Existence and estimation of the Hausdorff dimension of attractors for an epidemic model

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Abstract

We prove the existence of pullback and uniform attractors for the process associated to a non-autonomous SIR model, with several types of non-autonomous features. The Hausdorff dimension of the pullback attractor is also estimated. We illustrate some examples of pullback attractors by numerical simulations.

Keywords: SIR epidemic model, Non-autonomous equation, Attractor, Hausdorff dimension
Mathematics Subject Classifications (2010): 35B40, 37B55, 37N25, 92D30

1 Introduction and setting of the problem

Over the past one hundred years, mathematics have been used to understand and predict the spread of diseases. Almost all mathematical models of diseases start from the same basic premise: that the population can be subdivided into a set of distinct classes, dependent upon their experience with respect to the disease. One line of investigation classifies individuals as one of susceptible, infectious or recovered. Such a model is termed as an SIR model.

The first SIR model for the transmission of infectious diseases, which was introduced by Kermack and McKendrick [1] in 1927, is one of the fundamental models of mathematical epidemiology [2, 3, 4]. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptibles S, infectives I and recovereds R, of a constant total population.

Many generalizations of this model have been proposed and studied, for instance, to include age structure, time delays, spatial diffusion and variable infectivity; see, for example, [3, 5, 6]. More recently, non-autonomous versions of the SIR model and related epidemic systems for which the total population may vary in time have been investigated [7, 8]. These non-autonomous versions of the SIR model have attracted attention due to the seasonal variations in many diseases. For example, for the spread of infectus childhood diseases Thieme [8] cites arguments that the school system induces a time-hererogeneity in the per capita/capita infection rate, because the chain of infections is interrupted or at least weakened by the vacations and new individuals are recruited into a scene with higher infection risk at the beginning of each school year. This requires the inclusion of time variable coefficients or forcing terms in the models.

Here we study the SIR model introduced in [7]. The model is defined by the following three ordinary differential equations

$$\left.\begin{aligned}\frac{dS}{dt} &= aq(t) - aS + bI - \gamma\tfrac{SI}{N},\\ \frac{dI}{dt} &= -(a+b+c)I + \gamma\tfrac{SI}{N},\\ \frac{dR}{dt} &= cI - aR,\end{aligned}\right\} \tag{1}$$

where

$$N(t) = S(t) + I(t) + R(t),$$

with initial condition

$$S(t_0) = S_0, \quad I(t_0) = I_0, \quad R(t_0) = R_0, \tag{2}$$

where $t_0 \in \mathbb{R}$, the parameters a, b, c and γ are positive constants such that $2a > c$, $a > b + 2\gamma$ and $q : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function such that satisfies $q(t) \geq q^- > 0$ for all $t \in \mathbb{R}$ and

$$\int_{-\infty}^{t} e^{ls} q^2(s) ds < +\infty, \quad \forall t \in \mathbb{R}, \tag{3}$$

where $l := min\{2a - c, a - b - 2\gamma, 2a + b + c - 2\gamma\} > 0$.

There are basically two ways to define attraction of a compact and invariant non–autonomous set for a process on a metric space. The first, and perhaps more obvious, corresponds to the attraction in the sense of Lyapunov stability, which is called *forward attraction*, and involves a moving target, while the second, called *pullback attraction*, involves a fixed target set with progressively earlier starting time. In general, these two types of attraction are independent concepts, while for the autonomous case, they are equivalent. Physically, the pullback attractor provides a way to assess an asymptotic regime at time t (the time at which we observe the system) for a system starting to evolve from the remote past. The pullback dynamic contains interesting dynamical properties, which allow us to understand the forward attraction. The first aim of this paper is to show the existence of a pullback and a uniform attractor for the process associated to (1)-(2). The fact that q is non-autonomous is the main novelty of our problem.

The dynamics induced by the class of periodic, almost periodic or almost automorphic continuous functions is not robust to small changes in the forcing term in the sense that a bounded entire solution corresponding to a perturbed forcing term may not belong to this class. Then, Kloeden and Rodrigues presented in [9] an alternative extension of periodic and almost periodic functions. Namely, they introduce the class of functions consisting of uniformly continuous functions, defined on the real line and taking values in a Banach space, with the property that a bounded entire solution of a non autonomous ODE belongs to this class when the forcing term does. The fact that the forcing term belongs to the class more general than almost periodic in a non-autonomous system means that the external force of the phenomena modeled possesses different types of intensity along time. We also consider that (1) includes a forcing term which belongs to this class of functions introduced by Kloeden and Rodrigues in [9] .

On the other hand, the theory of topological dimension [10], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor. For investigating attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value. Recently in [11] Lyapunov-type functions are introduced into upper estimates for the Hausdorff dimension of negatively invariant sets of cocycles. For this purpose, the methods proposed in [12, 13, 14] are further developed. The second aim of this paper is to estimate the Hausdorff dimension of the pullback attractor of (1)-(2) using the recent method proposed by Leonov *et al.* in [11].

The structure of the paper is as follows. In Section 2 we briefly recall some abstract results about the theory of pullback and uniform attractors. In Section 3 we establish the existence and uniqueness of a positive solution for our model and we prove a continuous dependence result with respect to initial data. Some sufficient conditions ensuring the existence of such type of attractors for (1)-(2) are collected in Section 4. We consider that the differential equations of non-autonomous SIR model are subjected to a periodic forcing term in Section 5. In Section 6 we consider that (1) includes a forcing term which belongs to a class of functions more general than almost periodic. We use recent results proposed by Kloeden and Rodrigues [9] to prove that the solution of (1)-(2) belongs to this class when the forcing term does. Finally, in Section 7 we estimate the Hausdorff dimension of the pullback attractor associated to (1)-(2). We illustrate these results with some numerical simulations.

2 Abstract results on Pullback and Uniform Attractors

In this section we recall some abstract results on the theory of pullback attractors (see, e.g., [15, 16]) and uniform attractors (see [17]).

2.1 Processes and attractors

Let (X, d_X) be a metric space, and let us denote $\mathbb{R}_d^2 = \{(t, t_0) \in \mathbb{R}^2 : t_0 \leq t\}$.

A process on X is a mapping U such that $\mathbb{R}_d^2 \times X \ni (t, t_0, x) \mapsto U(t, t_0)x \in X$ with $U(t_0, t_0)x = x$ for any $(t_0, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, t_0)x) = U(t, t_0)x$ for any $t_0 \leq r \leq t$ and all $x \in X$.

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D} = \{D(t) : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$. Let $\mathcal{D}$ be a nonempty set of parameterized families of nonempty bounded sets $\widehat{D} = \{D(t) = D \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $D \subset X$ is a bounded set. In what follows, we will consider a fixed universe of attraction $\mathcal{D}$ and throughout our analysis the concepts of absorption and attraction will be referred to this fixed universe.

Definition 1 *It is said that $\widehat{D}_0 \subset \mathcal{P}(X)$ is pullback absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\widehat{t}_0(t, \widehat{D}) \leq t$ such that*

$$U(t, t_0)D(t_0) \subset D_0(t) \quad \textit{for all } t_0 \leq \widehat{t}_0(t, \widehat{D}).$$

Denote the omega-limit set of $\widehat{D}$ by

$$\Lambda(\widehat{D}, t) := \bigcap_{s \leq t} \overline{\bigcup_{t_0 \leq s} U(t, t_0)D(t_0)}^X \qquad \text{for all } t \in \mathbb{R}, \tag{4}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Definition 2 *The family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor associated to the continuous process U if is invariant, attracts every $\{D(t)\} \in \mathcal{D}$ and minimal in the sense that if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

The general result on the existence of pullback attractors is the following.

Theorem 3 *[Crauel et al. [18], Schmalfuss [16]] Assume that there exists a family of compact pullback absorbing sets $\{B(t)\}_{t\in\mathbb{R}}$. Then, the family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ defined by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D}, t\right)}^X,$$

is the pullback attractor, where $\Lambda\left(\widehat{D}, t\right)$ is the omega-limit set at time t of $\widehat{D} \in \mathcal{D}$, where $\mathcal{D}$ is the universe of fixed nonempty bounded subsets of X.

Another approach to the asymptotic dynamics of non-autonomous equations, the uniform attractor, has been developed by Chepyzhov and Vishik [17].

Definition 4 *A set $K \subseteq X$ is said to be uniformly (with respect to $t_0 \in \mathbb{R}$) attracting for the process $\{U(t,t_0)\}$ on X if for all $t_0 \in \mathbb{R}$ and for any bounded set $B \subset X$,*

$$\lim_{T \to +\infty} \left(\sup_{t_0 \in \mathbb{R}} \operatorname{dist}_X (U(T+t_0,t_0)B, K) \right) = 0.$$

Respectively, the process $\{U(t,t_0)\}$ is said to be uniformly asymptotically compact (with respect to $t_0 \in \mathbb{R}$) if there exists a compact uniformly (with respect to $t_0 \in \mathbb{R}$) attracting set of $\{U(t,t_0)\}$.

Definition 5 *A closed set $\mathcal{A}_1 \subseteq X$ is said to be a uniform (with respect to $t_0 \in \mathbb{R}$) attractor for a process $\{U(t,t_0)\}$ if it is the minimal closed uniformly (with respect to $t_0 \in \mathbb{R}$) attracting set for this process. Minimality is meant in the sense that any closed attracting set is contained in $\mathcal{A}_1$.*

Theorem 6 *[Chepyzhov and Vishik [17], Haraux [19]] If a process $\{U(t,t_0)\}$ is uniformly asymptotically (with respect to $t_0 \in \mathbb{R}$) compact, then it has the uniform (with respect to $t_0 \in \mathbb{R}$) attractor $\mathcal{A}_1$. The set $\mathcal{A}_1$ is compact in X.*

To describe the structures of uniform attractors and to perform a comparison with the pullback attractor we introduce the notions of complete trajectory of a process, kernel of a process and kernel section.

Definition 7 *A map $u : \mathbb{R} \to X$ is called a complete trajectory of a process $U(t,t_0)$ if*

$$U(t,t_0)u(t_0) = u(t) \quad \textit{for all } t \geq t_0, \quad t, t_0 \in \mathbb{R}.$$

Definition 8 *The kernel $\mathbb{K}$ of a process $U(t,t_0)$ consists of all of its bounded complete trajectories of the process $U(t,t_0)$.*

Definition 9 *The set*

$$\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathbb{K}\}$$

is said to be the kernel section at a time moment $t = s$, $s \in \mathbb{R}$.

These kernel sections are, essentially, the fibres of the pullback attractor.

Lemma 10 *If $U(\cdot,\cdot)$ has a uniform attractor $\mathcal{A}_1$, then*

$$\bigcup_{t \in \mathbb{R}} \mathcal{K}(t) \subseteq \mathcal{A}_1.$$

2.2 The structures of attractors for periodic processes

Let $\{U(t,t_0)\}$ be a periodic process, and let T be its period, i.e.,

$$U(t+T,t_0+T)=U(t,t_0) \ \ \forall t\geq t_0, t_0\in\mathbb{R}.$$

We can now state a theorem about attractors of periodic processes.

Theorem 11 *[Chepyzhov and Vishik [20]] Let $\{U(t,t_0)\}$ be a periodic uniformly (with respect to $t_0\in\mathbb{R}$) asymptotic compact and $(X\times\mathbb{T}^1)-$continuous process, where $\mathbb{T}^1=\mathbb{R}$ (mod T). Then, the process $\{U(t,t_0)\}$ has a uniform (with respect to $t_0\in\mathbb{R}$) attractor, $\mathcal{A}_1$, and is given by*

$$\mathcal{A}_1=\bigcup_{\sigma\in[0,T)}\mathcal{K}(\sigma),$$

where $\mathcal{K}(\sigma)$ is the kernel section of the process $\{U(t,t_0)\}$ at time $t=\sigma$.

3 Existence and uniqueness of solutions

We state and sketch the proof of a result on the existence and uniqueness of positive solutions of (1)–(2) for initial data in $\mathbb{R}^3_+$.

Theorem 12 *For any initial value $(S_0,I_0,R_0)\in\mathbb{R}^3_+$, there exists a unique positive solution of the problem (1)–(2), denoted by $u(t;t_0,u_0):=(S(t;t_0,(S_0,I_0,R_0)),I(t;t_0,(S_0,I_0,R_0)),R(t;t_0,(S_0,I_0,R_0)))$.*

Proof. Let $(S(t),I(t),R(t))$ be a solution of (1)–(2) with initial condition (S_0,I_0,R_0). If we denote

$$\begin{aligned} f_1(S,I,R,t) &:= aq(t)-aS+bI-\gamma\frac{SI}{N},\\ f_2(S,I,R,t) &:= -(a+b+c)I+\gamma\frac{SI}{N},\\ f_3(S,I,R,t) &:= cI-aR, \end{aligned}$$

it is easy to verify that non-negative initial data imply non-negative solutions using Theorem 2.1 from Chapter 5 in [21], since

$$f_1(0,I,R,t)\geq aq^-+bI>0,\ f_2(S,0,R,t)=0,\ f_3(S,I,0,t)=cI\geq 0.$$

On the other hand, there is a unique local solution of (1)–(2) since the coefficients of the equation are, in fact, globally Lipschitz for any given non-negative initial value (S_0,I_0,R_0) in $\mathbb{R}^3_+$, (this is true also for the nonlinear terms due to their special structure, which will be shown in the proof of Theorem 13 below), and this solution is a global solution one (12) is proved. ■

Now, thanks to the uniqueness of solution of (1)-(2), we can define a process $\{U(t,t_0),t_0\leq t\}$ in $\mathbb{R}^3_+$, by

$$U(t,t_0)u_0=u(t;t_0,u_0) \ \ \forall u_0\in\mathbb{R}^3_+. \tag{5}$$

3.1 Continuity in initial data

Theorem 13 *The process defined by (5) is continuous in* $\mathbb{R}^3_+$.

Proof. We denote

$$(\overline{S},\overline{I},\overline{R}) := (S_1, I_1, R_1) - (S_2, I_2, R_2),$$

where (S_1, I_1, R_1) is the solution of (1)–(2) for the initial condition (S_0^1, I_0^1, R_0^1) and (S_2, I_2, R_2) is the solution for the initial condition (S_0^2, I_0^2, R_0^2). Then, $(\overline{S},\overline{I},\overline{R})$ is the solution for the following problem

$$\left.\begin{aligned} \frac{d\overline{S}}{dt} &= -a\overline{S} + b\overline{I} - \gamma F_{1,2}, \\ \frac{d\overline{I}}{dt} &= -(a+b+c)\overline{I} + \gamma F_{1,2}, \\ \frac{d\overline{R}}{dt} &= c\overline{I} - a\overline{R}, \end{aligned}\right\}$$

with initial condition

$$\overline{S}(t_0) = S_0^1 - S_0^2,\ \overline{I}(t_0) = I_0^1 - I_0^2,\ \overline{R}(t_0) = R_0^1 - R_0^2$$

where

$$F_{1,2} := \frac{S_1 I_1}{N_1} - \frac{S_2 I_2}{N_2}. \tag{6}$$

Now

$$|F_{1,2}| \quad\leq\quad \left|\overline{S}\right| + \left|\frac{N_2}{N_1}\right| \left|\overline{I}\right| + \left|\frac{N_2}{N_1}\right| \left|\overline{N}\right|, \tag{7}$$

since $\frac{I_1}{N_1}$, $\frac{I_2}{N_2}$, $\frac{S_2}{N_2}$ take values in $[0,1]$. Similarly, interchanging the indices in the above derivation, we also have

$$|F_{1,2}| \leq \left|\overline{S}\right| + \left|\frac{N_1}{N_2}\right| \left|\overline{I}\right| + \left|\frac{N_1}{N_2}\right| \left|\overline{N}\right|. \tag{8}$$

Fix $t \geq t_0$. Applying (7) where $N_2(t) \leq N_1(t)$ and (8) where $N_1(t) \leq N_2(t)$, we obtain

$$|F_{1,2}| \leq \left|\overline{S}\right| + \left|\overline{I}\right| + \left|\overline{N}\right|, \tag{9}$$

and we can deduce

$$|F_{1,2}(t)|^2 \leq 3\left|\overline{S}(t)\right|^2 + 3\left|\overline{I}(t)\right|^2 + 3\left|\overline{N}(t)\right|^2. \tag{10}$$

Defining

$$\Sigma(t) := \left|\overline{S}(t)\right|^2 + \left|\overline{I}(t)\right|^2 + \left|\overline{R}(t)\right|^2,$$

and using (10), we obtain

$$\frac{d}{dt}\Sigma(t) + \rho\Sigma(t) \leq 6\gamma \left|\overline{N}(t)\right|^2, \tag{11}$$

for an appropriate nonzero constant ρ.

Now, we observe that $\overline{N} = \overline{S} + \overline{I} + \overline{R}$ satisfies

$$\frac{d\overline{N}}{dt} + a\overline{N} = 0.$$

We deduce that

$$\frac{d}{dt}\left|\overline{N}(t)\right|^2 = -2a\left|\overline{N}(t)\right|^2.$$

Multiplying by e^{at} and integrating between t_0 and t, we have

$$\left|\overline{N}(t)\right|^2 \quad \leq \quad e^{-2a(t-t_0)}\left|\overline{N}_0\right|^2 \leq \Sigma(t_0),$$

since $\left|\overline{N}_0\right|^2 \leq \Sigma(t_0)$. Thus from the differential inequality (11) we obtain

$$\frac{d}{dt}\Sigma(t) + \rho\Sigma(t) \leq 6\gamma\Sigma(t_0),$$

which we integrate between t_0 and t, to obtain

$$\Sigma(t) \leq \left[\left(1 - \frac{6\gamma}{\rho}\right)e^{-\rho(t-t_0)} + \frac{6\gamma}{\rho}\right]\Sigma(t_0)$$

Hence $\Sigma(t) \to 0$ as $\Sigma(t_0) \to 0$ for each $t \geq t_0$. Hence we have shown that the process defined by (5) is continuous in $\mathbb{R}_3^+$. ■

4 Pullback and uniform attractors

In this section, we will prove the existence of pullback and uniform attractors in $\mathbb{R}_+^3$ of our problem (1)-(2).

4.1 Pullback Attractor

First, we will show the existence of a pullback attractor in $\mathbb{R}_+^3$.

Proposition 14 *Assume that $2a > c$ and $a > b+2\gamma$. Then for any initial condition $u_0 \in \mathbb{R}_+^3$, the solution u of (1)-(2) satisfies*

$$|u(t;t_0,u_0)|^2 \leq e^{-l(t-t_0)}|u_0|^2 + ae^{-lt}\int_{-\infty}^{t} e^{ls}q^2(s)ds, \tag{12}$$

for all $t \geq t_0$, where $l := min\{2a-c, a-b-2\gamma, 2a+b+c-2\gamma\}$.

Proof. We deduce that

$$\frac{d}{dt}\left|u(t)\right|^2 = 2aq(t)S - 2a\left|u(t)\right|^2 + 2bIS + 2cIR - 2(b+c)I^2 - 2\gamma(\frac{S^2 I}{N} - \frac{SI^2}{N}).$$

We have

$$2aq(t)S \leq aq^2(t) + aS^2,$$
$$2bIS \leq bI^2 + bS^2,$$

and

$$2cIR \leq cI^2 + cR^2.$$

Then, taking into account that $0 \leq \frac{S}{N}, \frac{I}{N} \leq 1$, we can deduce

$$\frac{d}{dt}\left|u(t)\right|^2 + l\left|u(t)\right|^2 \leq aq^2(t), \tag{13}$$

where $l := min\{2a - c, a - b - 2\gamma, 2a + b + c - 2\gamma\} > 0$.

Multiplying (13) by e^{lt}, we obtain that

$$\frac{d}{dt}\left(e^{lt}\left|u(t)\right|^2\right) \leq ae^{lt}q^2(t).$$

Integrating between t_0 and t

$$\begin{aligned} e^{lt}\left|u(t)\right|^2 &\leq e^{lt_0}\left|u_0\right|^2 + a\int_{t_0}^{t} e^{ls}q^2(s)ds \qquad (14)\\ &\leq e^{lt_0}\left|u_0\right|^2 + a\int_{-\infty}^{t} e^{ls}q^2(s)ds, \end{aligned}$$

whence (12) follows. ■

We consider the universe of fixed nonempty bounded subsets of $\mathbb{R}^3_+$. Now, we prove that there exists a pullback absorbing family for the process $U(t, t_0)$ defined by (5).

Proposition 15 *Assume that* $2a > c$ *and* $a > b + 2\gamma$. *Let* q *satisfies (3). Then, the family* $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ *defined by* $D_0(t) = \overline{B}_{\mathbb{R}^3_+}(0, \rho_l(t))$, *where* $\rho_l(t)$ *is the nonnegative number given by*

$$\rho_l^2(t) = 1 + ae^{-lt}\int_{-\infty}^{t} e^{ls}q^2(s)ds,\ \forall t \in \mathbb{R},$$

is pullback absorbing family for the process U *defined by (5).*

Proof. Let $D \subset \mathbb{R}^3_+$ be bounded. Then, there exists $d > 0$ such that $|u_0| \leq d$ for all $u_0 \in D$. Thanks to Proposition 14, we deduce that for every $t_0 \leq t$ and any $u_0 \in D$,

$$\begin{aligned} |U(t,t_0)u_0|^2 &\leq e^{-lt}e^{lt_0}|u_0|^2 + ae^{-lt}\int_{-\infty}^{t} e^{ls}q^2(s)ds \\ &\leq e^{-lt}e^{lt_0}d^2 + ae^{-lt}\int_{-\infty}^{t} e^{ls}q^2(s)ds. \end{aligned}$$

If we consider $T(t,D) := l^{-1}\log(e^{lt}d^{-2})$, we have

$$|U(t,t_0)u_0|^2 \leq 1 + ae^{-lt}\int_{-\infty}^{t} e^{ls}q^2(s)ds,$$

for all $t_0 \leq T(t,D)$ and for all $u_0 \in D$.

Consequently the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{\mathbb{R}^3_+}(0, \rho_l(t))$ is pullback absorbing for the process U defined by (5). ∎

Now, as a direct consequence of the preceding results and Theorem 3, we have the existence of the pullback attractor for the process U defined by (5).

Theorem 16 *Under the assumptions in Proposition 15, the process U defined by (5) possesses a pullback attractor $\mathcal{A}$, which is given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D}, t\right)}. \tag{15}$$

4.2 Uniform Attractor

Now, we suppose that q is translation bounded in $L^2_{loc}(\mathbb{R};\mathbb{R})$, i.e.,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} q^2(s)ds < \infty \ . \tag{16}$$

In this subsection, using Theorem 6, we will prove that, under the assumption (16), the process $\{U(t,t_0)\}$ has a uniform (with respect to $t_0 \in \mathbb{R}$) attractor.

Remark 17 *Observe that assumption (16) implies (3).*

Proposition 18 *Assume that $2a > c$ and $a > b + 2\gamma$. Let q satisfies (16). Then, the process U defined by (5) is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact.*

Proof. Let $D \subset \mathbb{R}^3_+$ be bounded, and as in the proof of Proposition 15, let $d > 0$ such that $|u_0| \le d$ for all $u_0 \in D$. Using (14), we have for any $u_0 \in D$

$$\begin{aligned} |u(t;t_0,u_0)|^2 &\le e^{-l(t-t_0)}|u_0|^2 + a\int_{t_0}^t e^{-l(t-s)}q^2(s)ds \\ &\le e^{-l(t-t_0)}d^2 + a\int_{t_0}^t e^{-l(t-s)}q^2(s)ds, \end{aligned} \tag{17}$$

for all $t \ge t_0$. We estimate the integral on the right-hand side of (17), taking into account (16),

$$\begin{aligned} \int_{t_0}^t e^{-l(t-s)}q^2(s)ds \le \int_{-\infty}^t e^{-l(t-s)}q^2(s)ds &\le \sum_{n\ge 0}\int_{t-(n+1)}^{t-n} e^{-l(t-s)}q^2(s)ds \\ &\le \sum_{n\ge 0} e^{-nl}\int_{t-(n+1)}^{t-n} q^2(s)ds = C_1(1-e^{-l})^{-1}, \end{aligned}$$

where $C_1 := \sup_{t\in\mathbb{R}} \int_t^{t+1} q^2(s)ds < \infty$.

Then, we can deduce that there exists a positive constant C_l such that

$$|u(t;t_0,u_0)|^2 \le e^{-l(t-t_0)}d^2 + C_l.$$

Replacing t by $t+t_0$, we have

$$|u(t+t_0;t_0,u_0)|^2 \le e^{-lt}d^2 + C_l,$$

and if we consider $t \ge T(D) := \frac{\log d^2}{l}$, we obtain

$$|u(t+t_0;t_0,u_0)|^2 \le 1 + C_l,$$

for all t_0 and for all $u_0 \in D$.

Then, the set $B_0 := \overline{B}_{\mathbb{R}^3_+}(0, 1+C_l)$ is compact and uniformly (with respect to $t_0 \in \mathbb{R}$) attracting for the process U defined by (5). Therefore, the process U is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact. ■

We can now state a theorem about the existence of a uniform attractor of (1)-(2). Taking into account Theorem 6 and Lemma 10, we can deduce the following result.

Theorem 19 *Under the assumptions in Proposition 18, the process U defined by (5) has a uniform attractor $\mathcal{A}_1$, which is compact in $\mathbb{R}^3_+$. Moreover, we have the following relation:*

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subseteq \mathcal{A}_1, \tag{18}$$

where $\mathcal{A}(t)$ is given by (15).

5 Attractors for Periodic Equations

We consider (1)-(2) with $2a > c$, $a > b + 2\gamma$ and a T-periodic continuous function $q : \mathbb{R} \mapsto \mathbb{R}$.

5.1 Pullback Attractor

In this subsection we show that when we have a periodic nonlinear term we obtain that the pullback attractor is a periodic pullback attractor. We observe that q satisfies (3). In fact q satisfies (16).

Then, under the assumptions in Proposition 15, the process defined by (5) has a pullback attractor $\mathcal{A}$ which is given by (15).

Corollary 20 *Assume that q is a T-periodic continuous function. Then the process U defined by (5) is periodic with period T, that is*

$$U(t, t_0)u_0 = U(t + T, t_0 + T)u_0,$$

for all $t_0, t \in \mathbb{R}$, and the pullback attractor $\mathcal{A}(\cdot)$ is also periodic with period T.

Proof. We can deduce that $(X(\cdot; t_0, u_0), Y(\cdot; t_0, u_0), Z(\cdot; t_0, u_0)) := (S(\cdot + T; t_0 + T, u_0), I(\cdot + T; t_0 + T, u_0), R(\cdot + T; t_0 + T, u_0))$ is the unique solution of (1) with initial value u_0 at $t = t_0$ because

$$\begin{aligned} \frac{dX}{dt}(t) = \frac{dS}{dt}(t+T) = \frac{dS}{d\tau}(\tau) &= aq(\tau) - aS(\tau) + bI(\tau) - \gamma\frac{S(\tau)I(\tau)}{N(\tau)} \text{ where } \tau = t+T \\ &= aq(t) - aX(t) + bY(t) - \gamma\frac{X(t)Y(t)}{\overline{N}(t)}, \end{aligned}$$

by T-periodicity of q,

$$\begin{aligned} \frac{dY}{dt}(t) = \frac{dI}{dt}(t+T) = \frac{dI}{d\tau}(\tau) &= -(a+b+c)I(\tau) + \gamma\frac{S(\tau)I(\tau)}{N(\tau)} \text{ where } \tau = t+T \\ &= -(a+b+c)Y(t) + \gamma\frac{X(t)Y(t)}{\overline{N}(t)}, \end{aligned}$$

and

$$\begin{aligned} \frac{dZ}{dt}(t) = \frac{dR}{dt}(t+T) = \frac{dR}{d\tau}(\tau) &= cI(\tau) - aR(\tau) \text{ where } \tau = t+T \\ &= cY(t) - aZ(t), \end{aligned}$$

where $\overline{N} = X + Y + Z$.

Hence, we have

$$U(t, t_0)u_0 = U(t + T, t_0 + T)u_0,$$

for all $t \geq t_0$.

Replacing t_0 by $t_0 - t$, where $t \geq 0$, and t by t_0, we thus have

$$U(t_0, t_0 - t)u_0 = U(t_0 + T, t_0 + T - t)u_0,$$

so, by (4),

$$\begin{aligned}\Lambda(\widehat{D}, t_0) &:= \bigcap_{s \leq t_0} \overline{\bigcup_{t_0 - t \leq s} U(t_0, t_0 - t)D(t_0 - t)} \\ &= \bigcap_{s \leq t_0} \overline{\bigcup_{t_0 - t \leq s} U(t_0 + T, t_0 + T - t)D(t_0 - t)} = \Lambda(\widehat{D}, t_0 + T),\end{aligned}$$

and then

$$\mathcal{A}(t_0) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda\left(\widehat{D}, t_0\right)} = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda\left(\widehat{D}, t_0 + T\right)} = \mathcal{A}(t_0 + T).$$

Hence, $\mathcal{A}(\cdot)$ is also T-periodic. ■

In Figure 1 we exhibit a simulation showing the pullback attractor for (1)-(2) with q a periodic function. In this simulation, we used the following parameters and initial conditions values: $a = \frac{1}{2}$, $b = c = \gamma = \frac{1}{8}$, $q(t) = cos(t) + 2$, $S(-2000) = 1$, $I(-2000) = 1$ and $R(-2000) = 1$.

5.2 Uniform Attractor

In this subsection we show that when we have a periodic term we obtain a relation between the uniform attractor and the pullback attractor.

Proposition 21 *The process U defined by (5) is $(\mathbb{R}^3_+ \times \mathbb{T}^1, \mathbb{R}^3_+)$- continuous.*

Proof. We have to prove that for all fixed $t_0 \in \mathbb{R}$, $t \geq t_0$, the mapping $(u, t) \mapsto U(t, t_0)u$ is continuous from $\mathbb{R}^3_+ \times \mathbb{T}^1$ into $\mathbb{R}^3_+$. By the continuous dependence of solutions of (1)-(2) on initial values, we have that as the coefficients of (1) are locally Lipschitz, then the process U defined by (5) is $(\mathbb{R}^3_+ \times \mathbb{T}^1, \mathbb{R}^3_+)$- continuous. ■

We can now state a theorem about the existence of a uniform attractor of (1)-(2). Taking into account Theorem 11, we can deduce the following result.

Theorem 22 *Assume that $2a > c$, $a > b + 2\gamma$ and q is a T-periodic continuous function. Then, the set*

$$\mathcal{A}_1 = \bigcup_{\sigma \in [0,T)} \mathcal{A}(\sigma),$$

is a uniform (with respect to $t_0 \in \mathbb{R}$) attractor for the process U defined by (5), where $\{\mathcal{A}(\sigma)\}_{\sigma \in \mathbb{R}}$ is the pullback attractor of the process U defined by (5).

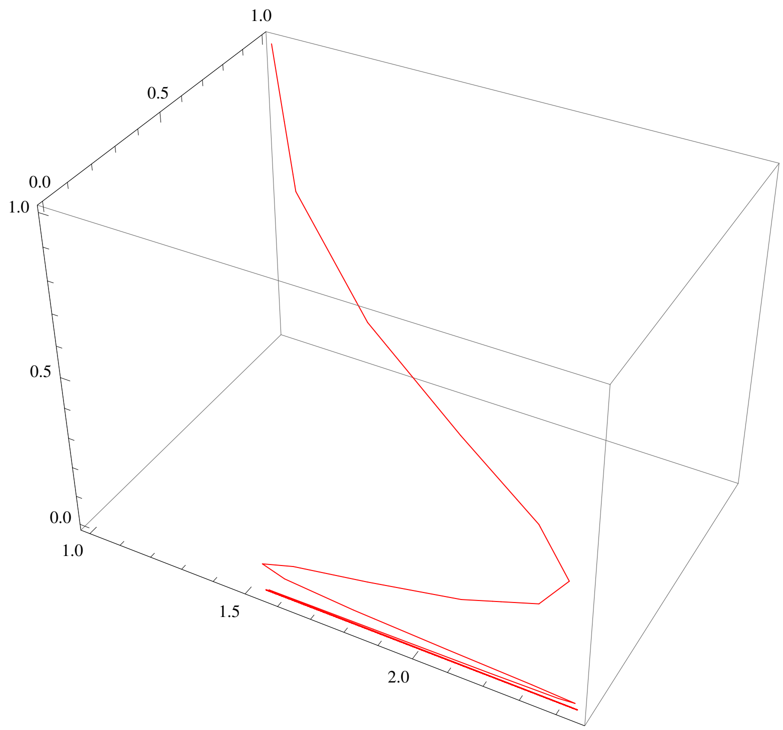

Figure 1: Numerical solution $(S(t), I(t), R(t))$

6 Pullback attractors for a class of ODEs more general than almost periodic

In this section we use a new class of functions and we generalize some results about periodic solutions of (1)-(2). We use recent results due to Kloeden and Rodrigues [9], where the authors introduced a class of functions which has the property that a bounded temporally global solution of a nonautonomous ordinary differential equation belongs to this class when the forcing term does. Let $BUC(\mathbb{R},\mathbb{R}^3_+)$ denotes the space of bounded and uniformly continuous functions $f:\mathbb{R}\to\mathbb{R}^3_+$, with the supremum norm. We consider as in [9] the following class of functions,

$$\mathcal{F} := \{f \in BUC(\mathbb{R},\mathbb{R}^3_+) : f \text{ has precompact range } \mathcal{R}(f)\}.$$

The class $\mathcal{F}$ includes periodic functions. We now consider the class $\mathcal{F}_{ODE}$ defined by

$$\mathcal{F}_{ODE} := \{f : \mathbb{R}\times\mathbb{R}^3_+ \to \mathbb{R}^3_+; \text{ is uniformly continuous in } t\in\mathbb{R}, \text{ uniformly in } (S,I,R) \\ \text{in compact subsets } C\subset\mathbb{R}^3_+, \text{with precompact range } \mathcal{R}_C(f)\},$$

where

$$\mathcal{R}_C(f) := \bigcup_{(S,I,R)\in C} \{f(t,S,I,R), t\in\mathbb{R}\}.$$

Functions in the class $\mathcal{F}$ belong trivially to the class $\mathcal{F}_{ODE}$. For our problem, we consider

$$f_1(t,S,I,R) := aq(t) - aS + bI - \gamma\frac{SI}{N}, \tag{19}$$

$$f_2(t,S,I,R) := -(a+b+c)I + \gamma\frac{SI}{N}, \tag{20}$$

$$f_3(t,S,I,R) := cI - aR, \tag{21}$$

and we suppose that

$$q \in BUC(\mathbb{R},\mathbb{R}). \tag{22}$$

Proposition 23 *Under assumption (22), f_1, f_2 and f_3 defined by (19)-(21) belong to the class $\mathcal{F}_{ODE}$.*

Proof. We prove that $f_1 \in \mathcal{F}_{ODE}$. We observe that f_2 and f_3 trivially belong to $\mathcal{F}_{ODE}$ since it does not depend on t.

First, we have to prove that f_1 is uniformly continuous in $t\in\mathbb{R}$, uniformly in (S,I,R) in compact subsets $C\subset\mathbb{R}^3_+$, i.e., we have to prove that there is a function $\alpha_0(\theta,C)$, $\alpha_0(\theta,C)\mapsto 0_+$ $(\theta\mapsto 0+)$ such that

$$|f_1(t_1,S_1,I_1,R_1) - f_1(t_2,S_2,I_2,R_2)| \le \alpha_0(|t_1-t_2|+|S_1-S_2|+|I_1-I_2|+|R_1-R_2|,C) \tag{23}$$

for all $(S_1, I_1, R_1), (S_2, I_2, R_2) \in C$, where $C \subset \mathbb{R}^3_+$ is a compact subset, and $t_1, t_2 \in \mathbb{R}$.

We deduce that

$$|f_1(t_1, S_1, I_1, R_1) - f_1(t_2, S_2, I_2, R_2)| \leq a\,|q(t_1) - q(t_2)| + a\,|S_1 - S_2| + b\,|I_1 - I_2| + \gamma\,|F_{1,2}|,$$

where $F_{1,2}$ is given by (6). Taking into account (9) and using (22), we have (23) with $\alpha_0(|t_1 - t_2| + |S_1 - S_2| + |I_1 - I_2| + |R_1 - R_2|, C) \mapsto 0_+$ $(|t_1 - t_2| + |S_1 - S_2| + |I_1 - I_2| + |R_1 - R_2| \mapsto 0+)$, so f_1 is uniformly continuous in $t \in \mathbb{R}$, uniformly in (S, I, R) in compact subsets $C \subset \mathbb{R}^3_+$.

Finally, thanks to (22), in particular we have that q is a bounded function in $t \in \mathbb{R}$, and we can deduce that $\mathcal{R}_C(f_1)$ is precompact, where $C \subset \mathbb{R}^3_+$ is a compact subset. Therefore, $f_1 \in \mathcal{F}_{ODE}$. ■

Then, we can write (1) as

$$\frac{du}{dt} = f(t, u), \; t \in \mathbb{R}, \tag{24}$$

with initial condition

$$u(t_0) = u_0, \tag{25}$$

where $u(t; t_0, u_0) := (S(t; t_0, (S_0, I_0, R_0)), I(t; t_0, (S_0, I_0, R_0)), R(t; t_0, (S_0, I_0, R_0)))$, $t_0 \in \mathbb{R}$ and $f(t, u) := (f_1(t, S, I, R), f_2(t, S, I, R), f_3(t, S, I, R))$ belongs to the class $\mathcal{F}_{ODE}$.

Thanks to Lemma 8 in [9], on account of the following Theorem, the components sets of the pullback attractor and its entire solutions are in fact uniformly continuous.

Theorem 24 *Assume that $2a > c$ and $a > b + 2\gamma$. Under assumption (22), problem (24)-(25) generates a process which possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ such that $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t)$ is precompact.*

Proof. Taking into account (22) we deduce that q satisfies (3) and (16). Then, thanks to Theorem 16, there exists the pullback attractor for the process defined by (5). On the other hand, using Theorem 19, we have (18). Therefore, $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t)$ is bounded and therefore $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t)$ is precompact. ■

Lemma 25 *Under the assumptions in Theorem 24 we have that (ϕ_1, ϕ_2, ϕ_3) belongs to the class $\mathcal{F}$ for every entire solution (ϕ_1, ϕ_2, ϕ_3) of the problem (24)-(25) taking values in the pullback attractor.*

In Figure 2 we present a simulation showing the pullback attractor for (24)-(25) where q satisfies (22). We used the following parameters and initial conditions values: $a = \frac{1}{2}$, $b = c = \gamma = \frac{1}{8}$, $q(t) = cos(t) + 2 + e^{-|t|}$, $S(-2000) = 1$, $I(-2000) = 1$ and $R(-2000) = 1$.

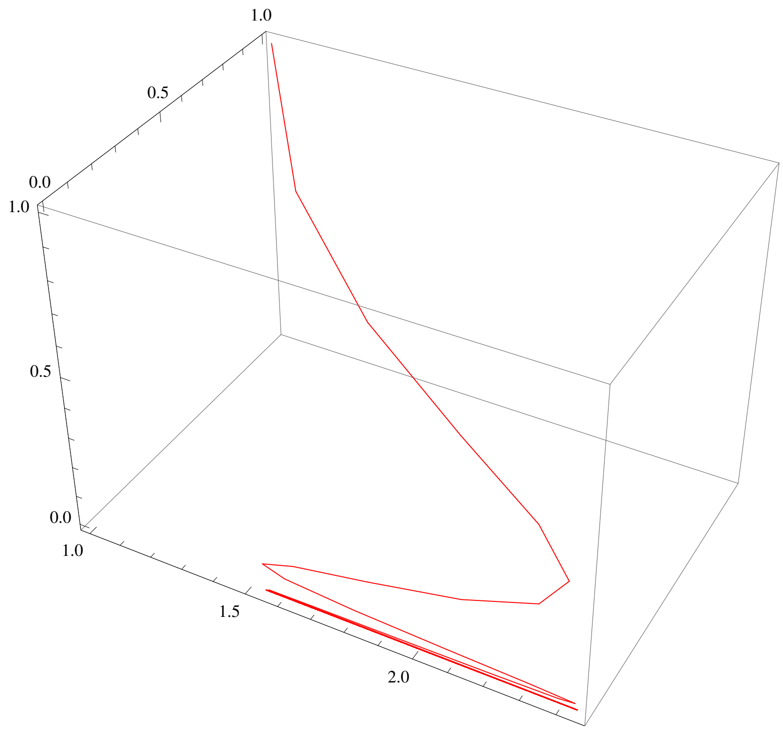

Figure 2: Numerical solution $(S(t), I(t), R(t))$

7 Upper Estimates for the Hausdorff Dimension of the Pullback Attractor

In this section we obtain an upper bound for the Hausdorff dimension of the pullback attractor of the process defined by (5). For this purpose, we use a method proposed by Leonov *et al.* in [11] in the framework of cocycle dynamical systems.

Assume that $q \in BUC(\mathbb{R}, \mathbb{R})$ and satisfies the following additional conditions:

(H1) Boundedness in time, i.e., there exists a nonnegative constant q_0 such that

$$|q(t)| \leq q_0, \quad \text{for all } t \in \mathbb{R}. \tag{26}$$

(H2) The hull of the function f denoting the right-hand side of (1), is a compact metric space, i.e., $\mathcal{H}(f) = \overline{\{f(t + \cdot, \cdot) : t \in \mathbb{R}\}}$ is a compact metric space.

Notice that if q is an almost periodic function, then q satisfies (26) and the hull $\mathcal{H}(f)$ is a compact metric space where the closure is taken in the uniform convergence topology (see [17] for more details).

In Section 4 we have proved that the solution mapping of (1)-(2) defines a process given by (5) which has a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}} \subset \mathbb{R}^3_+$ defined by (15).

Also we can obtain a cocycle by considering

$$\left.\begin{aligned} v' &= \mathbb{F}(\sigma_t p, v), \\ v(0) &= v_0 \in \mathbb{R}^3_+, \end{aligned}\right\} \tag{27}$$

where $p \in \mathcal{H}(f)$, $\mathbb{F}(p, v) := p(0, v)$ and σ is defined as the shift mapping $\sigma_t : \mathcal{H}(f) \mapsto \mathcal{H}(f)$ given by

$$\sigma_t \widetilde{f} := \widetilde{f}(\cdot + t, \cdot),$$

for $t \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{H}(f)$.

Then, the cocycle generated by (27) is given by

$$\varphi(t, p)v_0 = v(t; p, v_0),$$

where $v(t; p, v_0)$ denotes the solution of (27) with initial value v_0 at $t = 0$. If we take $p = f \in \mathcal{H}(f)$, then

$$\varphi(t, f)v_0 = v(t; f, v_0),$$

and (27) becomes

$$\left.\begin{aligned} v' &= \sigma_t f(0, v), \\ v(0) &= v_0 \in \mathbb{R}^3_+, \end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned} v' &= f(t, v), \\ v(0) &= v_0 \in \mathbb{R}^3_+, \end{aligned}\right\}$$

and we have

$$\varphi(t, f)v_0 = U(t, 0)v_0.$$

Then, our problem (1)-(2) generates a cocycle $(\{\varphi(t, p)\cdot\}_{p \in \mathcal{H}(f), t \in \mathbb{R}}, \mathbb{R}^3_+)$ over the base flow $(\{\sigma_t\}_{t \in \mathbb{R}}, \mathcal{H}(f))$, where

$$\varphi(t, \sigma_s f)v_0 = U(t + s, s)v_0. \tag{28}$$

Now, to estimate the Hausdorff dimension of the pullback attractor associated to the process defined by (5), we will use Theorem 2 in [11], which is stated in the framework of cocycle dynamical systems. Then, for the cocycle generated by our system, we need to verify:

i) There exists a family of compact sets $\{\widetilde{\mathcal{A}}(p)\}_{p \in \mathcal{H}(f)}$ which is negatively invariant for the cocycle defined by (28), i.e.

$$\widetilde{\mathcal{A}}(\sigma_t p) \subset \varphi(t, p)\widetilde{\mathcal{A}}(p), \text{ for all } p \in \mathcal{H}(f), t \geq 0.$$

ii) There exists a compact set $\widetilde{K} \subset \mathbb{R}^3_+$ such that

$$\overline{\bigcup_{p \in \mathcal{H}(f)} \widetilde{\mathcal{A}}(p)} \subset \widetilde{K}.$$

iii) There exists a continuous function $V : \mathcal{H}(f) \times \mathbb{R}^3_+ \to \mathbb{R}$ with derivatives $\frac{d}{dt}V(\sigma_t p, \varphi(t,p)u_0)$ along a given trajectory such that

$$\lambda_1(\sigma_t p, \varphi(t,p)u_0) + \lambda_2(\sigma_t p, \varphi(t,p)u_0) + s\lambda_3(\sigma_t p, \varphi(t,p)u_0) \\ + \frac{d}{dt}V(\sigma_t p, \varphi(t,p)u_0) < 0, \qquad (29)$$

for all $t \in \mathbb{R}$, $u_0 \in \widetilde{K}$, $p \in \mathcal{H}(f)$ and $s \in (0,1]$, where λ_i with $i = 1,2,3$ are the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (1) arranged in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Using the pullback attractor, $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, associated to the process defined by (5), we define the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(f)}$ by

$$\widetilde{\mathcal{A}}(p) = \begin{cases} \mathcal{A}(s) & \text{if } p = \sigma_s f, \\ \{x \in \mathbb{R}^3_+ : x = \lim_{t_n\to+\infty} x_{t_n},\ x_{t_n} \in \mathcal{A}(t_n)\} & \text{if } p \neq \sigma_s f, \end{cases} \qquad (30)$$

where $s \in \mathbb{R}$ and $p \in \mathcal{H}(f)$.

The set $\widetilde{\mathcal{A}}(p)$ is compact for any $p \in \mathcal{H}(f)$. Moreover, the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(f)}$ is negatively invariant. Indeed, if $p = \sigma_s f$, taking into account (28) and the fact that $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is invariant for the process U defined by (5), we obtain that $\varphi(t,p)\widetilde{\mathcal{A}}(p) = \widetilde{\mathcal{A}}(\sigma_t p)$ for all $t \geq 0$. If $p \neq \sigma_s f$, then $p = \lim_{t_n\to+\infty} \sigma_{t_n} f$ and it is easy to see that $\varphi(t,p)\widetilde{\mathcal{A}}(p) \supseteq \widetilde{\mathcal{A}}(\sigma_t p)$.

On the other hand, we can consider the following compact set

$$\widetilde{K} := \overline{\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)} \subset \mathbb{R}^3_+,$$

and we have that

$$\overline{\bigcup_{p \in \mathcal{H}(f)} \widetilde{\mathcal{A}}(p)} \subset \widetilde{K},$$

and, consequently, condition i)-ii) hold.

We can now establish our result on the estimate of the Haussdorff dimension of the pullback attractor for our model. We denote by $dim_H K$ the Hausdorff dimension of K. For simplicity, we assume that $c = 0$.

Theorem 26 *Assume that $c = 0$, $a > 0$, $a > b + 2\gamma$, and that $q \in BUC(\mathbb{R}, \mathbb{R})$ satisfies (H1)-(H2). Then the pullback attractor of the process U defined by (5) satisfies*

$$dim_H \mathcal{A}(t) \leq 3 - \frac{6a + 2b - 2\gamma}{2a + b - \gamma + m}, \tag{31}$$

for all $t \in \mathbb{R}$, where m is a positive number given by $m := 2b^2 + 12\gamma^2 + 6b\gamma + \frac{1}{4} + aq_0^2$.

Proof. We need to verify iii).

It is easy to see that the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (1) are

$$-a,$$

and

$$-a+\frac{1}{2}\left\{-b+\gamma(\frac{S}{N}-\frac{I}{N})\pm\sqrt{\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2}\right\}.$$

Hence, condition (29) can be written in the form

$$\begin{aligned} &2\frac{d}{dt}V_p(t,S,I,R) - 2a + (1+s)\left(-2a - b + \gamma(\frac{S}{N} - \frac{I}{N})\right) \\ &+ (1-s)\sqrt{\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2} < 0, \end{aligned} \tag{32}$$

for all $t \in \mathbb{R}$, $(S, I, R) \in \widetilde{K}$ and $p \in \mathcal{H}(f)$. Here,

$$V_p(t, S, I, R) \equiv V(\sigma_t p, \varphi(t, p)(S, I, R))$$

is a function defined for $(S, I, R) \in \widetilde{K}$, $p \in \mathcal{H}(f)$, and $t \in \mathbb{R}$ by the relation

$$V(\sigma_t p, S, I, R) := (1-s)\frac{1}{2}\left(\frac{1}{2}(S^2 + I^2 + R^2) + SI + Sq_0 + Iq_0\right).$$

Then

$$\frac{d}{dt}V_p = (1-s)\frac{1}{2}\left(aq(t)(S+I) - aq_0(S+I) + aq_0 q(t) - aR^2 - a(S+I)^2\right),$$

and, taking into account that $0 < \frac{S}{N}, \frac{I}{N} < 1$, inequality (32) is equivalent to the following

$$-2a + (-2a - b + \gamma)(1+s) + (1-s)\vartheta(t, S, I, R) < 0, \tag{33}$$

where

$$\vartheta(t,S,I,R) := \sqrt{\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2}$$
$$+ aq(t)(S+I) - aq_0(S+I) + aq_0q(t) - aR^2 - a(S+I)^2.$$

Let us denote

$$m := \max_{t,S,I,R} \vartheta(t,S,I,R).$$

We have iii) from (33), and due to Theorem 2 in [11] we obtain

$$dim_H \widetilde{\mathcal{A}}(p) \le 2 + \frac{m-4a-b+\gamma}{m+2a+b-\gamma} = 3 - \frac{6a+2b-2\gamma}{m+2a+b-\gamma}, \tag{34}$$

for all $p \in \mathcal{H}(f)$.

We have

$$\vartheta(t,S,I,R) = -\left(\gamma\sqrt{\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2} - \frac{1}{2\gamma}\right)^2$$
$$+ \gamma^2\left[\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2\right] + \frac{1}{4\gamma^2}$$
$$+ aq(t)(S+I) - aq_0(S+I) + aq_0q(t) - aR^2 - a(S+I)^2,$$

where $\gamma \neq 0$ is a varied parameter. Further,

$$\vartheta(t,S,I,R) \le \gamma^2\left[\left(-b+\gamma(\frac{S}{N}+\frac{I}{N}-2\frac{SI}{N^2})\right)^2+\left(b-\gamma(\frac{S}{N}-\frac{I}{N})\right)^2+2\left(\gamma\frac{SI}{N^2}\right)^2\right] + \frac{1}{4\gamma^2}$$
$$+ aq(t)(S+I) - aq_0(S+I) + aq_0q(t) - aR^2 - a(S+I)^2.$$

If we take the varied parameter $\gamma = 1$, taking into account (26) and $0 < \frac{S}{N}, \frac{I}{N} < 1$, then

$$\vartheta(t,S,I,R) \le 2b^2 + 12\gamma^2 + 6b\gamma + \frac{1}{4} + aq_0^2,$$

and (30) and (34) imply (31). ■

Remark 27 *For $a = \frac{1}{2}$, $b = \gamma = \frac{1}{8}$, $c = 0$, $q(t) = cos(t) + 2$, from the estimate (31), we obtain $dim_H \mathcal{A}(t) \le 2.51$ for all $t \in \mathbb{R}$.*

Acknowledgments

This work has been supported by Junta de Andalucía (Spain), Proyecto de Excelencia P12-FQM-2466.

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