Pullback attractors for a reaction-diffusion equation in a general nonempty open subset of $\mathbb{R}^N$ with non-autonomous forcing term in H^{-1}

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Abstract

The existence of minimal pullback attractors in $L^2(\Omega)$ for a non-autonomous reaction-diffusion equation, in the frameworks of universes of fixed bounded sets and that given by a tempered growth condition, is proved in this paper, when the domain Ω is a general nonempty open subset of $\mathbb{R}^N$, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. The main concept used in the proof is the asymptotic compactness of the process generated by the problem. The relation among these families is also discussed.

Keywords: Pullback attractor; asymptotic compactness; evolution process; non-autonomous reaction-diffusion equation

1 Introduction and setting of the problem

In the entire paper, we assume that $\Omega \subset \mathbb{R}^N$, where $N \geq 1$, is a given nonempty open set. Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u + \kappa u = f(u) + h(t), \text{ in } \Omega \times (\tau, +\infty), \\ u = 0, \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\kappa > 0$, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f \in C(\mathbb{R})$ satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, $l > 0$, and $p > 2$ such that

$$-\alpha_1 |s|^p \leq f(s)s \leq -\alpha_2 |s|^p, \tag{2}$$

$$(f(s) - f(r))(s-r) \leq l(s-r)^2 \quad \forall r, s \in \mathbb{R}. \tag{3}$$

Using (2), it follows that

$$|f(s)| \leq \alpha_1 |s|^{p-1} \quad \forall s \in \mathbb{R}. \tag{4}$$

The aim of this paper is to show the existence of minimal pullback attractors in the phase space $L^2(\Omega)$ for the problem (1) in the case of open domains. This, and the facts that the non-autonomous term h belongs to the space $L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and the nonlinear term f satisfies (3) with $l > 0$, are the main novelties of our problem.

The existence of the attractor for dissipative evolution equations has always relied on some kind of compactness of the process generated by such equations. Usually, the compactness is obtained through some regularization property of such equations together with the compact imbedding of the relevant Sobolev spaces. This approach is suitable only for bounded domains since Sobolev imbeddings are no longer compact otherwise.

Our aim here is to avoid weighted spaces by exploiting the energy equation valid for the problem (1) in order to obtain the so-called asymptotic compactness of the process. The concept of asymptotic compactness was already used in the non-autonomous case (see [Caraballo *et al.*(2006a)] and [Caraballo *et al.*(2006b)]), and was previously used in [Rosa(1998)] for the autonomous case.

Due to the non-autonomous character of our problem in this paper, we have to use an appropriate framework. Being possible to choose amongst several theories (skew-product flows, uniform attractors, trajectory attractors, pullback attractors) we will use the theory of pullback attractors since this allows for more generality in the non-autonomous terms (see [Anguiano *et al.*(2010)], [Caraballo *et al.*(2006a)], [Caraballo *et al.*(2006b)], [Marín-Rubio & Real(2009)], [Marín-Rubio & Real(2010)] for some results concerning pullback attractors and several reasons justifying the interest of using this theory).

It is also worth mentioning that our problem has received much attention over the last years, as we will recall now.

When Ω is bounded and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and is translation bounded, the existence of a pullback attractor in the space $H^1_0(\Omega)$ is proved in [Song & Wu(2007)]. In [Wang & Zhong(2008)], the existence of pullback attractor in $H^1_0(\Omega)$ is shown for a bounded domain and for $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. For a bounded domain Ω, and a translation bounded function $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the existence of a uniform attractor in $L^p(\Omega)$ is demonstrated in [Song & Zhong(2008)].

When Ω is unbounded, the reader can find similar results for several variants of our model in [Morillas & Valero(2005)], [Prizzi(2003)] and [Sun & Zhong, 2005] for the autonomous case, and in [Wang *et al.*(2007)] for the non-autonomous case where $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$.

When Ω is not necessarily bounded but satisfying the Poincaré inequality, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and f satisfies (2) and (3) with $l = 0$, the existence of a pullback attractor in $L^2(\Omega)$ is proved in [Anguiano *et al.*(2010)].

Our paper continues the line of investigation started in [Anguiano *et al.*(2010)]. But the fact that l is a positive constant in our model, implies that the techniques previously used in [Anguiano *et al.*(2010)] do not work in our case. Therefore, we use a different technique which allow us to obtain a more general result (namely Theorem 21).

We will provide in this paper a sufficient condition ensuring the existence of minimal pullback attractors in $L^2(\Omega)$ when the domain is a general nonempty open subset of $\mathbb{R}^N$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and f satisfies (2) and (3). A case that has not been considered in the literature yet, as far as we know.

The existence of a pullback attractor begs a number of questions: the structure of the attractor, dimension estimates or the dynamics on the attractor, among others. We would like to investigate in a future work some of these questions, since the focus of this one is on the existence of minimal pullback attractors.

The structure of the paper is as follows. In Section 2 we give a weak formulation of the problem, the concept of weak solution, and establish the existence and uniqueness of solution using the monotonicity method. A continuous dependence result with respect to initial data, which is the main key for the asymptotic compactness we will require later, is addressed in Section 3. There we use an energy method that strengthens the energy equality satisfied by the solutions. A brief recall on abstract results about the existence of minimal pullback attractors is given in Section 4. In Section 5, the main goals of proving the existence of different families of pullback attractors for different universes, and the relation among them under certain suitable assumption, are finally established.

2 Existence and uniqueness of solution

We state in this section a result on the existence and uniqueness of solution of problem (1). Instead of working directly with our equation, we will apply a general result which is a slight modification of Theorem 1.4, Chapter 2 in [Lions(1969)].

By $|\cdot|$, $\|\cdot\|_{H^{-1}(\Omega)}$ and $\|\cdot\|_{L^p(\Omega)}$ we denote the norms in the spaces $L^2(\Omega)$, $H^{-1}(\Omega)$ and $L^p(\Omega)$, respectively. By $\left(|\nabla\cdot|^2 + \kappa\,|\cdot|^2\right)^{1/2}$ we denote the norm in the space $H^1_0(\Omega)$. We will use $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$ or $[L^2(\Omega)]^N$, and $\langle\cdot,\cdot\rangle$ to denote either the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$ or between $L^{p'}(\Omega)$ and $L^p(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

Definition 1 *A weak solution of (1) is a function u, satisfying*

$$u \in C([\tau,\infty); L^2(\Omega)), \tag{5}$$

$$u \in L^2(\tau,T; H^1_0(\Omega)) \cap L^p(\tau,T; L^p(\Omega)) \quad \forall T > \tau, \tag{6}$$

$$\begin{cases} \dfrac{d}{dt}(u(t),v) + (\nabla u(t), \nabla v) + \kappa(u(t),v) = \langle f(u(t)),v\rangle + \langle h(t),v\rangle \\ \text{in } \mathcal{D}'(\tau,\infty), \text{ for all } v \in H^1_0(\Omega)\cap L^p(\Omega), \end{cases} \tag{7}$$

$$u(\tau) = u_\tau. \tag{8}$$

Theorem 2 *Assume that $\kappa > 0$, $f \in C(\mathbb{R})$ satisfies (2) and (3), and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Then, for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, there exists a unique solution $u(t) = u(t;\tau,u_\tau)$ of the problem (1). Moreover, this solution satisfies the energy equality*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |\nabla u(t)|^2 + \kappa|u(t)|^2 = \langle f(u(t)), u(t)\rangle + \langle h(t), u(t)\rangle, \ \ a.e.\ t > \tau. \tag{9}$$

Proof. The proof of this result is standard. For the sake of completeness, we give a sketch of a proof.

Let us consider the spaces $H = L^2(\Omega)$, $V_1 = H^1_0(\Omega)$ and $V_2 = L^p(\Omega) \cap L^2(\Omega)$ with $p > 2$. Denote $V_1' = H^{-1}(\Omega)$ and $V_2' = L^{p'}(\Omega) + L^2(\Omega)$.

Recall that $|\cdot|$ denotes the norm in H, by $\|\cdot\|_1 = \left(|\nabla\cdot|^2 + \kappa\,|\cdot|^2\right)^{1/2}$ we will denote the norm in V_1, and by $\|\cdot\|_2 = \|\cdot\|_{L^p(\Omega)} + |\cdot|$ the norm in V_2.

Now, we define a continuous symmetric linear operator $A_1 : V_1 \to V_1'$, given by

$$\langle A_1(v), w\rangle = (\nabla v, \nabla w) + \kappa(v, w), \ \forall v, w \in H^1_0(\Omega).$$

Let us denote

$$A_2(u) = -f(u), \quad h_1(t) = h(t), \quad h_2(t) = 0.$$

From (4) one deduces that $A_2 : V_2 \to V_2'$. On the other hand, we have that $h_1 \in L^2_{loc}(\mathbb{R}; V_1')$.

With this notation, and denoting $V = \cap_{i=1}^2 V_i$, $p_1 = 2$, $p_2 = p$, one has that (5)–(8) is equivalent to

$$u \in C([\tau,\infty); H), \quad u \in \bigcap_{i=1}^{2} L^{p_i}(\tau, T; V_i), \quad \text{for all } T > \tau, \tag{10}$$

$$u'(t) + \sum_{i=1}^{2} A_i(u(t)) = h(t) \quad \text{in } \mathcal{D}'(\tau,\infty; V'), \tag{11}$$

$$u(\tau) = u_\tau. \tag{12}$$

Applying a slight modification of [Lions(1969), Ch.2,Th.1.4], it is not difficult to see that problem (10)–(12) has a unique solution. Moreover, u satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}\,|u(t)|^2 + \sum_{i=1}^{2}\langle A_i(u(t)), u(t)\rangle_i = (h(t), u(t)) \quad \text{a.e. } t > \tau,$$

where $\langle\cdot,\cdot\rangle_i$ denotes the duality product between V_i' and V_i.

This last equality turns out to be just (9). ■

3 A continuous dependence result

In this section, we give a result on continuous dependence of the solutions of (1) with respect to the initial datum u_τ.

Theorem 3 *Assume that the assumptions of Theorem 2 are satisfied. Let* $\{u_\tau^{(n)}\}_{n\geq 1} \subset L^2(\Omega)$ *be a sequence such that*

$$u_\tau^{(n)} \rightharpoonup u_\tau \text{ weakly in } L^2(\Omega). \tag{13}$$

Let us denote $u^{(n)} = u(\cdot;\tau,u_\tau^{(n)})$ *and* $u = u(\cdot;\tau,u_\tau)$ *the corresponding weak solutions of (1). Then, for all* $T>\tau$,

$$\begin{aligned} u^{(n)} &\rightharpoonup u && \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), && (14)\\ u^{(n)} &\rightharpoonup u && \text{weakly in } L^p(\tau,T;L^p(\Omega)), \\ f(u^{(n)}) &\rightharpoonup f(u) && \text{weakly in } L^{p'}(\tau,T;L^{p'}(\Omega)), \\ u^{(n)}(t) &\rightharpoonup u(t) && \text{weakly in } L^2(\Omega), \text{ for all } t>\tau. && (15)\end{aligned}$$

If Ω *is a bounded set, then for all* $T>\tau$,

$$\begin{aligned} u^{(n)} &\to u && \text{strongly in } L^2(\tau,T;L^2(\Omega)), && (16)\\ u^{(n)}(t) &\to u(t) && \text{strongly in } L^2(\Omega), \text{ for all } t>\tau. && (17)\end{aligned}$$

Proof. For all but last of the above convergences we argue similarly to the proof of Proposition 4.1 in [Anguiano *et al.*(2010)]. For the last convergence, we use an energy method that strengthens the energy equality satisfied by the solutions.

From (16), we deduce that from every subsequence of $\{u^{(n)}\}$ we can extract a subsequence that we will denote by $\{u_\nu\}$, such that

$$|u_\nu(t)| \to |u(t)| \quad a.e. \text{ in } (\tau,T). \tag{18}$$

Let us define

$$J_\nu(t) = \frac{1}{2}|u_\nu(t)|^2 - \int_\tau^t \langle h(s), u_\nu(s)\rangle\, ds,$$

and

$$J(t) = \frac{1}{2}|u(t)|^2 - \int_\tau^t \langle h(s), u(s)\rangle\, ds,$$

for all $t\geq\tau$.

It is clear that J_ν and J are continuous functions. Also, from (14) and (18) we see that

$$J_\nu(t)\to J(t) \; a.e. \; t\in(\tau,T) \quad \text{as } \nu\to\infty. \tag{19}$$

On the other hand, taking into account (9) and (2), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_\nu(t)|^2 \leq \langle h(t), u_\nu(t)\rangle, \quad t>\tau.$$

Thus, for every ν, the function J_ν is a non-increasing function of t.

We are now in position to show that

$$J_\nu(t) \to J(t) \quad \text{strongly for all } t \in (\tau, T)\,. \tag{20}$$

Let $t \in (\tau, T)$ and $\varepsilon > 0$ be fixed. From (19) and the continuity of J, we can take $t' > t$ and $t'' < t$ such that

$$J_\nu(t') \to J(t') \quad \text{strongly as } \nu \to \infty, \tag{21}$$

$$J_\nu(t'') \to J(t'') \quad \text{strongly as } \nu \to \infty, \tag{22}$$

$$\left| J(t'') - J(t) \right| \leq \varepsilon, \tag{23}$$

and

$$\left| J(t) - J(t') \right| \leq \varepsilon. \tag{24}$$

As J_ν is a non-increasing function of t, we obtain

$$J_\nu(t') - J_\nu(t) \leq 0, \tag{25}$$

and

$$J_\nu(t'') - J_\nu(t) \geq 0, \tag{26}$$

for every ν. Using (23) and (26) we have

$$\begin{aligned} J_\nu(t) - J(t) &= J_\nu(t) - J_\nu(t'') + J_\nu(t'') - J(t'') \\ &\quad + J(t'') - J(t) \\ &\leq \left| J_\nu(t'') - J(t'') \right| + \varepsilon. \end{aligned} \tag{27}$$

Analogously, using (24) and (25) we obtain

$$\begin{aligned} J(t) - J_\nu(t) &= J(t) - J(t') + J(t') - J_\nu(t') \\ &\quad + J_\nu(t') - J_\nu(t) \\ &\leq \left| J(t') - J_\nu(t') \right| + \varepsilon. \end{aligned} \tag{28}$$

From (21), (22), (27) and (28), we have

$$\limsup_{\nu \to \infty} |J(t) - J_\nu(t)| \leq \varepsilon, \tag{29}$$

and therefore, as $\varepsilon > 0$ is arbitrary, (20) follows from (29). Thanks to (29), and taking into account (14), we deduce that

$$|u_\nu(t)| \to |u(t)| \ \text{ strongly } \ \forall t \in (\tau, T)\,,$$

and then, by (15), we obtain

$$u_\nu(t) \to u(t) \text{ strongly in } L^2(\Omega) \quad \forall t \in (\tau, T)\,.$$

Then from a standard contradiction argument combined with the fact that $T > \tau$ has been taken arbitrarily, we deduce that (17) holds. ■

4 Abstract results on minimal pullback attractors

In this section we remember some abstract results on pullback attractors theory. We present a resume of some results on the existence of minimal pullback attractors obtained in [García-Luengo *et al.*(2012)] (see also [Caraballo *et al.*(2006a)], [Caraballo *et al.*(2006b)], [Marín-Rubio & Real(2009)]). In particular, we consider the process U being closed (see below Definition 4).

Consider a given metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ with $U(\tau,\tau)x = x$ for any $(\tau,x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

Definition 4 *Let U be a process on X.*

a) U is said to be continuous if for any pair $\tau \leq t$, the mapping $U(t,\tau) : X \to X$ is continuous.

b) U is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Remark 5 *It is clear that every continuous process is closed. More generally, every strong-weak continuous process (see [Marín-Rubio & Real(2009)] for the definition) is a closed process.*

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ [observe that we do not require any additional condition on these sets as compactness or boundedness].

Definition 6 *We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 7 *It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \leq t$ such that*

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \textit{for all } \tau \leq \tau_0(t,\widehat{D}).$$

Observe that in the definition above $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Definition 8 *A process U on X is said to be pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$, i.e. if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Denote

$$\Lambda(\widehat{D}_0,t) := \bigcap_{s\le t}\overline{\bigcup_{\tau\le s} U(t,\tau)D_0(\tau)}^X \quad \text{for all } t\in\mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

We denote by $\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2) = \sup_{x\in\mathcal{O}_1}\inf_{y\in\mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\, \mathcal{O}_2\subset X.$$

We have the following result (cf. [García-Luengo *et al.*(2012)]) on existence of minimal pullback attractors.

Theorem 9 *Consider a closed process $U:\mathbb{R}^2_d\times X\to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0=\{D_0(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$ which is pullback $\mathcal{D}-$absorbing for U, and assume also that U is pullback $\widehat{D}_0-$asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}}=\{\mathcal{A}_{\mathcal{D}}(t):t\in\mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t)=\overline{\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},t)}^X \quad t\in\mathbb{R},$$

has the following properties:

(a) for any $t\in\mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and

$$\mathcal{A}_{\mathcal{D}}(t)\subset\Lambda(\widehat{D}_0,t),$$

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}-$attracting, i.e.

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t))=0 \quad \textit{for all } \widehat{D}\in\mathcal{D}, \quad t\in\mathbb{R},$$

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)=\mathcal{A}_{\mathcal{D}}(t)$ for all $\tau\le t$,

(d) if $\widehat{D}_0\in\mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t)=\Lambda(\widehat{D}_0,t)\subset\overline{D_0(t)}^X$, for all $t\in\mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C}=\{C(t)\ :\ t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D}=\{D(t):t\in\mathbb{R}\}\in\mathcal{D}$,

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),C(t))=0,$$

then $\mathcal{A}_{\mathcal{D}}(t)\subset C(t)$.

Remark 10 *Under the assumptions of Theorem 9, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}-$attractor for the process U.*

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e. if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}_F^X$, the corresponding minimal pullback $\mathcal{D}_F^X-$attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli, [Crauel *et al.*(1997), Th.1.1, p.311], and will be denoted $\mathcal{A}_{\mathcal{D}_F^X}$.

Now, it is easy to conclude the following result.

Corollary 11 *Under the assumptions of Theorem 9, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F^X$, then both attractors, $\mathcal{A}_{\mathcal{D}_F^X}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 12 *It can be proved (see [Marín-Rubio & Real(2009)]) that, under the assumptions of the preceding corollary, if, moreover, $\widehat{D}_0 \in \mathcal{D}$, and for some $T \in \mathbb{R}$ the set $\cup_{t \leq T} D_0(t)$ is a bounded subset of X, then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \leq T.$$

5 Existence of pullback attractors

Now, by the previous results, we are able to define correctly a process U on $L^2(\Omega)$ associated to (1), and to obtain the existence of minimal pullback attractors.

From the uniqueness of solution to problem (1), we can define a process in $L^2(\Omega)$. In addition, it is easy to prove that the process is continuous in $L^2(\Omega)$.

Proposition 13 *Assume that the assumptions of Theorem 2 are satisfied. Then, the family of maps $U(t, \tau) : L^2(\Omega) \to L^2(\Omega)$, with $\tau \leq t$, given by*

$$U(t, \tau)u_\tau = u(t), \tag{30}$$

where $u = u(\cdot; \tau, u_\tau)$ is the unique weak solution of (1), defines a continuous process on $L^2(\Omega)$.

Remark 14 *Observe that if* $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$*, then there exist* $N+1$ *functions* $h_0, h_1, ..., h_N$*, with* $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *for all* $0 \le i \le N$*, such that* $h = h_0 - \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$ *and* $\|h\|_{H^{-1}(\Omega)} = \left(\sum_{i=0}^N |h_i|^2\right)^{1/2}$.

We have the following result.

Lemma 15 *Under the assumptions of Theorem 2, the solution* u *of (1) satisfies*

$$|u(t)|^2 \le e^{-\kappa(t-\tau)}|u_\tau|^2 + 2e^{-\kappa t}\sum_{i=0}^N \int_\tau^t e^{\kappa s}|h_i(s)|^2\, ds, \tag{31}$$

for all $t \ge \tau$.

Proof. From (9), and taking into account (2), we obtain

$$\frac{d}{dt}\left(e^{\kappa t}|u(t)|^2\right) + 2e^{\kappa t}|\nabla u(t)|^2 + \kappa e^{\kappa t}|u(t)|^2 \le 2e^{\kappa t}\langle h(t), u(t)\rangle, \tag{32}$$

a.e. $t > \tau$, and then, observing that

$$2e^{\kappa t}\langle h(t), u(t)\rangle \le 2e^{\kappa t}\sum_{i=0}^N |h_i(t)|^2 + \frac{1}{2}e^{\kappa t}\left(|\nabla u(t)|^2 + \kappa|u(t)|^2\right), \tag{33}$$

we have in particular

$$\frac{d}{dt}\left(e^{\kappa t}|u(t)|^2\right) \le 2e^{\kappa t}\sum_{i=0}^N |h_i(t)|^2,$$

a.e. $t > \tau$.

Integrating in this last inequality, we obtain (31). ∎

Taking into account the estimate (31), we define the following universe.

Definition 16 *For any* $\kappa > 0$*, we will denote by* $\mathcal{D}_\kappa$ *the class of all families of nonempty subsets* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ *such that*

$$\lim_{\tau \to -\infty}\left(e^{\kappa\tau}\sup_{v \in D(\tau)}|v|^2\right) = 0.$$

Accordingly to the notation introduced in the Section 4, $\mathcal{D}_F^{L^2(\Omega)}$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $L^2(\Omega)$.

Remark 17 *Observe that* $\mathcal{D}_F^{L^2(\Omega)} \subset \mathcal{D}_\kappa$ *and that both are inclusion-closed.*

As an evident consequence of Lemma 15, we have the following result.

Corollary 18 *Assume that the assumptions of Theorem 2 are satisfied. Suppose moreover that*

$$\sum_{i=0}^{N} \int_{-\infty}^{0} e^{\kappa s} \left| h_i(s) \right|^2 ds < +\infty. \tag{34}$$

Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2(\Omega)}(0, R^{1/2}(t))$, the closed ball in $L^2(\Omega)$ of center zero and radius $R^{1/2}(t)$, where

$$R(t) = 1 + 2e^{-\kappa t} \sum_{i=0}^{N} \int_{-\infty}^{t} e^{\kappa s} \left| h_i(s) \right|^2 ds,$$

is pullback $\mathcal{D}_\kappa-$absorbing for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$ given by (30) (and therefore $\mathcal{D}_F^{L^2(\Omega)}-$absorbing too), and $\widehat{D}_0 \in \mathcal{D}_\kappa$.

The following result will be crucial in the proof of the existence of minimal pullback attractors for (1).

Lemma 19 *Under the assumptions in Corollary 18, for any real numbers $t_1 \le t_2$ and any $\varepsilon > 0$, there exist $T = T(t_1, t_2, \varepsilon, \widehat{D}_0) \le t_1$ and $M = M(t_1, t_2, \varepsilon, \widehat{D}_0) \ge 1$ verifying*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \ge 2m\}} u^2 (x,t)\, dx \le \varepsilon, \ \ \forall \tau \le T, \quad t \in [t_1, t_2], \quad m \ge M,$$

for any weak solution $u(t) = u(t; \tau, u_\tau)$ where $u_\tau \in D_0(\tau)$.

Proof. let $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$ and $u(t) = u(t; \tau, u_\tau) = U(t,\tau)u_\tau$ be fixed. We take a smooth function $\theta \in C^1([0, +\infty))$ verifying

$$0 \le \theta(s) \le 1,$$

$$\theta(s) = 0 \ \ \forall s \in [0, 1],$$

$$\theta(s) = 1 \ \ \forall s \ge 2.$$

Under the above assumptions on u_τ, f and h, if u is a weak solution of (1), the function $|\theta u(t)|^2 = \int_\Omega \theta^2 \left(\frac{|x|^2}{m^2} \right) u^2 (x,t)\, dx$ is absolutely continuous and $\frac{d}{dt} |\theta u|^2 = 2 \left\langle \frac{du}{dt}, \theta^2 u \right\rangle$ for a.a. t (see [Morillas & Valero(2005), Lemma 3]).

On the other hand (see for example [Brezis(1983), propositions IX.4 and IX.5]) observe that $\theta \left(\frac{|\cdot|^2_{\mathbb{R}^N}}{m^2} \right) u(\cdot, t) \in H^1_0(\Omega)$, a.e. in (τ, ∞), with

$$\partial_i \left(\theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u(x,t) \right) = \theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) \partial_i u(x,t) + \frac{2x_i}{m^2} \theta' \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u(x,t), \tag{35}$$

and the same is true replacing θ by θ^2.

Hence, we obtain for every $t \geq \tau$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx + \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|\nabla u(x,t)|^2\,dx \\
&+\frac{4}{m^2}\int_\Omega \theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)x\cdot\nabla u(x,t)dx \\
&+\kappa\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx \\
&=\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)f(u(x,t))u(x,t)dx + \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_0(x,t)u(x,t)dx \\
&+\sum_{i=1}^N\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i(x,t)\partial_i u(x,t)dx \\
&+\sum_{i=1}^N\int_\Omega \frac{4x_i}{m^2}\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)h_i(x,t)dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{36}
$$

From (2), we obtain

$$
I_1 \leq -\alpha_2\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|u(x,t)|^p\,dx \leq 0. \tag{37}
$$

Using the Cauchy-Schwarz inequality, we obtain

$$
I_2 \leq \frac{\kappa}{2}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx + \frac{1}{2\kappa}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_0^2(x,t)dx, \tag{38}
$$

and

$$
I_3 \leq \frac{1}{4}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|\nabla u(x,t)|^2\,dx + \sum_{i=1}^N\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx. \tag{39}
$$

Using that $\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) = 0$ if $|x|_{\mathbb{R}^N} > \sqrt{2}m$, $\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \leq C_{\theta'}$ for all x, and the Cauchy-Schwarz inequality, we obtain

$$
|I_4| \leq \frac{16}{m^2}C^2_{\theta'}N\int_\Omega u^2(x,t)dx + \sum_{i=1}^N\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx, \tag{40}
$$

where we have used that $|x|_{\mathbb{R}^N} \leq \sqrt{2}m < 2m$.

Moreover, we have

$$\left|\frac{4}{m^2}\int_\Omega \theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)x\cdot\nabla u(x,t)dx\right| \tag{41}$$
$$\leq \frac{4}{m}C_{\theta'}\int_\Omega u^2(x,t)dx+\frac{4}{m}C_{\theta'}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|\nabla u(x,t)|^2\,dx.$$

Assume that $\frac{3}{4}-\frac{4}{m}C_{\theta'}>0$ (and this is true for m large enough).

Then, from (36)-(41) we, in particular, deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx+\frac{\kappa}{2}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx \tag{42}$$
$$\leq\left(\frac{4}{m}C_{\theta'}+\frac{16}{m^2}C^2_{\theta'}N\right)\int_\Omega u^2(x,t)dx+\frac{1}{2\kappa}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_0^2(x,t)dx$$
$$+2\sum_{i=1}^N\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx.$$

Then from (42), if we denote $\widehat{C}:=8C_{\theta'}+32C^2_{\theta'}N$ and $C_1:=\max\left\{\frac{1}{\kappa},4\right\}$, and multiplying by $e^{\kappa t}$, we obtain

$$\frac{d}{dt}\left(e^{\kappa t}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx\right)\leq\frac{\widehat{C}}{m}e^{\kappa t}\int_\Omega u^2(x,t)dx$$
$$+C_1\sum_{i=0}^N e^{\kappa t}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx.$$

Integrating now between τ and t, and using the properties of θ, we have

$$\int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}}u^2(x,t)dx\leq e^{-\kappa t}e^{\kappa\tau}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u_\tau^2(x)\,dx \tag{43}$$
$$+\frac{\widehat{C}}{m}e^{-\kappa t}\int_\tau^t e^{\kappa s}|u(s)|^2\,ds$$
$$+C_1\sum_{i=0}^N e^{-\kappa t}\int_{-\infty}^t e^{\kappa s}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,s)dxds,$$

for all $t\geq\tau$ and for m large enough.

On the other hand, from (32) and (33), integrating between τ and t, we, in particular, have

$$\frac{\kappa}{2}\int_\tau^t e^{\kappa s}|u(s)|^2\,ds\leq e^{\kappa\tau}|u_\tau|^2+2\sum_{i=0}^N\int_\tau^t e^{\kappa s}|h_i(s)|^2\,ds.$$

Thus, if we take $u_\tau \in D_0(\tau)$, we obtain

$$\int_\tau^t e^{\kappa s}|u(s)|^2 ds \leq 2\kappa^{-1}e^{\kappa\tau}R(\tau) + 4\kappa^{-1}\sum_{i=0}^{N}\int_{-\infty}^{t} e^{\kappa s}\left|h_i(s)\right|^2 ds. \tag{44}$$

Let us fix $t_1 \leq t_2 \in \mathbb{R}$.
Observing that

$$\lim_{\tau\to-\infty} e^{\kappa\tau}R(\tau) = 0,$$

from (34) and (44), we deduce that there exists a constant $C(t_1, t_2)$ such that

$$e^{-\kappa t}\int_\tau^t e^{\kappa s}|u(s)|^2\, ds \leq C(t_1, t_2) \quad \forall\, t \in [t_1, t_2],\, \tau \leq t_1,$$

and therefore, by (43),

$$\int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}} u^2(x,t)dx \leq e^{-\kappa t}e^{\kappa\tau}R(\tau) + \frac{\widehat{C}}{m}C(t_1,t_2) \tag{45}$$
$$+ C_1\sum_{i=0}^{N} e^{-\kappa t}\int_{-\infty}^{t} e^{\kappa s}\int_\Omega \theta^2\left(\frac{|x|_{\mathbb{R}^N}^2}{m^2}\right) h_i^2(x,s)dxds,$$

for all $t \in [t_1, t_2]$ and for m large enough, where $\tau \leq t_1$ and $u_\tau \in D_0(\tau)$.

On the other hand, from (34) and Lebesgue's Dominated Convergence Theorem, for every $t \in [t_1, t_2]$ we obtain

$$\int_{-\infty}^{t} e^{\kappa s}\int_\Omega \theta^2\left(\frac{|x|_{\mathbb{R}^N}^2}{m^2}\right) h_i^2(x,s)dxds \tag{46}$$
$$\leq \int_{-\infty}^{t_2}\int_\Omega \chi_{\{|x|_{\mathbb{R}^N}\geq m\}} e^{\kappa s}h_i^2(x,s)dxds \longrightarrow 0 \text{ as } m \to \infty,$$

for all $i = 1, .., N$, where χ is the indicator function.

From (45) and (46) we deduce our lemma. ■

Next, we prove that the process U is pullback $\widehat{D}_0-$asymptotically compact.

Proposition 20 *Under the assumptions in Corollary 18, the process U defined by (30) is pullback asymptotically compact with respect to the family $\widehat{D}_0$ defined in that Corollary.*

Proof. Let us fix a sequence $\tau_n \to -\infty$, a sequence $u_{\tau_n} \in D_0(\tau_n)$, and $t \in \mathbb{R}$. We have to prove that from the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ we can extract a subsequence that converges in $L^2(\Omega)$.

As the family $\widehat{D}_0$ is pullback $\mathcal{D}_\kappa$-absorbing and $\tau_n \to -\infty$, there exists $n_0(t) \geq 1$ such that $\tau_n \leq t-1$ and

$$U(t-1,\tau_n)u_{\tau_n} \subset U(t-1,\tau_n)D_0(\tau_n) \subset D_0(t-1), \tag{47}$$

for all $n \geq n_0(t)$.

From (47), we deduce that there exists a subsequence $\left\{\left(\tau_{n'}, u_{\tau_{n'}}\right)\right\} \subset \{(\tau_n, u_{\tau_n})\}$, and $\varsigma_0 \in D_0(t-1)$, such that

$$U(t-1, \tau_{n'})u_{\tau_{n'}} \rightharpoonup \varsigma_0 \quad \text{weakly in } L^2(\Omega)\,. \tag{48}$$

Now, for all $m \in \mathbb{Z}$, $m \geq 1$, we denote

$$\Omega_m = \Omega \cap \left\{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m\right\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$.

Taking into account (48) and Theorem 3, we have

$$U(t, t-1)\left(U(t-1, \tau_{n'})u_{\tau_{n'}}\right) \rightarrow U(t, t-1)\varsigma_0 \quad \text{strongly in } L^2(\Omega_{2m}). \tag{49}$$

Let us denote $u_{n'} = u(\cdot; \tau_{n'}, u_{\tau_{n'}})$ and $u = u(\cdot; t-1, \varsigma_0)$ the corresponding weak solutions of (1). By Lemma 19, for any $\varepsilon > 0$ there exist $T = T(t-1, t, \varepsilon, \widehat{D}_0) \leq t-1$, and $M = M(t-1, t, \varepsilon, \widehat{D}_0) \geq 1$, such that

$$\begin{aligned}
&\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} \left(u_{n'}(x,t) - u(x,t)\right)^2 dx \\
&\leq 2\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u_{n'}^2(x,t) dx \\
&\qquad + 2\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x,t) dx \leq 4\varepsilon,
\end{aligned}$$

for all $m \geq M$ and any n' such that $\tau_{n'} \leq T$.

And this, together with (49), will imply the strong convergence in $L^2(\Omega)$ of $U(t, \tau_{n'})u_{\tau_{n'}}$ to $U(t, t-1)\varsigma_0$. ■

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \rightarrow L^2(\Omega)$ defined by (30).

Theorem 21 *Assume that $\kappa > 0$, $f \in C(\mathbb{R})$ satisfies (2) and (3), and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Suppose moreover that the condition (34) holds. Then, there exist the minimal pullback $\mathcal{D}_F^{L^2(\Omega)}$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^{L^2(\Omega)}} = \{\mathcal{A}_{\mathcal{D}_F^{L^2(\Omega)}}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}_\kappa$-attractor

$$\mathcal{A}_{\mathcal{D}_\kappa} = \{\mathcal{A}_{\mathcal{D}_\kappa}(t) : t \in \mathbb{R}\},$$

for the process U defined by (30). The family $\mathcal{A}_{\mathcal{D}_\kappa}$ belongs to $\mathcal{D}_\kappa$, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^{L^2(\Omega)}}(t) \subset \mathcal{A}_{\mathcal{D}_\kappa}(t) \subset \overline{B}_{L^2(\Omega)}(0, R^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

If moreover h satisfies

$$\sup_{s\leq 0}\left(e^{-\kappa s}\sum_{i=0}^{N}\int_{-\infty}^{s}e^{\kappa\theta}\left|h_i(\theta)\right|^2 d\theta\right) < +\infty, \tag{50}$$

then

$$\mathcal{A}_{\mathcal{D}_F^{L^2(\Omega)}}(t) = \mathcal{A}_{\mathcal{D}_\kappa}(t) \quad \textit{for all } t \in \mathbb{R}. \tag{51}$$

Proof. All but the last results are consequences of Theorem 9 and Corollary 11. Finally, (51) follows from (50) and Remark 12, taking into account the expression $R(t)$ given in Corollary 18. ■

Acknowledgments

I would like to thank the referees for their comments and suggestions. Also, I would like to thank my father, Julio, for his support in this stage of my career.

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