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Flows and colorings

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Synopsis

Tutte first introduced the dichromate of a graph in large part motivated by the fact that it contained the flow polynomial and chromatic polynomial as univariate specializations. The latter receives attention in Chapter 11. In this chapter we consider flows.

- Flows of graphs taking values in a finite abelian group, colorings and tensions (dual to flows).
- The flow polynomial, chromatic polynomial and dichromatic polynomial as specializations of the Tutte polynomial.
- Coloring-flow convolution formulas for the Tutte polynomial.
- "A-bicycles" (called bicycles when $\mathsf{A} = \mathbb{Z}_2$), the Tutte polynomial evaluated at $(-1,-1)$, $(e^{2\pi i/3}, e^{4\pi i/3})$ and $(i, -i)$.
- Tutte's flow conjectures.

12.1 Introduction

W. Tutte [1109] describes how he first "became acquainted with the Tutte polynomial" by looking for graph invariants with deletion–contraction recurrences such as the number of spanning trees (of a connected graph), the chromatic polynomial, and the flow polynomial. In this chapter we focus on evaluations of the Tutte polynomial $T(G; x, y)$ of a graph G that involve $\mathbb{Z}_k$-flows of G, for which there are often dual results in terms of vertex k-colorings of

G, or, equivalently, $\mathbb{Z}_k$-tensions of G. These evaluations are all at points on the hyperbola $(x-1)(y-1)=k$ with k a positive integer.

The duality between colorings and flows of a graph G consists in the orthogonality of A-tensions and A-flows of G for a finite abelian group A of order k. (See [107, 546] for the interaction of graph theory and linear algebra more generally.) While colorings and flows are dual notions for planar graphs—the tensions of a plane graph correspond to the flows of the dual plane graph—for graphs generally, duality resides at the level of the cycle and cocycle matroids of the graph. The set of tensions of a non-planar graph do not correspond to the set of flows of a graph (a result that goes back to H. Whitney in the 1930s). There is a consequent asymmetry between properties of colorings and flows for graphs (for example, loopless graphs may have arbitrarily high chromatic number, but bridgeless graphs have bounded flow number). Any information about flows that can be extracted from the Tutte polynomial extends our still incomplete picture of them.

After giving the fundamental specializations of the Tutte polynomial to the flow polynomial and chromatic polynomial, we describe more elaborate decompositions of the Tutte polynomial that for graphs can be expressed in terms of Hamming weights of flows and tensions, beginning with the classical expression of the Tutte polynomial as the dichromatic polynomial (effectively a Hamming weight enumerator for tensions) and its dual expansion as a weight enumerator for flows. (For a development of the relationship between the Tutte polynomial and weight enumerators of codes see Chapter 16.) We then consider the conjunction of flows and tensions in what are known as bicycles (for $\mathbb{Z}_2$-flows and $\mathbb{Z}_2$-tensions) and "A-bicycles" more generally (the intersection of the set of A-tensions and the set of A-flows). Evaluations of the Tutte polynomial at the points $(-1,-1)$, $(i,-i)$ and $(e^{2\pi i/3}, e^{4\pi i/3})$ yield the dimension of the A-bicycle space for $\mathsf{A}=\mathbb{Z}_2$ and $\mathsf{A}=\mathbb{Z}_3$, and some further information about the weight distribution of A-bicycles for evaluations at the last two points.

In our concluding section reporting open problems, we give an overview of Tutte's flow conjectures, which have motivated intensive research in the topic of flows on graphs ever since their formulation.

Complementary surveys of the Tutte polynomial that treat combinatorial interpretations of its evaluations involving flows include [247, 454, 1135, 1141].

12.2 Flows and the flow polynomial

An *orientation* ω of a graph $G=(V,E)$ assigns a direction to each edge $uv \in E$, either $u \xrightarrow{\omega} v$ or $u \xleftarrow{\omega} v$. For $U \subset V$, define $\omega^+(U)=\{uv \in E : u \in U, v \in V \setminus U,\ u \xrightarrow{\omega} v\}$ and $\omega^-(U)=\omega^+(V \setminus U)$. In particular, for a vertex $v \in V$ the set $\omega^+(v)$ consists of those edges directed out of v by the orientation ω and $\omega^-(v)$ is the set of edges directed into v.

For an additive abelian group A, scalar multiples of a $\{0,\pm 1\}$-vector by an element of A are defined by using the identities $0a = 0, 1a = a$ and $(-1)a = -a$ for each $a \in \mathsf{A}$. The abelian group A is a $\mathbb{Z}$-module. The set of mappings $\phi : E \to \mathsf{A}$ is denoted by A^E, and likewise A^V is the set of mappings $\kappa : V \to \mathsf{A}$.

Definition 12.1. Let G^ω be the graph $G = (V, E)$ with a fixed orientation ω of its edges. The *incidence matrix* of G^ω is the matrix $D = (D_{v,e}) \in \{0, \pm 1\}^{V \times E}$ whose (v, e)-entry is defined by

$$D_{v,e} = \begin{cases} +1 & \text{if } e \text{ is directed out of } v \text{ by } \omega, \\ -1 & \text{if } e \text{ is directed into } v \text{ by } \omega, \\ 0 & \text{if } e \text{ is not incident with } v, \text{ or } e \text{ is a loop on } v. \end{cases}$$

Definition 12.2. The incidence matrix of G^ω defines a linear transformation $D : \mathsf{A}^E \to \mathsf{A}^V$ between modules, called the *boundary mapping*, which for $\phi : E \to \mathsf{A}$ and $v \in V$ is given by

$$D\phi(v) = \sum_{\substack{e=uv \\ u \xleftarrow{\omega} v}} \phi(e) - \sum_{\substack{e=uv \\ u \xrightarrow{\omega} v}} \phi(e).$$

The transpose of the incidence matrix defines the *coboundary* map $D^T : \mathsf{A}^V \to \mathsf{A}^E$, which for $\kappa : V \to \mathsf{A}$ and edge $e = uv$ is given by

$$D^T\kappa(e) = \begin{cases} \kappa(v) - \kappa(u) & \text{if } u \xleftarrow{\omega} v, \\ \kappa(u) - \kappa(v) & \text{if } u \xrightarrow{\omega} v. \end{cases}$$

In other words, the boundary assigns the net value at each vertex for a given mapping $\phi : E \to \mathsf{A}$, and the coboundary assigns to each edge the difference (taken according to the edge orientation) of its endpoints in a given mapping $\kappa : V \to \mathsf{A}$.

Definition 12.3. Let G^ω be the graph G with a fixed orientation ω, let A be an additive abelian group, and let D the incidence matrix of G^ω. The set of A-*flows* of G^ω is the kernel of D and the set of A-*tensions* of G^ω is the image of D^T.

An A-flow of G^ω is thus a mapping $\phi : E \to \mathsf{A}$ such that Kirchhoff's law is satisfied at each vertex of G:

$$\sum_{e \in \omega^+(v)} \phi(e) - \sum_{e \in \omega^-(v)} \phi(e) = 0, \qquad \text{for each } v \in V.$$

Likewise, an A-tension of G^ω is a mapping $\theta : E \to \mathsf{A}$ such that for each closed walk on G the sum of values on forward edges is equal to the sum of values on backward edges, forward and backward being relative to the direction of

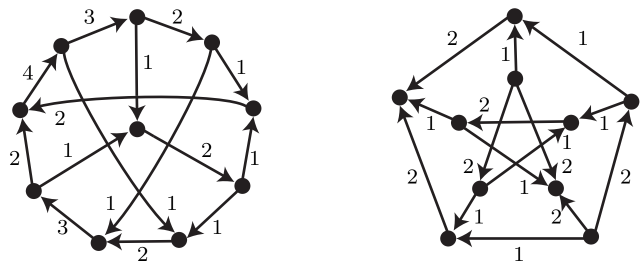

FIGURE 12.1: The Petersen graph, with on the left a nowhere-zero 5-flow and on the right a nowhere-zero $\mathbb{Z}_5$-flow.

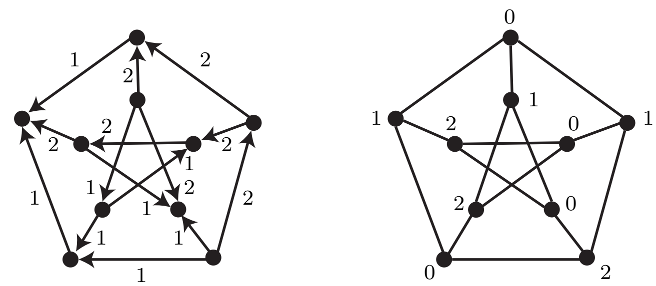

FIGURE 12.2: The Petersen graph, with on the left a nowhere-zero $\mathbb{Z}_3$-tension and a corresponding proper 3-coloring on the right.

traversal of the walk. To each A-tension θ there corresponds a set of $|\mathsf{A}|^{k(G)}$ vertex colorings $\kappa : V \to \mathsf{A}$ such that $D^T\kappa = \theta$.

The *support* of a mapping $\phi : E \to \mathsf{A}$ is defined by $\operatorname{supp}(\phi) = \{e \in E : \phi(e) \neq 0\}$.

Definition 12.4. A *nowhere-zero* A-flow of G^ω is an A-flow $\phi : E \to \mathsf{A}$ with the additional property that $\phi(e) \neq 0$ for every $e \in E$, that is, $\operatorname{supp}(\phi) = E$.

An *integer k-flow* of G^ω is a $\mathbb{Z}$-flow $\phi : E \to \mathbb{Z}$ such that $-k < \phi(e) < k$ for every $e \in E$. A *nowhere-zero* integer k-flow is an integer k-flow with the additional property that $\phi(e) \neq 0$ for every $e \in E$.

Remark 12.5. In the literature, the term 'k-flow' may mean either an integer k-flow or a $\mathbb{Z}_k$-flow, with the choice apparent from context. Since we discuss both integer k-flows and $\mathbb{Z}_k$-flows here, we make a clear distinction.

Example 12.6. Figure 12.1 illustrates a nowhere-zero integer k-flow with $k = 5$ and a nowhere-zero A-flow with $\mathsf{A} = \mathbb{Z}_5$ for the Petersen graph with orientation of edges as indicated. Figure 12.2 illustrates a nowhere-zero $\mathbb{Z}_3$-tension and a corresponding proper 3-coloring.

Although the definition of a (nowhere-zero) A-flow and a (nowhere-zero) integer k-flow of G^ω requires an orientation ω of G, the number of such (nowhere-zero) A-flows and (nowhere-zero) integer k-flows is independent of the orientation ω. Therefore, the number of (nowhere-zero) A-flows of G^ω is an invariant of G, and in the context of enumerating flows we may speak of flows of a graph G without specifying the orientation of the edges of G that is used.

The enumerative theory of integer k-flows is beyond the scope of this chapter, which focuses solely on enumerative results concerning nowhere-zero A-flows for a finite abelian group A. However, it is worth noting the following from [1094].

Proposition 12.7. *A graph G has a nowhere-zero $\mathbb{Z}_k$-flow if and only if G has a nowhere-zero integer k-flow.*

Consequently, if A and A' are finite additive abelian groups such that $|\mathsf{A}| \leq |\mathsf{A}'|$, then G having a nowhere-zero A-flow implies G also has a nowhere-zero A'-flow.

Definition 12.8. The *flow polynomial* of $G = (V, E)$ is defined by its evaluations at a positive integer k by

$$F(G;k) = \#\{\phi : E \to \mathbb{Z}_k : \phi \text{ a nowhere-zero } \mathbb{Z}_k\text{-flow of } G\}.$$

That the number of nowhere-zero $\mathbb{Z}_k$-flows is indeed a polynomial in k is a consequence of the following recurrence:

Proposition 12.9. *The flow polynomial of a graph satisfies*

$$F(G;k) = \begin{cases} F(G/e;k) - F(G\backslash e;k) & \textit{if } e \textit{ is ordinary,} \\ 0 & \textit{if } e \textit{ is a bridge,} \\ (k-1)F(G\backslash e;k) & \textit{if } e \textit{ is a loop,} \end{cases}$$

and multiplicativity over disjoint unions,

$$F(G_1 \sqcup G_2;k) = F(G_1;k)F(G_2;k).$$

The recurrence in Proposition 12.9 holds not only for the number of nowhere-zero $\mathbb{Z}_k$-flows of G but also for the number of nowhere-zero A-flows of G for any choice of finite additive abelian group A of order k, hence the following.

Corollary 12.10. *The number of nowhere-zero A-flows of a graph G depends only on $|\mathsf{A}|$ and not the structure of A as a group.*

Another consequence of Proposition 12.9 is that homeomorphic graphs (i.e., graphs that are isomorphic after suppression of all vertices of degree two) have the same flow polynomial.

Proposition 12.11. *Let $G = G_1 * G_2$ be a one-point join of graphs. Then*

$$F(G;k) = F(G_1;k)F(G_2;k).$$

Thus the flow polynomial of a graph G is the product of its values on the blocks (2-connected components) of G. As shown in [998], the flow polynomial of a graph may be further decomposed over vertex-cuts of size 2 or 3. The decompositions for vertex-cuts of sizes 1 or 2 yield by the deletion–contraction formula of Proposition 12.9 a similar decomposition of the flow polynomial of G when there is an edge e such that $G\backslash e$ is separable (see e.g. [638]).

An inductive deletion–contraction argument also establishes decompositions of the flow polynomial over small edge-cuts (from [998]):

Theorem 12.12. *Let G be a connected graph with an edge cut A of size $t \in \{2,3\}$. Let E_1 and E_2 be the edge sets of two components of $G\backslash A$, and $G_i = G/E_i$, for $= 1,2$. Then*

$$F(G;k) = \frac{F(G_1;k)F(G_2;k)}{F(\theta_t;k)},$$

where θ_t is the t-theta graph consisting of two vertices joined by t parallel edges.

Proposition 12.9 and the universality property of the Tutte polynomial (Theorem 2.21) imply the following result.

Theorem 12.13. *The flow polynomial of a graph is a specialization of the Tutte polynomial:*

$$F(G;k) = (-1)^{n(G)}\,T(G;0,1-k).$$

Remark 12.14. The analogously defined *integer flow polynomial*

$$F_{\mathbb{Z}}(G;k) = \#\{\phi : E \to \mathbb{Z} : \phi \text{ nowhere-zero integer } k\text{-flow of } G\}, \tag{12.1}$$

was only much later proved actually to be a polynomial by M. Kochol [699]. It is not a specialization of the Tutte polynomial. Furthermore, when not restricting to nowhere-zero flows, while the number of A-flows for $|\mathsf{A}| = k$ is equal to $k^{n(G)}$ (as can easily verified by from the deletion-contraction relation satisfied by the number of A-flows), the number of integer k-flows is a polynomial $F^0_{\mathbb{Z}}(G;k)$ which is as difficult to compute as $F_{\mathbb{Z}}(G;k)$.

12.3 Coloring-flow convolution formulas

Dual to the deletion–contraction recurrence for flows (Proposition 12.9) and the specialization of the Tutte polynomial to the flow polynomial (Theorem 12.13) are the following recurrence and specialization for colorings (for which see Chapter 11).

Proposition 12.15. *The chromatic polynomial of a graph G satisfies the recurrence relation*

$$\chi(G;k) = \chi(G\backslash e;k) - \chi(G/e;k), \tag{12.2}$$

for any edge e.

Theorem 12.16. *The chromatic polynomial of a graph is a specialization of the Tutte polynomial:*

$$\chi(G;k) = (-1)^{r(G)} k^{k(G)}\, T(G;1-k,0).$$

The dichromatic polynomial of Definition 2.1 can be expressed as a generating function for k-colorings of G (not necessarily proper) counted according to the number of monochromatic edges, that is, edges receiving the same color on their endpoints. (Edges are taken with their multiplicity when counting the number of monochromatic edges in the exponent of y.)

Proposition 12.17. *The dichromatic polynomial has the following expansion for positive integer k:*

$$Z(G;k,y-1) = \sum_{\kappa:V(G)\to[k]} y^{\#\{uv\in E(G):\kappa(u)=\kappa(v)\}}. \tag{12.3}$$

The right-hand side of Equation (12.3) is also known under the name of *bad coloring polynomial* [454], *monochrome polynomial* [1141], and *monochromial* [11]. When suitably parameterized $Z(G;k,y-1)$ gives the partition function of the k-state Potts model (see Chapter 20). Furthermore, by Theorem 2.6 we have the following identity.

Theorem 12.18. *For a graph $G=(V,E)$ and positive integer k,*

$$\sum_{\kappa:V\to[k]} y^{\#\{uv\in E:\kappa(u)=\kappa(v)\}} = k^{k(G)}(y-1)^{r(G)}\, T(G;\tfrac{k+y-1}{y-1},y).$$

For flows the dual result is as follows.

Theorem 12.19. *Let A be a finite additive abelian group of size k, and $G=(V,E)$ be a graph. Then*

$$\sum_{\substack{\phi:E\to\mathsf{A}\\ \phi\ \text{an}\ \mathsf{A}\text{-flow}}} x^{|E|-|\mathrm{supp}(\phi)|} = (x-1)^{n(G)}T(G;x,\tfrac{k+x-1}{x-1}).$$

Remark 12.20. The polynomial $k^{-k(G)}Z(G;k,y-1)$ is an invariant of the cycle matroid of G and is a generating function for $\mathbb{Z}_k$-tensions of G counted according to the size of their support. Tensions (see Definition 12.3) rather than vertex colorings are the proper object of study when considering the structure of the cycle matroid of a graph, and make coloring-flow duality for graphs more transparent. Tension-flow duality has been thoroughly explored in, for example, [376, 878].

Definition 12.21. Let $G = (V, E)$ be a graph. For $A \subseteq E$, let the *bond closure* of A, denoted $\mathrm{cl}(A)$, be the set of all edges both of whose endpoints belong to one and the same block of the spanning subgraph (V, A). The lattice of subsets A with $\mathrm{cl}(A) = A$ is called the *bond lattice* of the graph G, denoted by $\mathcal{L}(G)$.

The edges assigned the value zero in a tension (flow) once contracted (deleted) leave a nowhere-zero tension (flow) on the resulting graph whose edges are the support of the tension (flow). The support of a tension is a flat in the cycle matroid of G, that is, an element of the bond lattice of G. The support A of a tension has the property that if all but one element of a circuit C belongs to A then all of the edges of C do. The support A of a flow has the dual property that if all but one edge of a bond B belong to A then all of the edges of B do. Equivalently, $n(A \cup e) = n(A) + 1$ for all $e \in E \setminus A$. The support of a flow is a flat in the *bond matroid* of G (the dual of the cycle matroid of G).

Theorems 12.18 and 12.19 can be re-expressed as follows.

Theorem 12.22. *Let $G = (V, E)$ be a graph with bond lattice $\mathcal{L}(G)$ and circuit lattice $\mathcal{L}^*(G)$ (bond lattice of the dual of the cycle matroid of G). Then*

$$\sum_{A \in \mathcal{L}(G)} y^{|A|} \chi(G/A; k) = k^{k(G)} (y-1)^{r(G)} T(G; \tfrac{y-1+k}{y-1}, y).$$

Dually,

$$\sum_{A \in \mathcal{L}^*(G)} x^{|E|-|A|} F(G|A; k) = (x-1)^{n(G)} T(G; x, \tfrac{x-1+k}{x-1}),$$

where $G|A = G\backslash(E \setminus A)$ is the spanning subgraph on A.

Theorem 12.22 has a generalization to matroids (in which the characteristic polynomial replaces the chromatic polynomial—see Definition 4.136) due to H. Crapo [349] and Tutte [1097].

For a fixed graph G, let G_p denote the random subgraph of G obtained by deleting each edge of G independently with probability $1-p$. If f is a graph invariant, then the expected value of f on a random subgraph G_p of G is by definition equal to $\sum_{A \subseteq E} p^{|A|}(1-p)^{|E|-|A|} f(G|A)$. As an application of the expansions of Theorem 12.22 we have [1135]:

Theorem 12.23. *For a graph G and $0 < p \leq 1$, the random subgraph G_p has chromatic polynomial whose expectation is given by*

$$\mathbb{E}\, \chi(G_p; k) = (-p)^{r(G)} k^{k(G)} T(G; 1 - \tfrac{k}{p}, 1-p).$$

Dually, for a graph G and $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the random subgraph G_p has flow polynomial whose expectation is given by

$$\mathbb{E}\, F(G_p; k) = p^{r(G)} (1-2p)^{n(G)} T(G; \tfrac{1-p}{p}, 1 + \tfrac{kp}{1-2p}).$$

When $p = \frac{1}{2}$,
$$\mathbb{E}\, F(G_p; k) = k^{n(G)} 2^{-|E|}.$$

The convolution formula for Tutte polynomial of a matroid given independently in [704] and (implicitly) in [469] can be written for graphs in the form of a tension-flow convolution [960]:

Theorem 12.24. *For a graph* $G = (V, E)$,

$$T(G; x, y) = \sum_{A \subseteq E} T(G/A; x, 0)\, T(G|A; 0, y).$$

Equivalently, for positive integers k, k' *and* A *and* A' *additive abelian groups of orders* k *and* k' *respectively,*

$$T(G; 1-k, 1-k') = (-1)^{r(G)} \sum_{\substack{\theta\ \mathsf{A}\text{-}tension \\ \phi\ \mathsf{A}'\text{-}flow \\ \mathrm{supp}(\theta) \sqcup \mathrm{supp}(\phi) = E}} (-1)^{|\mathrm{supp}(\phi)|}.$$

A generalization of Theorem 12.24 was shown by J. Kung [735], which we state for graphs, but which holds, mutatis mutandis, for matroids more widely and for the multivariate Tutte polynomial in place of the Tutte polynomial.

Theorem 12.25. *For a graph* $G = (V, E)$ *and indeterminates* a, b, c, d,

$$\sum_{A \subseteq E} a^{r(E)-r(A)} (-d)^{|A|-r(A)} T(G|A; 1-a, 1-c) T(G/A; 1+b, 1+d) = T(G; 1+ab, 1+cd). \tag{12.4}$$

A pair of special cases of identity (12.4) follow upon taking $ac = 1$ or $bd = 1$, the first of which, given as identity (12.5) below, is used in [81] to prove a result about win vectors (score vectors of partial orientations).

Theorem 12.26. *For a graph* $G = (V, E)$,

$$\sum_{A \subseteq E} (z-1)^{|A|} (x-1)^{r(E)-r(A)} T(G|A; x, y) = (z-1)^{r(G)} z^{n(G)} T\left(G; (x-1)\tfrac{z}{z-1} + 1, (y-1)\tfrac{z-1}{z} + 1\right). \tag{12.5}$$

Dual to identity (12.5) *is*

$$\sum_{A \subseteq E} (z-1)^{|E \setminus A|} (y-1)^{|A|-r(A)} T(G/A; x, y) = z^{r(G)} (z-1)^{n(G)} T\left(G; (x-1)\tfrac{z-1}{z} + 1, (y-1)\tfrac{z}{z-1} + 1\right). \tag{12.6}$$

F. Jaeger [649] obtained for matroids representable over a finite field a version of the following convolution formula for $k = k'$ a prime power. V. Reiner [960] then extended it to prime powers $k \neq k'$, and for graphic matroids deduced its validity for arbitrary positive integers k and k':

Theorem 12.27. *Let k and k' be positive integers, A and A' additive abelian groups of orders k and k' respectively, and z an indeterminate. Then*

$$(1-z)^{r(G)} z^{n(G)} T\left(G; \tfrac{1+(k-1)z}{1-z}, \tfrac{1+(k'-1)(1-z)}{z}\right) = \sum_{\substack{\theta\ \mathsf{A}\text{-}tension \\ \phi\ \mathsf{A}'\text{-}flow \\ \mathrm{supp}(\theta)\cap\mathrm{supp}(\phi)=\emptyset}} z^{|\mathrm{supp}(\theta)|}(1-z)^{|\mathrm{supp}(\phi)|}.$$

Theorem 12.24 is a special case of Theorem 12.27 on taking the limit $z \to \infty$. F. Breuer and R. Sanyal [200] and B. Chen [296] derived related expansions of the Tutte polynomial at integer points.

12.4 A-bicycles

For a graph G with orientation ω and an abelian group A, recall from Definitions 12.1 and 12.2 that if $D : \mathsf{A}^E \to \mathsf{A}^V$ is the incidence mapping, then the set $\ker(D)$ consists of A-flows of G and the set $\mathrm{im}(D^T)$ consists of A-tensions of G (with a given orientation ω). The intersection $\ker(D)\cap\mathrm{im}(D^T)$ of the submodule of A-tensions with the submodule of A-flows is trivial if A is torsion-free, but reveals an interesting structure for finite abelian groups A. For $\mathsf{A} = \mathbb{Z}_2$ and $\mathsf{A} = \mathbb{Z}_3$ the dimension of this subspace is given in terms of evaluations of the Tutte polynomial at points $(-1,-1)$, $(i,-i)$ and $(e^{2\pi i/3}, e^{4\pi i/3})$, which are some of the special points at which the Tutte polynomial is not $\sharp P$-hard to compute [653].

We begin with $\mathsf{A} = \mathbb{Z}_2$, in which a $\mathbb{Z}_2$-flow (cycle) that is also a $\mathbb{Z}_2$-tension (cutset) is known as a *bicycle*. For more about bicycles see Sections 14.15–16 and 15.7 in [546], and for the usefulness of bicycles in relation to knots see Chapter 17 of the same reference. The following describes the *principal tripartition* [970] of edges of a graph.

Theorem 12.28. *Let e be an edge of a graph G. Then precisely one of the following holds:*

1. *e belongs to a bicycle,*
2. *e belongs to a cutset B such that $B \setminus \{e\}$ is a cycle,*
3. *e belongs to a cycle C such that $C \setminus \{e\}$ is a cutset.*

An edge e of G is of *bicycle-type*, *cut-type* or *flow-type* according as Items 1–3 holds in the statement of Theorem 12.28, respectively. A bridge is an edge of cut-type (take cut $B = \{e\}$ in Item 2) and a loop is an edge of flow-type (take cycle $\{e\}$ in Item 3). If G is planar then edges of bicycle-type in G remain of bicycle-type in G^*. By tension-flow duality, edges of cut-type in G are edges of flow-type in G^*, and similarly edges of flow-type in G^* are edges of cut-type in G^*.

Lemma 12.29. *Let $G = (V, E)$ be a graph with bicycle space of dimension $b(G)$, and let e be an edge of G. Then the graph invariant*

$$f(G) = (-1)^{|E|}(-2)^{b(G)}$$

satisfies

$$f(G) = \begin{cases} (-1)f(G/e) & e \text{ a bridge,} \\ (-1)f(G\backslash e) & e \text{ a loop,} \\ f(G/e) + f(G\backslash e) & e \text{ ordinary.} \end{cases}$$

Universality of the Tutte polynomial for deletion–contraction invariants (Theorem 2.21) yields the following polynomial-time computable evaluation of the Tutte polynomial [970]:

Theorem 12.30. *Let $G = (V, E)$ be a graph and let $b(G)$ denote the dimension of its bicycle space. Then*

$$(-1)^{|E|}(-2)^{b(G)} = T(G; -1, -1).$$

The point $(-1, -1)$ lies on the hyperbola $(x - 1)(y - 1) = 4$, so that by Theorem 12.19

$$T(G; -1, -1) = (-2)^{-n(G)} \sum_{\mathbb{Z}_2 \oplus \mathbb{Z}_2\text{-flows } \phi} (-1)^{|E| - |\mathrm{supp}(\phi)|}.$$

A $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-bicycle decomposes by projection into a pair of $\mathbb{Z}_2$-bicycles, and conversely such a pair of $\mathbb{Z}_2$-bicycles can be pieced together to make a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-bicycle. Hence there are precisely $(2^{b(G)})^2$ vectors that are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-bicycles, that is, they comprise a space of dimension $b(G)$ over $\mathbb{F}_4$. Hence Theorem 12.30 could also have stated that $T(G; -1, -1) = (-1)^{|E|}(-2)^{b(G)}$, where $b(G)$ is now defined as the dimension of the space of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-bicycles.

We now take $\mathsf{A} = \mathbb{Z}_3$ and $D : \mathbb{Z}_3^E \to \mathbb{Z}_3^V$ as the boundary mapping. An element of $\ker(D) \cap \mathrm{im}(D^T)$ is a $\mathbb{Z}_3$-*bicycle*. In other words, a $\mathbb{Z}_3$-bicycle is both a $\mathbb{Z}_3$-tension and a $\mathbb{Z}_3$-flow, and is self-orthogonal in $\mathbb{Z}_3^E$. Let $i = \sqrt{-1}$ and $j = e^{2\pi i/3}$. In [650] Jaeger proved by a deletion–contraction argument that $T(G; j, j^2) = \pm j^{|E| + \dim(\ker(D))}(i\sqrt{3})^{\dim(\ker(D) \cap \mathrm{im}(D^T))}$, using the *principal quadripartition* of the edges of a graph (a generalization to flows and tensions over finite fields of characteristic $\neq 2$ of the principal tripartition of Theorem 12.28). E. Gioan and M. Las Vergnas [533] provide a linear algebra proof that has the benefit of determining the sign.

Theorem 12.31. *Let $G = (V, E)$ be a graph and $j = e^{2\pi i/3}$. We have*

$$T(G; j, j^2) = (-1)^{q(G)} j^{|E|+n(G)} (i\sqrt{3})^{b_3(G)},$$

where $b_3(G)$ is the dimension of the space of $\mathbb{Z}_3$-bicycles of G, and $q(G)$ is the number of vectors with support size congruent to 2 *modulo* 3 *in any orthogonal basis for the space of $\mathbb{Z}_3$-flows of G.*

D. Vertigan [1118] proved that the Tutte polynomial evaluated at the point $(i, -i)$ on the hyperbola $(x-1)(y-1) = 2$ has the following interpretation:

Theorem 12.32. *Let G be a graph with bicycle dimension $b(G)$. Then*

$$|T(G; i, -i)| = \begin{cases} \sqrt{2}^{b(G)} & \text{if every bicycle has size a multiple of 4,} \\ 0 & \text{otherwise.} \end{cases}$$

As every bicycle has even size, the bicycles of size a multiple of 4 either comprise all bicycles, or exactly half of them. R. Pendavingh [913] determines the argument of the complex number $T(G; i, -i)$ in terms of a certain $\mathbb{Z}_4$-valued quadratic form canonically associated with G.

While the number of (nowhere-zero) A-flows and (nowhere-zero) A-tensions are independent of the structure of A, the same is not true in general of the number of A-bicycles, as shown in [98].

Definition 12.33. Let $D : \mathbb{Z}^E \to \mathbb{Z}^V$ be the integer boundary mapping and $D^T : \mathbb{Z}^V \to \mathbb{Z}^E$ the integer coboundary mapping. The *critical group* $K(G)$ of G (with arbitrary orientation ω) is defined as the quotient of integer lattices

$$K(G) = \mathbb{Z}^E / (\ker(D) \oplus \operatorname{im}(D^T)).$$

The invariant factors $n_1, \ldots, n_r$ of the critical group $K(G) \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ satisfy $n_1 \mid n_2 \mid \cdots \mid n_r$ and, by Kirchhoff's matrix-tree theorem, $n_1 n_2 \cdots n_r = T(G; 1, 1)$ (see for example [113]). For finite additive abelian group A and $n \in \mathbb{Z}$, let $\mathsf{A}^{(n)} = \{a \in \mathsf{A} : na = 0\}$.

Theorem 12.34. *Let G be a graph with a fixed arbitrary orientation and $n_1, \ldots, n_r$ be the invariant factors of the critical group of G. Then the module of A-bicycles of G has the following direct sum decomposition:*

$$\ker(D) \cap \operatorname{im}(D^T) \cong \mathsf{A}^{(n_1)} \oplus \mathsf{A}^{(n_2)} \oplus \cdots \oplus \mathsf{A}^{(n_r)}.$$

12.5 Open problems

As well as establishing an algebraic and homological approach to colorings and flows that is still being elaborated to this day, Tutte formulated a series of conjectures concerning flows, each of which reflects miraculous intuition and each of which still resists solution. Most of these ask about the existence of flows. As noted in Proposition 12.7 a graph has a nowhere-zero $\mathbb{Z}_k$-flow if and only if G has a nowhere-zero integer k-flow. Thus, these questions are typically stated just in terms of k-flows, without the need to specify which kind, and we follow that convention here.

Grötzsch's theorem [588] states that every triangle-free loopless planar graph is 3-colorable. By coloring-flow duality, this is equivalent to the statement that every 4-edge-connected planar graph has a nowhere-zero 3-flow. Tutte's 3-flow conjecture (first popularized in the 1970s [162] as formulated by Tutte in 1972) asserts that this is still true without the assumption of planarity.

Conjecture 12.35 (Tutte's 3-flow conjecture)**.** Every 4-edge-connected graph has a nowhere-zero 3-flow.

Kochol [698] showed that it suffices to prove Tutte's 3-flow conjecture for 5-edge-connected graphs.

To help find an oblique path up the steep hill to the summit of resolving Tutte's 3-flow conjecture, Jaeger [645] asked first whether there indeed exists some fixed integer k such that every k-edge-connected graph has a nowhere-zero 3-flow (Jaeger's weak 3-flow conjecture). Spectacular progress has recently been made by C. Thomassen [1068], who verified Jaeger's weak 3-flow conjecture by proving that every 8-edge-connected graph admits a nowhere-zero 3-flow. This was then improved by L. M. Lovász et al. [798], who proved that every 6-edge-connected graph has a nowhere-zero 3-flow.

Other references concerning the 3-flow conjecture include [645, 647, 652, 698, 1068, 1069]. For the 4-flow and 5-flow conjectures, to which we now turn, the reader may consult [645, 690, 967, 1000, 1094, 1096].

Conjecture 12.36 (Tutte's 4-flow conjecture)**.** Every bridgeless graph without a Petersen minor has a nowhere-zero 4-flow.

Jaeger [645] used the fact, due to Tutte and to C. Nash-Williams, that a 4-edge-connected graph has two edge-disjoint spanning trees in order to prove that every 4-edge-connected graph has a nowhere-zero 4-flow. As nowhere-zero 4-flows of a cubic graph correspond to proper edge 3-colorings, the 4-flow conjecture implies that every bridgeless cubic graph without a Petersen minor is 3-edge-colorable. The latter was another conjecture of Tutte. A proof was announced by N. Robertson, P. Seymour, and R. Thomas in [967], thereby giving a strengthening of the four color theorem, which is equivalent to the assertion that planar cubic graphs without a bridge are 3-edge-colorable.

Tutte conjectured that there is no obstruction to a nowhere-zero A-flow when $|\mathsf{A}| \geq 5$ other than a bridge (unlike cliques of order $k+1$ being obstructions to having a nowhere-zero k-tension).

Conjecture 12.37 (Tutte's 5-flow conjecture)**.** Every bridgeless graph has a nowhere-zero 5-flow.

Every bridgeless planar graph has a nowhere-zero 5-flow by coloring-flow duality and Heawood's theorem that every loopless planar graph is 5-colorable [604]. It is not difficult to show that a minimal counterexample to the 5-flow conjecture must be a *snark*, that is, a cyclically 4-edge-connected cubic graph without a proper edge 3-coloring and with girth at least 5; see [647]. (A graph is cyclically k-edge-connected if removing at most $k-1$ edges from G leaves at most one component containing a cycle.) U. Celmins [272] proved that a minimal counterexample must be a cyclically 5-edge-connected snark with girth at least 7. Kochol showed further that such a minimal counterexample snark must be cyclically 6-edge-connected in [700] and in [701, 702] that it must have girth at least 11.

A parameter measuring how far a bridgeless cubic graph is from being 3-edge-colorable is its *oddness*, defined as the minimum number of odd circuits in a 2-factor of G. Since G has an even number of vertices, its oddness is necessarily even. Furthermore, its oddness is 0 if and only if G is 3-edge-colorable. Jaeger [647] showed that cubic graphs with oddness at most 2 have a nowhere-zero 5-flow. Furthermore, a consequence of the main result in [1046] is that cyclically 7-edge-connected cubic graphs with oddness at most 4 have a nowhere-zero 5-flow. In [832] it is shown that a cyclically 6-edge-connected cubic graph G of oddness at most 4 has a nowhere-zero 5-flow.

Tutte made the 5-flow conjecture at a time when it was not even clear that there was a k such that every graph has a nowhere-zero k-flow. (The dual statement for the chromatic polynomial is patently false.) Jaeger [645] and P. Kilpatrick [690] independently proved using $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$-flows that every bridgeless graph has a nowhere-zero 8-flow. Later, Seymour [1000] proved using $\mathbb{Z}_2 \oplus \mathbb{Z}_3$-flows that every bridgeless graph has a nowhere-zero 6-flow.

Open Question 12.38. Are the real roots of the flow polynomial bounded above?

Tutte's flow conjectures ask for a characterization of those graphs for which the flow polynomial has respectively 3, 4 or 5 as a root. A conjecture of G. Birkhoff and D. Lewis [124], which is still open, states that the real roots of the chromatic polynomial of a planar graph do not exceed 4 (Birkhoff and Lewis did prove that 5 is an upper bound). The dual statement for the flow polynomial is that its real roots for a planar graph do not exceed 4. D. Welsh (as reported in [638]) conjectured that the latter statement is true without the assumption of planarity. G. Haggard et al. [596] produced counterexamples to Welsh's conjecture, including a generalized Petersen graph, and modified the conjecture to state that the real roots of the flow polynomial are at most 5. J. Jacobsen and J. Salas [643] thereupon produced, after extensive computations, another generalized Petersen graph whose roots exceed 5, and proposed that the real roots of the flow polynomial do not exceed 6. It is still not known, however, whether the real roots of the flow polynomial are bounded above at all. For further results on flow roots see the survey [638].