The transition between the Navier-Stokes equations to the Darcy equation in a thin porous medium

María Anguiano and Francisco Javier Suárez-Grau

Abstract. We consider a Newtonian flow in a thin porous medium Ω_ε of thickness ε which is perforated by periodically distributed solid cylinders of size a_ε. Generalizing [2], the fluid is described by the 3D incompressible Navier-Stokes system where the external force takes values in the space H^{-1}, and the porous medium considered has one of the most commonly used distribution of cylinders: hexagonal distribution. By means of an adaptation of the unfolding method, three different Darcy's laws are rigorously derived from this model depending on the magnitude a_ε with respect to ε.

Mathematics Subject Classification (2010). 76A20; 76M50; 35B27; 35Q30.

Keywords. Homogenization; Navier-Stokes equations; Darcy's law; porous medium; thin-film fluids.

1. Introduction

We consider a viscous fluid obeying the Navier-Stokes system in a thin porous medium Ω_ε of thickness ε which is perforated by periodically distributed solid cylinders of size a_ε. Here, ε and a_ε are dimensionless small parameters related to the thickness of the porous medium and to the interspatial distance between the cylinders, respectively. On the boundary of the solid cylinders, we prescribe Dirichlet boundary conditions. The aim of this work is to prove the convergence of the homogenization process depending on the magnitude a_ε with respect to ε.

María Anguiano has been supported by Junta de Andalucía (Spain), Proyecto de Excelencia P12-FQM-2466. Francisco Javier Suárez-Grau has been supported by Ministerio de Economía y Competitividad (Spain), Proyecto Excelencia MTM2014-53309-P.

We consider a porous medium with one of the most commonly used distribution of cylinders: hexagonal distribution. In order to define this distribution, we consider a domain ω which is a smooth, bounded, connected set in $\mathbb{R}^2$, which will be associated to a microstructure. Below we give in detail the microstructure associated to this distribution.

The domain: the microstructure associated to ω will be defined by using two types of regular meshes, one composed by rhombuses and another by hexagons. Thus, we define by $R' \subset \mathbb{R}^2$ the reference rhombus and by $H' \subset \mathbb{R}^2$ the reference hexagon, which have an area of $\sqrt{3}/2$ (see Figure 1 for more details).

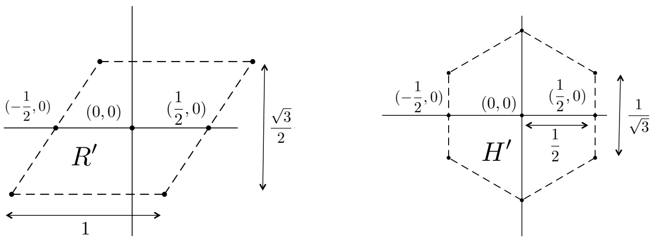

FIGURE 1. Views R' cell (left) and H' cell (right)

Taking into account that the hexagonal distribution of cylinders can be described by using meshes with rhombuses or hexagons (see Figure 4 for more details), we denote by $Y' = R'$ or H', which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \bigcup Y'_s = Y'$ and $Y'_f \bigcap Y'_s = \emptyset$). More precisely, we assume that Y'_s is a smooth, closed and connected set strictly included in Y'.

We denote the proportion of the material in the cell Y' by

$$\theta := \frac{|Y'_f|}{|Y'|} = \frac{2|Y'_f|}{\sqrt{3}}. \tag{1.1}$$

In order to go through every periodic cell in the hexagonal distribution, we introduce the parameter $k'(\ell') : \mathbb{Z}^2 \to \mathbb{Z}^2$, which is defined by using the hexagonal coordinate system (see Snyder *el al.* [8] for more details):

$$k_1(\ell') = \ell_1 + \frac{1}{2}\ell_2, \qquad k_2(\ell') = \frac{\sqrt{3}}{2}\ell_2\,, \quad \forall\, \ell' \in \mathbb{Z}^2\,.$$

For sake of simplicity, in the following we denote $k' = k'(\ell')$ omitting the dependence of $\ell' \in \mathbb{Z}^2$. Thus, the domain ω is covered by a regular mesh of size $a_\varepsilon\sqrt{3}/2$: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',a_\varepsilon} = a_\varepsilon k' + a_\varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},a_\varepsilon}$ and a solid part $Y'_{s_{k'},a_\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size a_ε. We also define $Y = Y' \times (0,1) \in \mathbb{R}^3$, and is divided in a fluid part Y_f and a solid part Y_s, and consequently $Y_{k',a_\varepsilon} = Y'_{k',a_\varepsilon} \times (0,1) \in \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},a_\varepsilon}$ and a solid part $Y_{s_{k'},a_\varepsilon}$ (see Figures 2 and 3 for more details).

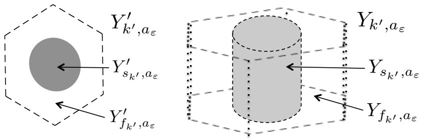

FIGURE 2. Views of a hexagonal periodic cell in 2D (left) and 3D (right)

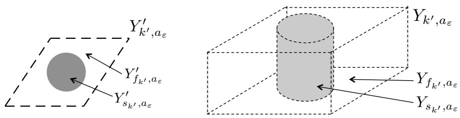

FIGURE 3. Views of a rhombohedral periodic cell in 2D (left) and 3D (right)

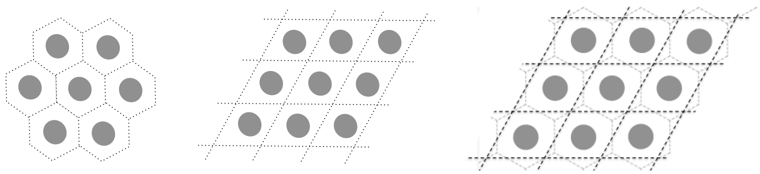

FIGURE 4. Views of the hexagonal mesh (left), the rhombohedral mesh (center) and both together (right)

Observe that the fluid part ω_ε of a porous medium with hexagonal distribution is defined by

$$\omega_\varepsilon = \omega \setminus \bigcup_{\ell' \in T_\varepsilon} Y'_{s_{k'},a_\varepsilon},$$

where $T_\varepsilon = \{\ell' \in \mathbb{Z}^2 : Y'_{k',a_\varepsilon} \cap \omega \neq \emptyset\}$. We will consider the open set $\Omega_\varepsilon \subset \mathbb{R}^3$ given by

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}.$$

Then Ω_ε denotes the whole fluid part in the thin film (see Figure 5 for more details).

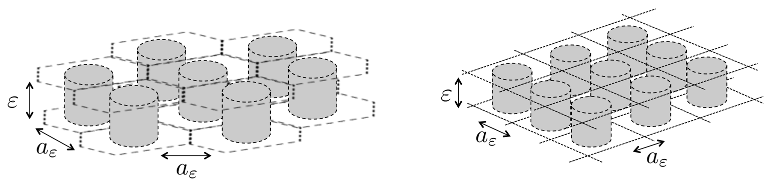

FIGURE 5. Views of Ω_ε by using the hexagonal mesh (left) and the rhombohedral mesh (right)

We define $\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1)$, $\Omega = \omega \times (0,1)$ and $Q_\varepsilon = \omega \times (0,\varepsilon)$. We have that

$$\widetilde{\Omega}_\varepsilon = \Omega \backslash \bigcup_{\ell' \in T_\varepsilon} Y_{s_{k'},a_\varepsilon} = \Omega \cap \bigcup_{\ell' \in T_\varepsilon} Y_{f_{k'},a_\varepsilon}. \tag{1.2}$$

The problem: let us consider the following Navier-Stokes system in Ω_ε:

$$\left\{ \begin{array}{rl} -\mu \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{array} \right. \tag{1.3}$$

where u_ε denotes the velocity field, p_ε is the (scalar) pressure, f_ε is the field of exterior body force and $\mu > 0$ is the viscosity.

In [2], we consider a non-Newtonian flow in a thin porous medium of thickness ε which is perforated by periodically distributed solid cylinders of size a_ε under a square distribution. The flow is described by the 3D incompressible Stokes system with a nonlinear viscosity, being a power of the shear rate (power law) of flow index $1 < p < +\infty$ and where the external force takes values in the space L^2. Applying an adaptation of the unfolding method, introduced by Cioranescu *et al.* [4], three types of Darcy's laws are obtained rigorously depending on the relation of a_ε with respect to ε. The Newtonian case, i.e. $p = 2$, has motived the fact of considering a much more general situation than the problem considered in [2]. In this sense, our aim in the present paper is to generalize three aspects: we consider that the flow is described by the 3D incompressible Navier-Stokes system, we suposse a larger space for the external force, namely f_ε belongs to H^{-1} and depends on ε, and we consider a porous medium with a hexagonal distribution of cylinders. Then, we show that the asymptotic behavior of the Navier-Stokes system depends on the parameter a_ε with respect to ε:

- If $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, i.e. when the cylinder height is similar to the distance between the cylinders, with λ the proportionality constant, we obtain a 2D Darcy law as an effective model with a permeability tensor and a function which depend on the parameter λ and are obtained through two local Stokes problems in 3D and a local Stokes problem in 3D with an external force with microstructure, respectively.
- If $a_\varepsilon \ll \varepsilon$, i.e. when the cylinder height is much larger than the distance between the cylinders, we obtain a 2D Darcy law as an effective model with a permeability tensor and a function which are obtained

through two local incomplete Stokes problems in 3D and a local incomplete Stokes problem in 3D with an external force with microstructure, respectively.

- If $a_\varepsilon \gg \varepsilon$, i.e. when the cylinder height is much smaller than the distance between the cylinders, we obtain a 2D Darcy law as an effective model with the permeability tensor and a function which are obtained by means of two local Hele-Shaw problems in 2D and a local Hele-Shaw problem in 2D with an external force with microstructure, respectively, which is a considerable simplification.

The inertial term coming from the Navier-Stokes system provides additional difficulties in the analysis, and we prove that this term does not appear in the effective problems. The fact of considering a more general external force, namely f_ε belonging to H^{-1} instead of L^2 and depending on ε, introduces additional difficulties and some changes with respect to the results in [2]. In particular, this allows us to consider general forces given in terms of the divergence of a tensor field. This changes the order of convergence of the velocity and the pressure and moreover, the limit external force does not appear explicitly in the effective problems. However, the external force gives rise to new functions in the effective problems which are defined by means of local problems with external forces with microstructure, also given in terms of the divergence of a tensor field. Finally, the hexagonal distribution of cylinders gives rise to a new proportion of the material which is $2/\sqrt{3}$ times the corresponding proportion in the square distribution, and new permeability tensors which are computed by means of local problems posed in the reference cylinder.

The behavior of the flow of Newtonian fluids through periodic arrays of cylinders has been studied extensively, mainly because of its importance in many applications in heat and mass transfer equipment. However, the literature on Newtonian thin film fluid flows through periodic arrays of cylinders is far less complete, although these problems have now become of great practical relevance because take place in a number of natural and industrial processes. This includes flow during manufacturing of fibre reinforced polymer composites with liquid moulding processes, passive mixing in microfluidic systems, and paper making.

Recently, the homogenization of the Navier-Stokes system in a thin porous medium with a square distribution has been studied in Fabricius *et al.* [5] by the multiscale expansion method which is a formal but powerful tool to analyse homogenization problems. They obtained the same three effective models depending on the relation of a_ε with respect to ε. They consider that the flow is only driven by the external pressure, but they claim that it is also possible to include the force term in the system. In this paper, we rigorously prove their claim assuming a general $f_\varepsilon \in H^{-1}$. Moreover, the particular case $a_\varepsilon \gg \varepsilon$ has been also studied by Zhengan and Hongxing in [11] where it is rigorously derived a 2D Darcy law with a permeability tensor depending on local Stokes problems. We remark that a more accurate 2D Darcy law with a permeability tensor depending on local Hele-Shaw problems in this particular case has been obtained in [5] by a formal method and in the present paper by the unfolding method (see Theorem 2.1-iii) for more details).

The paper is organized as follows. In Section 2, we state our main result, which is proved in Section 3.

2. Main result

Hereinafter the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

We suppose that the second member of (1.3), $f_\varepsilon \in H^{-1}(Q_\varepsilon)^3$, is of the form

$$f_\varepsilon(x) = (f'_\varepsilon(x), 0), \text{ a.e. } x \in Q_\varepsilon,$$

and there exists a positive constant $C > 0$ such that

$$\|f_\varepsilon\|_{H^{-1}(Q_\varepsilon)^3} \le C\varepsilon^{\frac{1}{2}}, \quad \forall \varepsilon > 0. \tag{2.1}$$

This choice of f_ε is usual when we deal with thin domains. Since the thickness of the domain, ε, is small then the vertical component of the force can be neglected.

Under the assumption of f_ε, it is well known that (1.3) has at least one weak solution $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$ (see Lions [6] for more details). The space $L_0^2(\Omega_\varepsilon)$ is the space of functions of $L^2(\Omega_\varepsilon)$ with null integral.

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3

$$y_3 = \frac{x_3}{\varepsilon}, \tag{2.2}$$

in order to have the functions defined in the open set with fixed height $\widetilde{\Omega}_\varepsilon$ defined by (1.2). Namely, we define $\tilde{u}_\varepsilon \in H_0^1(\widetilde{\Omega}_\varepsilon)^3$, $\tilde{p}_\varepsilon \in L_0^2(\widetilde{\Omega}_\varepsilon)$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \quad \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e.\ (x', y_3) \in \widetilde{\Omega}_\varepsilon,$$

and $\tilde{f}_\varepsilon \in H^{-1}(\Omega)^3$ by

$$\tilde{f}_\varepsilon(x', y_3) = f_\varepsilon(x', \varepsilon y_3) \ \ a.e.\ (x', y_3) \in \Omega.$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v = (v', v_3)$ and a scalar function w, we introduce the operators: D_ε, ∇_ε and div_ε, by

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } \ i = 1,2,3, \ j = 1,2, \quad (D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } \ i = 1,2,3,$$

$$\nabla_\varepsilon w = (\nabla_{x'} w, \frac{1}{\varepsilon}\partial_{y_3} w)^t, \quad \text{div}_\varepsilon v = \text{div}_{x'} v' + \frac{1}{\varepsilon}\partial_{y_3} v_3.$$

We denote by $L^2_\sharp(Y)$, $H^1_\sharp(Y)$, the functional spaces

$$L^2_\sharp(Y) = \Big\{ v \in L^2_{loc}(Y) \ : \quad \int_Y |v|^2 dy < +\infty, \\ v(y' + k'(\ell'), y_3) = v(y) \ \ \forall \ell' \in \mathbb{Z}^2, \text{ a.e. } y \in Y \Big\},$$

and

$$H^1_\sharp(Y)=\Big\{v\in H^1_{loc}(Y)\cap L^2_\sharp(Y):\int_Y|\nabla_y v|^2dy<+\infty\Big\}.$$

We denote by : the full contraction of two matrices: for $A=(a_{i,j})_{1\le i,j\le 2}$ and $B=(b_{i,j})_{1\le i,j\le 2}$, we have $A:B=\sum_{i,j=1}^2 a_{ij}b_{ij}$.

Using the transformation (2.2), the system (1.3) can be rewritten as

$$\left\{\begin{array}{r}-\mu\,\mathrm{div}_\varepsilon\,(D_\varepsilon\tilde u_\varepsilon)+(\tilde u_\varepsilon\cdot\nabla_\varepsilon)\tilde u_\varepsilon+\nabla_\varepsilon\tilde p_\varepsilon=\tilde f_\varepsilon \text{ in } \widetilde\Omega_\varepsilon,\\ \mathrm{div}_\varepsilon\,\tilde u_\varepsilon=0 \text{ in } \widetilde\Omega_\varepsilon,\\ \tilde u_\varepsilon=0 \text{ on } \partial\widetilde\Omega_\varepsilon.\end{array}\right.\tag{2.3}$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$. The sequence of solutions $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)\in H^1_0(\widetilde\Omega_\varepsilon)^3\times L^2_0(\widetilde\Omega_\varepsilon)$ is not defined in a fixed domain independent of ε but rather in a varying set $\widetilde\Omega_\varepsilon$. In order to pass the limit if ε tends to zero, convergences in fixed Sobolev spaces (defined in Ω) are used which requires first that $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ be extended to the whole domain Ω. Then, by definition, an extension $(\tilde u_\varepsilon,\,\tilde P_\varepsilon)\in H^1_0(\Omega)^3\times L^2_0(\Omega)$ of $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ is defined on Ω and coincides with $(\tilde u_\varepsilon,\,\tilde p_\varepsilon)$ on $\widetilde\Omega_\varepsilon$ (we will use the same notation, $\tilde u_\varepsilon$, for the velocity in $\widetilde\Omega_\varepsilon$ and its continuation in Ω).

Our main result is referred to the asymptotic behavior of the solution of (2.3) and is given by the following theorem.

Theorem 2.1. *We distinguish three cases depending on the relation between the parameter a_ε with respect to ε:*

i) *If $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$, then the extension $(\tilde u_\varepsilon/a_\varepsilon, a_\varepsilon\tilde P_\varepsilon)$ of a solution of (2.3) converges weakly to $(\tilde u,\tilde P)$ in $H^1_0(0,1;L^2(\omega)^3)\times L^2_0(\omega)$. Moreover, it holds that $(\tilde U,\tilde P)$, with $\tilde U_3=0$, is the unique solution of Darcy's law*

$$\left\{\begin{array}{l}\tilde U'(x')=-\dfrac{\theta}{\mu}\Big(A^\lambda\nabla_{x'}\tilde P(x')+b^\lambda(x')\Big)\ \text{ in }\ \omega,\\ \mathrm{div}_{x'}\tilde U'(x')=0\ \text{ in }\ \omega,\\ \tilde U'(x')\cdot n=0\ \text{ in }\ \partial\omega,\end{array}\right.\tag{2.4}$$

where $\tilde U(x')=\int_0^1\tilde u(x',y_3)\,dy_3$, θ is given by (1.1), and the symmetric and positive tensor $A^\lambda\in\mathbb R^{2\times2}$ and the function $b^\lambda:\mathbb R^2\to\mathbb R^2$ are defined by their entries

$$A^\lambda_{ij}=\frac{1}{|Y'_f|}\int_{Y_f}D_\lambda w^i(y):D_\lambda w^j(y)\,dy,$$
$$b^\lambda_i(x')=\frac{1}{|Y'_f|}\int_{Y_f}w_i(x',y)\,dy,\quad i,j=1,2\,.$$

For $i=1,2$, $w^i(y)$, with $\int_Y w^i_3\,dy=0$ and $w^i=0$ on $y_3=0,1$, denote the unique solutions in $H^1_\sharp(Y_f)^3$ of the local Stokes problems in

3D

$$\left\{\begin{array}{rcl} -\Delta_\lambda w^i + \nabla_\lambda q^i &=& e_i \quad in \;\; Y_f\,,\\ \mathrm{div}_\lambda w^i &=& 0 \quad in \;\; Y_f\,,\\ w^i &=& 0 \quad in \;\; \partial Y_s\,,\\ w^i, q^i \;\; Y' - periodic, && \end{array}\right.$$

and $w(x',y)$, with $w=0$ on $y_3=0,1$, denotes the unique solution in $L^2(\omega;H^1_\sharp(Y_f)^2)$ of the local Stokes problem in 3D

$$\left\{\begin{array}{rcl} -\Delta_\lambda w + \nabla_\lambda q &=& \mathrm{div}_\lambda \hat{G} \quad in \;\; \omega\times Y_f\,,\\ \mathrm{div}_\lambda w &=& 0 \quad in \;\; \omega\times Y_f\,,\\ w &=& 0 \quad in \;\; \omega\times\partial Y_s\,,\\ w, q \;\; Y' - periodic, && \end{array}\right.$$

where $D_\lambda = D_{y'} + \lambda\partial_{y_3}$, $\Delta_\lambda = \Delta_{y'} + \lambda\partial^2_{y_3}$, $\nabla_\lambda = \nabla_{y'} + \sqrt{\lambda}\partial_{y_3}$, $\mathrm{div}_\lambda = \mathrm{div}_{y'} + \lambda\partial_{y_3}$ and $\hat{G}(x',y)\in L^2_\sharp(\omega\times Y)^{2\times 3}$ is related to the external force f_ε (see the proof for more details).

ii) *If $a_\varepsilon \ll \varepsilon$, then the extension $(\tilde{u}_\varepsilon/a_\varepsilon, a_\varepsilon\tilde{P}_\varepsilon)$ of a solution of (2.3) converges weakly to $(\tilde{u},\tilde{P})$ in $L^2(\Omega)^3\times L^2_0(\omega)$ with $\tilde{u}=0$ on $y_3=0,1$. Moreover, it holds that $(\tilde{U},\tilde{P})$, with $\tilde{U}_3=0$, is the unique solution of Darcy's law*

$$\left\{\begin{array}{l} \tilde{U}'(x') = -\dfrac{\theta}{\mu}\left(A^0\nabla_{x'}\tilde{P}(x') + b^0(x')\right) \;\; in \;\; \omega,\\ \mathrm{div}_{x'}\tilde{U}'(x') = 0 \;\; in \;\; \omega,\\ \tilde{U}'(x')\cdot n = 0 \;\; in \;\; \partial\omega, \end{array}\right. \tag{2.5}$$

where $\tilde{U}(x') = \int_0^1 \tilde{u}(x',y_3)\,dy_3$, θ is given by (1.1), and the symmetric and positive tensor $A^0\in\mathbb{R}^{2\times 2}$ and the function $b^0:\mathbb{R}^2\to\mathbb{R}^2$ are defined by their entries

$$\begin{array}{l} A^0_{ij} = \dfrac{1}{|Y'_f|}\displaystyle\int_{Y_f} D_{y'}w^i(y) : D_{y'}w^j(y)\,dy,\\ b^0_i(x') = \dfrac{1}{|Y'_f|}\displaystyle\int_{Y_f} w_i(x',y)\,dy, \quad i=1,2\,. \end{array}$$

For $i=1,2$, $w^i(y)$, with $w^i=0$ on $y_3=0,1$, denote the unique solutions in $H^1_\sharp(Y_f)^2$ of the local incomplete Stokes problems in 3D

$$\left\{\begin{array}{rcl} -\Delta_{y'} w^i + \nabla_{y'} q^i &=& e_i \quad in \;\; Y_f,\\ \mathrm{div}_{y'} w^i &=& 0 \quad in \;\; Y_f,\\ w^i &=& 0 \quad in \;\; \partial Y_s,\\ w^i, q^i \;\; Y' - periodic, && \end{array}\right.$$

and $w(x',y)$, with $w=0$ on $y_3=0,1$, denotes the unique solution in $L^2(\omega;H^1_\sharp(Y_f)^2)$ of the local incomplete Stokes problem in 3D

$$\left\{\begin{array}{rcl} -\Delta_{y'} w + \nabla_{y'} q &=& \mathrm{div}_{y'} \hat{G}' \quad in \;\; \omega\times Y_f\,,\\ \mathrm{div}_{y'} w &=& 0 \quad in \;\; \omega\times Y_f\,,\\ w &=& 0 \quad in \;\; \omega\times\partial Y_s\,,\\ w, q \;\; Y' - periodic, && \end{array}\right.$$

where $\hat{G}'(x',y) \in L^2_\sharp(\omega \times Y)^{2\times 2}$ *is related to the external force* f_ε *(see the proof for more details).*

iii) *If* $a_\varepsilon \gg \varepsilon$*, then the extension* $(\tilde{u}_\varepsilon/\varepsilon, \varepsilon\tilde{P}_\varepsilon)$ *of a solution of (2.3) converges weakly to* $(\tilde{u}, \tilde{P})$ *in* $H^1(0,1;L^2(\omega)^3) \times L^2_0(\omega)$*. Moreover, it holds that* $(\tilde{U}, \tilde{P})$*, with* $\tilde{U}_3 = 0$*, is the unique solution of Darcy's law*

$$\begin{cases} \tilde{U}'(x') = -\dfrac{\theta}{12\mu}\left(A^\infty \nabla_{x'}\tilde{P}(x') + b^\infty(x')\right) \text{ in } \omega, \\ \operatorname{div}_{x'} \tilde{U}'(x') = 0 \text{ in } \omega, \\ \tilde{U}'(x')\cdot n = 0 \text{ in } \partial\omega, \end{cases} \tag{2.6}$$

where $\tilde{U}(x') = \int_0^1 \tilde{u}(x',y_3)\,dy_3$*,* θ *is given by (1.1), and the symmetric tensor* $A^\infty \in \mathbb{R}^{2\times 2}$ *and the function* $b^\infty : \mathbb{R}^2 \to \mathbb{R}^2$ *are defined by their entries*

$$A^\infty_{ij} = \frac{1}{|Y'_f|}\int_{Y'_f} (e^i + \nabla_{y'} q^i) e_j \, dy',$$
$$b^\infty_i(x') = \frac{1}{|Y'_f|}\int_{Y'_f} (G_3 + \nabla_{y'} q)\, dy', \quad i,j = 1,2\,.$$

For $i = 1,2$*,* $q^i(y')$ *denote the unique solutions in* $H^1_\sharp(Y'_f)$ *of the local Hele-Shaw problems in 2D,*

$$\begin{cases} \Delta_{y'} q^i = 0 & \text{in } Y'_f, \\ (\nabla_{y'} q^i + e_i)\cdot n = 0 & \text{in } \partial Y'_s, \\ q^i \ Y' - \text{periodic}, \end{cases}$$

and $q(x',y')$ *denotes the unique solution in* $L^2(\omega; H^1_\sharp(Y'_f))$ *of the local Hele-Shaw problem in 2D*

$$\begin{cases} -\Delta_{y'} q = \operatorname{div}_{y'} G_3 & \text{in } \omega \times Y'_f, \\ \nabla_{y'} q \cdot n = 0 = 0 & \text{in } \partial Y'_s, \\ q \quad Y' - \text{periodic}, \end{cases}$$

where $G_3(x',y') \in L^2_\sharp(\omega \times Y')^2$ *is related to the external force* f_ε *(see the proof for more details).*

3. Proof of the main result

In this section we prove our main result. In particular, Theorem 2.1 is proved by means of an adaptation of the unfolding method (see Arbogast *et al.* [3] and Cioranescu *et al.* [4]), which is strongly related to the two-scale convergence method (see Allaire [1] and Nguetseng [7]). To apply this method, *a priori* estimates are established and some compactness results are proved.

Some notations: in order to apply the unfolding method, we will need the following notation. For $\ell' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k'(\ell') \iff x' \in Y'_{k',1}\,. \tag{3.1}$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{\ell'\in\mathbb{Z}^2}\partial Y'_{k',1}$). Moreover, for every $a_\varepsilon > 0$, we have $\kappa\left(\frac{x'}{a_\varepsilon}\right) = k'(\ell') \iff x' \in Y'_{k',a_\varepsilon}$.

We will use $\langle\cdot,\cdot\rangle$ to denote the duality product between H^{-1} and H^1_0. We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line and by C a generic positive constant which can change from line to line.

A priori estimates: let us begin with a lemma on the Poincaré inequality in the domain $\widetilde{\Omega}_\varepsilon$.

Lemma 3.1. *There exists a constant C independent of ε, such that,*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$\|\tilde v\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le C a_\varepsilon \|D_\varepsilon \tilde v\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad \forall \tilde v \in H^1_0(\widetilde{\Omega}_\varepsilon)^3, \tag{3.2}$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\|\tilde v\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le C \varepsilon \|D_\varepsilon \tilde v\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad \forall \tilde v \in H^1_0(\widetilde{\Omega}_\varepsilon)^3. \tag{3.3}$$

Proof. The proof is similar to Lemma 4.2 and Remark 4.3 in [2] with $p = 2$. □

Remark 3.2. Observe that if $f_\varepsilon \in H^{-1}(Q_\varepsilon)^3$ such that $f_\varepsilon(x) = (f'_\varepsilon(x), 0)$, then there exist $f^0_\varepsilon, f^1_\varepsilon, f^2_\varepsilon, f^3_\varepsilon$ with $f^i_\varepsilon \in L^2(Q_\varepsilon)^2$ for all $0\le i\le 3$, such that $f'_\varepsilon = f^0_\varepsilon - \mathrm{div}_x G_\varepsilon$, where $G_\varepsilon = (f^1_\varepsilon, f^2_\varepsilon, f^3_\varepsilon) \in L^2(Q_\varepsilon)^{2\times 3}$, and

$$\|f_\varepsilon\|_{H^{-1}(Q_\varepsilon)^3} = \left(\sum_{i=0}^{3} \|f^i_\varepsilon\|^2_{L^2(Q_\varepsilon)^2}\right)^{1/2}. \tag{3.4}$$

We remark that $\mathrm{div}_x G_\varepsilon \in L^2(Q_\varepsilon)^2$ with entries $(\mathrm{div}_x G_\varepsilon)_k = \sum_{\ell=1}^3 \partial_{x_\ell} f^\ell_{\varepsilon,k}$, $k = 1, 2$. Moreover, applying the change of variables (2.2), we have that $\tilde f'_\varepsilon = \tilde f^0_\varepsilon - \mathrm{div}_\varepsilon \tilde G_\varepsilon$, where $\tilde f^i_\varepsilon \in L^2(\Omega)^2$ for all $0 \le i \le 3$, and $\tilde G_\varepsilon = (\tilde f^1_\varepsilon, \tilde f^2_\varepsilon, \tilde f^3_\varepsilon) \in L^2(\Omega)^{2\times 3}$.

Remark 3.3. Observe that thanks to (3.4), the assumption (2.1) can be written by

$$\|f^0_\varepsilon\|^2_{L^2(Q_\varepsilon)^2} + \|f^1_\varepsilon\|^2_{L^2(Q_\varepsilon)^2} + \|f^2_\varepsilon\|^2_{L^2(Q_\varepsilon)^2} + \|f^3_\varepsilon\|^2_{L^2(Q_\varepsilon)^2} \le C\varepsilon, \tag{3.5}$$

which implies, using the change of variables (2.2),

$$\|\tilde f^0_\varepsilon\|^2_{L^2(\Omega)^2} + \|\tilde f^1_\varepsilon\|^2_{L^2(\Omega)^2} + \|\tilde f^2_\varepsilon\|^2_{L^2(\Omega)^2} + \|\tilde f^3_\varepsilon\|^2_{L^2(\Omega)^2} \le C, \quad \forall \varepsilon > 0. \tag{3.6}$$

Let us obtain some *a priori* estimates for $\tilde u_\varepsilon$.

Lemma 3.4. *There exists a constant C independent of ε, such that if $\tilde u_\varepsilon \in H^1_0(\widetilde{\Omega}_\varepsilon)^3$ is the solution of (2.3), one has*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0<\lambda<+\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$\|\tilde u_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le C a_\varepsilon, \quad \|D_\varepsilon \tilde u_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le C, \tag{3.7}$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le C\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le C. \tag{3.8}$$

Proof. Multiplying by $\tilde{u}_\varepsilon$ in the first equation of (2.3) and integrating over $\widetilde{\Omega}_\varepsilon$, we have

$$\mu \left\|D_\varepsilon \tilde{u}_\varepsilon\right\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} = \langle f_\varepsilon, \tilde{u}_\varepsilon\rangle_{\widetilde{\Omega}_\varepsilon}. \tag{3.9}$$

Taking into account (3.4) and (3.6), we can deduce that

$$\|\tilde{f}_\varepsilon\|_{H^{-1}(\widetilde{\Omega}_\varepsilon)^3} = \left(\sum_{i=0}^{3} \|\tilde{f}^i_\varepsilon\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^2}\right)^{1/2} \le \left(\sum_{i=0}^{3} \|\tilde{f}^i_\varepsilon\|^2_{L^2(\Omega)^2}\right)^{1/2} \le C, \tag{3.10}$$

and we obtain

$$\langle \tilde{f}_\varepsilon, \tilde{u}_\varepsilon\rangle_{\widetilde{\Omega}_\varepsilon} \le C \left\|D\tilde{u}_\varepsilon\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le C \left\|D_\varepsilon \tilde{u}_\varepsilon\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}},$$

and by (3.9), we have

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \le C.$$

For the cases $a_\varepsilon \approx \varepsilon$ or $a_\varepsilon \ll \varepsilon$, taking into account (3.2), we obtain

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \le C a_\varepsilon.$$

For the case $a_\varepsilon \gg \varepsilon$, proceeding similarly with (3.3), we obtain the first estimate in (3.8). □

We turn to the case of the pressure. From equation (2.3), we easily obtain that $\nabla_\varepsilon \tilde{p}_\varepsilon$ is uniformly bounded in $H^{-1}(\widetilde{\Omega}_\varepsilon)^3$. Then, a well-known theorem of functional analysis (see, e.g. Proposition 1.2, Chapter I, [10]), states that $\tilde{p}_\varepsilon \in L^2(\widetilde{\Omega}_\varepsilon)$ with the following estimate

$$\|\tilde{p}_\varepsilon\|_{L^2_0(\widetilde{\Omega}_\varepsilon)} \le C(\widetilde{\Omega}_\varepsilon) \left\|\nabla_\varepsilon \tilde{p}_\varepsilon\right\|_{H^{-1}(\widetilde{\Omega}_\varepsilon)^3}.$$

In the above estimate, the constant depends on the domain $\widetilde{\Omega}_\varepsilon$, and thus may be not uniformly bounded when ε goes to zero. Then, the idea is to extend the pressure to the whole domain Ω.

The Extension of $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain Ω: we will extend the solution $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain Ω. It is easy to extend the velocity by zero in $\Omega\backslash\widetilde{\Omega}_\varepsilon$ (this is compatible with the Dirichlet boundary condition on $\partial\widetilde{\Omega}_\varepsilon$). We will use the same notation, $\tilde{u}_\varepsilon$, for the velocity in $\widetilde{\Omega}_\varepsilon$ and its continuation in Ω. We note that the extension $\tilde{u}_\varepsilon$ belongs to $H^1_0(\Omega)^3$.

Now, we give some properties of the restricted operator from $H^1_0(\Omega)^3$ into $H^1_0(\widetilde{\Omega}_\varepsilon)^3$ preserving divergence-free vectors. First, we extend the technique of Tartar [9] to the case of a thin domain, i.e. we define and give some properties of the restricted operator R_ε from $H^1_0(Q_\varepsilon)^3$ into $H^1_0(\Omega_\varepsilon)^3$.

Lemma 3.5. *There exists a (restriction) operator R_ε acting from $H^1_0(Q_\varepsilon)^3$ into $H^1_0(\Omega_\varepsilon)^3$ such that*

1. $R_\varepsilon v = v$, *if $v \in H^1_0(\Omega_\varepsilon)^3$ (elements of $H^1_0(\Omega_\varepsilon)^3$ are continuated by* 0 *to Q_ε)*

2. $\operatorname{div}(R_\varepsilon v) = 0$ *in* Ω_ε, *if* $\operatorname{div} v = 0$ *in* Q_ε
3. *For any* $v \in H_0^1(Q_\varepsilon)^3$ *(the constant* C *is independent of* v *and* ε*),*
   i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, *or* $a_\varepsilon \ll \varepsilon$, *then*

$$\begin{aligned} \|R_\varepsilon v\|_{L^2(\Omega_\varepsilon)^3} &\leq C\|v\|_{H_0^1(Q_\varepsilon)^3}, \\ \|DR_\varepsilon v\|_{L^2(\Omega_\varepsilon)^{3\times 3}} &\leq C\frac{1}{a_\varepsilon}\|v\|_{H_0^1(Q_\varepsilon)^3}, \end{aligned}$$

   ii) *if* $a_\varepsilon \gg \varepsilon$, *then*

$$\begin{aligned} \|R_\varepsilon v\|_{L^2(\Omega_\varepsilon)^3} &\leq C\|v\|_{H_0^1(Q_\varepsilon)^3}, \\ \|DR_\varepsilon v\|_{L^2(\Omega_\varepsilon)^{3\times 3}} &\leq C\frac{1}{\varepsilon}\|v\|_{H_0^1(Q_\varepsilon)^3}. \end{aligned}$$

Proof. We argue similarly to the proof of Lemma 4.5 in [2] with $p = 2$. □

Now, using the restricted operator R_ε defined in a thin domain, we define the restricted operator from $H_0^1(\Omega)^3$ into $H_0^1(\widetilde{\Omega}_\varepsilon)^3$ and give some properties.

Lemma 3.6. *Let us define* $\tilde{R}_\varepsilon \tilde{v} = R_\varepsilon v$ *for any* $\tilde{v} \in H_0^1(\Omega)^3$, *where* $\tilde{v}(x', y_3) = v(x', \varepsilon y_3)$ *and* R_ε *is defined in Lemma* 3.5*. Then, there exists a constant* C*, independent of* $\tilde{v}$ *and* ε*, such that*

1. $\tilde{R}_\varepsilon \tilde{v} = \tilde{v}$, *if* $\tilde{v} \in H_0^1(\widetilde{\Omega}_\varepsilon)^3$ *(elements of* $H_0^1(\widetilde{\Omega}_\varepsilon)^3$ *are continuated by* 0 *to* Ω*)*
2. $\operatorname{div}_\varepsilon(\tilde{R}_\varepsilon \tilde{v}) = 0$ *in* $\widetilde{\Omega}_\varepsilon$, *if* $\operatorname{div}_\varepsilon \tilde{v} = 0$ *in* Ω
3. *For any* $\tilde{v} \in H_0^1(\Omega)^3$ *(the constant* C *is independent of* $\tilde{v}$ *and* ε*),*
   i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, *or* $a_\varepsilon \ll \varepsilon$, *then*

$$\begin{aligned} \left\|\tilde{R}_\varepsilon \tilde{v}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} &\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3}, \\ \left\|D_\varepsilon \tilde{R}_\varepsilon \tilde{v}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} &\leq C\frac{1}{a_\varepsilon}\|\tilde{v}\|_{H_0^1(\Omega)^3}. \end{aligned}$$

   ii) *if* $a_\varepsilon \gg \varepsilon$, *then*

$$\begin{aligned} \left\|\tilde{R}_\varepsilon \tilde{v}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} &\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3}, \\ \left\|D_\varepsilon \tilde{R}_\varepsilon \tilde{v}\right\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} &\leq C\frac{1}{\varepsilon}\|\tilde{v}\|_{H_0^1(\Omega)^3}. \end{aligned}$$

Proof. Considering the change of variables given in (2.2) and the estimates given in Lemma 3.5, we obtain the desired result. □

Lemma 3.7. *There exists a constant* C *independent of* ε*, such that the extension* $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ *of the solution* $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ *of* (2.3) *satisfies*

i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, *or* $a_\varepsilon \ll \varepsilon$,

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)^3} \leq Ca_\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C, \quad \left\|\tilde{P}_\varepsilon\right\|_{L_0^2(\Omega)} \leq C\frac{1}{a_\varepsilon}, \tag{3.11}$$

ii) *if* $a_\varepsilon \gg \varepsilon$,

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C, \quad \left\|\tilde{P}_\varepsilon\right\|_{L_0^2(\Omega)} \leq C\frac{1}{\varepsilon}. \tag{3.12}$$

Proof. Taking into account Lemma 3.4, it is clear that, after extension, the two first estimates in (3.11) and (3.12) hold.

The mapping R_ε defined in Lemma 3.5 allows us to extend the pressure p_ε to Q_ε introducing F_ε in $H^{-1}(Q_\varepsilon)^3$:

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = \langle \nabla p_\varepsilon, R_\varepsilon(\varphi)\rangle_{\Omega_\varepsilon}\,, \quad \text{for any } \varphi \in H_0^1(Q_\varepsilon)^3\,. \tag{3.13}$$

We calcule the right hand side of (3.13) by using (1.3) and we have

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = -\mu \int_{\Omega_\varepsilon} Du_\varepsilon : DR_\varepsilon(\varphi)\, dx + \langle f_\varepsilon, R_\varepsilon(\varphi)\rangle_{\Omega_\varepsilon} - \int_{\Omega_\varepsilon} (u_\varepsilon\cdot\nabla)u_\varepsilon\, R_\varepsilon(\varphi)\, dx\,. \tag{3.14}$$

Moreover, $\operatorname{div}\varphi = 0$ implies

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = 0\,,$$

and the DeRham theorem gives the existence of P_ε in $L_0^2(Q_\varepsilon)$ with $F_\varepsilon = \nabla P_\varepsilon$.

We get for any $\tilde\varphi \in W_0^{1,p}(\Omega)^3$, using the change of variables (2.2),

$$\begin{aligned}\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde\varphi\rangle_\Omega &= -\int_\Omega \tilde P_\varepsilon \operatorname{div}_\varepsilon \tilde\varphi\, dx' dy_3\\ &= -\varepsilon^{-1}\int_{Q_\varepsilon} P_\varepsilon \operatorname{div}\varphi\, dx = \varepsilon^{-1}\langle \nabla P_\varepsilon, \varphi\rangle_{Q_\varepsilon}\,.\end{aligned}$$

Then, using the identification (3.14) of F_ε,

$$\begin{aligned}\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde\varphi\rangle_\Omega &= \varepsilon^{-1}\Big(-\mu\int_{\Omega_\varepsilon} Du_\varepsilon : DR_\varepsilon(\varphi)\, dx + \langle f_\varepsilon, R_\varepsilon(\varphi)\rangle_{\Omega_\varepsilon}\\ &\qquad -\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon R_\varepsilon(\varphi)\, dx\Big)\,,\end{aligned}$$

and applying the change of variables (2.2),

$$\begin{aligned}\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde\varphi\rangle_\Omega = \ & -\mu\int_{\tilde\Omega_\varepsilon} D_\varepsilon \tilde u_\varepsilon : D_\varepsilon \tilde R_\varepsilon(\tilde\varphi)\, dx' dy_3 + \langle \tilde f_\varepsilon, \tilde R_\varepsilon(\tilde\varphi)\rangle_{\tilde\Omega_\varepsilon}\\ & -\int_{\tilde\Omega_\varepsilon}(\tilde u_\varepsilon\cdot\nabla_\varepsilon)\tilde u_\varepsilon \tilde R_\varepsilon(\tilde\varphi)\, dx' dy_3\,,\end{aligned} \tag{3.15}$$

where $\tilde R_\varepsilon$ is given in Lemma 3.6.

Now, we estimate the right-hand side of (3.15). First, we consider $a_\varepsilon \approx \varepsilon$ or $a_\varepsilon \ll \varepsilon$.

Using (3.7) and Lemma 3.6, we get

$$\begin{aligned}\left|\mu\int_{\tilde\Omega_\varepsilon} D_\varepsilon \tilde u_\varepsilon : D_\varepsilon \tilde R_\varepsilon(\tilde\varphi)\, dx' dy_3\right| \leq \ & C\|D_\varepsilon \tilde R_\varepsilon(\tilde\varphi)\|_{L^2(\tilde\Omega_\varepsilon)^{3\times 3}}\\ & \leq C\frac{1}{a_\varepsilon}\|\tilde\varphi\|_{H_0^1(\Omega)^3}\,,\end{aligned} \tag{3.16}$$

and by (3.10) and Lemma 3.6, we have

$$\begin{aligned}\left|\langle \tilde f_\varepsilon, \tilde R_\varepsilon(\tilde\varphi)\rangle_{\tilde\Omega_\varepsilon}\right| \leq \ & C\|D\tilde R_\varepsilon(\tilde\varphi)\|_{L^2(\tilde\Omega_\varepsilon)^{3\times 3}}\\ \leq \ & C\|D_\varepsilon \tilde R_\varepsilon(\tilde\varphi)\|_{L^2(\tilde\Omega_\varepsilon)^{3\times 3}} \leq C\frac{1}{a_\varepsilon}\|\tilde\varphi\|_{H_0^1(\Omega)^3}\,.\end{aligned} \tag{3.17}$$

The inertial term can be written by

$$\int_{\widetilde{\Omega}_\varepsilon}(\tilde{u}_\varepsilon\cdot\nabla_\varepsilon)\tilde{u}_\varepsilon\,\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3=-\int_{\widetilde{\Omega}_\varepsilon}\tilde{u}_\varepsilon\tilde{\otimes}\tilde{u}_\varepsilon:D_{x'}\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3 \tag{3.18}$$
$$+\frac{1}{\varepsilon}\left(\int_{\widetilde{\Omega}_\varepsilon}\partial_{y_3}\tilde{u}_{\varepsilon,3}\tilde{u}_\varepsilon\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3+\int_{\widetilde{\Omega}_\varepsilon}\tilde{u}_{\varepsilon,3}\partial_{y_3}\tilde{u}_\varepsilon\,\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3\right),$$

where $(u\tilde{\otimes}w)_{ij}=u_iw_j$, $i=1,2$, $j=1,2,3$.

We consider separately the two terms in the right-hand side of (3.18):

(i) The estimate of the first part of the right-hand side of (3.18) has the form

$$\|\tilde{u}_\varepsilon\|^2_{L^4(\widetilde{\Omega}_\varepsilon)^3}\|D_{x'}\tilde{R}_\varepsilon(\tilde{\varphi})\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 2}}\,.$$

We introduce the interpolation parameter $\theta=\frac{1}{4}$, where $\frac{1}{4}=\frac{\theta}{2}+\frac{1-\theta}{6}$, and we have the interpolation

$$\|\tilde{u}_\varepsilon\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\le\|\tilde{u}_\varepsilon\|^\theta_{L^2(\widetilde{\Omega}_\varepsilon)^3}\|\tilde{u}_\varepsilon\|^{1-\theta}_{L^6(\widetilde{\Omega}_\varepsilon)^3}\,,$$

and by the the Sobolev embedding, $H^1_0\hookrightarrow L^6$, and the estimate (3.7), we obtain

$$\|\tilde{u}_\varepsilon\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\le\quad\|\tilde{u}_\varepsilon\|^\theta_{L^2(\widetilde{\Omega}_\varepsilon)^3}\|D\tilde{u}_\varepsilon\|^{1-\theta}_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\le C\,a_\varepsilon^{\frac{1}{4}}.$$

Then by Lemma 3.6,

$$\left|\int_{\widetilde{\Omega}_\varepsilon}\tilde{u}_\varepsilon\tilde{\otimes}\tilde{u}_\varepsilon:D_{x'}\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3\right|\le C\,a_\varepsilon^{-\frac{1}{2}}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}\,.$$

(ii) The estimate of the second part of the right-hand side of (3.18) has the form

$$\frac{C}{\varepsilon}\|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\|\tilde{u}_\varepsilon\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\|\tilde{R}_\varepsilon(\tilde{\varphi})\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\,.$$

Working as in item (i) and using Lemma 3.6, we have

$$\|\tilde{u}_\varepsilon\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\le C\,a_\varepsilon^{\frac{1}{4}}\,,$$

and

$$\|\tilde{R}_\varepsilon(\tilde{\varphi})\|_{L^4(\widetilde{\Omega}_\varepsilon)^3}\le\|\tilde{R}_\varepsilon(\tilde{\varphi})\|^\theta_{L^2(\widetilde{\Omega}_\varepsilon)^3}\|D\tilde{R}_\varepsilon(\tilde{\varphi})\|^{1-\theta}_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\le Ca_\varepsilon^{-\frac{3}{4}}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3},$$

and by estimate (3.7), we get

$$\frac{1}{\varepsilon}\left|\int_{\widetilde{\Omega}_\varepsilon}\left(\partial_{y_3}\tilde{u}_{\varepsilon,3}\tilde{u}_\varepsilon\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3+\int_{\widetilde{\Omega}_\varepsilon}\tilde{u}_{\varepsilon,3}\partial_{y_3}\tilde{u}_\varepsilon\,\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3\right)\right|$$
$$\le C\,a_\varepsilon^{-\frac{1}{2}}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}\,.$$

Then, from (3.18) we can deduce that

$$\left|\int_{\widetilde{\Omega}_\varepsilon}(\tilde{u}_\varepsilon\cdot\nabla_\varepsilon)\tilde{u}_\varepsilon\,\tilde{R}_\varepsilon(\tilde{\varphi})\,dx'dy_3\right|\le C\,a_\varepsilon^{-\frac{1}{2}}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}\,,$$

Taking into account the previous estimate, (3.16) and (3.17) in (3.15), we have

$$|\langle\nabla_\varepsilon\tilde{P}_\varepsilon,\tilde{\varphi}\rangle_\Omega|\le C\frac{1}{a_\varepsilon}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}\,,$$

and so we have the third estimate in (3.11)

Finally, if we argue similarly for the case $a_\varepsilon \gg \varepsilon$, we obtain the third estimate in (3.12). □

Adaptation of the Unfolding method: let us introduce the adaption of the unfolding method in which we divide the domain Ω in hexagonal cylinders or parallelepipeds of base of size $a_\varepsilon\sqrt{3}/2$ and vertical length 1. For this purpose, given the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$, $\tilde{u}_\varepsilon$ also extended by zero outside of Ω, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by

$$\hat{u}_\varepsilon(x', y) = \tilde{u}_\varepsilon\left(a_\varepsilon\kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e. } (x', y) \in \omega \times Y, \tag{3.19}$$

$$\hat{P}_\varepsilon(x', y) = \tilde{P}_\varepsilon\left(a_\varepsilon\kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \text{a.e. } (x', y) \in \omega \times Y, \tag{3.20}$$

where the function κ is defined in (3.1).

Remark 3.8. For $\ell' \in T_\varepsilon$, the restriction of $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ to $Y'_{k',a_\varepsilon} \times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ by using the change of variables

$$y' = \frac{x' - a_\varepsilon k'}{a_\varepsilon},$$

which transforms Y_{k',a_ε} into Y.

Let us obtain some *a priori* estimates for the sequences $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$.

Lemma 3.9. *There exists a constant C independent of ε, such that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (3.19)-(3.20) satisfies*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \ll \varepsilon$,*

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 2}} \leq Ca_\varepsilon, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon,$$

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\, a_\varepsilon, \qquad \left\|\hat{P}_\varepsilon\right\|_{L_0^2(\omega\times Y)} \leq C\,\frac{1}{a_\varepsilon},$$

ii) *if $a_\varepsilon \gg \varepsilon$,*

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 2}} \leq Ca_\varepsilon, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon,$$

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\,\varepsilon, \quad \left\|\hat{P}_\varepsilon\right\|_{L_0^2(\omega\times Y)} \leq C\,\frac{1}{\varepsilon}.$$

Proof. Using Lemma 3.7 and reasoning as in the proof of Lemma 4.9 in [2], we have the desired result.

□

Some compactness results: in this section we obtain some compactness results about the behavior of the sequences $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ and $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ satisfying *a priori* estimates given in Lemma 3.7 and Lemma 3.9 respectively. If we argue similarly to Section 5 in [2], we obtain the following results.

Lemma 3.10. *For a subsequence of ε still denoted by ε, there exists $\tilde{P} \in L^2_0(\Omega)$, independent of y_3, such that*

i) *if $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, or $a_\varepsilon \ll \varepsilon$, then*

$$a_\varepsilon \tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ in } L^2_0(\Omega),$$

ii) *if $a_\varepsilon \gg \varepsilon$, then*

$$\varepsilon \tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ in } L^2_0(\Omega).$$

Lemma 3.11. *For a subsequence of ε still denoted by ε,*

i) *if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then there exists $\tilde{u} \in H^1(0,1;L^2(\omega)^3)$ where $\tilde{u}_3 = 0$, and $\tilde{u} = 0$ on $y_3 = 0,1$, such that*

$$\frac{\tilde{u}_\varepsilon}{a_\varepsilon} \rightharpoonup (\tilde{u}', 0) \text{ in } H^1(0,1;L^2(\omega)^3),$$

ii) *if $a_\varepsilon \ll \varepsilon$, then there exists $\tilde{u} \in L^2(\Omega)^3$ where $\tilde{u}_3 = 0$, and $\tilde{u} = 0$ on $y_3 = 0,1$, such that*

$$\frac{\tilde{u}_\varepsilon}{a_\varepsilon} \rightharpoonup (\tilde{u}', 0) \text{ in } L^2(\Omega)^3,$$

iii) *if $a_\varepsilon \gg \varepsilon$, then there exists $\tilde{u} \in H^1(0,1;L^2(\omega)^3)$ where $\tilde{u}_3 = 0$, and $\tilde{u} = 0$ on $y_3 = 0,1$, such that*

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup (\tilde{u}', 0) \text{ in } H^1(0,1;L^2(\omega)^3).$$

Moreover, in every case

$$\mathrm{div}_{x'}\left(\int_0^1 \tilde{u}'(x', y_3)dy_3\right) = 0 \text{ in } \omega, \quad \left(\int_0^1 \tilde{u}'(x', y_3)dy_3\right)\cdot n = 0 \text{ on } \partial\omega.$$

Lemma 3.12. *For a subsequence of ε still denoted by ε,*

i) *if $a_\varepsilon \approx \varepsilon$ with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, then there exists $\hat{u} \in L^2(\omega; H^1_\sharp(Y)^3)$, with $\hat{u} = 0$ on $\omega \times Y_s$ and on $y_3 = 0,1$, such that*

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon} \rightharpoonup \hat{u} \text{ in } L^2(\omega; H^1(Y)^3),$$

$$\mathrm{div}_\lambda \hat{u} = 0 \text{ in } \omega \times Y,$$

where $\mathrm{div}_\lambda = \mathrm{div}_{y'} + \lambda\partial_{y_3}$,

ii) *if $a_\varepsilon \ll \varepsilon$, then there exists $\hat{u} \in L^2(\Omega; H^1_\sharp(Y')^3)$, with $\hat{u} = 0$ on $\omega \times Y_s$ and on $y_3 = 0,1$, such that*

$$\frac{\hat{u}_\varepsilon}{a_\varepsilon} \rightharpoonup \hat{u} \text{ in } L^2(\Omega; H^1(Y')^3),$$

$$\mathrm{div}_{y'} \hat{u}' = 0 \text{ in } \omega \times Y,$$

iii) *if* $a_\varepsilon \gg \varepsilon$, *then there exists* $\hat{u} \in H^1(0,1; L^2_\sharp(\omega \times Y')^3)$, *with* $\hat{u} = 0$ *on* $\omega \times Y_s$ *and on* $y_3 = 0, 1$, *such that*

$$\frac{\hat{u}_\varepsilon}{\varepsilon} \rightharpoonup (\hat{u}', 0) \text{ in } H^1(0,1; L^2(\omega \times Y')^3),$$

$$\mathrm{div}_{y'} \hat{u}' = 0 \text{ in } \omega \times Y.$$

Moreover, in every case

$$\int_0^1 \tilde{u}(x', y_3)\, dy_3 = \frac{1}{|Y'|} \int_Y \hat{u}(x', y)\, dy,$$

and

$$\mathrm{div}_{x'} \left(\int_Y \hat{u}'(x', y) dy \right) = 0 \text{ in } \omega, \quad \left(\int_Y \hat{u}'(x', y) dy \right) \cdot n = 0 \text{ on } \partial\omega.$$

Lemma 3.13. *For a subsequence of* ε *still denoted by* ε, *there exists* $\hat{P} \in L^2_0(\omega \times Y)$ *such that*

i) *if* $a_\varepsilon \approx \varepsilon$, *with* $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$, *or* $a_\varepsilon \ll \varepsilon$, *then*

$$a_\varepsilon \hat{P}_\varepsilon \rightharpoonup \hat{P} \text{ in } L^2_0(\omega \times Y), \tag{3.21}$$

ii) *if* $a_\varepsilon \gg \varepsilon$, *then*

$$\varepsilon \hat{P}_\varepsilon \rightharpoonup \hat{P} \text{ in } L^2_0(\omega \times Y). \tag{3.22}$$

Proof of Theorem 2.1. We will multiply system (2.3) by a test function having the form of the limit $\hat{u}$ (as explicated in Lemma 3.12), and we will use the convergences given in the previous section in order to identify the effective model in every case.

First of all, we choose a test function $v(x', y) \in \mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$ with $v(x', y) = 0 \in \omega \times Y_s$ (thus, $v(x', x'/a_\varepsilon, y_3) \in H^1_0(\widetilde{\Omega}_\varepsilon)^3$). Multiplying (2.3) by $v(x', x'/a_\varepsilon, y_3)$, integrating by parts, and taking into account that reasoning as in the proof of Lemma 3.7 for the inertial term we get

$$\left| \int_{\widetilde{\Omega}_\varepsilon} (\tilde{u}_\varepsilon \cdot \nabla_\varepsilon) \tilde{u}_\varepsilon\, v\, dx' dy_3 \right| \leq C\, a_\varepsilon^{\frac{1}{4}} \|v\|_{H^1_0(\widetilde{\Omega}_\varepsilon)^3},$$

then we have

$$\begin{aligned} &\mu \int_\Omega D_{x'} \tilde{u}_\varepsilon : D_{x'} v\, dx' dy_3 + \frac{\mu}{a_\varepsilon} \int_\Omega D_{x'} \tilde{u}_\varepsilon : D_{y'} v\, dx' dy_3 \\ &+ \frac{\mu}{\varepsilon^2} \int_\Omega \partial_{y_3} \tilde{u}_\varepsilon : \partial_{y_3} v\, dx' dy_3 - \int_\Omega \tilde{P}_\varepsilon\, \mathrm{div}_{x'} v'\, dx' dy_3 \\ &- \frac{1}{a_\varepsilon} \int_\Omega \tilde{P}_\varepsilon\, \mathrm{div}_{y'} v'\, dx' dy_3 - \frac{1}{\varepsilon} \int_\Omega \tilde{P}_\varepsilon\, \partial_{y_3} v_3\, dx' dy_3 = \langle \tilde{f}_\varepsilon, v \rangle_{\widetilde{\Omega}_\varepsilon} + O_\varepsilon. \end{aligned} \tag{3.23}$$

We analyze the second member of (3.23). Taking into account Remark 3.2, we observe that

$$\begin{aligned}\langle \tilde f_\varepsilon, v\rangle_{\widetilde\Omega_\varepsilon} = &\int_{\widetilde\Omega_\varepsilon} \tilde f^0_\varepsilon\cdot v'\,dx'dy_3 + \int_{\widetilde\Omega_\varepsilon}\tilde G'_\varepsilon : D_{x'}v'\,dx'dy_3\\ &+\frac{1}{\varepsilon}\int_{\widetilde\Omega_\varepsilon}\tilde G_{\varepsilon,3}\cdot\partial_{y_3}v'\,dx'dy_3 + \frac{1}{a_\varepsilon}\int_{\widetilde\Omega_\varepsilon}\tilde G'_\varepsilon : D_{y'}v'\,dx'dy_3,\end{aligned}\tag{3.24}$$

where $\tilde G'_\varepsilon = (\tilde f^1_\varepsilon, \tilde f^2_\varepsilon)\in L^2(\Omega)^{2\times 2}$ and $\tilde G_{\varepsilon,3} = \tilde f^3_\varepsilon \in L^2(\Omega)^2$.

We define $\hat f^i_\varepsilon$, $0\le i\le 3$, by

$$\hat f^i_\varepsilon(x',y) = \tilde f^i_\varepsilon\left(a_\varepsilon\kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right),\quad \text{a.e. } (x',y)\in\omega\times Y,$$

where the function κ is defined in (3.1). Reasoning as in Lemma 3.9 and taking into account (3.5), there exists a constant C independent of ε, such that

$$\|\hat f^i_\varepsilon\|_{L^2(\omega\times Y)^2}\le C,\qquad 0\le i\le 3,$$

and then for a subsequence of ε still denoted by ε, there exists $\hat f^i\in L^2_\sharp(\omega\times Y)^2$, $0\le i\le 3$, such that

$$\hat f^i_\varepsilon \rightharpoonup \hat f^i \text{ in } L^2(\omega\times Y)^2,\qquad 0\le i\le 3.\tag{3.25}$$

Now, by the change of variables given in Remark 3.8, from (3.23) and (3.24), we obtain

$$\begin{aligned}&\frac{\mu}{a_\varepsilon^2}\int_{\omega\times Y} D_{y'}\hat u'_\varepsilon : D_{y'}v'\,dx'dy + \frac{\mu}{\varepsilon^2}\int_{\omega\times Y}\partial_{y_3}\hat u'_\varepsilon : \partial_{y_3}v'\,dx'dy\\ &-\int_{\omega\times Y}\hat P_\varepsilon\,\mathrm{div}_{x'}v'\,dx'dy - \frac{1}{a_\varepsilon}\int_{\omega\times Y}\hat P_\varepsilon\,\mathrm{div}_{y'}v'\,dx'dy\\ &=\int_{\omega\times Y}\hat f^0_\varepsilon\cdot v'\,dx'dy + \int_{\omega\times Y}\hat G'_\varepsilon : D_{x'}v'\,dx'dy\\ &+\frac{1}{\varepsilon}\int_{\omega\times Y}\hat G_{\varepsilon,3} : \partial_{y_3}v'\,dx'dy + \frac{1}{a_\varepsilon}\int_{\omega\times Y}\hat G'_\varepsilon : D_{y'}v'\,dx'dy + O_\varepsilon,\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}&\frac{\mu}{a_\varepsilon^2}\int_{\omega\times Y}\nabla_{y'}\hat u_{\varepsilon,3}\cdot\nabla_{y'}v_3\,dx'dy + \frac{\mu}{\varepsilon^2}\int_{\omega\times Y}\partial_{y_3}\hat u_{\varepsilon,3}\cdot\partial_{y_3}v_3\,dx'dy\\ &-\frac{1}{\varepsilon}\int_{\omega\times Y}\hat P_\varepsilon\,\partial_{y_3}v_3\,dx'dy + O_\varepsilon = 0,\end{aligned}\tag{3.27}$$

where $\hat G'_\varepsilon = (\hat f^1_\varepsilon, \hat f^2_\varepsilon)\in L^2(\omega\times Y)^{2\times 2}$, $\hat G_{\varepsilon,3} = \hat f^3_\varepsilon\in L^2(\omega\times Y)^2$.

This variational formulation will be useful in the following steps. We proceed similarly to the proof of Theorem 6.1 in [2], so we will only give some details.

Step 1. Critical case $a_\varepsilon\approx\varepsilon$, with $a_\varepsilon/\varepsilon\to\lambda$, $0<\lambda<+\infty$. We choose a test function $v_\varepsilon = (a_\varepsilon v', \lambda\varepsilon v_3)$ in (3.26)-(3.27), passing to the limit using the convergence (3.25), Lemma 3.12-i) and the convergence (3.21), we obtain

$$
\begin{aligned}
&\mu \int_{\omega\times Y_f} D_{y'}\hat{u} : D_{y'}v\,dx'dy + \mu\,\lambda \int_{\omega\times Y_f} \partial_{y_3}\hat{u} : \partial_{y_3}v\,dx'dy \\
&= \int_{\omega\times Y_f} \hat{G}' : D_{y'}v'\,dx'dy + \lambda \int_{\omega\times Y_f} \hat{G}_3 : \partial_{y_3}v'\,dx'dy\,,
\end{aligned}
$$

where $\hat{G}' = (\hat{f}^1, \hat{f}^2) \in L^2_\sharp(\omega\times Y)^{2\times 2}$ and $\hat{G}_3 = \hat{f}^3 \in L^2_\sharp(\omega\times Y)^2$. The previous variational formulation is equivalent to the effective system

$$
\begin{cases}
-\mu\Delta_\lambda\hat{u} + \nabla_\lambda\hat{q} = -\mathrm{div}_\lambda\hat{G} - \nabla_{x'}\tilde{P} \ \text{ in } \ \omega\times Y_f, \\
\mathrm{div}_\lambda\hat{u} = 0 \ \text{ in } \ \omega\times Y_f, \\
\hat{u} = 0 \ \text{ in } \ \omega\times Y_s, \quad \hat{u} = 0 \ \text{ on } y_3 = 0, 1, \\
\mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x', y)dy\right) = 0 \text{ in } \ \omega, \\
\left(\displaystyle\int_Y \hat{u}'(x', y)dy\right)\cdot n = 0 \text{ on } \ \partial\omega, \\
y' \to \hat{u}(x', y), \hat{q}(x', y) \quad Y' - \text{periodic},
\end{cases}
\tag{3.28}
$$

where $\Delta_\lambda = \Delta_{y'} + \lambda\partial^2_{y_3}$, $\nabla_\lambda = \nabla_{y'} + \sqrt{\lambda}\partial_{y_3}$ and $\mathrm{div}_\lambda = \mathrm{div}_{y'} + \lambda\partial_{y_3}$. Observe that $\mathrm{div}_\lambda\hat{G} \in L^2_\sharp(\omega\times Y)^2$ given by its entries $(\mathrm{div}_\lambda\hat{G})_k = \sum_{\ell=1}^2 \partial_{y_\ell}\hat{f}^\ell_k + \lambda\partial_{y_3}\hat{f}^3_k$, $k = 1, 2$.

Step 2. Subcritical case $a_\varepsilon \ll \varepsilon$. We choose a test function of the form $v_\varepsilon = (a_\varepsilon v', a_\varepsilon\varepsilon v_3)$ in (3.26)-(3.27), passing to the limit using the convergence (3.25), Lemma 3.12-ii) and the convergence (3.21), we obtain

$$
\mu \int_{\omega\times Y_f} D_{y'}\hat{u}' : D_{y'}v' = \int_{\omega\times Y_f} \hat{G}' : D_{y'}v'\,dx'dy\,.
$$

The previous variational formulation is equivalent to the effective system

$$
\begin{cases}
-\mu\Delta_{y'}\hat{u}' + \nabla_{y'}\hat{q} = -\mathrm{div}_{y'}\hat{G}' - \nabla_{x'}\tilde{P} \quad \text{ in } \ \omega\times Y_f, \\
\mathrm{div}_{y'}\hat{u}' = 0 \quad \text{ in } \ \omega\times Y_f, \\
\hat{u}' = 0 \text{ in } \ \omega\times Y_s, \quad \hat{u} = 0 \text{ on } y_3 = 0, 1, \\
\mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x', y)dy\right) = 0 \text{ in } \ \omega, \\
\left(\displaystyle\int_Y \hat{u}'(x', y)dy\right)\cdot n = 0 \text{ on } \ \partial\omega, \\
y' \to \hat{u}'(x', y), \hat{q}(x', y) \quad Y' - \text{periodic}.
\end{cases}
\tag{3.29}
$$

Observe that $\mathrm{div}_{y'}\hat{G}' \in L^2_\sharp(\omega\times Y)^2$ given by its entries $(\mathrm{div}_{y'}\hat{G}')_k = \sum_{\ell=1}^2 \partial_{y_\ell}\hat{f}^\ell_k$, $k = 1, 2$.

Step 3. Supercritical case $a_\varepsilon \gg \varepsilon$. We choose a test function $v_\varepsilon = (\varepsilon v', 0)$ in (3.26)-(3.27), passing to the limit using the convergence (3.25), Lemma

3.12-iii) and the convergence (3.22), we obtain

$$\mu \int_{\omega\times Y} \partial_{y_3}\hat{u}' : \partial_{y_3}v'dx'dy = \int_{\omega\times Y_f} \hat{G}_3 : \partial_{y_3}v'\,dx'dy\,.$$

This variational formulation is equivalent to the effective system

$$\begin{cases} -\mu\partial_{y_3}^2\hat{u}' + \nabla_{y'}\hat{q} = -\nabla_{x'}\tilde{P} - \partial_{y_3}\hat{G}_3 \quad \text{in } \omega\times Y_f, \\ \mathrm{div}_{y'}\hat{u}' = 0 \quad \text{in } \omega\times Y_f, \quad \hat{u} = 0 \text{ on } y_3 = 0,1, \\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x',y)dy\right) = 0 \text{ in } \omega, \\ \left(\displaystyle\int_Y \hat{u}'(x',y)dy\right)\cdot n = 0 \text{ on } \partial\omega, \\ y' \to \hat{u}'(x',y), \hat{q}(x',y') \quad Y' - \text{periodic}. \end{cases} \tag{3.30}$$

Observe that $\partial_{y_3}\hat{G}_3 \in L^2_\sharp(\omega\times Y)^2$ given by its entries $(\partial_{y_3}\hat{G}_3)_k = \hat{f}^3_k$, $k = 1,2$.

Step 4. In this final step, we will eliminate the microscopic variable y in the effective problem. The derivation of (2.4), (2.5) and (2.6) from the effective problems (3.28), (3.29) and (3.30) respectively, is an easy algebra exercise. Let us point that problems (2.4), (2.5) and (2.6) are well-posed problems since they are simply second order elliptic equations for the pressure $\tilde{P}$ (with Neumann boundary condition).

As is well-known, the local problems are also well-posed with periodic boundary condition, and it is easily checked, by integration by parts, that

$$A^\lambda_{ij} = \int_{Y_f} D_\lambda w^i(y) : D_\lambda w^j(y)\,dy = \int_{Y_f} w^i(y)e_j dy, \quad i = 1,2,\ j = 1,2,3.$$

Observe that condition $\int_Y w^i_3\,dy = 0$, $i = 1,2$, implies that $A^\lambda_{i3} = 0$. Then $A^\lambda \in \mathbb{R}^{2\times 2}$ and the definition implies that A^λ is symmetric and positive definite. Remark that (3.30) can be expressed by means of local problems of the same type for functions w^i and w. These problems can be solved as the classical derivation of the Reynolds equation, and w^i and w can be explicitly obtained by means of local problems for q^i and q, respectively. As consequence A^∞ and b^∞ are given, for $i,j = 1,2$, by

$$A^\infty_{ij} = \frac{1}{|Y'_f|}\int_{Y'_f} (e^i + \nabla_{y'}q^i)e_j\,dy', \quad b^\infty_i(x') = \frac{1}{|Y'_f|}\int_{Y'_f} (G_3 + \nabla_{y'}q)\,dy',$$

where $G_3(x',y') = 6\int_0^1 \hat{G}_3(x',y)dy_3 - 12\int_0^1 \left(\int_0^{y_3} \hat{G}_3(x',y',s)ds\right)dy_3$ and $\hat{G}_3$ is defined in *Step 1.* □

References

[1] Allaire, G.: Homogenization and two-scale convergence. SIAM J. Math. Anal. **23**, 1482-1518 (1992)

[2] Anguiano, M., Suárez-Grau, F.J.: Homogenization of an incompressible non-Newtonian flow through a thin porous medium. ZAMP-Journal of Applied Mathematics and Physics. **68:45** (2017)

[3] Arbogast,T., Douglas J.R., J., Hornung,U.: Derivation of the double porosity model of single phase flow via homogenization theory. SIAM J. Math. Anal. **21**, 823-836 (1990)

[4] Cioranescu, D., Damlamian, A., Griso, G.: Periodic unfolding and homogenization. C.R. Acad. Sci. Paris Ser. I. **335**, 99-104 (2002)

[5] Fabricius, J., Hellström, J.G.I., Lundström, T.S., Miroshnikova, E., Wall, P.: Darcy's Law for Flow in a Periodic Thin Porous Medium Confined Between Two Parallel Plates. Transp. Porous Med. **115**, 473-493 (2016)

[6] Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris (1969)

[7] Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, 608-623 (1989)

[8] Snyder, W.A., Qi, H., Sander, W.: Coordinate system for hexagonal pixels. Proc. SPIE 3661, Medical Imaging 1999: Image Processing. (1999)

[9] Tartar, L.: Incompressible fluid flow in a porous medium convergence of the homogenization process. Appendix to Lecture Notes in Physics, 127. Springer-Velag, Berlin (1980)

[10] Temam, R.: Navier-Stokes equations. North Holland, Amsterdam (1977).

[11] Zhengan, Y., Hongxing, Z.: Homogenization of the Navier-Stokes flow in porous medium with thin film. Acta Mathematica Scientia. **28**, 863-974 (2008)

María Anguiano
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
P. O. Box 1160, 41080-Sevilla
Spain
e-mail: `anguiano@us.es`

Francisco Javier Suárez-Grau
Departamento de Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas.
Universidad de Sevilla
41012-Sevilla
Spain
e-mail: `fjsgrau@us.es`