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AN INTEGRAL AND MRAC–BASED APPROACH TO THE ADAPTIVE STABILISATION OF A CLASS OF LINEAR TIME–DELAY SYSTEMS WITH UNKNOWN PARAMETERS

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The design of a novel strategy based on the model reference adaptive control method for the stabilisation of a second-order linear time-delay system with unknown parameters is presented. The proposed approach is developed under the assumption that only one state of the system is available, and the sign of the control gain is known. First, the integral operator is applied to obtain a new representation of the original system, where the whole state is known. The use of the integral operator decomposes the control problem into two subproblems that are solved by using the model reference adaptive control method and the backstepping procedure. The effectiveness of the proposed approach is illustrated through an academic example and a practical application case regarding a chemical reactor recycle system.

Keywords: MRAC, time-delay systems, backstepping, integral operator.

1. Introduction

Time delays are parametric descriptions in systems where the process involves transportation of material, energy or information. The infinite-dimensional property of time-delay systems has given rise to modifications of the existing stability analysis techniques and control methods developed for delay-free systems. Stabilisation and control of time-delay systems is a topic of growing interest due to the variety of practical applications that involve time lags. Many research works have been devoted to the control of time-delay systems, using, for example, the sliding-modes technique (Mathiyalagan and Sangeetha, 2019), predictor approaches (Nguyen,

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2018), Lyapunov-based techniques (Fang and Park, 2013; Hua *et al.*, 2022), adaptive control (Jia *et al.*, 2018), flatness-based control (Bekcheva *et al.*, 2017), attractive ellipsoid method-based control (Mera *et al.*, 2014), among others.

The task is further complicated when the system parameters are unknown and the state is partially available for measurements. Under this scenario, adaptive control techniques constitute efficient solutions. This paper presents a control design method for the regulation and trajectory tracking problem of a class of linear time-invariant time-delay systems with parametric uncertainties based on the model reference adaptive control (MRAC) method applying an integral operator. This approach is an efficient tool for dealing with parametric uncertainties in various dynamic systems, including linear and nonlinear, single and multivariable, continuous and discrete, deterministic and stochastic ones (Åström and Wittenmark, 2008).

Some adaptive control techniques have been developed to estimate parameters in time-delay systems. For instance in the works of Orlov *et al.* (2003; 2009), the synthesis of an adaptive parameter identifier for linear dynamic systems with delays in the state and in the control input is presented. The system state is assumed to be available for measurements; necessary and sufficient conditions for the system parameters and delays to be identifiable are provided. Once the parameter identifiability is guaranteed, the proposed adaptive identifier achieves simultaneous online identification of the system parameters and delays. Theoretical results are supported by numerical simulations (Orlov *et al.*, 2003) and an experimental study case is provided (Orlov *et al.*, 2009).

In the work of Yuan *et al.* (2019), an adaptive control scheme for switched time-delay systems that can handle impulsive behaviour in both states and time-varying delays is proposed. The key idea is to construct a Lyapunov function with a time-varying coefficient that guarantees that it will be non-increasing at the switching instants. A practical application example is used to illustrate the effectiveness of the method.

Yuan *et al.* (2018) also address the robust adaptive stabilisation of uncertain switched time-varying delay systems with unknown disturbances in a high-order form. The proposed design guarantees global asymptotic stability for arbitrary switching. A numerical example illustrates the effectiveness of the proposed method.

In the work of Evesque *et al.* (2003), the problem of adaptive control in the presence of large time delays is considered. The proposed control consists of a reduced-order controller that depends on the relative degree of the plant, rather than its order, combined with a Posicast control structure. It is shown that the architecture is amenable to adaptation, leading to stability within a bounded domain for a small time delay. Stable adaptive laws are generated by using high-order tuners and a Lyapunov–Krasvoskii functional that guarantees the closed-loop stability.

Niculescu and Annaswamy (2003) propose a continuous-time adaptive control for systems whose relative degree does not exceed two. The controller structure corresponds to a modified Smith controller to handle plants that may be open-loop unstable. A Lyapunov–Krasovskii functional derived for a model transformation of the original system is used to derive stability properties of the closed-loop system, which results in semi-global stability despite the time delay, and leads to the asymptotic convergence of the output error to zero. The controller is also shown to be robust against disturbances.

The design of MRAC for a class of linear multi-input multi-output (MIMO) systems with delays has been addressed by Mirkin and Gutman (2005; 2010). First (Mirkin and Gutman, 2005), the trajectory tracking task for a class of invariant time-delay systems is solved through a controller whose structure is established on an error equation parameterisation based on the standard MRAC structure for plants without delay. The proposed control technique includes an additional adaptive feedforward control component as an output of a dynamical system driven by the reference signal. Further (Mirkin and Gutman, 2010), an output feedback adaptive control scheme that uses feedback actions based on the Lyapunov–Krasovskii approach is applied to design the adaptation algorithms and to prove the stability of a non-linear system with an unknown time-varying state delay with external disturbances.

Adaptive control techniques have been developed also for systems with a time delay in the input signal, (see, e.g., Yao *et al.*, 2019; Mirkin *et al.*, 2008). In the work of Yao *et al.* (2019), the adaptive control problem for nonlinear systems with unknown parameters is considered; the Padé approximation method is used to deal with the input delay. A controller is designed based on the backstepping technique that guarantees the boundedness of all the closed-loop signals; the tracking error converges to the predefined stable dynamics. Finally, a simulation example is given to verify the effectiveness of the proposed scheme. Mirkin *et al.* (2008) develop a state feedback Lyapunov-based design of direct MRAC for a class of non-linear systems with an external disturbance and with both input and state delays. The procedure is based on reference trajectories prediction; the proposed controller attempts to anticipate the future states. The stability analysis is based on a Lyapunov–Krasovskii-type functional. In the work of Bresch-Pietri *et al.* (2012), an adaptive control scheme based on a backstepping transformation for linear uncertain time-delay systems is presented. The proposal, based on a Lyapunov analysis, is capable of addressing a collection of classic problems of equilibrium regulation. The effectiveness of the proposed approach is illustrated through numerical simulations of a second-order plant in which there is no knowledge of the actuator state.

It is important to point out that the aforementioned research works propose adaptive control schemes to solve stabilisation or trajectory tracking problems by estimating the parameters of uncertain time-delay systems. All of them assume that the whole state of the system is available. Unlike what can be found in the literature, this paper proposes a simple solution to the problem of regulation and trajectory tracking for a class of second-order linear time-delay systems with unknown parameters in which total access to the states is unavailable. The proposed method, based on MRAC together with the backstepping procedure, does not allow the estimation of the actual parameters of the system, but it guarantees the convergence of the state to the desired reference signal. Considering that only the state $x(t)$ is available, and, inspired by the work of Aguilar-Ibañez *et al*. (2021), the integral operator is used to reconstruct the non-available state. Numerical examples are provided to highlight the effectiveness of the proposed method.

The contributions of this paper regarding the existing literature can be summarised in the following points:

- To the best of the authors' knowledge, the problem stated here has not been addressed before, i.e., there are no research works that propose a solution for the stabilisation of a time-delay system considering the lack of access to the whole state and under unknown system parameters.
- The use of an observer or state estimator is not required since the full state of the offered system representation is available. This representation is derived from the integration over time of the original functional differential equations describing the second-order time-delay system.
- The proposed solution is simple since the proposed representation, given by a cascade connection of two subsystems (see Eqns. (6) and (7)), allows splitting the control problem into two stages. In the first one, an MRAC-based method, along with an appropriate stability analysis using a Lyapunov–Krasovskii functional, is applied to guarantee the asymptotic stability of the first subsystem. In the second stage, the backstepping technique allows stabilising the overall system.

Throughout the paper, the following notation is used: matrix transposition is denoted by the superscript T, $\mathbb{R}^n$ denotes the n-dimensional Euclidean vector space, the Euclidean norm for vectors and the induced norm for matrices are used, both denoted by $||\cdot||$, and $\mathcal{C}[t_0 - h, t_0] \to \mathbb{R}^n$ is the Banach space of continuous functions with the norm $||x_t|| = \max_{\theta\in[t_0-h,t_0]} |x(t+\theta)|$.

2. Problem statement

Consider a second-order linear time-delay system of the form

$$\ddot{x}(t) = a_1 x(t) + a_h x(t-h) + a_2 \dot{x}(t) + bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}$ is the system state variable, $u(t) \in \mathbb{R}$ is the system input, h is a known and constant time delay, a_1, a_h and $a_2 \in \mathbb{R}$ and $b > 0 \in \mathbb{R}$ are unknown constants. The initial conditions of the system are represented by the continuously differentiable function $\phi(\theta), \forall\theta \in [-2h, 0]$.

In the system (1), it is assumed that the state $\dot{x}(t)$ is not available for measurements. In practical application problems, it is well known that the derivative of the state (velocity) is not always available due to sensor limitations. Adding velocity sensors can increase the cost and complexity of a control system. Even when velocity measurements are available, they can be noisy or prone to errors, making them unreliable for control purposes. In practice, it is more convenient to rely on position measurements and use Luenberger or reduced-order observers to reconstruct the non-available state. In the proposed approach, instead of using state observers, the integration operator is applied as explained in Section 2.1.

In this work, the trajectory tracking control problem is addressed. The objective is to establish a control law such that $x(t) \to x_{\mathrm{ref}}(t)$, where $x_{\mathrm{ref}}(t)$ is a given smooth reference trajectory. The proposed adaptive controller design approach is based on the results for delay-free systems presented by Aguilar-Ibañez *et al.* (2021; 2013) and Ortega *et al*. (2003). Critical assumptions to achieve the control goals are presented below.

Assumption 1. The first and second time derivatives of the reference trajectory $x_{\mathrm{ref}}(t)$ are bounded and continuous.

Assumption 2. The initial condition is continuous and is such that $\int_{t_0-2h}^{t_0-h} \phi(s)\,\mathrm{d}s$ is bounded.

Assumption 3. The state $x(t)$ and the delayed state $x(t-h)$ are available for all $t > 0$.

To address the adaptive tracking problem, the following error function is considered:

$$\begin{aligned} e(t) &= x(t) - x_{\mathrm{ref}}(t), \\ \dot{e}(t) &= \dot{x}(t) - \dot{x}_{\mathrm{ref}}(t). \end{aligned} \tag{2}$$

2.1. Control design. This section presents the controller design to address the adaptive trajectory tracking problem of the system (1).

In order to reach the control goal, the non-available state $\dot{x}(t)$ is reconstructed through the use of the integration operator over Eqn. (1). Hence, integrating both the sides of Eqn. (1), from $t_0 - h$ to t, the following expression is obtained:

$$\begin{aligned}\dot{x}(t) =& a_1 \int_{t_0-h}^{t} x(s)\,\mathrm{d}s + a_h \int_{t_0-h}^{t} x(s-h)\,\mathrm{d}s \\ &+ a_2 x(t) + a_3 + b v(t),\end{aligned}$$

where $a_3 = -a_2 x(t_0-h) + \dot{x}(t_0-h)$ and the new control $v(t)$ is given by

$$v(t) = \int_{t_0-h}^{t} u(s)\,\mathrm{d}s.$$

Then it is possible to rewrite the error equation as

$$\begin{aligned}\dot{e}(t) =& a_1 \int_{t_0-h}^{t} x(s)\,\mathrm{d}s + a_h \int_{t_0-h}^{t} x(s-h)\,\mathrm{d}s \\ &+ a_2 x(t) + a_3 + b v(t) - \dot{x}_{\mathrm{ref}}(t).\end{aligned} \tag{3}$$

Note that Eqn. (3) can be written as

$$\dot{e}(t) = a^T H(t, t_0 - h) + b v(t), \tag{4}$$

where a is a vector of unknown parameters given by

$$a = \begin{bmatrix} a_1 & a_h & a_2 & a_3 & 1 \end{bmatrix}^T,$$

and $H(t, t_0 - h)$ is defined as

$$\begin{aligned}&H(t, t_0 - h) \\ &= \Big[\int_{t_0-h}^{t} x(s)\,\mathrm{d}s \quad \int_{t_0-h}^{t} x(s-h)\,\mathrm{d}s \\ &\quad x(t) \;\; 1 \;\; -\dot{x}_{\mathrm{ref}}(t) \Big]^T.\end{aligned} \tag{5}$$

Hence, the extended system is written as

$$\dot{e}(t) = a^T H(t, t_0 - h) + b v(t), \tag{6}$$

$$\dot{v}(t) = u(t). \tag{7}$$

The integrator backstepping technique will be applied to the system (6)–(7). This system can be viewed as a cascade connection of two components; one is (6), with $v(t)$ as input, and the other is (7). Then, a state feedback control law $v(t) = \sigma(x)$ such that the origin of

$$\dot{e}(t) = a^T H(t, t_0 - h) + b\sigma(x) \tag{8}$$

is asymptotically stable must be designed.

Since $H(t, t_0 - h)$ is available for measurements, it is possible to apply the MRAC method as follows.

Consider the following parameterisation:

$$b^{-1}\left(-a^T H - e(t) - \int_{t_0-h}^{t} e(s)\,\mathrm{d}s \right) = H^T \theta_*, \tag{9}$$

where $\theta_* \in \mathbb{R}^5$ is the ideal vector, in which all the parameters are constant and known. Define the adaptive controller $\sigma(x)$ as

$$\sigma(x) = H^T(t, t_0 - h)\widehat{\theta}, \tag{10}$$

where $\widehat{\theta}$ is an estimate of the ideal vector θ_*. Then, in order to find the evolution of $\widehat{\theta}$, Eqn. (8) can be written in terms of Eqns. (9) and (10) as

$$\begin{aligned}\dot{e}(t) = & -e(t) - \int_{t_0-h}^{t} e(s)\,\mathrm{d}s \\ &+ b H^T(t, t_0 - h)\Delta\theta,\end{aligned} \tag{11}$$

where $\Delta\theta = \widehat{\theta} - \theta_*$.

The next step consists in finding the adaptive evolution of the parameters in vector $\widehat{\theta}$ such that the trajectories of the system (11) satisfy $e(t) \to 0$ and $\dot{e}(t) \to 0$.

It is well known that the Lyapunov method is effective in determining the stability of a given system. For a delay-free system, this requires the construction of a Lyapunov function $V(t, x(t))$, which could be seen as a measure that quantifies the deviation of the state $x(t)$ from the trivial solution 0. In the case of time-delay systems, the state at time t required to specify the future evolution of the system beyond t is the value of $x(t)$ in the interval $[t - h, t]$, i.e., $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$. Then, it is natural to expect that, for a time-delay system, the corresponding Lyapunov function should be a functional $V(t, x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0. Such a functional is known as a Lyapunov–Krasovskii one (Gu *et al.*, 2003).

The Lyapunov–Krasovskii theorem given below states the conditions to guarantee the asymptotic stability of a time-delay system.

Theorem 1. (Gu *et al.*, 2003, Proposition 5.2, p. 148) *A time-delay system is asymptotically stable if there exists a bounded quadratic Lyapunov–Krasovskii functional $V(\phi)$ such that, for some $\epsilon > 0$, it satisfies*

$$V(\phi) \geq \epsilon ||\phi(0)||^2,$$

and its derivative along the system trajectories,

$$\dot{V}(\phi) = \dot{V}(x_t)\Big|_{x_t=\phi},$$

satisfies

$$\dot{V}(\phi) \leq -\epsilon ||\phi(0)||^2.$$

The proof of Theorem 1 can be found in the work of Gu *et al*. (2003).

Now, in view of Theorem 1, to analyse the stability of the system (11) and find the adaptive evolution of the parameters in vector $\widehat{\theta}$, the following Lyapunov–Krasovskii functional is considered:

$$V_0 = \frac{1}{2b}e^2(t) + \frac{1}{2b}\left(\int_{t_0-h}^{t} e(s)\,\mathrm{d}s\right)^2 + \frac{1}{2}\Delta\theta^T\Lambda\Delta\theta, \tag{12}$$

with a constant $\Lambda > 0$. As the ideal vector θ_* is constant, the time derivative of V_0 along the trajectories of the system (11) leads to

$$\dot{V}_0 = -\frac{1}{b}e^2(t) + e(t)H^T(t,t_0-h)\Delta\theta + \dot{\widehat{\theta}}^T\Lambda\Delta\theta. \tag{13}$$

The following adaptation function:

$$\dot{\widehat{\theta}}^T = -e(t)\Lambda^{-1}H^T(t,t_0-h) \tag{14}$$

allows one to express Eqn. (13) as

$$\dot{V}_0 = -\frac{1}{b}e^2(t). \tag{15}$$

It can be concluded that $e(t) \in L_\infty \cap L_2$, $\Delta\theta \in L_\infty$ and so is $\widehat{\theta}$, and $\int_{t_0-h}^{t} e(s)\,\mathrm{d}s \in L_\infty$, as $t \to \infty$. Subsequently, it is demonstrated that each term of $\dot{e}(t)$ defined in (11) is bounded. Assumption 1, together with $e(t) \in L_\infty$ and $\int_{t_0-h}^{t} e(s)\,\mathrm{d}s \in L_\infty$, implies that $\int_{t_0-h}^{t} x(s)\,\mathrm{d}s \in L_\infty$ and all the elements of $H(t,t_0-h)$, given in (5), are bounded. For the second element of $H(t,t_0-h)$, note that $\int_{t_0-h}^{t} x(s-h)\,\mathrm{d}s = \int_{t_0-2h}^{t_0-h} \phi(s)\,\mathrm{d}s + \int_{t_0-h}^{t-h} x(s)\,\mathrm{d}s$, where the second term has already been proven to be bounded and, if Assumption 2 holds, the first term is also bounded. Therefore, it can be concluded that $\dot{e}(t) \in L_\infty$. Finally, from Barbalat's lemma with (15) and $\dot{e}(t) \in L_\infty$, it is concluded that $e(t) \to 0$. Moreover, $\dot{\widehat{\theta}} \in L_\infty$ with $\dot{\widehat{\theta}} \to 0$, thus ensuring that all the signals are bounded.

Proposition 1 given below summarises the above idea.

Proposition 1. *Consider the system (8). Then, the MRAC law*

$$\sigma(x) = H^T(t,t_0-h)\widehat{\theta}, \tag{16}$$

where $\widehat{\theta}$ *evolves according to*

$$\dot{\widehat{\theta}} = -e(t)\Lambda^{-1}H(t,t_0-h) = \varpi(t,t_0-h),$$

ensures, for any initial condition, that $\lim_{t\to\infty} e(t) = 0$ *and* $\lim_{t\to\infty} \dot{e}(t) = 0$*, with all signals bounded.*

Remark 1. Note that there are no restrictions on the magnitude of the plant delay since the stability analysis is time-delay independent. That is, the negativity of the Lypaunov–Krasovskii derivative is guaranteed regardless of the delay value by appropriately choosing the adaptation function.

Once the control law $\sigma(x)$ guarantees the asymptotic stability of the system (6), the next step consists in designing a control law $u(t)$ to stabilise the overall system (6)–(7). By adding and subtracting $b\sigma(x)$ on the right-hand side of Eqn. (6), the equivalent representation is obtained

$$\dot{e}(t) = [a^T H(t,t_0-h) + b\sigma(x)] + b[v(t) - \sigma(x)], \tag{17}$$

$$\dot{v}(t) = u(t), \tag{18}$$

where $v = \int_{t_0-h}^{t} u(s)\,\mathrm{d}s$.

Changing the variable $\xi = v - \sigma(x)$, Eqns. (17) and (18) can be rewritten as

$$\dot{e}(t) = [a^T H(t,t_0-h) + b\sigma(x)] + b\xi, \tag{19}$$

$$\dot{\xi}(t) = z, \tag{20}$$

where $z = u - \dot{\sigma}$.

The system (19)–(20) is similar to the initial system (6)–(7), except that now the first component has an asymptotically stable origin when the input ξ is zero. This feature allows us to design the adaptive control $u(t)$ that stabilises the extended system (6)–(7). Proposition 2 states a proposal for such an adaptive control $u(t)$.

Proposition 2. *Consider the system* (19)–(20)*, with an asymptotically stable origin when the input* ξ *is zero and its dynamics defined in Proposition 1.*

Consider also the errors defined as $\xi = v - \sigma(x)$ and $\widetilde{p} = \widehat{p} - p \in \mathbb{R}^5$*, where*

$$p = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \widehat{p} = \begin{bmatrix} \widehat{a} \\ \widehat{b} \end{bmatrix}.$$

Then, the adaptive control

$$u(t) = -\kappa_0\xi - \Phi(t,t_0-h), \tag{21}$$

where

$$\begin{aligned}\Phi(t,t_0-h) = {}& e(t) - x\widehat{\theta}_1 - x(t-h)\widehat{\theta}_h \\ & - \widehat{\theta}_2\left(H^T(t,t_0-h)\widehat{a} + \widehat{b}v + \dot{x}_{\mathit{ref}}(t)\right) \\ & - H^T(t,t_0-h)\varpi(t,t_0-h) \\ & + \ddot{x}_{\mathit{ref}}(t)\widehat{\theta}_4,\end{aligned} \tag{22}$$

$$\widehat{\theta} = [\widehat{\theta}_1 \;\; \widehat{\theta}_h \;\; \widehat{\theta}_2 \;\; \widehat{\theta}_3 \;\; \widehat{\theta}_4]^T, \tag{23}$$

and

$$\dot{\widehat{p}} = -\kappa_1 \widehat{\theta}_2 \begin{bmatrix} H(t, t_0 - h) \\ v \end{bmatrix} \xi, \tag{24}$$

with κ_0, $\kappa_1 > 0$, ensures the stability property of the system (11), with $e(t) \to 0$ as long as $t \to \infty$, for the extended system with (global) boundedness of all signals.

Proof. Substituting Eqns. (9) and (16) into (19) yields

$$\begin{aligned} \dot{e}(t) = &-e(t) - \int_{t_0-h}^{t} e(s)\,\mathrm{d}s \\ &+ bH^T(t, t_0 - h)\Delta\theta + b\xi. \end{aligned} \tag{25}$$

To make the convergence analysis, the following Lyapunov–Krasovskii functional is proposed:

$$V_T = V_0 + \frac{1}{2\kappa_1}||\widetilde{p}||^2 + \frac{1}{2}\xi^2. \tag{26}$$

Hence, the time derivative of V_T along the trajectories of (25) satisfies, after using once again Eqn. (14), the following:

$$\begin{aligned} \dot{V}_T = &\dot{V}_0 + e(t)\xi + \frac{1}{\kappa_1}\widetilde{p}^T\dot{\widehat{p}} \\ &+ \xi\Big(u - \dot{H}^T(t, t_0 - h)\widehat{\theta} \\ &-H^T(t, t_0 - h)\varpi(t, t_0 - h)\Big). \end{aligned} \tag{27}$$

Notice that in view of (23) we get

$$\begin{aligned} \dot{H}^T(t, t - h)\widehat{\theta} = &\, x(t)\widehat{\theta}_1 + x(t - h)\widehat{\theta}_h \\ &+ \dot{x}(t)\widehat{\theta}_2 - \ddot{x}_{\mathrm{ref}}(t)\widehat{\theta}_4. \end{aligned} \tag{28}$$

On the other hand, $\dot{x}(t)$ can be written in terms of $\widehat{a}$, $\widehat{b}$ and $\widetilde{p}$ as

$$\begin{aligned} \dot{x}(t) = &\,\widehat{a}^T H + \widehat{b}v \\ &- \widetilde{p}^T \begin{bmatrix} H(t, t_0 - h) \\ v \end{bmatrix} + \dot{x}_{\mathrm{ref}}(t). \end{aligned} \tag{29}$$

Then, by substituting (15), (28) and (29) into (27), and in view of (21)–(24) , we deduce that

$$\dot{V}_T = -\frac{1}{b}e^2(t) - \kappa_0\xi^2.$$

Once again, it is concluded that $e(t) \to 0$ and $\xi \to 0$, as long as $t \to \infty$, with the set of signals $\{\widetilde{p}, \phi, e, \xi\} \in L_\infty$. According to the definition of $\widetilde{p}$, one can also conclude that $\widehat{p}^T = [\widehat{a}^T, \widehat{b}]$ is bounded. ∎

Figure 1 shows a simplified block diagram to illustrate the general idea of the proposed control approach.

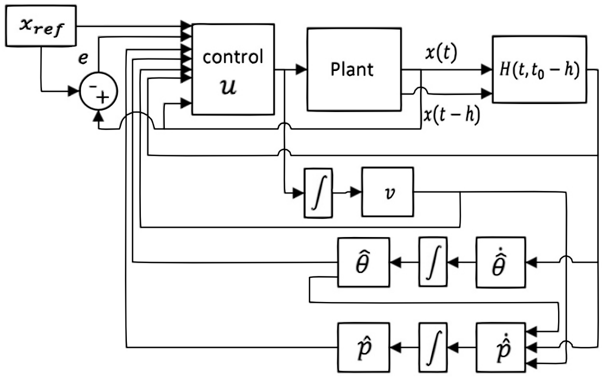


Fig. 1. Simplified block diagram of the proposed control approach.

Remark 2. In order to implement the proposed control scheme, the plant under consideration should have the structure defined in Eqn. (1), which plays a significant role in physics and engineering since it allows modelling several physical systems like mechanical and electromechanical oscillators, electric circuits, and structural vibrations.

3. Numerical examples

3.1. Academic example. Consider the unstable time-delay system with unknown parameters described by Eqn. (1). For simulations, the following parameters are considered: $a_1 = 2$, $a_h = 1$, $a_2 = 8$, $b = 2$. The time delay is $h = 2$, and the initial conditions are given by $(\phi(t),\ \dot{\phi}(t)) = (-1,\ 2)$.

The control task consists in reaching the equilibrium point $(x(t),\ \dot{x}(t)) = (5, 0)$. By using Propositions 1 and 2, the control objective is achieved. Figure 2 shows the obtained simulation results. The controller gains are chosen as $\kappa_0 = 25$, $\kappa_1 = 0.6$. It should be highlighted that the controller effectively stabilises the second-order system with a negligible steady-sate error. Note in Fig. 3 that, even when the parameter estimates converge to constant values after $t = 2.5$, they do not correspond to the actual parameters values. In Fig. 4(a) one can see the behaviour of the control signal. Figure 4(b) shows the error signal, whose magnitude is rapidly approaching zero.

Consider now the trajectory tracking problem where $x_{\mathrm{ref}}(t) = \sin 5t$. Propositions 1 and 2 are applied to achieve the stated control task. Figures 5 and 6 show the obtained simulation results, considering the controller gains $\kappa_0 = 25$, $\kappa_1 = 0.5$.

Note that the controller effectively makes the unstable second-order system reach the desired reference.

Figures 6(a) and (b) show the controller trajectory and the error of the system, respectively. Note that although the error does not converges to zero, it remains

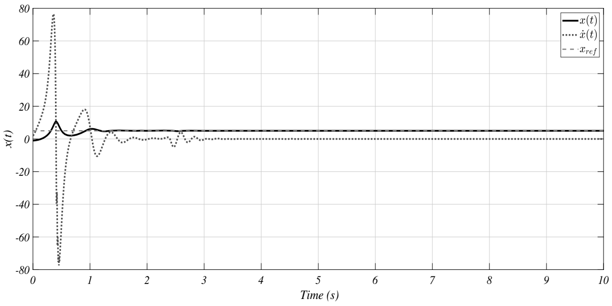
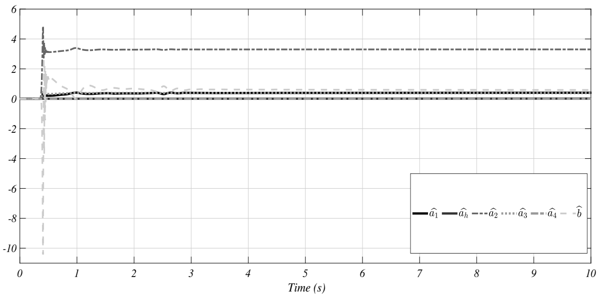
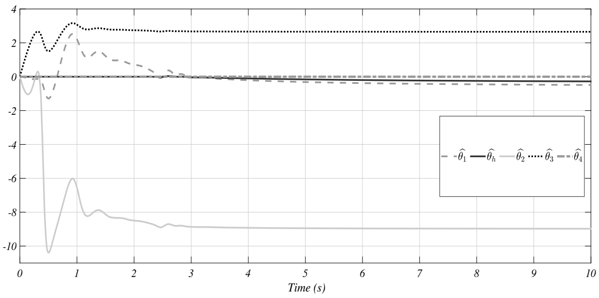


Fig. 2. Trajectories of the controlled system.

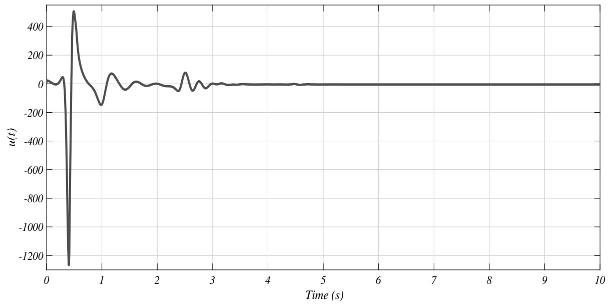


(a) $\hat{p}$

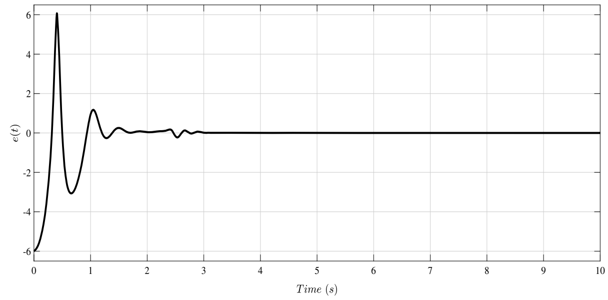


(b) $\hat{\theta}$

Fig. 3. Estimated parameters.



(a) control signal $u(t)$



(b) error signal

Fig. 4. Control and error signals obtained in simulation.

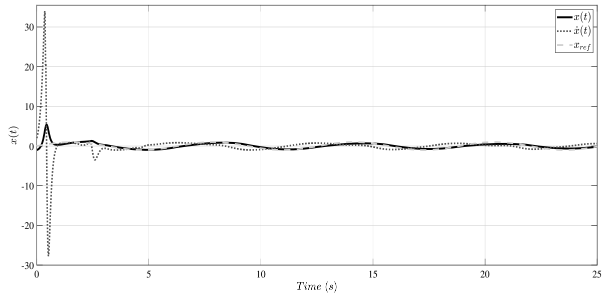
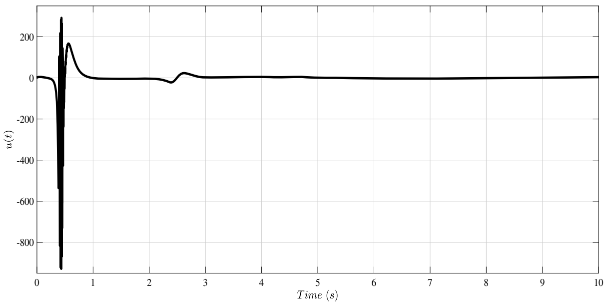


Fig. 5. Trajectories of the controlled system, reaching the reference $x_{\text{ref}}(t) = \sin 5t$.

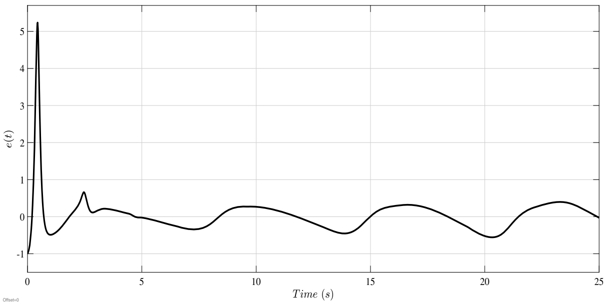
close to this value in spite of the instability of the plant.

One condition that might be seen as a strong restriction of the proposed approach is the requirement of knowing the time-delay value of the plant. In the following example, it is shown that the proposed control scheme is robust against uncertainties in the time delay. Simulations of the system are developed by assuming that the time-delay value used in the adaptation function (14) does not correspond to the delay-value of the plant, i.e., the plant has a delay $h = 2$ while in the adaptation function we consider $h = 1.85$. Figure 7 (a) shows the output of the controlled system. It is evident that, despite not using the exact value of the plant delay, the control task is achieved. However, the magnitude of the control signal is considerably increased (see Fig. 7 (b)).

As mentioned before, there are no research works that propose a solution for the stabilisation of a time-delay system considering the lack of access to the whole state and under unknown system parameters. However, in order to assess the performance of the proposed controller, a comparison with the controller proposed by Aguilar-Ibañez *et al.* (2021) was developed. Numerical simulations results of the system under consideration in closed loop with the controller presented by Aguilar-Ibañez *et al.* (2021) are shown in Fig. 8. To apply this controller, the time delay was set to zero. Note that the system trajectories converge to the reference values in a short period of time. However, the controller exhibits oscillations of significant magnitude, which are inconvenient from a practical perspective.

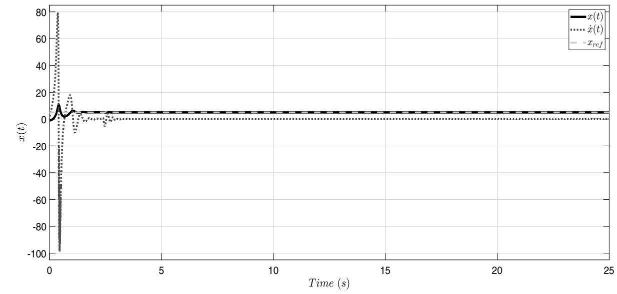


(a) control signal $u(t)$

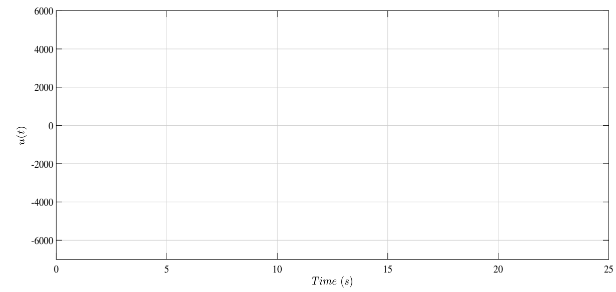


(b) error signal

Fig. 6. Trajectory tracking problem: control and error signals.



(a) output



(b) control signal

Fig. 7. Trajectories of the controlled system under an uncertainty in the delay.

3.2. Practical application example: A chemical reactor recycle system. Chemical reactor recycle systems are often used in chemical industry when the reaction is autocatalytic (i.e., if one of the reaction products is also a catalyst for the same or a coupled reaction), or when it is required to keep isothermal operation of the reactor. Recycle reactors allow increasing the overall conversion and give rise to costs reduction. In these systems several operations are involved, the main being the separation of the input to be recycled from the yields and the transportation through pipes. These operations introduce time delays in the system.

The study case addressed by Phoojaruenchanachai *et al.* (1998) as well as Ali and Mahmoud (2022) is used to illustrate the effectiveness of the proposed approach. It is important to recall that even the overall model does not match the second order time-delay system under consideration; the proposed analysis can be applied to stabilise one of the system states.

In this example, an irreversible reaction $A \to B$ with a negligible heat effect takes place in the two-stage reactor system. Temperature is maintained constant; thus only the composition of product streams from two reactors, c_1, c_2 needs to be controlled. The material balance equations for the reactor system include uncertainties in the system (δk_1 and δk_2). The control objective is to steer c_1 and c_2 to a given set point (c_{1s}, c_{2s}). The system states are defined as $x_1 = c_1 - c_{1s}$ and $x_2 = c_2 - c_{2s}$; then, $(c_1, c_2) \to (c_{1s}, c_{2s})$ whenever $(x_1, x_2) \to (0, 0)$.

The recycle reactor model is given by the time-delay matrix equation (30) describing the dynamics of the two system states:

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + Bu(t), \qquad (30)$$

where

$$A = \begin{bmatrix} -2.3333 + \delta k_1 & 0 \\ 0.25 & -3 + \delta k_2 \end{bmatrix},$$

$$A_h = \begin{bmatrix} 0 & 0.25 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix},$$

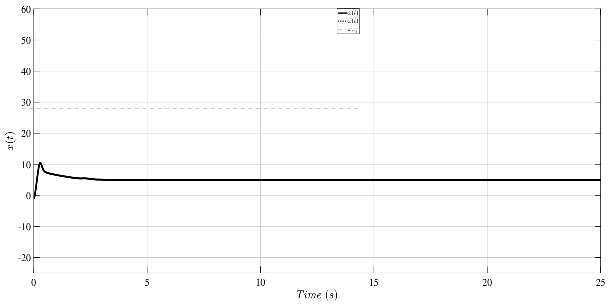
with $h = 1$, $c_{1s} = 0.5$, $c_{2s} = 1$, $\delta k_1 = 5$ and $\delta k_2 = 0.5$.

The proposed control approach is used to stabilise the state x_2 of the chemical system, i.e, the output of the system is defined as $y(t) = Cx(t)$, where $C = [0 \quad 1]$. The transfer function of the system (30) allows determining the relation between the state $x_2(t)$ and the control input $u(t)$. This relation gives rise to the equation

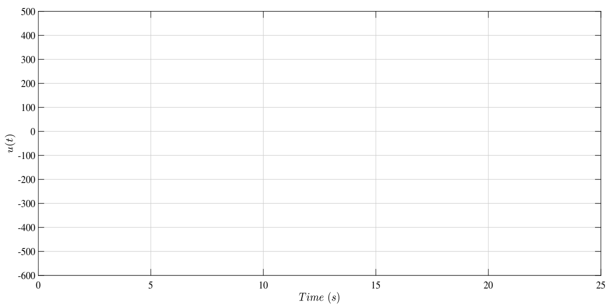
$$\ddot{x}_2(t) = a_1 x_2(t) + a_h x_2(t-h) + a_2 \dot{x}_2(t) + bu(t), \quad (31)$$

where $a_1 = -(\delta k_1 - 2.3333)(\delta k_2 - 3)$, $a_h = 0.0625$, $a_2 = \delta k_1 + \delta k_2 - 5.3333$ and $b = 0.1$.

Note that the dynamics of this system state are described by a second-order time-delay equation of the form (1). To implement the proposed control control approach, it is assumed that the parameters



(a) output



(b) control signal

Fig. 8. Trajectories of the system defined in Section 3.1 in closed loop with the controller proposed by Aguilar-Ibañez *et al*. (2021).

of (31) are unknown. The control objective, as discussed above, consists in reaching the equilibrium point $(x_2(t),\ \dot{x}_2(t)) = (0, 0)$.

The controller gains are chosen as $\kappa_0 = 100$, $\kappa_1 = 10$. The effectiveness of the proposed approach in stabilising the reactor system is illustrated in Fig. 9. Note that in about 8 seconds the system trajectories are driven close to the equilibrium. Figure 10 shows the corresponding control signal.

4. Conclusions

In this paper, the adaptive stabilisation problem of an uncertain second-order linear time-delay system was solved. The proposed approach was developed under the assumption that the system position is always available and the sign of the control gain is known. The integral operator was applied to obtain a new representation of the original system, where the whole state is known. The use of the integral operator allows one to divide the original control problem into two subproblems: one consists in using the MRAC method to stabilise the new uncertain subsystem, the other applies the backstepping procedure to stabilise the whole extended system. It should pointed out that the stabilisation of the extended system is equivalent to the stabilisation of the original uncertain system.

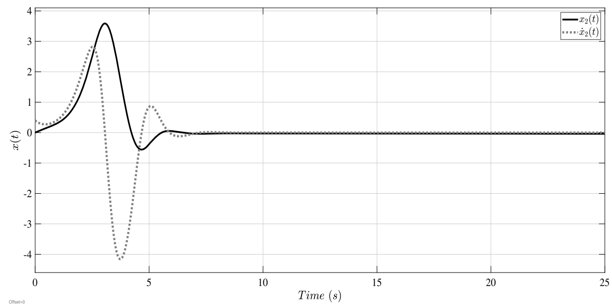


Fig. 9. Trajectories of the controlled system.

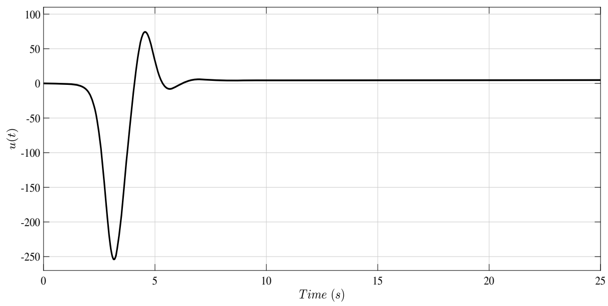


Fig. 10. Control signal $u(t)$ applied to the reactor system.

Numerical simulations of an academic and a practical application example demonstrate the effectiveness of the proposed approach, both for the regulation task and for the trajectory tracking problem. Future work contemplates the design of stabilising controllers based on the proposed scheme for a more general class of systems (higher-order and MIMO systems, among others).

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