = \sum_{i=1}^{2} \tilde{u}_i(t,x',y_3)w^i(y')\,.$$

Proof. First of all, we choose a test function $v(x',y)\in\mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with $v(x',y) = 0 \in \omega\times Y_s$ (thus, $v(x', x'/a_\varepsilon, y_3)\in H^1_0(\widetilde{\Omega}_\varepsilon)^3$). Multiplying (3.7) by $v(x', x'/a_\varepsilon, y_3)$ and integrating by parts, we have

$$\begin{array}{l} \dfrac{d}{dt}\left(\displaystyle\int_\Omega \tilde{u}_\varepsilon(t)\cdot v\,dx'dy_3\right) + \mu\displaystyle\int_\Omega D_{x'}\tilde{u}_\varepsilon(t) : D_{x'}v\,dx'dy_3 + \dfrac{\mu}{a_\varepsilon}\displaystyle\int_\Omega D_{x'}\tilde{u}_\varepsilon(t) : D_{y'}v\,dx'dy_3 \\ +\dfrac{\mu}{\varepsilon^2}\displaystyle\int_\Omega \partial_{y_3}\tilde{u}_\varepsilon(t) : \partial_{y_3}v\,dx'dy_3 - \displaystyle\int_\Omega \tilde{P}_\varepsilon(t)\operatorname{div}_{x'}v'\,dx'dy_3 - \dfrac{1}{a_\varepsilon}\displaystyle\int_\Omega \tilde{P}_\varepsilon(t)\operatorname{div}_{y'}v'\,dx'dy_3 \\ -\dfrac{1}{\varepsilon}\displaystyle\int_\Omega \tilde{P}_\varepsilon(t)\,\partial_{y_3}v_3\,dx'dy_3 = \displaystyle\int_\Omega f'(t)\cdot v'\,dx'dy_3\,, \end{array}$$

in $\mathcal{D}'(0,T)$. By the change of variables given in Remark 4.5, we obtain

$$\begin{array}{l} \dfrac{d}{dt}\left(\displaystyle\int_{\omega\times Y} \hat{u}'_\varepsilon(t)\cdot v'\,dx'dy\right) + \dfrac{\mu}{a_\varepsilon^2}\displaystyle\int_{\omega\times Y} D_{y'}\hat{u}'_\varepsilon(t) : D_{y'}v'\,dx'dy + \dfrac{\mu}{\varepsilon^2}\displaystyle\int_{\omega\times Y} \partial_{y_3}\hat{u}'_\varepsilon(t) : \partial_{y_3}v'\,dx'dy \\ -\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon(t)\operatorname{div}_{x'}v'\,dx'dy - \dfrac{1}{a_\varepsilon}\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon(t)\operatorname{div}_{y'}v'\,dx'dy = \displaystyle\int_{\omega\times Y} f'(t)\cdot v'\,dx'dy + O_\varepsilon\,, \end{array}$$

and

$$\begin{array}{l} \dfrac{d}{dt}\left(\displaystyle\int_{\omega\times Y} \hat{u}_{\varepsilon,3}(t)\cdot v_3\,dx'dy\right) + \dfrac{\mu}{a_\varepsilon^2}\displaystyle\int_{\omega\times Y} \nabla_{y'}\hat{u}_{\varepsilon,3}(t)\cdot\nabla_{y'}v_3\,dx'dy + \dfrac{\mu}{\varepsilon^2}\displaystyle\int_{\omega\times Y} \partial_{y_3}\hat{u}_{\varepsilon,3}(t) : \partial_{y_3}v\,dx'dy \\ -\dfrac{1}{\varepsilon}\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon(t)\,\partial_{y_3}v_3\,dx'dy + O_\varepsilon = 0\,. \end{array}$$

We consider $\varphi\in C^1_c([0,T])$ such that $\varphi(T) = 0$ and $\varphi(0)\neq 0$. Multiplying by φ and integrating between 0 and T, we have

$$\begin{array}{l} -\varphi(0)\displaystyle\int_{\omega\times Y} (\hat{u}^0_\varepsilon)'\cdot v'\,dx'dy - \displaystyle\int_0^T \dfrac{d}{dt}\varphi(t)\displaystyle\int_{\omega\times Y} \hat{u}'_\varepsilon(t)\cdot v'\,dx'dydt \\ +\dfrac{\mu}{a_\varepsilon^2}\displaystyle\int_0^T \varphi(t)\displaystyle\int_{\omega\times Y} D_{y'}\hat{u}'_\varepsilon(t) : D_{y'}v'\,dx'dydt + \dfrac{\mu}{\varepsilon^2}\displaystyle\int_0^T \varphi(t)\displaystyle\int_{\omega\times Y} \partial_{y_3}\hat{u}'_\varepsilon(t) : \partial_{y_3}v'\,dx'dydt \\ -\displaystyle\int_0^T \varphi(t)\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon(t)\operatorname{div}_{x'}v'\,dx'dydt - \dfrac{1}{a_\varepsilon}\displaystyle\int_0^T \varphi(t)\displaystyle\int_{\omega\times Y} \hat{P}_\varepsilon(t)\operatorname{div}_{y'}v'\,dx'dydt \\ = \displaystyle\int_0^T \varphi(t)\displaystyle\int_{\omega\times Y} f'(t)\cdot v'\,dx'dydt + O_\varepsilon\,, \end{array} \tag{6.70}$$

and

$$
\begin{aligned}
&-\varphi(0)\int_{\omega\times Y}\hat u^0_{\varepsilon,3}\cdot v_3\,dx'dy-\int_0^T\frac{d}{dt}\varphi(t)\int_{\omega\times Y}\hat u_{\varepsilon,3}(t)\cdot v_3\,dx'dydt\\
&+\frac{\mu}{a_\varepsilon^2}\int_0^T\varphi(t)\int_{\omega\times Y}\nabla_{y'}\hat u_{\varepsilon,3}(t):\nabla_{y'}v_3\,dx'dydt+\frac{\mu}{\varepsilon^2}\int_0^T\varphi(t)\int_{\omega\times Y}\partial_{y_3}\hat u_{\varepsilon,3}(t)\cdot\partial_{y_3}v\,dx'dydt\\
&-\frac{1}{\varepsilon}\int_0^T\varphi(t)\int_{\omega\times Y}\hat P_\varepsilon(t)\,\partial_{y_3}v_3\,dx'dydt+O_\varepsilon=0\,.
\end{aligned}
\tag{6.71}
$$

This variational formulation will be useful in the following steps.

We proceed in three steps.
Step 1. Critical case $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$.

First, we prove that $\hat P$ does not depend on the microscopic variable y. To do this, we consider as test function $a_\varepsilon v'(x',x'/a_\varepsilon,y_3)$ in (6.70) and $\varepsilon v_3(x',x'/a_\varepsilon,y_3)$ in (6.71), which gives

$$
\begin{aligned}
&-a_\varepsilon\varphi(0)\int_{\omega\times Y}(\hat u^0_\varepsilon)'\cdot v'\,dx'dy-a_\varepsilon\int_0^T\frac{d}{dt}\varphi(t)\int_{\omega\times Y}\hat u'_\varepsilon(t)\cdot v'\,dx'dydt\\
&+\frac{\mu}{a_\varepsilon}\int_0^T\varphi(t)\int_{\omega\times Y}D_{y'}\hat u'_\varepsilon(t):D_{y'}v'\,dx'dydt+\mu\frac{a_\varepsilon}{\varepsilon^2}\int_0^T\varphi(t)\int_{\omega\times Y}\partial_{y_3}\hat u'_\varepsilon(t):\partial_{y_3}v'\,dx'dydt\\
&-\int_0^T\varphi(t)\int_{\omega\times Y}\hat P_\varepsilon(t)\mathrm{div}_{y'}v'\,dx'dydt=a_\varepsilon\int_0^T\varphi(t)\int_{\omega\times Y}f'(t)\cdot v'\,dx'dydt+O_\varepsilon\,,
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon\varphi(0)\int_{\omega\times Y}\hat u^0_{\varepsilon,3}\cdot v_3\,dx'dy-\varepsilon\int_0^T\frac{d}{dt}\varphi(t)\int_{\omega\times Y}\hat u_{\varepsilon,3}(t)\cdot v_3\,dx'dydt\\
&+\mu\frac{\varepsilon}{a_\varepsilon^2}\int_0^T\varphi(t)\int_{\omega\times Y}\nabla_{y'}\hat u_{\varepsilon,3}(t)\cdot\nabla_{y'}v_3\,dx'dydt+\frac{\mu}{\varepsilon}\int_0^T\varphi(t)\int_{\omega\times Y}\partial_{y_3}\hat u_{\varepsilon,3}(t)\cdot\partial_{y_3}v_3\,dx'dydt\\
&-\int_0^T\varphi(t)\int_{\omega\times Y}\hat P_\varepsilon(t)\,\partial_{y_3}v_3\,dx'dydt+O_\varepsilon=0\,.
\end{aligned}
$$

We pass to the limit when ε tends to zero and using Remark 5.5, convergences (5.42), (5.44) and (5.45) with

$$
v\frac{d}{dt}\varphi(t)\in L^1(0,T;L^2(\omega\times Y))^3,\quad v\varphi(t)\in L^2(0,T;L^2(\omega;H^1(Y)^3)),\tag{6.72}
$$

we have

$$
\int_0^T\varphi(t)\int_{\omega\times Y}\hat P(t)\mathrm{div}_y v(x',y)\,dx'dydt=0,
$$

which shows that $\hat P$ does not depend on y.

Now, we choose a test function $v_\varepsilon=(v'(x',x'/a_\varepsilon,y_3),\lambda\,\varepsilon/a_\varepsilon\,v_3(x',x'/a_\varepsilon,y_3))$ in (6.70)-(6.71) with $v(x',y)\varphi(t)=0$ in $(0,T)\times\omega\times Y_s$, and satisfying incompressibility conditions (5.46) and (5.47), i.e. $\mathrm{div}_\lambda(v(x',y)\varphi(t))=0$ in $(0,T)\times\omega\times Y$ and $\mathrm{div}_{x'}(\int_Y v'(x',y)\varphi(t)\,dy)=0$ in $(0,T)\times\omega$ respectively. Taking into account (6.72), we pass to the limit when ε tends to zero and using Remark 5.5 and the convergences (5.42), (5.44) and (5.45), we have

$$
\begin{aligned}
&\mu\int_0^T\varphi(t)\int_{\omega\times Y}D_{y'}\hat u(t):D_{y'}v\,dx'dydt+\mu\lambda^2\int_0^T\varphi(t)\int_{\omega\times Y}\partial_{y_3}\hat u(t):\partial_{y_3}v\,dx'dydt\\
&=\int_0^T\varphi(t)\int_{\omega\times Y}f'(t)\cdot v'\,dx'dydt\,.
\end{aligned}
\tag{6.73}
$$

By density (6.73) holds for every function w in the Hilbert space V defined by

$$V = \left\{ \begin{array}{l} w(t,x',y) \in L^2(0,T;L^2(\omega;H^1_\sharp(Y)^3)), \quad \text{such that} \\ \operatorname{div}_\lambda w(t,x',y) = 0 \ \text{ in } (0,T)\times\omega\times Y, \quad \operatorname{div}_{x'}\left(\displaystyle\int_Y w(t,x',y)\,dy\right) = 0 \ \text{ in } (0,T)\times\omega, \\ w(t,x',y) = 0 \ \text{ in } (0,T)\times\omega\times Y_s, \quad \left(\displaystyle\int_Y w(t,x',y)\,dy\right)\cdot n = 0 \ \text{ on } (0,T)\times\omega \end{array} \right\}.$$

By Lax-Milgram lemma, the variational formulation (6.73) in the Hilbert space V admits a unique solution $\hat{u}$ in V. Reasoning as in [10], the orthogonal of V with respect to the usual scalar product in $L^2((0,T)\times\omega\times Y)$ is made of gradients of the form $\nabla_{x'}q(t,x') + \nabla_\lambda \hat{q}(t,x',y)$, with $q(t,x') \in L^2(0,T;L^2(\omega)/\mathbb{R})$ and $\hat{q}(t,x',y) \in L^2(0,T;L^2(\omega;H^1_\sharp(Y)))$. Therefore, by integration by parts, the variational formulation (6.73) is equivalent to the homogenized system defined in (6.67). It remains to prove that the pressure $\tilde{P}(t,x')$ arising as a Lagrange multiplier of the incompressibility constraint $\operatorname{div}_{x'}(\int_Y \hat{u}(t,x',y)dy) = 0$ is the same as the limit of the pressure $\tilde{P}_\varepsilon$. This can be easily done by multiplying equation (3.7) by a test function with $\operatorname{div}_\lambda$ equal to zero, and identifying limits. Since (6.67) admits a unique solution, then the complete sequence $(\hat{u}_\varepsilon/a_\varepsilon^2, \hat{P}_\varepsilon)$ converges to the unique solution $(\hat{u}(t,x',y), \tilde{P}(t,x'))$. This gives the desired result. Finally, observe that (5.61) and $\tilde{u}_3 = 0$, we have that $\int_Y \hat{u}_3 dy = 0$.

Step 2. Supercritical case $a_\varepsilon \gg \varepsilon$.

First, we show that $\hat{P}$ does not depend on the vertical variable y_3. To do this, we consider as test function $\varepsilon v_3(x', x'/a_\varepsilon, y_3)$ in (6.71). Taking into account (6.72), we pass to the limit when ε tends to zero and using Remark 5.5, the convergences (5.42), (5.48) and (5.49) and the second estimate in (5.58), we get

$$\int_0^T \varphi(t) \int_{\omega\times Y} \hat{P}(t)\,\partial_{y_3} v_3 \, dx'dydt = 0.$$

This shows that $\hat{P}$ does not depend on y_3.

Since we have to consider test function v' in (6.70) reflecting the behavior of $\hat{u}'$, then we consider test functions v' independent of y_3. Thus, let us now prove that $\hat{P}$ does not depend on the microscopic variable y'. For this, we take now as test function $a_\varepsilon v'(x', x'/a_\varepsilon)$ in (6.70). Taking into account (6.72), we pass to the limit when ε tends to zero and using Remark 5.5, the convergences (5.42), (5.48) and (5.49), we get

$$\int_0^T \varphi(t) \int_{\omega\times Y'} \hat{P}(t)\operatorname{div}_{y'} v' \, dx'dy'dt = 0,$$

which implies that $\hat{P}$ does not depend on y'. Thus, we conclude that $\hat{P}$ does not depend on the entire variable y.

Finally, we take as test function $v'(x', x'/a_\varepsilon)$ in (6.70) satisfying (5.50) and (5.51), i.e., we consider $\operatorname{div}_{y'}(v'(x',y')\varphi(t)) = 0$ in $(0,T)\times\omega\times Y'$ and $\operatorname{div}_{x'}(\int_{Y'} v'(x',y')\varphi(t)\,dy') = 0$ in $(0,T)\times\omega$ respectively. Passing to the limit and taking into account that $\hat{P}$ does not depend on y and that $\hat{u}'$, v' do not depend on y_3, we get

$$\mu \int_0^T \varphi(t) \int_{\omega\times Y'} D_{y'}\hat{u}'_\varepsilon(t) : D_{y'} v' \, dx'dy'dt = \int_0^T \varphi(t) \int_{\omega\times Y'} f'(t)\cdot v' \, dx'dy'dt. \qquad (6.74)$$

By density, and reasoning as in Step 1, the variational formulation (6.74) is equivalent to the homogenized system (6.68) (it is easy to see that $\hat{u}_3 = 0$).

Step 3. Subcritical case $a_\varepsilon \ll \varepsilon$.

First, we show that $\hat{P}$ does not depend on the vertical variable y_3. To do this, we consider as test function $a_\varepsilon v_3(x', x'/a_\varepsilon, y_3)$ in (6.71). Taking into account (6.72), we pass to the limit when ε tends to zero and using Remark 5.5, the convergences (5.43), (5.52), (5.53) and (5.54), we get

$$\int_0^T \varphi(t) \int_{\omega\times Y} \hat{P}(t)\, \partial_{y_3} v_3 \, dx' dy dt = 0.$$

This shows that $\hat{P}$ does not depend on y_3.

Now, we consider as test function $(a_\varepsilon^2/\varepsilon) v'(x', x'/a_\varepsilon, y_3)$ in (6.70). Passing to the limit, we have

$$\int_0^T \varphi(t) \int_{\omega\times Y} \hat{P}(t)\, \mathrm{div}_{y'} v' \, dx' dy dt = 0,$$

which shows that $\hat{P}$ does not depend on y', and so $\hat{P}$ only depends on x'.

Taking into account convergences (5.38)-(5.39) and (5.54), and free divergence conditions (5.40)-(5.41) and (5.55), we choose in (3.7) the following test function

$$\begin{cases} v'_\varepsilon(x', y_3) = v'(x', y_3) + \frac{a_\varepsilon}{\varepsilon} \phi'(x', x'/a_\varepsilon, y_3), \\ v_{\varepsilon,3}(x', y_3) = \varepsilon v_3(x', y_3) + a_\varepsilon \phi_3(x', x'/a_\varepsilon, y_3), \end{cases}$$

such that

$$\begin{gathered} \mathrm{div}_{x'}(v'(x', y_3)\varphi(t)) + \partial_{y_3}(v_3(x', y_3)\varphi(t)) = 0, \quad \text{ in } (0,T)\times\Omega, \\ \mathrm{div}_{x'}\left(\int_0^1 v'(x', y_3)\varphi(t) dy_3\right) = 0, \quad \text{ in } (0,T)\times\omega, \\ \mathrm{div}_{y'}(\phi'(x', y)\varphi(t)) = 0, \quad \text{ in } (0,T)\times\omega\times Y. \end{gathered}$$

Integrating by parts, applying the change of variables given in Remark 4.5 in the integrals involving the test functions ϕ, multiplying by φ and integrating between 0 and T, we obtain

$$\begin{aligned} &-\varphi(0)\int_\Omega (\tilde{u}_\varepsilon^0)' \cdot v' \, dx' dy_3 - \int_0^T \frac{d}{dt}\varphi(t) \int_\Omega \tilde{u}'_\varepsilon(t) \cdot v' \, dx' dy_3 dt \\ &-\varepsilon\varphi(0)\int_\Omega \tilde{u}_{\varepsilon,3}^0 \cdot v_3 \, dx' dy_3 - \varepsilon \int_0^T \frac{d}{dt}\varphi(t) \int_\Omega \tilde{u}_{\varepsilon,3}(t) \cdot v_3 \, dx' dy_3 dt \\ &\frac{\mu}{\varepsilon^2}\int_0^T \varphi(t) \int_\Omega \partial_{y_3}\tilde{u}'_\varepsilon(t) : \partial_{y_3} v' \, dx' dy_3 dt + \frac{\mu}{\varepsilon a_\varepsilon}\int_0^T \varphi(t) \int_{\omega\times Y} D_{y'}\hat{u}'_\varepsilon(t) : D_{y'}\phi' \, dx' dy dt \\ &-\frac{a_\varepsilon}{\varepsilon}\int_0^T \varphi(t) \int_{\omega\times Y} \hat{P}_\varepsilon(t)\, \mathrm{div}_{x'}\phi' \, dx' dy dt - \frac{a_\varepsilon}{\varepsilon}\int_0^T \varphi(t) \int_{\omega\times Y} \hat{P}_\varepsilon(t)\, \partial_{y_3}\phi_3 \, dx' dy dt \\ &= \int_0^T \varphi(t) \int_\Omega f'(t) \cdot v' \, dx' dy_3 dt + O_\varepsilon. \end{aligned} \tag{6.75}$$

From the first estimate in (4.25), up to a subsequence and using a semicontinuity argument, there exists $\hat{w} \in L^\infty(0,T;L^2(\Omega))^3$ such that

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} \hat{w} \text{ in } \; L^\infty(0,T;L^2(\Omega))^3,$$

and from (4.17) in Ω, it is easy to deduce that there exists $\tilde{u}^0 \in L^2(\Omega)^3$, such that

$$\frac{\tilde{u}^0_\varepsilon}{\varepsilon} \rightharpoonup \tilde{u}^0 \text{ in } \; L^2(\Omega)^3.$$

Taking into account

$$v\frac{d}{dt}\varphi(t) \in L^1(0,T;L^2(\Omega))^3, \quad v\varphi(t) \in L^2(0,T;H^1(0,1;L^2(\omega))^3), \quad \phi\,\varphi(t) \in L^2(0,T;L^2(\omega;H^1(Y)^3)),$$

we pass to the limit when ε tends to zero and using the above convergences, convergences (5.38) and (5.54) for the velocity, and (5.43) for the pressure and taking into account that $\hat{P}$ does not depend on y_3, we get

$$\begin{aligned}
&\mu\int_0^T\varphi(t)\int_\Omega \partial_{y_3}\tilde{u}'(t) : \partial_{y_3}v'\,dx'dy_3dt + \mu\int_0^T\varphi(t)\int_{\omega\times Y} D_{y'}\hat{u}'(t) : D_{y'}\phi'\,dx'dydt \\
&-\int_0^T\varphi(t)\int_{\omega\times Y}\hat{P}(t)\,\mathrm{div}_{x'}\phi'\,dx'dydt = \int_0^T\varphi(t)\int_\Omega f'(t)\cdot v'\,dx'dy_3dt.
\end{aligned} \tag{6.76}$$

Let us now obtain a problem for $\tilde{u}'$ eliminating $\hat{u}'$ and $\hat{P}$ in (6.76). For this purpose, we define

$$\hat{u}'(t,x',y) = \sum_{i=1}^2 \tilde{u}_i(t,x',y_3)w^i(y'), \quad \phi'(x',y)\varphi(t) = \sum_{i=1}^2 v_i(x',y_3)\varphi(t)w^i(y'),$$

where w^i, $i=1,2$ are the unique solution of the local problems (3.11). Then, (6.76) reads as follows

$$\mu\int_0^T\varphi(t)\int_\Omega \partial_{y_3}\tilde{u}'(t) : \partial_{y_3}v'\,dx'dy_3dt + \mu A\int_0^T\varphi(t)\int_\Omega \tilde{u}'(t)\cdot v'\,dx'dy_3dt = \int_0^T\varphi(t)\int_\Omega f'(t)\cdot v'\,dx'dy_3dt\,,$$

where A is defined in (3.10).

By density, and reasoning as in Step 1, this variational formulation is equivalent to the homogenized system (6.69). □

In the final step, we will eliminate the microscopic variable y in the homogenized problem. This is the focus of the Theorem 3.1.

Proof of Theorem 3.1. In the cases $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$ or $a_\varepsilon \gg \varepsilon$, the derivation of (3.8) and (3.9) from the homogenized problems (6.67) and (6.68) respectively, is an easy algebra exercise. Let us point that problems (3.8) and (3.9) are well-posed problems since it is simply second order elliptic equations for the pressure $\tilde{P}$ (with Neumann boundary condition). As is well-known, the local problems are also well-posed with periodic boundary condition, and it is easily checked, by integration by parts, that

$$A^\lambda_{ij} = \int_{Y_f} D_\lambda w^i(y) : D_\lambda w^j(y)\,dy = \int_{Y_f} w^i(y)e_j dy, \quad i=1,2, \quad j=1,2,3.$$

Observe that condition $\int_Y w_3^i dy = 0$, $i = 1, 2$, implies that $A_{i3}^\lambda = 0$. Then $A^\lambda \in \mathbb{R}^{2\times 2}$ and the definition implies that A^λ is symmetric and positive definite (analogously for A).

Observe that the case $a_\varepsilon \ll \varepsilon$ is already proved in the proof of Theorem 6.1.

□

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