H^2-boundedness of the pullback attractor for the non-autonomous SIR equations with diffusion

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Abstract

We prove some regularity results for the pullback attractor of a non-autonomous SIR model with diffusion in a bounded domain Ω of $\mathbb{R}^d$ where $d \geq 1$. We show a regularity result for the unique solution of the problem. We establish a general result about $H^2(\Omega)^3$-boundedness of invariant sets for the associate evolution process. Then, as a consequence, we deduce that the pullback attractor of the non-autonomous system of SIR equations with diffusion is bounded in $H^2(\Omega)^3$.

Keywords: SIR epidemic model with diffusion; nonautonomous dynamical systems; pullback attractors; invariant sets; H^2-regularity

1 Introduction and setting of the problem

The emerging and reemerging diseases have led to a revived interest in infectious diseases. Mathematical models have become important tools in analyzing the spread and control of infectious diseases. Mathematical epidemiology seems to have grown exponentially starting in the middle of the 20th century (the first edition in 1957 of Bailey's book [5] is an important landmark), so that a tremendous variety of models have now been formulated, mathematically analyzed, and applied to infectious diseases.

The SIR model for the transmission of infectious diseases, which was introduced by Kermack and McKendrick [13] in 1927, is one of the fundamental models of mathematical epidemiology [1, 6]. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptibles S, infectives I and recovereds R, of a constant total population.

There is a strong biological motivation to include time-dependent terms into epidemiological models, for instance temporally varying forcing is typical of seasonal variation of a disease [12, 20]. Recently, nonautonomous versions of the SIR model and related epidemic systems for which the total population may vary in time have been investigated [11, 14, 15, 17, 22, 23].

We consider a classical and well-known model from mathematical epidemiology in the form of the SIR equations (cf., e.g., [14, 16]), with diffusion, in which a temporal forcing term is considered.

Let us introduce the model we will be involved with in this paper. Let $\Omega \subset \mathbb{R}^d$, where $d \geq 1$, be a bounded domain with a smooth boundary $\partial\Omega$.

We consider the following problem for a temporally-forced SIR model with diffusion

$$\left.\begin{aligned} \frac{\partial S}{\partial t} - \Delta S &= aq(t) - aS + bI - \gamma \tfrac{SI}{N}, \\ \frac{\partial I}{\partial t} - \Delta I &= -(a+b+c)I + \gamma \tfrac{SI}{N}, \\ \frac{\partial R}{\partial t} - \Delta R &= cI - aR, \end{aligned}\right\} \tag{1}$$

where

$$N(t) = S(t) + I(t) + R(t),$$

with the Dirichlet boundary condition

$$S(x,t) = I(x,t) = R(x,t) = 0 \text{ on } \quad \partial\Omega \times (t_0, +\infty) \tag{2}$$

and initial condition

$$S(x,t_0) = S_0(x), \quad I(x,t_0) = I_0(x), \quad R(x,t_0) = R_0(x) \text{ for } \; x \in \Omega, \tag{3}$$

where $t_0 \in \mathbb{R}$ and the parameters a, b, c and γ are positive constants such that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian. The temporal forcing term is given by a continuous function $q : \mathbb{R} \to \mathbb{R}$ taking positive bounded values, i.e. $q(t) \in [q^-, q^+]$ for all $t \in \mathbb{R}$ where $0 < q^- \leq q^+$, such that $q' \in L^2_{loc}\left(\mathbb{R}; L^2(\Omega)\right)$.

Before to continue with the setting of the problem, let us introduce some notation that will be useful in the sequel. $L^2(\Omega)$ denotes the space of square integrable real valued functions defined on Ω with the norm $|\cdot|_{L^2(\Omega)}$ corresponding to the scalar product defined

$$(u,v) = \int_\Omega u \cdot v dx$$

for all u, $v \in L^2(\Omega)$, while $H^1_0(\Omega)$ denotes the space of such functions satisfying the Dirichlet boundary condition that have square integrable generalized derivatives with the scalar product

$$((u,v)) := (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v dx$$

for all $u, v \in H^1_0(\Omega)$ and the norm $||\cdot|| := |\nabla \cdot|_{L^2(\Omega)}$. We will denote by $\langle \cdot, \cdot \rangle$ the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

In addition, X_3 denotes the space of functions $(u_1,u_2,u_3)\in L^2(\Omega)^3$ with the scalar product

$$((u_1,u_2,u_3),(v_1,v_2,v_3))=(u_1,v_1)+(u_2,v_2)+(u_3,v_3)\,,$$

and norm

$$|(u_1,u_2,u_3)|_{L^2(\Omega)}=|u_1|_{L^2(\Omega)}+|u_2|_{L^2(\Omega)}+|u_3|_{L^2(\Omega)}$$

for all (u_1,u_2,u_3),$(v_1,v_2,v_3)\in X_3$, while Y_3 denotes the space of functions $(u_1,u_2,u_3)\in H_0^1(\Omega)^3$ with the scalar product

$$(((u_1,u_2,u_3),(v_1,v_2,v_3)))=((u_1,v_1))+((u_2,v_2))+((u_3,v_3))\,,$$

and norm

$$||(u_1,u_2,u_3)||=||u_1||+||u_2||+||u_3||$$

for all (u_1,u_2,u_3),$(v_1,v_2,v_3)\in Y_3$. Finally, let X_3^+ be the subspace of non-negative functions in X_3 and Y_3^+ be the subspace of non-negative functions in Y_3. We will denote the subspace of non-negative functions in $H^2(\Omega)^3$ by Z_3^+.

In Anguiano and Kloeden [3] we discussed several issues concerning problem (1)-(3). We first proved the existence and uniqueness of positive solutions of (1)-(3) for initial data in X_3^+. The globally defined nonnegative solutions of (1)–(3) generate a nonautonomous 2-parameter semigroup or process in the Banach space X_3^+, i.e., a family of mappings $U_{t,t_0}:X_3^+\to X_3^+$ with $t\geq t_0$ in $\mathbb{R}$ satisfying

$$U_{t_0,t_0}x=x,\quad U_{t,t_0}x=U_{t,r}\circ U_{r,t_0}x$$

for all $t_0\leq r\leq t$ and $x\in X_3^+$. In [3, Proposition 1] we established that the 2-parameter family of mappings $U_{t,t_0}:X_3^+\to X_3^+$, $t_0\leq t$, given by

$$U_{t,t_0}(S_0,I_0,R_0)=(S(t),I(t),R(t)),$$

where $(S(t),I(t),R(t))$ is the unique positive solution of (1)–(3) with the initial value (S_0,I_0,R_0) defines a continuous process on X_3^+. Then, we studied the asymptotic behavior of this process in the framework of pullback attractors. Recall that a pullback attractor for the process U (e.g., cf. [7, 8, 9]) in the space X_3^+ is a family $\mathcal{A}=\{A(t),t\in\mathbb{R}\}$ of nonempty compact subsets of X_3^+, which is invariant in the sense that

$$U_{t,t_0}A(t_0)=A(t)\qquad\text{for all } t\geq t_0$$

and pullback attracts bounded subsets D of X_3^+, i.e.,

$$\mathrm{dist}_{X_3^+}\left(U_{t,t_0}D,A(t)\right)\to 0\quad\text{as}\quad t_0\to-\infty,$$

where we denote by $\mathrm{dist}_{X_3^+}(\mathcal{O}_1,\mathcal{O}_2)$ the Hausdorff semi-distance in X_3^+ between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_{X_3^+}(\mathcal{O}_1,\mathcal{O}_2)=\sup_{x\in\mathcal{O}_1}\inf_{y\in\mathcal{O}_2}d_{X_3^+}(x,y)\quad\text{for }\mathcal{O}_1,\,\mathcal{O}_2\subset X_3^+.$$

Namely, in [3, Theorem 6.2.] we establish that the process U_{t,t_0} has a unique pullback attractor.

However, as far as we know, there are no results in the literature concerning regularity of the pullback attractor as we will consider in this paper (for similar results for the reaction-diffusion equations see [2, 4], and for the Navier-Stokes equations see [10]). The regularity results on the solutions and the attractors (that we obtain here) might be useful in the future in order to implement new methods to seek solutions of more general problems by different arguments, to gain attraction in higher norms, or for numerical purposes.

The structure of the paper is as follows. In Section 2 we establish a regularity result for the unique positive solution to problem (1)-(3). In Section 3 we prove some results which, in particular, imply that, under suitable assumptions, the pullback attractor $\mathcal{A}$ for U_{t,t_0} satisfies that $A(t)$ is a bounded subset of $Z_3^+ \cap Y_3^+$, for every $t \in \mathbb{R}$.

2 A regularity result

In this section, we prove a regularity result for the positive solution to (1)-(3), whose existence and uniqueness are guaranteed in [3].

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the linear operator associated with the negative Laplacian. The operator A is symmetric, coercive and continuous. Since the space $H_0^1(\Omega)$ is included in $L^2(\Omega)$ with compact injection, as a consequence of the Hilbert-Schmidt Theorem there exists a nondecreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues of A with zero Dirichlet boundary condition in Ω, with $\lim_{j\to\infty} \lambda_j = +\infty$ and there exists an orthonormal basis of Hilbert $\{w_j : \ j \geq 1\}$ of $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ with $V_n := span \, \{w_j : \ 1 \leq j \leq n\}$ densely embedded in $H_0^1(\Omega)$, such that

$$Aw_j = \lambda_j w_j \text{ for all } j \geq 1.$$

For each integer $n \geq 1$, we denote by $(S_n(t), I_n(t), R_n(t)) = (S_n(t; t_0, S_0), I_n(t; t_0, I_0), R_n(t; t_0, R_0))$ the Galerkin approximation of the solution $(S(t; t_0, S_0), I(t; t_0, I_0), R(t; t_0, R_0))$ of (1)-(3), which is given by

$$S_n(t) = \sum_{j=1}^{n} \gamma_{nj}^1(t) w_j, \quad I_n(t) = \sum_{j=1}^{n} \gamma_{nj}^2(t) w_j, \quad R_n(t) = \sum_{j=1}^{n} \gamma_{nj}^3(t) w_j, \tag{4}$$

and is the solution of

$$\frac{d}{dt}\left(S_n(t), w_j\right) = \langle \Delta S_n(t), w_j \rangle + \left(f_1(S_n(t), I_n(t), R_n(t), t), w_j\right), \quad \forall w_j \in V_n \tag{5}$$

$$\frac{d}{dt}\left(I_n(t), w_j\right) = \langle \Delta I_n(t), w_j \rangle + \left(f_2(S_n(t), I_n(t), R_n(t), t), w_j\right), \quad \forall w_j \in V_n \tag{6}$$

$$\frac{d}{dt}\left(R_n(t), w_j\right) = \langle \Delta R_n(t), w_j \rangle + \left(f_3(S_n(t), I_n(t), R_n(t), t), w_j\right), \quad \forall w_j \in V_n \tag{7}$$

with initial data

$$(S_n(t_0), w_j) = (S_0, w_j)\,, (I_n(t_0), w_j) = (I_0, w_j)\,, (R_n(t_0), w_j) = (R_0, w_j)\,, \quad (8)$$

for all $w_j \in V_n$, where

$$\gamma^1_{nj}(t) = (S_n(t), w_j), \quad \gamma^2_{nj}(t) = (I_n(t), w_j), \quad \gamma^3_{nj}(t) = (R_n(t), w_j).$$

We denote

$$\begin{aligned} f_1(S_n(t), I_n(t), R_n(t), t) &:= aq(t) - aS_n(t) + bI_n(t) - \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\ f_2(S_n(t), I_n(t), R_n(t), t) &:= -(a+b+c)I_n(t) + \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\ f_3(S_n(t), I_n(t), R_n(t), t) &:= cI_n(t) - aR_n(t), \end{aligned}$$

where $N_n(t) = S_n(t) + I_n(t) + R_n(t)$.

Remark 1 *It can be proved that $S_n(t; t_0, S_0), I_n(t; t_0, I_0), R_n(t; t_0, R_0) \geq 0$ for all $t \geq t_0$ (see, for instance, Theorem 2.1 from Chapter 5 in [19]).*

On the other hand, if we denote

$$D(A) = \left\{ v \in H^1_0(\Omega) : Av \in L^2(\Omega) \right\},$$

with the scalar product

$$(v, w)_{D(A)} = (Av, Aw) \quad \forall v, w \in D(A),$$

then $D(A)$ is a Hilbert space, and $D(A)$ is included in $H^1_0(\Omega)$ with continuous and dense injection. Let $D(A)^+$ be the subspace of non-negative functions in $D(A)$.

Remark 2 *We note that if $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain, then we have that $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, and moreover the norm induced by $(\cdot, \cdot)_{D(A)}$ in $D(A)$ and the norm of $H^2(\Omega)$ are equivalent.*

Next result gives some preliminary (and standard) uniform estimates for the Galerkin approximations defined above. We include its proof for clarity since it will be used in the sequel.

Lemma 3 *Assume that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$ where λ_1 is the first eigenvalue of the operator A on the domain Ω with Dirichlet boundary condition. For any bounded set $B \subset X_3^+$, consider $(S_0, I_0, R_0) \in B$. Then, the sequence of corresponding solutions to the Galerkin scheme (4)-(8) is bounded in $L^2(t_0, T; Y_3^+)$ for all $T > t_0$.*

Proof. Multiplying by γ_{nj}^1 in (5), by γ_{nj}^2 in (6), by γ_{nj}^3 in (7) and summing from $j = 1$ to n, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\Big(|S_n(r)|^2_{L^2(\Omega)}+|I_n(r)|^2_{L^2(\Omega)}+|R_n(r)|^2_{L^2(\Omega)}\Big)+\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2 \\ &= \ (f_1(S_n(r),I_n(r),R_n(r),r),S_n(r))+(f_2(S_n(r),I_n(r),R_n(r),r),I_n(r)) \\ &+ \ (f_3(S_n(r),I_n(r),R_n(r),r),R_n(r)) . \end{aligned} \tag{9}$$

We have

$$\begin{aligned} (f_1(S_n(r),I_n(r),R_n(r),r),S_n(r)) \ &\le \ \frac{a}{4}(q^+)^2\,|\Omega| + (\frac{b}{2}+\gamma)\,|S_n(r)|^2_{L^2(\Omega)} \\ &+ \ \frac{b}{2}\,|I_n(r)|^2_{L^2(\Omega)} , \end{aligned} \tag{10}$$

where

$$\left(\frac{S_n(r)I_n(r)}{N_n(r)},S_n(r)\right) \le \int_\Omega \left|\frac{I_n(r,x)}{N_n(r,x)}\right| |S_n(r,x)|^2\,dx \le |S_n(r)|^2_{L^2(\Omega)} ,$$

since $|I_n/N_n| \le 1$.

We deduce

$$(f_2(S_n(r),I_n(r),R_n(r),r),I_n(r)) \le \gamma\,|I_n(r)|^2_{L^2(\Omega)} , \tag{11}$$

since $|S_n/N_n| \le 1$ and we also obtain

$$(f_3(S_n(r),I_n(r),R_n(r),r),R_n(r)) \le \frac{c}{2}\,|I_n(r)|^2_{L^2(\Omega)} + \frac{c}{2}\,|R_n(r)|^2_{L^2(\Omega)} . \tag{12}$$

Taking into account (10)-(12) in (9), we can deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\Big(|S_n(r)|^2_{L^2(\Omega)}+|I_n(r)|^2_{L^2(\Omega)}+|R_n(r)|^2_{L^2(\Omega)}\Big)+\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2 \\ &\le \ \frac{a}{4}(q^+)^2\,|\Omega|+\left(\frac{b+c}{2}+\gamma\right)\Big(|S_n(r)|^2_{L^2(\Omega)}+|I_n(r)|^2_{L^2(\Omega)}+|R_n(r)|^2_{L^2(\Omega)}\Big) . \end{aligned}$$

By the Poincaré inequality, we have

$$\begin{aligned} &\frac{d}{dr}\Big(|S_n(r)|^2_{L^2(\Omega)}+|I_n(r)|^2_{L^2(\Omega)}+|R_n(r)|^2_{L^2(\Omega)}\Big) \\ &+(2-\lambda_1^{-1}(b+c+2\gamma))\Big(\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\Big) \le \frac{a}{2}(q^+)^2\,|\Omega| , \end{aligned} \tag{13}$$

provided $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$.

Integrating between t_0 and T, the result follows. ■

Now we may establish a regularity result for the solution to the problem.

Theorem 4 *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain and assume that $\gamma+\frac{b}{2}+\frac{c}{2} < \lambda_1$ where λ_1 is the first eigenvalue of the operator A on the domain Ω with Dirichlet boundary condition. Then, for any initial condition $(S_0, I_0, R_0) \in Y_3^+$, the solution to (1)-(3), whose existence and uniqueness are guaranteed in [3], satisfies in addition that $(S, I, R) \in C\left([t_0, T]; Y_3^+\right) \cap L^2\left(t_0, T; (D(A)^+)^3\right)$, for all $T > t_0$.*

Proof. Let $(S_0, I_0, R_0) \in Y_3^+$, and we consider the basis of Hilbert $\{w_j : j \geq 1\}$ of $L^2(\Omega)$ formed by the eigenfunctions associated with eigenvalues of the operator A with zero Dirichlet boundary condition in Ω. It is not difficult to conclude that $w_j \in D(A)$.

For each integer $n \geq 1$, we consider the sequence $\{(S_n, I_n, R_n)\}$ defined by (4)-(8).

Multiplying by the derivative $(\gamma_{nj}^1)'$ in (5), by the derivative $(\gamma_{nj}^2)'$ in (6), by the derivative $(\gamma_{nj}^3)'$ in (7) and summing from $j = 1$ to n,

$$\begin{aligned}
&|S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dr}\Big(\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2\Big) \\
&= \ (f_1(S_n(r), I_n(r), R_n(r), r), S_n'(r)) + (f_2(S_n(r), I_n(r), R_n(r), r), I_n'(r)) \\
&+ \ (f_3(S_n(r), I_n(r), R_n(r), r), R_n'(r)) \, . \qquad (14)
\end{aligned}$$

We obtain

$$\begin{aligned}
(f_1(S_n(r), I_n(r), R_n(r), r), S_n'(r)) &\leq a^2 (q^+)^2 |\Omega| + \frac{1}{2} |S_n'(r)|^2_{L^2(\Omega)} \qquad (15) \\
&+ 3(a^2 + \gamma^2) |S_n(r)|^2_{L^2(\Omega)} + 3b^2 |I_n(r)|^2_{L^2(\Omega)},
\end{aligned}$$

where

$$\begin{aligned}
\left(\gamma \frac{S_n(r) I_n(r)}{N_n(r)}, S_n'(r)\right) &\leq \int_\Omega \gamma \left|\frac{I_n(r,x)}{N_n(r,x)}\right| |S_n(r,x)| \, |S_n'(r,x)| \, dx \\
&\leq 3\gamma^2 |S_n(r)|^2_{L^2(\Omega)} + \frac{1}{12} |S_n'(r)|^2_{L^2(\Omega)} \, ,
\end{aligned}$$

since $|I_n/N_n| \leq 1$.

We have

$$\begin{aligned}
(f_2(S_n(r), I_n(r), R_n(r), r), I_n'(r)) \ &\leq \ ((a+b+c)^2 + \gamma^2) |I_n(r)|^2_{L^2(\Omega)} \qquad (16) \\
&+ \ \frac{1}{2} |I_n'(r)|^2_{L^2(\Omega)},
\end{aligned}$$

since $|S_n/N_n| \leq 1$.

We also obtain

$$\begin{aligned}
(f_3(S_n(r), I_n(r), R_n(r), r), R_n'(r)) &\leq \frac{1}{2} |R_n'(r)|^2_{L^2(\Omega)} + c^2 |I_n(r)|^2_{L^2(\Omega)} \qquad (17) \\
&+ a^2 |R_n(r)|^2_{L^2(\Omega)} \, .
\end{aligned}$$

Taking into account (15)-(17) in (14), we have

$$\begin{aligned}
&|S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} + \frac{d}{dr}\left(\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2\right) \\
&\leq \ 2a^2 (q^+)^2 |\Omega| + k_1 \left(|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)}\right) \qquad (18)
\end{aligned}$$

for an appropriate positive constant k_1.

Integrating now between t_0 and t, we obtain

$$\begin{aligned}
&\int_{t_0}^{t}\Big(|S_n'(\theta)|^2_{L^2(\Omega)}+|I_n'(\theta)|^2_{L^2(\Omega)}+|R_n'(\theta)|^2_{L^2(\Omega)}\Big)d\theta+\|S_n(t)\|^2+\|I_n(t)\|^2+\|R_n(t)\|^2\\
\leq\ & \|S_0\|^2+\|I_0\|^2+\|R_0\|^2+2a^2(q^+)^2\,|\Omega|\,(t-t_0)\\
+\ & k_1\int_{t_0}^{t}\Big(|S_n(\theta)|^2_{L^2(\Omega)}+|I_n(\theta)|^2_{L^2(\Omega)}+|R_n(\theta)|^2_{L^2(\Omega)}\Big)d\theta. \qquad (19)
\end{aligned}$$

Using the Poincaré inequality in (13) and integrating between t_0 and t, in particular, we obtain

$$\begin{aligned}
&\int_{t_0}^{t}\Big(|S_n(\theta)|^2_{L^2(\Omega)}+|I_n(\theta)|^2_{L^2(\Omega)}+|R_n(\theta)|^2_{L^2(\Omega)}\Big)d\theta \qquad (20)\\
\leq\ & C\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}+\frac{a}{2}(q^+)^2\,|\Omega|\,(t-t_0)\right),\ \ \forall n\geq 1,
\end{aligned}$$

where $C(\lambda_1,b,c,\gamma):=(2\lambda_1-b-c-2\gamma)^{-1}$.

Taking into account (20) in (19), we deduce

$$\begin{aligned}
&\int_{t_0}^{t}\Big(|S_n'(\theta)|^2_{L^2(\Omega)}+|I_n'(\theta)|^2_{L^2(\Omega)}+|R_n'(\theta)|^2_{L^2(\Omega)}\Big)d\theta+\|S_n(t)\|^2+\|I_n(t)\|^2+\|R_n(t)\|^2\\
\leq\ & \left(1+k_1\lambda_1^{-1}C\right)\left(\|S_0\|^2+\|I_0\|^2+\|R_0\|^2\right)+(q^+)^2\,|\Omega|\,(t-t_0)(2a^2+\frac{a}{2}k_1C),
\end{aligned}$$

for all $t_0\leq t$.

It follows that $\{(S_n,I_n,R_n)\}$ is bounded in $L^\infty\left(t_0,T;Y_3^+\right)$ and $\{S_n',I_n',R_n'\}$ is bounded in $L^2\left(t_0,T;X_3\right)$, for all $T>t_0$.

Taking into account the uniqueness of solution, it is not difficult to conclude that the sequence $\{(S_n,I_n,R_n)\}$ converges weakly-star in $L^\infty\left(t_0,T;Y_3^+\right)$ to the solution (S,I,R) of (1)-(3), and we also obtain that $(S',I',R')\in L^2\left(t_0,T;X_3\right)$.

Now, we will see that the Galerkin sequence $\{(S_n,I_n,R_n)\}$ is bounded in $L^2\left(t_0,T;(D(A)^+)^3\right)$, and in which case we will have that $(S,I,R)\in L^2\left(t_0,T;(D(A)^+)^3\right)$.

As $(S,I,R)\in L^\infty\left(t_0,T;Y_3^+\right)$ and $(S',I',R')\in L^2\left(t_0,T;X_3\right)$, by Theorem 2.1 in [21], we can deduce that $(S,I,R)\in C\left([t_0,T]\,;Y_3^+\right)$.

To prove that the sequence $\{(S_n,I_n,R_n)\}$ is bounded in $L^2\left(t_0,T;(D(A)^+)^3\right)$, multiplying in (5) by $\lambda_j\gamma_{nj}^1$, in (6) by $\lambda_j\gamma_{nj}^2$ and in (7) by $\lambda_j\gamma_{nj}^3$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing once more from $j=1$ to n, we have

$$\begin{aligned}
&(S_n'(r),\Delta S_n(r))+(I_n'(r),\Delta I_n(r))+(R_n'(r),\Delta R_n(r)) \qquad (21)\\
=\ & |\Delta S_n(r)|^2_{L^2(\Omega)}+|\Delta I_n(r)|^2_{L^2(\Omega)}+|\Delta R_n(r)|^2_{L^2(\Omega)}\\
+\ & (f_1(S_n(r),I_n(r),R_n(r),r),\Delta S_n(r))+(f_2(S_n(r),I_n(r),R_n(r),r),\Delta I_n(r))\\
+\ & (f_3(S_n(r),I_n(r),R_n(r),r),\Delta R_n(r))\,.
\end{aligned}$$

We obtain

$$-\left(f_1(S_n(r),I_n(r),R_n(r),r),\Delta S_n(r)\right) \leq 2a^2(q^+)^2\,|\Omega| + \frac{1}{2}|\Delta S_n(r)|^2_{L^2(\Omega)} + 2(a^2+\gamma^2)|S_n(r)|^2_{L^2(\Omega)} + 2b^2|I_n(r)|^2_{L^2(\Omega)} \tag{22}$$

where

$$\left(\gamma\frac{S_n(r)I_n(r)}{N_n(r)},\Delta S_n(r)\right) \leq \int_\Omega \gamma\left|\frac{I_n(r,x)}{N_n(r,x)}\right| |S_n(r,x)|\,|\Delta S_n(r,x)|\,dx$$
$$\leq 2\gamma^2\,|S_n(r)|^2_{L^2(\Omega)} + \frac{1}{8}\,|\Delta S_n(r)|^2_{L^2(\Omega)},$$

since $|I_n/N_n| \leq 1$ and similarly, we have

$$-\left(f_2(S_n(r),I_n(r),R_n(r),r),\Delta I_n(r)\right) \leq \frac{1}{2}\,|\Delta I_n(r)|^2_{L^2(\Omega)} + \left((a+b+c)^2+\gamma^2\right)|I_n(r)|^2_{L^2(\Omega)}, \tag{23}$$

since $|S_n/N_n| \leq 1$.

We also obtain

$$-\left(f_3(S_n(r),I_n(r),R_n(r),r),\Delta R_n(r)\right) \leq \frac{1}{2}\,|\Delta R_n(r)|^2_{L^2(\Omega)} + c^2\,|I_n(r)|^2_{L^2(\Omega)} + a^2\,|R_n(r)|^2_{L^2(\Omega)}. \tag{24}$$

Taking into account (22)-(24) in (21), we have

$$\frac{d}{dr}\left(\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\right)+|\Delta S_n(r)|^2_{L^2(\Omega)}+|\Delta I_n(r)|^2_{L^2(\Omega)}+|\Delta R_n(r)|^2_{L^2(\Omega)}$$
$$\leq \quad 4a^2(q^+)^2\,|\Omega| + 2\lambda_1^{-1}k_2\left(\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\right),$$

for all $r \geq t_0$ and for an appropriate positive constant k_2.

Finally, integrating the last inequality between t_0 and T, and taking into account that $\{(S_n,I_n,R_n)\}$ is bounded in $L^\infty\left(t_0,T;Y_3^+\right)$, we can deduce that $\{(S_n,I_n,R_n)\}$ is also bounded in $L^2\left(t_0,T;(D(A)^+)^3\right)$. ■

Taking into account the uniqueness of solution of (1)-(3) and using Lemma 3, it is not difficult to conclude the following remark.

Remark 5 *Under the assumptions in Theorem 4, for any initial condition $(S_0,I_0,R_0) \in X_3^+$, the solution (S,I,R) of (1)-(3) satisfies*

$$(S,I,R) \in L^2(t_0,T;Y_3^+),$$

for all $T > t_0$.

As a consequence of Theorem 4 and Remark 5, we can now establish the following result.

Theorem 6 *Under the assumptions in Theorem 4, for any initial condition $(S_0,I_0,R_0) \in X_3^+$, the solution (S,I,R) of (1)-(3) satisfies*

$$(S,I,R) \in C\left((t_0,T]\,;Y_3^+\right),$$

for all $T > t_0$.

3 H^2-boundedness of invariants sets

In this section we will prove that under suitable assumptions, every family of bounded subsets of X_3^+ which is invariant for the process U, is in fact bounded in Z_3^+.

First, we recall a lemma (see [18]) which is necessary for the proof of our result.

Lemma 7 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$.*

Then, $u(t) \in X$ for all $t \in [t_0, T]$ and

$$\|u(t)\|_X \leq \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)} \qquad \forall t \in [t_0, T].$$

We first prove the following result

Proposition 8 *Assume the assumptions in Theorem 4. Then, for any bounded set $B \subset X_3^+$, any $t_0 \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > t_0 + \varepsilon$, the set $\{(S_n(r; t_0, S_0), I_n(r; t_0, I_0), R_n(r; t_0, R_0)) : r \in [t_0+\varepsilon, t],\ (S_0, I_0, R_0) \in B,\ n \geq 1\}$, is a bounded subset of Y_3^+.*

Proof. Let us fix a bounded set $B \subset X_3^+$, $t_0 \in \mathbb{R}$, $\varepsilon > 0$, $t > t_0 + \varepsilon$, and $(S_0, I_0, R_0) \in B$.

Integrating (18) between $s \in [t_0, r]$ and $r \leq t$, we obtain

$$\begin{aligned}
&\int_s^r\Big(|S_n'(\theta)|^2_{L^2(\Omega)}+|I_n'(\theta)|^2_{L^2(\Omega)}+|R_n'(\theta)|^2_{L^2(\Omega)}\Big)d\theta+\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\\
&\leq \|S_n(s)\|^2+\|I_n(s)\|^2+\|R_n(s)\|^2+2a^2(q^+)^2\,|\Omega|\,(t-t_0)\\
&+ k_1\int_{t_0}^t\Big(|S_n(\theta)|^2_{L^2(\Omega)}+|I_n(\theta)|^2_{L^2(\Omega)}+|R_n(\theta)|^2_{L^2(\Omega)}\Big)d\theta.
\end{aligned}$$

Using (20), we have

$$\begin{aligned}
&\int_s^r\Big(|S_n'(\theta)|^2_{L^2(\Omega)}+|I_n'(\theta)|^2_{L^2(\Omega)}+|R_n'(\theta)|^2_{L^2(\Omega)}\Big)d\theta+\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\\
&\leq \|S_n(s)\|^2+\|I_n(s)\|^2+\|R_n(s)\|^2+(q^+)^2\,|\Omega|\,(t-t_0)(2a^2+\frac{a}{2}k_1C)\\
&+ k_1C\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{25}$$

for all $s \in [t_0, r]$, and any $r \in [t_0, t]$.

Integrating in this last inequality with respect to s from t_0 to r, we, in particular, obtain

$$\begin{aligned}
&(r-t_0)\left(\|S_n(r)\|^2+\|I_n(r)\|^2+\|R_n(r)\|^2\right) \qquad (26)\\
\le &\int_{t_0}^{t}\!\left(\|S_n(s)\|^2+\|I_n(s)\|^2+\|R_n(s)\|^2\right)ds+(q^+)^2\,|\Omega|\,(t-t_0)^2(2a^2+\frac{a}{2}k_1C)\\
+\ & k_1C\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}\right)(t-t_0).
\end{aligned}$$

for all $r\in[t_0,t]$, and for any $n\ge 1$.

Now, integrating (13) between t_0 and r, we obtain

$$\begin{aligned}
&|S_n(r)|^2_{L^2(\Omega)}+|I_n(r)|^2_{L^2(\Omega)}+|R_n(r)|^2_{L^2(\Omega)} \qquad (27)\\
+\ &(2-\lambda_1^{-1}(b+c+2\gamma))\int_{t_0}^{r}\left(\|S_n(s)\|^2+\|I_n(s)\|^2+\|R_n(s)\|^2\right)ds\\
\le\ &|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}+\frac{a}{2}(q^+)^2\,|\Omega|\,(t-t_0),
\end{aligned}$$

for all $r\in[t_0,t]$, $n\ge 1$.

From (26) and (27), our result holds. ■

Corollary 9 *Under the assumptions in Proposition 8, for any bounded set* $B\subset X_3^+$*, any* $t_0\in\mathbb{R}$*, any* $\varepsilon>0$*, and any* $t>t_0+\varepsilon$*, the set* $\bigcup_{r\in[t_0+\varepsilon,t]}U_{r,t_0}B$ *is a bounded subset of* Y_3^+.

Proof. This is a straightforward consequence of Lemma 7, Proposition 8 and the fact that the Galerkin sequence $(S_n(\cdot;t_0,S_0),I_n(\cdot;t_0,I_0),R_n(\cdot;t_0,R_0))$ converges weakly to the unique solution to (1)-(3) $(S(\cdot;t_0,S_0),I(\cdot;t_0,I_0),R(\cdot;t_0,R_0))$ in $L^2(t_0,t;Y_3^+)$. ■

Proposition 10 *Under the assumptions in Proposition 8, suppose moreover that* $q'\in L^2_{loc}\left(\mathbb{R};L^2(\Omega)\right)$*. Then, for any bounded set* $B\subset X_3^+$*, any* $t_0\in\mathbb{R}$*, any* $\varepsilon>0$*, and any* $t>t_0+\varepsilon$*, the set* $\{(S_n(r;t_0,S_0),I_n(r;t_0,I_0),R_n(r;t_0,R_0)):r\in[t_0+\varepsilon,t],\ (S_0,I_0,R_0)\in B,\ n\ge 1\}$ *is a bounded subset of* Z_3^+.

Proof. Let us fix a bounded set $B\subset X_3^+$, $t_0\in\mathbb{R}$, $\varepsilon>0$, $t>t_0+\varepsilon$, and $(S_0,I_0,R_0)\in B$.

As we are assuming that $q'\in L^2_{loc}\left(\mathbb{R};L^2(\Omega)\right)$, we can differentiate with respect to time in (5), and then, multiplying by $(\gamma^1_{nj})'$, and summing from $j=1$ to n, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dr}\left(|S_n'(r)|^2_{L^2(\Omega)}\right)+\|S_n'(r)\|^2\\
=\ &\left(aq'(r)-aS_n'(r)+bI_n'(r)-\gamma\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)',S_n'(r)\right). \qquad (28)
\end{aligned}$$

Now, we differentiate with respect to time in (6), and then, multiplying by $(\gamma_{nj}^2)'$, and summing from $j=1$ to n, we have

$$\frac{1}{2}\frac{d}{dr}\left(|I_n'(r)|^2_{L^2(\Omega)}\right)+\|I_n'(r)\|^2=\left(-(a+b+c)I_n'(r)+\gamma\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)', I_n'(r)\right). \quad (29)$$

On the other hand, we differentiate with respect to time in (7), and then, multiplying by $(\gamma_{nj}^3)'$, and summing from $j=1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dr}\left(|R_n'(r)|^2_{L^2(\Omega)}\right)+\|R_n'(r)\|^2=\left(cI_n'(r)-aR_n'(r),R_n'(r)\right). \quad (30)$$

We observe that

$$\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)'=S_n'(r)\frac{I_n(r)}{N_n(r)}+\frac{S_n(r)}{N_n(r)}I_n'(r)-\frac{S_n(r)I_n(r)}{N_n^2(r)}N_n'(r). \quad (31)$$

Using (31) we obtain

$$\begin{aligned}&\left(aq'(r)-aS_n'(r)+bI_n'(r)-\gamma\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)', S_n'(r)\right) \quad (32)\\ \leq\ & \frac{1}{2}a\,|q'(r)|^2_{L^2(\Omega)}+\frac{1}{2}(a+7\gamma+b)\,|S_n'(r)|^2_{L^2(\Omega)}\\ +\ & (\gamma+\frac{b}{2})\,|I_n'(r)|^2_{L^2(\Omega)}+\frac{\gamma}{2}\,|R_n'(r)|^2_{L^2(\Omega)}\end{aligned}$$

where

$$\left(\gamma\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)', S_n'(r)\right)\leq\frac{7\gamma}{2}\,|S_n'(r)|^2_{L^2(\Omega)}+\gamma\,|I_n'(r)|^2_{L^2(\Omega)}+\frac{\gamma}{2}\,|R_n'(r)|^2_{L^2(\Omega)},$$

since $|I_n/N_n|\,,|S_n/N_n|\leq 1$ and similarly, we have

$$\begin{aligned}&\left(-(a+b+c)I_n'(r)+\gamma\left(\frac{S_n(r)I_n(r)}{N_n(r)}\right)', I_n'(r)\right) \quad (33)\\ \leq\ & \gamma\,|S_n'(r)|^2_{L^2(\Omega)}+\frac{7\gamma}{2}\,|I_n'(r)|^2_{L^2(\Omega)}+\frac{\gamma}{2}\,|R_n'(r)|^2_{L^2(\Omega)}.\end{aligned}$$

We also obtain

$$\left(cI_n'(r)-aR_n'(r),R_n'(r)\right)\leq\frac{c}{2}\,|I_n'(r)|^2_{L^2(\Omega)}+\frac{c}{2}\,|R_n'(r)|^2_{L^2(\Omega)}. \quad (34)$$

Taking into account (32) in (28), (33) in (29) and (34) in (30), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dr}\left(|S_n'(r)|^2_{L^2(\Omega)}+|I_n'(r)|^2_{L^2(\Omega)}+|R_n'(r)|^2_{L^2(\Omega)}\right)+\|S_n'(r)\|^2+\|I_n'(r)\|^2+\|R_n'(r)\|^2\\ \leq\ & \frac{1}{2}a\,|q'(r)|^2_{L^2(\Omega)}+k_3\left(|S_n'(r)|^2_{L^2(\Omega)}+|I_n'(r)|^2_{L^2(\Omega)}+|R_n'(r)|^2_{L^2(\Omega)}\right),\end{aligned}$$

for an appropriate positive constant k_3.

In particular, integrating in the last inequality, it follows

$$\begin{aligned}
& |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \\
\leq \;& |S_n'(s)|^2_{L^2(\Omega)} + |I_n'(s)|^2_{L^2(\Omega)} + |R_n'(s)|^2_{L^2(\Omega)} + a\int_{t_0+\varepsilon/2}^{t} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta \\
+ \;& 2k_3 \int_{t_0+\varepsilon/2}^{t} \left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)} \right) d\theta,
\end{aligned}$$

for all $t_0 + \varepsilon/2 \leq s \leq r \leq t$.

Now, integrating with respect to s between $t_0 + \varepsilon/2$ and r,

$$\begin{aligned}
& (r - t_0 - \varepsilon/2) \left(|S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \right) \\
& \leq [2k_3(t - t_0 - \varepsilon/2) + 1] \int_{t_0+\varepsilon/2}^{t} \left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)} \right) d\theta \\
& + \; (r - t_0 - \varepsilon/2) a \int_{t_0+\varepsilon/2}^{t} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta,
\end{aligned}$$

for all $t_0 + \varepsilon/2 \leq r \leq t$, and, in particular,

$$\begin{aligned}
& |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \\
& \leq 2\varepsilon^{-1} [2k_3(t - t_0 - \varepsilon/2) + 1] \int_{t_0+\varepsilon/2}^{t} \left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)} \right) d\theta \\
& + \; a \int_{t_0+\varepsilon/2}^{t} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta,
\end{aligned} \tag{35}$$

for all $r \in [t_0 + \varepsilon, t]$.

On the other hand, taking into account (22)-(24) in (21), we have

$$\begin{aligned}
& \frac{1}{4} \left(|\Delta S_n(r)|^2_{L^2(\Omega)} + |\Delta I_n(r)|^2_{L^2(\Omega)} + |\Delta R_n(r)|^2_{L^2(\Omega)} \right) \\
\leq \;& |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} + 2a^2 (q^+)^2 |\Omega| \\
+ \;& k_2 \left(|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)} \right),
\end{aligned} \tag{36}$$

for all $r \geq t_0$.

Finally, observe that by (25)

$$\begin{aligned}
& \int_{t_0+\varepsilon/2}^{t} \left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)} \right) d\theta \\
\leq \;& \|S_n(t_0 + \varepsilon/2)\|^2 + \|I_n(t_0 + \varepsilon/2)\|^2 + \|R_n(t_0 + \varepsilon/2)\|^2 \\
+ \;& (q^+)^2 |\Omega| (t - t_0)(2a^2 + \frac{a}{2} k_1 C) + k_1 C \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} \right).
\end{aligned} \tag{37}$$

The result is a direct consequence of Proposition 8 and estimates (35), (36) and (37). ■

Corollary 11 *Under the assumptions of Proposition 10, for any bounded set* $B \subset X_3^+$*, any* $t_0 \in \mathbb{R}$*, any* $\varepsilon > 0$*, and any* $t > t_0 + \varepsilon$*, the set* $\bigcup_{r \in [t_0+\varepsilon, t]} U_{r,t_0} B$ *is a bounded subset of* Z_3^+.

Proof. This follows from Lemma 7, propositions 8 and 10, and the facts that the sequence $(S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution to (1)-(3) $(S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$ in $L^2(t_0, t; (D(A)^+)^3)$ (see Lemma 3 and propositions 8 and 10) and that $(S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0)) \in C([t_0 + \varepsilon, t]; Y_3^+)$ (see Theorem 6). ■

As a direct consequence of the above results, we can now establish our main result.

Theorem 12 *Under the assumptions in Proposition 10, if* $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ *is a family of bounded subsets of* X_3^+*, such that* $U_{t,t_0} A(t_0) = A(t)$ *for any* $t_0 \leq t$*, then for any* $T_1 < T_2$*, the set* $\bigcup_{t \in [T_1, T_2]} A(t)$ *is a bounded subset of* $Z_3^+ \cap Y_3^+$.

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