# Shortcut sets for Euclidean graphs

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## Abstract

A *Euclidean graph* $G$ is the locus of a rectilinear embedding of a planar graph in the Euclidean plane. A *shortcut set* $\mathcal{S}$ is a collection of segments with endpoints on $G$ such that the Euclidean graph obtained from $G$ by adding the segments in $\mathcal{S}$ has smaller diameter than $G$. The minimum cardinality of a shortcut set is the *shortcut number* $\operatorname{scn}(G)$. In this work, we first provide a tight upper bound on $\operatorname{scn}(G)$. We then show that it is possible, in polynomial time, to determine if $\operatorname{scn}(G) = 1$ and, in that case, to construct a shortcut set that minimizes the diameter among all possible shortcut sets. Finally, we compute the shortcut number in some families of Euclidean graphs.

## Introduction

The set of points $G$ in the Euclidean plane lying on a rectilinear embedding of a certain planar graph is called a *Euclidean graph*. We write $p \in G$ for a point $p$ on $G$, and distinguish the vertices of the underlying planar graph saying that $u \in V(G)$. The *edge set* $E(G)$ of $G$ is the collection of straight-line segments $uv$ such that $u, v \in V(G)$ are adjacent in the underlying planar graph. The *length* of an edge is the Euclidean distance between its two endpoints. A path between $p, q \in G$ is a sequence $pu_1 \dots u_k q$ such that $u_1u_2, \dots, u_{k-1}u_k \in E(G)$, $p$ is a point on an edge ($\neq u_1u_2$) incident with $u_1$, and $q$ is a point on an edge ($\neq u_{k-1}u_k$) incident with $u_k$. The *distance* $d(p, q)$ between points $p$ and $q$ is the length of a shortest $p$-$q$ path in $G$. The *diameter* of $G$ is $\operatorname{diam}(G) = \max_{p,q \in G} d(p, q)$, and the pair $p, q \in G$ is called *antipodal* whenever they are at distance $\operatorname{diam}(G)$. As usual $\delta(u)$ denotes the *degree* of vertex $u$.

We define a *shortcut segment* for a Euclidean graph $G$ as a segment $pq$ with $p, q \in G$ satisfying that $\operatorname{diam}(G \cup pq) < \operatorname{diam}(G)$, and it is called *simple* if $G \cap pq = \{p, q\}$. We want to point out that this definition comes from that given by Yang in [1], which includes as a possibility the equality of the diameters of those two Euclidean graphs, but we believe that our definition captures better the intuitive idea of shortcut, and it allows us to enrich the problem with new elements.

Since one can easily find Euclidean graphs that have no shortcut segment (a triangle, for example), an interesting question arises: is it possible to reduce the diameter of any Euclidean graph by adding a finite number of segments? With this idea in mind, we define a *shortcut set* for $G$ as a finite set $\mathcal{S}$ of straight line segments with endpoints on $G$ such that $\operatorname{diam}(G \cup (\cup_{s \in \mathcal{S}} s)) < \operatorname{diam}(G)$. The *shortcut number* $\operatorname{scn}(G)$ is the cardinality of a minimum shortcut set for $G$, and $\mathcal{S}$ is *optimal* if $|\mathcal{S}| = \operatorname{scn}(G)$ and it minimizes $\operatorname{diam}(G \cup (\cup_{s \in \mathcal{S}} s))$ among all shortcut sets of size $\operatorname{scn}(G)$. The Euclidean graphs $G$ here considered are those that have no two antipodal vertices with dilation 1 (otherwise $G$ does not admit a shortcut set). From now on, we assume this condition to be satisfied.

In [1], the author deals with shortcut segments for Euclidean chains (i.e., Euclidean graphs obtained from paths). He designs three different approximation algorithms to compute optimal shortcut segments for that family of Euclidean graphs. Further, he obtains necessary and sufficient conditions on the uniqueness of those shortcut segments in certain subfamilies, and studies the ratio $\operatorname{diam}(C \cup pq)/\operatorname{diam}(C)$ for any Euclidean chain $C$. In particular, we extend here some of his results to general graphs. To the best of our knowledge, there are no other studies on shortcut sets and so our work initiates the study of these sets in general Euclidean graphs. We want to stress that our problem can be viewed as a variant of the Diameter-Optimal $k$-Augmentation Problem for edge-weighted geometric graphs, where one has to insert $k$ additional edges to a plane geometric graph in order to minimize the diameter of the resulting graph. Even for this case in which the endpoints of the added segments are vertices of the graph, there are very few results as it is pointed out in [2] (where other references are given).

## 1 An upper bound on the shortcut number

In this section, we provide an upper bound on $\operatorname{scn}(G)$ by constructing, in polynomial time, an adequate shortcut set. This leads us to show that one can al-

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ways find an optimal shortcut set.

**Theorem 1** *Let $G$ be a Euclidean graph with $n_1$ pendant vertices. Then, there exists a shortcut set for $G$ of cardinality at most $2|E(G)|-n_1$ that can be computed in polynomial time. Consequently,* $\text{scn}(G) \leq 2|E(G)| - n_1$*, and moreover, this bound is tight.*

**Proof.** (Sketch) Let $u \in V(G)$ be a vertex of degree $\delta(u) = r \geq 2$ and let $u_0, ..., u_{r-1}$ be the set of its neighbors sorted clockwise. Let $\ell_i$ be the line containing the edge $e_i = uu_i$ with $i \in \{0, ..., r-1\}$, and let $H_i^+$ be the right half-space considering the positive direction from $u$ to $u_i$. For each $i$, let $\varphi(u_i) = u_j$ (if it exists) such that $u_j \in H_i^+$ and $u_{j+1} \notin H_i^+$ (where indices are taken modulo $r$). Clearly, the angle formed by edges $e_i$ and $e_j$ is smaller than $\pi$, and so we can take a segment $s_i$ intersecting all edges $e_k$ for every $k$ so that $i \leq k \leq j$; Figure 1 illustrates this fact. Further, $s_i$ shortens all paths formed by any two of those edges, and it can be placed sufficiently close to $u$ in order not to increase $\text{diam}(G)$.

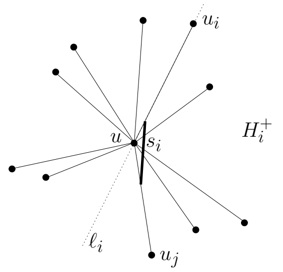


Figure 1: Segment $s_i$ (depicted as a thick segment) cuts all edges $e_k$ with $i \leq k \leq j$.

In this way, we can construct a shortcut set for $G$ of cardinality at most $\sum_{u \in V(G), \delta(u) \geq 2} \delta(u) = 2|E(G)| - n_1$. Thus, we obtain the desired bound whose equality is attained by the *star* $S_n$ with odd $n$, given by $V(S_n) = \{u, u_0, ..., u_{n-1}\}$, $E(G) = \{uu_i : 0 \leq i \leq n-1\}$, and every $u_i$ placed at $(\cos(\frac{2\pi}{n}i), \sin(\frac{2\pi}{n}i))$. □

**Corollary 2** *Every Euclidean graph admits an optimal shortcut set.*

## 2 Euclidean graphs $G$ with $\text{scn}(G) = 1$

The aim of this section is to show that it is possible to identify in polynomial time whether $\text{scn}(G) = 1$ and, in that case, to find an optimal shortcut set (see Theorem 5). To do this, we first need to prove that the diameter of a Euclidean graph $G$ can be computed in polynomial time. We begin with a technical lemma.

**Lemma 3** *Let $p, q \in G$ be an antipodal pair placed in two different non-pendant edges, say $e = uv$ and $e' = u'v'$. Then, there exist two different shortest $p$-$q$ paths, say $P_1$ and $P_2$, such that either $u, u' \in P_1$ and $v, v' \in P_2$, or $u, v' \in P_1$ and $v, u' \in P_2$ (see Figure 2).*

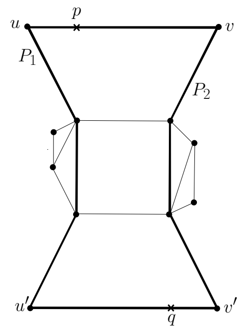


Figure 2: Paths $P_1$ and $P_2$ (depicted with thick edges) containing respectively vertices $u, u'$ and $v, v'$.

Note that, in general the diameter of the Euclidean graph is given by points that may not belong to $V(G)$, which entails a substantial difficulty for finding $\text{diam}(G)$. However, the proof of the following lemma shows that this parameter can be computed in polynomial time by only considering the distances between elements of $V(G)$.

**Lemma 4** *The diameter of a Euclidean graph with $n$ vertices can be computed in polynomial time.*

**Proof.** (Sketch) For a graph $G$ on $n$ vertices, our aim is to find an antipodal pair $p, q \in G$. In a first step, we compute the distance between any pair of vertices of $V(G)$. After this, we consider the case when $p$ (similar for $q$) lies on some pendant edge, say $e = uv$, with $\delta(u) = 1$ . We have,

$$d(p,q) = \max_{r \in u'v'} d(u,r) = \frac{d(u,u') + d(u',v') + d(v',v)}{2}$$

since $d(u', v') = |u'v'|$ (note that $e'$ cannot be a pendant edge since this would lead us to the case $p, q \in V(G)$). So, as a second step, we find all furthest points to the pendant vertices of $V(G)$ using only the information obtained in the first step.

Finally, we assume that $p$ and $q$ lie on two non-pendant edges $e = uv$ and $e' = u'v'$, respectively. To compute $d(p, q)$, we only need to consider the shortest $p$-$q$ paths $P_1$ and $P_2$ provided by Lemma 3, and again the distances computed in the first step. More precisely,

$$d(p,q) = \min\left\{\frac{d(u,v) + d(v,v') + d(v',u') + d(u',u)}{2}, \right.$$
$$\left. , \frac{d(u,v) + d(v,u') + d(u',v') + d(v',u)}{2}\right\}$$

Thus, $\text{diam}(G)$ corresponds with the maximum over all values obtained in each step. □

**Theorem 5** *For every Euclidean graph $G$, it is possible to determine in polynomial time whether* $\mathrm{scn}(G) = 1$ *and, in that case, to construct an optimal shortcut set for $G$.*

**Proof.** Let us first construct two vertical lines $\alpha$ and $\beta$ such that $G$ lies inside the strip that they define. To each vertex $u \in V(G)$, we assign two horizontal segments with an extreme in $w$ and the other in, respectively, $\alpha$ and $\beta$. The set of all those horizontal segments is denoted by $\mathcal{H}$.

For a pair of edges $e, e' \in E(G)$ and a line $\ell$ crossing both of them, we define the class $\mathcal{P}(\ell)$ as the set of lines intersecting exactly the same segments of $\mathcal{H}$ that are intersected by $\ell$. Thus, we have classified the lines that cross edges $e$ and $e'$ by this relation. Note that a line $m \in \mathcal{P}(\ell)$ crosses $e$ and $e'$ in two points, say $e_m$,$e'_m$, and these points define a segment that we denote by $s_m$. Throughout this proof, we shall consider $\mathcal{P}(\ell)$ as a set of lines as well as a set of points in segments defined by those lines. An example of the region of the plane formed by points on segments $s_m$ with $m \in \mathcal{P}(\ell)$ is depicted in Figure 3.

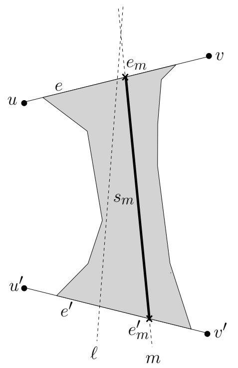


Figure 3: The region in grey corresponds to the points on segments defined by $\mathcal{P}(\ell)$ and edges $e, e'$.

We now show that the number of families $\mathcal{P}(\ell)$ per pair of edges $e, e'$ is of order $O(n^2)$. For each pair of vertices $u, v \in V(G)$, we construct a line $\ell_{u^+v^+}$ parallel to segment $uv$ leaving $u$ and $v$ to its left, and satisfying that any other such a line closer to $uv$ crosses the same segments of $\mathcal{H}$. In the same way, we construct the lines $\ell_{u^+v^-}$ (leaving $u$ to its left and $v$ to its right), $\ell_{u^-v^+}$ ($u$ to its right and $v$ to its left) and $\ell_{u^-v^-}$ (both to its right). Observe that if a line $\ell$ intersects a subset of segments of $\mathcal{H}$ then there exist vertices $u, v \in V(G)$ such that one among the above defined lines, say $\ell_{u^+v^-}$, crosses exactly the same segments of $\mathcal{H}$, and so $\mathcal{P}(\ell) = \mathcal{P}(\ell_{u^+v^-})$. Thus, there are four different families of lines $\mathcal{P}(\ell)$ associated with every pair $u, v \in V(G)$, and so the total number of families is of order $O(n^2)$.

On the other hand, let $e = uv$ and $e' = u'v'$ be two edges of $G$, and let $\mathcal{P}(\ell)$ be a family of lines intersecting both $e$ and $e'$. We define the map $f_\ell^{(e,e')} : \mathcal{P}(\ell) \longrightarrow \mathbb{R}$ as $f_\ell^{(e,e')}(m) = \mathrm{diam}(G \cup s_m)$. To compute the minimum of this map, we shall see that it is only necessary to consider segments intersecting the border of $\mathcal{P}(\ell)$. Indeed, let $m \in \mathcal{P}(\ell)$ be a line so that $s_m$ does not intersect the border of $\mathcal{P}(\ell)$. It is clear that at least one of the four segments depicted in Figure 4 has image lower than $s_m$.

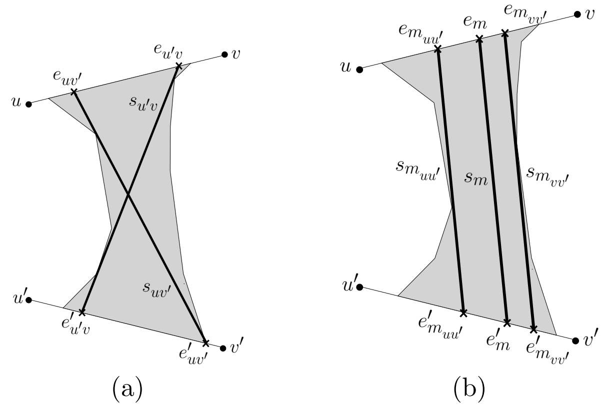


Figure 4: (a) Segments $s_{uv'}$ and $s_{u'v}$, and (b) segments $s_{m_{uu'}}$ and $s_{m_{vv'}}$.

Finally, observe that if $s_m$ with $m \in \mathcal{P}(\ell)$ is a shortest segment between $e$ and $e'$ going through a fixed point $p$ on the border of the region induced by $\mathcal{P}(\ell)$, then $s_m$ must be orthogonal to the bisector of the angle formed by $e$ and $e'$, whenever these edges are not parallel (see Figure 5). Otherwise $s_m$ is orthogonal to both $e$ and $e'$.

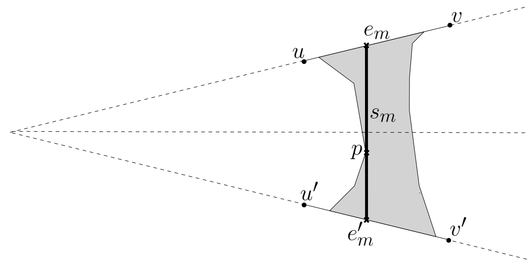


Figure 5: A shortest segment joining $e$ and $e'$ intersecting the border of $\mathcal{P}(\ell)$.

Therefore, in order to find the minimum of $f_\ell^{(e,e')}$ it suffices to compute the images of the lines containing the following segments: $s_{uv'}$ and $s_{u'v}$, the two segments orthogonal to the bisector of the angle between $e$ and $e'$ (or orthogonal to both $e$ and $e'$ if they are parallel) that intersects the border of $\mathcal{P}(\ell)$ and are inside $\mathcal{P}(\ell)$, and the segments inside $\mathcal{P}(\ell)$ containing a segment of the border of $\mathcal{P}(\ell)$. Thus, the line with the smallest image is the minimum of $f_\ell^{(e,e')}$ and, by Lemma 4, this can be computed in polynomial time. Consequently, we can also obtain in polynomial time the minimum value of $f_\ell^{(e,e')}$ over all pairs $e, e'$ and classes $\mathcal{P}(\ell)$, which is attained by a certain segment say $s_m$. If this minimum is smaller than $\text{diam}(G)$ then segment $s_m$ forms an optimal shortcut set for $G$ (thereby showing that $\text{scn}(G) = 1$); otherwise $\text{scn}(G) > 1$. □

## 3 Special families of graphs

Due to the hardness of computing the shortcut number, it is natural to restrict the problem to certain graph classes. Indeed, as well as Yang studies shortcuts of Euclidean chains in [1], we compute this parameter in Euclidean graphs generated from polygons and embeddings of $K_4$.

First, we consider *polygons* $P$ as Euclidean graphs that come from planar embeddings of cycles, and provide their shortcut number.

**Proposition 6** *For every polygon $P$, the following statements hold.*

1. *$P$ does not admit a simple shortcut segment.*
2. *If $P$ is convex then* $\text{scn}(P) = 2$.
3. *If $P$ is non-convex then* $\text{scn}(P) = 1$.

**Proof.** (Sketch) First, observe that if $P$ is a convex polygon then any segment $pq$ with $p, q \in P$ and $P \cap pq = \{p, q\}$ splits $P$ into two paths, say $P_1$ and $P_2$. Thus, let $p_1$ and $p_2$ be the midpoints of $P_1$ and $P_2$, respectively (see Figure 6(a)). It is trivial to check that $p_1$ and $p_2$ remain at distance $\text{diam}(P)$.

On the other hand, if $P$ is non-convex then there is some pocket given by two vertices, say $u, v \in P$. We consider a segment going through $u$ and very close to $v$ that produces three intersection points with $P$: vertex $u$, a point $r$ outside the pocket, and a point $r'$ inside the pocket; Figure 6(b) illustrates this situation. We can prove that $ur$ is a shortcut segment just checking that for any antipodal pair $p, q \in P \cup ur$ there is a shortest $p$-$q$ path shortened by $ur$. Finally, Figure 7 shows that $\text{scn}(P) \leq 2$. □

The only non-trivial Euclidean graphs $G$ obtained from a planar embedding of a complete graph $K_n$ appear for $n = 3$ and $n = 4$. Since the case $n = 3$ has been already studied in Proposition 6, it only remains to consider the case $n = 4$; with an analysis of the antipodal points we can prove the following proposition.

**Proposition 7** *For any Euclidean graph $G$ obtained from a planar embedding of $K_4$, it holds that $scn(G) = 1$.*

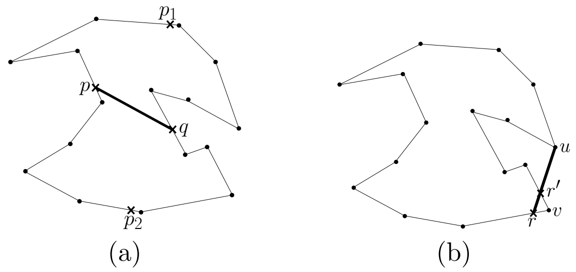


Figure 6: (a) Polygons do not admit simple shortcut segments, and (b) non-convex polygons always admit shortcut segments.

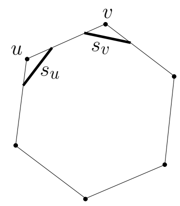


Figure 7: A convex polygon and one of its shortcut sets (depicted with thick edges).

## 4 Concluding remarks and open questions

We have studied shortcut sets of Euclidean graphs, designing first a method to construct, in polynomial time, shortcut sets of bounded size for Euclidean graphs $G$ that have no two antipodal vertices with dilation 1 (which is a necessary condition to guarantee the existence of a shortcut set). This method yields a tight upper bound on $\text{scn}(G)$ that is used to prove that $G$ has an optimal shortcut set. We then show that Euclidean graphs $G$ with $\text{scn}(G) = 1$ can be identified in polynomial time and, in that case, we construct an optimal shortcut set for $G$. Finally, we have computed the shortcut number of polygons, and all possible rectilinear embeddings of $K_4$.

It would be interesting to reduce the complexity of the algorithm provided in the proof of Theorem 5. Also, it is unknown if the problem of deciding whether $\text{scn}(G) = k$, for fixed $k \geq 2$, is polynomial. Finally, it would be interesting to explore the shortcut number of Euclidean graphs generated from other classes of graphs; for instance: trees, 2-connected outerplanar graphs, and planar graphs with high symmetry.

## References

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