

A NEW APPROACH TO MULTIVARIATE LIFETIME  
DISTRIBUTIONS BASED ON THE EXCESS-WEALTH  
CONCEPT: AN APPLICATION IN TUMOR GROWTH

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Mathematics for acceptance a thesis entitled **“A New Approach to Multivariate Lifetime distributions based on the Excess-Wealth Concept: An Application in Tumor Growth”** by **María del Rosario Rodríguez Griñolo** in partial fulfillment of the requirements for the degree of **Doctor in the Department of Statistics and Operations Research.**

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*To my parents: Carmeli and Miguel.*





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# Preface

Lifetime distributions are of great importance in the theory of stochastic modelling, renewal theory, reliability and survival analysis. As was pointed out by Kochar and Wiens (1987), the ageing of a physical or biological system is known as the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer one. Many criteria of ageing (e.g. IFR, DMRL and NBUE notions) have been developed in the literature over many years and have been characterized by different methods. Analogous multivariate versions have also been suggested and studied. It is worth mentioning that many of these methods are included within the framework of stochastic ordering.

This work fundamentally deals with two problems: the problem of studying new multivariate ageing notions and the problem of characterizing them by means of a multivariate dispersion function. Two concepts are taken as a starting point: the multivariate quantiles and the upper-corrected orthant. The first concept, also known as standard construction, was introduced for the first time by O'Brien (1975) and it has been widely used in simulation theory as well as in multivariate stochastic orderings. The second concept was given by Fernández-Ponce and Suárez-Lloréns (2003). They provided the notion of the upper-corrected orthant associated with the standard construction and obtained the important result that the cumulated probability in this region does not depend on the distribution. These concepts, together with the work by Fernández-Ponce *et al.* (1998), where the univariate ageing notions are characterized through the excess-wealth function, gave the basis for the development of this research. Obviously, numerous significant results in the reliability and stochastic

order areas are considered, in order to achieve our proposals.

This work is structured in four chapters. Chapter 1 is introductory and presents the state-of-the-art in univariate and multivariate ageing notions. In particular, some works in which stochastic comparisons are used to characterize these notions are summarized. Quantiles and their generalizations, as well as the univariate excess-wealth function are also considered in this chapter.

Chapter 2 aims to give a multivariate excess-wealth concept, based jointly on the work by Fernández-Ponce *et al.*(1998) and Fernández-Ponce and Suárez-Lloréns (2003). It starts by introducing notions and preliminaries which will be used through the work. Then, given the importance of the upper-corrected orthant in this research, attention is focused on providing new results about this concept. The relationships between the support of a random vector, the upper-corrected orthant and the right-upper orthant at a point, are established. Finally, in the last two sections, the multivariate excess-wealth function and the multivariate excess-wealth order are studied. This function is defined in terms of the upper-corrected orthant and it is shown that it preserves the same properties which are verified by the univariate version.

The definitions of new multivariate ageing properties are the topic of Chapter 3. From new generalizations of the hazard rate and mean residual life functions, multivariate versions of the IFR, DMRL and NBUE ageing notions are presented, together with the chain of implications between them. Following the development in Fernández-Ponce *et al.* (1998), characterizations of this new property are given in terms of the multivariate excess-wealth function. Finally, these multivariate notions allow the definition of a new ordering to compare the ageing of two random vectors.

Finally, in Chapter 4, an application of a particular multivariate lifetime distribution is considered in oncology. Patient age and tumor size at spontaneous detection of the tumor, play an important role in the prevention of cancer. There is increasing interest in the early detection of chronic diseases, with the expectation that earlier diagnosis, combined with therapy, result in more cures and longer survivals. The process of tumor development can be explained in terms of patient age at the onset

of the tumor (time from the patient is born until the first tumor cell appears) and the sojourn time (time from the first tumor cell appearing until the detection of the tumor). A non-deterministic exponential model that relates the sojourn time to the tumor size at spontaneous detection is suggested and studied in this chapter. In the process of estimating the parameters in this model, a constraint is used which represents an inherent multivariate ageing property of the lifetime distributions considered. The proposed model is illustrated using two real databases.

Sevilla, April 2010.





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# CHAPTER 1

## Introduction

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### 1.1 Univariate life distributions and generalizations

In reliability theory, lifetime of systems and components are frequently studied through univariate concepts of ageing. There exist several concepts of statistical aging for studying life distributions. The most usual reliability measures associated with a nonnegative random variable  $X$ , which usually represents the life-length of a unit or system, are the mean residual life and the failure rate (hazard) function. Their definitions are given below. Let  $X$  be a nonnegative univariate random variable with survival function  $\bar{F}_X$  and finite mean  $\mu_X$ . The mean residual life function of  $X$  is defined as

$$\mu_X(x) = \frac{\int_x^\infty \bar{F}_X(t) dt}{\bar{F}_X(x)}, \quad x \in \mathbb{R}^+$$

and the failure (hazard) rate function of  $X$  is defined as

$$r_X(x) = \lim_{h \rightarrow 0} \frac{P\{x < X \leq x + h | X > x\}}{h} = \frac{f_X(x)}{\bar{F}_X(x)}, \quad x \in \mathbb{R}^+.$$

When  $F$  is absolutely continuous, the failure rate can alternatively be expressed as

$$r_X(x) = \frac{\partial\{-\log[\bar{F}_X(x)]\}}{\partial x}, \quad x \in \mathbb{R}^+.$$

Among the univariate life distribution classes that have been extensively examined in the literature, the following are considered: increasing failure rate (IFR), decreasing mean residual life (DMRL) and new better than used in expectation (NBUE). The definitions of these classes and their dual classes are recalled here (see, for example, Barlow and Proshan, 1975 and Kochar and Wiens, 1987).

**Definition 1.1.1.** Let  $X$  be a nonnegative univariate random variable with finite mean  $\mu_X$ , mean residual life function  $\mu_X(x)$  and failure rate  $r_X(x)$ . Then,

- i)  $X$  is said to have a *new better[worse] than used in expectation* (NBUE)[NWUE] distribution if  $\mu_X(x) \leq [\geq]\mu_X$  for all  $x \geq 0$ .
- ii)  $X$  is said to have a *decreasing [increasing] mean residual life* (DMRL)[IMRL] distribution if  $\mu_X(x)$  is decreasing [increasing].
- iii)  $X$  is said to have an *increasing[decreasing] failure rate* (IFR) [DFR] distribution if  $r_X(x)$  is increasing [decreasing].

It is well-known (see Kochar and Wiens, 1987) that

$$\text{IFR} \Rightarrow \text{DMRL} \Rightarrow \text{NBUE}.$$

Exponential distribution is the most fundamental distribution in reliability theory. Note that the hazard rate of the exponential distribution with parameter  $\lambda$  is just  $\lambda$  and its mean residual life is  $1/\lambda$ . Thus, one of its properties, which makes it especially important, is that the remaining life of the used component is independent

of its initial age. That is, this distribution has the “memoryless” property, which ensures that it belongs to each above defined class, with equality in each defined inequality. Moreover, is the only life distribution which verifies this property.

The univariate ageing properties given in Definition 1.1.1 have been characterized by different approaches, including stochastic comparisons. Stochastic comparisons are approaches for making comparisons among various probability distributions. Shaked and Shanthikumar (2007) is an excellent source on this topic. Barlow and Proschan (1975) established several equivalent relationships between the classifications of lifetime distributions (particularly the IFR and NBUE distributions and their dual versions) and various partial orderings. Kochar and Wiens (1987) focused on the NBUE and DMRL distributions. Kochar (1989) examined the extension of the DMRL and related partial orderings of the life distribution. Fagiuoli and Pellerey (1993) defined new partial orders and discussed the relevance of these orderings in ageing properties classification. Belzunce *et al.* (1996) gave characterizations of the IFR(DFR) and DMRL(IMRL) classes, for a non-negative random variable, in terms of the dispersive orders and using the residual lifetime  $X_t = (X - t|X > t)$  for all  $t \in \{t : P(X > t) > 0\}$ . Later, Pellerey and Shaked (1997), aging by means of dispersive order and the residual lifetime, extended the results in Belzunce *et al.* (1996) but using a different method and when the random variable need not be non-negative. Di Crescenzo (1999) used dual orders to characterize some lifetime distributions.

Another approach for characterizing lifetime distributions is by using some particular functions. For example, Fernández-Ponce *et al.* (1998) gave the relationship between some classes of lifetime distribution and the univariate excess-wealth function. The definition and some properties of this function will be recalled in the next section.

Some more works on ageing property are emphasized here. Di Crescenzo and Pellerey (1998) studied lifetimes of components operating in random environment and Di Crescenzo (1999) provided some preservation results of ageing properties under a proportional reversed hazards model. Lillo (2000) gave an approach to identify properties associated with measures for ageing (the mean residual life and the failure rate functions). Lillo (2005) also studied the relationship between the mean and median residual life functions and characterized some criteria for ageing by means of the median residual life function. Klar and Müller (2003) gave a new class of life distributions and showed that this new class can be characterized by expected remaining lifetimes after a family of random times, thus generalizing the notion of NBUE. Wei and Hu (2007) characterized several ageing notions by using the spacing between record values.

These ageing notions have been generalized in the multivariate case by several authors. For this, various multivariate analogues of the hazard or failure rate have been considered. Basu (1971) and Puri and Rubin (1974) extended the hazard rate concept to the multivariate case by a single scalar quantity calculated as the quotient between the joint density function and the joint survival function. On the other hand, Johnson and Kotz (1975) affirmed that the basic idea underlying the univariate definition is that of the rate of decrease in survivors with an increase in the value of  $X$  and when there are two or more variates this rate depends on which variate is changed. Therefore, a different rate for each variate is necessary. Based on this idea, they defined the concept of the multivariate hazard gradient and extended the IFR property. In Marshall (1975), some properties of the multivariate hazard gradient are studied, and some examples are given to show the usefulness of the hazard gradient

in characterizing distributions. Roy (1994) and Roy (2001) introduced and studied multivariate versions of the IFR, DMRL and NBUE properties, together with their chain of implication and gave some equivalent definitions and characterization results. It also worth mentioning others papers where the multivariate lifetime distributions have been studied and characterized, for example, the paper by Block and Sampson (1988); Shaked and Shanthikumar (1987); Shaked and Shanthikumar, (1990); Scarsini and Shaked (1999); Hu *et al.* (2001a) and Hu and Wei (2001b), Bassan *et al.* (2002); Bassan and Spizzichino (2005a, 2005b), among others.

## 1.2 The univariate excess-wealth function

Let  $X$  be a random variable with a finite mean  $\mu_X$ , distribution function  $F_X$ , survival function  $\bar{F}_X = 1 - F_X$  and quantile function  $F_X^-$  defined as  $F_X^-(u) = \inf_x\{F(x) \geq u\}$  for all  $u \in (0, 1)$ . Fernández-Ponce *et al.* (1998) and, independently, Shaked and Shanthikumar (1998) defined the excess-wealth (ew) function (or right-spread function) and studied some of its important properties. The ew function of  $X$  is defined as

$$\begin{aligned}
 S_X^+(u) &= E[(X - F_X^-(u))^+] \\
 &= E\{\max[X - F_X^-(u), 0]\} \\
 &= \int_{F_X^-(u)}^{\infty} \bar{F}_X(t) dt \quad \text{for all } u \in [0, 1].
 \end{aligned}
 \tag{1.2.1}$$

The ew function has been named in different ways in the literature and has been used for different purposes in diverse fields such as actuarial science, insurance, economics and reliability. Fernández-Ponce *et al.* (1998) used the ew function to characterize different classes of ageing distribution which are really important in reliability theory

and survival analysis. In particular, these authors characterized the IFR, DMRL and NBUE. These characterizations are obtained comparing, in different stochastic sense, the underline distribution with the negative exponential distribution. Note that the ew function of the negative exponential distribution with parameter  $\lambda$  at a point  $u \in (0, 1)$  is  $\frac{1-u}{\lambda}$ . Again using the ew function, Fernández-Ponce *et al.* (1996) proposed a test statistic for testing exponentiality versus the DMRL alternative and another test for testing exponentiality versus the NBUE alternative.

From (1.2.1), the ew function can be interpreted as a measure of dispersion to the right of every quantile  $F_X^-(u)$ . So, this function is considered as a tool that defines a dispersive order and compares two distributions in terms of their variability. In the literature, several partial orders have been defined to compare the dispersion of two distributions. Examples of these orders are the convex ( $\leq_{cx}$ ), increasing [decreasing] convex ( $\leq_{icx}$ )[ $\leq_{dcx}$ ] and dispersive order ( $\leq_{dis}$ ). The definitions of these orders are recalled. See Shaked and Shanthikumar (2007) for a comprehensive discussion of these and other stochastic orders, as well as theirs properties and the relationships between them.

**Definition 1.2.1.** Let  $X$  and  $Y$  be two random variables with distribution functions  $F_X$  and  $F_Y$ , respectively. Then,  $X$  is said to be smaller than  $Y$  in the sense of

- 1) the convex order ( $X \leq_{cx} Y$ ) if  $E[h(X)] \leq E[h(Y)]$  for all convex function for which the expectations exist.
- 2) the increasing [decreasing] convex order ( $X \leq_{icx} [\leq_{dcx}] Y$ ) if  $E[h(X)] \leq E[h(Y)]$  for all increasing [decreasing] convex function for which the expectations exist.
- 3) the dispersive order ( $X \leq_{disp} Y$ ) if  $F_X^-(\beta) - F_X^-(\alpha) \leq F_Y^-(\beta) - F_Y^-(\alpha)$  for all  $0 < \alpha < \beta < 1$ .

The excess-wealth order (or equivalently, right-spread order) was defined independently by Fernández-Ponce *et al.* (1998) and Shaked and Shathikumar (1998). The definition is the following.

**Definition 1.2.2.** Let  $X$  and  $Y$  be two random variables. It is said that  $X$  is smaller than  $Y$  in the excess-wealth order (denoted by  $X \leq_{ew} Y$ ) if

$$S_X^+(u) \leq S_Y^+(u) \text{ for all } u \in (0, 1).$$

Fernández-Ponce *et al.* (1998) based the definition of this order on the observation (see Muñoz-Pérez, 1990) that

$$X \leq_{disp} Y \Leftrightarrow (X - F_X^-(u))^+ \leq_{st} (Y - F_Y^-(u))^+ \text{ for all } u \in (0, 1) \quad (1.2.2)$$

where  $\leq_{st}$  denote the usual stochastic order. Recall that, given two random variables  $X$  and  $Y$ , then  $X \leq_{st} Y$  if  $E[h(x)] \leq E[h(Y)]$  for all increasing function  $h$  for which the expectations exist. Obviously, the ew order is weaker than the dispersive order (see Fernández-Ponce *et al.* ,1998 and Shaked and Shathikumar, 1998).

In the past few years, many works have been devoted to studying the ew order and its relationship with several variability orders. Kochar and Carrière (1997) showed that two random variables are equivalent in terms of ew order if and only if either are identically distributed or they differ by a location parameter. Assuming that  $X$  and  $Y$  are two random variables with finite mean and 0 as the common left endpoint of their supports, they also gave an alternative short proof of the implication

$$X \leq_{ew} Y \Rightarrow X \leq_{icx} Y \quad (1.2.3)$$

which was proved by Shaked and Shantikumar (1998) in a quite involved and lengthy manner. Furthermore, these authors showed that if  $X_1$  and  $X_2$  ( $Y_1$  and  $Y_2$ ) be two

independent copies of  $X$  ( $Y$ ) then, the relationship  $X \leq_{ew} Y \Rightarrow X_1 - X_2 \leq_{cx} Y_1 - Y_2$  holds. Belzunce (1999) gave a characterization of ew order similar to (1.2.2) when the  $st$  order is replaced by the icx order. That is,

$$X \leq_{ew} Y \Leftrightarrow (X - F_X^-(u))^+ \leq_{icx} (Y - F_Y^-(u))^+ \text{ for all } u \in (0, 1), \quad (1.2.4)$$

and he gave some applications of this characterization in reliability theory. Kochar *et al.* (2002) showed that the ew order is closed under increasing convex transformation, which was a significant extension of the result (1.2.3). Some applications of the ew order in reliability are also considered in this paper. For instance, they show that if  $X \leq_{ew} Y$ , then a parallel system of  $n$  components having independent lifetimes which are copies of  $Y$  has a larger lifetime in the sense of the ew order than a similar system of  $n$  components having independent lifetimes which are copies of  $X$ . Furthermore, if  $Y$  is an exponential random variable and  $X$  has an NBUE distribution with the left endpoint of its support being 0, then upper bounds on the mean and on the variance of the lifetime of the parallel system having independent lifetime which are copies of  $X$  can be obtained. Later, Li (2006) showed that the total life of a parallel system with i.i.d exponential components is smaller in the ew order than an exponential life with the same mean as the system. They also gave upper bound for the mean and the variance of the life length of a parallel system with i.i.d. NBUE components.

Applications of the ew function and ew order can also be found in the literature about insurance and actuarial science. It is usually interesting to make risk comparisons and therefore measures which consider the right-tail risk have been defined. Denuit and Vermandele (1999) considered a slight modification of the ew function and the ew order and applied them in these fields. In the context of insurance, the ew transformation can be thought of as describing the situation of a reinsurer in a stop-loss



treaty where the deductible  $d^u$  is chosen in such a way that the probability for the claim amount being smaller than  $d^u$  is equal to  $u$ , that is  $F_X^-(d^u) = u$ . Considering two risks  $X$  and  $Y$ , then when  $X \leq_{ew} Y$ , the reinsurer will prefer to cover  $X$  than  $Y$  by a stop-loss treaty with deductibles  $d_1^u$  and  $d_2^u$  chosen in such way that  $F_X^-(d_1^u) = u$  and  $F_Y^-(d_2^u) = u$ , because the ratio of the expected reinsurance benefit to the expected total claims is always smaller with  $X$  than with  $Y$  (see Denuit and Vermandele, 1999 for details). In actuarial science, the ew function  $S_X^+(u)$  is usually called expected shortfall at level  $u$  and represents the expected shortfall of the portfolio with loss  $X$  and solvency capital requirement  $F_X^-(u)$ . Recall that the shortfall of the portfolio with loss  $X$  and solvency capital requirement  $F_X^-(u)$  is defined as  $\max[X - F_X^-(u), 0]$  and it can be interpreted as that part of the loss that cannot be covered by the company (see Denuit *et al.*, 2005 and Dhaene *et al.*, 2008). Recently, Sordo (2008) introduced a class of risk measures which include the expected shortfall as particular case and characterized the comparison of random variable according to the measures in this class in terms of the ew order. The relationship of the ew order to the usual stochastic order is derived in Sordo (2009a). Moreover, Sordo proved that if two random variables have the same finite support and are ordered in the sense of the ew order, then these variables have the same distribution. Sordo (2009b) gave another characterization of the ew order in terms of a class of measure which is defined considering convex real functions. It is worth noting that whereas Sordo (2008) characterized the excess-wealth order by means of the spread of a risk throughout its distribution, in Sordo (2009b) he focused on the tail risk.

### 1.3 On quantiles and their multivariate generalizations

It is well-known that, given a value  $u \in (0, 1)$ , the  $u$ th quantile of a univariate distribution  $X$  is a point that partitions the support of  $X$  into two sets such that the probability of the set to the left of the quantile is approximately  $u$  and the probability of the set to the right of the quantile is approximately  $(1 - u)$ . If  $X$  is a random variable with absolutely continuous distribution  $F_X$ , then the quantile function at a point  $u \in (0, 1)$ , denoted by  $F_X^-(u)$ , is defined as

$$F_X^-(u) = \inf\{x : F(x) \geq u\} \text{ for all } u \in [0, 1]. \quad (1.3.1)$$

Quantiles of univariate data are frequently used in the construction of descriptive statistics, for example, the median, the interquartile range and several measures of skewness and kurtosis based on percentiles. Perhaps, the most important particular case of the quantile is the median, given that the median is the central point which minimizes the average of the absolute deviations. That is, if  $X_1, \dots, X_n$  is a sample of observations of real-values, then the sample median is the argument which minimizes the sum  $\sum_{i=1}^m |X_i - \Theta|$ . So, it can be considered as a nonparametric and robust estimate for the centre of a distribution.

Many of the works on generalizing quantiles to multivariate distributions have concentrated on the particular case of the median. Here, some generalization of the concept of median into higher dimensional settings are reviewed. In 1909, Weber defined the  $L_1$  median by minimizing the multivariate version of the absolute residuals. It was proved that this median has uniqueness properties, however, it does not possess the

property of affine equivariance. An interesting alternative to the  $L_1$  median was provided by Oja (1983) who gave a class of measures which includes a generalization of the univariate median. This author defined the multivariate simplex median by minimizing the sum of volumes of simplices with vertices on the observations. In spite of Oja's median not having the uniqueness properties of the  $L_1$  median, it has the advantage of affine equivariance (see Oja, 1983 for further details). Later, and based on the definition of Oja's simplex median, Brown and Hettmansperger (1987, 1989) introduced another notion of bivariate quantile. Liu (1988,1990) defined the simplicial depth median maximizing an empirical simplicial depth function. Their definition was motivated by the idea that the univariate median can be characterized by its lying in the greatest number of intervals constructed from the data points. That is, it can be viewed as being deep inside the data cloud. An excellent review of these works is given by Small (1990).

Chaudhuri (1996) investigated a notion of quantiles based on the geometric configuration of the multivariate data clouds. These geometric quantiles are defined minimizing the sum of the extended loss function which was used by Koenker and Bassett (1978) to estimate the  $uth$  regression quantile in a linear regression setup. Chakraborty and Chaudhuri (1999) emphasized three fundamental properties of the univariate median and gave an excellent review of which of these properties are preserved by different versions of the multivariate median. The robustness of the transformation retransformation medians was also considered in detail by these authors.

However, the above definitions of multivariate quantile do not satisfy the kind of probability cumulation condition given in (1.3.1) because their definitions do not involve the cumulative probability distribution. This fact motivated Chen and Welsh

(2002) to define the bivariate quantiles as points which satisfy natural generalizations of the cumulative probability. These author gave a first definition fixing a natural direction in  $\mathbb{R}^2$  from south to north. Then they developed a second definition of bivariate quantile points which allows the distribution of the variable to specify the appropriate direction.

Another concept, closely connected with the univariate quantile function, is the  $\mathbf{u}$ -quantile. Let  $\mathbf{X}$  be a random vector in  $\mathbb{R}^n$  with absolutely continuous distribution function (cdf)  $F(\cdot)$ . The multivariate  $\mathbf{u}$ -quantile for  $\mathbf{X}$ , or the regression representation, was introduced by O'Brien (1975), Arjas and Lehtonen (1978) and Rüschendorf (1981). The definition is as follows. Let  $\mathbf{u}_n = (u_1, \dots, u_n)$  be a vector in  $[0, 1]^n$ , the multivariate  $\mathbf{u}$ -quantile for  $\mathbf{X}$ , denoted by  $\hat{\mathbf{x}}(\mathbf{u}_n)$ , is defined as

$$\begin{aligned} \hat{x}_1(\mathbf{u}_1) &= F_{X_1}^-(u_1), \\ \hat{x}_2(\mathbf{u}_2) &= F_{X_2|X_1=\hat{x}_1(u_1)}^-(u_2), \\ &\vdots \\ \hat{x}_n(\mathbf{u}_n) &= F_{X_n|\bigcap_{j=1}^{n-1} X_j=\hat{x}_j(u_j)}^-(u_n), \end{aligned} \tag{1.3.2}$$

where  $F^-(u) = \inf\{x : F(x) \geq u\}$  and  $\mathbf{u}_i = (u_1, \dots, u_i)$  for all  $i = 1, \dots, n$ .

This last construction, named standard construction, is widely used in simulation theory and can be interpreted as an extension of the quantile function on  $\mathbb{R}^n$ . Note that the standard construction depends on the choice of the ordering of the marginal distributions. This concept has been frequently used in studying the dispersion of a multivariate random variable. Rüschendorf (1981) used it for the stochastic comparison of risks with respect to supermodular ordering which is of particular interest in

many applications. Li *et al.* (1996) used this representation as a tool for the construction of multivariate distributions with given non-overlapping multivariate marginals. Shaked and Shanthikumar (1997) also proposed the standard construction as a useful tool for the stochastic comparison of random vectors. Müller and Scarsini (2001) used the multivariate  $\mathbf{u}$ -quantile for the stochastic comparison of random vectors with a common copula. Shaked and Shanthikumar (1998) defined a multivariate dispersive order based on a particular transformation by means of this construction. More recently, Fernández-Ponce and Suárez-Lloréns (2003) used the standard construction to define a new multivariate dispersive order. This new order is defined through the concept of conditional quantiles which are more widely separated. That is, let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vectors in  $^n$ ,  $\mathbf{X}$  is said to be less than  $\mathbf{Y}$  in a dispersion sense, denoted as  $\mathbf{X} \leq_{Disp} \mathbf{Y}$ , if

$$\| \hat{\mathbf{x}}(\mathbf{v}) - \hat{\mathbf{x}}(\mathbf{u}) \|_2 \leq \| \hat{\mathbf{y}}(\mathbf{v}) - \hat{\mathbf{y}}(\mathbf{u}) \|_2$$

for all  $\mathbf{u}$  and  $\mathbf{v}$  in  $(0, 1)^n$ . This concept depends on the permutation of the margins for multivariate distributions and it is a generalization of the well-known univariate d. o. ( $\leq_{disp}$ ). They provided a characterization through a multivariate contraction function under several regularity conditions. This order possesses an excellent property which is that the contraction function which is used for its characterization is a unique function.

As will be seen in following chapters, the notion of  $\mathbf{u}$ -quantile will be a fundamental tool throughout this work.

## 1.4 General Objectives

The general objective of this work is to define and study new multivariate ageing properties following the guidelines given by Fernández-Ponce and Suárez-Lloréns (2003). We draw on the concepts in Fernández-Ponce and Suárez-Lloréns (2003) to generalize the development in Fernández-Ponce *et al.* (1998) about univariate lifetime distribution.

Following the classical development in multivariate research in the reliability field, it will be necessary:

- to provide new results and properties of the corrected orthant given by Fernández-Ponce and Suárez-Lloréns (2003).
- to generalize the excess-wealth function to the multivariate case and study what univariate properties remain in higher dimension.
- to define, using the multivariate excess-wealth (*mew*) function, the corresponding ordering and study the relationship with other dispersive orders defined in the literature.
- to define new multivariate ageing properties and to characterize them by means of the *mew* function as that was developed by Fernández-Ponce *et al.* (1998) in the univariate case.

This work is organized as follows. In Chapter 2 some necessary notions and definitions are summarized which will be used later. The upper-corrected orthant of a random vector at a point is introduced and some particular examples are studied. Interesting properties of this orthant in terms of the  $\mathbf{u}$ -quantiles are obtained. The conditionally

increasing[decreasing] in sequence (CIS)[CDS] property, which plays an important role throughout this work, is characterized by means of the upper-corrected orthant and the monotonicity of the  $\mathbf{u}$ -quantile. The *mew* function together with some of its properties is also defined. At the end of the chapter a new multivariate ordering is considered. In Chapter 3, attention is focused on new multivariate ageing properties. Different multivariate ageing properties are defined and characterized by the *mew* function. Following a development similar to that in the univariate case, several ordering for multivariate lifetime distributions are studied and used to characterize the multivariate ageing through the vector with independent exponential components. Finally, in Chapter 4 an application of a particular ageing property is studied in oncology.

#### 1.4. GENERAL OBJECTIVES

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## CHAPTER 2

# A Generalization of the Excess-Wealth Concept

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### Abstract

In the univariate case, the excess-wealth function or right spread function has been used to characterize some aging notions and to define a dispersive ordering to compare two probability distributions in terms of their variability. In this chapter, the concept of excess-wealth function is extended to the multivariate case in terms of the upper-corrected orthant and the multivariate  $\mathbf{u}$ -quantile. Also studied is a multivariate order, based on these functions, and which is weaker than the multivariate dispersive order.

## 2.1 Introduction

Comparisons among univariate random variables in some stochastic sense have been extensively studied by many authors during the last thirty years. Many applications of these stochastic orderings exist, from economic theory to reliability and queuing theory (see Barlow and Proshan, 1978; Stoyan, 1983; Shaked and Shanthikumar, 2007). In particular, variability orders for univariate distributions have found a profound interest among researchers. Among these types of orders, the dispersion ordering (d.o.) has been well studied. Two interesting papers on the d.o. are those by Shaked (1982) and Deshpandé and Kochar (1983). This order was characterized through the number of crossings and the corresponding changes of sign for the distribution functions (see Hickey, 1986; Muñoz-Pérez, 1990 and Saunders, 1984). Dispersion ordering has also been studied for lifetime distributions with aging properties. Bartoszewicz (1995) characterized the d.o. using the Total Time on Test (TTT) transforms and found a relationship between the order based on the mean residual lives and the d.o. (see also Bartoszewicz, 1997). Kochar (1996) studied the d.o. among order statistics from DFR distributions. Also, as was pointed out in Section 1.1, different characterizations for IFR and DFR random variables can be seen in Pellerey and Shaked (1997). Kochar and Wiens (1987) defined other types of lifetime distributions weaker than IFR ones. Thus, a characterization in the dispersion sense for these distributions needed a new d.o. weaker than the classic d.o. For this reason, Fernández-Ponce *et al.* (1998) gave the concept of right-spread function which characterized the aging notions in Kochar and Wiens (1987). In a parallel direction and independently, Shaked and Shanthikumar (1998) defined the same function which they called the excess-wealth function. Both papers contain the analysis of a new weak d.o. termed excess-wealth order

by Shaked and Shanthikumar (1998) and by right-spread order by Fernández-Ponce *et al.* (1998) . Later, several authors characterized lifetime distributions using this partial order (see Section 1.2).

Several interesting results have also been shown for multivariate distributions. Shaked and Shanthikumar (1997) studied supermodular stochastic orders and positive dependence of random vectors. They applied the results to problems of optimal assembly of reliability systems, optimal allocation of minimal repair efforts, and optimal allocation of reliability systems. Bassan and Spizzichino (1999) compared distributions of residual lifetimes of dependent components of different ages. Their approach yielded several multivariate notions which were based on one-dimensional stochastic comparisons. Müller and Scarsini (2001) found conditions under the stochastic order for random vectors which implied that any positive linear combinations of the components of one of them is dominated in the convex order by the same positive linear combination of the components for the other random vector. This problem had a motivation in the comparison of portfolios in terms of risks. The conditions for the above dominance would concern the dependence structure of the two random vectors  $\mathbf{X}$  and  $\mathbf{Y}$ , i.e. the two random vectors would have a common copula and would be conditionally increasing. Rüschendorf (2004) extended some of the results on the comparison of multivariate risk vectors with respect to supermodular and related orderings. Colangelo *et al.* (2005) studied new notions of positive dependence with are associated to multivariate stochastic orders of positive dependence.

However, the multivariate d.o. has not been studied so extensively. It is worth mentioning the papers of Giovagnoli and Wynn (1995), Kosheroy and Mosler (1997) and more recently that of Fernández-Ponce and Suárez-Lloréns (2003) in this area.

This chapter is organized as follows. In Section 2 some notation and preliminaries are given. In Section 3, we further study some properties of the multivariate quantiles which were introduced in the Fernández-Ponce and Suárez-Lloréns (2003) paper. In Section 4, we extend the concept of the excess-wealth function to the multivariate case. In the last section the excess-wealth order for multivariate distributions is defined and is shown to be weaker than the multivariate d.o..

## 2.2 Notation and preliminaries

Some notations are given here which will be used throughout the chapter. Fundamentally, random vectors will be dealt with which take on values in  $\mathbb{R}^n$ . The space  $\mathbb{R}^n$  is endowed with the usual componentwise partial order, which is defined as follows. Let  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  be two vectors in  $\mathbb{R}^n$ ; and therefore  $\mathbf{x} \leq \mathbf{y}$  if  $x_i \leq y_i$  for  $i = 1, \dots, n$ . Throughout the chapter “increasing” means “non-decreasing” and “decreasing” means “non-increasing”. Particularly, a function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be an increasing function when  $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$  for  $\mathbf{x} \leq \mathbf{y}$ . The notation  $\sim_{st}$  stands for equality in law. The vector of ones will be denoted by  $\mathbf{1}$ , i.e.  $\mathbf{1} = (1, \dots, 1)$  and the corresponding zeros by  $\mathbf{0}$ . The dimension of  $\mathbf{1}$  will be clear from the expression in which it appears. The multiple integral  $\int_A F(t_1, \dots, t_n) dt_1 \dots dt_n$  will be denoted as  $\int_A F(\mathbf{t}) d\mathbf{t}$ . The dimension of a random vector is clear from the context and unless otherwise stated it is assumed that it is  $n$ .

Let  $\mathbf{X}$  be a random vector in  $\mathbb{R}^n$  with distribution function (cdf)  $F_{\mathbf{X}}(\cdot)$ . Fernández-Ponce and Suárez-Lloréns (2003) defined several concepts for a multivariate random vector which will be used later. The first concept is the multivariate  $\mathbf{x}$ -rate vector, denoted by  $\left[ \dot{\mathbf{x}}(\mathbf{x}) \right]$ , and defined as  $\dot{\mathbf{x}}(x_1) = P(X_1 \leq x_1), \dots, \dot{\mathbf{x}}(x_n) =$

$P(X_n \leq x_n \mid \bigcap_{j=1}^{n-1} X_j = x_j)$ . The second concept is the right-upper orthant at a point  $\mathbf{z}$ , denoted by  $C(\mathbf{z})$ , and it is defined as  $C(\mathbf{z}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{z} \leq \mathbf{x}\}$ . Finally, the upper-corrected orthant at point  $\mathbf{z}$  for the random variable  $\mathbf{X}$ , denoted as  $R_{\mathbf{X}}(\mathbf{z})$ , is defined as

$$R_{\mathbf{X}}(\mathbf{z}) = \{\mathbf{x} \in \mathbb{R}^n : x_1 \geq F_{X_1}^-[\hat{x}_1^*(z_1)], \dots, x_n \geq F_{X_n | \bigcap_{j=1}^{n-1} X_j = x_j}^-[\hat{x}_n^*(z_n)]\}.$$

In fact, only vectors of nonnegative variables with unlimited supports on the right were considered by Fernández-Ponce and Suárez-Lloréns (2003). As we will also consider vectors of lifetimes having limited supports, we will consider a different notion, that is a generalization of the upper-corrected orthant to our case. For its definition it should be recalled that the support of a random vector  $\mathbf{X}$  is defined as  $Supp(\mathbf{X}) = \{\mathbf{x} \in \mathbb{R}^n : P[\mathbf{X} \in B_{\mathbf{x}}(\varepsilon)] > 0 \text{ for all } \varepsilon > 0\}$  where  $B_{\mathbf{x}}(\varepsilon)$  is the centred ball at  $\mathbf{x}$  with radius  $\varepsilon$ .

**Definition 2.2.1.** Given a random vector  $\mathbf{X}$ , its upper-corrected orthant at  $\mathbf{z} \in Supp(\mathbf{X})$  is defined as

$$R_{\mathbf{X}}(\mathbf{z}) = \{\mathbf{x} \in Supp(\mathbf{X}) : x_1 \geq F_{X_1}^-[\hat{x}_1^*(z_1)], \dots, x_n \geq F_{X_n | \bigcap_{j=1}^{n-1} X_j = x_j}^-[\hat{x}_n^*(z_n)]\}.$$

It is easily shown that if  $\mathbf{X}$  is a random vector with independent components then the upper-corrected orthant at  $\mathbf{z}$  coincide with the intersection between  $Supp(\mathbf{X})$  and the corresponding right-upper orthant at  $\mathbf{z}$ .

**Example 2.2.1.** Let  $\mathbf{X} = (X_1, X_2)$  be a bivariate vector with  $Supp(\mathbf{X}) = A \cup B$  and a joint density function given by

$$f(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if } (x_1, x_2) \in A \\ 1 & \text{if } (x_1, x_2) \in B \\ 0 & \text{otherwise} \end{cases}$$

## 2.2. NOTATION AND PRELIMINARIES

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where  $A$  and  $B$  are:

$$\begin{aligned} A &= \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\} \\ B &= \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1 \leq 2, 0 \leq x_2 \leq 2 - x_1\} \end{aligned}$$

The marginal  $X_1$  has the following distribution function:

$$F_{X_1}(x_1) = \begin{cases} 0 & \text{if } x_1 < 0 \\ \frac{1}{2}x_1 & \text{if } x_1 \in [0, 1) \\ 1 + (2 - x_1)\left(\frac{x_1}{2} - 1\right) & \text{if } x_1 \in [1, 2) \\ 1 & \text{if } x_1 \geq 2 \end{cases}$$

and the conditioned variable  $X_2|X_1 = x_1$  has a uniform distribution  $U[0, 1]$  if  $x_1 \in [0, 1)$  and a uniform distribution  $U[0, 2 - x_1]$  if  $x_1 \in [1, 2]$ .

For this particular bivariate vector, the upper-corrected orthant for  $\mathbf{z} = (z_1, z_2)$  is the following set:

$$R_{\mathbf{X}}(\mathbf{z}) = \begin{cases} R_1(\mathbf{z}) & \text{if } \mathbf{z} \in A \\ R_2(\mathbf{z}) & \text{if } \mathbf{z} \in B \end{cases}$$

where

$$\begin{aligned} R_1(\mathbf{z}) &= \{\mathbf{x} \in \text{Supp}(\mathbf{X}) : \{x_1 \in [z_1, 1]; x_2 \in [z_2, 1]\} \cup \\ &\quad \{x_1 \in [1, 2]; x_2 \in [z_2(2 - x_1), 2 - x_1]\}\} \\ R_2(\mathbf{z}) &= \left\{ \mathbf{x} \in \text{Supp}(\mathbf{X}) : x_1 \in [z_1, 2]; x_2 \in \left[ \frac{z_2}{2 - z_1}(2 - x_1), 2 - x_1 \right] \right\} \end{aligned}$$

The upper-corrected orthant of  $\mathbf{X}$  in  $\mathbf{z} = (0.6, 0.6)$  is represented in Figure 2.1 (a). In order to point out the concept of upper-corrected orthant, it is also displayed in different points in Figure 2.1 (b). ▲

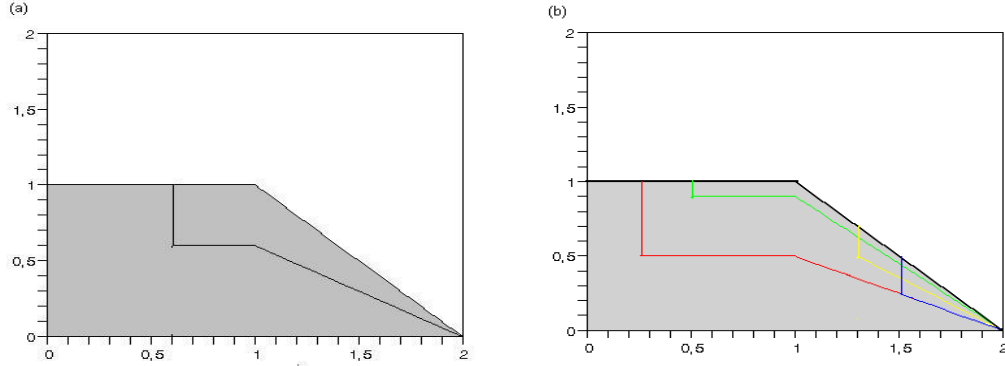


Figure 2.1: (a) The upper-corrected orthant of  $\mathbf{X}$  in  $(0.6, 0.6)$  (b) The upper-corrected orthant of  $\mathbf{X}$  in  $(0.25, 0.5)$  (red line),  $(0.5, 0.9)$  (green line),  $(1.3, 0.5)$  (yellow line) and  $(1.5, 0.25)$  (blue line).

**Remark 2.2.1.** Note that if  $\mathbf{t}_1 \leq \mathbf{t}_2$  then it could not be held that  $R_{\mathbf{X}}(\mathbf{t}_2) \subset R_{\mathbf{X}}(\mathbf{t}_1)$ . For example, let  $\mathbf{X}$  be a bivariate random vector with joint density function given by

$$f_{\mathbf{X}}(\mathbf{t}) = \begin{cases} 2/3 & \text{if } \mathbf{t} \text{ is in the triangle with vertices } (0, 0), (0, 1) \text{ and } (1, 1); \\ 4/3 & \text{if } \mathbf{t} \text{ is in the triangle with vertices } (0, 0), (1, 0) \text{ and } (1, 1); \\ 0 & \text{otherwise} \end{cases}$$

Let  $\mathbf{t}_1 = (\frac{1}{2}, \frac{1}{2})$ . By straightforward computations, it is verified that

$$R_{\mathbf{X}}(\mathbf{t}_1) = \left\{ x \geq \frac{1}{2}, y \geq \frac{x+1}{3} \right\}.$$

Now, consider  $\mathbf{t}_2 = (\frac{2}{3}, \frac{1}{2})$ . It holds that  $\mathbf{t}_1 \leq \mathbf{t}_2$ , and  $\mathbf{t}_2 \in R_{\mathbf{X}}(\mathbf{t}_2)$ . But,  $\mathbf{t}_2 \notin R_{\mathbf{X}}(\mathbf{t}_1)$ .

Thus  $R_{\mathbf{X}}(\mathbf{t}_2) \not\subset R_{\mathbf{X}}(\mathbf{t}_1)$ . ▲

From now on, assume that the following regularity conditions (RC) are verified by every cdf  $F$ .

1.  $F$  is a continuous function.
2. The  $\widehat{\mathbf{x}}(\mathbf{u})$  vector is differentiable at each component. That is, the derivatives

$$\frac{\partial \widehat{x}_i(\mathbf{u}_i)}{\partial u_i} \text{ exists for all } i = 1, \dots, n.$$

3. The conditional distribution of  $X_i$  to  $X_1, \dots, X_{i-1}$  ( $F_{i|1, \dots, i-1}$ ) is a continuous and strictly increasing function for  $i = 1, \dots, n$ . For convenience  $F_{1|0} = F_1$ .
4.  $F_{X_i|X_1, \dots, X_{i-1}}^-(0) < \infty$  for all  $i = 1, \dots, n$ .

It is easy to verify that under the regularity conditions above there exists a biunivocal relationship between vectors  $\mathbf{u} \in [0, 1]^n$  and the points  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ .

The proof of the next result directly follows from Fernández-Ponce and Suárez-Lloréns (2003). This result is the main reason of interest in the notion of upper-corrected orthant.

**Proposition 2.2.2.** *Let  $\mathbf{X}$  be a random vector. Then*

$$P \left\{ \mathbf{X} \in R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u}_n)] \right\} = \prod_{j=1}^n (1 - u_j).$$

This result means that the probability associated with the upper-corrected orthant at the  $\mathbf{u}$ -quantile does not depend on the distribution function.

## 2.3 The upper-corrected orthant: some new properties

As it can be seen in Proposition 2.2.2, the upper-corrected orthant plays a similar role in the multivariate setting as the upper quantile-interval for univariate distributions, being  $P[X \in R_X(\hat{x}(u))] = P[X \geq F_X^-(u)] = 1 - u$  in the univariate case. Thus it is very interesting to obtain several properties for the upper-corrected orthant.

**Proposition 2.3.1.** *Let  $\mathbf{u}_n$  and  $\mathbf{v}_n$  be two vectors in  $(0, 1)^n$  with  $\mathbf{u}_n \leq \mathbf{v}_n$ . Then*

$$R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{v}_n)] \subset R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u}_n)].$$



**Proof.** The proof is by mathematical induction. If  $u_1 \leq v_1$ , then  $\widehat{x}_1(u_1) \leq \widehat{x}_1(v_1)$ . Hence the proposition is obviously true for  $n = 1$ . Suppose now that the proposition is true for  $n = m$ . To complete the induction argument, the same conclusion with  $n = m + 1$  has to be proved. Let  $\mathbf{u}_{m+1}$  and  $\mathbf{v}_{m+1}$  be two vectors in  $(0, 1)^{m+1}$  with  $\mathbf{u}_{m+1} \leq \mathbf{v}_{m+1}$ . If  $\mathbf{t}_{m+1} \in R_{\mathbf{X}_{m+1}}[\widehat{\mathbf{x}}(\mathbf{v}_{m+1})]$  then

$$\mathbf{t}_m \in R_{\mathbf{X}_m}[\widehat{\mathbf{x}}(\mathbf{v}_m)] \text{ and } t_{m+1} \geq \widehat{x}_{m+1|\mathbf{t}_m}(v_{m+1}).$$

Consequently,  $\mathbf{t}_m \in R_{\mathbf{X}_m}[\widehat{\mathbf{x}}(\mathbf{u}_m)]$  and  $t_{m+1} \geq \widehat{x}_{m+1|\mathbf{t}_m}(u_{m+1})$ . Thus the result is obtained.  $\square$

This proposition means that the upper-corrected orthant is a decreasing set in terms of the  $\mathbf{u}$ -quantile. Note that this result generalizes an equivalent property for the upper intervals at the corresponding  $u$ -quantiles in the univariate case.

Now, the relationship between the support of  $\mathbf{X}$  and the concept of the upper-corrected orthant are given.

**Proposition 2.3.2.** *Let  $\mathbf{X}$  be a random vector verifying the (RC). Then,*

$$R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})] = \text{Supp}(\mathbf{X}).$$

**Proof.** This proposition is only proved for bivariate random vectors. By using the induction argument, it can trivially be shown for multivariate random vectors. Obviously, the inclusion  $R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})] \subset \text{Supp}(\mathbf{X})$  is verified.

Conversely, assume that  $\mathbf{t}_2 \in \text{Supp}(\mathbf{X})$ , i.e.  $P[\mathbf{X} \in B_{\mathbf{t}_2}(\varepsilon)] > 0$  for all  $\varepsilon > 0$ . Let  $C_{\mathbf{t}_2}$  be a square such that  $B_{\mathbf{t}_2}(\varepsilon) \subset C_{\mathbf{t}_2}$  and  $\mu(C_{\mathbf{t}_2}) = 4\varepsilon^2$ , with  $\mu(\cdot)$  the Lebesgue measure in  $\mathbb{R}^2$ . Hence it is obtained that

$$P\{X_1 \in (t_1 - \varepsilon, t_1 + \varepsilon]\} > P(\mathbf{X} \in C_{\mathbf{t}_2}) > 0 \text{ and } t_1 > F_1^-(0).$$

### 2.3. THE UPPER-CORRECTED ORTHANT: SOME NEW PROPERTIES

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However if it were held that  $t_2 < F_{2|t}^-(0)$  for all  $t \in (t_1 - \varepsilon', t_1 + \varepsilon']$  and for all  $0 < \varepsilon' < \delta = t_1 - F_1^-(0)$  then  $P[\mathbf{X} \in C_{\mathbf{t}}(\delta)] = 0$  would be verified, which is an impossible equality. Consequently, by reduction to the absurd, it can be obtained a value  $t(\varepsilon')$  in  $(t_1 - \varepsilon', t_1 + \varepsilon']$  for each  $\varepsilon' < \delta$  such that  $t_2 \geq F_{2|t(\varepsilon')}^-(0)$ . Thus by taking  $\varepsilon' \downarrow 0$  and by using (3) in RC, it is obtained that  $t_2 \geq F_{2|t_1}^-(0)$ . Hence  $\mathbf{t}_2 \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(0)]$ , i.e.  $Supp(\mathbf{X}) \subset R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})]$ . And the result is proved with this last inclusion.  $\square$

**Proposition 2.3.3.** *Let  $\mathbf{X}$  be a random vector and  $\mathbf{x}$  be a point in  $Supp(\mathbf{X})$ . If  $\mathbf{t}$  is a point in  $R_{\mathbf{X}}(\mathbf{x})$ , then  $R_{\mathbf{X}}(\mathbf{t}) \subset R_{\mathbf{X}}(\mathbf{x})$ .*

**Proof.** It is easily shown that if  $\mathbf{x}$  and  $\mathbf{t}$  are points in  $Supp(\mathbf{X})$  then there exist only two vectors  $\mathbf{u}_n, \mathbf{v}_n \in (0, 1)^n$  such that  $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{u}_n)$  and  $\mathbf{t} = \widehat{\mathbf{x}}(\mathbf{v}_n)$ .

If  $\mathbf{t}$  is a point in  $R_{\mathbf{X}}(\mathbf{x})$ ,

$$t_1 \geq \widehat{x}_1(u_1), \quad t_2 \geq \widehat{x}_{2|t_1}(u_2), \dots, t_n \geq \widehat{x}_{n|t_{n-1}}(u_n).$$

Then  $\mathbf{t}_i = \widehat{\mathbf{x}}_{i|\mathbf{t}_{i-1}}(v_i) \geq \widehat{\mathbf{x}}_{i|\mathbf{t}_{i-1}}(u_i)$  for  $i = 2, \dots, n$ . Therefore, from (3) in RC, it is obtained that  $\mathbf{u}_n \leq \mathbf{v}_n$ . And the result follows by using Proposition 2.3.1.  $\square$

It is well-known that the univariate quantile function is an increasing function, i.e.  $u \leq v$  if, and only if  $F^-(u) \leq F^-(v)$ . This property is not verified by the  $\mathbf{u}$ -quantile for random vectors, in general. However if a type of dependence is held then this property can be verified. This type of dependence must be based on the growth of the corresponding conditional distributions. We refer to the CIS property given in the following definition.

**Definition 2.3.1.** A random vector  $\mathbf{X} = (X_1, \dots, X_n)$  is said to be conditionally increasing (decreasing) in sequence, CIS(CDS), if  $X_i$  is stochastically increasing (decreasing) in  $X_1, \dots, X_{i-1}$  for  $i = 2, \dots, n$ . That is,

$$(X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \leq_{st} [\geq_{st}] (X_i | X_1 = x'_1, \dots, X_{i-1} = x'_{i-1})$$

whenever  $x_j \leq x'_j, j = 1, \dots, i-1$ .

Before giving the condition under which the monotony of the  $\mathbf{u}$ -quantile holds, it is necessary to give some previous results where the CIS(CDS) property is characterized in terms of the upper-corrected orthant.

**Theorem 2.3.4.**  $\mathbf{X}$  is a CIS random vector if and only if  $R_{\mathbf{X}}(\mathbf{x}) \subset C(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Proof.** First, we prove the sufficient condition. The proof will be by mathematical induction. The proposition is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ . We now need to show that it is true for  $n = m + 1$ . if  $\widehat{\mathbf{x}}(\mathbf{u}_m) \preceq \mathbf{t}_m$  then, given that  $\mathbf{X}$  is CIS,

$$\widehat{x}_{m+1|\widehat{\mathbf{x}}_m(\mathbf{u}_m)}(u_{m+1}) \leq \widehat{x}_{m+1|\mathbf{t}_m}(u_{m+1})$$

However, we know that if  $\mathbf{t}_{m+1} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}_{m+1})]$  then  $t_{m+1} \geq \widehat{x}_{m+1|\mathbf{t}_m}(u_{m+1})$ . Therefore  $\mathbf{t}_{m+1} \in C[\widehat{\mathbf{x}}(\mathbf{u}_{m+1})]$ . Hence we have completed the proof by the induction argument.

The necessary condition is also proved by mathematical induction. The proposition is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ . It is sufficient to prove that

$$F_{X_{m+1}|\bigcap_{j=1}^m X_j=x_j}(x_{m+1}) \geq F_{X_{m+1}|\bigcap_{j=1}^m X_j=t_j}(x_{m+1}) \text{ for all } x_j \leq t_j, j = 1, \dots, m \tag{2.3.1}$$

### 2.3. THE UPPER-CORRECTED ORTHANT: SOME NEW PROPERTIES

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If  $\mathbf{t}_{m+1} \in R_{\mathbf{X}}(\mathbf{x})$ , then  $t_s \geq F_{X_s | \bigcap_{j=1}^{s-1} X_j = t_j}^- [F_{X_s | \bigcap_{j=1}^{s-1} X_j = x_j}(x_s)]$  for  $s = 1, \dots, m+1$ . In particular, if  $t_{m+1} = F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}^- [F_{X_{m+1} | \bigcap_{j=1}^m X_j = x_j}(x_m)]$ , given that  $R_{\mathbf{X}}(\mathbf{x}) \subset C(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$ , it holds that

$$F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}^{-1} [F_{X_{m+1} | \bigcap_{j=1}^m X_j = x_j}(x_m)] \geq x_{m+1}. \quad (2.3.2)$$

Now, from  $F_{X_{m+1} | \bigcap_{j=1}^m X_j = x_j}(x)$  is an increasing function in  $x$ , by solving the inequality (2.3.2), the inequality (2.3.1) is obtained.  $\square$

**Theorem 2.3.5.**  $\mathbf{X}$  is a CDS random vector if and only if  $C(\mathbf{x}) \cap \text{Supp}(\mathbf{X}) \subset R_{\mathbf{X}}(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$

**Proof.** First, we prove the sufficient conditions. The proof will be obtained by mathematical induction. The proposition is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ . Now we need to show that it is true for  $n = m+1$ . If  $\mathbf{t}_{m+1} \in C(\mathbf{x}_{m+1}) \cap \text{Supp}(\mathbf{X})$ , then  $\mathbf{t}_{m+1} \geq \mathbf{x}_{m+1}$ . Since  $\mathbf{X}$  is CDS, it follows that

$$F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(x_{m+1}) \geq F_{X_{m+1} | \bigcap_{j=1}^m X_j = x_j}(x_{m+1}) \text{ for all } x_j \leq t_j, j = 1, \dots, m \quad (2.3.3)$$

On the other hand, given that  $F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(x)$  is increasing in  $x$ ,

$$F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(t_{m+1}) \geq F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(x_{m+1}). \quad (2.3.4)$$

From (2.3.3) and (2.3.4) it holds that

$$F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(t_{m+1}) \geq F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}(x_{m+1}).$$

And given that  $F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}^-$  is an increasing function, then

$$t_{m+1} \geq F_{X_{m+1} | \bigcap_{j=1}^m X_j = t_j}^- [F_{X_{m+1} | \bigcap_{j=1}^m X_j = x_j}(x_{m+1})].$$

Therefore,  $\mathbf{t}_{m+1} \in R_{\mathbf{X}}(\mathbf{x})$ .

The necessary condition is also proved by mathematical induction. The proposition is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ , then it is sufficient to prove that

$$F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(x_{m+1}) \geq F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1}) \text{ for all } x_j \leq t_j, j = 1, \dots, m \quad (2.3.5)$$

If  $\mathbf{t}_{m+1} \in C(\mathbf{x}) \cap \text{Supp}(\mathbf{X}) \subset R_{\mathbf{X}}(\mathbf{x})$ , then for  $s = 1, \dots, m + 1$ ,  $t_s \geq x_s$  and  $t_s \geq F_{X_s|\cap_{j=1}^{s-1} X_j=t_j}^- [F_{X_s|\cap_{j=1}^{s-1} X_j=x_j}(x_s)]$ . In particular, if  $t_{m+1} = x_{m+1}$ , it holds that

$$x_{m+1} \geq F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}^- [F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})],$$

and, given that  $F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(x)$  is an increasing function in  $x$ , the inequality (2.3.5) is obtained. Thus,  $\mathbf{X}$  is CDS. □

Now, we study the monotonicity of the  $\widehat{\mathbf{x}}(\mathbf{u})$  when the vector  $\mathbf{X}$  has the CIS or CDS property.

**Theorem 2.3.6.**  $\mathbf{X}$  is a CIS random vector if and only if  $\widehat{\mathbf{x}}(\mathbf{u}) \preceq \widehat{\mathbf{x}}(\mathbf{v})$  whenever  $\mathbf{u} \preceq \mathbf{v}$ .

**Proof.** For the sufficient condition, see Rubinstein *et al.* (1985).

The necessary condition will be by mathematical induction. From Theorem 2.3.4, it is sufficient to prove that  $R_{\mathbf{X}}(\mathbf{x}) \subset C(\mathbf{x})$ . The statement is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ . Let  $\mathbf{t}_{m+1} \in R_{\mathbf{X}}(\mathbf{x})$ , that is,

$$t_s \geq F_{X_s|\cap_{j=1}^{s-1} X_j=t_j}^- [F_{X_s|\cap_{j=1}^{s-1} X_j=x_j}(x_s)] \text{ for } s = 1, \dots, m + 1.$$

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In particular, for  $s = m + 1$ , it holds

$$t_{m+1} \geq F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}^- [F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})]$$

and, given that  $F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(x)$  is increasing in  $x$ , it follows

$$F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(t_{m+1}) \geq F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1}).$$

Now, let  $t_j = \widehat{\mathbf{x}}(\mathbf{v}_j)$ ,  $x_j = \widehat{\mathbf{x}}(\mathbf{u}_j)$  for  $j = 1, \dots, m$  and  $v_{m+1} = F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(t_{m+1})$  and  $u_{m+1} = F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})$ . Then by hypothesis,  $\widehat{\mathbf{x}}(\mathbf{v}_{m+1}) \geq \widehat{\mathbf{x}}(\mathbf{u}_{m+1})$ , that is

$$F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}^{-1} [F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}(t_{m+1})] \geq F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{u}_j)}^{-1} [F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{u}_j)}(x_{m+1})].$$

and therefore,  $t_{m+1} \geq x_{m+1}$  and the result is obtained.  $\square$

**Theorem 2.3.7.**  $\mathbf{X}$  is a CDS random vector if and only if  $\mathbf{u} \leq \mathbf{v}$  for all  $\widehat{\mathbf{x}}(\mathbf{u}) \preceq \widehat{\mathbf{x}}(\mathbf{v})$ .

**Proof.** Suppose that  $\mathbf{X}$  is a CDS random vector. The proof will be by mathematical induction. The proposition is obviously true for  $n = 1$ . Assume that the proposition is true for  $n = m$ . If  $\mathbf{X}$  is a CDS random vector, then

$$F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{u}_j)}^- (u_{m+1}) \geq F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}^- (u_{m+1}) \quad (2.3.6)$$

for all  $\mathbf{u}_j \leq \mathbf{v}_j, j = 1, \dots, m$ . If  $\widehat{\mathbf{x}}(\mathbf{u}_{m+1}) \preceq \widehat{\mathbf{x}}(\mathbf{v}_{m+1})$ , then by definition

$$F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{u}_j)}^- (u_{m+1}) \leq F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}^- (v_{m+1}). \quad (2.3.7)$$

Finally, from (2.3.6) and (2.3.7) and given that  $F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}^-(u)$  is increasing in  $u$ , it follows that  $v_{m+1} \geq u_{m+1}$  and the result is obtained.

Conversely, suppose that  $\widehat{\mathbf{x}}(\mathbf{u}_{m+1}) \preceq \widehat{\mathbf{x}}(\mathbf{v}_{m+1})$  implies that  $\mathbf{u}_{m+1} \leq \mathbf{v}_{m+1}$ . In the light of Theorem 2.3.5, it is sufficient to prove that  $C(\mathbf{x}) \subset R_{\mathbf{X}}(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^m$ .

Let  $\mathbf{t}_{m+1} \in C(\mathbf{x})$ , that is,

$$t_s \geq x_s \text{ for } s = 1, \dots, m + 1. \quad (2.3.8)$$

In particular, for  $s = m + 1$  and from the regularity conditions, the inequality (2.3.8) is equivalent to

$$F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}^- [F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(t_{m+1})] \geq F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}^- [F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})]. \quad (2.3.9)$$

Now, if  $t_j = \widehat{\mathbf{x}}(\mathbf{v}_j)$ ,  $x_j = \widehat{\mathbf{x}}(\mathbf{u}_j)$  for  $j = 1, \dots, m$  and  $v_{m+1} = F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(t_{m+1})$ ,  $u_{m+1} = F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})$ , the inequality (2.3.9) can be written

$$F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{v}_j)}^-(v_{m+1}) \geq F_{X_{m+1}|\cap_{j=1}^m X_j=\widehat{\mathbf{x}}(\mathbf{u}_j)}^-(u_{m+1}). \quad (2.3.10)$$

Moreover, by hypothesis the above inequality implies that  $v_{m+1} \geq u_{m+1}$ , and given that  $F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}(x)$  is increasing in  $x$ , it follows that

$$t_{m+1} \geq F_{X_{m+1}|\cap_{j=1}^m X_j=t_j}^- [F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}(x_{m+1})].$$

Therefore,  $\mathbf{t}_{m+1} \in R_{\mathbf{X}}(\mathbf{x})$  and the result is obtained. □

In Remark 2.2.1, it was proved that if  $\mathbf{x} \leq \mathbf{x}'$ , then it could not be held that  $R_{\mathbf{X}}(\mathbf{x}') \subseteq R_{\mathbf{X}}(\mathbf{x})$ . However, if  $\mathbf{X}$  has the CDS property, it can be proved that this last inclusion holds. Recall that, by definition,  $\mathbf{x} \leq \mathbf{x}'$  if and only if  $C(\mathbf{x}') \subseteq C(\mathbf{x})$ . So, the following

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results give the relationship between the right-upper orthant and the upper-corrected orthant.

**Proposition 2.3.8.** *Let  $\mathbf{X}$  be a CDS random vector. If  $C(\mathbf{x}') \subseteq C(\mathbf{x})$ , then  $R_{\mathbf{X}}(\mathbf{x}') \subseteq R_{\mathbf{X}}(\mathbf{x})$ .*

**Proof.** From the Proposition 2.3.3, it is sufficient to prove that  $\mathbf{x}' \in R_{\mathbf{X}}(\mathbf{x})$ .

For  $n = 2$ , by definition,

$$R_{\mathbf{X}}(\mathbf{x}) = \{\mathbf{z} \in \text{Supp}(\mathbf{X}) : z_1 \geq F_{X_1}^-[\check{x}_1(x_1)], z_2 \geq F_{X_2|X_1=z_1}^-[\check{x}_2(x_2)]\}.$$

Now, if  $\mathbf{x} = (x_1, x_2) \leq \mathbf{x}' = (x'_1, x'_2)$ , it holds that

$$\begin{aligned} x'_1 &\geq F_{X_1}^-[\check{x}_1(x_1)] \\ x'_2 &\geq F_{X_2|X_1=x_1}^-[\check{x}_2(x_2)] \geq F_{X_2|X_1=x'_1}^-[\check{x}_2(x_2)] \end{aligned}$$

where the last inequality follows due to of fact that  $\mathbf{X}$  is CDS. Therefore,  $\mathbf{x}' \in R_{\mathbf{X}}(\mathbf{x})$ .

Now, assume that the proposition is true for  $n = m$ , we need to show that for  $m + 1$ . Let  $\mathbf{x}' = (x'_1, \dots, x'_m, x'_{m+1}) \in R_{\mathbf{X}}(\mathbf{x})$ . If  $\mathbf{x} = (x_1, \dots, x_m, x_{m+1}) \leq \mathbf{x}' = (x'_1, \dots, x'_m, x'_{m+1})$ , it holds that

$$x'_{m+1} \geq F_{X_{m+1}|\cap_{j=1}^m X_j=x_j}^-[\check{x}_{m+1}(x_{m+1})] \geq F_{X_m|\cap_{j=1}^m X_j=x'_j}^-[\check{x}_{m+1}(x_{m+1})]$$

so, by mathematical induction, the assumption is obtained.  $\square$

**Proposition 2.3.9.** *If  $\mathbf{X}$  is CIS and  $R_{\mathbf{X}}(\mathbf{x}') \subseteq R_{\mathbf{X}}(\mathbf{x})$ , then  $C(\mathbf{x}') \subseteq C(\mathbf{x})$ .*

**Proof.** For  $n = 2$ . If  $R(\mathbf{x}') \subset R(\mathbf{x})$ , it holds that  $x' \in R(\mathbf{x})$ . Then,  $x'_1 \geq x_1$  and



$$x'_2 \geq F_{X_2|X_1=x'_1}^- [F_{X_2|X_1=x_1}(x_2)] \quad (2.3.11)$$

Since  $\mathbf{X}$  is CIS,  $X_2|X_1 = x_1 \leq_{st} X_2|X_1 \leq x'_1$ . Therefore,

$$F_{X_2|X_1=x_1}(x_2) \geq F_{X_2|X_1=x'_1}(x_2) \quad (2.3.12)$$

Thus, from (2.3.11) and (2.3.12), it follows that

$$F_{X_2|X_1=x'_1}(x'_2) \geq F_{X_2|X_1=x'_1}(x_2).$$

Finally, given that  $F_{X_2|X_1=x'_1}(\cdot)$  is an increasing function in  $(\cdot)$ ,  $x'_2 \geq x_2$ , and the result is obtained.  $\square$

For concluding this section, we show the relationship between the upper- corrected orthant of two random vectors with the same dependence structure in the sense they have the same copula. A copula  $C$  is a cumulative distribution function with uniform marginals on  $[0,1]$ , Furthermore, it is shown that if  $H$  is an  $n$ -dimensional distribution function with marginals  $F_1, \dots, F_n$ , then there exists an  $n$ -copula  $C$  such that for all  $\mathbf{x} \in \mathbb{R}^n$ , it holds that

$$H(x_1, \dots, x_n) = C[F_1(x_1), \dots, F_n(x_n)].$$

Moreover, if  $F_1, \dots, F_n$  are continuous, then  $C$  is unique. For more details about copulas, see Nelsen (1999).

**Proposition 2.3.10.** *Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two bivariate random vectors with the same copula  $C$ . Then there exist two real strictly increasing functions,  $h_1$  and  $h_2$ , such that*

$$\mathbf{h}[R_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{u}))] = R_{\mathbf{Y}}[\widehat{\mathbf{y}}(\mathbf{u})] \text{ for all } \mathbf{u} \in [0, 1]^2 \text{ and } \mathbf{h} = (h_1, h_2).$$

**Proof.** It is well-known that if  $\mathbf{X}$  and  $\mathbf{Y}$  have the same copula  $C$  then there exist two strictly increasing functions,  $h_1$  and  $h_2$ , such that  $\mathbf{Y} \sim_{st} (h_1(X_1), h_2(X_2))$ . Thus it is easily shown that

$$F_{Y_1}^-(u_1) = h_1(F_{X_1}^-(u_1))$$

$$F_{Y_2|Y_1=F_{Y_1}^-(u_1)}^-(u_2) = h_2[F_{X_2|X_1=F_{X_1}^-(u_1)}^-(u_2)]$$

Therefore the result is directly obtained.  $\square$

## 2.4 The multivariate excess-wealth function

In this section, the univariate excess-wealth function given in Section 1.2 is generalized to the multivariate case. It is also studied what properties of the ew function are preserved in higher dimensions.

We start introducing the concept of the multiple expectation associated with a random vector  $\mathbf{X}$ .

**Definition 2.4.1.** Let  $\mathbf{X}$  be a nonnegative random vector, then the multiple expectation associated with  $\mathbf{X}$ , when it exists, is defined as the real value

$$\bar{\mu}_{\mathbf{X}} = \int_{Supp(\mathbf{X})} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] dt.$$

Note the multivariate expectation associated with  $\mathbf{X}$  depends on the ordering of the marginal distributions because of the definition of the upper-corrected orthant.

More especially, it is interesting to obtain an expression for the bivariate expectation function easier to hand. Let  $\mathbf{X} = (X_1, X_2)$  be a bidimensional random vector with density function  $f_{\mathbf{X}}(x_1, x_2)$ . Obviously, it holds that

$$\hat{\mathbf{x}}(u_1, u_2) = [F_{X_1}^-(u_1), F_{2|F_{X_1}^-(u_1)}^-(u_2)]$$

and

$$R_{\mathbf{X}}(\hat{\mathbf{x}}(u_1, u_2)) = \{\mathbf{x} \in \text{Supp}(\mathbf{X}) : x_1 \geq F_{X_1}^-(u_1), x_2 \geq F_{X_2|X_1=x_1}^-(u_2)\}.$$

Furthermore, let  $\mathbf{t} = (t_1, t_2)$  be a point in  $\text{Supp}(\mathbf{X})$  then it is easily shown that

$$\begin{aligned} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] &= \int_{t_1}^{+\infty} \int_{F_{X_2|X_1=w_1}^-(F_{X_2|X_1=t_1}(t_2))}^{+\infty} f(w_1, w_2) dw_2 dw_1 \\ &= \int_{t_1}^{+\infty} f_{X_1}(w_1) \left[ \int_{F_{X_2|X_1=w_1}^-(F_{X_2|X_1=t_1}(t_2))}^{+\infty} f_{X_2|X_1=w_1}(w_2) dw_2 \right] dw_1. \\ &= \int_{t_1}^{+\infty} f_{X_1}(w_1) \left[ \bar{F}_{X_2|X_1=w_1}(F_{X_2|X_1=t_1}^-(F_{X_2|X_1=t_1}(t_2))) dw_2 \right] dw_1 \\ &= \int_{t_1}^{+\infty} f_{X_1}(w_1) \bar{F}_{X_2|X_1=t_1}(t_2) dw_1 \\ &= P(X_1 > t_1)P(X_2 > t_2|X_1 = t_1). \end{aligned} \tag{2.4.1}$$

Observing the equality (2.4.1), the upper-corrected orthant in a point  $\mathbf{t}$  could be defined as the set  $R_{\mathbf{X}}(\mathbf{t})$  such that  $P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] = P(X_1 > t_1)P(X_2 > t_2|X_1 = t_1)$ . Consequently, under straightforward calculations is easily obtained by using (2.4.1) that

$$\bar{\mu}_{\mathbf{X}} = \nu_{\mathbf{X}} - \int_0^{+\infty} F_1(t)F_{X_2|X_1=t}^-(0)dt,$$

where  $\nu_{\mathbf{X}} = \int_0^{+\infty} F_1(t)E[X_2|X_1 = t]dt$ . Particularly, if  $\mathbf{X}$  represents a non-negative lifetime random variable then

$$\bar{\mu}_{\mathbf{X}} \leq \nu_{\mathbf{X}},$$

and the equality is held if  $\text{Supp}(\mathbf{X}) = [0, +\infty)^2$ .

Now, we give the definition of the multivariate excess-wealth function.

**Definition 2.4.2.** Let  $\mathbf{X}$  be a nonnegative random vector with finite multiple expectation. The multivariate excess-wealth function associated to  $\mathbf{X}$  is defined as

$$S_{\mathbf{X}}^+(\mathbf{u}) = \int_{R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] dt.$$

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By using (2.4.1), for all  $(u_1, u_2) \in (0, 1)^2$ , the bivariate excess-wealth function can be expressed as

$$S_{\mathbf{X}}^+(u_1, u_2) = \int_{F_1^-(u_1)}^{\infty} \bar{F}_1(t_1) \cdot S_{X_2|X_1=t_1}^+(u_2) dt_1. \quad (2.4.2)$$

Several interesting properties for the multivariate excess-wealth function can be shown as in the univariate case.

**Proposition 2.4.1.** *Let  $\mathbf{X}$  be a nonnegative random vector with finite multiple expectation, then*

i)  $S_{\mathbf{X}}^+(\mathbf{u})$  is a decreasing and a shift invariant function.

ii)  $0 \leq S_{\mathbf{X}}^+(\mathbf{u}) \leq S_{\mathbf{X}}^+(\mathbf{0}) = \bar{\mu}_{\mathbf{X}}$  for all  $\mathbf{u}$  in  $[0, 1]^n$ .

iii) If the components of  $\mathbf{X}$  are independent, then  $S_{\mathbf{X}}^+(\mathbf{u}) = \prod_{i=1}^n S_{X_i}^+(u_i)$ .

**Proof.**

i) Firstly, by using the proposition 2.3.1, it follows that if  $\mathbf{u}_n \preceq \mathbf{v}_n$ , then

$$R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{v}_n)] \subset R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}_n)].$$

From this, it immediately follows that the multivariate excess-wealth function is decreasing.

Finally, it is easy to show that

$$R_{\mathbf{X}+\mathbf{c}}[\widehat{\mathbf{x}} + \mathbf{c}(\mathbf{u}_n)] = R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}_n)] + \mathbf{c} \text{ and } R_{\mathbf{X}+\mathbf{c}}[(\mathbf{t}_n)] = R_{\mathbf{X}}[(\mathbf{t}_n - \mathbf{c})] + \mathbf{c}$$

Consequently, the multivariate excess-wealth function is a shift invariant function.

ii) -iii) The proofs are trivial and therefore are omitted.

**Example 2.4.2.** *The bivariate vector as defined in Example 2.2.1 is considered. By straightforward calculus it is possible to show that the bivariate excess-wealth function of this vector has the following expression:*

$$S_{\mathbf{X}}^+(u_1, u_2) = \begin{cases} f_1(u_1, u_2) & \text{if } 0 \leq u_1 \leq 1/2 \\ f_2(u_1, u_2) & \text{if } 1/2 \leq u_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where  $0 \leq u_2 \leq 1$  and

$$\begin{aligned} f_1(u_1, u_2) &= \frac{1}{2} \left( -\frac{1}{4} + \frac{1}{2}u_2 - \frac{1}{4}u_2^2 \right) (1 - 4u_1^2) + \frac{9}{16} - u_1 - u_2(1 - 2u_1) \\ &\quad + \frac{1}{2}u_2^2(1 - 2u_1) - \frac{1}{8}u_2 + \frac{1}{16}u_2^2. \end{aligned}$$

$$\begin{aligned} f_2(u_1, u_2) &= -0.5, u_2 + u_2 u_1 - 0.5 u_2, u_1^2 + 0.25 - 0.5 u_1 + 0.25 u_1^2 + 0.25 u_2^2 \\ &\quad - 0.5 u_2^2 u_1 + 0.25 u_2^2 u_1^2. \end{aligned}$$

In Figure 2.2 we can see the graphic of the excess-wealth function for the vector  $\mathbf{X}$ .

▲

Now, we give the relationship between the multivariate excess-wealth function and the probability associated to the upper-corrected orthant at the  $\mathbf{u}$ -quantile.

**Proposition 2.4.3.** *Let  $\mathbf{X} = (X_1, X_2)$  be a bidimensional random vector with density function  $f_{\mathbf{X}}(x_1, x_2)$  and excess-wealth function  $S_{\mathbf{X}}^+(u_1, u_2)$ . Then*

$$P\{\mathbf{X} \in R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]\} = f_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})] \frac{\partial^2 S_{\mathbf{X}}^+(u_1, u_2)}{\partial u_2 \partial u_1}$$

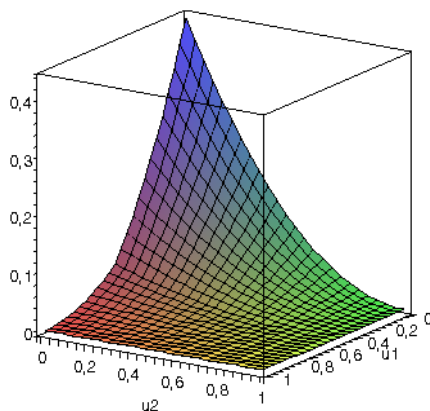


Figure 2.2: The excess-wealth function of  $\mathbf{X}$ .

**Proof.** Differentiating (2.4.2), it is easily shown that

$$\frac{\partial^2 S_{\mathbf{X}}^+(u_1, u_2)}{\partial u_2 \partial u_1} = \frac{P(\mathbf{X} \in R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})])}{f_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))} \text{ for all } (u_1, u_2) \text{ in } (0, 1)^2. \quad (2.4.3)$$

Consequently, the result is obtained.  $\square$

## 2.5 The multivariate excess-wealth ordering

In this section, the multivariate excess-wealth ordering is defined and some of its properties are studied, as that was done by Fernández-Ponce *et al.* (1998) and Shaked and Shanthikumar (1998) in the univariate case. Moreover, we give the relationship between this order and the multivariate d.o. defined by Fernández-Ponce and Suárez-Lloréns (2003).

**Definition 2.5.1.** Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two nonnegative random vectors with finite multiple expectations. Then,  $\mathbf{X}$  is said to be smaller than  $\mathbf{Y}$  in the sense of multivariate

excess-wealth order ( $\mathbf{X} \leq_{ew} \mathbf{Y}$ ) if

$$S_{\mathbf{X}}^+(\mathbf{u}) \leq S_{\mathbf{Y}}^+(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } (0, 1)^n$$

The notation  $\sim_{ew}$  means that  $(\mathbf{X} \leq_{ew} \mathbf{Y})$  and  $(\mathbf{Y} \leq_{ew} \mathbf{X})$ . The following theorem gives some closure results.

**Theorem 2.5.1. i)** *Let  $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(m)}$  be a set of independent random vectors and let  $\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)}$  be another set of independent random vectors. If  $\mathbf{X}^{(i)} \leq_{ew} \mathbf{Y}^{(i)}$  for  $i = 1, \dots, m$  then  $(\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(m)}) \leq_{ew} (\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)})$ .*

**ii)** *Let  $\{\mathbf{X}^{(j)} : j = 1, 2, \dots\}$  and  $\{\mathbf{Y}^{(j)} : j = 1, 2, \dots\}$  be two sequences of random vectors such that  $\mathbf{X}^{(j)} \rightarrow_{st} \mathbf{X}$  and  $\mathbf{Y}^{(j)} \rightarrow_{st} \mathbf{Y}$  as  $j \rightarrow \infty$ , where  $\rightarrow_{st}$  denotes convergence in distribution. If  $\mathbf{X}^{(j)} \leq_{ew} \mathbf{Y}^{(j)}$  for  $j = 1, 2, \dots$  then  $\mathbf{X} \leq_{ew} \mathbf{Y}$ .*

**Proof.**

**i)** It is trivial by using Definition 2.5.1 and **iii)** of the Proposition 2.4.1 and by mathematical induction on the dimension of each of the independent random vectors  $\mathbf{X}^{(i)}$  and  $\mathbf{Y}^{(i)}$ .

**ii)** First, we prove that if  $\mathbf{X}^{(j)} \rightarrow_{st} \mathbf{X}$  then  $\widehat{\mathbf{x}}^{(j)}(\mathbf{u}) \rightarrow \widehat{\mathbf{x}}(\mathbf{u})$ .

Let  $\widehat{\mathbf{x}}^{(j)}(\mathbf{u}) = [\widehat{\mathbf{x}}_1^{(j)}(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_n^{(j)}(\mathbf{u}_n)]$  the multivariate  $\mathbf{u}$ -quantile, where  $\widehat{\mathbf{x}}_s^{(j)}(\mathbf{u}_s) = [F_{X_s | \bigcap_{r=1}^{s-1} \mathbf{X}_r = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}^{(j)}(u_s)]^-$  for  $s = 1, \dots, n$ ,  $F^-(u) = \inf\{x : F(x) \geq u\}$  and  $\mathbf{u}_s = (u_1, \dots, u_s)$  for all  $s = 1, \dots, n$ .

Assume that  $\mathbf{X}^{(j)} \rightarrow_{st} \mathbf{X}$ , that is,  $\lim_{j \rightarrow \infty} F^{(j)}(x_1, \dots, x_n) = F(x_1, \dots, x_n)$  where  $F^{(j)}$  and  $F$  are the joint distribution functions of  $\mathbf{X}^j$  and  $\mathbf{X}$ , respectively. We have to prove that  $\widehat{\mathbf{x}}_s^{(j)}(\mathbf{u}_s) \rightarrow \widehat{\mathbf{x}}_s(\mathbf{u}_s)$  for  $s = 1, 2, \dots, n$ . The proof is by mathematical induction. For  $n = 1$ ,

$$\begin{aligned}
 \lim_{j \rightarrow \infty} [F_{X_1}^{(j)}(u_1)]^- &= \lim_{j \rightarrow \infty} \inf[x_1 : F_{X_1}^{(j)}(x_1) > u_1] \\
 &= \inf[x_1 : \lim_{j \rightarrow \infty} (F_{X_1}^{(j)}(x_1)) > u_1] \\
 &= \inf[x_1 : F_{X_1}(x_1) \geq (u_1)] = [F_{X_1}^-(u_1)]
 \end{aligned}$$

where the second equality follows from the distribution function is absolutely continuous. Therefore, the proposition holds for  $n = 1$ . Now, assume that the proposition is true for  $n - 1$ . We need to show that it is true for  $n$ .

$$\begin{aligned}
 \lim_{j \rightarrow \infty} \widehat{\mathbf{x}}_n^{(j)}(\mathbf{u}_n) &= \lim_{j \rightarrow \infty} [F_{X_n | \cap_{r=1}^{n-1} \mathbf{X}_r^{(j) = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(u_n)]^- \\
 &= \lim_{j \rightarrow \infty} \inf[x_n : F_{X_n | \cap_{r=1}^{n-1} \mathbf{X}_r^{(j) = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(x_n) > u_n] \quad (2.5.1)
 \end{aligned}$$

Given that  $F_{X_s | \cap_{r=1}^{s-1} \mathbf{X}_r^{(j) = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(x_s)$  is an absolutely continuous and strictly increasing function for all  $j$  and for  $s = 1, \dots, n$ , the right-side in (2.5.1) can be written as

$$\inf[x_n : \lim_{j \rightarrow \infty} F_{X_n | \cap_{r=1}^{n-1} \mathbf{X}_r^{(j) = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(x_n) > u_n]. \quad (2.5.2)$$

Now, by applying the monotone convergence theorem and as the stochastic convergence is closed under marginalization, the following equality is obtained

$$\begin{aligned}
 \lim_{j \rightarrow \infty} F_{X_n | \cap_{r=1}^{n-1} \mathbf{X}_r^{(j) = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(x_n) &= \lim_{j \rightarrow \infty} \int_{-\infty}^{x_n} \frac{f^{(j)}[\widehat{\mathbf{x}}_1^{(j)}(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}^{(j)}(\mathbf{u}_{n-1}), t]}{f^{(j)}[\widehat{\mathbf{x}}_1^{(j)}(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}^{(j)}(\mathbf{u}_{n-1})]} dt \\
 &= \int_{-\infty}^{x_n} \lim_{j \rightarrow \infty} \frac{f^{(j)}[\widehat{\mathbf{x}}_1^{(j)}(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}^{(j)}(\mathbf{u}_{n-1}), t]}{f^{(j)}[\widehat{\mathbf{x}}_1^{(j)}(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}^{(j)}(\mathbf{u}_{n-1})]} dt \\
 &= \int_{-\infty}^{x_n} \frac{f[\widehat{\mathbf{x}}_1(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}(\mathbf{u}_{n-1}), t]}{f[\widehat{\mathbf{x}}_1(\mathbf{u}_1), \dots, \widehat{\mathbf{x}}_{n-1}(\mathbf{u}_{n-1})]} dt \\
 &= F_{X_n | \cap_{r=1}^{n-1} \mathbf{X}_r = \widehat{\mathbf{x}}_r(\mathbf{u}_r)}(x_n). \quad (2.5.3)
 \end{aligned}$$



Consequently, from (2.5.2) and (2.5.3), it holds that

$$\begin{aligned} \lim_{j \rightarrow \infty} \widehat{\mathbf{x}}_n^{(j)}(\mathbf{u}_n) &= \inf[x_n : \lim_{j \rightarrow \infty} F_{X_n | \bigcap_{r=1}^{n-1} \mathbf{X}_r^{(j)} = \widehat{\mathbf{x}}_r^{(j)}(\mathbf{u}_r)}(x_n) > u_n] \\ &= \inf[x_n : F_{X_n | \bigcap_{r=1}^{n-1} \mathbf{X}_r = \widehat{\mathbf{x}}_r(\mathbf{u}_r)}(x_n) > u_n] \\ &= \widehat{\mathbf{x}}_n(\mathbf{u}_n) \end{aligned}$$

and by mathematical induction, it is proved that  $\widehat{\mathbf{x}}^{(j)}(\mathbf{u}) \rightarrow \widehat{\mathbf{x}}(\mathbf{u})$ .

It is easy to prove that if  $\mathbf{X}^{(j)} \rightarrow_{st} \mathbf{X}$  then  $R_{\mathbf{X}^{(j)}}[\widehat{\mathbf{x}}^{(j)}(\mathbf{u})] \rightarrow R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]$  for all  $\mathbf{u}$  in  $(0, 1)^n$ . Now by using the monotone convergence theorem and the fact that  $\bar{\mu}_{\mathbf{X}^{(j)}} < +\infty$  for all  $j$ , it follows that  $S_{\mathbf{X}^{(j)}}^+(\mathbf{u}) \rightarrow S_{\mathbf{X}}^+(\mathbf{u})$ . Thus the result is obtained. □

An interesting property is now proved for the bivariate case which can be easily generalized for any dimension. This result was proved in the univariate case by Kochar and Carrière (1997).

**Theorem 2.5.2.** *Let  $\mathbf{X} = (X_1, X_2)$  and  $\mathbf{Y} = (Y_1, Y_2)$  be two random vectors.*

1. *If  $\mathbf{X} \sim_{st} \mathbf{Y} + \mathbf{c}$  then  $\mathbf{X} \sim_{ew} \mathbf{Y}$*
2. *If  $\mathbf{X} \sim_{ew} \mathbf{Y}$  then  $f[\widehat{\mathbf{x}}(\mathbf{u})] = g[\widehat{\mathbf{y}}(\mathbf{u})]$  for all  $\mathbf{u}$  in  $(0, 1)^2$  where  $f(\cdot)$  and  $g(\cdot)$  are the corresponding density functions for  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively.*

**Proof.**

1. Trivial, since the multivariate excess-wealth function is shift invariant.

2. Suppose  $\mathbf{X} \sim_{ew} \mathbf{Y}$ , i.e. it holds

$$\int_{R_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] d\mathbf{t} = \int_{R_{\mathbf{Y}}[\hat{\mathbf{y}}(\mathbf{u})]} P[\mathbf{Y} \in R_{\mathbf{Y}}(\mathbf{t})] d\mathbf{t}. \quad (2.5.4)$$

Since  $\mathbf{X}$  and  $\mathbf{Y}$  satisfy the regularity conditions, it follows that the quantiles  $\hat{\mathbf{x}}(\mathbf{u})$  and  $\hat{\mathbf{y}}(\mathbf{u})$  are differentiable with respect to  $\mathbf{u} = (u_1, u_2)$  at each component. Consequently, differentiating both side of (2.5.4) with respect to  $\mathbf{u} = (u_1, u_2)$  we get

$$\frac{\partial \hat{x}_1(u_1)}{\partial u_1} \cdot \frac{\partial \hat{x}_{2|\hat{x}_1(u_1)}(u_2)}{\partial u_2} = \frac{\partial \hat{y}_1(u_1)}{\partial u_1} \cdot \frac{\partial \hat{y}_{2|\hat{y}_1(u_1)}(u_2)}{\partial u_2}$$

for all  $(u_1, u_2)$  in  $(0, 1)^2$ . That is

$$f_{X_1}[\hat{x}_1(u_1)] \cdot f_{X_2|\hat{x}_1(u_1)}[\hat{x}_{2|\hat{x}_1(u_1)}(u_2)] = g_{Y_1}[\hat{y}_1(u_1)] \cdot g_{Y_2|\hat{y}_1(u_1)}[\hat{y}_{2|\hat{y}_1(u_1)}(u_2)]$$

or equivalently  $f_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})] = g_{\mathbf{Y}}[\hat{\mathbf{y}}(\mathbf{u})]$  for all  $\mathbf{u}$  in  $(0, 1)^2$ .

□

The following example shows that the multivariate excess-wealth order is weaker than the multivariate dispersive order ( $\leq_{Disp}$ ) given by Fernández-Ponce and Suárez-Lloréns (2003).

**Example 2.5.3.** *Suppose that  $\mathbf{X} = (X_1, X_2)$  is a bivariate exponential random vector such that  $X_1$  and  $X_2$  are independent,  $X_1 \sim_{st} \text{Exp}(2)$  and  $X_2 \sim_{st} \text{Exp}(1)$ . Similarly, assume  $\mathbf{Y} = (Y_1, Y_2)$  is a bivariate exponential random vector such that  $Y_1$  and  $Y_2$  are independent with  $Y_1 \sim_{st} \text{Exp}(1/3)$  and  $Y_2 \sim_{st} \text{Exp}(3)$ . Consequently, it holds that*

$$X_1 <_{Disp} Y_1 \text{ and } Y_2 <_{Disp} X_2$$

which implies, from Corollary 3.3 in Fernández-Ponce and Suárez-Lloréns (2003), that  $\mathbf{X} \not\prec_{Disp} \mathbf{Y}$ . Furthermore, by using **iii**) of Proposition 2.4.1, it holds

$$S_{\mathbf{X}}^+(u_1, u_2) = \frac{1}{2}(1 - u_1)(1 - u_2) \leq (1 - u_1)(1 - u_2) = S_{\mathbf{Y}}^+(u_1, u_2)$$

for all  $(u_1, u_2)$  in  $(0, 1)^2$ , i.e.  $\mathbf{X} \leq_{ew} \mathbf{Y}$ .

We finish this section giving a sufficient condition for the excess-wealth order between two vectors with the same copula.

**Theorem 2.5.4.** *Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two absolutely continuous random vectors with the same copula  $C$ . If  $X_i \leq_{disp} Y_i$  for all  $i$ , then  $\mathbf{X} \leq_{ew} \mathbf{Y}$ .*

**Proof.** By using Theorem 2 in Arias-Nicolás *et al* (2005) we know that there exist two expansion functions,  $h_1$  and  $h_2$ , such that

$$h_1(X_1) \sim_{st} Y_1 \text{ and } h_2(X_2) \sim_{st} Y_2.$$

From (2.4.2), it follows

$$\begin{aligned} S_{\mathbf{Y}}^+(u_1, u_2) &= \int_{F_{Y_1}^-(u_1)}^{\infty} \bar{F}_{Y_1}(t) S_{Y_2|Y_1=t}^+(u_2) dt \\ &= \int_{h_1(F_{X_1}^-(u_1))}^{\infty} P(X_1 > h_1^{-1}(t)) S_{h_2(X_2)|X_1=h_1^{-1}(t)}^+(u_2) dt \\ &= \int_{F_{X_1}^-(u_1)}^{\infty} P(X_1 > x) S_{h_2(X_2)|X_1=x}^+(u_2) dh_1(x) \\ &\geq \int_{F_{X_1}^-(u_1)}^{\infty} P(X_1 > x) S_{h_2(X_2)|X_1=x}^+(u_2) dx \end{aligned} \tag{2.5.5}$$

where (2.5.5) follows directly by using the fact that  $h_1$  is an expansion function, that is  $\frac{dh_1(x)}{dx} \geq 1$ .

Now, note that

$$\begin{aligned}
 S_{h_2(X_2)|X_1=x}^+(u_2) &= \int_{F_{h_2(X_2)|X_1=x}^-(u_2)}^{\infty} P(h_2(X_2) > t | X_1 = x) dt \\
 &= \int_{h_2^{-1}(F_{X_2|X_1=x}^-(u_2))}^{\infty} P(X_2 > h_2^{-1}(t) | X_1 = x) dt \\
 &= \int_{F_{X_2|X_1=x}^-(u_2)}^{\infty} P(X_2 > w | X_1 = x) dh_2(w) \\
 &\geq \int_{F_{X_2|X_1=x}^-(u_2)}^{\infty} P(X_2 > w | X_1 = x) d(w) \\
 &= S_{X_2|X_1=x}^+(u_2)
 \end{aligned} \tag{2.5.6}$$

Therefore, from (2.5.5) and (2.5.6), it follow that  $S_{\mathbf{Y}}^+(u_1, u_2) \geq S_{\mathbf{X}}^+(u_1, u_2)$ . □

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## CHAPTER 3

# Multivariate Lifetime Distributions

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### Abstract

The lifetime distributions are of great importance in the theory of stochastic modeling, renewal theory, reliability analysis and quality control. In the literature, several lifetime distributions and their properties have been defined. Characterizations of these distributions, based on some different orderings, have been also studied. In this chapter, new properties for multivariate lifetime distributions, based on the upper-corrected orthant, are defined and some characterizations in terms of the multivariate excess wealth functions are studied.

### 3.1 New characterizations of lifetime distributions

In this section, we propose a multivariate version of the univariate aging properties, based on the concept of upper-corrected orthant, together with their chain of implication. The relationship between these properties and the multivariate excess wealth function are given following the ideas in Fernández-Ponce *et al.* (1998) for the univariate case.

First we give new generalizations of the univariate mean residual life and the failure rate. Let  $\mathbf{X}$  be a nonnegative random vector. By using Proposition 2.3.3, it is easy to see that if  $\mathbf{X}$  is an  $n$ -dimensional random variable and  $\mathbf{t} \in R_{\mathbf{X}}(\mathbf{x})$  then

$$P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t}) | \mathbf{X} \in R_{\mathbf{X}}(\mathbf{x})] = \frac{P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})]}{P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{x})]}$$

This equality enables us to define the total expected residual life in  $\mathbf{x}$  in the following form.

**Definition 3.1.1.** Let  $\mathbf{X}$  be an  $n$ -dimensional random variable. The total expected residual life of  $\mathbf{X}$  in  $\mathbf{x}$  is defined as the following real value,

$$\bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \frac{1}{P\{\mathbf{X} \in R_{\mathbf{X}}(\mathbf{x})\}} \int_{R_{\mathbf{X}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{X}}(t)] dt \quad (3.1.1)$$

for all  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ .

Note there exists a closed relationship between the total expected residual life function and the multivariate excess wealth function. In fact, we can see that

$$\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u})) = \frac{S_{\mathbf{x}}^+(\mathbf{u})}{\prod_{i=1}^n (1 - u_i)}. \quad (3.1.2)$$

Now, a new multivariate version of the failure rate function is considered.

**Definition 3.1.2.** Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be an  $n$ -dimensional random variable.

The multivariate corrected failure rate of  $\mathbf{X}$  is given by

$$r_c(\mathbf{z}) = \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{P[\mathbf{X} \in C(\mathbf{z}, \mathbf{h}) | \mathbf{X} \in R_{\mathbf{X}}(\mathbf{z})]}{h_1 \cdots h_n}$$

where,  $\mathbf{h} = (h_1, \dots, h_n)$ ,  $\mathbf{0} = (0, \dots, 0)$  and  $C(\mathbf{z}, \mathbf{h})$  is the set given by

$$C(\mathbf{z}, \mathbf{h}) = \{\mathbf{x} \in \mathbb{R}^n : x_1 \in [z_1, z_1 + h_1], x_j \in [\varphi_1(x_j, \mathbf{z}); \varphi_2(x_j, \mathbf{z})] \text{ for } j = 2, \dots, n\}$$

where

$$\varphi_1(x_j, \mathbf{z}) = F_{X_j | \bigcap_{i=1}^{j-1} X_i = x_i}^{-1}(F_{X_j | \bigcap_{i=1}^{j-1} X_i = z_i}(z_j))$$

and

$$\varphi_2(x_j, \mathbf{z}) = F_{X_j | \bigcap_{i=1}^{j-1} X_i = x_i}^{-1}(F_{X_j | \bigcap_{i=1}^{j-1} X_i = z_i}(z_j + h_j)).$$

Equivalently, the multivariate corrected failure rate can be expressed as

$$r_c(\mathbf{z}) = \frac{f(\mathbf{z})}{P[(\mathbf{X} \in R_{\mathbf{X}}(\mathbf{z}))]} \quad (3.1.3)$$

In fact, for the bivariate case,

$$\begin{aligned} r_c(\mathbf{z}) &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{P[\mathbf{X} \in C(\mathbf{z}, \mathbf{h}) | \mathbf{X} \in R_{\mathbf{X}}(\mathbf{z})]}{h_1 \cdot h_2} \\ &= \frac{1}{P[(\mathbf{X} \in R_{\mathbf{X}}(\mathbf{z}))]} \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{P[\mathbf{X} \in C(\mathbf{z}, \mathbf{h})]}{h_1 \cdot h_2} \end{aligned} \quad (3.1.4)$$

The limit in (3.1.4) can be expressed as

$$\begin{aligned} \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{P[\mathbf{X} \in C(\mathbf{z}, \mathbf{h})]}{h_1 \cdot h_2} &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{1}{h_1 \cdot h_2} \int_{z_1}^{z_1+h_1} \int_{\varphi_1(x_1, \mathbf{z})}^{\varphi_2(x_1, \mathbf{z})} f(x_1, x_2) dx_2 dx_1 \\ &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{1}{h_1 \cdot h_2} \int_{z_1}^{z_1+h_1} f_{X_1}(x_1) \int_{\varphi_1(x_1, \mathbf{z})}^{\varphi_2(x_1, \mathbf{z})} f_{X_2 | X_1 = x_1}(x_2) dx_2 dx_1 \\ &= \lim_{h_1 \rightarrow 0} \frac{1}{h_1} \int_{z_1}^{z_1+h_1} f_{X_1}(x_1) \lim_{h_2 \rightarrow 0} \frac{\Delta_h F_{X_2 | X_1 = z_1}(z_2)}{h_2} dt_2 \\ &= \lim_{h_1 \rightarrow 0} \frac{1}{h_1} \int_{z_1}^{z_1+h_1} f_{X_1}(x_1) f_{X_2 | X_1 = z_1}(z_2) dx_1 \\ &= f_{X_1}(z_1) f_{X_2 | X_1 = z_1}(z_2) \\ &= f(\mathbf{z}) \end{aligned}$$

### 3.1. NEW CHARACTERIZATIONS OF LIFETIME DISTRIBUTIONS

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where  $\Delta_h F_{X_2|X_1=z_1}(z_2) = F_{X_2|X_1=z_1}(z_2 + h_2) - F_{X_2|X_1=z_1}(z_2)$ . So, we obtained the equation (3.1.3).

Note that for the bidimensional case, it follows  $r_c(z_1, z_2) = r_{X_1}(z_1)r_{X_2|X_1=z_1}(z_2)$ . In general, it holds that

$$r_c(\mathbf{z}) = \prod_{i=1}^n r_{X_i|\cap_{j=1}^{i-1} X_j=z_j}(z_i). \quad (3.1.5)$$

Note also, that in the univariate case it holds  $S_X^+(u) = \int_u^1 \frac{1}{r_X[\hat{\mathbf{x}}(z)]} dz$ . It is easy to see, from the Definition 2.4.2, that  $S_{\mathbf{X}}^+(\mathbf{u})$  can be written as

$$S_{\mathbf{X}}^+(\mathbf{u}) = \int_{u_1}^1 \dots \int_{u_n}^1 [r_c(\hat{\mathbf{x}}(\mathbf{z}))]^{-1} d\mathbf{z}. \quad (3.1.6)$$

In the equation below, we can see how (3.1.6) is obtained for the bivariate case.

$$\begin{aligned} S_{\mathbf{X}}^+(\mathbf{u}) &= \int_{F_1^{-1}(u_1)}^{+\infty} \int_{F_{X_2|X_1=t_1}^-(u_2)}^{+\infty} P(X_1 > t_1)P(X_2 > t_2|X_1 = t_1) dt_2 dt_1 \\ &= \int_{u_1}^1 \int_{u_2}^1 (1-z)(1-w) \frac{dz dw}{f_1[F_1^{-1}(z)]f_{X_2|X_1=F_1^{-1}(z)}[F_{X_2|X_1=F_1^{-1}(z)}^{-1}(w)]} \\ &= \int_{u_1}^1 \int_{u_2}^1 \frac{(1-z)(1-w)}{f_{\mathbf{X}}[\hat{\mathbf{x}}(z, w)]} dz dw \\ &= \int_{u_1}^1 \int_{u_2}^1 \frac{1}{\{r_c[\hat{\mathbf{x}}(z, w)]\}} dz dw \end{aligned} \quad (3.1.7)$$

where the first equality follows by changing  $z = F_1(t_1)$  and  $w = F_{2|x_1=F_1^{-1}(z)}(t_2)$ .

Note that making this change, it follows  $t_1 = F_1^{-1}(z)$ ,  $t_2 = F_{2|x_1=F_1^{-1}(z)}^{-1}(w)$  and the



Jacobian matrix is

$$\begin{aligned}
 J &= \begin{bmatrix} \frac{\partial}{\partial z} F_1^{-1}(z) & 0 \\ \frac{\partial}{\partial z} F_{2|x_1=F_1^{-1}(z)}^{-1}(w) & \frac{\partial}{\partial w} F_{2|x_1=F_1^{-1}(z)}^{-1}(w) \end{bmatrix} \\
 &= \frac{\partial}{\partial z} F_1^{-1}(z) \frac{\partial}{\partial w} F_{2|x_1=F_1^{-1}(z)}^{-1}(w) \\
 &= \frac{1}{f(F_1^{-1}(z)) f_{2|x_1=F_1^{-1}(z)}(F_{2|x_1=F_1^{-1}(z)}^{-1}(w))}.
 \end{aligned}$$

The next result shows that the only bivariate vector with exponential marginals and a constant corrected multivariate failure rate is that whose components are independent. This result can be generalized on the multivariate case.

**Theorem 3.1.1.** *Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative random vector with exponential marginals. Then  $\mathbf{X}$  has a constant  $r_c(\mathbf{x})$  if and only if  $X_1$  and  $X_2$  are independent.*

**Proof.** Obviously, if  $X_1$  and  $X_2$  are independent and have exponential distributions, it holds that  $r_c(\mathbf{x})$  is constant.

Now, assume that  $X_1$  and  $X_2$  have exponential distributions. If  $r_c(\mathbf{x}) = c$  for all  $\mathbf{x} = (x_1, x_2)$  then from (3.1.5) it follows that  $r_{X_2|X_1=x_1}(x_2) = c/\lambda_1$ , where  $\lambda_1$  is the parameter of the distribution of  $X_1$ . Given that the only univariate distribution with a constant failure rate is the exponential distribution, it follows that, for all  $x_1$ ,  $X_2|X_1 = x_1$  has an exponential distribution with parameter  $c/\lambda_1$ . That is,  $X_1$  and  $X_2$  are independent. □

For marginals not necessary exponentials, we get the following result.

**Theorem 3.1.2.** *Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative random vector. If the corrected failure rate of  $\mathbf{X}$  is constant, that is  $r_c(\mathbf{x}) = c$ , then  $r_{X_2|X_1}(x_2) = h(x_1)$  does not depend on  $x_2$ .*

**Proof.** Suppose the corrected failure rate of  $\mathbf{X}$  is constant. Then, it follows that

$$r_c(\mathbf{x}) = r_{X_1}(x_1)r_{X_2|X_1=x_1}(x_2) = c.$$

Thus,  $r_{X_2|X_1=x_1}(x_2) = c/r_{X_1}(x_1) = h(x_1)$ , where  $h(x)$  is a function which does not depend on  $x_2$ . □

The next example shows that the corrected failure rate does not determine the distribution of  $\mathbf{X}$ .

**Example 3.1.3.** Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative random vector such that  $X_1$  has density and survival functions given, respectively, by

$$f_{X_1}(x_1) = x_1 e^{-\frac{x_1^2}{2}} \text{ and } \bar{F}_{X_1}(x_1) = e^{-\frac{x_1^2}{2}} \text{ for } x_1 > 0,$$

and  $X_2|X_1 = x_1$  has an exponential distribution with parameter  $\frac{1}{x_1}$ . The corrected failure rate of  $\mathbf{X}$  is

$$r_c(x, y) = r_{X_1}(x_1)r_{X_2|X_1=x_1}(x_2) = 1.$$

Therefore, there exists a vector with marginals not exponential and dependent such that its  $r_c$  is constant. Thus, this example shows that the corrected failure rate does not determine the distribution of  $\mathbf{X}$ , given that it is possible to find two vectors with the same  $r_c$  (for example, the vector with i.i.d marginals and with distribution  $Exp(1)$ ). △

Similarly to the univariate case and considering the above definitions, we can define new concepts of “multivariate lifetime distributions”.

**Definition 3.1.3.** Let  $\mathbf{X}$  be a nonnegative  $n$ -dimensional random variable satisfying the regularity conditions (RC) and with finite total expectation residual life  $\bar{\mu}_{\mathbf{X}}(\mathbf{x})$  and multivariate failure rate function  $r_c(\mathbf{x})$ .

- (a)  $\mathbf{X}$  is said to have a Corrected Multivariate Increasing [Decreasing] Failure Rate (CMIFR)[CMDFR] distribution if the function  $r_c(\mathbf{x})$  is increasing [decreasing] in  $\mathbf{x}$  for all  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ .
- (b)  $\mathbf{X}$  is said to have a Corrected Multivariate Decreasing [Increasing] Mean Residual Life (CMDMRL)[CMIMRL] distribution if  $\bar{\mu}_X(\mathbf{x})$  is decreasing [increasing] in  $\mathbf{x}$  for all  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ .
- (c)  $\mathbf{X}$  is said to have a Corrected Multivariate New [Worse] Better than Used in Expectation (CMNBUE)[CMNWUE] distribution if

$$\nu_{\mathbf{X}} \geq [\leq] \bar{\mu}_{\mathbf{X}}(\mathbf{x}). \quad (3.1.8)$$

In the univariate case the multivariate corrected failure rate,  $r_c(\mathbf{x})$ , and the total expected residual life,  $\bar{\mu}_{\mathbf{X}}(\mathbf{x})$  coincide with the hazard rate function  $r_X(x)$  and the mean residual life,  $\mu_X(x)$ , respectively. Therefore, the concepts of corrected multivariate lifetime distributions and the usual concept of univariate lifetime distributions are equivalent.

**Example 3.1.4.** *Let  $(X_1, X_2)$  be the nonnegative random vector defined as in Example 2.2.1. The total expected residual life of  $\mathbf{X}$  is given by*

$$\bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{(x_2-1)(2x_1^2-8x_1+7)}{8(x_1-2)} & \text{if } (x_1, x_2) \in A \\ \frac{1}{8}(x_1-2)(x_1+x_2-2) & \text{if } (x_1, x_2) \in B \end{cases}$$

*In particular,  $\bar{\mu}_{\mathbf{X}}(\mathbf{0}) = \bar{\mu}_{\mathbf{X}} = 7/16$ . Moreover, the total expected residual life is decreasing in  $\mathbf{x}$ . In fact, if  $(x_1, x_2) \in A$ ,*

$$\frac{\partial}{\partial x_1} \bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \frac{(x_2-1)(2x_1^2-8x_1+9)}{8(x_1-2)^2} \quad \text{and} \quad \frac{\partial}{\partial x_2} \bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \frac{2x_1^2-8x_1+7}{8(x_1-2)}$$

which are negative for all  $(x_1, x_2) \in A$  and if  $(x_1, x_2) \in B$ ,

$$\frac{\partial}{\partial x_1} \bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \frac{1}{8}(2x_1 + x_2 - 4) \quad \text{and} \quad \frac{\partial}{\partial x_2} \bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \frac{1}{8}(x_1 - 2)$$

which are also negative for all  $(x_1, x_2) \in B$ . Thus, the random vector  $\mathbf{X}$  is CMDMRL.

Moreover, given that  $\bar{\mu}_{\mathbf{X}}(\mathbf{x})$  is decreasing, for all  $\mathbf{x} \geq 0$ ,  $\nu_{\mathbf{X}} = \bar{\mu}_{\mathbf{X}} \geq \bar{\mu}_{\mathbf{X}}(\mathbf{x})$ , that is,  $\mathbf{X}$  is also CMNBUE.

In the same way, it can be proved that the bivariate corrected failure rate of  $\mathbf{X}$  given by  $r_c(\mathbf{x}) = \frac{1}{(2-x_1)(1-x_2)}$  if  $\mathbf{x} \in A$  and  $r_c(\mathbf{x}) = \frac{1}{(1-x_1/2)(2-x_1-x_2)}$  if  $\mathbf{x} \in B$ , is increasing in  $\mathbf{x}$ . Thus, the bivariate vector  $\mathbf{X}$  is CMIFR.

△

**Example 3.1.5.** Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative random vector such that  $X_1$  has a uniform distribution  $U[0, 1]$  and the conditional variable  $X_2|X_1 = x_1$  has a uniform distribution  $U[0, x_1]$  if  $x_1 \in [0, 0.5)$  and has a uniform distribution  $U[1 - x_1, 1]$  if  $x_1 \in [0.5, 1]$ .

The upper-corrected orthant for  $\mathbf{x} = (x_1, x_2)$  is the following set:

$$R_{\mathbf{X}}(\mathbf{x}) = \begin{cases} R_1(\mathbf{x}) & \text{if } 0 \leq x_1 < 0.5 \text{ and } 0 \leq x_2 < x_1 \\ R_2(\mathbf{x}) & \text{if } 0.5 \leq x_1 < 1 \text{ and } 1 - x_1 \leq x_2 < 1 \end{cases}$$

where

$$\begin{aligned} R_1(\mathbf{x}) &= \left\{ \mathbf{t} \in \text{Supp}(\mathbf{X}) : t_1 \in [x_1, 0.5]; t_2 \in \left[ \frac{x_2}{x_1} t_1, t_1 \right] \right\} \cup \\ &\quad \left\{ \mathbf{t} \in \text{Supp}(\mathbf{X}) : t_1 \in [0.5, 1]; t_2 \in \left[ 1 + t_1 \left( \frac{x_2}{x_1} - 1 \right), 1 \right] \right\} \\ R_2(\mathbf{x}) &= \left\{ \mathbf{t} \in \text{Supp}(\mathbf{X}) : t_1 \in [x_1, 1]; t_2 \in \left[ 1 + \frac{t_1}{x_1} (x_2 - 1), 1 \right] \right\} \end{aligned}$$

The total expected residual life of  $\mathbf{X}$  is given by

$$\bar{\mu}_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \bar{\mu}_1(\mathbf{x}) & \text{if } 0 < x_1 < 0.5 \text{ and } 0 < x_2 < x_1 \\ \bar{\mu}_2(\mathbf{x}) & \text{if } 0.5 \leq x_1 < 1 \text{ and } 1 - x_1 \leq x_2 < 1 \end{cases}$$

where

$$\begin{aligned} \bar{\mu}_1(\mathbf{x}) &= \frac{-\frac{x_1^5}{6} + x_1^4(\frac{1}{4} + \frac{x_2}{3}) + \frac{x_1 x_2}{6} + x_1^3(-\frac{1}{2} - \frac{x_2}{6})x_2 - \frac{x_2^2}{12} + x_1^2(-\frac{1}{12} + \frac{x_2^2}{4})}{(x_1 - 1)x_1(x_1 - x_2)} \\ \bar{\mu}_2(\mathbf{x}) &= \frac{(x_1 - 1)^2(1 + 2x_1)(x_2 - 1)^2}{12(1 - x_1)x_1(1 - x_2)} \end{aligned}$$

The total expected residual life function is not increasing or decreasing. In fact, if  $\mathbf{x}_1 = (0.3, 0.2)$ ,  $\mathbf{x}_2 = (0.4, 0.3)$  and  $\mathbf{x}_3 = (0.7, 0.8)$ , it holds that  $\bar{\mu}(\mathbf{x}_1) = 0.0311 > 0.022 = \bar{\mu}(\mathbf{x}_2)$  and  $\bar{\mu}(\mathbf{x}_2) = 0.022 < 0.036 = \bar{\mu}(\mathbf{x}_3)$ . Therefore, the random vector  $\mathbf{X}$  has not a CMDMRL or CMIMRL distribution. However, after some calculations, it can be shown that  $\nu_{\mathbf{X}} = 0.125 \geq \bar{\mu}_{\mathbf{X}}(\mathbf{x})$  for all  $\mathbf{x}$ , so it holds that  $\mathbf{X}$  has a CMNBUE distribution.

The bivariate corrected failure rate of  $\mathbf{X}$  is given by

$$r_c(\mathbf{x}) = \begin{cases} \frac{1}{(1-x_1)(x_1-x_2)} & \text{if } \mathbf{x} \in A \\ \frac{1}{(1-x_1)(1-x_2)} & \text{if } \mathbf{x} \in B \end{cases}$$

For  $\mathbf{x}_1 = (0.3, 0.2)$ ,  $\mathbf{x}_2 = (0.4, 0.3)$  and  $\mathbf{x}_3 = (0.6, 0.8)$ , it holds that  $r_c(\mathbf{x}_1) < r_c(\mathbf{x}_2)$  and  $r_c(\mathbf{x}_3) < r_c(\mathbf{x}_2)$ . Therefore, the random vector  $\mathbf{X}$  does not have a CMIFR or CMDFR distribution.

△

Two properties of closure for the CMNBUE, CMDMRL and CMIFR distributions are now given.

**Proposition 3.1.6.** 1) If  $\mathbf{X}$  and  $\mathbf{Y}$  are CMNBUE [CMDMRL, CMIFR] distributions and  $\mathbf{X}, \mathbf{Y}$  are independent, then  $(\mathbf{X}, \mathbf{Y})$  is a CMNBUE [CMDMRL, CMIFR] distribution.

2) If  $\mathbf{X}$  is a CMNBUE [CMDMRL, CMIFR] distribution, then the vector  $\mathbf{aX} = (a_1X_1, \dots, a_nX_n)$  is a CMNBUE [CMDMRL, CMIFR] distribution.

**Proof.**

1) Let  $\mathbf{X} = (X_1, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, \dots, Y_m)$  be two independent CMNBUE vectors and  $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ . It is easily proved that

$$P[\mathbf{Z} \in R_{\mathbf{z}}(\mathbf{z})] = P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{z}_1)]P[\mathbf{Y} \in R_{\mathbf{y}}(\mathbf{z}_2)] \quad (3.1.10)$$

and

$$\int_{R_{\mathbf{z}}(\mathbf{z})} P[\mathbf{Z} \in R_{\mathbf{z}}(\mathbf{t})]d\mathbf{t} = \int_{R_{\mathbf{x}}(\mathbf{z}_1)} P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{t}_1)]d\mathbf{t}_1 \int_{R_{\mathbf{y}}(\mathbf{z}_2)} P[\mathbf{Y} \in R_{\mathbf{y}}(\mathbf{t}_2)]d\mathbf{t}_2 \quad (3.1.11)$$

where  $\mathbf{z} = (z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m})$ ,  $\mathbf{z}_1 = (z_1, \dots, z_n)$  and  $\mathbf{z}_2 = (z_{n+1}, \dots, z_{n+m})$ .

From (3.1.10) and (3.1.11) it follows

$$\bar{\mu}_{\mathbf{Z}}(\mathbf{z}) = \bar{\mu}_{\mathbf{X}}(\mathbf{z}_1)\bar{\mu}_{\mathbf{Y}}(\mathbf{z}_2), \quad (3.1.12)$$

and given that  $\mathbf{X}$  and  $\mathbf{Y}$  are CMNBUE, it holds  $\bar{\mu}_{\mathbf{Z}}(\mathbf{z}) \leq \nu_{\mathbf{X}}\nu_{\mathbf{Y}} = \nu_{\mathbf{Z}}$ . Therefore,  $\mathbf{Z}$  is CMNBUE.

Now, we consider that  $\mathbf{X}$  and  $\mathbf{Y}$  are CMDMRL. Then  $\bar{\mu}_{\mathbf{X}}(\mathbf{z}_1)$  and  $\bar{\mu}_{\mathbf{Y}}(\mathbf{z}_2)$  are decreasing in  $\mathbf{z}_1$  and  $\mathbf{z}_2$ , respectively. So, from (3.1.12) it holds that  $\bar{\mu}_{\mathbf{Z}}(\mathbf{z})$  is decreasing in  $\mathbf{z}$ . That is,  $\mathbf{Z}$  is CMDMRL.

Finally, if  $\mathbf{X}$  and  $\mathbf{Y}$  are CMIFR, then  $r_{c,\mathbf{Z}}(\mathbf{z}) = r_{c,\mathbf{X}}(\mathbf{z}_1)r_{c,\mathbf{Y}}(\mathbf{z}_2)$  is decreasing in  $\mathbf{z}$  and therefore, the assumption is obtained.

2) Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a CMNBUE distribution and  $\mathbf{a} = (a_1, \dots, a_n)$  with  $a_i > 0$  for  $i = 1, \dots, n$ . Denote  $\mathbf{Y} = (a_1 X_1, \dots, a_n X_n)$ . It is easily proved that

$$P[\mathbf{Y} \in R_{\mathbf{Y}}(\mathbf{t})] = P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t}')] \quad (3.1.13)$$

where  $\mathbf{t}' = (t_1/a_1, \dots, t_n/a_n)$ , and

$$\int_{R_{\mathbf{Y}}(\mathbf{y})} P[\mathbf{Y} \in R_{\mathbf{Y}}(\mathbf{t})] d\mathbf{t} = \prod_{i=1}^n a_i \int_{R_{\mathbf{X}}(\mathbf{y}')} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{w})] d\mathbf{w} \quad (3.1.14)$$

where  $\mathbf{y}' = (y_1/a_1, \dots, y_n/a_n)$ .

From (3.1.13) and (3.1.14) it follows

$$\bar{\mu}_{\mathbf{Y}}(\mathbf{y}) = \prod_{i=1}^n a_i \bar{\mu}_{\mathbf{X}}(\mathbf{y}'), \quad (3.1.15)$$

and given that  $\mathbf{X}$  is CMNBUE, it holds  $\bar{\mu}_{\mathbf{Y}}(\mathbf{y}) \leq \prod_{i=1}^n a_i \nu_{\mathbf{X}} = \nu_{\mathbf{Y}}$  for all  $\mathbf{y}$ . Thus,  $\mathbf{Y}$  is CMNBUE.

Now, we consider that  $\mathbf{X}$  is CMDMRL. Then  $\bar{\mu}_{\mathbf{X}}(\mathbf{x})$  is decreasing in  $\mathbf{x}$ . So, from (3.1.15) it holds that  $\bar{\mu}_{\mathbf{Y}}(\mathbf{y})$  is decreasing in  $\mathbf{y}$ . That is,  $\mathbf{Y}$  is CMDMRL.

Finally, If  $\mathbf{X}$  is CMIFR, then  $r_{c,\mathbf{Y}}(\mathbf{y}) = \frac{1}{\prod_{i=1}^n a_i} r_{c,\mathbf{X}}(\mathbf{y}')$  is also increasing in  $\mathbf{y}$  and therefore, the assumption is obtained.

□

A property of the CMNBUE distributions is now given. It will be showed that if  $X_1$  and  $X_2$  have NBUE distributions and  $\varphi$  is a nonnegative function, the random vector  $(X_1, \varphi(X_1) + X_2)$  has a CMNBUE distribution. However, if the NBUE property is replaced by the DMRL or IFR property, then, in general, it could not be held that the vector  $(X_1, \varphi(X_1) + X_2)$  has a CMDMRL or CMIFR distribution, respectively.

**Theorem 3.1.7.** *Let  $X_1$  and  $X_2$  be two independent and NBUE univariate distributions and let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative function. Hence  $(Z_1, Z_2) = (X_1, \varphi(X_1) + X_2)$  has a CMNBUE distribution.*

**Proof.** Let  $F_1$  and  $F_2$  be the corresponding distribution functions of the random variables  $X_1$  and  $X_2$ , respectively. It is easily shown that

$$\begin{aligned} \nu_{(Z_1, Z_2)} &= \int_0^\infty \bar{F}_1(t) \mathbb{E}(\varphi(X_1) + X_2 | X_1 = t) dt \\ &= \int_0^\infty \varphi(t) \bar{F}_1(t) dt + \mathbb{E}(X_1) \mathbb{E}(X_2). \end{aligned} \quad (3.1.16)$$

Obviously, the  $Supp(Z_1, Z_2) = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0, z_2 - \varphi(z_1) > 0\}$ .

Then, using (3.1.1), if  $(z_1, z_2) \in Supp(Z_1, Z_2)$ , it is obtained that

$$\begin{aligned} \bar{\mu}_{(Z_1, Z_2)}(z_1, z_2) &= \frac{1}{\bar{F}_1(z_1) \bar{F}_2(z_2 - \varphi(z_1))} \int_{z_1}^\infty \bar{F}_1(t_1) dt_1 \int_{z_2 - \varphi(z_1)}^\infty \bar{F}_2(t_2) dt_2 \\ &= \mathbb{E}(X_1 - z_1 | X_1 > z_1) \mathbb{E}[X_2 - (z_2 - \varphi(z_1)) | X_2 > (z_2 - \varphi(z_1))] \end{aligned}$$

for all  $z_2 > \varphi(z_1)$ . Furthermore, by using the fact that  $X_1$  and  $X_2$  have NBUE distributions,  $\mathbb{E}(X_1 - z_1 | X_1 > z_1) \mathbb{E}[X_2 - (z_2 - \varphi(z_1)) | X_2 > (z_2 - \varphi(z_1))] \leq \mathbb{E}(X_1) \mathbb{E}(X_2)$  holds.

Consequently, from (3.1.16) and using the fact that  $\varphi(X)$  is a nonnegative function, it is easily deduced that

$$\nu_{(Z_1, Z_2)} > \mathbb{E}(X_1) \mathbb{E}(X_2) \geq \bar{\mu}_{(Z_1, Z_2)}(z_1, z_2) \text{ for all } (z_1, z_2) \text{ in } Supp(Z_1, Z_2).$$

Thus, the random vector  $(Z_1, Z_2) = (X_1, \varphi(X_1) + X_2)$  is CMNBUE by using (c) in Definition 3.1.3.  $\square$



**Remark 3.1.1.** Note that if  $X_1$  and  $X_2$  are two independent and IFR univariate distributions, it could not be held that  $\mathbf{Z} = (Z_1, Z_2) = (X_1, \varphi(X_1) + X_2)$  has a CMIFR distribution. For example, let  $X_i$  be a random variable with a uniform distribution on  $[0, 1]$ , for  $i = 1, 2$  and let  $\varphi$  be the identity function. The bivariate corrected failure rate of  $\mathbf{Z}$  is given by

$$\begin{aligned} r_c(z_1, z_2) &= \frac{f_{\mathbf{Z}}(z_1, z_2)}{P[\mathbf{Z} \in R_{\mathbf{Z}}(z_1, z_2)]} = \frac{f_{X_1}(z_1)f_{X_2}(z_2 - z_1)}{F_1(z_1)\bar{F}_2(z_2 - z_1)} \\ &= r_{x_1}(z_1)r_{x_2}(z_2 - z_1) \\ &= \frac{1}{(1 - z_1)} \frac{1}{(1 - z_2 + z_1)}. \end{aligned}$$

Now, if  $\mathbf{z} = (0.1, 0.8)$  and  $\mathbf{z}' = (0.5, 0.9)$  it holds that  $r_c(\mathbf{z}) \geq r_c(\mathbf{z}')$ , and therefore  $r_c(\mathbf{z})$  is not an increasing function.

Similarly, we can prove that if  $X_1$  and  $X_2$  are two independent and DMRL univariate distributions, it could not be held that  $\mathbf{Z} = (Z_1, Z_2) = (X_1, X_1 + X_2)$  has a CMDMRL distribution. In fact, after simple calculus, one can see that

$$\bar{\mu}_{(Z_1, Z_2)}(z_1, z_2) = \frac{(1 - z_1)(1 - z_2 + z_1)}{4}.$$

If  $\mathbf{z} = (0.1, 0.7)$  and  $\mathbf{z}' = (0.3, 0.7)$  it holds that  $\bar{\mu}_{(Z_1, Z_2)}(\mathbf{z}) \leq \bar{\mu}_{(Z_1, Z_2)}(\mathbf{z}')$ , and therefore  $\bar{\mu}_{(Z_1, Z_2)}(\mathbf{z})$  is not a decreasing function in  $\mathbf{z}$ .

▲

The following results give a necessary and sufficient condition for the CMNBUE property of  $\mathbf{X}$ , based on the multivariate excess-wealth function. It can be considered as a generalization of Corollary 3.1. (c) in Fernández-Ponce *et al.* (1998).

**Theorem 3.1.8.** Let  $\mathbf{X}$  be a random vector verifying the regularity conditions (RC).  $\mathbf{X}$  is a CMNBUE distribution if and only if

$$\mathcal{S}^+_{\mathbf{X}}(\mathbf{u}) \leq \nu_{\mathbf{X}} \prod_{j=1}^n (1 - u_j) \text{ for all } \mathbf{u} \in [0, 1]^n.$$

**Proof.** Firstly, assume that  $\mathbf{X}$  has a CMNBUE distribution. By (c) in Definition 3.1.3,  $\nu_{\mathbf{X}} \geq \bar{\mu}_{\mathbf{X}}(\mathbf{x})$  holds for all  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ . From (RC) defined in Section 2.2, it is also known that for each  $\mathbf{x} \in \text{Supp}(\mathbf{X})$ , only one vector  $\mathbf{u} \in [0, 1]^n$  exists such that  $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{u})$ . Now, by using the equality (3.1.2) and the Proposition 2.2.2, the result holds. Conversely, if  $\mathcal{S}^+_{\mathbf{X}}(\mathbf{u}) \leq \nu_{\mathbf{X}} \prod_{j=1}^n (1 - u_j)$  for all  $\mathbf{u} \in [0, 1]^n$  and  $\mathbf{X}$  verifies the regularity conditions, then it is easily seen that  $\mathbf{X}$  has a CMNBUE distribution.  $\square$

Analogous results to Theorem 3.1.8 can be shown for the CMDMRL and CMIRF properties. However, additional conditions should be considered. This is due to the aging properties are defined in the support of the vector  $\mathbf{X}$  while the multivariate excess wealth function is defined in  $\mathbf{u} \in [0, 1]^n$ . As we can see in (3.1.2) and (3.1.7) it is possible to establish a relationship between the function  $S_{\mathbf{X}}^+$  and the functions  $\bar{\mu}_{\mathbf{X}}$  and  $r_c$  evaluated in the  $\mathbf{u}$ -quantile, respectively. Because of this, to obtain results where the CMDMRL and CMIRF properties and the function  $S_{\mathbf{X}}^+$  are related, it is necessary to impose that the vector  $\mathbf{X}$  is CDS or CIS.

The previous problem can be solved by considering the multivariate aging notions based on the quantile approach, so obtaining the following definitions.

**Definition 3.1.4.** Let  $\mathbf{X}$  be a non-negative random variable with finite total expectation  $\bar{\mu}_{\mathbf{X}}$  and multivariate failure rate function  $r_c(\mathbf{x})$ .

- (a)  $\mathbf{X}$  is said to have a Quantile-Corrected Multivariate Increasing [Decreasing] Failure Rate (Q-CMIFR)[Q-CMDFR] distribution if the function  $r_c[\widehat{\mathbf{x}}(\mathbf{u})]$  is increasing [decreasing] in  $\mathbf{u}$  for all  $\mathbf{u} \in [0, 1]^n$ .
- (b)  $\mathbf{X}$  is said to have a Quantile-Corrected Multivariate Decreasing [Increasing]

Mean Residual Life (Q-CMDMRL)[Q-CMIMRL] distribution if  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]$  is decreasing [increasing] in  $\mathbf{u}$  for all  $\mathbf{u} \in [0, 1]^n$ , i.e. if  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{v})] \leq \bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]$  for all  $\mathbf{u} \leq \mathbf{v}$ .

(c)  $\mathbf{X}$  is said to have a Quantile-Corrected Multivariate New [Worse] Better than Used in Expectation (Q-CMNBUE)[Q-CMNBUE] distribution if

$$\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})] \geq [\leq] \bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] \text{ for all } \mathbf{u} \in [0, 1]^n. \quad (3.1.17)$$

where  $\mathbf{0}$  denotes the vector of zeros, i.e.  $\mathbf{0} = (0, \dots, 0)$ .

For the prosecution we recall the definition of different kinds of convexity. We also need some preliminary results. The proof of the first is based on elemental calculus and therefore omitted.

**Definition 3.1.5.** Let  $f$  be a  $\mathbb{R}$ -valued function on  $\mathbb{R}^n$ . The function  $f$  is said to be a componentwise convex function if it is convex in each argument when the other arguments are held fixed.

**Definition 3.1.6.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be directionally convex if for any  $\mathbf{x}_i \in \mathbb{R}^n, i = 1, 2, 3, 4$  such that  $\mathbf{x}_1 \leq \mathbf{x}_2, \mathbf{x}_3 \leq \mathbf{x}_4$  and  $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$ , then

$$f(\mathbf{x}_1) + f(\mathbf{x}_4) \geq f(\mathbf{x}_2) + f(\mathbf{x}_3).$$

If the gradient vector  $f'(\mathbf{x}) = [f'_1(\mathbf{x}), \dots, f'_n(\mathbf{x})]$  exists, then  $f$  is directionally convex if and only if  $f'$  is increasing (see it was shown in Brunk, 1964)

**Proposition 3.1.9.** Let  $H(\mathbf{x}) = \int_{x_1}^{c_1} \dots \int_{x_n}^{c_n} h(\mathbf{t})d\mathbf{t}$  for some constants  $c_i$ , and  $x_i \leq c_i$ . Then  $H(\mathbf{x})$  is directionally convex if and only if  $h(\mathbf{x})$  is decreasing in  $\mathbf{x}$ .

**Proposition 3.1.10.** Let  $h : \mathbb{R}^{+n} \rightarrow \mathbb{R}^+$  be decreasing in each argument. Then  $\frac{\int_{x_1}^{c_1} \dots \int_{x_n}^{c_n} h(\mathbf{t})d\mathbf{t}}{\prod_i (c_i - x_i)}$  is decreasing in  $\mathbf{x} = (x_1, \dots, x_n)$ .

**Proof.** Let  $G(i, \mathbf{x}) = \int_0^{\mathbf{c}} \mathbf{1}_{[\mathbf{x}, \mathbf{c}]}(\mathbf{t})g(i, \mathbf{t})d\mathbf{t}$ , where

$$g(i, \mathbf{t}) = \begin{cases} h(\mathbf{t}) & \text{if } i = 1, \\ 1 & \text{if } i = 2, \end{cases}$$

and

$$\mathbf{1}_{[\mathbf{x}, \mathbf{c}]}(\mathbf{t}) = \begin{cases} 0 & \text{if } \mathbf{t} \notin [\mathbf{x}, \mathbf{c}], \\ 1 & \text{if } \mathbf{t} \in [\mathbf{x}, \mathbf{c}]. \end{cases}$$

Recall that a positive real function  $f(\mathbf{x}, \mathbf{y})$  defined on  $\mathcal{X} \times \mathcal{Y}$  is said to be  $\text{MTP}_2$  in  $(\mathbf{x}, \mathbf{y})$  if the ratio  $\frac{f(\mathbf{x}_1, \mathbf{y})}{f(\mathbf{x}_2, \mathbf{y})}$  is decreasing in  $\mathbf{y}$  whenever  $\mathbf{x}_1 \leq \mathbf{x}_2$ . It is easy to verify that, under the assumptions on  $h$ , the function  $g(i, \mathbf{t})$  defined above is  $\text{MTP}_2$  in  $(i, \mathbf{t})$ , while the function  $\mathbf{1}_{[\mathbf{x}, \mathbf{c}]}(\mathbf{t})$  is clearly  $\text{MTP}_2$  in  $(\mathbf{x}, \mathbf{t})$  for every fixed  $\mathbf{c}$ . Thus the multivariate version of the Basic Composition Formula can be applied (see Karlin and Rinott (1980) for details), obtaining that  $G(i, \mathbf{x})$  is  $\text{MTP}_2$  in  $(i, \mathbf{x})$ , i.e., that

$$\frac{G(1, \mathbf{x})}{G(2, \mathbf{x})} = \frac{\int_{x_1}^{c_1} \cdots \int_{x_n}^{c_n} h(\mathbf{t})d\mathbf{t}}{\prod_i (c_i - x_i)}$$

is decreasing in  $\mathbf{x}$ . □

The following result gives the relationship between the convexity of  $S_{\mathbf{X}}^+(\mathbf{u})$  given in Definition 3.1.6 and the Q-CMIFRL property. This result can be considered as a generalization of Corollary 3.1 (a) in Fernández-Ponce *et al.* (1998), where the authors proved that a univariate random variable  $X$  is IFR if, and only if, the univariate associated excess-wealth function is a convex function.

**Corollary 3.1.11.** *Let  $\mathbf{X}$  be a random vector satisfying the regularity conditions (RC).  $\mathbf{X}$  is Q-CMIFRL if and only if  $S_{\mathbf{X}}^+(\mathbf{u})$  is a directionally convex function.*

**Proof.** Trivial by using (3.1.6) and Proposition 3.1.9. □

The next corollary is immediately obtained since the directional convexity implies the componentwise convexity.

**Corollary 3.1.12.** *Let  $\mathbf{X}$  be a random vector satisfying the regularity conditions (RC). If  $\mathbf{X}$  is Q-CMIFR then  $S_{\mathbf{X}}^+(\mathbf{u})$  is a componentwise convex function.*

The relationship between the Q-CMDMRL property and the  $S_{\mathbf{X}}^+(\mathbf{u})$  function is given in the following results.

**Theorem 3.1.13.**  *$\mathbf{X}$  is Q-CMDMRL if and only if  $\frac{S_{\mathbf{X}}^+(\mathbf{u})}{\prod_i(1-u_i)}$  is decreasing in  $\mathbf{u}$ .*

**Proof.** The results is obtained from equality  $P[\mathbf{X} \in R_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))] = \prod_i(1-u_i)$  and equation (3.1.2). □

For the next theorems, we need to give some previous definitions and results about different sorts of convexity from a point  $\mathbf{x}_0$ .

**Definition 3.1.7.** Let  $f$  be a  $\mathbb{R}$ -valued function on  $\mathbb{R}^n$ . It is said to be an  $n$ -convex function from  $\mathbf{x}_0 \in \mathbb{R}^n$  if for any finite collection of different points  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  with  $\mathbf{x}_i \leq \mathbf{x}_{i+1}, i = 1, \dots, n$  and  $\mathbf{x}_{n+1} = \mathbf{x}_0$ , it holds that

$$f\left(\sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{n+1} \lambda_i f(\mathbf{x}_i) \text{ for } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{n+1} \lambda_i = 1$$

**Definition 3.1.8.** Let  $f$  be a  $\mathbb{R}$ -valued function on  $\mathbb{R}^n$ . It is said to be a convex function from  $\mathbf{x}_0 \in \mathbb{R}^n$  if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{x}_0) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{x}_0)$$

when  $\mathbf{x} \leq \mathbf{x}_0$  and for all  $\lambda \in [0, 1]$ .

**Remark 3.1.2.** *Note that there is no relation between the Definition 3.1.7 and the Definition 3.1.8. In fact, in Definition 3.1.7, the convexity from a point  $\mathbf{x}_0$  is defined for a point  $\mathbf{x}$ , while the  $n$ -convex given in Definition 3.1.8 is defined for a collection of  $n$  different points.*

The following result shows that the convexity from a point  $\mathbf{x}_0$  is closed under increasing and convex transformations.

**Proposition 3.1.14.** *Let  $f$  be a  $\mathbb{R}$ -valued function on  $\mathbb{R}^n$  and let  $\varphi$  be an  $\mathbb{R}$ -valued function on  $\mathbb{R}$ . If  $f$  is a convex function from  $\mathbf{x}_0$  and  $\varphi$  is an increasing and convex function, then  $\varphi \circ f$  is a convex function from  $\mathbf{x}_0$ .*

**Proof.** By using the fact that  $f$  is a convex function from  $\mathbf{x}_0$  and  $\varphi$  is an increasing function, it holds

$$\begin{aligned} (\varphi \circ f)(\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}_0) &= \varphi [f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}_0)] \\ &\leq \varphi [\lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{x}_0)]. \end{aligned}$$

Thus, by using the fact that  $\varphi$  is a convex function, we obtain

$$(\varphi \circ f)(\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}_0) \leq \lambda[\varphi \circ f(\mathbf{x})] + (1 - \lambda)[\varphi \circ f(\mathbf{x}_0)].$$

Consequently,  $(\varphi \circ f)$  is a convex function from  $\mathbf{x}_0$ . □

**Example 3.1.15.** *Let  $\phi : (0, 1)^2 \mapsto (0, 1)$  be the function such that for  $(u_1, u_2) \in (0, 1)^2$ , we have  $\phi(u_1, u_2) = (1 - u_1)^{k_1}(1 - u_2)^{k_2}$ . It is easily shown that  $\phi(\cdot)$  is convex from  $\mathbf{e} = (1, 1)$  if, and only if,  $k_1 + k_2 \geq 2$  and  $k_i > 0$  for  $i = 1, 2$ .*

The relationship between the Q-CMDMRL properties and the convexity of  $S_{\mathbf{X}}^+$  is stated in the following results.

**Theorem 3.1.16.** *Let  $\mathbf{X}$  be a random vector verifying the regularity conditions (RC). If  $S_{\mathbf{X}}^+(\mathbf{u})$  is a  $n$ -convex function from  $\mathbf{e}$  then  $\mathbf{X}$  has a  $Q$ -CMDMRL distribution.*

**Proof.** Let  $\mathbf{u} = (u_1, u_2) \leq \mathbf{v} = (v_1, v_2)$ . Define  $\Delta\mathbf{u} = (1 - u_1)(1 - u_2)$ ,  $\Delta\mathbf{v} = (1 - v_1)(1 - v_2)$ ,  $\Delta\mathbf{v} \leq \Delta\mathbf{u}$ . Let  $\mathbf{w}$  be such that  $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + (\Delta\mathbf{u} - \Delta\mathbf{v})\mathbf{e}$ , where

$$\alpha = \min \left\{ \frac{(\Delta\mathbf{u} - \Delta\mathbf{v})(1 - v_2)}{v_2 - u_2}, \Delta\mathbf{v}, \frac{(\Delta\mathbf{u} - \Delta\mathbf{v})(1 - v_1)}{v_1 - u_1} \right\}. \quad (3.1.18)$$

Let see that  $\mathbf{w} \leq \mathbf{v}$ . In fact,

$$\begin{aligned} \alpha\mathbf{u} + \beta\mathbf{v} + (\Delta\mathbf{u} - \Delta\mathbf{v})\mathbf{e} &\leq \mathbf{v} \Leftrightarrow \\ \alpha\mathbf{u} + (\Delta\mathbf{u} - \Delta\mathbf{v})\mathbf{e} &\leq (1 - \beta)\mathbf{v} \Leftrightarrow \\ \frac{\alpha\mathbf{u}}{\alpha + (\Delta\mathbf{u} - \Delta\mathbf{v})} + \frac{(\Delta\mathbf{u} - \Delta\mathbf{v})\mathbf{e}}{\alpha + (\Delta\mathbf{u} - \Delta\mathbf{v})} &\leq \mathbf{v} \end{aligned}$$

But, the last inequality is true since  $\alpha$  is chosen as in (3.1.18). Therefore, from  $S_{\mathbf{X}}^+(\mathbf{u})$  is a decreasing function,  $S_{\mathbf{X}}^+(\mathbf{w}) \geq S_{\mathbf{X}}^+(\mathbf{v})$  and, in light of the fact that  $S_{\mathbf{X}}^+(\cdot)$  is  $n$ -convex from  $\mathbf{e}$ ,

$$\begin{aligned} S_{\mathbf{X}}^+(\mathbf{w}) &\leq \alpha S_{\mathbf{X}}^+(\mathbf{u}) + \beta S_{\mathbf{X}}^+(\mathbf{v}) \Leftrightarrow \\ (1 - \beta)S_{\mathbf{X}}^+(\mathbf{v}) &\leq \alpha S_{\mathbf{X}}^+(\mathbf{u}) \Leftrightarrow \\ S_{\mathbf{X}}^+(\mathbf{v}) &\leq \frac{\alpha}{\alpha + \Delta\mathbf{u} - \Delta\mathbf{v}} S_{\mathbf{X}}^+(\mathbf{u}) \leq \frac{\Delta\mathbf{v}}{\Delta\mathbf{u}} S_{\mathbf{X}}^+(\mathbf{u}) \end{aligned} \quad (3.1.19)$$

Therefore, from (3.1.19) and (3.1.2), it follows that  $\bar{\mu}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{v})] \leq \bar{\mu}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})]$ , for all  $\mathbf{u} \leq \mathbf{v}$ , that is,  $\mathbf{X}$  has a  $Q$ -CMDMRL distribution.  $\square$

**Theorem 3.1.17.** *Let  $\mathbf{X}$  be a random vector verifying the regularity conditions (RC). If  $\mathbf{X}$  has a  $Q$ -CMDMRL distribution then  $S_{\mathbf{X}}^+(\mathbf{u})$  is a convex function from  $\mathbf{e}$ .*

**Proof.** From (3.1.2),  $\mathbf{X}$  has a Q-CMDRML distribution if and only if

$$\frac{S_{\mathbf{X}}^+(\mathbf{u})}{P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]\}}$$

is a decreasing function in  $\mathbf{u}$ . Therefore, for all  $\lambda \in [0, 1]$  and  $\mathbf{u}^* = \lambda\mathbf{u} + (1 - \lambda)\mathbf{e} > \mathbf{u}$ , it holds

$$\frac{S_{\mathbf{X}}^+(\mathbf{u}^*)}{P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}^*)]\}} \leq \frac{S_{\mathbf{X}}^+(\mathbf{u})}{P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]\}}. \quad (3.1.20)$$

Using Proposition 2.2.2 it is clearly shown that

$$P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}^*)]\} = \lambda^n \prod_{j=1}^n (1 - u_j)$$

or equivalently,

$$\frac{P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u}^*)]\}}{P\{\mathbf{X} \in R_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})]\}} = \lambda^n.$$

Thus, the inequality (3.1.20) holds if and only if  $S_{\mathbf{X}}^+(\mathbf{u}^*) \leq \lambda^n S_{\mathbf{X}}^+(\mathbf{u})$ . Being  $\lambda^n S_{\mathbf{X}}^+(\mathbf{u}) \leq \lambda S_{\mathbf{X}}^+(\mathbf{u}) + (1 - \lambda)S_{\mathbf{X}}^+(\mathbf{e})$ , it follows that  $S_{\mathbf{X}}^+(\mathbf{u})$  is a convex function from  $\mathbf{e}$ .  $\square$

Finally the same result in Theorem 3.1.8 is given for the Q-CMNBUE property.

**Theorem 3.1.18.**  $\mathbf{X}$  is Q-CMNBUE if and only if  $\frac{S_{\mathbf{X}}^+(\mathbf{u})}{\prod_i (1 - u_i)} \leq S_{\mathbf{X}}^+(\mathbf{0})$  for all  $\mathbf{u} \geq \mathbf{0}$ .

**Proof.** It is enough to observe that for all  $\mathbf{u} \in [0, 1]^n$  it is  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] = \frac{S_{\mathbf{X}}^+(\mathbf{u})}{\prod_i (1 - u_i)}$ , and that, in particular, it holds  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})] = S_{\mathbf{X}}^+(\mathbf{0})$ . The assertion now follows from definition above of Q-CMNBUE property.  $\square$

**Remark 3.1.3.** *The previous results can be held for the notions of CMIFR, CMDMRL and CMNBUE if additional conditions are considered. For the different results, these conditions are the following:*



- i) The sufficient condition in Corollary 3.1.11 is verified if Q-CMIFR is replaced for the CMIFR property whenever  $\mathbf{X}$  is CIS. However, the necessary condition is held for the property CMIFR whenever  $\mathbf{X}$  is CDS. Because of this, the Corollary 3.1.12 holds whenever  $\mathbf{X}$  is CIS.*
- ii) The sufficient condition in Theorem 3.1.13 is verified if Q-CMDMRL is replaced for the CMDMRL property whenever  $\mathbf{X}$  is CIS. However, the necessary condition is held for the property CMDMRL whenever  $\mathbf{X}$  is CDS.*
- iii) Theorem 3.1.16 is verified if Q-CMDMRL is replaced for the CMDMRL property whenever  $\mathbf{X}$  is CDS.*
- iV) Theorem 3.1.17 is verified if Q-CMIFR is replaced for the CMIFR property whenever  $\mathbf{X}$  is CIS.*

### 3.1.1 The corrected hazard gradient

As was seen in Section 1.1, the univariate failure rate can be defined in terms of a derivative and in terms of a limit. This failure rate was extended to the bivariate case by using different methods. The first, uses the bivariate failure rate function defined by Basu (1971) by

$$\begin{aligned} r(x_1, x_2) &= \lim_{h_1 \rightarrow 0, h_2 \rightarrow 0} \frac{P\{x_1 < X_1 \leq x_1 + h_1; x_2 < X_2 \leq x_2 + h_2 | X_1 > x_1; X_2 > x_2\}}{h_1 h_2} \\ &= \frac{f(x_1, x_2)}{\bar{F}(x_1, x_2)}. \end{aligned}$$

The second option uses the hazard gradient defined by Johnson and Krotz (1975) by

$$h(x_1, x_2) = [h_1(x_1, x_2), h_2(x_1, x_2)]$$

where

$$h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \ln \bar{F}(x_1, x_2) \text{ for } i = 1, 2.$$

In Definition 3.1.2, the corrected multivariate failure rate is considered in a similar way to the bivariate failure rate defined by Basu (1971) and some results are given. However, in this subsection, a new definition of corrected hazard rate, similar of the hazard rate defined by Johnson and Kotz (1975) is given.

Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be a non-negative random vector with a partial differentiable survival function  $\bar{F}$ , that is, the partial derivatives  $\frac{\partial \bar{F}(x_1, \dots, x_n)}{\partial x_i}$  exists for all  $i = 1, \dots, n$ . The function  $R(x_1, \dots, x_n) = -\ln\{P[\mathbf{X} \in R_{\mathbf{X}}(x_1, \dots, x_n)]\}$  is called corrected hazard function of  $\mathbf{X}$  and the corrected hazard gradient of  $\mathbf{X}$  is defined by the following vector of partial derivatives

$$h_c(\mathbf{x}) = (R_1(\mathbf{x}), \dots, R_n(\mathbf{x}))$$

where  $R_i(\mathbf{x}) = R_i(x_1, \dots, x_n) = \frac{\partial R(x_1, \dots, x_n)}{\partial x_i}$  for  $i = 1, 2, \dots, n$ .

For simplicity, we only work in the bivariate case. All results can easily be generalized for any dimension. From now on, we consider  $\mathbf{X} = (X_1, X_2)$  a bivariate random vector with corrected hazard function given by

$$\begin{aligned} R(x_1, x_2) &= -\ln[P(X_1 > x_1)P(X_2 > x_2|X_1 = x_1)] \\ &= -\ln P(X_1 > x_1) - \ln P(X_2 > x_2|X_1 = x_1). \end{aligned}$$

The corrected hazard gradient has the following expression:

$$\begin{aligned} R_1(x_1, x_2) &= \frac{\partial(-\ln P(X_1 > x_1))}{\partial x_1} - \frac{\partial \ln P(X_2 > x_2 | X_1 = x_1)}{\partial x_1} \\ &= r_1(x_1) - \hat{r}(x_2 | x_1), \end{aligned}$$

$$\begin{aligned} R_2(x_1, x_2) &= \frac{\partial -\ln P(X_2 > x_2 | X_1 = x_1)}{\partial x_2} \\ &= r(x_2 | x_1) \end{aligned}$$

where  $r_1(x_1)$  and  $r(x_2 | x_1)$  are the hazard rate of  $X_1$  and  $X_2 | X_1 = x_1$ , respectively and  $\hat{r}(x_2 | x_1) = \frac{\partial \ln P(X_2 > x_2 | X_1 = x_1)}{\partial x_1}$ .

Considering this generalization of the hazard rate function, we extend the univariate IFR property following the idea in Johnson and Kotz (1975).

**Definition 3.1.9.** Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative bivariate random vector with corrected hazard gradient  $h_c(\mathbf{x}) = (R_1(\mathbf{x}), R_2(\mathbf{x}))$ . The vector  $\mathbf{X}$  is said to have a Corrected Bivariate Increasing Hazard Rate (CBIHR) distribution if  $R_i(\mathbf{x})$  is increasing in  $x_i$  for all  $\mathbf{x} = (x_1, x_2)$  and each  $i$ .

**Example 3.1.19.** Let  $\mathbf{X}$  be the bivariate random vector given in Example 2.2.1. If  $\mathbf{x} \in A$ , the corrected hazard gradient  $h_c(\mathbf{x}) = (R_1(\mathbf{x}), R_2(\mathbf{x}))$  is defined as

$$R_i(\mathbf{x}) = \frac{1}{1 - x_i}, \text{ for } i = 1, 2$$

and if  $\mathbf{x} \in B$ , then

$$R_1(\mathbf{x}) = \frac{x_2}{(2 - x_1 - x_2)(2 - x_1)} - 1 \text{ and } R_2(\mathbf{x}) = \frac{1}{2 - x_1 - x_2}.$$

After straightforward calculus, it can be proved that  $R_i, i = 1, 2$  is increasing in  $x_i$  for all  $\mathbf{x} = (x_1, x_2)$  and each  $\mathbf{x}_i$ , that is,  $\mathbf{X}$  has a CBIHR distribution.  $\triangle$

Similarly to the corrected failure rate function, it can be proved that the only bivariate random vector with exponential marginal and a constant corrected hazard rate gradient is that with independent components.

**Theorem 3.1.20.** *Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative random vector. Then  $\mathbf{X}$  has a constant corrected hazard rate gradient if and only if  $X_1$  and  $X_2$  are independent and exponential.*

**Proof.** Obviously, if  $X_1$  and  $X_2$  are independent and have exponential distributions, it holds that  $h_i(x_1, x_2)$  is constant for  $i = 1, 2$ .

Now, assume that  $h_i(x_1, x_2) = c$ . Then

$$\frac{\partial \ln P[\mathbf{X} \in R_{\mathbf{X}}(x_1, x_2)]}{\partial x_i} = -c_i \text{ for } i = 1, 2,$$

or equivalently,

$$P[\mathbf{X} \in R_{\mathbf{X}}(x_1, x_2)] = e^{-c_1 x_1} g(x_2)$$

$$P[\mathbf{X} \in R_{\mathbf{X}}(x_1, x_2)] = e^{-c_2 x_2} g(x_1)$$

This implies that

$$P[\mathbf{X} \in R_{\mathbf{X}}(x_1, x_2)] \sim e^{-c_1 x_1 - c_2 x_2}.$$

Moreover, given that  $P[\mathbf{X} \in R_{\mathbf{X}}(x_1, x_2)] = \bar{F}_1(x_1) \bar{F}_{X_2|X_1=x_1}(x_2)$  it follows that  $X_1$  and  $X_2$  are independently and exponentially distributed.  $\square$

In order to see the relationship between the CBIHR property and the CBIFR and CBDMRL properties, we provide some definitions and results of a generalization of  $n$ -convex functions.

**Definition 3.1.10.** Let  $u_0, u_1, \dots, u_n$  be of class  $C^n[a, b]$ . Then  $\{u_i\}_{i=0}^n$  is an Extended Completed Tchebycheff (ECT) system on  $[a, b]$  if and only if for  $k = 0, 1, \dots, n$  we have  $W(u_0, u_1, \dots, u_k) > 0$  on  $[a, b]$ , where  $W(u_0, u_1, \dots, u_k)$  denotes the Wronskian of the functions  $u_0, u_1, \dots, u_k$ , i.e.

$$W(u_0, u_1, \dots, u_n) = \begin{vmatrix} u_0 & u_0' & \dots & u_0^{(k)} \\ u_1 & u_1' & \dots & u_1^{(k)} \\ \vdots & \vdots & \dots & \vdots \\ u_k & u_k' & \dots & u_k^{(k)} \end{vmatrix}$$

**Example 3.1.21.** Let  $u_i(t) = t^i, i = 0, 1, \dots, n$ . Then  $\{u_i(t)\}_{i=0}^n$  is an ECT system on  $[a, b]$ . It is sufficient to note that for all  $k = 0, 1, \dots, n$ , the Wronskian  $W(u_0, u_1, \dots, u_k)$  is the determinant of a lower triangular matrix with nonnegative diagonal elements. △

**Definition 3.1.11.** Let  $\bar{F}(x|\theta)$  be a survival function which depends on a parameter  $\theta \in \Theta \subseteq \mathbb{R}^+$ . It is said that  $\bar{F}(x|\theta)$  is a Wronskian survival function with respect to  $\theta$  if

- (i) the system of function  $\{\bar{F}(x|\theta); \frac{\partial \bar{F}(x|\theta)}{\partial \theta}\}$  is an ECT system, and
- (ii) the system of function  $\{\bar{F}(x|\theta); \frac{-\partial \bar{F}(x|\theta)}{\partial x}\}$  is an ECT system.

**Proposition 3.1.22.**  $\bar{F}(x|\theta)$  is a Wronskian survival function with respect to  $\theta$  if, and only if,

- (i)  $\ln \bar{F}(x|\theta)$  is a convex function in  $\theta$ .
- (ii) The function  $r(x|\theta)$  is a nondecreasing function in  $\theta$ .

**Proof.**

- (i) From Definition 3.1.22,  $\bar{F}(x|\theta)$  is a Wronskian survival function with respect to  $\theta$  if, and only if,  $W(\{\bar{F}(x|\theta); \frac{\partial}{\partial\theta}\bar{F}(x|\theta)\}) > 0$ .

Now, by taking  $\varphi(x, \theta) = \ln\bar{F}(x|\theta)$ , it is obtained that

$$\frac{\partial\varphi(x, \theta)}{\partial\theta} = \frac{\frac{\partial\bar{F}(x|\theta)}{\partial\theta}}{\bar{F}(x|\theta)}$$

and

$$\begin{aligned} \frac{\partial^2\varphi(x, \theta)}{\partial\theta^2} &= \frac{\bar{F}(x|\theta)\frac{\partial^2\bar{F}(x|\theta)}{\partial\theta^2} - \left(\frac{\partial\bar{F}(x|\theta)}{\partial\theta}\right)^2}{[\bar{F}(x|\theta)]^2} \\ &= \frac{W(\{\bar{F}(x|\theta); \frac{\partial}{\partial\theta}\bar{F}(x|\theta)\})}{[\bar{F}(x|\theta)]^2} > 0 \end{aligned}$$

obtaining the result.

- (ii) The proof is similar to (i) and therefore is omitted.

□

**Example 3.1.23.** Let  $X_1 \rightsquigarrow U[0, 1]$  and  $X_2|X_1 = t_1 \rightsquigarrow U[t_1, 1]$ . Then,  $\bar{F}_{X_2|X_1=t_1}$  is a Wronskian survival function with respect to  $X_1 = t_1$ . In fact, it is easy to prove that

$$\ln P(X_2 > t_2 | X_1 = t_1) = \ln \frac{1-t_2}{1-t_1} \text{ is convex in } t_1$$

and

$$r(t_2|t_1) = \frac{1/(1-t_1)}{(1-t_2)/(1-t_1)} = \frac{1}{1-t_2} \text{ for all } t_2 > t_1 \text{ is constant in } t_1.$$

Therefore, from Proposition 3.1.22 the assertion is obtained.

△

A property of the Wronskian functions is that if  $X_1, \dots, X_n$  are independent and identically distributed random variables with Wronskian survival function, then the

minimum of this set also has a Wronskian survival function. It can be shown in the next proposition.

**Proposition 3.1.24.** *Let  $X_1, \dots, X_n$  be independent and identically distributed random variables with Wronskian survival function  $\bar{F}(x|\theta)$ . Then,  $Y = \min\{X_1, \dots, X_n\}$  has a Wronskian survival function.*

**Proof.** Let  $Y = \min\{X_1, \dots, X_n\}$ . The functions  $\ln \bar{F}_Y(y|\theta)$  and  $r(y|\theta)$  are given, respectively, by

$$\begin{aligned} \ln \bar{F}_Y(y|\theta) &= \ln[\bar{F}(y|\theta)]^n \\ &= n \ln \bar{F}(y|\theta) \end{aligned}$$

and

$$\begin{aligned} r(y|\theta) &= n \frac{[\bar{F}(y|\theta)]^{n-1}}{[\bar{F}(y|\theta)]^n} f(y|\theta) \\ &= n \frac{f(y|\theta)}{\bar{F}(y|\theta)} \\ &= nr(y|\theta). \end{aligned}$$

Given that  $\bar{F}(x|\theta)$  is a Wronskian survival function with respect to  $\theta$ , it is easily showed that  $\ln \bar{F}_Y(y|\theta)$  is a convex function and  $r(y|\theta)$  is a nondecreasing function in  $\theta$ . Therefore, from Proposition 3.1.22 it follows that  $Y$  has a Wronskian survival function . □

For the next statement, we recall the definition of an MTP2 function.

**Definition 3.1.12.** A function  $f(\mathbf{x})$  defined on  $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \dots \times \mathcal{R}_n$  where each  $\mathcal{R}_i$  is totally ordered, is said to be Multivariate Totally Positive of order 2 (MTP2) if

$$f(\mathbf{x} \vee \mathbf{y})f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x})f(\mathbf{y})$$

where, for all  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$

$$\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \max(x_2, y_2) \dots \max(x_n, y_n))$$

and

$$\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \min(x_2, y_2) \dots (x_n, y_n)).$$

Note that the above definition is equivalent to saying that the determinant of the matrix

$$\begin{pmatrix} f(\mathbf{x}, \mathbf{y}) & f(\mathbf{x}, \mathbf{y}') \\ f(\mathbf{x}', \mathbf{y}) & f(\mathbf{x}', \mathbf{y}') \end{pmatrix}$$

is nonnegative for all choices  $\mathbf{x} < \mathbf{x}'$  and  $\mathbf{y} < \mathbf{y}'$ .

One of the operations that preserve the MTP2 property is the composition formula (see for example Karlin and Rinott, 1980). That is, supposing that  $f(\mathbf{x}, \mathbf{y})$  is MTP2 in  $\mathcal{R}_1 \times \mathcal{R}_2$  and  $g(\mathbf{y}, \mathbf{z})$  is MTP2 over  $\mathcal{R}_2 \times \mathcal{R}_3$ , where  $\mathcal{R}_1, \mathcal{R}_2$  and  $\mathcal{R}_3$  are subsets of possibly different Euclidean spaces, then

$$h(\mathbf{x}, \mathbf{z}) = \int f(\mathbf{x}, \mathbf{y})g(\mathbf{y}, \mathbf{z})d\mathbf{z} \text{ is MTP2 over } \mathcal{R}_1 \times \mathcal{R}_3.$$

Now, we can give the following relationship between the BIHR property and the aging properties considered in Definition 3.1.3

**Proposition 3.1.25.** *Let  $\mathbf{X} = (X_1, X_2)$  be a nonnegative bidimensional random vector such that*

(i)  $\mathbf{X}$  is CDS

(ii) The survival function of  $X_2|X_1 = t_1$  is Wronskian respect to  $t_1$ , for all  $t_1 \in \mathbb{R}^+$ .

If  $\mathbf{X}$  is CBIHR, then  $\mathbf{X}$  is CBDMRL.



**Proof.** If  $\mathbf{X}$  is CBIHR, by Definition 3.1.9 we know that

$$R_1(t_1, t_2) = r_1(t_1) - \frac{\partial}{\partial t_1} \ln P(X_2 > t_2 | X_1 = t_1) \quad \text{and} \quad R_2(t_1, t_2) = r(t_2 | t_1)$$

are nondecreasing functions in  $t_1$  and  $t_2$ , respectively. By using (ii) and from Proposition 3.1.22, it is immediately obtained that  $r_1(t_1)$  is a nondecreasing function. Therefore, the corrected bivariate failure rate function

$$r_c(t_1, t_2) = \frac{f(t_1, t_2)}{P[\mathbf{X} \in R(t_1, t_2)]} = r_1(t_1) \cdot r(t_2 | t_1)$$

is a nondecreasing function in  $(t_1, t_2)$ , that is,  $\mathbf{X}$  is *CBIFR*.

Now, we define the following function:

$$h(i, \mathbf{y}) = \begin{cases} f(\mathbf{y}) & \text{if } i = 2 \\ P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{y})] & \text{if } i = 1 \end{cases}, \quad g(\mathbf{y}, \mathbf{x}) = \mathbf{1}_{\{\mathbf{y} \in R_{\mathbf{x}}(\mathbf{x})\}}.$$

It holds that

$$\frac{h(1, \mathbf{y})}{h(2, \mathbf{y})} \geq \frac{h(1, \mathbf{y}')}{h(2, \mathbf{y}')} \quad \text{with } \mathbf{y} \leq \mathbf{y}' \text{ since } r_c(\mathbf{y}) \text{ is increasing in } \mathbf{y}.$$

Thus  $h(i, \mathbf{y})$  is MTP2 in  $(i, \mathbf{y})$ .

On the other hand, since  $\mathbf{X}$  is CDS and from Proposition 2.3.8, if  $\mathbf{x}_1 \leq \mathbf{x}_2$  and  $\mathbf{y}_1 \leq \mathbf{y}_2$ , it follows that  $R_{\mathbf{x}}(\mathbf{x}_2) \subseteq R_{\mathbf{X}}(\mathbf{x}_1)$  and  $R_{\mathbf{x}}(\mathbf{y}_2) \subseteq R_{\mathbf{x}}(\mathbf{y}_1)$ . Therefore, it holds

$$\mathbf{1}_{\{\mathbf{y}_1 \in R_{\mathbf{x}}(\mathbf{x}_1)\}} \mathbf{1}_{\{\mathbf{y}_2 \in R_{\mathbf{x}}(\mathbf{x}_2)\}} \geq \mathbf{1}_{\{\mathbf{y}_1 \in R_{\mathbf{x}}(\mathbf{x}_2)\}} \mathbf{1}_{\{\mathbf{y}_2 \in R_{\mathbf{x}}(\mathbf{x}_1)\}}.$$

Thus,  $g(\mathbf{y}, \mathbf{x})$  is MTP2.

Now, by applying the composition formula it follows that

$$H(i, \mathbf{x}) = \int h(i, \mathbf{y}) g(\mathbf{y}, \mathbf{x}) d\mathbf{y} = \int_{R_{\mathbf{x}}(\mathbf{x})} h(i, \mathbf{y}) d\mathbf{y}$$

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is MTP2 for  $i = 1, 2$ . Therefore

$$\begin{aligned} \frac{H(1, \mathbf{x})}{H(2, \mathbf{x})} &= \frac{\int_{R_{\mathbf{x}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{y})] d\mathbf{y}}{\int_{R_{\mathbf{x}}(\mathbf{x})} f(\mathbf{y}) d\mathbf{y}} \\ &= \frac{\int_{R_{\mathbf{x}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{y})] d\mathbf{z}}{P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{x})]} \end{aligned}$$

is decreasing in  $\mathbf{x}$ . Thus, the assertion is obtained.  $\square$

**Remark 3.1.4.** *Note that we have proved in the previous proposition that if  $\mathbf{X}$  has a CBIHR distribution then  $\mathbf{X}$  also has a CBIFR distribution, whenever  $X_2|X_1 = t_1$  is Wronskian with respect to  $t_1$ . Moreover, if  $\mathbf{X}$  is CDS and has a CBIHR distribution, then  $\mathbf{X}$  has a CBDMRL distribution. In the next section, we give other relationships among the lifetime property for the multivariate case.*

## 3.2 Relations between different multivariate lifetime distributions

Similarly to the univariate case, the same relationship between the multivariate lifetime distributions can be given. Here, we state the relationship between the lifetime distributions given in the above section.

**Proposition 3.2.1.** *If  $\mathbf{X}$  is CDS and has a CMIFR distribution, then  $\mathbf{X}$  has a CMDMRL distribution.*

**Proof.** It should be shown that  $\bar{\mu}_{\mathbf{X}}(\mathbf{x})$  is decreasing in  $\mathbf{x}$  for all  $\mathbf{x} \in \text{Supp}(\mathbf{X})$  or equivalently,  $\frac{\int_{R_{\mathbf{X}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{z})] d\mathbf{z}}{P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{x})]}$  is decreasing in  $\mathbf{x}$ .

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Let  $h(i, \mathbf{z})$  and  $g(\mathbf{z}, \mathbf{x})$  be two functions defined by

$$h(i, \mathbf{z}) = \begin{cases} f(\mathbf{z}) & \text{if } i = 2 \\ P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{z})] & \text{if } i = 1 \end{cases}, \quad g(\mathbf{z}, \mathbf{x}) = \mathbf{1}_{\{\mathbf{z} \in R_{\mathbf{x}}(\mathbf{x})\}}.$$

It is easy to prove that  $h(i, \mathbf{z})$  is MTP2 in the sense of the Definition 3.1.12 because  $\mathbf{X}$  is CMIFR, i.e., because  $r_c(\mathbf{z}) = \frac{f_{\mathbf{X}}(\mathbf{z})}{P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{z})]}$  is increasing in  $\mathbf{z}$ . Moreover, given that  $X$  is CDS and by the Proposition 2.3.8, it holds that  $g(\mathbf{z}, \mathbf{x})$  is also MTP2. Now, by applying the composition formula it follows that

$$H(i, \mathbf{x}) = \int h(i, \mathbf{z})g(\mathbf{z}, \mathbf{x})d\mathbf{z} = \int_{R_{\mathbf{x}}(\mathbf{x})} h(i, \mathbf{z})d\mathbf{z}$$

is MTP2, and therefore

$$\begin{aligned} \frac{H(1, \mathbf{x})}{H(2, \mathbf{x})} &= \frac{\int_{R_{\mathbf{x}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{z})]d\mathbf{z}}{\int_{R_{\mathbf{x}}(\mathbf{x})} f(\mathbf{z})d\mathbf{z}} \\ &= \frac{\int_{R_{\mathbf{x}}(\mathbf{x})} P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{z})]d\mathbf{z}}{P[\mathbf{X} \in R_{\mathbf{x}}(\mathbf{x})]} \end{aligned}$$

is decreasing in  $\mathbf{x}$ . Thus, the assertion is obtained. □

**Proposition 3.2.2.** *If  $\mathbf{X}$  has a CMDMRL distribution, then  $\mathbf{X}$  has a CMNBUE distribution.*

**Proof.** The assertion follows immediately from the definitions of the two classes.

□

**Proposition 3.2.3.** *If  $\mathbf{X}$  has a Q-CMIFR distribution, then  $\mathbf{X}$  has a Q-CMDMRL distribution.*

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**Proof.** From Definition 3.1.4,  $\mathbf{X}$  is Q-CMDMRL if the function

$$\bar{\mu}_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{u})) = \frac{S_{\mathbf{X}}^+(\mathbf{u})}{\prod_{j=1}^n (1 - u_j)} \text{ is decreasing in } \mathbf{u} \text{ for all } \mathbf{u} \in (0, 1)^n.$$

As was seen in (3.1.6),  $S_{\mathbf{X}}^+(\mathbf{u})$  can be written as  $S_{\mathbf{X}}^+(\mathbf{u}) = \int_{u_1}^1 \cdots \int_{u_n}^1 \{r_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{z})]\}^{-1} d\mathbf{z}$ .

Since  $X$  is Q-CMIFR,  $[r_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{z}))]^{-1}$  is decreasing in  $\mathbf{z}$  for  $\mathbf{z} \in [0, 1]^n$  and from Proposition 3.1.10

$$\frac{S_{\mathbf{X}}^+(u)}{\prod_{j=1}^n (1 - u_j)} = \frac{\int_{u_1}^1 \cdots \int_{u_n}^1 [r_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{z}))]^{-1} d\mathbf{z}}{\prod_{j=1}^n (1 - u_j)}$$

is also decreasing in  $\mathbf{u}$ . Therefore, the assertion is obtained.  $\square$

**Proposition 3.2.4.** *If  $\mathbf{X}$  has a Q-CMDMRL distribution, then  $\mathbf{X}$  has a Q-CMNBUE distribution.*

**Proof.** Immediate and therefore it is omitted.  $\square$

Finally, some relationships among the lifetime distribution given in Definition 3.1.3 and 3.1.4 are stated.

**Proposition 3.2.5.** *The following assertions hold*

- i) If  $\mathbf{X}$  is CIS and has a CMIFR distribution, then  $\mathbf{X}$  has a Q-CMIFR distribution.*
- ii) If  $\mathbf{X}$  is CDS and has a Q-CMIFR distribution, then  $\mathbf{X}$  has a CMIFR distribution.*

**Proof.**

- (i)* Let  $\mathbf{u}$  and  $\mathbf{v}$  be such that  $\mathbf{u} \leq \mathbf{v}$ . Since  $\mathbf{X}$  is CIS, then  $\widehat{\mathbf{x}}(\mathbf{u}) \leq \widehat{\mathbf{x}}(\mathbf{v})$ . By using that  $\mathbf{X}$  is CMIFR, it follows that  $r_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] \leq r_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{v})]$  for all  $\mathbf{u} \leq \mathbf{v}$ . Thus,  $\mathbf{X}$  is Q-CMIFR.

(ii) The proof of this part is based on the same lines.

□

**Proposition 3.2.6.** *The following assertions hold*

- i) *If  $\mathbf{X}$  is CIS and has a CMDMRL distribution, then  $\mathbf{X}$  has a Q-CMDMRL distribution.*
- ii) *If  $\mathbf{X}$  is CDS and has a Q-CMDMRL distribution, then  $\mathbf{X}$  has a CMDMRL distribution.*

**Proof.**

- i) Assume that  $\mathbf{X}$  has a CMDMRL distribution. Let  $\mathbf{u}$  and  $\mathbf{v}$  be two vectors in  $[0, 1]^n$ , such that  $\mathbf{u} \leq \mathbf{v}$ . By using the fact that  $\mathbf{X}$  is CIS, it holds  $\widehat{\mathbf{x}}(\mathbf{u}) \leq \widehat{\mathbf{x}}(\mathbf{v})$ . Thus,  $\mu_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] \geq \mu_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{v})]$  for all  $\mathbf{u} \leq \mathbf{v}$ . Consequently,  $\mathbf{X}$  has a Q-CMDMRL distribution.
- ii) Let  $\mathbf{x}$  and  $\mathbf{y}$  be two points in  $Supp(\mathbf{X})$  such that  $\mathbf{x} \leq \mathbf{y}$ . There exist two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $[0, 1]^n$  such that  $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{u})$  and  $\mathbf{y} = \widehat{\mathbf{x}}(\mathbf{v})$ . Given that  $\mathbf{X}$  is CDS, applying Theorem 2.3.7, it holds that  $\mathbf{u} \leq \mathbf{v}$ . Moreover, from  $\mathbf{X}$  has a Q-CMDMRL distribution, it follow  $\mu_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] \geq \mu_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{v})]$ . Thus,  $\mathbf{X}$  has a CMDMRL distribution.

□

**Proposition 3.2.7.** *If  $\mathbf{X}$  has a Q-CMNBUE distribution , then  $\mathbf{X}$  has a CMNBUE distribution.*

**Proof.** If  $\mathbf{X}$  has a Q-CMNBUE distribution, then  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{u})] \leq \bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})]$  for all  $\mathbf{u} \in [0, 1]^n$ . Now, from the regularity conditions, for each  $\mathbf{u} \in [0, 1]^n$ , there exists only one point  $\mathbf{x} \in \text{Supp}(\mathbf{X})$  such that  $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{u})$  and given that  $\bar{\mu}_{\mathbf{X}}[\widehat{\mathbf{x}}(\mathbf{0})] = \bar{\mu}_{\mathbf{X}} \leq \nu_{\mathbf{X}}$ , it follows that  $\bar{\mu}_{\mathbf{X}}(\mathbf{x}) \leq \nu_{\mathbf{X}}$  for all  $\mathbf{x}$  and therefore, the result is obtained.  $\square$

**Remark 3.2.1.** Note that if the  $\text{Supp}(\mathbf{X}) = C(\mathbf{0})$ , then  $\bar{\mu}_{\mathbf{X}} = \nu_{\mathbf{X}}$  and, therefore, the Q-CMNBUE and CMNBUE properties are equivalent.

Consequently, the following relationship among the previous aging notions is held:

$$\begin{array}{ccccc}
 CMIFR & \xrightarrow{CDS} & CMDMRL & \longrightarrow & CMNBUE. \\
 \text{CDS} \uparrow \downarrow \text{CIS} & & \text{CDS} \uparrow \downarrow \text{CIS} & & \uparrow \\
 Q - CMIFR & \longrightarrow & Q - CMDMRL & \longrightarrow & Q - CMNBUE
 \end{array}$$

Note that the properties CIS and CDS assure that if  $\mathbf{u}, \mathbf{v} \in [0, 1]^n$  with  $\mathbf{u} \leq \mathbf{v}$ , then  $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{u}) \leq \widehat{\mathbf{x}}(\mathbf{v}) = \mathbf{y}$  with  $\mathbf{x}, \mathbf{y} \in \text{Supp}(\mathbf{X})$  and viceversa, respectively. These conditions are necessary given that the CMIFR, CMDMRL and CMNBUE properties are defined on  $\text{Supp}(\mathbf{X})$  and, however, the Q-CMIFR, Q-CMDMRL and Q-CMNBUE properties are defined on  $[0, 1]^n$ .

### 3.3 The aging properties for order statistics

As seen in Definition 2.2.1. the upper-corrected orthant depends on the ordering of the marginal distribution and, therefore, all properties or results given in terms of the upper-corrected orthant also depend on this ordering. This property is especially useful to study random vectors where the marginals have a natural order.

In light of this fact, it is interesting to study the multivariate aging properties of the vector of order statistics. Let  $X_1, X_2, \dots, X_n$  be independent random variable and let  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  denote the corresponding order statistics. Order statistics play an important role in statistic and reliability theory (see, for instance, David, 1970, Balakrishnan and Rao 1998a, 1998b). In reliability theory,  $k$ -out-of- $n$  systems functions if, and only if, at least  $k$  of its  $n$  components functions. That is, the distribution function of the lifetime of this system is the same as that of the  $(n - k + 1)$ st order statistic in a set of  $n$  nonnegative random variables. That shows that the study of  $k$ -out-of- $n$  systems is equivalent to the study of order statistics. In particular, the parallel and series systems are 1-out-of- $n$  and  $n$ -out-of- $n$  systems and their distribution functions are the same as that of  $X_{(1)}$  and  $X_{(n)}$ , respectively.

In this section, our proposal is to study the aging properties of the bivariate vector  $\mathbf{X} = (X_{(1)}, X_{(2)})$  where  $X_{(1)} = \min\{X_1, X_2\}$  and  $X_{(2)} = \max\{X_1, X_2\}$  when  $X_1$  and  $X_2$  are two independent and identically distributed (i.i.d) random variables with a common distribution function which possesses different univariate ageing properties. From now on, suppose that  $X_1$  and  $X_2$  are two i.i.d. random variables with a common distribution function  $F(\cdot)$ , survival function  $\bar{F}(\cdot)$ , failure rate  $r(\cdot)$  and excess-wealth function  $\mathcal{S}^+(\cdot)$ . It is known that

$$\begin{aligned} f_{X_{(1)}}(t_1) &= 2f(t_1)\bar{F}(t_1), \\ \bar{F}_{X_{(1)}}(x_1) &= [\bar{F}(x_1)]^2, \\ f_{(X_{(1)}, X_{(2)})}(t_1, t_2) &= 2f(t_1)f(t_2) \text{ whenever } t_1 \leq t_2, \\ \bar{F}_{X_{(2)}|X_{(1)}=t_1}(t_2) &= \frac{\bar{F}(t_2)}{\bar{F}(t_1)} \text{ whenever } t_1 \leq t_2. \end{aligned}$$

Note that  $\mathbf{X}$  is CIS, given that  $\bar{F}_{X_{(2)}|X_{(1)}=t_1}(t_2)$  is increasing in  $t_1$ .

With straightforward calculation it can be verified that for  $t_1 \leq t_2$

$$R_{\mathbf{X}}(t_1, t_2) = \left\{ (x, y) \in \text{Supp}(\mathbf{X}) : x \geq t_1, y \geq \bar{F}^{-} \left( \frac{\bar{F}(x)\bar{F}(t_2)}{\bar{F}(t_1)} \right) \right\},$$

$$P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})] = \bar{F}(t_1)\bar{F}(t_2) \text{ if } t_1 \leq t_2,$$

$$r_c(t_1, t_2) = \begin{cases} 0 & \text{if } t_2 < t_1 \\ 2r(t_1)r(t_2) & \text{if } t_2 \geq t_1, \end{cases}$$

$$\nu_{\mathbf{X}} = \int_0^{+\infty} [\bar{F}(t_1)]^2 E[X_{(2)}|X_{(1)} = t_1] dt_1$$

and

$$\bar{\mu}_{\mathbf{X}}(t_1, t_2) = \frac{1}{P[\mathbf{X} \in R_{\mathbf{X}}(\mathbf{t})]} \int_{t_1}^{\infty} dx \int_{\bar{F}^{-} \left( \frac{\bar{F}(x)\bar{F}(t_2)}{\bar{F}(t_1)} \right)}^{\infty} P[\mathbf{X} \in R_{\mathbf{X}}(x, y)] dy.$$

Moreover, it is easy to prove that

$$\hat{x}_{(1)}(u_1) = F^{-} [1 - (1 - u_1)^{1/2}],$$

$$\hat{x}_{(2)|t_1} = F^{-} [u_2 + (1 - u_2)F(t_1)]$$

where  $F^{-}(u) = \inf\{x : F(x) \geq u\}$  and

$$S_{\mathbf{X}}^+(u_1, u_2) = \int_{u_1}^1 (1 - v_1)^{1/2} S^+ [1 - (1 - u_2)(1 - v_1)^{1/2}] dF^{-} [1 - (1 - v_1)^{1/2}].$$

Consider that  $F(\cdot)$  has the IFR aging property. Then, the corrected hazard rate of  $\mathbf{X}$ , given by  $r_c(t_1, t_2) = 2r(t_1)r(t_2)$ , is clearly increasing in  $(t_1, t_2)$ . Therefore, the vector  $\mathbf{X} = (X_{(1)}, X_{(2)})$  is CMIFR. Furthermore,  $\mathbf{X}$  is also CBIHR. In fact, for  $t_1 \leq t_2$

$$r_{(1)}(t_1) = \frac{f_{(1)}(t_1)}{\bar{F}_{(1)}(t_1)} = 2 \frac{f(t_1)}{\bar{F}(t_1)} = 2r(t_1)$$



$$\begin{aligned}
 \hat{r}(t_2|t_1) &= \frac{\partial \ln \bar{F}_{X_{(2)}|X_{(1)}=t_1}(t_2)}{\partial t_1} \\
 &= \frac{\partial}{\partial t_1} \ln \frac{\bar{F}(t_2)}{\bar{F}(t_1)} = -\frac{\partial}{\partial t_1} \ln \bar{F}(t_1) \\
 &= r(t_1) \\
 r(t_2|t_1) &= -\frac{\partial}{\partial t_2} \ln \frac{\bar{F}(t_2)}{\bar{F}(t_1)} \\
 &= r(t_2).
 \end{aligned}$$

Therefore,  $R_1(t_1, t_2) = r_{(1)}(t_1) - \hat{r}(t_2|t_1) = r(t_1)$  and  $R_2(t_1, t_2) = r(t_2|t_1) = r(t_2)$  are increasing in  $(t_1, t_2)$ .

As the vector  $\mathbf{X}$  is CIS, from i) in Proposition 3.2.5, it follows that  $\mathbf{X}$  also has a Q-CMIFR distribution and from Proposition 3.2.3 and 3.2.4,  $\mathbf{X}$  has a Q-CMDMRL and Q-CMNBUE distribution. Finally, using Proposition 3.2.7 it is proved that  $\mathbf{X}$  has a CMNBUE distribution.

In the particular case that  $F(\cdot)$  is the negative exponential distribution with parameter  $\lambda$ , it is easy to check that

$$\bar{\mu}_{\mathbf{X}} = \frac{1}{2\lambda^2}, \quad \nu_{\mathbf{X}} = \frac{3}{4\lambda^2}$$

$$\hat{x}_1(u) = \frac{1}{\lambda} \ln [(1-u)^{-1/2}],$$

$$\hat{x}_2(u, v) = \begin{cases} \frac{1}{\lambda} \ln \left[ \frac{(1-u)^{-1/2}}{(1-v)} \right] & \text{if } u \geq 1 - (1-v)^2, \\ 0 & \text{if } u < 1 - (1-v)^2, \end{cases}$$

and

$$S_{\mathbf{X}}^+(u, v) = \begin{cases} \frac{(1-u)(1-v)}{2\lambda^2} & \text{if } u \geq 1 - (1-v)^2, \\ 0 & \text{if } u < 1 - (1-v)^2. \end{cases}$$

### 3.4 Orders for multivariate lifetime distributions

The univariate ageing properties NBUE, DMRL, IFR and so on, have been used to compare the relative ageing of two arbitrary life distributions. We review some of these comparisons. (see Kochar and Wiens (1987) and Kochar (1989) for more details.).

**Definition 3.4.1.** Let  $X$  and  $Y$  be two random variables with finite means and strictly increasing on their support. Let  $r_X$  ( $r_Y$ ) and  $\mu_X$  ( $\mu_Y$ ) be the failure rate function and the mean residual life of  $X$  ( $Y$ ), respectively.

(i)  $X$  is said to be more IFR than  $Y$  ( $X \preceq_{IFR} Y$ ) if

$$\frac{r_X[F_X^-(u)]}{r_Y[F_Y^-(u)]} \text{ is non-decreasing in } u \in (0, 1) \quad (3.4.1)$$

(ii)  $X$  is said to be more DMRL than  $Y$  ( $X \preceq_{DMRL} Y$ ) if

$$\frac{\mu_X[F_X^-(u)]}{\mu_Y[F_Y^-(u)]} \text{ is non-increasing in } u \in (0, 1) \quad (3.4.2)$$

(iii)  $X$  is said to be more NBUE than  $Y$  ( $X \preceq_{NBUE} Y$ ) if

$$\frac{\mu_X[F_X^-(u)]}{\mu_Y[F_Y^-(u)]} \leq \frac{E[X]}{E[Y]} \text{ for all } u \in (0, 1) \quad (3.4.3)$$

All these orderings have the property that if  $\bar{F}_Y(x) = e^{-\lambda x}$  is the exponential distribution, then

$$X \leq_{\mathcal{P}} Y \text{ if and only if } F_X \text{ has the property } \mathcal{P}$$

for  $\mathcal{P} \in \{ \text{IFR, DMRL, NBUE} \}$ .

Now, we define the corresponding multivariate corrected orderings by using the definition of multivariate ageing properties. We use these orderings to characterize the

Q-CMIFR, Q-CMDMRL and Q-CMNBUE properties by means of the multivariate exponential distribution with independent and exponential marginals.

**Definition 3.4.2.** Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vectors with finite multiple expectation and strictly increasing on their support. Let  $r_{\mathbf{X}}$  ( $r_{\mathbf{Y}}$ ) and  $\bar{\mu}_{\mathbf{X}}$  ( $\bar{\mu}_{\mathbf{Y}}$ ) be the multivariate failure rate function and the total expected residual life of  $\mathbf{X}$  ( $\mathbf{Y}$ ), respectively.

(i)  $\mathbf{X}$  is said to be more Q-CMIFR than  $\mathbf{Y}$  ( $\mathbf{X} \preceq_{Q-CMIFR} \mathbf{Y}$ ) if

$$\frac{r_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{r_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))} \text{ is non-decreasing in } \mathbf{u} \in (0, 1)^n. \quad (3.4.4)$$

(ii)  $\mathbf{X}$  is said to be more Q-CMDMRL than  $\mathbf{Y}$  ( $\mathbf{X} \preceq_{Q-CMDMRL} \mathbf{Y}$ ) if

$$\frac{\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))} \text{ is non-increasing in } \mathbf{u} \in (0, 1)^n. \quad (3.4.5)$$

(iii)  $\mathbf{X}$  is said to be more Q-CMNBUE than  $\mathbf{Y}$  ( $\mathbf{X} \preceq_{Q-CMNBUE} \mathbf{Y}$ ) if

$$\frac{\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))} \leq \frac{\bar{\mu}_{\mathbf{X}}}{\bar{\mu}_{\mathbf{Y}}} \text{ for all } \mathbf{u} \in (0, 1)^n. \quad (3.4.6)$$

**Theorem 3.4.1.** *Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vectors with finite multiple expectation and strictly increasing on their support. If  $\bar{F}_{\mathbf{Y}}(\mathbf{y}) = e^{-\sum_{i=1}^n \lambda_i y_i}$  is the multivariate exponential distribution with independent and exponential marginals, then*

- i)  $\mathbf{X} \preceq_{Q-CMIFR} \mathbf{Y}$  if, and only if,  $\mathbf{X}$  is a Q-CMIFR distribution.*
- ii)  $\mathbf{X} \preceq_{Q-CMDMRL} \mathbf{Y}$  if, and only if,  $\mathbf{X}$  is a Q-CMDMRL distribution.*
- iii)  $\mathbf{X} \preceq_{Q-CMNBUE} \mathbf{Y}$  if, and only if,  $\mathbf{X}$  is a Q-CMNBUE distribution.*

**Proof.**

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- i) From (3.4.4),  $\mathbf{X} \preceq_{Q-CMIFR} \mathbf{Y}$  if, and only if,  $\frac{r_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{r_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))}$  is non-decreasing in  $\mathbf{u} \in (0, 1)^n$ . Given that  $Y$  has independent and exponential marginals, from Theorem 3.1.1 it follows  $r_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u})) = \prod_{i=1}^n \lambda_i = c$ . Therefore,

$$\mathbf{X} \preceq_{Q-CMIFR} \mathbf{Y} \text{ if and only if } \frac{r_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{c} \text{ is non-decreasing in } \mathbf{u} \in (0, 1)^n.$$

or equivalently, see Definition 3.1.4,

$$\mathbf{X} \preceq_{Q-CMIFR} \mathbf{Y} \text{ if and only if } \mathbf{X} \text{ is a Q-CMIFR distribution.}$$

- ii) From (3.4.5),  $\mathbf{X} \preceq_{CMDMRL} \mathbf{Y}$  if and only if  $\frac{\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))}$  is non-increasing in  $\mathbf{u} \in (0, 1)^n$ . Given that  $Y$  has independent and exponential marginals, it follows that  $\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u})) = \frac{1}{\prod_{i=1}^n \lambda_i} = c$  for all  $u \in (0, 1)^n$ . Therefore,

$$\mathbf{X} \preceq_{Q-CMDMRL} \mathbf{Y} \text{ if and only if } c\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u})) \text{ is non-increasing in } \mathbf{u} \in (0, 1)^n,$$

Or equivalently, see Definition 3.1.4,

$$\mathbf{X} \preceq_{Q-CMDMRL} \mathbf{Y} \text{ if and only if } \mathbf{X} \text{ is a Q-CMDMRL distribution.}$$

- iii) From (3.4.6),  $\mathbf{X} \preceq_{Q-CMNBUE} \mathbf{Y}$  if and only if  $\frac{\bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u}))}{\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u}))} \leq \frac{\bar{\mu}_{\mathbf{X}}}{\bar{\mu}_{\mathbf{Y}}}$  for all  $\mathbf{u} \in (0, 1)^n$ . Given that  $Y$  has independent and exponential marginals, it follows that  $\bar{\mu}_{\mathbf{Y}}(\hat{\mathbf{y}}(\mathbf{u})) = \bar{\mu}_{\mathbf{Y}}$ . Therefore,

$$\mathbf{X} \preceq_{Q-CMNBUE} \mathbf{Y} \text{ if and only if } \bar{\mu}_{\mathbf{X}}(\hat{\mathbf{x}}(\mathbf{u})) \leq \bar{\mu}_{\mathbf{X}} \text{ for all } \mathbf{u} \in (0, 1)^n.$$

Or equivalently, see Definition 3.1.4,

$$\mathbf{X} \preceq_{Q-CMNBUE} \mathbf{Y} \text{ if and only if } \mathbf{X} \text{ is a Q-CMNBUE distribution.}$$

□

Fernández-Ponce *et al.* (1998) have shown that the univariate NBUE, DMRL and IFR partial orderings given in Definition 3.4.1 can be conveniently expressed in terms of their univariate excess-wealth function. In the next theorem, we express the corresponding multivariate orderings in terms of their multivariate excess-wealth function following the development in Fernández-Ponce *et al.* (1998).

**Theorem 3.4.2.** *Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vector satisfying the regularity conditions and with multivariate excess-wealth functions  $S_{\mathbf{X}}^+(\mathbf{u})$  and  $S_{\mathbf{Y}}^+(\mathbf{u})$ , respectively.*

*Then*

(i)

$$\mathbf{X} \leq_{Q-CMIFR} \mathbf{Y} \iff \frac{\frac{\partial^n}{\partial u_1 \dots \partial u_n} S_{\mathbf{X}}^+(\mathbf{u})}{\frac{\partial^n}{\partial u_1 \dots \partial u_n} S_{\mathbf{Y}}^+(\mathbf{u})} \text{ is non-increasing in } \mathbf{u}.$$

(ii)

$$\mathbf{X} \leq_{Q-CMDMRL} \mathbf{Y} \iff \frac{S_{\mathbf{X}}^+(\mathbf{u})}{S_{\mathbf{Y}}^+(\mathbf{u})} \text{ is non-decreasing in } \mathbf{u}.$$

(iii)

$$\mathbf{X} \leq_{Q-CMNBUE} \mathbf{Y} \iff \frac{S_{\mathbf{X}}^+(\mathbf{u})}{S_{\mathbf{Y}}^+(\mathbf{u})} \leq \frac{\bar{\mu}_{\mathbf{X}}}{\bar{\mu}_{\mathbf{Y}}}.$$

**Proof.**

(i) From equation (2.4.3),

$$\frac{\partial^n S_{\mathbf{X}}^+(\mathbf{u})}{\partial u_n \dots \partial u_1} = [r_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{u}))]^{-1}.$$

The required result follows from this, the regularity conditions and (3.4.4).

(ii) From (3.1.2)

$$\bar{\mu}_{\mathbf{X}}(\widehat{\mathbf{x}}(\mathbf{u})) = \frac{S_{\mathbf{X}}^+(\mathbf{u})}{\prod_{i=1}^n (1 - u_i)}.$$

The required result follows from this, the regularity conditions and (3.4.5).

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(iii) The required result follows from (3.1.2), the regularity conditions and (3.4.5).

□

Finally, we conclude this section with another multivariate stochastic order, the CMMRL order. As was mentioned in Section 3.1, equation (3.1.2), there is a closed relationship between the total expected residual life evaluated in  $\mathbf{u}$ -quantile and the multivariate excess-wealth function. We show that, under some conditions, multivariate excess-wealth ordering implies the generalized multivariate mean residual life (CMMRL) ordering. First, we give the definition of the CMMRL ordering.

**Definition 3.4.3.** A random vector  $\mathbf{X}$  is said to be smaller than another random vector  $\mathbf{Y}$  in the corrected multivariate mean residual life ( $\mathbf{X} \preceq_{CMMRL} \mathbf{Y}$ ) if

$$\bar{\mu}_{\mathbf{X}}(\mathbf{t}) \leq \bar{\mu}_{\mathbf{Y}}(\mathbf{t}) \text{ for all } \mathbf{t}.$$

For the next statement, recall the definition of the strong stochastic order (see Shaked and Shanthikumar (1994), Chapter 6).

**Definition 3.4.4.** Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vectors. It is said that  $\mathbf{X}$  is smaller than  $\mathbf{Y}$  in the strong stochastic order ( $\mathbf{X} \leq_{sst} \mathbf{Y}$ ) if

$$X_1 \leq_{st} Y_1$$

$$[X_2|X_1 = x_1] \leq_{st} [Y_2|Y_1 = y_1] \text{ whenever } x_1 \leq y_1$$

and in general, for  $i = 2, \dots, n$ ,

$$[X_i|X_1 = x_1, \dots, X_{i-1} = x_{i-1}] \leq_{st} [Y_i|Y_1 = y_1, \dots, Y_{i-1} = y_{i-1}]$$

whenever  $x_j \leq y_j, j = 1, 2, \dots, i - 1$ .

**Theorem 3.4.3.** *Let  $\mathbf{X}$  and  $\mathbf{Y}$  be two random vectors satisfying the regularity conditions and such that  $\mathbf{X} \leq_{sst} \mathbf{Y}$ .*

- a) *If  $\mathbf{X} \leq_{CMMRL} \mathbf{Y}$  and either  $\mathbf{X}$  or  $\mathbf{Y}$  is CMIMRL, then  $\mathbf{X} \leq_{ew} \mathbf{Y}$ .*
- b) *If  $\mathbf{X} \leq_{ew} \mathbf{Y}$  and either  $\mathbf{X}$  or  $\mathbf{Y}$  is CMDMRL, then  $\mathbf{X} \leq_{CMMRL} \mathbf{Y}$ .*

**Proof.**

- (a) Since  $\mathbf{X} \leq_{sst} \mathbf{Y}$ , it follows that  $\hat{\mathbf{x}}(\mathbf{u}) \leq \hat{\mathbf{y}}(\mathbf{u})$  for all  $\mathbf{u}$  in  $(0, 1)^n$ . Suppose that  $\mathbf{Y}$  is CMIMFR. Then  $\bar{\mu}_{\mathbf{Y}}[\hat{\mathbf{y}}(\mathbf{u})] \leq \bar{\mu}_{\mathbf{Y}}[\hat{\mathbf{x}}(\mathbf{u})]$ . If  $\mathbf{X} \leq_{CMMRL} \mathbf{Y}$  it holds that  $\bar{\mu}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})] \leq \bar{\mu}_{\mathbf{Y}}[\hat{\mathbf{x}}(\mathbf{u})]$  for all  $\mathbf{u}$ . Therefore,

$$\bar{\mu}_{\mathbf{X}}[\hat{\mathbf{x}}(\mathbf{u})] \leq \bar{\mu}_{\mathbf{Y}}[\hat{\mathbf{y}}(\mathbf{u})] \text{ for all } \mathbf{u} \in (0, 1)^n. \quad (3.4.7)$$

Finally, from the inequalities (3.4.7) and (3.1.2), the assertion is obtained.

- (b) The proof in this part is on the same lines and, therefore, it is omitted.

□

This theorem generalizes a result in Fernández-Ponce *et al.*(1998) where the univariate excess-wealth order and the mrl order are related. Considering again the relationship in (3.1.2), a sufficient condition under which Q-CMNBUE ordering implies *ew* ordering can be given.

**Theorem 3.4.4.** *If  $\mathbf{X} \leq_{Q-CMNBUE} \mathbf{Y}$  and  $\bar{\mu}_{\mathbf{X}} \leq \bar{\mu}_{\mathbf{Y}}$ , then  $\mathbf{X} \leq_{ew} \mathbf{Y}$ .*

**Proof.** It follows from (3.1.2) that  $\mathbf{X} \leq_{Q-CMNBUE} \mathbf{Y}$  if, and only if, for all  $\mathbf{u} \in (0, 1)^n$

$$\frac{S_{\mathbf{X}}^+(\mathbf{u})}{S_{\mathbf{Y}}^+(\mathbf{u})} \leq \frac{\bar{\mu}_{\mathbf{X}}}{\bar{\mu}_{\mathbf{Y}}}.$$

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Since  $\bar{\mu}_{\mathbf{X}} \leq \bar{\mu}_{\mathbf{Y}}$ , it follows  $S_{\mathbf{X}}^+(\mathbf{u}) \leq S_{\mathbf{Y}}^+(\mathbf{u})$  for all  $\mathbf{u} \in (0, 1)^n$ . Hence, the result.  $\square$



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## CHAPTER 4

# An Interesting Application

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### Abstract

Patient age and tumor size at the spontaneous detection of the tumor play an important role in the prevention of cancer. Tumor size is one of the most powerful predictors of tumor behavior in cancer. Hence, it is incorporated in almost all clinical reports. In describing the natural history of cancer, the process of tumor development can be explained in terms of the age at tumor onset (time from the patient is born until the first tumor cell appears) and the sojourn time (time from when the first tumor cell appearing until the detection of the disaster). A non-deterministic exponential model that relates the sojourn time to the tumor size at spontaneous detection is studied. We use a constraint in this model which represents an inherent multivariate aging property of the lifetime distributions considered. The proposed model is illustrated using two real databases.

## 4.1 A brief history

The natural history of cancer is of great scientific interest in its own right. It can be characterized by parameters of initiation, promotion and progression stages of tumor development, with the structure of the latter stage being dependent on a specific mechanism of cancer detection. There is increasing interest in early detection of chronic diseases with the expectation that earlier diagnosis combined with therapy results in more cures and longer survivals.

Possible changes in post-treatment survival may be affected by a number of clinical covariates which can be measured at the time of diagnosis and treatment. For example, the multivariate distribution of covariates at the time of diagnosis provides a link between the natural history of breast cancer and post-treatment (cancer-specific) survival. As a result, a great number of trials have been carried out in cancer sites, especially breast, colon, prostate and lung cancers. These trials generate data which can be used to estimate the sensitivity of the examinations, the sojourn time distribution of the preclinical state, and other characteristics of the screening cohort. Estimates of these parameters are important in planning public health programmes in order to extend proven benefit to large populations, as well as in designing future early-detection trials. Several authors have developed different models to study early detection trials during the last decades. There are many papers in the literature on this topic. For example, Albert *et al.* (1978a) and Albert *et al.* (1978b) developed a comprehensive model of the natural history of cancer in a general population based setting which included a cohort setting as a special case. Tsodikov and Müller (1998) developed a model of carcinogenesis for fractionated and continuous exposures. They

introduced a surface statistical model by making assumptions about the hazard function of the time of tumor latency. Bartoszyński *et al.* (2001) gave a wide spectrum of problems associated with stochastic modeling of cancer detection. They discussed marginal distributions of tumor size and age at detection, as well as associated estimation problems, and also gave the joint distribution of those two random variables and their randomized counterpart. Explicit formulas for the marginal distributions of tumor size and age of an individual at detection were shown not to be sufficient for the complete utilization of the information contained in the corresponding sample observations for estimation of parameters describing the natural history of the disease; their joint distribution is required in order to develop pertinent methods for statistical inference based on maximum likelihood. However, this joint distribution is partially known in real problems.

Gregori *et al.* (2002) introduced a new goodness-of-fit test based on the concept of hazard rate, point processes and martingale theory. This test is designed for two-stage stochastic models of carcinogenesis. From the statistical point of view, the basic problems of data analysis in carcinogenesis studies are no different from those in parametric survival analysis. Tsodikov *et al.* (2003) considered the utility of the bounded cumulative hazard model in cure rate estimation which is an appealing alternative to the widely used two-component mixture model.

More recently, it is worth mentioning the following papers. McIntosh and Urban (2003) proposed computationally simple longitudinal screening algorithms to model the behaviour of a biomarker (substances that can be found in the body when cancer is present, for example the PSA tumor marker in prostate cancer and the CA 15 – 3 tumor marker in breast cancer) that can be implemented with data that is obtainable

in a short period of time. Davidov and Zelen (2004) studied over-diagnosis in early-detection programmes (over-diagnosis refers to the situation where screening detects a disease that would have otherwise been undetected in a person's lifetime), in which disease would not have been diagnosed because the individual would have died of other causes prior to its clinical onset. They analysed an idealized early-detection programme and derived the mathematical expression for the probability of overdiagnosis. Shen and Zelen (2005) assumed that the schedule of a screening programme was periodic and that the sojourn time in the preclinical state had a piecewise density function. Modeling the preclinical sojourn time distribution as a piecewise density function resulted in robust estimation of the distribution function. They estimated the piecewise density function and the examination sensitivity using both generalized least squares and maximum likelihood methods.

Hanin and Yakovlev (2007) studied the problem of identifiability of the joint distribution of age and tumor size at detection in the presence of an arbitrary screening schedule. Several identifiability results had been reported before by Albert *et al.* (1978a), Albert *et al.* (1978b) and Bartoszynski *et al.* (2001), but all of them were concerned with a model of cancer detection in absence of screening. Hanin and Yakovlev (2007) completed the model given by Bartoszynski *et al.* (2001) describing the impact of screening on the natural history of cancer.

Ghosh (2006) laid out a framework for the analysis of data on tumor size and metastases with covariates. Considering the equivalence of the observed data structure with those from the field of survival analysis, the author characterized non-parametric maximum likelihood estimators of the distribution for tumor size at which metastasis transitions occur and their associated asymptotic properties. He presented two

scenarios where the distribution of tumor size is identifiable (in the case I scenario detection of the cancers is not affected by the presence of metastases and in the case II scenario cancers are detected immediately when the metastasis occurs). These correspond to the situations in which tumor size is treated as a right-censored and an interval-censored random variable, respectively. Later, under the case I scenario, Ghosh(2008) modeled the effect of the tumor size on the risk of metastasis using a binary regression model with monotonicity constraints and developed general inference procedures for this model.

## 4.2 Modeling the age and tumor size at detection

The literature on mathematical modeling and optimization of cancer screening is extensive with almost all of the published work having been focused on the time history of cancer. According to a conventional staging of cancer, it progresses through the following phases

1. Tumor latency that ranges from the birth of an individual to the appearance of the first clonogenic tumor cell. Such a random event is called onset of disease and will be denoted by  $T$ .
2. Once the tumor emerges (and therefore becomes detectable), it enters the pre-clinical stage which starts at the moment of tumor detection. The time spent in the preclinical stage is referred to as sojourn time and is denoted by  $W$ .
3. A treated tumor progresses through a post-treatment stage that results either in cure or death or tumor recurrence.

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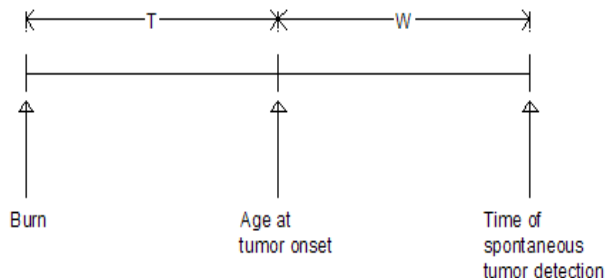


Figure 4.1: Time history of cancer

Progression of the disease through the aforementioned stages is viewed as irreversible. For each individual, the natural history of the disease is characterized by a random vector  $Z = (T, W)$ . Associated with a population of individuals at time  $t$  is a vector  $(Z, A(t))$  where  $A(t)$  is the age for an individual at time  $t$ .

Albert *et al.* (1978a) and Albert *et al.* (1978b) studied the age and tumor size at detection. They assumed that the models for tumor detectability can be synthesized by first modeling the behavior of tumor growth over time and superimposing a model for detection probability as a function of tumor size. Hanin and Boucher (1999) used this model to approach the problem of optimal cancer screening.

As mentioned earlier, let  $T$  be the age at tumor onset and  $W$  be the time of spontaneous detection of the tumor measured from the onset of disease. Define a random variable  $S$  to represent tumor size (the number of cells in a tumor) at spontaneous detection. Hanin and Boucher (1999) supposed that the law of tumor growth is described by a deterministic function  $f : [0, \infty) \rightarrow [1, \infty)$  with  $f(0) = 1$ , such that  $S = f(W)$ . It is assumed that

- (a) random variables  $T$  and  $W$  are absolutely continuous and independent.

(b) function  $f$  is differentiable and  $f' > 0$ .

(c) the hazard rate for spontaneous detection of the tumor is proportional to the current tumor size with a non-negative coefficient. That is,  $r_W(w) = \alpha S(w) = \alpha f(w)$ , where  $\alpha$  is a non-negative constant.

From the assumption (c), it follows that the survival function,  $\bar{G}_W(w)$ , for the random variable  $W$  is given by

$$\begin{aligned}\bar{G}_W(w) &= \exp \left\{ - \int_0^w r_W(u) du \right\} \\ &= \exp \left\{ -\alpha \int_0^w f(u) du \right\} \\ &= e^{-\alpha \Phi(w)}\end{aligned}$$

where  $\Phi(t) = \int_0^t f(u) du$ .

In like manner, the survival function of the tumor size  $S = f(W)$  is

$$\bar{F}_S(s) = \bar{G}_W[\Psi(s)] = e^{-\alpha \Phi(\Psi(s))} \quad (4.2.1)$$

where  $\Psi(t)$  is the inverse function for  $f(t)$ . In the special case of deterministic exponential growth with rate  $\lambda_0$ , it follows from (4.2.1) that tumor size at detection  $S$  has a translated exponential distribution with unknown parameter  $\lambda_0$ . So, the survival function of  $S$  is given by

$$\bar{F}_S(s) = \exp\{-\lambda_0(s - 1)\} \text{ for } s \geq 1.$$

Sample values of the random vector  $Y = (S, T + W)$  with components interpreted as tumor size at spontaneous detection and age, respectively, can be observed. In the particular case of exponential tumor growth, the joint density function for the random variable  $Y$  is easily obtained (see Bartoszynski *et al.*, 2001). However, natural questions arise for this model of tumor incidence:

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- ▶ Can this formulation be a real model for the relationship between  $S$  and  $W$ ?
- ▶ Furthermore, the condition **c)** on  $f$  is too strong since it is assumed that  $W$  has an IFR distribution. This condition can be extended by imposing another survival property.

Under these considerations, our objectives are the following.

1. To estimate the parameters in a non-deterministic model which relates the variables  $S$  and  $W$ .
2. To analyse the unknown random variable  $T$  plus a unknown random delay, which will be denoted by  $\Delta$ .

For our model, the following hypothesis are considered.

- C1. The unknown variables  $W$  and  $T$  have NBUE distributions.

Note that this property is an intuitive survival property for the random variable  $W$ , as well as the random variable  $T$ . In fact,  $W$  has an NBUE distribution if  $E(W) \geq E(W - w | W > w)$  holds for all  $w$  in  $\mathbb{R}$ . This inequality indicates that the mean time of spontaneous detection of tumor measured from the onset of disease is greater than or equal to the mean residual time of spontaneous tumor detection by assuming that this time is greater than  $w$ . Similarly, if  $T$  has an NBUE distribution, then the mean time from the birth of an individual to the appearance of the first tumor cell is greater than, or equal to, the mean residual time to the appearance of the first tumor cell by assuming that this time is greater than  $t$ .



From a nonparametric point of view, it should be noted that the NBUE property of  $W$  is more general than the IFR property of  $W$  which is assumed by Bartoszyński *et al* (2001). Thus, under this hypothesis, more possible distributions are included to describe the variable  $W$ .

C2. The distribution of tumor size  $S$  is unknown but it is assumed that the logarithm of  $S$  has an NBUE distribution.

C3. In contrast to Bartoszyński *et al* (2001), a non-deterministic exponential tumor growth is assumed. So, the relationship between  $S$  and  $W$  can be modeled as

$$W = \alpha + \beta \ln S + \Delta \tag{4.2.2}$$

where  $\beta > 0$  and  $\Delta$  is a random delay which is independent of  $S$ .

C4. It is assumed that  $T + \Delta$  has an NBUE distribution.

Consequently, we have a random sample of size  $n$  from a homogeneous population:

$$\mathbf{y}_i = (s_i, v_i) \quad i = 1, \dots, n \text{ with } v_i = t_i + w_i \text{ and } w_i = \alpha + \beta \ln s_i + \delta_i$$

and where  $s_i$  and  $v_i$  represent the tumor size and the age at detection for the  $i$ th patient, respectively. Note that the values of  $t_i$  and  $w_i$  are completely unknown.

From the relationship  $v_i = t_i + \alpha + \beta \ln s_i + \delta_i$  for all  $i$ , it follows that

$$t_i + \delta_i = v_i - \alpha - \beta \ln s_i \text{ for } i. \tag{4.2.3}$$

For simplicity, the value  $t_i + \delta_i$  will be denoted by  $\eta_i$  for  $i = 1, \dots, n$  as sample values of the variable  $T + \Delta$ , where  $T$  is the time at onset and  $\Delta$  is the nonnegative random time in (4.2.2).

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It is easy to show, by using Theorem 3.1.7 and taking  $X_1 = \ln S$ ,  $X_2 = T + \alpha + \Delta$  and  $\phi(X_1) = \beta * X_1$ , that the random vector  $(\ln S, T + \alpha + \beta \ln S + \Delta)$  has a CMNBUE distribution. That means that the multiple expectation associated to the vector with component logarithm of tumor size and age at detections is greater than or equal to the total expected residual life of this vector in any particular value of the vector  $(\ln s, v)$ . This property enables a condition to be established in the problem of estimating the parameters  $\alpha$  and  $\beta$  in the model given in 4.2.2.

Therefore, from a conservative point of view, to estimate the above parameters is equivalent to estimating  $\alpha$  and  $\beta$  such that the sum  $\sum_{i=1}^n \eta_i$  is minimum under the constraints  $\eta_i > 0$  and  $(\ln S, T + \alpha + \beta \ln S + \Delta) \in \mathcal{F}_{CMNBUE}$  for all  $i$ , where the notation  $(\ln s_i, t_i + \alpha + \beta \ln s_i + \delta_i) \in \mathcal{F}_{CMNBUE}$  for all  $i$  indicates that the random sample is obtained from a CMNBUE multivariate distribution. This problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{i=1}^n \eta_i \\ \text{s.t.} \quad & \eta_i \geq 0 \quad \text{for all } i \\ & (\ln s_i, t_i + \alpha + \beta \ln s_i + \delta_i) \in \mathcal{F}_{CMNBUE} \text{ for all } i \end{aligned} \quad (4.2.4)$$

In order to break down the model more easily, the restriction (4.2.4) must be expressed in a different way. Next, it is shown that, using Theorem 3.1.8, it holds that

$$\begin{aligned} & (\ln S, T + \alpha + \beta \ln S + \Delta) \text{ is CMNBUE if, and only if,} \\ & \frac{S_{\ln S}^+(u_1)}{\mathbb{E}(\ln S)(1 - u_1)} \cdot \frac{S_{T+\Delta}^+(u_2)}{\mathbb{E}(T + \Delta)(1 - u_2)} \leq 1 + \frac{\alpha}{\mathbb{E}(T + \Delta)} + \beta \frac{\mathbb{E}(\ln^2 S)}{\mathbb{E}(\ln S)\mathbb{E}(T + \Delta)} \end{aligned} \quad (4.2.5)$$

for all  $(u_1, u_2) \in [0, 1]^2$ .

Denote  $L = \ln S$ , and consider  $\mathbf{H} = (L, \alpha + \beta L + T + \Delta)$ , then by definition

$$S_{\mathbf{H}}^+(u_1, u_2) = \int_{F_L^-(u_1)}^{\infty} \bar{F}_L(t_1) \cdot S_{\alpha + \beta L + T + \Delta | L = t_1}^+(u_2) dt_1. \quad (4.2.6)$$

First, we obtained the expression of  $S_{\alpha+\beta L+T+\Delta|L=t_1}^+(u_2)$ . By definition, we know that

$$S_{\alpha+\beta L+T+\Delta|L=t_1}^+(u_2) = \int_{F_{\alpha+\beta L+T+\Delta|L=t_1}^-(u_2)}^{\infty} \bar{F}_{\alpha+\beta L+T+\Delta|L=t_1}(t_2) dt_2. \quad (4.2.7)$$

Then, using the following trivial equality

$$\bar{F}_{\alpha+\beta L+T+\Delta|L=t_1}(t_2) = \bar{F}_{T+\Delta}(t_2 - \alpha - \beta t_1) \quad (4.2.8)$$

and replacing (4.2.8) in (4.2.7), it follows that

$$\begin{aligned} S_{\alpha+\beta L+T+\Delta|L=t_1}^+(u_2) &= \int_{F_{\alpha+\beta L+T+\Delta|L=t_1}^-(u_2)}^{\infty} \bar{F}_{T+\Delta}(t_2 - \alpha - \beta t_1) dt_2 \\ &= \int_{F_{T+\Delta}^-(u_2) + \alpha + \beta t_1}^{\infty} \bar{F}_{T+\Delta}(t_2 - \alpha - \beta t_1) dt_2 \\ &= \int_{F_{T+\Delta}^-(u_2)}^{\infty} \bar{F}_{T+\Delta}(r) dr \\ &= S_{T+\Delta}^+(u_2) \end{aligned} \quad (4.2.9)$$

Thus, by replacing (4.2.9) in (4.2.6), it holds that

$$\begin{aligned} S_{\mathbf{H}}^+(u_1, u_2) &= \int_{F_L^-(u_1)}^{\infty} \bar{F}_L(t_1) \cdot S_{T+\Delta}^+(u_2) dt_1 \\ &= S_L^+(u_1) S_{T+\Delta}^+(u_2). \end{aligned}$$

Now, let the expression of  $\nu_{\mathbf{H}}$  be obtained. By definition, we have that

$$\begin{aligned} \nu_{(L, \alpha+\beta L+T+\Delta)} &= \int_0^{+\infty} \bar{F}_L(t_1) E[\alpha + \beta L + T + \Delta | L = t_1] dt_1 \\ &= \int_0^{+\infty} \bar{F}_L(t_1) [\alpha + \beta t_1 + E[T + \Delta]] dt_1 \\ &= \alpha E[L] + \beta \int_0^{+\infty} \bar{F}_L(t_1) t_1 dt_1 + E[T + \Delta] E[L] \end{aligned} \quad (4.2.10)$$

where

$$\begin{aligned}
 \int_0^{+\infty} \bar{F}_L(t_1)t_1 dt_1 &= \int_0^{+\infty} \int_{t_1}^{+\infty} t_1 f_L(y) dy dt_1 \\
 &= \int_0^{+\infty} \int_0^y t_1 f_L(y) dt_1 dy \\
 &= \int_0^{+\infty} f_L(y) \frac{y^2}{2} dy \\
 &= \frac{1}{2} E[L^2]
 \end{aligned} \tag{4.2.11}$$

Finally, if (4.2.11) is replaced in (4.2.10), it follows that

$$\begin{aligned}
 \nu_{(L, \alpha + \beta L + T + \Delta)} &= \alpha E[L] + \beta \frac{E[L^2]}{2} + E[T + \Delta] E[L] \\
 &= E[L] E[T + \Delta] \left\{ 1 + \frac{\alpha}{E[T + \Delta]} + \beta \frac{E[L^2]}{2E[L]E[T + \Delta]} \right\}
 \end{aligned}$$

Therefore, from Theorem 3.1.8,

$\mathbf{H} = (\ln S, T + \alpha + \beta \ln S + \Delta)$  is CMNBUE if and only if

$$S_{\mathbf{H}}^+(u_1, u_2) \leq \nu_{\mathbf{H}}(1 - u_1)(1 - u_2) \text{ for all } (u_1, u_2) \in (0, 1)^2$$

or, equivalently, if

$$\frac{S_{\ln S}^+(u_1)}{(1 - u_1)E[\ln S]} \frac{S_{T+\Delta}^+(u_2)}{(1 - u_2)E[T + \Delta]} \leq 1 + \frac{\alpha}{E[T + \Delta]} + \beta \frac{E[\ln^2 S]}{E[\ln S]E[T + \Delta]}$$

for all  $(u_1, u_2) \in [0, 1]^2$ , and the inequality (4.2.5) is obtained.

In particular, the inequality (4.2.5) is held for  $(0, 0)$ . Moreover, by assumption, the variables  $\ln S$  and  $T + \Delta$  have NBUE distributions, so their corresponding univariate excess wealth functions  $S_{\ln S}^+(u_1)$  and  $S_{T+\Delta}^+(u_2)$  are decreasing for all  $u_1$  and  $u_2$ . Moreover, it holds that  $S_{\ln S}^+(u_1) \leq E(\ln S)(1 - u_1)$  for all  $u_1$ ,  $S_{T+\Delta}^+(u_2) \leq E(T + \Delta)(1 - u_2)$  for all  $u_2$  and  $S_{\ln S}^+(0) = E(\ln S)$ ,  $S_{T+\Delta}^+(0) = E(T + \Delta)$  (see Fernández-Ponce *et al.*,

1998). Therefore, by taking into account that  $E(T + \Delta) > 0$ , the inequality (4.2.5) is equivalent to

$$0 \leq \alpha + \beta \frac{\mathbb{E}(\ln^2 S)}{\mathbb{E}(\ln S)}.$$

Thus, the problem for estimating  $\alpha$  and  $\beta$  is the following linear programming problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^n \eta_i \\ \text{s.t.} \quad & \eta_i \geq 0 \text{ for all } i \\ & 0 \leq \alpha + \beta \frac{\mathbb{E}(\ln^2 S)}{\mathbb{E}(\ln S)}. \end{aligned}$$

Given that the distribution of  $\ln S$  is unknown, the values  $\mathbb{E}(\ln^2 S)$  and  $\mathbb{E}(\ln S)$  are also unknown. Considering the asymptotic property of  $\frac{\sum_{i=1}^n \ln^2 s_i}{\sum_{i=1}^n \ln s_i}$  as an estimator of  $\frac{\mathbb{E}(\ln^2 S)}{\mathbb{E}(\ln S)}$ , this problem can be solved by replacing the last constraint with the following inequality

$$0 \leq \alpha + \beta \frac{\sum_{i=1}^n \ln^2 s_i}{\sum_{i=1}^n \ln s_i}. \tag{4.2.12}$$

Note that our model is not exactly a parametric regression model since the ‘residual’ random variable is non-negative. Neither can non-regular regression (see Smith, 1994) be used since the hypothesis on the tail of the distribution function for the errors cannot be verified for NBUE distributions in general. Consequently, this problem must be solved by using a different model.

Recall that quantile regression is a regression model in which a specified conditional quantile of the outcome variable is expressed as a linear function of subject characteristics. This is in contrast to the ordinary least squares regression, in which the

## 4.2. MODELING THE AGE AND TUMOR SIZE AT DETECTION

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mean of a continuous response variable is expressed as a linear function of a set of independent variables. If  $\theta$  is a parameter in  $(0, 1)$ , the  $\theta$ th regression quantile is denoted by  $B^*(\theta)$  and is the solution set to

$$\min_{\alpha, \beta} \sum_{i=1}^n c(y_i - (\alpha + \beta x_i) : \theta)$$

where

$$c(e : \theta) = \begin{cases} \theta|e| & \text{if } e > 0 \\ (1 - \theta)|e| & \text{if } e < 0 \end{cases}$$

For the special case of median regression, with  $\theta = 0.5$ , it reduces to the least absolute deviation estimation. Positive residuals are weighted more heavily than negative residuals if one wants to model a quantile that lies above the 0.5th quantile, while the converse is true if one wishes to model a quantile that lies below the 0.5 quantile. In particular, the smallest regression quantile is denoted by  $\underline{B}^*$  and it is defined by

$$\underline{B}^* = \{(\alpha, \beta) : (\alpha, \beta) \in B^*(\theta) \text{ and } \text{sign}(y_i - \alpha - \beta x_i) \geq 0, i = 1, \dots, n\}.$$

The specified quantile of the distribution of the dependent variable, conditional on the predictor variable values is given by

$$Q_{y_i}(\theta|x_i) = \alpha(\theta) + \beta(\theta)x_i,$$

where  $Q_{y_i}(\theta|\cdot)$  denotes the  $\theta$ -quantile of the conditional distribution of  $y_i$ . Therefore, the regression parameter  $\beta(\theta)$  denotes how the specified quantile changes with a one-unit change in  $x$ . In contrast to the least squares regression, the coefficients for the quantile regression cannot be explicitly estimated, given that an closed-form expression does not exist . An iterative algorithm must be used to estimate the parameters (see Koenker and Basset, 1978 and Koenker and D'Orey, 1987).

Now, note that the values of  $\alpha$  and  $\beta$  that minimize the sum of  $\eta_i$  for all  $\eta_i > 0$  can be obtained by solving the problem of the smallest regression quantile when the values  $\eta_i = v_i - \alpha - \beta \ln s_i$  for  $i = 1, \dots, n$  are considered as the values of the residual variable.

Thus, it can be concluded that, from a conservative point of view, estimating the parameters in our model is equivalent to solving a particular problem of quantile regression. Moreover, it is possible to analyse the unknown variable  $T + \Delta$  by means of the residuals obtained in this quantile regression.

## 4.3 An application to real datasets

### 4.3.1 Example 1. German breast cancer study data

#### Materials and Methods

From July 1984 to December 1989, the German Breast Cancer Study Group initially recruited 720 patients with primary node positive breast cancer into the Comprehensive Cohort Study (Schmoor *et al.*, 1996). Some of the variables considered for each patient were: date of diagnosis, patient's age at diagnosis, tumor size (tumor diameter in mm), tumor grade and number of nodes involved. The study itself which was realized by this group is not of interest for the present purpose. Our attention is focused on two variables: age at detection and tumor size. The 686 patients who completed the data for the standard factors of age and tumor size are analysed in this study.

Tumor size, initially given in mm, was transformed into the number of tumor cells per  $cm^3$ . For this, we assume the tumor is a symmetric ball in  $\mathbb{R}^3$  and consider that

### 4.3. AN APPLICATION TO REAL DATASETS

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approximately  $10^{12}$  tumor cells exist per  $\text{cm}^3$ , (see Spanish Society of Medical website, <http://www.seom.org/>). A descriptive analysis was realized for tumor size as well as patient's age at detection. The NBUE property for both variables was confirmed by using a statistical test given by Fernández-Ponce *et al.* (1996). The plots of their univariate excess-wealth functions were also used to recognize this property.

The parameters in the model that relates the variables sojourn time ( $W$ ) and tumor size ( $S$ ) were estimated by using the model proposed in the above section. Minimum quantile regression results in a line that represents the relationship between the logarithm of tumor size and the age at detection. The residuals of this regression are considered as the estimation of the age at tumor onset plus a random delay for each patient.

All statistical analyses were performed using **R** software. In particular, the *quantreg* package was used to solve the problems of quantile regression (see Appendix A in Koenker, 2005 and website <http://cran.r-project.org/web/packages/quantreg/index.html>).

### Results

The mean and median patient age in this series was 53 years (interquartile range: 46-61) with a standard deviation of 10.12. The mean tumor size was  $2.5317 \times 10^{16}$  tumor cells (interquartile range:  $4.1888 \times 10^{15} - 2.2449 \times 10^{16}$ ) with a standard deviation of  $5.9614 \times 10^{16}$ . Considering the variable logarithm of tumor size, it was obtained that the mean logarithm of tumor size is 29.8999 (interquartile range: (20.0634–30.7422)), the median was 29.73286 and the standard deviation 1.3765.

The box-and-whisker plots of the age and the logarithm of tumor size are shown in Figure 4.2, respectively. The  $p$ -values for the Shapiro-Wilk normality test was 0.00



for both cases, therefore the normality hypothesis is rejected for both variables.

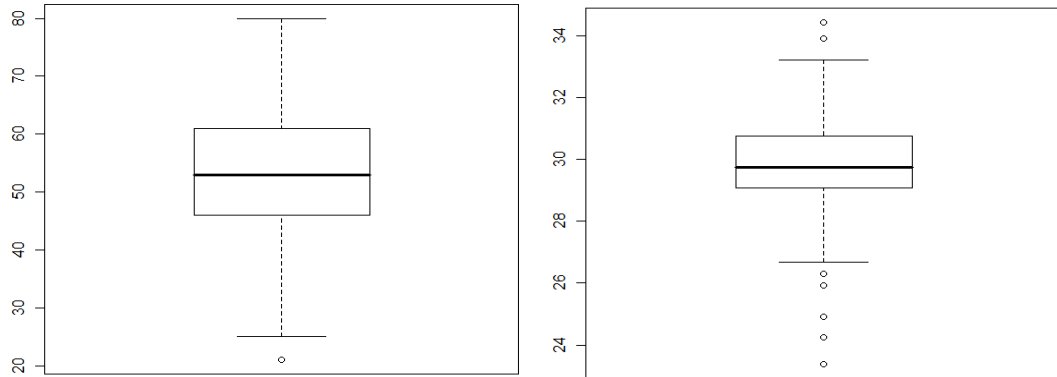


Figure 4.2: Box-and-whisker plots of age (left) and logarithm of tumor size (right).

The plots of the empirical excess-wealth (ew) functions of the age and logarithm of tumor size are given in Figure 4.3. It can be observed that the empirical ew function of the age variable is under the line from the point  $(1, 0)$  to point  $(0, \mu_V)$ , where  $\mu_V$  is the sample mean age. Note that this line corresponds to the ew function of an exponential distribution with parameter  $\lambda = \mu_V$ . Similarly, this occurs for the logarithm of tumor size. This property is always verified for all variables having an NBUE distribution. Moreover, the NBUE property of age and the logarithm of tumor size were checked by means of the test for NBUE alternatives given by the Fernández-Ponce *et al.* (1996). The statistic for this test,  $\Psi(F_n(t))$ , is based on the empirical ew function and it has really interesting asymptotic properties. For patient's age  $\Psi(F_n(t)) = 0.4741$  is obtained and  $\Psi(F_n(t)) = 0.4942$  is obtained for the logarithm of tumor size. Therefore, both random variables have an NBUE distribution (see critical values for NBUE alternatives in Table 4.1). In contrast with these variables, the tumor size variable  $S$  does not have this ageing property, but it has the dual

### 4.3. AN APPLICATION TO REAL DATASETS

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property. In Figure 4.3 c), it can be seen that the empirical ew function is on the line from the point  $(1,0)$  to point  $(0, \mu_S)$ , where  $\mu_S$  is the sample mean of tumor size.

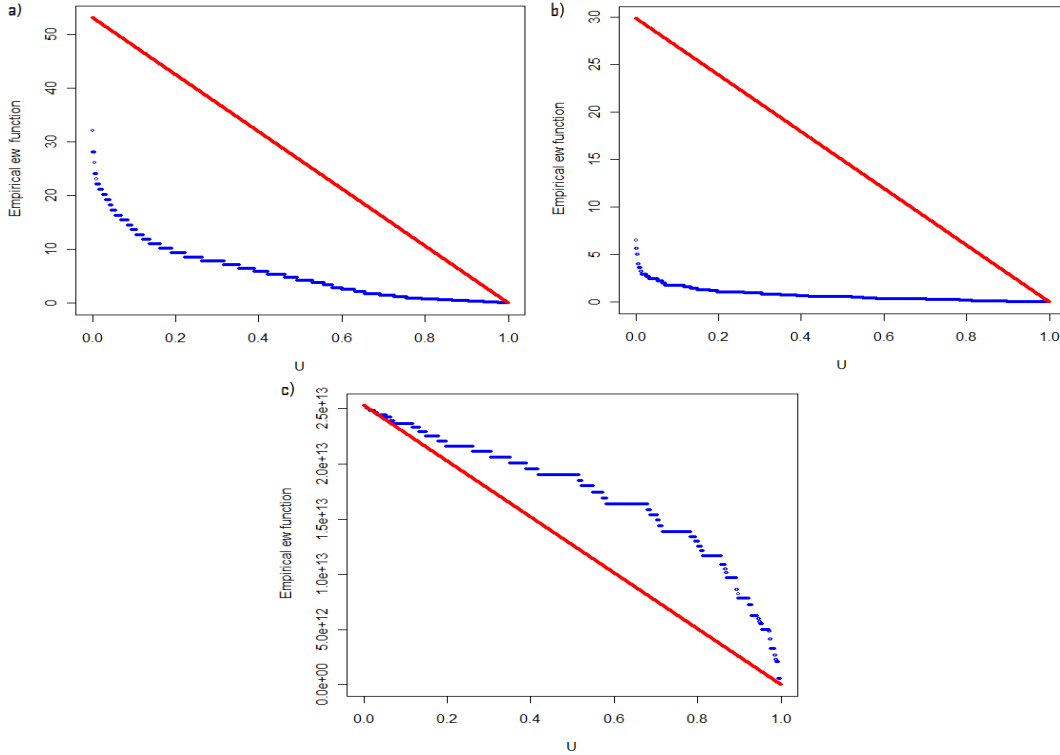


Figure 4.3: (a) Empirical ew function of age (blue line) and empirical ew function for exponential with parameter equal to mean age (b) Empirical ew function of logarithm tumor size (blue line) and empirical ew function for exponential with parameter equal to mean logarithm tumor size (c) Empirical ew function of tumor size (blue line) and empirical ew function for exponential with parameter equal to mean tumor size.

When the constraint  $(\ln S, T + \alpha + \beta \ln S + \Delta) \in \mathcal{F}_{CMNBUE}$  was not included in the quantile regression problem, the estimates intercept and slope of the smallest quantile regression line were  $\hat{\alpha} = -36.035670$  and  $\hat{\beta} = 2.022514$ , respectively. It should be added that the smallest regression quantile is obtained for  $\theta = 0.0001$  (for values of

### 4.3. AN APPLICATION TO REAL DATASETS

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$n$	$\alpha$		
	0.1	0.05	0.01
400	0.0549	0.0732	0.0925
450	0.0544	0.0715	0.0910
475	0.0543	0.0661	0.0876
500	0.0532	0.0654	0.0818
600	0.0434	0.0570	0.0737
650	0.0425	0.0535	0.0736
700	0.0401	0.0532	0.0721

Table 4.1: Critical value for NBUE alternative with different significance levels and sample size.

$\theta$  bigger than 0.0001, some residuals are negative and for values of  $\theta$  smaller than 0.0001, they are obtained the same residuals and the same quantile regression line).

The mean and the standard deviation for the residual variable obtained from the quantile regression were 28.5640 and 10.4756, respectively. A 95 percent confidence interval for this mean was (27.77813; 29.34988) and the standard error was 0.4002. Recall that, from a conservative point of view, the  $i$ th residual in the quantile regression is considered as the estimation of the value  $t_i + \delta_i$ . Therefore, it would be said that the mean age at appearance of the first tumor cell plus a random delay was approximately 28.5640 years, when the CMNBUE constraint is not considered. It was also proved that the variable  $T + \Delta$  has an NBUE distribution, given that the value of statistic for the test for NBUE alternatives was  $\Psi(F_n(t)) = 0.4305$ .

Figure 4.4 shows the scatterplot of the logarithm of tumor size vs the age of the patients at detection. Superimposed on the plot is the 0.0001-quantile regression line in blue, when the CMNBUE constraint is not considered.

On the other hand, if the property  $(\ln S, T + \alpha + \beta \ln S + \Delta) \in \mathcal{F}_{CMNBUE}$  is included as a constraint in the quantile regression problem, the estimated intercept and slope are

### 4.3. AN APPLICATION TO REAL DATASETS

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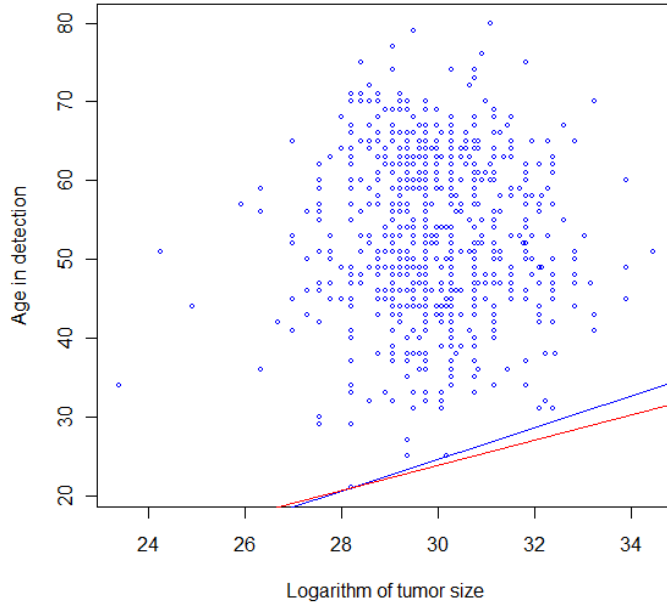


Figure 4.4: Scatterplot of the age at detection vs the logarithm of the tumor size. The red line is the 0.0001-quantile regression line considering the CMNBUE constraint and the blue line is the 0.0001-quantile regression line without considering the CMNBUE constraint.

$\hat{\alpha} = -23.80051$  and  $\hat{\beta} = 1.58865$ , respectively. The mean and the standard deviation for the residual variable obtained from the quantile regression were 29.3038 and 10.3288, respectively. A 95 percent confidence interval for this mean was (28.52852 30.07825 ) and the standard error was 0.3946. Therefore, if the multivariate ageing property is considered, it could be concluded that the mean age at the first tumor cell appearance plus a random delay is approximately 29.30 years. In Figure 4.4, the 0.0001-quantile regression line when the CMNBUE constraint is considered is also drawn in red.

Table 4.2 shows different estimated quantiles for the variable  $T + \Delta$  when the CMNBUE property is not, and is, considered in the quantile regression problem.

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	$Q(0.05)$	$Q(0.25)$	$Q(0.5)$	$Q(0.75)$	$Q(0.95)$
Case 1	10.81	20.89	28.54	36.90	45.00
Case 2	11.96	21.85	29.38	37.56	45.35

Table 4.2: Quantiles of the estimated variable  $T + \Delta$  when the CMNBUE property of vector  $(\ln S, T + \alpha + \beta \ln S + \Delta)$  is not considered (case 1) and is considered (case 2) as a constraint in the quantile regression problem.

Considering the estimations of the parameters  $\alpha, \beta$  having included the multivariate ageing property in the linear program problem, the patients's age at detection is approximate for different tumor sizes by using the relationship  $\widehat{V} = \widehat{\alpha} + \widehat{\beta} \ln S + \overline{T + \Delta}$ , where  $\overline{T + \Delta}$  is the estimation of the mean of the variable  $T + \Delta$  obtained as the mean of the residuals in the quantile regression. Table 4.3 shows the results. In the third and forth columns, the age at detection is also given when the variable  $T + \Delta$  is estimated by the lower and upper values of the confidence interval for  $\overline{T + \delta}$ , respectively.

Finally, a descriptive analysis is also made when the logarithm of tumor size variable is put into groups. Estimations for the variable  $T + \Delta$  are shown in Table 4.4 for different logarithm of tumor size groups. Note that for low values of the logarithm of the tumor size, the estimated quantiles of the variable  $T + \Delta$  when the CMNBUE constraint is not considered are higher than these quantiles when this constraint is considered. This inequality is inverted for high values of the logarithm of the tumor size.

Considering only the residuals for values of the logarithm of tumor size in the first group in Table 4.4 (that is,  $\ln S$  between 23 and 28, o approximately, the tumor diameter having values between 3 and 14 mm), the age at detection is also approximate

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Tumor size ( <i>mm</i> )	$\widehat{V}$	$\widehat{V}_l$	$\widehat{V}_u$	Tumor size ( <i>mm</i> )	$\widehat{V}$	$\widehat{V}_l$	$\widehat{V}_u$
3	42.63	41.86	43.41	16	50.61	49.84	51.39
4	44.00	43.23	44.78	18	51.17	50.40	51.95
5	45.07	44.29	45.84	20	51.67	50.90	52.45
6	45.94	45.16	46.71	25	52.74	51.96	53.51
7	46.81	46.03	47.58	30	53.61	52.83	54.38
8	47.31	46.53	48.08	40	54.98	54.20	55.75
9	47.87	47.09	48.64	50	56.04	55.27	56.82
10	48.37	47.60	49.15	60	56.91	56.14	57.69
12	49.24	48.46	50.01	70	57.64	56.87	58.42
14	49.98	49.21	50.76	80	58.28	57.51	59.06

Table 4.3: Approximate age at detection for different values of tumor size using the relationship  $V = \alpha + \beta \ln S + T + \Delta$ .  $\widehat{V}$  represents the approximate age at detection when  $\overline{T + \Delta}$  is considered as an estimator of the variable  $T + \Delta$ .  $\widehat{V}_l$  and  $\widehat{V}_u$  represent the approximate age at detection when the variable  $T + \Delta$  is estimated by the lower and upper values of the confidence interval for  $\overline{T + \Delta}$ , respectively.

when the CMNBUE constraint is considered. See Table 4.5. In this case, a 95 percent confidence interval for the mean of  $T + \Delta$  is (28.93157, 34.02738).

### Conclusions

Data from the Comprehensive Cohort Study performed by the German Breast Cancer Study Group are analysed in order to estimate the parameters in the non-deterministic model which describes the tumor growth when the patient's age at detection and the tumor size are known. Several descriptive statistics of the time onset variable plus a random delay were also obtained. For this propose, the model studied in Section 4.2 was applied. In order to realize how the multivariate ageing CMNBUE property influences in our results, the linear programming problem is solved in two different cases. In the first case the CMNBUE property is not included as a constraint in the problem whereas it is included in the second case.

### 4.3. AN APPLICATION TO REAL DATASETS

Logarithm of tumor size $\ln S$	Case 1		Case 2	
	Estimated residual mean $\overline{T + \Delta}$	Estimated residual median $\mathcal{Q}_{T+\Delta}(0.5)$	Estimated residual mean $\overline{T + \Delta}$	Estimated residual median $\mathcal{Q}_{T+\Delta}(0.5)$
23-28	31.9400	31.11791	31.47947	30.87292
28-30	30.93755	30.92949	31.38098	31.39608
30-33	25.52009	24.94253	26.76001	25.96104

Table 4.4: Estimated residual mean and median for different values of the logarithm of tumor size when the CMNBUE property is not (case 1) and is (case 2) considered as a constraint.

The hypothesis of the model are actually confirmed. The initially assumed NBUE property of the age  $V$  and the logarithm of tumor size  $\ln S$  are checked as well as the NBUE property of the estimated variable  $T + \Delta$ .

Under a non-deterministic exponential tumor growth, and from a conservative point of view, the estimations of parameters in the model (4.2.2) were  $\hat{\alpha} = -36.0356$  and  $\hat{\beta} = 2.0225$  when the multivariate property was not included as a constraint and  $\hat{\alpha} = -23.80051$  and  $\hat{\beta} = 1.58865$  when this property was included.

Estimations of the values  $t_i + \delta_i$ , defined as in (4.2.3), were obtained as the residuals of the smallest quantile regression of the response variable  $V$  and the independent variable  $\ln S$ . If the residuals in the regression are analysed without considering a classification of the value of the logarithm of the tumor size, then it is shown that the multivariate ageing property CMNBUE of vector  $(\ln S, T + \alpha + \beta \ln S + \Delta)$  causes higher estimations of the time of appearance of the first tumor cell plus a random delay (see Table 4.2). However, if the residuals are analysed putting the logarithm of tumor size variable into groups, then it can be observed that the multivariate ageing property causes lower estimation for the logarithm of tumor size less than a particular value (see Table 4.4). Graphically, this fact can be observed by means of the quantile regression lines in Figure 4.4. The lines cut each other at the point 28.2 and for

### 4.3. AN APPLICATION TO REAL DATASETS

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Tumor size ( <i>mm</i> )	$\widehat{V}_{\overline{T+\Delta}}$	$\widehat{V}_{lower}$	$\widehat{V}_{upper}$
3	44.80	42.26	47.36
4	46.17	43.63	48.73
5	47.23	44.70	49.79
6	48.10	45.56	50.66
7	48.84	46.30	51.40
8	49.47	46.94	52.03
9	50.04	47.50	52.59
10	50.54	48.00	53.09
12	51.41	48.87	53.96
14	52.14	49.60	54.70

Table 4.5: Estimations of age in detection for different values of tumor size considering only tumors with diameter between 3 and 14 mm.  $\widehat{V}_{\overline{T+\Delta}}$  represents the estimated age in detection when  $\overline{T+\Delta}$  is considered as an estimator of the variable  $T+\Delta$ .  $\widehat{V}_{lower}$  and  $\widehat{V}_{upper}$  represent the estimated age in detection when the variable  $T+\Delta$  is estimated by the lower and upper values of the confidence interval for  $\overline{T+\Delta}$ , respectively.

values of logarithm of tumor size smaller than this point, the quantile regression line considering the CMNBUE property is on the quantile regression line obtained without including this property.

Given that the CMNBUE property appears as an inherent property of the vector  $(\ln S, V)$ , attention was focused on the approximate age at detection, when this constraint is considered. Some interesting conclusions about the age at detection can be obtained from Table 4.3. Mammography is a well-known test commonly used in breast cancer screening. The main advantage of mammography is that it can find tumors that are too small to palpated and allows an early diagnosis to be obtained. However, the mammography has some disadvantages. False-negatives test results frequently occur in young women because, in this group, the breast tissue is more dense. The same occurs with the false-positives. They are more common in younger



women, women with a family history of breast cancer or women who have had previous breast biopsies. In addition, mammography exposes the breast to radiation which is considered as a risk factor for breast cancer. For women younger than 50 years of age, the risks from a radiation exposure or a false-positive test result may be greater than the benefits provided by annual mammogram screening. In light of these disadvantages of mammography in breast cancer screening, the test is recommended every two years in women between the ages of 50 and 74 years. (See, for example, the National Cancer Institute (USA) website <http://www.cancer.gov>, and the Spanish Society of Medical Oncology website <http://www.seom.org/>). However, the results in Table 4.3 show that tumors of a smaller size (tumor diameter between 3 mm and 14 mm) are detected at ages of less than 50 years. It is worth noting that the 40 percent of women in this study were younger than 50 years of age. Clear evidence is shown for the need to consider breast cancer screening in women younger than 50 years of age. However, given the disadvantages presented by mammography, other tests should be considered for younger women. This fact would justify the time and capital investment needed in research for new techniques to detect breast cancer and to consider younger women for breast cancer screening.

#### **4.3.2 Example 2. Prostate cancer data**

##### **Materials and Methods**

The data used in this example came from the Veterans Administration Cooperative Urological Research Group's (VACURG) studies (see Bay (1972) and Bay (1973) and the references therein). The VACURG was organized in 1960. The original group consisted of 14 hospitals, all of which had full-time urologists on their staffs at the

### 4.3. AN APPLICATION TO REAL DATASETS

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time. The purpose of the group was to conduct large-scale, prospective, statistically randomized clinical trials of treatment of urological disorders. There were three consecutive studies of treatment for cancer of the prostate. In the first the patients were admitted from 1960 until 1967, in the second study, from 1967 to 1969 and in the third study the patients were admitted from 1969 to 1972. The main purpose of this group was a study of the common treatments then in use for cancer of the prostate, since there was no general agreement about the best way to treat these patients. For our study, we only consider those patients that were admitted in the second study with stage III and IV tumors (stage III tumors are locally extended without evidence of distant metastasis and stage IV tumors have distant metastases). The data were taken from Andrews and Herzberg (1985). Initially, There were 506 patients but we only considered the 475 patients who completed the data for the age and tumor size variables.

Tumor size, initially given in  $cm^2$  from rectal examination, was transformed to the number of tumor cells per  $cm^3$ . For this, the tumor was assumed to be a symmetric ball in  $\mathbb{R}^3$  and it was considered that there were approximately  $10^{12}$  tumor cells per  $cm^3$ , (see Spanish Society of Medical Oncology website, <http://www.seom.org/>). A descriptive analysis was made for tumor size as well as patient's age at detection. The NBUE property for both variables was confirmed by using a statistical test given by Fernández-Ponce *et al.* (1996). The plots of their univariate excess-wealth functions were also used to recognize this property.

The smallest quantile regression results in a line that represents the relationship between the logarithm of tumor size and the age at detection. The residuals of this regression are considered as the estimation of the age at tumor onset plus a random

delay for each patient.

All statistical analyses were performed using **R** software. In particular, the *quantreg* package was used to solve the problems of quantile regression (see Appendix A in Koenker, 2005 and website <http://cran.r-project.org/web/packages/quantreg/index.html>).

### Results

The median age and the mean were 73 and 71.56 year, respectively (interquartile range: 70-76) and standard deviation was 6.98. The mean tumor size was  $5.2194 \times 10^{13}$  tumor cells (interquartile range:  $8.4104 \times 10^{12} - 7.2392 \times 10^{13}$ ) and standard deviation:  $6.4931 \times 10^{13}$ . Considering the logarithm of tumor size variable, it was obtained that the mean logarithm of tumor size was 30.8064 (interquartile range: 29.7605-31.9131), the median was 30.9431 and the standard deviation was 1.4023.

The box-and-whisker plots of the age and the logarithm of tumor size are shown in Figure 4.5, respectively. The  $p$ -values for the Shapiro-Wilk normality test was 0.00 for both cases, therefore the normality hypothesis is rejected for both variables.

The plots of the empirical excess-wealth (ew) functions of the age and logarithm of tumor size are given in Figure 4.6. It can be observed that the empirical ew function of the age variable is under the line from the point (1,0) to point (0,  $\mu_V$ ), where  $\mu_V$  is the sample mean age. Note that this line corresponds to the ew function of an exponential distribution with parameter  $\lambda = \mu_V$ . Similarly, it occurs for the logarithm of tumor size. This property is always verified for all variables having an NBUE distribution. Moreover, the NBUE property of age and the logarithm of tumor size was checked by means of the test for NBUE alternatives given by the Fernández-Ponce *et al.* (1996). The statistic for this test,  $\Psi(F_n(t))$ , is based on the empirical ew distribution and

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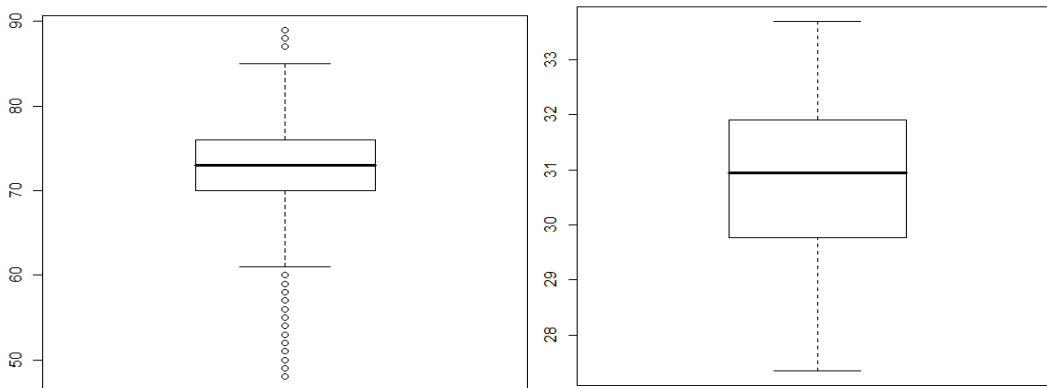


Figure 4.5: Box-and-whisker plots of age (left) and logarithm of tumor size (right).

it has really interesting asymptotic properties. It is obtained that  $\Psi(F_n(t)) = 0.4901$  for age and  $\Psi(F_n(t)) = 0.4949$  for logarithm of tumor size. Therefore, both random variables have an NBUE distribution (see critical values for NBUE alternatives in Table 4.1). In contrast with these variables, the variable tumor size  $S$  does not have this ageing property, but the dual property. In Figure 4.6 c), it can be seen that the empirical ew function is on the line from the point  $(1,0)$  to point  $(0, \mu_S)$ , where  $\mu_S$  is the sample mean of tumor size.

When the constraint  $(\ln S, T + \alpha + \beta \ln S + \Delta) \in \mathcal{F}_{CMNBUE}$  was not included in the quantile regression problem, the estimates intercept and slope of the minimum quantile regression line were 35.4520 and 0.4327, respectively.

The mean and the standard deviation for the residual variable obtained from the quantile regression was 22.78415 and 7.011316, respectively. A 95 percent confidence interval for this mean was (22.15201; 23.41628) and standard error was 0.3217. Recall that, from a conservative point of view, the  $i$ th residual in the quantile regression is considered as the estimation of the value  $t_i + \delta_i$ . Therefore, it would be said that the

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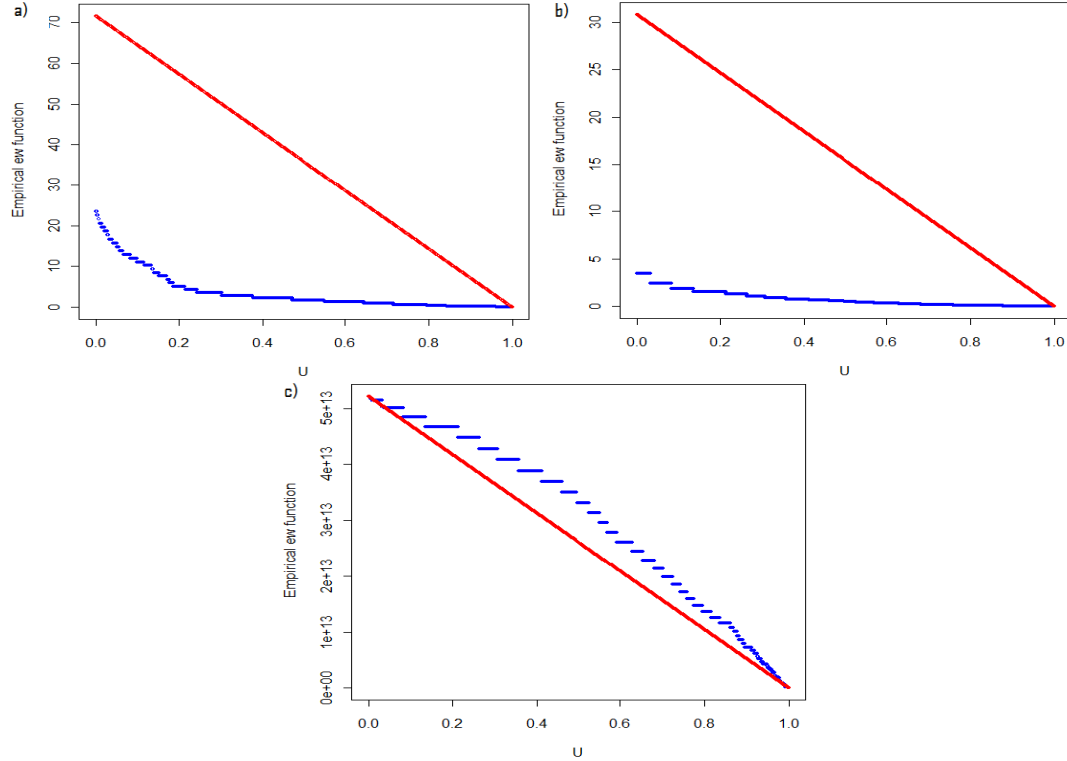


Figure 4.6: (a) Empirical ew function of age (blue line) and empirical ew function for exponential with parameter equal to mean age (b) Empirical ew function of logarithm tumor size (blue line) and empirical ew function for exponential with parameter equal to mean logarithm tumor size (c) Empirical ew function of tumor size (blue line) and empirical ew function for exponential with parameter equal to mean tumor size.

mean age at appearance of the first tumor cell plus a random delay was approximately 22.78415 years, when the CMNBUE constraint is not considered. It was also proved that the variable  $T + \Delta$  has an NBUE distribution, given that the value of the statistic for the test for NBUE alternatives was  $\Psi(F_n(t)) = 0.4562$ .

The Scatterplot of the logarithm of tumor size vs the age of the patients at detection is given in Figure 4.7. Superimposed on the plot is the 0.0001-quantile regression line in red.

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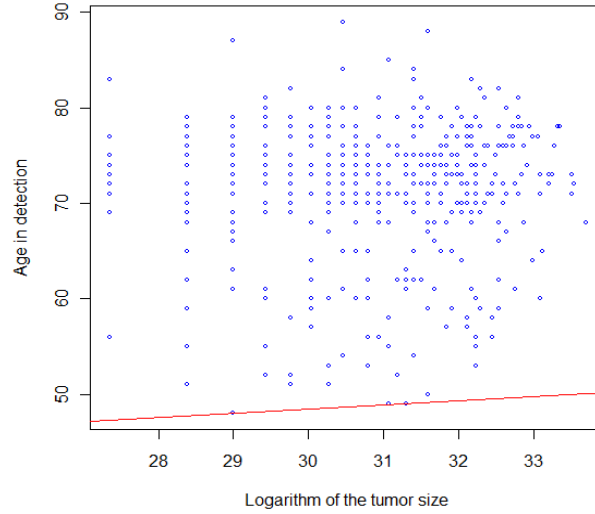


Figure 4.7: Scatterplot of the age at detection vs the logarithm of the tumor size. The line red is the 0.0001-quantile regression line without considering the CMNBUE property as a constraint

If the property  $(\ln S, T + \alpha + \beta \ln S + \Delta) \in \mathcal{F}_{CMNBUE}$  is included as a constraint in the quantile regression problem, the estimated intercept and slope are the same as that in the above case. This is due to the data initially verify the CMNBUE property. Note that the estimated values of  $\alpha$  and  $\beta$  are positive and therefore they verify the constraint (4.2.12). Therefore, for this particular set of data, all results obtained when the multivariate ageing property is included as a constraint in the problem are the same as that when this property is not included.

Considering the estimations of the parameters  $\alpha$ ,  $\beta$  and the estimation of the mean of the variable  $T + \Delta$  obtained as the mean of the residuals in the quantile regression, the patients's age at detection is approximate for different tumor sizes using the relationship  $\widehat{V} = \widehat{\alpha} + \widehat{\beta} \ln S + \overline{T + \Delta}$ . Table 4.6 shows the results. In the third and fourth columns, the approximate age at detection is also given when the variable

$T + \Delta$  is estimated by the lower and upper values of the confidence interval for  $\overline{T + \delta}$ , respectively.

Tumor size ( <i>mm</i> )	$\hat{V}$	$\hat{V}_l$	$\hat{V}_u$
11	70.03	69.40	70.67
12	70.14	69.52	70.78
13	70.25	69.62	70.88
14	70.35	69.72	70.98
15	70.43	69.81	71.07
16	70.52	69.89	71.15
17	70.60	69.97	71.23
18	70.67	70.04	71.31
19	70.74	70.11	71.38
20	70.81	70.18	71.44
30	71.33	70.71	71.97
40	71.71	71.08	72.34
50	72.00	71.37	72.63

Table 4.6: Approximate age at detection for different values of tumor size using the relationship  $V = \alpha + \beta \ln S + T + \Delta$ .  $\hat{V}$  represents the approximate age at detection when  $\overline{T + \Delta}$  is considered as an estimator of the variable  $T + \Delta$ .  $\hat{V}_l$  and  $\hat{V}_u$  represent the approximate age at detection when the variable  $T + \Delta$  is estimated by the lower and upper values of the confidence interval for  $\overline{T + \Delta}$ , respectively.

### Conclusions

Data from the Veterans Administration Cooperative Urological Research Group's (VACURG) studies were analysed in order to estimate the parameters in the non-deterministic model which describes the tumor growth when the patient's age at detection and the tumor size are known. Several descriptive statistics of the time onset variable plus a random delay were also obtained. For this propose, the model studied in Section 4.2 was applied. In order to realize how the multivariate ageing property

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CMNBUE influences in our results, the linear programming problem is solved in two different cases. In the first case the CMNBUE property is not included as a constraint in the problem whereas it is included in the second case. It is shown that the obtained results are the same in both cases given that the considered set of data initially verified this multivariate ageing property.

The hypothesis of the model are actually confirmed. The initially assumed NBUE property of the age variable  $V$  and the logarithm of tumor size variable  $\ln S$  are checked as well as the NBUE property of the estimated variable  $T + \Delta$ .

Under a non-deterministic exponential tumor growth, and from a conservative point of view, the estimations of parameters in the model (4.2.2) were  $\hat{\alpha} = 35.4520$  and  $\hat{\beta} = 0.4328$ . Estimations of the values  $t_i + \delta_i$  were obtained as the residuals of the smallest quantile regression of the response variable  $V$  and the independent variable  $\ln S$ . As a result of these estimations, mean age of appearance of the first tumor cell plus a random delay was approximately 22.78 years.

The patients' ages at detection were also approximate for different values of tumor size. The results in Table 4.6 are consistent with the current research on prostate cancer. In most men with prostate cancer, the disease grows very slowly and patients do not experience symptoms during the early stage. Therefore this kind of cancer is usually detected in older men (around 70 years of age). Different clinical trials to prevent and to screen for prostate cancer had been, and are being, developed. Two screening tests are commonly used to detect prostate cancer in the absence of symptoms. These are the digital rectal examination (DRE), in which a doctor palpates the prostate through the rectum to find hard or lumpy areas, and a blood test that detects a substance made by the prostate called prostate-specific antigen (PSA). Together, these



tests can detect many prostate cancers that have not produced symptoms. However, the decision to screen can be controversial. Prostate cancer can develop into a fatal, painful disease, but it can also develop so slowly that it may never cause problems during the man's life. The majority of men with early prostate cancer diagnosis have a good rate of survival after their diagnosis. Even without treatment, many of these men may not die of prostate cancer, but rather will live with it until they eventually die of some other cause. Neither of the screening tests for prostate cancer is perfect. The DRE and PSA are associated with false positives and false negatives. Using the DRE and PSA together will miss fewer cancers but also increases the number of false positives and subsequent biopsies in men without cancer. Moreover, treatment for prostate cancer with surgery or radiation often leaves patients with sexual impotence and urinary incontinence. Different organizations are supporting research to learn more about screening for prostate cancer and to determine whether screening with PSA tests and DREs reduces the death rate from this disease. Researchers are also assessing the risks of screening. (see the National Cancer Institute and Sociedad (USA) website ,<http://www.cancer.gov/cancertopics/types/prostate> and the Spanish Society of Medical Oncology website, <http://www.seom.org/>).

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# Conclusions

This last epigraph briefly summarizes the conclusions which are obtained in this work. As was pointed out in the Introduction, the univariate ageing properties and the stochastic orderings are closely connected. In the same way, an close link exists between the ageing properties of a random variable and its ew function. An important element which is presented in this relationship is the quantile function. The aim of this work is to generalize these relationships to the multivariate case.

In Chapter 2, the upper-corrected orthant concept, introduced by Fernández-Ponce and Suárez-Llorenz (2003) is studied in depth. This concept together with the  $\mathbf{u}$ -quantile allow the multivariate version of the ew function to be defined. That was shown that interesting properties of the univariate ew function are preserved in the multivariate case. Using the multivariate ew function, a new multivariate dispersive order was defined and proved to be weaker than that defined by Fernández-Ponce and Suárez-Llorenz (2003). As a particular case, this order between vectors with the same dependence structure is studied, that is, vectors which have the same copula.

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Chapter 3 contains the fundamental results of our work. The concepts of median residual life and failure rate are generalized in terms of the probability of the upper-corrected orthant. The new multivariate versions of the NBUE, DMRL and IFR distributions are defined and a chain of implications established between them. Given that the multivariate ageing properties are defined in support of the considered vector and that the mew function is defined in the region  $[0, 1]^n$ , additional conditions of the vector, e.g. the CIS property, had to be considered to get a characterization of the multivariate lifetime distributions in terms of the mew function. Finally, orderings for multivariate life distribution are presented.

In Chapter 4, the CMNBUE property is applied in oncology. This property appears as an inherent property of the vector whose components describe the logarithm of the tumor size and the age of the patient at detection. We studied the effect of this property when the parameters in a non-deterministic model which describe the growth of tumor size are estimated in two real database. This work reflects the need to develop research in all scientific fields in order to decrease the minimum patient'age in official medical examinations.

Finally, in considering the numerous applications of the ageing property and the excess-wealth function in the univariate dimension, in areas such as economics, reliability and survival analysis, in the future, it is in our interests to generate these applications in multivariate cases.

# Bibliography

- [1] Albert, A., Gertman, P.M. and Louis, T.A. (1978 a). Screening for the early detection of cancer. I. The temporal natural history of a progressive disease state. *Mathematical Biosciences*, **40**, 1–59.
- [2] Albert, A., Gertman, P.M., Louis, T.A. and Liu, S.I. (1978 b). Screening for the early detection of cancer. II. The impact of the screening on the natural history of the disease. *Mathematical Biosciences*, **40**, 61–109.
- [3] Andrews, D.F, and Herzberg, A.M. (1985). *DATA. A Collection of Problems from Many Fields for the Student and Research Worker*. Springer-Verlag, New York .
- [4] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992). *A First Course in Order Statistics*. John Wiley & Sons, New York .
- [5] Allison, R.A. and Foster, J.E.. (2004). Measuring health inequality using qualitative data, *Jour. of Health Economics.*, **23**, 3, 505–524.
- [6] Arjas, E. and Lehtonen, T. (1978). Approximating many server queues by means of single server queues, *Math. Oper. Res.*, **3**, 205–223.

## BIBLIOGRAPHY

---

- [7] Bagai, I. and Kochar, S.C. (1986). On tail-ordering and comparison of failure rates. *Communications in Statistics-Theory and Methods*, **15**, 4, 1377-1388.
- [8] Balakrishnan, N., and Rao, C.R. (1998a). *Handbook of Statistics. Order statistics: Theory & Methods*. North-Holland, The Netherlands.
- [9] Balakrishnan, N., and Rao, C.R. (1998b). *Handbook of Statistics- Order statistics: Applications*. North-Holland, The Netherlands.
- [10] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, (Holt, Rinehart and Winston.) NewYork.
- [11] Bartoszewicz, J. (1985). Moment inequalities for order statistics from ordered families of distributions. *Metrika*, **32**, 383-389.
- [12] Bartoszewicz, J. (1985). Dispersive ordering and monotone failure rate distributions, *Advances in Applied Probability*, **17**, 472-474.
- [13] Bartoszewicz, J. (1987). A note on dispersive ordering defined by hazard functions. *Statistics and Probability Letters*, **6**, 13-16.
- [14] Bartoszewicz, J. (1995). Stochastic order relations and the total time on test transform, *Statistics and Probability Letters*, **22**, 103-110.
- [15] Bartoszewicz, J. (1997). Dispersive functions and stochastic orders, *Applicationes Mathematicae*, **24**, 429-444.
- [16] Bartoszyński, R., Edler, L., Hanin, L., Kopp-Schneider, A., Pavlova, L., Tsodikov, A., Zorin, A. and Yakolev, A.Y. (2001). Modeling cancer detection:

- tumor size as a source of information on unoboservable stages of carcinogenesis. *Mathematical Biosciences*, **171**, 2, 113–142.
- [17] Bassan, B., Kochar, S. and Spizzichino, F. (2002). Some bivariate notions of IFR and DMRL and related properties, *Journal of Applied Probability*, **39**, 3, 533–544.
- [18] Bassan, B. and Spizzichino, F. (1999). Stochastic comparisons for residual lifetimes and Bayesian notions of multivariate ageing, *Advances in Applied Probability*, **31**, 1078–1094.
- [19] Bassan, B. and Spizzichino, F. (2005a). Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, *Journal of Multivariate Analysis*, **93**, 313–339.
- [20] Bassan, B. and Spizzichino, F. (2005b). Bivariate survival models with Clayton aging functions, *Insurance: Mathematics and Economics*, **37**, 1, 6–12.
- [21] Bassett, G.W.(1988). A property of the observations fit by the extreme regression quantiles. *Computational Statistics and Data Analysis*, **6**, 353–359.
- [22] Basu, A.P. (1971). Bivariate Failure rate, *J, Amer. Statist. Assoc.*, **66**, 103–104.
- [23] Byar, D.P. (1972). Treatment of prostatic cancer: studies by the Veterans Administration cooperative urological research group. *Journal of Urban Health : Bulletin of the New York Academy of Medicine* .; **48**, 5, 751766.
- [24] Byar, D.P. (1973). The veterans administration cooperative urological research group’s studies of cancer of the prostate *Cancer* **32**, 5, 1126–1130.

## BIBLIOGRAPHY

---

- [25] Belzunce, F.; Caldel, J. and Ruiz, J.M. (1996). Dispersive ordering and Characterizations of aging classes, *Statistics and Probability Letters*, **54**, 321–327.
- [26] Belzunce, F. (1999). On a characterization of right spread order by the increasing convex order, *Statistics and Probability Letters*, **45**, 2, 103–110.
- [27] Belzunce, F.; Pinar, J.F. and Ruiz, J.M. (2001). A family of tests for right spread order, *Statistics and Probability Letters*, **54**, 1, 79–92.
- [28] Boland, P., Joag-Dev, K., Hollanader, M. and Kocher. S. (1996). Bivariate dependence properties of order statistics, *Journal of Multivariate Analysis*, **56**,1, 75–89.
- [29] Block, H. W. and Sampson, A. R. (1988). Conditionally Ordered Distributions, *Journal of Multivariate Analysis*, **27**, 91–104.
- [30] Brown, B. M. and Hettmansperger, T. P. (1987). Affine invariant rank methods in the bivariate location model, *Journal of the Royal Statistical Society B*, **49**, 301–310.
- [31] Brown, B. M. and Hettmansperger, T. P. (1989). An affine invariant bivariate version of the sign test, *Journal of the Royal Statistical Society B*, **51**, 117–126.
- [32] Brunk, H. D. (1964). Integral inequalities for functions with nondecreasing increments, *Pacific Journal of Mathematics*, **14**, 783–793.
- [33] Chaudhuri, P. (1996). On a geometric notion of quantiles for multivariate data, *Journal of the American Statistical Association*, **91**, No 434, 862–872.



- [34] Chakraborty, B. and Chaudhuri, P. (1999). A note on the robustness of multivariate medians, *Statistics and Probability Letters*, **45**, 3, 269–276.
- [35] Chen, L. A. and Welsh, A. H. (2002). Distribution-function-based bivariate quantiles, *Journal of Multivariate Analysis*, **83**, 1, 208–231.
- [36] Colangelo, A.; Scarsini, M. and Shaked, M. (2005). Some notions of multivariate positive dependence, *Insurance: Mathematics and Economics*, **37**, 1, 13-26.
- [37] Dhaene, J. ; Laeven, R. J. A. ; Vanduffel, S. ; Darkiewicz, G. and Goovaerts, M.J. (2008). Can a coherent risk measure be too subadditive?, *Journal of Risk and Insurance*, **75**, No. 2, 365-386.
- [38] David, H.A. (1970). *Order statistics*. John Wiley & Sons, New York. Second edition 1981.
- [39] Davidov, O. and Zelen, M. (2004). Overdiagnosis in early detection programs. *Biostatistics*, **5** (4), 603-613.
- [40] Denuit, M. and Vermandele, C. (1999). Lorenz and excess wealth orders, with applications in reinsurance theory. *Scandinavian Actuarial Journal*, **2** , 170-185.
- [41] Denuit, M. ; Dhaene, J; Goovaerts, M. and Kaas, R. (2005). *Actuarial Theory for Dependent Risks*. New York: Wiley.
- [42] Deshpandé, J.V. and Kochar, S.C. (1983). Dispersive ordering is the same as tail-ordering, *Advances in Applied Probability*, **15**, 686–687.

## BIBLIOGRAPHY

---

- [43] Dharmadhikari, A.D., Ghosh, J.K. and Kuber, M.M. (1997). On NBUE property of one component system supported by an inactive standby and a repair facility, *Journal of Statistical Planning and Inference*, **64**, 1, 1–8.
- [44] Di Crescenzo, A. and Pellerey, F. (1998) On lifetimes in random environments *Naval Research Logistics*, **45**, 365 –375.
- [45] Di Crescenzo, A. (1999) Dual stochastic orderings describing ageing properties of devices of unknown age *Stochastic Models*, **15**, 3, 561 –576.
- [46] Di Crescenzo, A. (2000) Some results on the proportional reversed hazards model *Statistics and Probability Letters*, **50**, 313 –321.
- [47] Di Crescenzo, A. and Longobardi, M. (2006) On the NBUE ageing notion within the competing risk model *Journal of Statistical Planning and Inference*, **136**, 1838 –1654.
- [48] Fagioli, E. and Pellerey, F. (1993). New partial orderings and applications, *Naval Research Logistics*, **40**, 6, 829–842.
- [49] Fernández-Ponce, J.M., Infante-Macas, R. and Muñoz-Pérez, J. (1996). Characterization of lifetime distributions based on a quantile dispersion measure, *Computational Statistics & Data Analysis*, **21**, 5, 547–561.
- [50] Fernández-Ponce, J.M., Kochar, S.C. and Muñoz-Pérez, J. (1998). Partial orderings of distributions based on right-spread functions, *Journal of Applied Probability*, **35**, 1, 221–228.

- [51] Fernández-Ponce, J.M. and Suárez-Lloréns, A. (2003). A multivariate dispersion ordering based on quantiles more widely separated. *Journal of Multivariate Analysis*, **85**, 1, 40-53.
- [52] Giovagnoli, A. and Wynn, H.P. (1995). Multivariate dispersion orderings. *Statistics and Probability Letters*, **22**, 4, 325-332.
- [53] Ghosh, D. (2006). Modelling tumour biology progression relationships in screening trials. *Statistics in Medicine*, **25**, 11, 1872-1884
- [54] Ghosh, D.; Banerjee, M and Biswas, P. (2008). Inference for constrained estimation of tumor size distributions. *Biometrics*, **64**, 4, 1009–1017.
- [55] Gregori, G.; Hanin, L.; Luebeck, G.; Moolgavkar, S. and Yakovlev, A. (2002). Testing goodness of fit for stochastic models of carcinogenesis. *Mathematical Biosciences*, **175**, 1, 13–29.
- [56] Hanin, L.G. and Boucher, K.M. (1999). Identifiability of parameters in the Yakovlev-Polig model of carcinogenesis. *Math. Biosci.*, **160**, 1, 1–24.
- [57] Harville, D.A. (1997). *Matrix Algebra from a Statistician's Perspective*. Springer-Verlag, New York, Inc.
- [58] Hanin, L.G. and Boucher, K.M. (1999). Identifiability of parameters in the Yakovlev-Polig model of carcinogenesis. *Math. Biosci.*, **160**, 1, 1–24.
- [59] Hanin, L.G. (2002). Identification problem for stochastic models with application to carcinogenesis, cancer detection and radiation biology, *Discrete Dynamics in Nature and Society*, **7**, 3, 177–189.

## BIBLIOGRAPHY

---

- [60] Hanin, L. G.; Yakovlev, A. (2007) Identifiability of the joint distribution of age and tumor size at detection in the presence of screening. *Mathematical Bioscience*, **208**, no. 2, 644–657.
- [61] Hickey, R.J. (1986). Concepts of dispersion in distributions: a comparative note, *Journal of Applied Probability*, **23**, 914–921.
- [62] Hu, T. ; Khaledi, B.E. and Shaked, M. (2001a). Multivariate hazard rate orders, *Journal of Multivariate Analysis*, **84**, 1, 173–189.
- [63] Hu, T. and Wei, Y. (2001b). Multivariate stochastic comparisons of inspection and repair policies, *Statistics and Probability Letters*, **53**, 3, 315–324.
- [64] Johnson, W. and Geisser, S. (1983). A predictive view of the detection and characterization of influential observations in regression analysis. *Journal of the American Statistical Association*, **78**, Number 381, 137-144.
- [65] Johnson, N.L and Kotz., S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis* **5**, 56-66.
- [66] Karlin, S. and Rinott, Y. (1980). Classes of orderings of measures and related correlation inequalities, *Journal of Multivariate Analysis*, **10**, 4, 467–498.
- [67] Klar, B. and Mller, A. (2003). Characterizations of classes of lifetime distributions generalizing the NBUE class *Journal of Applied Probability*, **40**, 1, 20–32.
- [68] Kochar, S.C. and Wiens, D.P. (1987). Partial orderings of life distributions with respect to their aging properties, *Naval Research Logistics*, **34**, 6, 823–829.

- [69] Kochar, S.C. (1989). On extensions of DMRL and related partial orderings of life distributions, *Communications in Statistics–Stochastic Models*, **5**, 2, 235–245.
- [70] Kochar, S.C. (1996). Dispersive ordering of order statistics. *Statistics and Probability Letters*, **27**,3, 271-274.
- [71] Kochar, S.C. and Carrière, K.C. (1997). Connections among various variability orderings, *Statistics and Probability Letters*, **35**, 4, 327–333.
- [72] Kochar, S.C., Xiaohu, Li and Shaked, M. (2002). The total time on test transform and the excess wealth stochastic orders of distributions, *Advances in Applied Probability*, **34**, (4), 826–845.
- [73] Koenker, R W (2005). *Quantile Regression*. Econometric Society Monographs. Cambridge University Press. New York .
- [74] Koenker, R W and Bassett, Jr. G. (1978). Regression quantiles, *Econometrica*, **46**, (1), 33–50.
- [75] Koenker, R W and Bassett, Jr. G. (1982). Robust tests for heterocedasticity based on regression quantiles, *Econometria*, **50**, 43–61.
- [76] Koenker, R W and D’Orey, V. (1987). Computing regression quantiles, *Applied Statistics*, **36**, (3), 383–393.
- [77] Koshevoy, G.A. and Mosler, K. (1997). Multivariate Gini Indices, *Journal of Multivariate Analysis*, **60**, 2, 252–276.

## BIBLIOGRAPHY

---

- [78] Li, H.; Scarsini, M. and Shaked, M. (1996). Linkages: a tool for the construction of multivariate distributions with given nonoverlapping multivariate marginals. *Journal of Multivariate Analysis*, **56**, 20–41.
- [79] Li, X. and Shaked, M. (2004). The observed total time on test and the observed excess wealth. *Statistics and Probability Letters*, **68**, 3, 247-258.
- [80] Li, X. (2006). A note on the right spread order associated with exponential distributions. *Journal of Systems Science and Complexity*, **19**, 1, 108-2113.
- [81] Lillo, R.E. (2000). Note on relations between criteria for ageing. *Reliability Engineering and System Safety*, **67**, 129–133.
- [82] Lillo, R.E. (2005). On the median residual lifetime and its ageing properties: a characterization theorem and applications. *Naval Research Logistics*, **52**, 370–380.
- [83] Liu, R.Y. (1988). On a notion of simplicial depth. *Proceedings of the National Academy of Sciences*, **85**, 6, 1732-1734.
- [84] Liu, R.Y. (1990). On a notion of data depth based on random simplices. *Annals of Statistics*, **18**, 1, 405-414.
- [85] McIntosh, M.W. and Urban, N. (2003). A parametric empirical Bayes method for cancer screening using longitudinal observations of a biomarker *Biostatistics*, **4**, 27-40.
- [86] Marshall, A.W. (1975). Some comments on the hazard gradient, *Stochastic Processes and their Applications*, **3**, 3, 293-300.

- [87] Muñoz-Pérez, J. (1990). Dispersive ordering by the spread function, *Statistics and Probability Letters*, **10**, 5, 407–410.
- [88] Müller, A. and Scarsini, M.(2001). “Stochastic comparison of random vectors with a common copula.”, *Math. Oper. Res.* , **26**, 4, 723-740.
- [89] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. Chischester: John Wiley Sons, Ltd .
- [90] Nelsen , R. B. (1999). *An Introduction to Copulas*. New York: Springer-Verlang.
- [91] O’Brien, G. (1975). The comparison method for stochastic processes, *Ann. Prob.*, **3**, 1, 80–88.
- [92] Oja, H. (1983). Descriptive statistics for multivariate distributions. *Statistics and Probability Letters*, **1**, 327-332.
- [93] Pecaric, J.E.; Prochan, F. and Tong, Y.L. (1992), *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc.
- [94] Pellerey, F. and Shaked, M. (1997). Characterizations of the IFR and DFR aging notions by means of the dispersive order, *Statistics and Probability Letters*, **33**, 4, 389–393.
- [95] Puri, P.S. and Rubin, H. (1974). On a characterization of the family of distributions with constant multivariate failure rates, *The Annals of Probability*, **2**, 738–740.
- [96] Rojo, J. and He, G.Z. (1991). New properties and characterizations of the dispersive ordering, *Statistics and Probability Letters*, **11**, 365-372.

## BIBLIOGRAPHY

---

- [97] Roy, D. (1994). Classification of life distributions in multivariate models. *IEEE Transactions on Reliability*, **43**, 2, 224-229.
- [98] Roy, D. (2001). Some properties of a classification system for multivariate life distributions. *IEEE Transactions on Reliability*, **50**, 2, 214-220.
- [99] Rüschemdorf, L. (1981). Stochastically ordered distributions and monotonicity of the OC-function of sequential probability ratio tests, *Math. Operationsforsch Statist.*, **12**, 3, 327–338.
- [100] Rubinstein, R.Y., Samorodnitsky, G. and Shaked, M. (1985). Antithetic variates, multivariate dependence and simulation of stochastic systems. *Management Science*, **31**, 1, 66–77.
- [101] Rüschemdorf, L. (2004). Comparison of multivariate risks and positive dependence, *Journal of Applied Probability*, **41**, 2, 391–406.
- [102] Saunders, D.J. (1984). Dispersive ordering of distributions, *Advances in Applied Probability*, **16**, 693–694.
- [103] Scarsini, M. and Shaked, M. (1999). Distributions with known initial hazard rate functions, *Journal of Statistical Planning and Inference*, **78**, 1, 39–55.
- [104] Shaked, M. (1982). Dispersive ordering of distributions. *Journal of Applied Probability*, **19**, 310-320.
- [105] Shaked, M. (1985). Ordering distributions by dispersions, *in: Encyclopaedia of Statistical Sciences* Johnson and Kotz, ed., **Vol. 6**, 485-490.



- [106] Shaked, M. and Shanthikumar, J.G. (1987). Multivariate hazard rates and stochastic ordering, *Advances in Applied Probability*, **19**, 123–137.
- [107] Shaked, M. and Shanthikumar, J.G. (1990). Multivariate stochastic orderings and positive dependence in reliability theory, *Math. Oper. Res.*, **15**, 3, 545–552.
- [108] Shaked, M. and Shanthikumar, G. (1997). Supermodular stochastic orders and positive dependence of random vectors, *Journal of Multivariate Analysis*, **61**, 1, 86–101.
- [109] Shaked, M. and Shanthikumar, J.G. (1998). Two variability orders, *Probability in the Engineering and Informational Sciences*, **12**, 1, 1–23.
- [110] Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer Series in Statistics, Springer.
- [111] Shen, Y. and Zelen, M. (2005). Robust modeling in screening studies: estimation of sensitivity and preclinical sojourn time distribution. *Biostatistics*, **6**(4), 604–614.
- [112] Schmoor, C., Olschewski, M. and Schumacher, M. (1996). Randomized and non-randomized patients in clinical trials: experiences with comprehensive cohort studies.. *Statistics in Medicine*, **15**, 3, 263–271.
- [113] Small, C. G. (1990). A survey of multidimensional medians. *International Statistical Review*, **58**, 3, 263–277.
- [114] Smith, R. L. (1994). Non-regular regression. *Biometrika*, **81**,1 173–183.

## BIBLIOGRAPHY

---

- [115] Sordo, M.A. (2008). Characterizations of classes of risk measures by dispersive orders. *Insurance: Mathematics and Economics*, **42**, 3, 1028–1034.
- [116] Sordo, M.A. (2009a). On the relationship of location-independent riskier order to the usual stochastic order. *Statistics and Probability Letters*, **79**, 2, 155–157.
- [117] Sordo, M.A. (2009b). Comparing tail variabilities of risks by means of the excess wealth order. *Insurance: Mathematics and Economics*, **45**, 3, 466–469.
- [118] Stoyan, D. (1983). *Comparison Methods for Queues and Other Stochastic Models*, (Wiley, New York).
- [119] Tsodikov, A.D.; Ibrahim, J.G. and Yakovlev, A.Y. (2003). Estimating cure rates from survival data: an alternative to two-component mixture models, *Journal of the American Statistical Association*, **Vol. 98**, No. 464, 1063–1078.
- [120] Tsodikov, A. and Müller, W.A. (1998), Modeling carcinogenesis under a time-changing exposure. *Mathematical Biosciences*, **152**, 2, 179–191.
- [121] Weber, Y. (1909), *Über den Standort der Industrien*, Tübingen. *English translation by Friedrich, C.J. (1929), Alfred Weber's Theory of Location of Industries*, University of Chicago Press. *Stochastic Models*, **23**, 4, 575–591.
- [122] Wei, Y. and Hu, T. (2007), Characterizations of aging classes in terms of spacings between record values. *Stochastic Models*, **23**, 4, 575–591.
- [123] Spanish Society of Medical Oncology, <http://www.seom.org/>
- [124] <http://www.r-project.org/>