Computing stress singularities in transversely isotropic multimaterial corners by means of explicit expressions of the orthonormalized Stroh-eigenvectors.

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Abstract: Composite materials reinforced by unidirectional long fibers behave macroscopically as homogeneous transversely isotropic linear elastic materials. A general, accurate and computationally efficient procedure for the evaluation of singularity exponents and singular functions characterizing singular stress fields in multimaterial corners involving this kind of material is presented in this paper. To take full advantage of the sextic Stroh formalism of anisotropic elasticity applied to this particular problem, the complete set of explicit expressions of the eigenvalues and eigenvectors of the real 6×6 fundamental elasticity matrix **N** has been deduced for all the non-degenerate and degenerate (repeated roots of the sextic Stroh equation) cases. These expressions will also facilitate further applications of the Stroh formalism to these materials. Several numerical examples of singularity analysis of multimaterial corners appearing in adhesively bonded joints and damaged cross-ply laminates of composite materials are presented.

Key words: corner singularities, transversely isotropic materials, Stroh formalism, composites, adhesively bonded joints, delamination.

1. Introduction.

The Linear Theory of Elasticity predicts, in general, singular (unbounded) stresses at discontinuities in geometry, material properties and boundary conditions. Configurations where different materials converge at one point can be easily found, for instance, in composite structures, adhesively bonded joints and microelectronics. These configurations, called *multimaterial corners* or cross points, are potential places where failure can initiate due to these singular stress fields.

In the present paper special interest is focused on piecewise homogeneous multimaterial corners involving transversely isotropic materials, representing composite laminas reinforced by unidirectional long fibers, and subjected to a generalized plane strain state. These laminas play an important role in composite applications due to their high specific stiffness and strength.

According to a general mathematical formulation of a corner problem, which can be found in Kondratev [1], Costabel and Dauge [2] and Nicaise and Sändig [3], the displacement components u_i (i=1,2,3) in the neighbourhood of the corner tip, where a polar coordinate systems (r,θ) is considered, can be written as a sum of terms, each one of them having the following form:

$$u_i(r,\theta) = K\, r^{\delta} \log^{p} r\, g_i(\theta,\delta,p), \quad p = 0,1,\ldots \qquad (1)$$

where (real or complex) numbers δ are called *singularity exponents* and functions g_i are called *singular* (or *characteristic*) *functions*. Both δ and g_i depend only on the local corner configuration (geometry, materials and boundary conditions at the corner tip) and are usually evaluated by means of a quadratic or a nonlinear eigenvalue problem, where δ are given by eigenvalues and g_i are obtained from the associate eigenvectors. The logarithmic terms are present in (1) if and only if the algebraic multiplicity of an eigenvalue is greater that its geometric multiplicity. This happens, for instance, in the transition from two real to two complex conjugates roots, a very particular case that is not considered in the present study, only non logarithmic singularities being considered in what follows.

In the asymptotic series expansion of an elastic solution in the corner tip neighbourhood, the coefficients K of the terms in the form (1) for displacements, and of the corresponding terms for stresses, are called *generalized stress intensity factors*, which depend on the overall problem formulation (the global domain, materials and boundary conditions).

A general, accurate and computationally efficient procedure for the evaluation of singularity exponents δ and singular functions g_i should: a) cover all kind of homogeneous transversely isotropic linearly elastic materials at any spatial orientation, b) consider any finite number of homogeneous wedges converging at the corner tip and perfectly bonded between them, c) consider all kind of standard homogeneous boundary conditions, and d) provide, in general, expressions as most analytic and compact as possible.

Although in the past several general excellent procedures for singularity analysis of anisotropic linear elastic multimaterial corners have been developed, none of them combined all the above listed features. Let us mention, at least, a few outstanding computational procedures implemented: Leguillon and Sanchez-Palencia [4] constructed eigenvalue problems by a finite element discretization in the angular variable; Papadakis and Babuška [5] obtained eigenvalue problems by using a numerical solver for systems of ordinary differential equations and a shooting technique; Yosibash [6] applied finite element discretizations of a modified Steklov formulation in an angular sector of an annulus; Costabel et al. [7] particularized the general solution basis for the corresponding system of ordinary differential equations in the angular variable deduced in [2] to the elastic corner problem, and constructed the eigenvalue problems in an analytic way, with an exception of the computation of the roots of a characteristic equation for each homogeneous material included in the corner. While the former three procedures are numerically based, not working with a closed form expression of the eigenvalue problem, the later one is analytically based, starting from a closed form representation of the elastic solution at a corner and presenting a closed form expression of the eigenvalues problem. Any analytically based procedure for the corner singularity analysis is expected to be faster than numerically based ones and allowing also a more detailed analysis of the position and nature of singularity exponents (eigenvalues), in particular the relation of its algebraic and geometric multiplicities. A disadvantage of the procedure developed by Costabel et al. [7] is that it does not cover all kind of anisotropic materials (in particular not the degenerate ones) and that the size of the eigenvalue problem depends on the number of homogeneous wedges included in the corner.

The present paper is aimed to develop a general procedure which satisfies the above mentioned requirements for the evaluation of singularity exponents δ and singular functions g_i.

It appears that the most suitable analytic approach to formulate the eigenvalue problem corresponding to a multimaterial corner in a compact form is based on the so-called Lekhnitskii-Eshelby-Stroh complex variable formalism of two dimensional anisotropic elasticity (see Lekhnitskii [8], Eshelby *et al.* [9], Stroh [10,11], (see Ting [12], for a comprehensive review), and in the following referred to as Stroh formalism. Ting [13], Wu [14], Barroso *et al.* [15], Yin [16] and Hwu *et al.* [17] showed that this formalism is a powerful analytical tool for the singularity analysis including anisotropic materials, and thus it also provides the theoretical basis for the present work.

The elastic solution representation in the Stroh formalism is based on knowledge of the eigenvectors of the real 6×6 *fundamental elasticity matrix* **N**, which is associated with a specific orientation of the coordinate system with respect to the material. To take full advantage of the fundamental orthogonality and closure relations of the Stroh formalism the eigenvectors have to be properly orthogonalized and normalized.

Several authors have studied the Stroh formalism applied to tranversely isotropic materials. Tanuma [18] obtained the surface impedance tensor (associated with the linear relationship between displacements and tractions given at a surface), and Nakamura and Tanuma [19] and Ting and Lee [20] introduced independently new explicit closed forms of the 3D fundamental solution; for further developments of this solution see Távara *et al.* [21]. The complete set of explicit expressions of the orthonormalized eigenvectors is, to the authors' knowledge, not available for these materials at present. This is perhaps because, although it is possible to deduce them in a relatively straightforward way, the deduction is somewhat tedious as requires lengthy algebraic calculations. Tanuma [18] obtained explicit expressions of some of the Stroh eigenvectors for transversely isotropic materials in the dual coordinate systems as intermediate results in his procedure for the evaluation of the surface impedance tensor. Nevertheless, not all of them (the eigenvectors) were obtained in Tanuma's work (it was not necessary for the final result) and those obtained were neither normalized nor orthogonalized (again because it was not necessary for the final result). It is also important to mention that many applications do need the orthonormalized expressions of the Stroh eigenvectors in the coordinate system used in the present work and although the relationship between both coordinate systems is straightforward, lengthy and involved calculations are required for the final expressions to be explicitly obtained, as mentioned above.

Tanuma [18] showed how the relative orientation of a transversely isotropic material can make the matrix **N** mathematically degenerate (some of the eigenvalues are equal and the associate eigenvectors linearly dependent), leading to numerical instabilities when a critical relative orientation is approached, due to the fact that the eigenvectors are not continuously defined in the transition from the non-degenerate to degenerate cases. The explicit expressions of the eigenvectors obtained in the present work include, in addition to the mathematically non-degenerate cases, all mathematically degenerate cases for these materials in the framework of the Stroh formalism, the orthonormalization procedure being more cumbersome in these cases than in the non-degenerate cases. Using the results presented, no limitations will appear in applying the Stroh formalism to both non-degenerate and degenerate transversely isotropic materials.

A key aspect of this work is that it deals with the concept of degenerate materials in the framework of Stroh formalism, having repeated roots of the sextic Stroh equation with the algebraic multiplicity of the root (eigenvalue) greater that its geometric multiplicity. It has to be stressed that if the expressions for the degenerate materials are not available, only approximate results can be obtained. This problem has been typically overcome using slight perturbations of the values of the actual material constants (working in this way with a mathematically non-degenerate material). However, the accuracy of the results obtained by this procedure is not being known *a priori*.

The structure of the present paper is as follows. A brief review of the general Stroh formalism is presented in Section 2. Some relevant relations of the Stroh formalism when applied to transversely isotropic materials are introduced in Section 3, the main objective being to clarify how the eigenvectors transform between several coordinate systems used in different approaches of this formalism, an aspect not sufficiently treated in the literature. The complete set of new explicit expressions of the orthonormalized eigenvectors (associated with one of these systems) of the Stroh formalism for these materials deduced in the present work is introduced in the Appendix, together with some observations concerning the procedures used in their deduction. Although this is one of the main results of the paper, it has been decided to include it in the Appendix for the sake of an easy readability of the paper, due to the purely algebraic character of the developments. In addition to this, the inclusion in an Appendix allows the content to have an independent structure (ideas, numbering of equations,...) which makes it easier to be used by other researchers working in the Stroh formalism with transversely isotropic materials. Finally, an analytically based approach to carry out singularity analysis of linear elastic multimaterial corners is presented in Section 4, where some numerical studies of real corner configurations in adhesively bonded joints between metals and composites and damaged cross-ply laminates are presented as well.

2. The Stroh formalism

In a fixed rectangular coordinate system x_i (i=1,2,3), the equilibrium equations for a homogeneous lineraly elastic anisotropic material, with elastic constants C_{ijkl}, can be expressed in terms of the displacements u_i (i=1,2,3) as:

$$C_{ijkl} u_{k,lj} = 0\,, \tag{2}$$

where as usual the comma denotes differentiation and repeated indices denote summation. If displacements u_i only depend on the coordinates x_1 and x_2, a situation known as a *generalized plane strain state*, then (2) has a general solution expressed through an analytic function *f* of a complex variable *z*, which is a linear combination of x_1 and x_2.

$$u_i = a_i f(z)\,, \quad z = x_1 + p x_2\,, \tag{3}$$

where the complex vector a_i and the complex constant p are to be determined. The stresses can be obtained by means of the stress function vector, φ_i (i=1,2,3), which can be expressed as:

$$\varphi_i = b_i f(z), \tag{4}$$

with b_i being a complex vector, through the expressions: $\sigma_{i1} = -\varphi_{i,2}$, $\sigma_{i2} = \varphi_{i,1}$. Then, p and a_i defined in (3) should satisfy the following eigenrelation and the corresponding characteristic equation:

$$\left\{\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)p + \mathbf{T}p^2\right\}\mathbf{a} = \mathbf{0} \;\Rightarrow\; \left|\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)p + \mathbf{T}p^2\right| = 0 \tag{5}$$

where **Q**, **R** and **T** are real 3×3 matrices given by the elastic stiffnesses of the material, C_{ijkl}, as follows:

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2} \text{ and } T_{ik} = C_{i2k2}. \tag{6}$$

The vector b_i in (4) is related to a_i by the relation

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\,\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a} \tag{7}$$

The above relations can be rewritten to a standard eigensystem, which defines the real (6×6) Fundamental Elasticity Matrix **N** in terms of the stiffness constants of the material:

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}. \tag{8}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \; \mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{-T} - \mathbf{Q} \tag{9}$$

Eigenequation (8) defines six eigenvalues p_α and eigenvectors $\boldsymbol{\xi}_\alpha^T = \left[\mathbf{a}_\alpha^T, \mathbf{b}_\alpha^T\right]$ $(\alpha = 1,\ldots,6)$, providing the values of the constant p and vectors $\mathbf{a}$ and $\mathbf{b}$ in (3) and (4). Note that the eigenvalues p_α in (8) are also roots of the characteristic equation in (5).

Due to the positive definite character of the strain energy, the eigenvalues of **N** are complex, and p_α and $p_{\alpha+3}$ (α=1, 2, 3), can be considered as three pairs of complex conjugate numbers.

The relation of the algebraic and geometric multiplicities of the eigenvalues of **N** defines the structure of the general solution for stresses and displacements. If the algebraic and geometric multiplicities of the eigenvalues p_α are equal, and in particular if the eigenvalues p_α are all distinct, the general solution for displacements, u_i, and stress functions, φ_i, is expressed through the superposition of the six solutions from (3) and (4) respectively, where the overbar denotes the complex conjugate.

$$\mathbf{u} = \sum_{\alpha=1}^{3} \left\{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\right\}, \tag{10}$$

$$\boldsymbol{\varphi} = \sum_{\alpha=1}^{3} \left\{ \mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha) \right\}. \tag{11}$$

We refer to this kind of material as a mathematically non-degenerate one.

Although not being a requirement, in many theoretical developments, $f_\alpha(z_\alpha)$ have the same functional form, namely $f_\alpha(z_\alpha) = f(z_\alpha)q_\alpha$ (α=1,2,3) and $f_{\alpha+3}(\bar{z}_\alpha) = \bar{f}(\bar{z}_\alpha)\bar{q}_\alpha$ (α=1,2,3), where q_α are arbitrary complex constants.

If, however, some of the eigenvalues p_α are repeated and there are not linearly independent associated eigenvectors ξ_α (geometrical multiplicity is less than the algebraic one), the corresponding material is referred to as mathematically degenerate, and expressions in (10) and (11) are not valid. The structure of the formalism in the degenerate cases is much more cumbersome when compared to the non-degenerate case, see Ting & Hwu [22] and Wang & Ting [23], and also Ting [12]. Examples of applications of the Stroh formalism in problems with the simultaneous presence of non-degenerate and degenerate materials of any kind were presented by Barroso *et al.* [15], see also Yin [16].

One of the facts that make the Stroh formalism elegant and powerful is the fulfilment of the orthogonality and closure relations, which connect **a** and **b**. This is why the eigenvectors will be properly orthogonalized and normalized in the present work. Defining the Stroh matrices $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$, orthogonality and closure relations take the following compact form:

$$\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = \mathbf{I}, \tag{12}$$

with **X** and $\mathbf{X}^{-1}$ given as:

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = [\xi_1, \xi_2, \xi_3], \ \mathbf{X}^{-1} = \begin{bmatrix} \boldsymbol{\Gamma}\mathbf{B}^T & \boldsymbol{\Gamma}\mathbf{A}^T \\ \boldsymbol{\Gamma}\bar{\mathbf{B}}^T & \boldsymbol{\Gamma}\bar{\mathbf{A}}^T \end{bmatrix} = [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3], \tag{13}$$

where $\boldsymbol{\Gamma}$ is a Boolean matrix, defined for the degeneracy cases of **N** following Ting [24] as:

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{14}$$

non-degenerate   degenerate (p_2=p_3)   extraordinary degenerate

It should be pointed out that Γ, which is usually introduced for the degenerate case p_1=p_2 (see Ting [12]), is used here for the case p_2=p_3.

3. Stroh formalism for transversely isotropic materials.

As above, a reference coordinate system x_i (i=1,2,3) will be associated with the directions which define the generalized plane strain state. Let C_{ijkl} denote the tensor of elastic stiffnesses of a transversely isotropic material associated with this coordinate system. The objective of this section is to obtain explicit analytic expressions of the Stroh eigenvalues and orthonormalized eigenvectors of this material which are associated with the coordinate system x_i.

A material with a transversely isotropic elastic behaviour has a plane in which all directions have the same elastic properties or, equivalently, has an axis perpendicular to this plane, with an elastic rotational symmetry. Let us define a rectangular coordinate system attached to the material $\hat{x}_i$ (i=1,2,3), with the $\hat{x}_3$ axis coinciding with this rotational symmetry axis. In fibrous materials which behave as transversely isotropic materials, the axis $\hat{x}_3$ coincides with the fiber direction. In contracted Voigt notation the fourth order tensor C_{ijkl} (i,j,k,l=1,2,3) is represented by a symmetric matrix C_{IJ} (I,J=1,...,6). Then in the material coordinate system $\hat{x}_i$ we can write (following the nomenclature by Clements [25], and Tanuma, [18]):

$$\hat{C}_{IJ} = \begin{bmatrix} A & N & F & 0 & 0 & 0 \\ N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(A-N) \end{bmatrix}. \tag{15}$$

The five elastic constants that define the elastic behaviour of the transversely isotropic material ($A = \hat{C}_{11} = \hat{C}_{1111}$, $C = \hat{C}_{33} = \hat{C}_{3333}$, $N = \hat{C}_{12} = \hat{C}_{1122}$, $F = \hat{C}_{13} = \hat{C}_{1133}$ and $L = \hat{C}_{44} = \hat{C}_{55} = \hat{C}_{1313} = \hat{C}_{2323}$) must fulfil the following conditions, to ensure that the strain energy is positive (Tanuma, [18]):

$$L > 0, \quad \tfrac{1}{2}(A-N) > 0, \quad A + C + N > 0 \text{ and } (A+N)C > 2F^2. \tag{16}$$

For further considerations in the present work, the following coordinate systems are introduced:

- The previously introduced coordinate system attached to the material, $\hat{x}_i$, in which $\hat{x}_3$ is the elastic rotational symmetry axis, see Figure 1.a). The direction of this axis is defined by the vector $\hat{\mathbf{f}} = [0,0,1]^T$ and the elastic stiffness matrix is denoted by $\hat{C}_{ijkl}$. For fibrous materials which behave as transversely isotropic materials, this axis, and subsequently the vector $\hat{\mathbf{f}}$, coincides with the fiber direction, Figure 1 a).
- A coordinate system, x_i^*, which represents an intermediate coordinate system between that attached to the material and that associated with the directions which define the generalized plane strain state, Figure 1.b). The elastic stiffness matrix is denoted by C_{ijkl}^* and rotation matrix $\mathbf{\Omega}_2(\phi)$ and the rotated vector $\mathbf{f}^* = \mathbf{\Omega}_2(\phi) \cdot \hat{\mathbf{f}}$ being:

$$\mathbf{\Omega}_2(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}, \qquad \mathbf{f}^* = \mathbf{\Omega}_2(\phi)\cdot\hat{\mathbf{f}} = \begin{bmatrix} -\sin\phi \\ 0 \\ \cos\phi \end{bmatrix} \tag{17}$$

- Finally, a coordinate system, x_i, associated with the directions where the generalized plane strain state is defined, the elastic solution then being independent of x_3 $u_i = u_i(x_1, x_2)$, see Figure 1.c). The elastic stiffness matrix is denoted by C_{ijkl} and rotation matrix $\mathbf{\Omega}_3(\theta)$ and the rotated vector $\mathbf{f} = \mathbf{\Omega}_3(\theta)\cdot\mathbf{f}^*$ (which coincides with the fibre direction in fibrous materials) being:

$$\mathbf{\Omega}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{f} = \mathbf{\Omega}_3(\theta)\cdot\mathbf{f}^* = \begin{bmatrix} -\cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \tag{18}$$

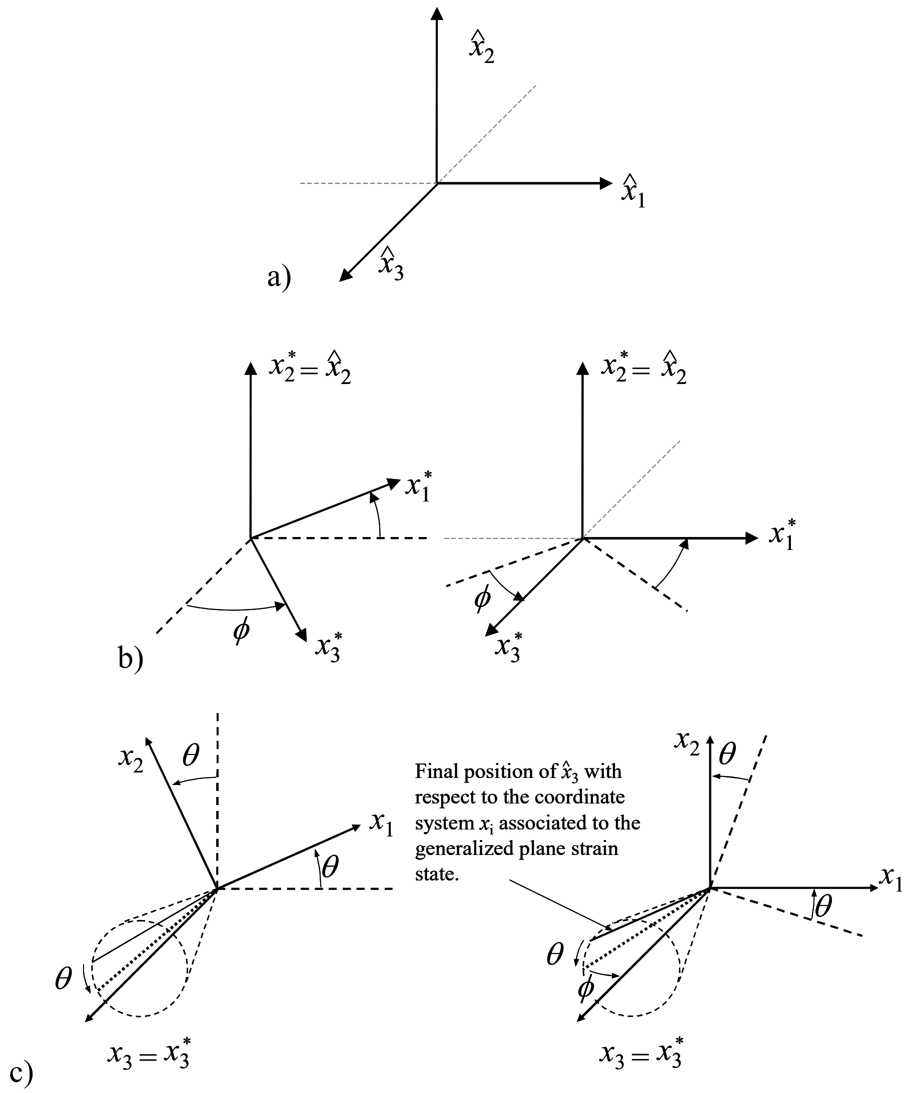

Figure 1. a) Coordinate system $\hat{x}_i$ attached to the material, b) Coordinate system x_i^* (rotation by ϕ around $\hat{x}_2 \equiv x_2^*$), c) Coordinate system x_i (rotation by θ around $x_3^* \equiv x_3$).

The coordinate system x_i^* is rotated with respect to the system $\hat{x}_i$ by an angle ϕ around the axis $\hat{x}_2 \equiv x_2^*$, $\Omega_2(\phi)$ being the rotation matrix that relates both systems. At the same time, the coordinate system x_i is rotated with respect to x_i^* by an angle θ around $x_3^* \equiv x_3$, $\Omega_3(\theta)$ being the rotation matrix which connects both systems.

In order to apply the Stroh formalism in the coordinate system which defines the generalized plane strain state, x_i, the elastic constant tensor, $\hat{C}_{ijkl}$, has to be transformed to this coordinate system, which can be done using the rotation matrices $\mathbf{\Omega}_2(\phi)$ and $\mathbf{\Omega}_3(\theta)$. Note that a third rotation around $\hat{x}_3$ is not necessary when considering transversely isotropic materials, due to the fact that $\hat{x}_3$ coincides with the rotational symmetry axis. Then, the matrices **Q**, **R** and **T** in (6), which lead to the evaluation of p_α using (5), are obtained from C_{ijkl}, already expressed in the coordinate system x_i.

Matrices $\mathbf{A} = \mathbf{A}(\phi,\theta)$ and $\mathbf{B} = \mathbf{B}(\phi,\theta)$ and the eigenvalues $p_\alpha(\phi,\theta)$ for a transversely isotropic material, with a generic orientation defined by ϕ and θ with respect to the coordinate system x_i, see Figure 1c, can be obtained from the matrices $\mathbf{A}(\phi,0)$ and $\mathbf{B}(\phi,0)$ and the eigenvalues $p_\alpha(\phi,0)$ of the same material, with an orientation ϕ and θ=0, see Figure 1b, due to the fact that **A** and **B** behave as tensors of order 1 when rotating around x_3 axis, Ting [26]. For the sake of brevity, we denote these matrices as $\mathbf{A}^*$ and $\mathbf{B}^*$ and their corresponding eigenvalues as p^*. The corresponding relations are:

$$\mathbf{A}(\phi,\theta) = \mathbf{\Omega}_3(\theta)\mathbf{A}(\phi,0) = \mathbf{\Omega}_3(\theta)\mathbf{A}^*, \tag{19}$$

$$\mathbf{B}(\phi,\theta) = \mathbf{\Omega}_3(\theta)\mathbf{B}(\phi,0) = \mathbf{\Omega}_3(\theta)\mathbf{B}^*, \tag{20}$$

$$p_\alpha(\phi,\theta) = \frac{\left(p_\alpha(\phi,0)\cos\theta - \sin\theta\right)}{\left(p_\alpha(\phi,0)\sin\theta + \cos\theta\right)},\ p_\alpha(\phi,0) = p_\alpha^*. \tag{21}$$

Thus, to obtain **A**, **B** and p_α, we need to express, in the x_i^* axes, the elastic constants tensor C_{ijkl}^* of a transversely isotropic material with the rotational symmetry axis situated in the $x_1^* - x_3^*$ plane, rotated an angle -ϕ with respect to x_3^* (Figure 1b). Then, it is necessary to evaluate $\mathbf{A}^*$, $\mathbf{B}^*$ and p_α^*, and finally, to use equations (19-21) to obtain the general expressions of **A**, **B** and p_α.

The first step of the above mentioned procedure simply represents a rotation of $\hat{C}_{ijkl}$, or in contracted notation $\hat{C}_{IJ}$, with respect to $\hat{x}_2 \equiv x_2^*$, a rotation that can be written as:

$$C_{ijkl}^* = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{lt}\hat{C}_{pqrt}, \tag{22}$$

where $\Omega_{ij} = [\mathbf{\Omega}_2(\phi)]_{ij}$ is introduced in (17). Defining $\Omega_{1j} = m_j$, it is possible to write

$$
\begin{aligned}
Q_{ik}^{*} &= C_{i1k1}^{*} = \Omega_{ip}(\hat{C}_{pqrt}m_q m_t)\Omega_{kr} = \left\lfloor \mathbf{\Omega}_2(\phi)\hat{\mathbf{Q}}(\phi)\mathbf{\Omega}_2^T(\phi)\right\rfloor_{ik},\\
R_{ik}^{*} &= C_{i1k2}^{*} = \Omega_{ip}(\hat{C}_{pqr2}m_q)\Omega_{kr} = \left\lfloor \mathbf{\Omega}_2(\phi)\hat{\mathbf{R}}(\phi)\mathbf{\Omega}_2^T(\phi)\right\rfloor_{ik},\\
T_{ik}^{*} &= C_{i2k2}^{*} = \Omega_{ip}(\hat{C}_{p2r2})\Omega_{kr} = \left\lfloor \mathbf{\Omega}_2(\phi)\hat{\mathbf{T}}(\phi)\mathbf{\Omega}_2^T(\phi)\right\rfloor_{ik}.
\end{aligned} \tag{23}
$$

An application of matrices $\hat{\mathbf{Q}}(\phi)$, $\hat{\mathbf{R}}(\phi)$ and $\hat{\mathbf{T}}(\phi)$ defined in (23) will be discussed later on. Using an analogue of (5) with $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$ the eigenvalues $p_\alpha^* = p_\alpha(\phi,0)$ are obtained. Then, the associated $\mathbf{a}^*$ and $\mathbf{b}^*$ can be evaluated by:

$$
\left\{\mathbf{Q}^* + (\mathbf{R}^* + \mathbf{R}^{*T})p_\alpha^* + \mathbf{T}^* p_\alpha^{*2}\right\}\mathbf{a}_\alpha^* = \mathbf{0}, \tag{24}
$$

$$
\mathbf{b}_\alpha^* = (\mathbf{R}^{*T} + p_\alpha^*\mathbf{T}^*)\,\mathbf{a}_\alpha^* = -\frac{1}{p_\alpha^*}(\mathbf{Q}^* + p_\alpha^*\mathbf{R}^*)\mathbf{a}_\alpha^*. \tag{25}
$$

Thus, using (24-25) we evaluate $\mathbf{A}^* = \mathbf{A}(\phi,0)$ and $\mathbf{B}^* = \mathbf{B}(\phi,0)$, and then, applying (19-20), we finally get the desired expressions of $\mathbf{A}(\phi,\theta)$ and $\mathbf{B}(\phi,\theta)$. Note that expressions (24-25) are only suitable for materials with a non-degenerate matrix **N**; see (A18) and (A28) (or Ting, [12]) for the corresponding expressions for a degenerate matrix **N**.

Tanuma [18], dealing with the evaluation of the surface impedance tensor, $\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1}$, worked using dual coordinate systems where in the representation of an elastic solution the components of the displacement and stress function vectors refer to $\hat{x}_i$ coordinate system and the position vector refers to x_i^* coordinate system. He presented some non-normalized explicit expressions of **a** and **b**, using these dual coordinate systems. Therefore, keeping in mind that the final expressions of the eigenvectors **a** and **b** in the present work must fulfil orthogonality relations, must be expressed in a coordinate system fixed to the axes which define the generalized plane strain state and must cover any generic orientation of the transversely isotropic material, it is possible, in cases where Tanuma's results for **a** and **b** are available, to proceed in an alternative way. First the expressions for **a** and **b** obtained by Tanuma in the dual coordinate systems are changed to the system x_i^*, then the normalization procedure is applied (a procedure which is different for different degeneration cases) and finally the complete 3D space orientation of the rotational symmetry axis is covered through (19-20), equations that apply for rotations around $x_3^* \equiv x_3$.

It is easy to obtain the relationship between the eigenvectors in both coordinate systems, the dual one used by Tanuma (denoted in (26-27) by a superindex D referring to "*Dual*") and the system x_i^* denoted with a superscript *. Note that the vectors $\mathbf{a}^D$ and $\mathbf{b}^D$ are obtained by means of expressions analogous to (24-25) using $\hat{\mathbf{Q}}(\phi)$, $\hat{\mathbf{R}}(\phi)$ and $\hat{\mathbf{T}}(\phi)$, instead of $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$, with the same eigenvalues p_α^*. Then, this relationship can be written as:

$$
\mathbf{a}_\alpha^* = \mathbf{\Omega}_2(\phi)\mathbf{a}_\alpha^D, \tag{26}
$$

$$
\mathbf{b}_\alpha^* = \mathbf{\Omega}_2(\phi)\mathbf{b}_\alpha^D, \tag{27}
$$

where $\mathbf{\Omega}_2(\phi)$ is defined in Figure 1.b. Note that $\mathbf{a}^D$ and $\mathbf{b}^D$ depend on ϕ through its associated $\hat{\mathbf{Q}}(\phi)$, $\hat{\mathbf{R}}(\phi)$ and $\hat{\mathbf{T}}(\phi)$ given in (23).

The complete set of new explicit expressions of the orthonormalized eigenvectors $\mathbf{A}^* = [a_1^*, a_2^*, a_3^*]$ and $\mathbf{B}^* = [b_1^*, b_2^*, b_3^*]$ is presented in Appendix. Although $\mathbf{A}^*$ and $\mathbf{B}^*$ have been obtained in a relatively straightforward way by well-established procedures (Ting [12]), using, in the cases where it was suitable, the previous results by Tanuma [18], lengthy algebraic calculations have been required and an independent checking of the results obtained have been carried out to warranty their correctness. It is considered that the result is of relevant technical interest, as it may encourage further applications of the Stroh formalism to transversely isotropic materials. It is worth mentioning that all possible cases, including mathematically degenerate cases, have been analysed, now overcoming the numerical instabilities that typically appear in approximation of eigenvectors of the fundamental elasticity matrix $\mathbf{N}$ in these cases.

Recall that it only remains to apply (19-20) to the final expressions of $\mathbf{A}^*$ and $\mathbf{B}^*$ which are presented in Appendix in order to achieve the final expressions of $\mathbf{A}(\phi,\theta)$ and $\mathbf{B}(\phi,\theta)$.

4. Singularity analysis of multimaterial corners involving transversely isotropic materials

As an application of the theoretical results presented in the previous section and Appendix, an analysis of singular stress fields appearing at the tip of a multimaterial anisotropic corner, which involves linear elastic transversely isotropic materials subjected to a generalized plane strain state, is briefly outlined.

4.1. Characteristic equation of the multimaterial corner.

Let a fixed polar coordinate system (r,θ) be defined at the corner tip. Consider a multimaterial corner formed by N>1 homogeneous elastic wedges, as shown in Figure 2, given by polar sectors defined by angles (θ_{i-1},θ_i), i=1,...,N, $(\theta_{i-1}<\theta_i)$. Perfectly bonded interfaces, verifying equilibrium and compatibility conditions, are considered between adjacent wedges.

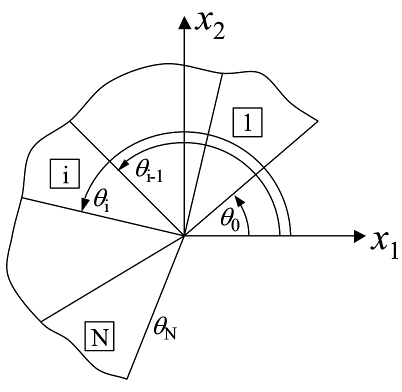

Figure 2. Multimaterial corner.

The procedure described below is based on a previous work of the present authors, Barroso *et al* [15], further developing an original idea by Ting [13]. As was mentioned in the Introduction, the following fields in displacement ($u_i(r,\theta) \approx r^{\delta} g_i(\theta,\delta)$) (*i,j*=1,2,3), and stresses ($\sigma_{ij}(r,\theta) \approx r^{\delta-1} f_{ij}(\theta,\delta)$) (*i,j*=1,2,3), are considered to appear in the neighbourhood of the corner (for $r \to 0_+$). We refer to the value $1-\delta$ as the order of stress singularity.

The representation of the elastic solution associated with the singular exponent at the neighbourhood of the corner tip (for $r \to 0_+$) can be expressed as:

$$\mathbf{w}(r,\theta) = r^{\delta}\mathbf{X}\mathbf{Z}(\theta,\delta)\mathbf{t}, \qquad \text{where } \mathbf{w}(r,\theta) = \begin{bmatrix} \mathbf{u}(r,\theta) \\ \boldsymbol{\varphi}(r,\theta) \end{bmatrix} \text{ and } \mathbf{t} = \begin{bmatrix} \mathbf{q} \\ \tilde{\mathbf{q}} \end{bmatrix}, \tag{28}$$

$\mathbf{u}(r,\theta)$ being the displacement vector and $\boldsymbol{\varphi}(r,\theta)$ the stress function vector, $\mathbf{q} = (q_1, q_2, q_3)^T$, $\tilde{\mathbf{q}} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)^T$, q_α and $\tilde{q}_\alpha$ being arbitrary real or complex constants, $\mathbf{X}$ is defined in (13) and $\mathbf{Z}(\theta,\delta)$ and its inverse $[\mathbf{Z}(\theta,\delta)]^{-1}$ are known analytical functions of the singularity exponent δ and the angle θ associated with a material wedge, having been obtained in Ting [13] for anisotropic non-degenerate materials and in Barroso *et al* [15] for anisotropic degenerate and extraordinary-degenerate materials. The structure of $\mathbf{Z}(\theta,\delta)$ is:

$$\mathbf{Z}(\theta,\delta) = \begin{bmatrix} \boldsymbol{\Psi}(p_*,\theta,\delta) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}(\bar{p}_*,\theta,\delta) \end{bmatrix}, \tag{29}$$

where for non-degenerate transversely isotropic materials $\boldsymbol{\Psi}(p_*,\theta,\delta)$ takes the diagonal form:

$$\boldsymbol{\Psi}(p_*,\theta,\delta) = \text{diag}\left[\zeta_1^{\delta}(\theta), \zeta_2^{\delta}(\theta), \zeta_3^{\delta}(\theta)\right] \text{ and } \zeta_\alpha^{\delta}(\theta) = (\cos\theta + p_\alpha \sin\theta)^{\delta}, \tag{30}$$

whereas for degenerate transversely isotropic materials (taking $p_2 = p_3$) it is a non-symmetric matrix:

$$\boldsymbol{\Psi}(p_*,\theta,\delta) = \begin{bmatrix} \zeta_1^{\delta}(\theta) & 0 & 0 \\ 0 & \zeta_2^{\delta}(\theta) & \Lambda(p_2,\theta,\delta)\zeta_2^{\delta}(\theta) \\ 0 & 0 & \zeta_2^{\delta}(\theta) \end{bmatrix}, \tag{31}$$

where $\Lambda(p_2,\theta,\delta) = \dfrac{\delta \sin\theta}{\zeta_2(\theta)}$ and $\zeta_\alpha(\theta) = \cos\theta + p_\alpha \sin\theta$.

Applying the continuity conditions introduced by the hypothesis of perfect interfaces (perfect bonding) between the adjacent wedges, $\mathbf{w}_i(r,\theta_i) = \mathbf{w}_{i+1}(r,\theta_i)$ $i = (1,\dots,N-1)$, it is easy to obtain the following relationship:

$$\mathbf{w}_N(r,\theta_N)=\mathbf{K}_N\mathbf{w}_1(r,\theta_0)\,. \tag{32}$$

where matrix $\mathbf{K}_N$ is defined as:

$$\mathbf{K}_N=\mathbf{E}_N\cdot\mathbf{E}_{N-1}\cdot\ldots\mathbf{E}_2\cdot\mathbf{E}_1\,, \tag{33}$$

and $\mathbf{E}_i$ is the transfer matrix defined for the *i-th* material wedge as it relates the pertinent displacements and the stress function vectors, associated with a singularity exponent δ, between the external faces, θ_{i-1} and θ_i, of the *i-th* material wedge. These transfer matrices $\mathbf{E}_i$ can be written as:

$$\mathbf{E}_i=\mathbf{X}\mathbf{Z}(\theta_i,\delta)\left[\mathbf{Z}(\theta_{i-1},\delta)\right]^{-1}\mathbf{X}^{-1}\,, \tag{34}$$

Thus, $\mathbf{K}_N$ in (32) represents in fact the transfer matrix, associated with a singularity exponent δ, for the whole multimaterial corner, as it relates the elastic variables between its external faces (defined by angles θ_0 and θ_N).

Let us define

$$\hat{\mathbf{K}}_N=\mathbf{D}_{BC}(\theta_N)\,\mathbf{K}_N\,\mathbf{D}^T_{BC}(\theta_0)\,, \tag{35}$$

where $\mathbf{D}_{BC}(\theta_N)$ and $\mathbf{D}_{BC}(\theta_0)$ represent homogeneous orthogonal boundary conditions along the external faces, at angles $\theta=\theta_0$ and $\theta=\theta_N$ of the corner, see Mantič *et al* [27] and Barroso *et al* [15] for their definitions. $\mathbf{D}_{BC}(\theta)$ is an orthogonal (6×6) matrix, thus $\mathbf{D}^{-1}_{BC}=\mathbf{D}^T_{BC}$, and $\mathbf{D}_{BC}\mathbf{D}^T_{BC}=\mathbf{I}$, the structure of $\mathbf{D}_{BC}(\theta)$ being:

$$\mathbf{D}_{BC}(\theta)=\begin{bmatrix}\mathbf{D}_u(\theta) & \mathbf{D}_\varphi(\theta)\\ \mathbf{D}_\varphi(\theta) & \mathbf{D}_u(\theta)\end{bmatrix}. \tag{36}$$

Some examples of $\mathbf{D}_u(\theta)$ and $\mathbf{D}_\varphi(\theta)$ follow: $\mathbf{D}_u(\theta)=\mathbf{0}$, $\mathbf{D}_\varphi(\theta)=\mathbf{I}$ for a free boundary and $\mathbf{D}_u(\theta)=\mathbf{I}$, $\mathbf{D}_\varphi(\theta)=\mathbf{0}$ for a fixed one. Expressions of $\mathbf{D}_u(\theta)$ and $\mathbf{D}_\varphi(\theta)$ for other homogeneous boundary conditions can be found in Mantič *et al*. [27] and Barroso *et al*. [15].

Using the boundary condition matrix $\mathbf{D}_{BC}(\theta)$ introduced in (36), we get:

$$\hat{\mathbf{w}}(r,\theta)=\mathbf{D}_{BC}(\theta)\,\mathbf{w}(r,\theta)=\begin{bmatrix}\mathbf{w}_P(r,\theta)\\ \mathbf{w}_U(r,\theta)\end{bmatrix}, \tag{37}$$

where $\mathbf{w}_P(r,\theta)$ and $\mathbf{w}_U(r,\theta)$ respectively denote the vectors of prescribed and unknown components of $\mathbf{w}(r,\theta)$.

Applying the orthogonality property of $\mathbf{D}_{BC}(\theta)$ and relations (35) and (37), we can finally write:

$$\begin{bmatrix} \mathbf{w}_P(r,\theta_N) \\ \mathbf{w}_U(r,\theta_N) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{K}}_N^{(1)} & \hat{\mathbf{K}}_N^{(2)} \\ \hat{\mathbf{K}}_N^{(3)} & \hat{\mathbf{K}}_N^{(4)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_P(r,\theta_0) \\ \mathbf{w}_U(r,\theta_0) \end{bmatrix} \text{ or in a compact form } \hat{\mathbf{w}}(r,\theta_N) = \hat{\mathbf{K}}_N \, \hat{\mathbf{w}}(r,\theta_0)\,. \quad (38)$$

After applying the homogeneous boundary conditions in the linear system in (38) which can be written as: $\mathbf{w}_P(r,\theta) = \mathbf{0}$ for $\theta=\theta_0$ and θ_N, the following identity is obtained:

$$\hat{\mathbf{K}}_N^{(2)}(\delta)\,\mathbf{w}_U(r,\theta_0) = \mathbf{0}\,. \quad (39)$$

This homogeneous linear system has a non-trivial solution if and only if the characteristic equation, obtained from the following complex 3×3 determinant is fulfilled.

$$\left|\hat{\mathbf{K}}_N^{(2)}(\delta)\right| = \mathbf{0}\,, \quad (40)$$

With the explicit expressions of **A** and **B** for any non-degenerate or degenerate case of transversely isotropic materials (see Appendix), the characteristic equation in (40) can be easily evaluated and, by finding its roots, the stress singularity orders δ-1 can be determined.

Let us mention that from a physical point of view the transfer matrix $\mathbf{E}_i$ in (34) should be continuously dependent on the wedge material properties although the form of its representation using **X** and $\mathbf{Z}(\theta,\delta)$ matrices changes between different classes of material degeneracy. An explicit proof of this continuous character of $\mathbf{E}_i$, and also the issue of the possibility of a continuous expression for $\mathbf{E}_i$, is still lacking to the best of the authors' knowledge. One implication of the continuous character of the transfer matrices $\mathbf{E}_i$ (i=1,...,N) is the continuity of the singularity order δ-1 with respect to a continuous variation of material properties of wedges in a multimaterial corner.

4.2. Numerical examples.

With the general approach presented in Section 4.1 and the explicit expressions for **A** and **B** obtained in this work for transversely isotropic materials (in Appendix), the analysis of multimaterial corners involving any kind of transversely isotropic materials (mathematically degenerate or not) can be performed in a straightforward way. Two applications will be presented, the first dealing with adhesively bonded joints between metallic and composite materials, while the second one deals with the analysis of cracks appearing in cross-ply laminates.

All numerical evaluations in Sections 4.2.1 and 4.2.2 have been obtained using a program implemented in *Mathematica* (Wolfram [28]) by the present authors, which uses all the expressions introduced in the Appendix as well as the corresponding expressions for orthotropic and isotropic materials. Further details can be found in Barroso *et al* [15].

4.2.1. Application to adhesively bonded joints.

Let us consider the following configuration of an adhesively bonded joint between a composite laminate and a metal sheet. The laminate shown in Figure 3 is composed by three plies (0º/45º/90º) and is adhesively bonded to a metal sheet by means of an adhesive layer. In such a geometrical configuration, if the loads do not change in x_3 direction, a generalized plane strain state can be used to approximate the elastic state far from the free edges of the specimen, and the Stroh formalism is applicable.

It should be mentioned that the angle of each ply defined in Figure 3 agrees with the angle ϕ definition used in Figure 1 and in Appendix, but is different from that usually considered in composite applications, where the 0º lamina means that fibers are parallel to the load.

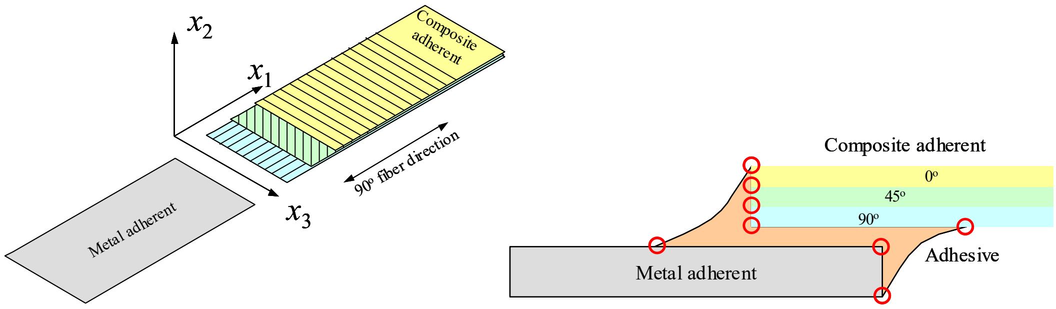

Figure 3. Metal to composite adhesively bonded joint.

The multimaterial corners where singular stress fields may appear in the adhesively bonded joint analyzed here are marked with a circle in Figure 3.

Unidirectional laminas, and laminates having the same direction in all laminas, are typically considered, and it will be done so in this paper, as transversally isotropic in an equivalent homogeneous representation. One person familiarized with the properties of unidirectional laminas of actual materials used in the aeronautical industry might argue that suppliers of these materials give properties that do not, strictly speaking, satisfy the former assumption. Thus, for instance, with reference to the material used in this paper, AS4/8552, a typical carbon-epoxy composite, the following properties are given by the supplier with the fiber direction coincident with x_1: E_{11}=141 GPa, E_{22}=E_{33}=9.58 GPa, G_{12}=G_{13}=5 GPa, G_{23}=3.5 GPa, ν_{12}=ν_{13}=0.3, ν_{23}=0.32.

Although the structure of this set of properties coincides with that of a transversely isotropic material, the values of these properties do not satisfy strictly the relation of this structure. Thus, the value of G_{23} corresponding to the assumed isotropic plane ought to satisfy the relation of an isotropic material $G=E/2(1+\nu)$, where $E_{22}=E_{33}=E$ and $\nu_{23}=\nu$. However, if the values of E and ν are substituted in the former expression, it leads to G_{23}=3.629 GPa, a value which is slightly different from that given by the supplier (3.5 GPa).

This difference is due basically to the intrinsic heterogeneous character of the composite material which may originate that properties obtained directly from a test of the whole laminate do not strictly satisfy relations derived from the homogeneity assumptions and obtained from different tests. The difference, which affects only to G_{23} value, is in this case of a 3.5% and will be omitted in the analysis, taking for G_{23} the value that satisfies transversally isotropic behaviour.

In any case, it has been verified, using the theoretical results and the developed code presented in Barroso *et al* [15] and in the present work, that this small difference in the elastic values considering an orthotropic or a transversely isotropic behaviour leads to close results of a corner singularity analysis. Nonetheless, it is worth mentioning that there is a great difference in the mathematical treatment between both mechanical behaviours when using the Stroh formalism.

For the case of transversely isotropic properties considered here for each single ply, the present work provides explicit expressions of eigenvalues and eigenvectors of the Stroh formalism for all cases, degenerate and non-degenerate. This leads, for instance, to the use of the expressions of Case 1.a (in Appendix), equations (A.13-A.14), for the corners involving the degenerate transversely isotropic layer oriented in x_3 direction ($\phi = 0°$), and Cases 2.2 and 3.2 (in Appendix) for corners involving non-degenerate materials, see (A.47) and (A.49-A.50). These expressions are normalized, the evaluation of $\mathbf{X}^{-1}$ by (13) then being direct and easy.

Finally, with reference to the other properties of the materials involved in the joint, the following properties have been taken for the aluminium as metal adherent: E=68.6 GPa and ν=0.3, while an epoxy-based adhesive has been taken with isotropic elastic properties: E=3 GPa and ν=0.3.

In Figure 4 a detail of the end of the overlap of a real adhesive joint between a carbon-epoxy composite [90º,0º,90º] and aluminum is shown.

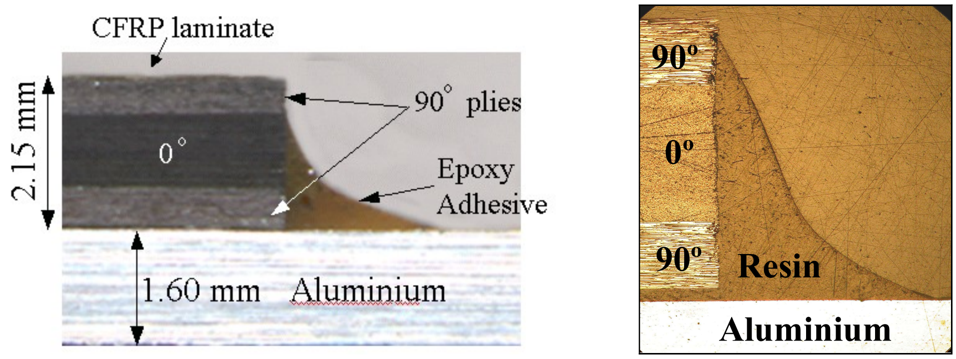

Figure 4. Details of a lap joint between a composite panel and a metal sheet.

Figure 5 shows the order of stress singularities (Re and Im denoting the real and imaginary parts of 1-δ respectively) for a typical closed three-material corner which appears in the scheme of Figure 3 and the photograph in Figure 4. In this example, transversely isotropic elastic properties given above have been assumed for the plies. One of the plies of the laminate has a fixed angle of $\phi = 0°$, which makes the material mathematically degenerate, while the orientation of the other ply varies from 0º to 90º. The half space on the left hand side is occupied by the adhesive.

Thus, in this particular example, the adhesive which behaves as an isotropic material corresponds to Case 1.b.2) (see Appendix), the bottom lamina with a fixed $\phi = 0°$

orientation corresponds to Case 1.a), and the top lamina with an arbitrary orientation $\phi \neq 0^{\text{o}}$ corresponds to Case 3.

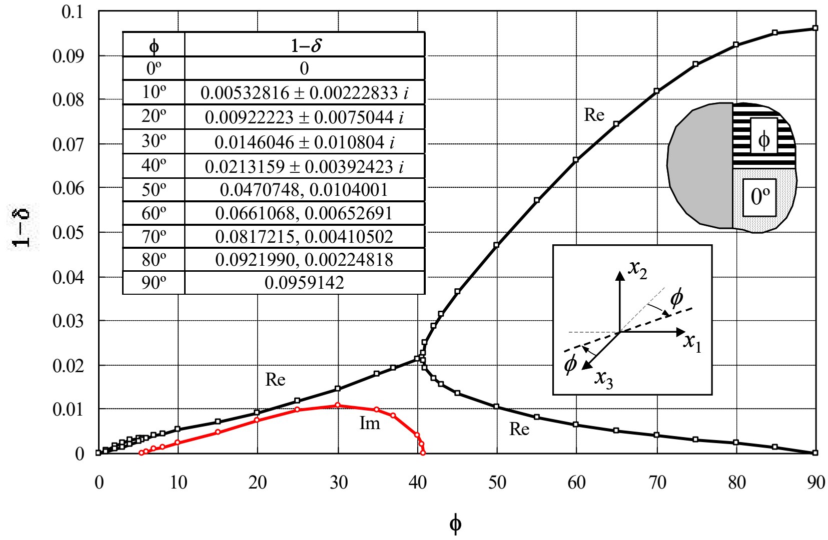

Figure 5. Order of stress singularities in a three-material corner involving a mathematically degenerate transversely isotropic material.

It can be appreciated in Figure 5 that, for $\phi = 0^{\text{o}}$, no corner configuration appears, no singular mode then being obtained. Two real orders of stress singularities exist until ϕ=5.5º is reached, turning then into two complex conjugates until ϕ=40.5º, after which two real values are again obtained except in the cross-ply 0º-90º configuration where there is only one singularity mode.

The asymptotic stress and displacement fields can also be computed easily using (28) and (39), see Barroso *et al.* [15] for details.

Let us consider another example with a similar corner configuration to that shown in Fig. 5 but with a 45º lamina fixed. In this case none of the materials is mathematically degenerate (except for values of ϕ=0º and 180º) because of their spatial orientation.

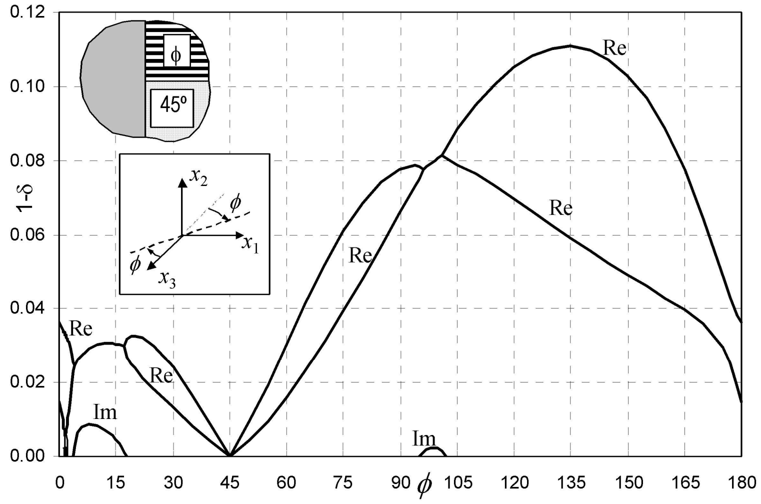

Figure 6. Orders of stress singularities in a three-material corner.

It can be appreciated in Figure 6 that when a 45º angle is reached, the corner configuration disappears, together with the singular character of the asymptotic stress field. For the particular case of a 90º angle in the upper lamina (see Figure 7 and also Figure 3) two order of stress singularities, both of them real, appear (Figure 6), namely: $1-\delta=0.074347$ and 0.063487.

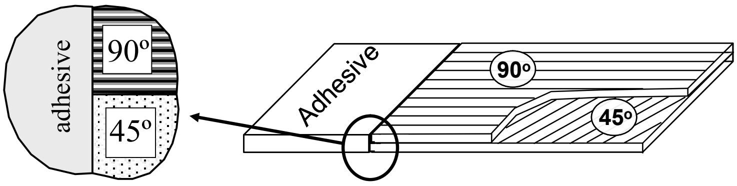

Figure 7. A three-material isotropic-transversely isotropic corner.

For the first singular mode ($1-\delta=0.074347$) the asymptotic stress and displacement fields, shown in Figure 8, have been obtained using equation (28), with r=1.

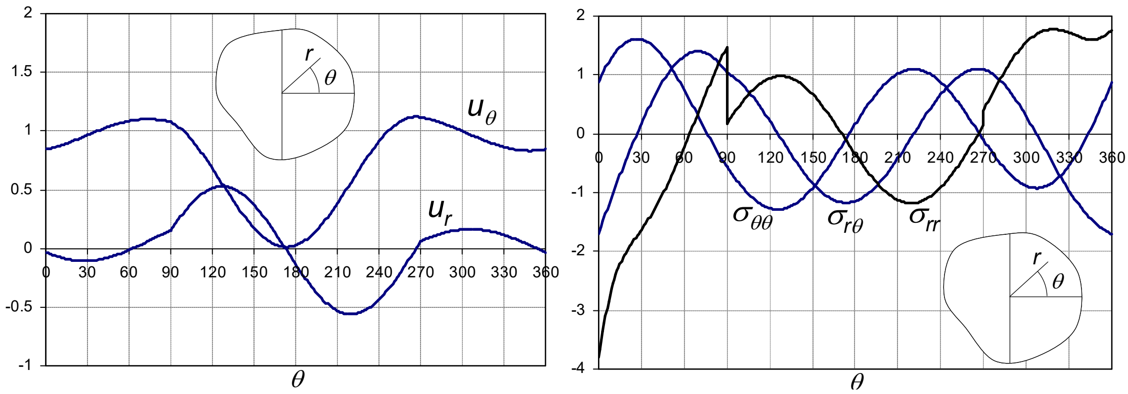

Figure 8. Displacements (left) and stresses (right) in a three-material corner.

Both displacements and stresses (in Figure 8) have been standardized according to Pageau *et al.* [29] in such a way that $\sigma_{\theta\theta}|_{\theta=0}=1/(2\pi)^{1-\delta}$. It can be observed that σ_{rr} stress component and the slope of some displacements and stress components, as functions of θ, are not continuous at the interfaces between materials (θ=0º, 90º and 270º). The elastic fields, associated with a singular stress state at a corner, obtained using the present approach will help to study failure mechanisms that typically appear in composite material structures, such as delamination.

4.2.2. Application to cracks appearing in cross-ply laminates.

Figure 9 shows schematically a $[0/90]_s$ damaged laminate under tension. The first damage is expected to be the nucleation and growth of a crack in the 90º ply transverse to the load. When this transverse crack reaches the 0º ply and before continuing through the interface, the multimaterial corner shown schematically in Figure 9b is obtained. The stress analysis and failure progression is extensively studied in Blázquez *et al.* [30].

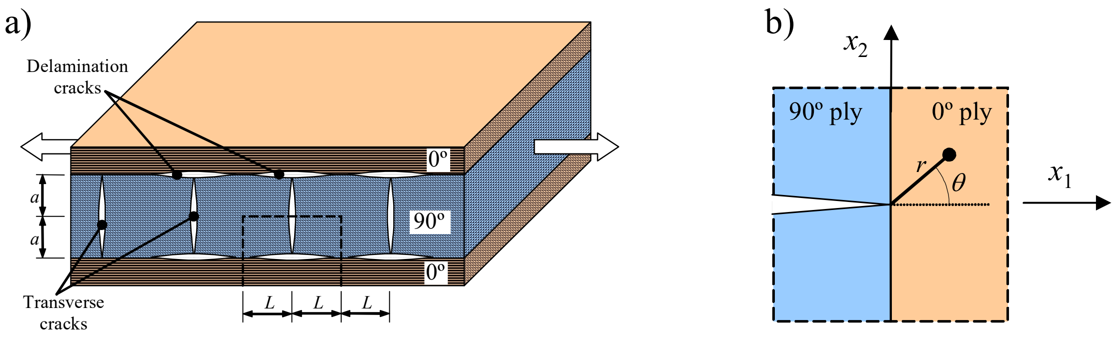

Figure 9. Cross ply laminate with failure progression (a) and configuration when the crack reaches the 0º ply (b).

The values of elastic constants of the unidirectional glass fiber material, considering a transversely isotropic constitutive law (the axis 1 defining the fiber direction and plane 2-3 being the isotropy plane perpendicular to the fiber direction), are: $E_{22}=E_{33}=E=16.2$ GPa and $\nu_{23}=0.4$ for the in-plane constants, and $E_{11}=45.6$ GPa, $\nu_{12}=\nu_{13}=0.278$ and $G_{12}=G_{13}=5.83$ GPa for the out-of-plane constants.

For this three-material corner configuration (Figure 9b), the 90º ply has $\phi=0º$ (according to Figure 1) in both material wedges (from 90º to 180º, and from 180º to 270º), therefore being mathematically degenerate. Using the expressions of the Appendix (Case 1.a for the 90º ply material wedges, and Case 3.2 for the 0º ply material wedge) and the *Mathematica* code developed in Barroso *et al.* [15], the following three numerical solutions are obtained:

$$\begin{aligned}&\text{root } 1=\lambda_1=1-\delta_1=0.500606791803375 \text{ (antiplane mode)}\\&\text{root } 2=\lambda_2=1-\delta_2=0.499748666269924 \text{ (in-plane antisymmetric mode)}\\&\text{root } 3=\lambda_3=1-\delta_3=0.435787998869871 \text{ (in-plane symmetric mode)}\end{aligned} \quad (41)$$

The numerical solution obtained for the antiplane mode in (41) has a 15 digit precision, which is unrealistic from an engineering point of view but it can be useful for researchers as an accurate solution of a benchmark problem. This solution was succesfully compared with the analytical solution by Mantič *et al.* [31]. Thus, the values obtained for the two in-plane modes will be also considered as the reference values in what follows.

If the explicit expressions of the **A** and **B** Stroh matrices obtained in this paper for the degenerate cases, see Appendix, were not available, only a numerical approximation could be obtained by using the non-degenerate expressions, for the 90º plies, considering a value of ϕ close to 0º. Figure 10 shows the numerical solution using this alternative, the dashed lines representing the reference values for the $\phi=0º$ orientation (computed by means of the expressions of the Appendix for the degenerate cases) and the horizontal axis representing the fiber orientation. When $\phi=0º$ is approached, the numerical solution becomes unstable (using *Mathematica* with its standard precision). The solution corresponding to root 2 is detailed with a zoom in the vertical axis. Both axes in Figure 10 have the same meaning but different scales, arrows on the curves indicating which axis is to be used. In particular, roots 1 and 2 use the right vertical scale and root 3 the left vertical scale.

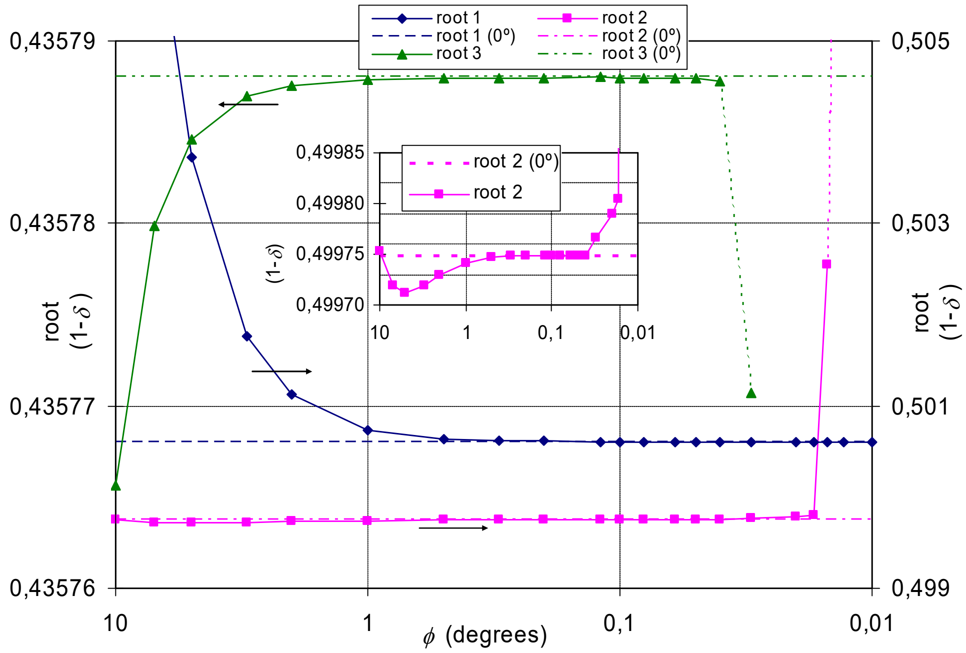

Figure 10. Evolution of the order of stress singularities as ϕ=0º is approached.

The relative errors (in %) with respect to the reference values (obtained by using the explicit expressions for the degenerate cases) are depicted in Figure 11. It can be observed how the value of the numerical solutions for the in-plane roots tends to the value for the ϕ=0º configuration, as $\phi \to 0$º up to a certain value of ϕ where the solution shows numerical instabilities when the degenerate situation is approached. The antiplane solution, which is that showing a greater interval of ϕ corresponding to a stable behaviour, shows the last stable value for ϕ=0.01º, the result being strongly unstable for ϕ=0.009º and giving very large errors (out of the scale used in Figure 11).

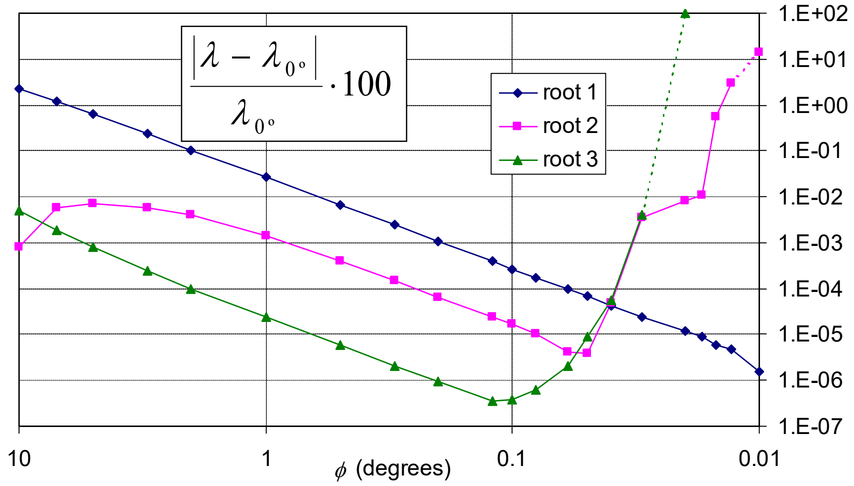

Figure 11. Relative errors of the order of stress singularities with respect to the ϕ=0º reference solution.

It can be seen in Figure 11 how large errors can be obtained if the angle ϕ defining the non-degenerate approximate solution is too close to or too far from ϕ=0º defining the degenerate case. The value for the angle giving the minimum error is *a priori* unknown if the solution for ϕ=0º is not available. The relative errors in the stress singularity order $[|\lambda_{num} - \lambda_{exact}|/\lambda_{exact}]$ (λ_{num} and λ_{exact}, respectively, being the numerical approximation and the exact value of the stress singularity order) are amplified when computing the relative error in stresses close to the corner tip ($|r^{-\lambda_{num}} - r^{-\lambda_{exact}}|/r^{-\lambda_{exact}}$, $\sigma \approx r^{-\lambda}$, with λ=1-δ), this error depending on the relative error of the stress singularity order itself and the distance r to the corner tip. It can be easily verified that, for roots in (41), relative errors in stresses are larger than the relative errors in the stress singularity order for r<0.1.

If values of stresses very close to the corner tip have to be used, as in many standard Fracture Mechanics analyses, significant errors could then appear. The usefulness of the obtained explicit expressions, covering all possible material and geometrical configurations, is then fully confirmed.

5. Conclusions

The complex variable Stroh formalism of anisotropic elasticity has been applied to develop a general, accurate and computationally efficient procedure for the analysis of elastic singularities at the multimaterial corners involving transversely isotropic materials, subjected to generalized plane strain states. This procedure: a) covers all kind of homogeneous transversely isotropic linearly elastic materials at any spatial orientation, b) considers any finite number of homogeneous wedges converging at the corner tip and perfectly bonded between them, c) considers all kind of standard homogeneous boundary conditions, and d) provides expressions as most analytic and compact as possible. Applications to adhesive bonded joints between composite laminates and metals, and damaged cross-ply laminates, have been presented, involving non-degenerate and degenerate transversely isotropic materials.

The procedure presented requires a knowledge of the eigenvalues p_α and eigenvectors $\xi_\alpha^T = \lfloor \mathbf{a}_\alpha^T, \mathbf{b}_\alpha^T \rfloor$ (α=1,…,6) of the fundamental elasticity matrix **N** in the Stroh formalism for transversely isotropic materials in an explicit form. In order to obtain a general procedure for corner singularity analysis, all mathematically degenerate cases of **N** have been taken into account in the analysis presented, so there is no limitation either in the orientation of the material or in the particular values of the elastic constants.

The final expressions of the eigenvalues and eigenvectors are expressed in terms of the stiffness constants of the material and the angles defining the spatial orientation of the material. In all the cases studied, matrices $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ have been properly orthogonalized and normalized in order to fulfil the important orthogonality and closure relations of the Stroh formalism. The explicit expressions of **A** and **B** have been presented in the Appendix as they can be useful for other applications of Stroh formalism with transversely isotropic materials.

The new explicit expressions of matrices **A** and **B** deduced are directly applicable in those representations of elastic variables in the Stroh formalism which are associated with the coordinate system defined by the generalized plane strain state (see Ting [12]), an approach typically followed in engineering applications. Thus, it is expected that these expressions will contribute to further developments in the Stroh formalism applied to stress analysis of composite materials modelled like transversely isotropic materials, because of the fact that exact expressions for degenerate cases can now be used. The numerical uncertainties and instabilities which usually appear when trying to model degenerate cases as limit cases of non-degenerate materials can then from now on be overcome.

It has been shown that large errors in the order of stress singularities can appear when using the expressions for non-degenerate cases together with a perturbation of the real material configuration (e.g. by modifying the fiber orientation) instead of the pertinent expressions for the actual degenerate configuration. Moreover, these errors, and also an adequate perturbation of the real configuration giving minimum errors, are *a priori* unknown, which confirms the usefulness of the expressions deduced in the present work for engineering applications. Results with high accuracy have been reported and can be used as benchmarks by other researchers.

6. Acknowledgements

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APPENDIX: Orthonormalized expressions of a and b

As mentioned in the main text, once the eigenvalues and eigenvectors of a transversely isotropic material with an orientation defined by $\theta = 0$ and a generic angle ϕ around

x_2 in the $x_1 - x_3$ plane are known, it is easy, by means of (19-21), to obtain the respective values of a generally oriented transversely isotropic material. Thus, we will focus our attention on materials with $\theta = 0$. Transforming $\hat{C}_{ijks}$ in (15) considering a rotation around $\hat{x}_2$ by an angle ϕ, using (23) and writing the characteristic equation (5) for $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$ defined in (23), we arrive at an analogoue of (5) in terms of $\hat{\mathbf{Q}}(\phi)$, $\hat{\mathbf{R}}(\phi)$ and $\hat{\mathbf{T}}(\phi)$, defined also in (23), finally obtaining the three eigenvalues of the material in terms of ϕ and the elastic constants of the material:

$$p_1^* = i\left(\cos^2\phi + \frac{2L}{A-N}\sin^2\phi\right)^{1/2}, \tag{A.1}$$

$$p_2^* = i\left(\cos^2\phi + \sin^2\phi\left[\left(AC - F^2 - 2FL\right) + \sqrt{(AC-(F+2L)^2)(AC-F^2)}\right]\frac{1}{2AL}\right)^{1/2}, \tag{A.2}$$

$$p_3^* = i\left(\cos^2\phi + \sin^2\phi\left[\left(AC - F^2 - 2FL\right) - \sqrt{(AC-(F+2L)^2)(AC-F^2)}\right]\frac{1}{2AL}\right)^{1/2}. \tag{A.3}$$

Following the classification of **N** in Ting [24], and depending on the particular values of ϕ and A, N, C, F and L, the following cases appear, Tanuma [18].

1. *$\mathbf{N}^*$ is non-semisimple if and only if*

   1.a) $\phi=0$ or $\phi=\pm\pi$, or (A.4)

   1.b) $\sqrt{AC} - F - 2L = 0$, with two different subcases: (A.5)

   1.b.1) $\dfrac{2L}{A-N} \neq \sqrt{\dfrac{C}{A}}$, or (A.6)

   1.b.2.) $\dfrac{2L}{A-N} = \sqrt{\dfrac{C}{A}}$. (A.7)

2. *With $\phi \neq 0$, π and $\sqrt{AC} - F - 2L \neq 0$, $\mathbf{N}^*$ is semisimple if and only if*

$$AL\left(\frac{2L}{A-N}\right)^2 - (AC - F^2 - 2FL)\frac{2L}{A-N} + CL = 0. \tag{A.8}$$

3. *In any other case, $\mathbf{N}^*$ is simple.*

With the expressions in (A.1-A.3), it is not possible to find an extraordinary degenerate matrix $\mathbf{N}^*$ (three equal eigenvalues and only one linearly independent eigenvector) for transversely isotropic materials. Thus, the following two non-semisimple (or degenerate) cases can be considered for transversely isotropic materials:

- D1 case, with two identical eigenvalues ($p_2^* = p_3^*$) and two linearly independent eigenvectors. The two linearly independent eigenvectors are $\boldsymbol{\xi}_1^*$ and $\boldsymbol{\xi}_2^*$, while the generalized eigenvector is $\boldsymbol{\xi}_3^*$, then

$$\mathbf{N}^*\boldsymbol{\xi}_1^* = p_1^*\boldsymbol{\xi}_1^*,\ \ \mathbf{N}^*\boldsymbol{\xi}_2^* = p_2^*\boldsymbol{\xi}_2^*,\ \ \mathbf{N}^*\boldsymbol{\xi}_3^* = p_2^*\boldsymbol{\xi}_3^* + \boldsymbol{\xi}_2^*. \tag{A.9}$$

- D2 case, with three identical eigenvalues ($p_1^* = p_2^* = p_3^* = p$) and two linearly independent eigenvectors. The two linearly independent eigenvectors are $\boldsymbol{\xi}_1^*$ and $\boldsymbol{\xi}_2^*$, while the generalized eigenvector is $\boldsymbol{\xi}_3^*$, then

$$\mathbf{N}^*\boldsymbol{\xi}_1^* = p\boldsymbol{\xi}_1^*,\ \ \mathbf{N}^*\boldsymbol{\xi}_2^* = p\boldsymbol{\xi}_2^*,\ \ \mathbf{N}^*\boldsymbol{\xi}_3^* = p\boldsymbol{\xi}_3^* + \boldsymbol{\xi}_2^*. \tag{A.10}$$

Recall that $\mathbf{a}^*$ and $\mathbf{b}^*$ define the components of $\mathbf{A}^*=\mathbf{A}(\phi,0)$ and $\mathbf{B}^*=\mathbf{B}(\phi,0)$.

For the sake of simplicity, in the following expressions in this Appendix the superscript $^{(*)}$ in the roots p_α^* will be omitted.

Case 1.a.: $\phi=0$ or $\phi=\pm\pi$.

In this case we have a triple eigenvalue $p_1 = p_2 = p_3 = p = i$ and two linearly independent eigenvectors, and thus it is a D2 case, see (A.10). Particularly in this case, expressions of $\mathbf{a}^*$ and $\mathbf{b}^*$ are identical to $\mathbf{a}^D$ and $\mathbf{b}^D$ according to (26-27), as $\boldsymbol{\Omega}_2(\phi) = \mathbf{I}$, see (17). The corresponding expressions of $\mathbf{a}^*$ and $\mathbf{b}^*$ can be directly obtained using the results by Tanuma [18]. The normalization procedure is defined by the following expressions (Ting [12], Chapter 5):

$$\boldsymbol{\xi}_1^* = k_1\boldsymbol{\xi}_1^{*0},\ \ \boldsymbol{\xi}_2^* = k_2\boldsymbol{\xi}_2^{*0},\ \ \boldsymbol{\xi}_3^* = k_2\boldsymbol{\xi}_3^{*0} + k_3\boldsymbol{\xi}_2^{*0}, \tag{A.11}$$

$$k_1^2 = \left[(\boldsymbol{\xi}_1^{*0})^T\hat{\mathbf{I}}\boldsymbol{\xi}_1^{*0}\right]^{-1},\ \ k_2^2 = \left[(\boldsymbol{\xi}_2^{*0})^T\hat{\mathbf{I}}\boldsymbol{\xi}_3^{*0}\right]^{-1},\ \ k_3 = -\frac{k_2^3}{2}\left[(\boldsymbol{\xi}_3^{*0})^T\hat{\mathbf{I}}\boldsymbol{\xi}_3^{*0}\right], \tag{A.12}$$

where $\boldsymbol{\xi}_\alpha^{*0}$ are the unnormalized eigenvectors obtained by (26-27) from those presented by Tanuma [18], $\boldsymbol{\xi}_\alpha^*$ the normalized ones and $\hat{\mathbf{I}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix}$ with $\mathbf{I}$ being the identity matrix (3×3). The normalized eigenvectors are given by:

$$\mathbf{a}_1^* = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{\sqrt{2iL}} \end{bmatrix},\ \ \mathbf{a}_2^* = k\begin{bmatrix} \pm 1 \\ i \\ 0 \end{bmatrix},\ \ \mathbf{a}_3^* = k\left(\frac{3A-N}{2(A+N)}\right)\begin{bmatrix} \mp i \\ -1 \\ 0 \end{bmatrix}, \tag{A.13}$$

$$\mathbf{b}_1^* = \begin{bmatrix} 0 \\ 0 \\ \dfrac{iL}{\sqrt{2iL}} \end{bmatrix}, \quad \mathbf{b}_2^* = k(A-N)\begin{bmatrix} \pm i \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3^* = k\left(\frac{A-N}{2}\right)\begin{bmatrix} \pm 1 \\ -i \\ 0 \end{bmatrix}, \tag{A.14}$$

$$k = \left[\frac{A+N}{4A(A-N)}\right]^{1/2} > 0\,. \tag{A.15}$$

The sign ± depends on the value of the $\cos\phi=\pm 1$. In this case, the elastic rotational symmetry axis coincides with x_3, so $\mathbf{\Omega}_3(\theta)$ in (19-20) has no effect, and thus $\mathbf{a}^*=\mathbf{a}$ and $\mathbf{b}^*=\mathbf{b}$.

Case 1.b.1: $\sqrt{AC}-F-2L=0$ and $\dfrac{2L}{A-N} \neq \sqrt{\dfrac{C}{A}}$.

The eigenvalues (A.1-A.3) are:

$$p_1 = i\left(\cos^2\phi + \frac{2L}{A-N}\sin^2\phi\right)^{1/2}, \tag{A.16}$$

$$p_2 = p_3 = i\left(\cos^2\phi + \sqrt{\frac{C}{A}}\sin^2\phi\right)^{1/2}. \tag{A.17}$$

There exists only one linearly independent eigenvector associated with $p_2 = p_3$, and thus it is a D1 case, see (A.9). The eigenvectors can be obtained using (26-27) and the results by Tanuma [18]. The vector $\mathbf{b}_3^*$ is obtained through the relationship between the generalized eigenvector $\boldsymbol{\xi}_3^*$ and the eigenvector $\boldsymbol{\xi}_2^*$, for the degenerate case D1, which takes the form, see Ting [12]:

$$\mathbf{b}_3^* = \mathbf{T}^*\mathbf{a}_2^* + \left[\mathbf{R}^{*T} + p_2\mathbf{T}^*\right]\mathbf{a}_3^*. \tag{A.18}$$

Following the normalization procedure applied as in the previous case using (A.11-A.12), we obtain:

$$\mathbf{a}_1^* = k_1\begin{bmatrix} \cos\phi \\ -\dfrac{\cos\phi}{p_1} \\ \sin\phi \end{bmatrix}, \quad \mathbf{b}_1^* = k_1\cos\phi(A-N)\begin{bmatrix} p_1 \\ -1 \\ \dfrac{\cot\phi}{2p_1}\left[p_1^2(\tan^2\phi-1)-1\right] \end{bmatrix}, \tag{A.19}$$

$$\mathbf{a}_2^* = k_2\begin{bmatrix} 1 \\ p_2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2^* = k_2 p_1^2(A-N)\begin{bmatrix} -p_2 \\ 1 \\ p_2\cot\phi\left(1+\dfrac{1}{p_1^2}\right) \end{bmatrix}, \tag{A.20}$$

$$\mathbf{a}_3^* = \frac{k_2}{2p_2\sin^2\phi(F+L)}\begin{bmatrix} 4Ap_2^2\cos^2\phi + \Delta \\ -p_2\Delta \\ 4Ap_2^2\cos\phi\sin\phi \end{bmatrix}, \qquad \text{(A.21.1)}$$

$$\mathbf{b}_3^* = \frac{k_2(A-N)}{(F+L)\sin^2\phi}\begin{bmatrix} Ap_2^2\cos^2\phi - \frac{p_1^2\sin^2\phi}{2}(F+L) \\ \frac{1}{p_2}\left[-Ap_2^2\cos^2\phi - \frac{p_1^2\sin^2\phi}{2}(F+L)\right] \\ \cot\phi\left[Ap_2^2\sin^2\phi - \frac{(1+p_1^2)}{2}(\Delta + 2Ap_2^2)\right] \end{bmatrix}, \qquad \text{(A.21.2)}$$

$$k_1 = \frac{i}{\sin\phi}\sqrt{\frac{p_1}{2L}}, \quad k_2 = \frac{i}{2p_2}\sqrt{\frac{F+L}{AL}}, \quad \Delta = A\cos^2\phi + L\sin^2\phi - Ap_2^2. \qquad \text{(A.22)}$$

It is still necessary to apply (19-20) to the obtained expressions of $\mathbf{A}^*$ and $\mathbf{B}^*$ to take into account a rotation by θ around x_3.

Case 1.b.2: $\sqrt{AC} - F - 2L = 0$ and $\frac{2L}{A-N} = \sqrt{\frac{C}{A}}$.

Under these conditions, p_1 equals $p_2 = p_3$, see (A.16-A.17), so a triple eigenvalue appears with two linearly independent associated eigenvectors. Thus, this is a D2 case, as was obtained in Case 1.a, but with the difference that in this case the eigenvalues are not necessarily the imaginary unit *i*. Notice that isotropic materials are covered by the present case with $p_1 = p_2 = p_3 = p = i$.

The normalization procedure of this case demands more attention, as will be seen below. Although the eigenvectors associated with p_1 and p_2 are the same as those obtained in the previous case 1.b.1, we will obtain them by the general procedure outlined in Section 3, and not using results by Tanuma [18], together with equations (26-27). The comparison of this case (1.b.2) and the previous one (1.b.1) will help toward a better understanding of both approaches.

Matrices $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$ (17) for this particular case are defined as:

$$\mathbf{Q}^* = \begin{bmatrix} Ap^4 & 0 & -\cot\phi\, Ap^2(1+p^2) \\ 0 & -\tfrac{1}{2}(A-N)p^2 & 0 \\ -\cot\phi\, Ap^2(1+p^2) & 0 & \cot^2\phi\, A(1+p^2)^2 + L \end{bmatrix}, \qquad \text{(A.23)}$$

$$\mathbf{T}^* = \begin{bmatrix} -\tfrac{1}{2}(A-N)p^2 & 0 & \tfrac{1}{2}(A-N)\cot\phi(1+p^2) \\ 0 & A & 0 \\ \tfrac{1}{2}(A-N)\cot\phi(1+p^2) & 0 & \cot^2\phi\tfrac{1}{2}(A-N)(\tan^2\phi - 1 - p^2) \end{bmatrix}, \qquad \text{(A.24)}$$

$$\mathbf{R}^* = \begin{bmatrix} 0 & -Np^2 & 0 \\ -\frac{1}{2}(A-N)p^2 & 0 & \frac{1}{2}(A-N)\cot\phi(1+p^2) \\ 0 & \cot\phi N(1+p^2) & 0 \end{bmatrix}. \qquad \text{(A.25)}$$

The eigenrelation to obtain $\mathbf{a}^*$ (24) with the above $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$ can be written, after appropriate simplifications, as:

$$\tfrac{1}{2}(A+N)\begin{bmatrix} p^4 & -p^3 & -\cot\phi(1+p^2)p^2 \\ -p^3 & p^2 & \cot\phi(1+p^2)p \\ -\cot\phi(1+p^2)p^2 & \cot\phi(1+p^2)p & \cot^2\phi(1+p^2)^2 \end{bmatrix}\mathbf{a}^* = 0\,. \qquad \text{(A.26)}$$

Solving (A.26), we obtain:

$$\mathbf{a}_1^{*0} = \begin{bmatrix} \cos\phi \\ -\dfrac{\cos\phi}{p} \\ sin\phi \end{bmatrix}, \quad \mathbf{a}_2^{*0} = \begin{bmatrix} 1 \\ p \\ 0 \end{bmatrix}, \qquad \text{(A.27)}$$

while for $\mathbf{a}_3^*$ we apply, Ting [12], the equation,

$$-\left\{\mathbf{Q}^* + (\mathbf{R}^* + \mathbf{R}^{*T})p_2 + \mathbf{T}^* p_2^2\right\}\mathbf{a}_3^* = \left\{2p_2\mathbf{T}^* + \mathbf{R}^* + \mathbf{R}^{*T}\right\}\mathbf{a}_2^*, \qquad \text{(A.28)}$$

obtaining:

$$\mathbf{a}_3^{*0} = \begin{bmatrix} \dfrac{3A-N}{p(A+N)} \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(A.29)}$$

In (A.27) and (A.29) the "0" in the superscript denotes that no normalization has been applied for the moment. It can be observed that $\mathbf{a}_1^{*0}$ and $\mathbf{a}_2^{*0}$ (A.27) are equal to $\mathbf{a}_1^*$ (A.19)$_1$ and $\mathbf{a}_2^*$ (A.20)$_1$ respectively, except for the normalization factor. The expression of $\mathbf{a}_3^*$ changes from the previous case, as does the normalization procedure.

Using (25) and (A.18) we obtain ($\mathbf{b}_1^{*0}, \mathbf{b}_2^{*0}$) and ($\mathbf{b}_3^{*0}$) respectively,

$$\mathbf{b}_1^{*0} = (A-N)\cos\phi\begin{bmatrix} p \\ -1 \\ \dfrac{\cot\phi}{2p}\left[p^2(\tan^2\phi - 1) - 1\right] \end{bmatrix}, \quad \mathbf{b}_2^{*0} = (A-N)p\begin{bmatrix} -p^2 \\ p \\ \cot\phi(1+p^2) \end{bmatrix}, \qquad \text{(A.30)}$$

$$\mathbf{b}_3^{*0} = \left(\frac{A-N}{A+N}\right)\begin{bmatrix} -2Ap^2 \\ (A-N)p \\ 2A\cot\phi\,(1+p^2) \end{bmatrix}. \qquad \text{(A.31)}$$

It can be observed that $\boldsymbol{\xi}_1^{*0}$ and $\boldsymbol{\eta}_2^{*0}$ ($\boldsymbol{\eta}_2^T = (\mathbf{b}_3^T, \mathbf{a}_3^T)$ for this D2 case) are not orthogonal to each other. Through a linear combination it is possible to make an orthogonal base (the "⊥" in the superscript denotes an orthogonalized vector),

$$\boldsymbol{\xi}_3^{*0\perp} = \boldsymbol{\xi}_1^{*0} + k\boldsymbol{\xi}_3^{*0} \;\Rightarrow\; (\boldsymbol{\xi}_1^{*0})^T \cdot \boldsymbol{\eta}_2^{*0\perp} = 0 \;\Rightarrow\; k = \left(\frac{A+N}{A-N}\right)\frac{L\sin^2\phi}{2Ap\cos\phi}. \tag{A.32}$$

In order to continue to satisfy relations (25) and (A.18) between the new $\mathbf{a}_3^{*0\perp}$, $\mathbf{b}_3^{*0\perp}$ and $\mathbf{a}_2^{*0}$, $\mathbf{b}_2^{*0}$, it follows that:

$$\mathbf{a}_2^{*0\perp} = k\mathbf{a}_2^{*0} \quad \text{and} \quad \mathbf{b}_2^{*0\perp} = k\mathbf{b}_2^{*0}. \tag{A.33}$$

With $\boldsymbol{\xi}_1^{*0}$, $\boldsymbol{\xi}_2^{*0\perp}$ and $\boldsymbol{\xi}_3^{*0\perp}$ all orthogonal, we perform the normalization procedure in a similar way to Case 1.a (A.11-A.12), as we only have one generalized eigenvector.

The final expressions of the components of $\mathbf{A}^*$ and $\mathbf{B}^*$, already normalized, are defined by:

$$p = p_1 = p_2 = p_3 = i\left(\cos^2\phi + \frac{2L}{A-N}\sin^2\phi\right)^{1/2}, \tag{A.34}$$

$$\mathbf{a}_1^* = k_1\begin{bmatrix}\cos\phi \\ -\dfrac{\cos\phi}{p} \\ \sin\phi\end{bmatrix},\; \mathbf{b}_1^* = k_1\cos\phi(A-N)\begin{bmatrix} p \\ -1 \\ \dfrac{\cot\phi}{2p}\left[p^2(\tan^2\phi - 1) - 1\right]\end{bmatrix}, \tag{A.35}$$

$$\mathbf{a}_2^* = k_2 k\begin{bmatrix}1 \\ p \\ 0\end{bmatrix},\; \mathbf{b}_2^* = k_2 k p^2 (A-N)\begin{bmatrix} -p \\ 1 \\ p\cot\phi\left(1+\dfrac{1}{p^2}\right)\end{bmatrix}, \tag{A.36}$$

$$\mathbf{a}_3^* = k_2\begin{bmatrix}\Psi + \cos\phi\left(1+\dfrac{1}{2p^2}\right) \\ -p\Psi - \dfrac{\cos\phi}{2p} \\ \sin\phi\end{bmatrix}, \mathbf{b}_3^* = k_2\begin{bmatrix}\left(\dfrac{A-N}{2}\right)p\cos\phi - \dfrac{(A+N)Lp\sin^2\phi}{4A\cos\phi} \\ -\left(\dfrac{A-N}{2}\right)\cos\phi - \dfrac{(A+N)L\sin^2\phi}{4A\cos\phi} \\ \left(\dfrac{A-N}{2}\right)p\sin\phi + \dfrac{(A+N)L(1+p^2)\sin\phi}{4Ap}\end{bmatrix}, \tag{A.37}$$

$$k = \left(\frac{A+N}{A-N}\right)\frac{L\sin^2\phi}{2Ap\cos\phi},\; k_1 = \frac{i}{\sin\phi}\sqrt{\frac{p}{2L}},\; k_2 = \frac{i\cos\phi}{L\sin^2\phi}\sqrt{\frac{A(A-N)}{A+N}}, \tag{A.38}$$

$$\Psi = \frac{(3A-N)L\sin^2\phi}{(A-N)4Ap^2\cos\phi}. \tag{A.39}$$

When comparing with the previous case, we can see how $\mathbf{a}_1^*$ and $\mathbf{b}_1^*$ are identical, see (A.19) and (A.35), and how $\mathbf{a}_2^*$ and $\mathbf{b}_2^*$ are also equal except for the normalization factor *k*, see (A.20) and (A.36). The generalized eigenvectors ξ_3^* are different in both cases.

The semisimple and simple cases will be analyzed together, and following Tanuma [18] this analysis will be performed separately for vanishing and non-vanishing value of *F*+*L*.

Cases 2.1 and 3.1: Semisimple and simple cases with $F + L = 0$.

For anisotropic materials having a fundamental elasticity matrix $\mathbf{N}^*$ with three linearly independent eigenvectors, $\mathbf{N}^*$ being simple when ($p_1 \neq p_2 \neq p_3 \neq p_1$) and semisimple when ($p_1 = p_2 \neq p_3$), the structure of matrices $\mathbf{A}^*$ and $\mathbf{B}^*$ is well known, see Ting [12], Chapter 6. For the particular case of simple and semisimple cases of transversely isotropic materials, with $F + L = 0$, we have the following normalization factors:

$$k_\alpha^2 = \left[2(\mathbf{a}_\alpha^* \cdot \mathbf{b}_\alpha^*)\right]^{-1}. \tag{A.40}$$

For the semisimple case (Case 2), (A.8) holds and:

$$p_1 = p_2 = i\left(\cos^2\phi + \frac{2L}{A-N}\sin^2\phi\right)^{1/2}, \quad p_3 = i\left(\cos^2\phi + \frac{L}{A}\sin^2\phi\right)^{1/2}. \tag{A.41}$$

For the simple case (Case 3):

$$p_1 = i\left(\cos^2\phi + \frac{2L}{A-N}\sin^2\phi\right)^{1/2}, p_2 = i\left(\cos^2\phi + \frac{C}{L}\sin^2\phi\right)^{1/2}, p_3 = i\left(\cos^2\phi + \frac{L}{A}\sin^2\phi\right)^{1/2} \tag{A.42}$$

For both cases (simple and semisimple) together with (A.41-A.42), we have:

$$\mathbf{a}_1^* = k_1\frac{\cos^2\phi}{\sin\phi}\begin{bmatrix} -1 \\ 1/p_1 \\ -\tan\phi \end{bmatrix}, \quad \mathbf{b}_1^* = k_1\frac{(A-N)\cos^2\phi}{\sin\phi}\begin{bmatrix} -p_1 \\ 1 \\ \dfrac{\cos^2\phi(1+2p_1^2) - p_1^2}{2p_1\cos\phi\sin\phi} \end{bmatrix}, \tag{A.43}$$

$$\mathbf{a}_2^* = k_2\begin{bmatrix} \sin\phi \\ 0 \\ -\cos\phi \end{bmatrix}, \quad \mathbf{b}_2^* = k_2\begin{bmatrix} Lp_2\sin\phi \\ -L\sin\phi \\ -Lp_2\cos\phi \end{bmatrix}, \tag{A.44}$$

$$\mathbf{a}_3^* = k_3 \begin{bmatrix} \cos^2\phi \\ p_3 \\ \sin\phi\cos\phi \end{bmatrix}, \quad \mathbf{b}_3^* = k_3(\cos^2\phi - p_1^2)\left(\frac{A-N}{2}\right)\begin{bmatrix} p_3 \\ -1 \\ p_3\cot\phi\left(\dfrac{2}{\cos^2\phi - p_1^2} - 1\right) \end{bmatrix}, \tag{A.45}$$

$$k_1 = \frac{i}{\cos\phi}\sqrt{\frac{p_1}{2L}}, \quad k_2 = \frac{1}{\sqrt{2Lp_2}}, \quad k_3 = \frac{i}{\sin\phi\sqrt{2p_3L}}. \tag{A.46}$$

Cases 2.2 and 3.2: Semisimple and simple cases with $F + L \neq 0$.

The eigenvalues are those in (A.1-A.3), with $p_1 = p_2$ for the semisimple case. The vectors $\mathbf{a}_\alpha^*$ and $\mathbf{b}_\alpha^*$ can be taken from Ting [12], chapter 6, or Tanuma [18] together with (26-27), and with the normalization factors defined in (A.40), we finally obtain:

$$\mathbf{a}_1^* = k_1\frac{\cos^2\phi}{\sin\phi}\begin{bmatrix} -1 \\ 1/p_1 \\ -\tan\phi \end{bmatrix}, \quad \mathbf{b}_1^* = k_1\frac{(A-N)\cos^2\phi}{\sin\phi}\begin{bmatrix} -p_1 \\ 1 \\ \dfrac{\cos^2\phi(1+2p_1^2) - p_1^2}{2p_1\cos\phi\sin\phi} \end{bmatrix}, \tag{A.47}$$

$$k_1 = \frac{i}{\cos\phi}\sqrt{\frac{p_1}{2L}}, \tag{A.48}$$

and for $i = 2,3$, we have:

$$\mathbf{a}_i^* = k_i\begin{bmatrix} -(F+L)\cos^2\phi\sin\phi + A\sin\phi(\cos^2\phi + p_i^2) + L\sin^3\phi \\ -(F+L)p_i\sin\phi \\ -(F+L)\cos\phi\sin^2\phi - A\cos\phi(\cos^2\phi + p_i^2) - L\cos\phi\sin^2\phi \end{bmatrix}, \tag{A.49}$$

$$\mathbf{b}_i^* = k_i\begin{bmatrix} -(A-N)(F+L)p_i\cos^2\phi\sin\phi - Lp_i\sin\phi(F\sin^2\phi - A\cos^2\phi - Ap_i^2) \\ \left[\{AF - (F+L)N\}\cos^2\phi + FL\sin^2\phi - ALp_i^2\right]\sin\phi \\ -(A-N)(F+L)p_i\cos\phi\sin^2\phi + Lp_i\cos\phi(F\sin^2\phi - A\cos^2\phi - Ap_i^2) \end{bmatrix}, \tag{A.50}$$

$$k_i = \left(2p_iAL(\cos^2\phi + p_i^2)\{A(\cos^2\phi + p_i^2) + 2L\sin^2\phi\} - 2p_iFL\sin^4\phi(F+2L)\right)^{-1/2}. \tag{A.51}$$

Once $\mathbf{A}^*$ and $\mathbf{B}^*$ have been evaluated for all the possible cases for a transversely isotropic material, it only remains to apply the transformation law of **a** and **b** for a rotation around x_3^* axis, see (19-20), to have the final expressions of $\mathbf{A}=\mathbf{A}(\phi,\theta)$ and $\mathbf{B}=\mathbf{B}(\phi,\theta)$ in any generic orientation of the material with respect to the generalized plane strain configuration.