Breaking symmetries of graphs with resolving sets *

D. Garijo[1], A. González[1], and A. Márquez[1]

Department of Applied Mathematics I, University of Seville, Spain
{dgarijo,gonzalezh,almar}@us.es

Abstract. We undertake a study on the maximum value of the difference between the metric dimension and the determining number of a graph as a function of its order. Our results include lower and upper bounds on that maximum, and exact computations when restricting to some specific families of graphs. Although our technique is mainly based on locating-dominating sets, it also requires very diverse tools and relationships with well-known objects in graph theory; among them: a classical result in graph domination by Ore, a Ramsey-type result by ErdHos and Szekeres, a polynomial time algorithm to compute distinguishing sets and dominating sets of twin-free graphs, k-dominating sets, and matchings.

Key words: determining set, determining number, resolving set, metric dimension, locating-dominating set.

1 Introduction and preliminaries

In this paper we focuss on two graph parameters that have attracted much attention in recent years: the determining number and the metric dimension. Concretely, we deal with the following question proposed by Boutin [2]: *Can the difference between the determining number and the metric dimension of a graph be arbitrarily large?* We begin with some definitions and notations.

Let G be a finite, connected, undirected, and simple graph[2]. The *stabilizer* of a set $S \subseteq V(G)$ is $\mathrm{Stab}(S) = \{\phi \in \mathrm{Aut}(G) : \phi(u) = u, \forall u \in S\}$, and S is a

* The first and second authors are partially supported by projects 2010/FQM-164 and 2011/FQM-164. The third author is partially supported by the ESF EUROCORES programme EuroGIGA ComPoSe IP04 MICINN Project EUI-EURC-2011-4306, and projects 2010/FQM-164 and 2011/FQM-164.

[2] The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively; the order of G is $n = |V(G)|$. As usual, $\overline{G}$ denotes the complement of G. An *automorphism* of G is a bijective mapping of $V(G)$ onto itself such that $f(u)f(v) \in E(G)$ if and only if $uv \in E(G)$. The *automorphism group* of G is written as $\mathrm{Aut}(G)$, and its identity element is id_G. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest u-v path. Finally, we write $N(u)$ for the open neighbourhood of a vertex $u \in V(G)$.

determining set of G if $\mathrm{Stab}(S)$ is trivial. The minimum cardinality of a determining set is the *determining number* of G, denoted by $\mathrm{Det}(G)$. On the other hand, a vertex $u \in V(G)$ *resolves* a pair $\{x, y\} \subseteq V(G)$ if $d(u, x) \neq d(u, y)$. When every pair of vertices of G is resolved by some vertex in S, it is said that S is a *resolving set* of G. The minimum cardinality of a resolving set is the *metric dimension* of G, written as $\dim(G)$.

Determining sets of graphs are particular cases of *bases* of permutation groups, defined by Sims [9] in 1971 as subsets of elements whose stabilizer is trivial. Much later, Boutin [2] and Erwin and Harary [5] used the terms determining set and *fixing set*, respectively, to refer to a base of the automorphism group of a graph. Also in the 1970s, Harary and Melter [7], and independently Slater [8], introduced the notion of resolving set. For more references on these topics, we refer the reader to the survey of Bailey and Cameron [1].

The above-mentioned question posed by Boutin comes from the fact that every resolving set of a graph G is also a determining set, and so $\mathrm{Det}(G) \leq \dim(G)$ (see [2,5]). To study this question, we define the function $(\dim - \mathrm{Det})(n)$ as the maximum value of $\dim(G) - \mathrm{Det}(G)$ over all graphs G of order n, and develop a technique based on locating-dominating sets, whose definition (recalled from [11]) is provided below. The following result, written in terms of our function, is the best approach to date on the problem.

Proposition 1. [3] *For every $n \geq 8$,*

$$\lfloor \frac{2}{5} n \rfloor - 2 \leq (\dim - \mathrm{Det})(n) \leq n - 2.$$

A pair $\{x, y\} \subseteq V(G)$ is *distinguished* by a vertex $u \in V(G)$ if either $u \in \{x, y\}$ or $N(x) \cap \{u\} \neq N(y) \cap \{u\}$, and a set $D \subseteq V(G)$ is a *distinguishing set* of G if every pair of $V(G)$ is distinguished by some vertex in D. If D is also a *dominating set*, i.e., $N(x) \cap D \neq \emptyset$ for every $x \in V(G) \setminus D$, then D is a *locating-dominating set*. The minimum cardinality of a locating-dominating set is the *locating-domination number* of G, denoted by $\lambda(G)$.

Note that distinguishing sets and locating-dominating sets are in essence the same concept: one can easily check that every distinguishing set becomes a dominating set by adding at most one vertex. Thus,

Remark 1. Let D be a distinguishing set of a graph G. Then, $\lambda(G) \leq |D| + 1$.

Every locating-dominating set is clearly a resolving set, and so $\mathrm{Det}(G) \leq \dim(G) \leq \lambda(G)$ for any graph G. Let $(\lambda - \mathrm{Det})(n)$ and $\lambda(n)$ be the maximum values of, respectively, $\lambda(G) - \mathrm{Det}(G)$ and $\lambda(G)$ over all graphs G of order n. Obviously,

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda(n).$$

The function $\lambda(n)$ equals $n-1$ (attained by the complete graph K_n) but the non-trivial restriction of this function to the class $\mathcal{C}^*$ of twin-free graphs (i.e., graphs that do not contain twin vertices) denoted by $\lambda_{|\mathcal{C}^*}(n)$ is fundamental in this work.

The paper is organized as follows. In Section 2, we provide lower bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ by constructing appropriate families of graphs. Section 3 establishes first that $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ are bounded above by $\lambda_{|\mathcal{C}^*}(n)$ and then provides two upper bounds on this last function. The first bound is obtained by combining a variant of a classical theorem in domination theory due to Ore [8] and a Ramsey-type result of ErdHos and Szekeres [4]. The second upper bound on $\lambda_{|\mathcal{C}^*}(n)$ comes from a greedy algorithm to compute distinguishing sets and determining sets of bounded size, which in addition gives an upper bound on the determining number of a twin-free graph. In Section 4, we study the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ restricted to the class $\mathcal{C}_4$ of graphs not containing the cycle C_4 as a subgraph, and its subclass $\mathcal{T}$ of trees. For this purpose, we use two well-known invariants of graphs: the k-domination number and the matching number. We conclude in Section 5 with some remarks and open problems.

2 Lower bounds on $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$

Cáceres et al. [3] used the wheel graph $W_{1,n}$ to obtain the lower bound on $(\dim - \mathrm{Det})(n)$ given in Proposition 1. In order to get a better approach, we construct two adequate families of graphs that, in addition, give a lower bound on the function $(\lambda - \mathrm{Det})(n)$.

Let T_r, $r \geq 6$, be a tree that consists of a path $(u_1, ..., u_r)$ and a pendant vertex u_0 adjacent to u_3, and let G_r be the corona product $T_r \circ K_1$, i.e., the graph with vertex set $V(G_r) = \{u_0, u_1, ..., u_r, v_0, v_1, ..., v_r\}$ and edge set $E(G_r) = E(T_r) \cup \{u_i v_i : i \in \{0, 1, ..., r\}\}$. By adding another pendant vertex v_0' to u_0 in G_r we obtain the graph H_r (see Figure 1).

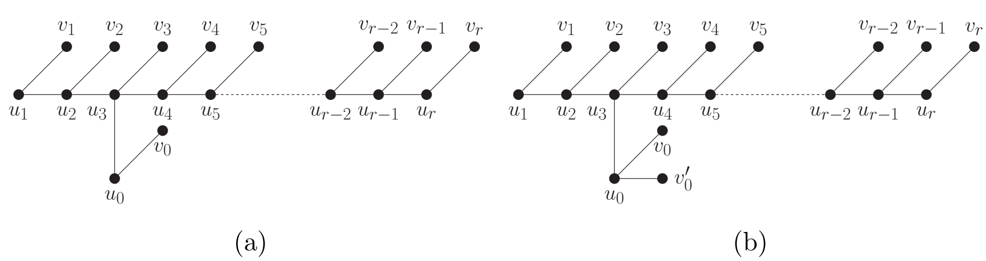

Fig. 1: The graphs (a) G_r and (b) H_r.

The following lemma is the key tool to obtain lower bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$.

Lemma 1. *For every $r \geq 6$, the following statements hold:*

(i) $\mathrm{Det}(G_r) = 0$ *and* $\mathrm{Det}(H_r) = 1$.
(ii) $\dim(\overline{G}_r) = r$ *and* $\dim(\overline{H}_r) = r + 1$.
(iii) $\lambda(G_r) = r + 1$ *and* $\lambda(H_r) = r + 2$.

With this lemma in hand, one can prove that G_r, H_r, and their complements (for appropriate r) are the above-mentioned adequate families of graphs which provide the following lower bounds.

Theorem 1. *For every $n \geq 14$,*

$$(\dim - \mathrm{Det})(n) \geq \lfloor \frac{n}{2} \rfloor - 1 \qquad \textit{and} \qquad (\lambda - \mathrm{Det})(n) \geq \lfloor \frac{n}{2} \rfloor.$$

We shall exhibit large classes of graphs in which the restrictions of $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ do not exceed $\frac{n}{2}$. Thus, we believe that the preceding bounds are in fact the exact values of our functions.

Conjecture 1. There exists a positive integer n_0 such that, for every $n \geq n_0$,

$$(\dim - \mathrm{Det})(n) = \lfloor \frac{n}{2} \rfloor - 1 \qquad \textit{and} \qquad (\lambda - \mathrm{Det})(n) = \lfloor \frac{n}{2} \rfloor.$$

3 Upper bounds on $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$

In order to obtain explicit upper bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$, our first step is to prove that they are bounded above by $\lambda_{|\mathcal{C}^*}(n)$. To do this, we associate to every graph G a twin-free graph $\widetilde{G}$, from which one can extract locating-dominating sets of G. We then prove that $\lambda(G) - \mathrm{Det}(G) \leq \lambda(\widetilde{G})$, whose maximum taken over all graphs of order n leads us to the desired inequality. This process comprises a complex machinery of technical results that we omit for the sake of brevity.

Theorem 2. *For every $n \geq 4$,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|\mathcal{C}^*}(n).$$

Theorems 4 and 2 give $\lfloor \frac{n}{2} \rfloor \leq \lambda_{|\mathcal{C}^*}(n)$. Further, in [6] we find numerous conditions for a twin-free graph G to satisfy $\lambda(G) \leq \lfloor \frac{n}{2} \rfloor$ (here we will indicate only some of them). Thus, we believe that the following conjecture, which implies most of Conjecture 1, is true.

Conjecture 2. There exists a positive integer n_1 such that, for every $n \geq n_1$,

$$\lambda_{|\mathcal{C}^*}(n) = \lfloor \frac{n}{2} \rfloor.$$

In the following two subsections, we are concerned with obtaining upper bounds on $\lambda_{|\mathcal{C}^*}(n)$ since, by Theorem 2, these are also bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$.

3.1 From minimal dominating sets to locating-dominating sets

A set $D \subseteq V(G)$ is a *minimal dominating set* if no proper subset of D is a dominating set of G. The following theorem, due to Ore [8], is one of the first results in the field of graph domination. Theorem 4 below is a variant of this result for twin-free graphs.

Theorem 3. [8] *Let G be a graph without isolated vertices and let $D \subseteq V(G)$ be a minimal dominating set of G. Then, $V(G) \setminus D$ is a dominating set of G.*

Theorem 4. *Let G be a twin-free graph and let $D \subseteq V(G)$ be a minimal dominating set of G. Then, $V(G) \setminus D$ is a locating-dominating set of G.*

The preceding theorem implies that minimal dominating sets of twin-free graphs G provide bounds on $\lambda(G)$. By showing that minimal dominating sets can be constructed from independent sets and cliques of maximum size, we reach the following corollary that supports Conjecture 2. Recall that the *independence number* $\alpha(G)$ and the *clique number* $\omega(G)$ are, respectively, the maximum cardinality of an independent set and a clique of G.

Corollary 1. *Let G be a twin-free graph. Then, $\lambda(G) \leq n - \max\{\alpha(G), \omega(G) - 1\}$. In particular, $\lambda(G) \leq \frac{n}{2}$ when either $\alpha(G) \geq \frac{n}{2}$ or $\omega(G) \geq \frac{n}{2} + 1$.*

ErdHos and Szekeres [4] proved that every graph of order n contains either a clique or an independent set of cardinality at least $\lceil \frac{\log_2 n}{2} \rceil$. Applying this result to Corollary 1, we obtain an upper bound on $\lambda_{|_{\mathcal{C}^*}}(n)$ that, on account of Theorem 2, improves significantly the upper bound of Proposition 1, due to Cáceres et al. [3].

Corollary 2. *For every $n \geq 4$,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|_{\mathcal{C}^*}}(n) \leq n - \lceil \frac{\log_2 n}{2} \rceil + 1.$$

3.2 A greedy algorithm for twin-free graphs

Our second upper bound on $\lambda_{|_{\mathcal{C}^*}}(n)$, that is one of the main contributions of this work, comes from a polynomial time algorithm that produces distinguishing sets of twin-free graphs. In addition, this algorithm computes determining sets of bounded size, and thus an upper bound on the determining number of a twin-free graph. To present this algorithm, we first provide some notation.

For any set $D \subseteq V(G)$, let us define a relation on $V(G)$ given by $u \sim_D v$ if and only if either $u = v$ or $\{u, v\}$ is distinguished by no vertex of D. It is easy to check that this is an equivalence relation, and so we denote by $[u]_D$ the set of vertices $v \in V(G)$ such that $u \sim_D v$. Thus, the sets D, $D^1 = \{u \in V(G) \setminus D : |[u]_D| = 1\}$ and $D^{>1} = V(G) \setminus (D \cup D^1)$ form a partition of $V(G)$, where any of these sets may be empty.

The following greedy algorithm gives a partition of $V(G)$ into three sets so that, combining them properly, one obtains distinguishing sets and determining sets of G of bounded size, as stated in Lemma 2 below.

Algorithm 1

Input: A twin-free graph G and a vertex $u_0 \in V(G)$.
Output: An appropriate partition of $V(G)$ into three subsets A, B, C.

$A \leftarrow \{u_0\}$
$B \leftarrow A^1$
$C \leftarrow A^{>1}$
while $\exists\, u, x, y \in C$ such that $[x]_A = [y]_A$ and $[x]_{A\cup\{u\}} \neq [y]_{A\cup\{u\}}$ **do**
  $A \leftarrow A \cup \{u\}$
  $B \leftarrow A^1$
  $C \leftarrow A^{>1}$
end
return A, B, C

Lemma 2. *Let A, B, C be the sets obtained by application of Algorithm 1 to a twin-free graph G and a vertex $u_0 \in V(G)$. Then, the following statements hold:*

(i) $A \cup B$, $A \cup C$ and $B \cup C$ are distinguishing sets of G.
(ii) A and $B \cup C$ are determining sets of G.

The pigeonhole principle ensures that one set among $A\cup B, A\cup C, B\cup C$ has cardinality at most $\lfloor \frac{2}{3}n \rfloor$. Then, by Lemma 2 and Remark 1, we obtain the following theorem that improves the upper bound of Corollary 2, and consequently the upper bound of Proposition 1 by Cáceres et al. [3].

Theorem 5. *Let G be a twin-free graph of order $n \geq 4$. Then,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|\mathcal{C}^*}(n) \leq \lfloor \frac{2}{3} n \rfloor + 1.$$

We want to point out that, this result and Theorem 4 give, as far as we know, the best bounds on the function $(\dim - \mathrm{Det})(n)$.

By applying again the pigeonhole principle, one gets that either A or $B\cup C$ has cardinality at most $\lfloor \frac{n}{2} \rfloor$ and so, by Lemma 2, we obtain the following bound on the determining number of a twin-free graph.

Theorem 6. *Let G be a twin-free graph of order $n \geq 4$. Then,*

$$\mathrm{Det}(G) \leq \lfloor \frac{n}{2} \rfloor.$$

4 Restriction to specific families of graphs

Let $(\dim - \mathrm{Det})_{|\mathcal{C}}(n)$ and $(\lambda - \mathrm{Det})_{|\mathcal{C}}(n)$ be the restrictions of our functions to a class of graphs $\mathcal{C}$. Here, we study these restrictions to the classes $\mathcal{C}_4$ and $\mathcal{T}$. To do this, we relate the locating-domination number to two well-known graph parameters: the k-domination number and the matching number.

Given a positive integer k, a set $D \subseteq V(G)$ is said to be a *k-dominating set* if $|N(x) \cap D| \geq k$ for every $x \in V(G) \setminus D$. The minimum cardinality of a k-dominating set is the *k-domination number* of G, denoted by $\gamma_k(G)$. On the other hand, the *matching number* of G, written as $\alpha'(G)$, is the cardinality of a maximum matching in G.

Let $\mathcal{K}_{2,k}$ denote the class of graphs not containing $K_{2,k}$ as a (not necessarily induced) subgraph (observe that $\mathcal{K}_{2,2} = \mathcal{C}_4$). The following proposition gives two bounds on the locating-domination number of a graph of $\mathcal{K}_{2,k}$ in terms of, respectively, its k-domination number and its matching number.

Proposition 2. *For every $G \in \mathcal{K}_{2,k}$ of order $n \geq 4$, the following statements hold:*

(i) $\lambda(G) \leq \gamma_k(G)$.
(ii) $\lambda(G) \leq \alpha'(G)$ *whenever* $k = 2$ *and* $G \in \mathcal{C}^*$.

As an application of this proposition, we obtain the exact value of the function $(\lambda - \mathrm{Det})_{|\mathcal{C}_4}(n)$ and give bounds on $(\dim - \mathrm{Det})_{|\mathcal{C}_4}(n)$.

Theorem 7. *For every $n \geq 49$, it holds that*

$$\lfloor \frac{2}{7} n \rfloor \leq (\dim - \mathrm{Det})_{|\mathcal{C}_4}(n) \leq \lfloor \frac{n}{2} \rfloor \quad \textit{and} \quad (\lambda - \mathrm{Det})_{|\mathcal{C}_4}(n) = \lfloor \frac{n}{2} \rfloor.$$

Regarding the family of trees, Cáceres et al. [3] proved that $(\dim - \mathrm{Det})_{|\mathcal{T}}(n) = \Omega(\sqrt{n})$. The following theorem provides the exact value of this function (thereby closing the study initiated by those authors) and also the value of $(\lambda - \mathrm{Det})_{|\mathcal{T}}(n)$.

Theorem 8. *For every $n \geq 49$, it holds that*

$$(\dim - \mathrm{Det})_{|\mathcal{T}}(n) = \lfloor \frac{2}{7} n \rfloor \quad \textit{and} \quad (\lambda - \mathrm{Det})_{|\mathcal{T}}(n) = \lfloor \frac{n}{2} \rfloor.$$

Observe that Theorems 7 and 8 also support Conjecture 2.

5 Concluding remarks

We develop a technique to study the function $(\dim - \mathrm{Det})(n)$ that involves two other functions related to locating-dominating sets: $(\lambda - \mathrm{Det})(n)$ and $\lambda_{|\mathcal{C}^*}(n)$. This technique uses tools that go from results by Ore, and ErdHos and Szekeres to matchings, k-domination, and the design of a polynomial time algorithm to obtain distinguishing sets and determining sets of twin-free graphs. We want to stress that our technique requires many auxiliary results that are of independent interest; here, they are omitted for the sake of brevity but we refer the interested reader to [6] for more details.

It would be interesting to settle Conjectures 1 and 2, which deal with the exact values of our functions. Further, it remains open the computation of the function $(\dim - \mathrm{Det})_{|\mathcal{C}_4}(n)$. It would be also interesting to find specific families of graphs where the restrictions of $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ may be computed. Finally, the maximum value of the difference between the metric dimension and the locating-domination number is still unknown and a study on this function may be proposed.

References

[1] R. F. Bailey and P. J. Cameron. Base size, metric dimension and other invariants of groups and graphs. *Bull. Lond. Math. Soc.*, 43(2):209–242, 2011.

[2] D. L. Boutin. Identifying graph automorphisms using determining sets. *Electron. J. Combin.*, 13(1):Research Paper 78, 12, 2006.

[3] J. Cáceres, D. Garijo, M. L. Puertas, and C. Seara. On the determining number and the metric dimension of graphs. *Electron. J. Combin.*, 17(1):Research Paper 63, 20, 2010.

[4] P. ErdHos and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[5] D. Erwin and F. Harary. Destroying automorphisms by fixing nodes. *Discrete Math.*, 306(24):3244–3252, 2006.

[6] D. Garijo, A. González, and A. Márquez. Resolving sets for breaking symmetries of graphs. Preprint. 2014. arXiv:1401.3686v1.

[7] F. Harary and R. A. Melter. On the metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.

[8] O. Ore. *Theory of graphs.* American Mathematical Society Colloquium Publications, Vol. 38. American Mathematical Society, Providence, R.I., 1962.

[9] C. C. Sims. Determining the conjugacy classes of a permutation group. In *Computers in algebra and number theory*, pages 191–195. SIAM–AMS Proc., Vol. IV. Amer. Math. Soc., Providence, R.I., 1971.

[10] P. J. Slater. Leaves of trees. *Congressus Numerantium*, 14:549–559, 1975.

[11] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.*, 22(4):445–455, 1988.