An Approach to Ballistic Deposition Based on Membrane Computing

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Ballistic Deposition was proposed by Vold [10] and Sutherland [9] as a model for colloidal aggregation. These early works were later extended to simulate the process of vapour deposition. In general, Ballistic Deposition models involve $(d + 1)$-dimensional particles which rain down sequentially at random onto a d-dimensional substrate; when a particle arrives on the existing agglomeration of deposited particles, it sticks to the first particle it contacts, which may result in lateral growth. In this paper we present a first P system model for Ballistic Deposition with $d = 1$.

1 INTRODUCTION

Some recent discoveries on the dynamical process of surface growth have encouraged the scientific community to revisit the study of systems exhibiting rough interfaces. In Nature, there exist many examples of rough interfaces, actually, *all* surfaces in Nature can be seen as rough surfaces, since the concept of roughness is associated with the scale of observation and surfaces on Nature are far from be smooth if observed at appropriate scale.

The propagation of forest fires [5], the growth of a colony of bacteria [3] or the propagation of reaction fronts in catalysed reactions [1] are real-world examples where the *frontier* between two media are far from being smooth. In these cases, the interfaces can be hardly modelled with Euclidean geometry

and it is necessary to consider new tools in order to handle them. Moreover, in that cases, we are interested not only in the morphology of the interfaces from a static point of view, but in the dynamics of how the interface develops in time.

This paper is devoted to the study of a process of formation of rough surfaces called *Ballistic Deposition* (BD). To this aim, we will explore the capability of some Membrane Computing devices as tools for modelling BD in a discrete approach.

The paper is organised as follows: first the Ballistic Deposition model is briefly described. In Section 3, *deposition P systems* are presented and following this model, a P system simulating the dynamics of Ballistic Deposition is presented in Section 4. Some conclusions are presented in Section 5. The paper ends with the Bibliography and an Appendix with the proof of our main result.

2 BALLISTIC DEPOSITION

In Nature, some interfaces are formed as result of a deposition process, other shrink due to erosion. A typical example of deposition process is the random fall of snowflakes on the ground floor. The randomness in the deposition process leads to a rough surface.

There exist many deposition models which try to represent different natural process. The simplest way to define such models is on a lattice where particles are deposited onto a surface oriented perpendicular to the particle trajectories, but other versions have been also investigated* . Ballistic Deposition (BD) was proposed by Vold [10] and Sutherland [9] as a model for colloidal aggregation. These early works were later extended to simulate the process of vapour deposition. In this model, a particle is released from a position above the surface. The particle follows a straight vertical trajectory until it reaches the surface, whereupon it sticks (see Figure 1).

In general, Ballistic Deposition models involve $(d+1)$-dimensional particles which rain down sequentially at random onto a d-dimensional substrate; when a particle arrives on the existing agglomeration of deposited particles, it sticks to the first particle it contacts, which may result in lateral growth. Many mathematical models exist in order to describe Ballistic Depositions. Here we follow M.D. Penrose in [7], where all particles are assumed identical.

In Penrose's mathematical model, the substrate is $\mathbb{R}^d \times \{0\}$, identified with $\mathbb{R}^d$ or some subregion thereof. All particles are $(d+1)$-dimensional

* A good starting point for the study of depositions is [2].

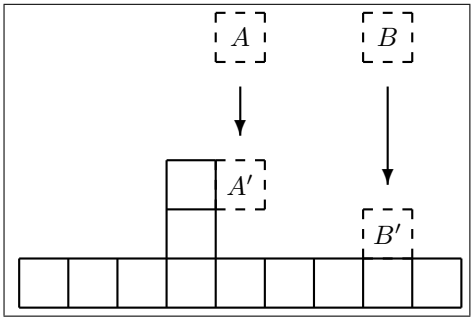

Figure 1
Ballistic Deposition

solids. Particles arrive sequentially at random positions in $\mathbb{R}^d$. When a particle reaches a position $x \in \mathbb{R}^d$, it slides down the ray $\{x\} \times [0, \infty)$ until the particle hits a position adjacent to either the substrate or a previously deposited particle where is permanently fixed. The difference between lattice and continuum models is that in the lattice model the positions at which particle arrive are restricted to be in the integer lattice $\mathbb{Z}^d$.

Let $\mathbf{0}$ denote the origin in $\mathbb{Z}^d$. A *displacement function* is a mapping D from $\mathbb{Z}^d$ to $[-\infty, \infty)$ verifying:

- $D(\mathbf{0}) = 1$
- The set $\mathcal{N} = \{x \in \mathbb{Z}^d \ : \ D(x) \neq -\infty\}$ is finite but has at least two elements (one of which is the origin)

For $x \in \mathbb{Z}^d$, let $\mathcal{N}_x = \{x + y \ : \ y \in \mathcal{N}\}$ and $\mathcal{N}_x^* = \{x - y \ : \ y \in \mathcal{N}\}$. The set $\mathcal{N}$ is a neighbourhood of the origin and $\mathcal{N}_x$ is a neighbourhood of x. The idea of a displacement function is that if a particle arrives at $z \in \mathcal{N}_x$ then it cannot slide down the ray $z \times [0, +\infty)$ below the position at height $D(z - x)$. In this way, if $h(x, t)$ measures the height of the interface at site x at time t then $h(x, t+1) = \max\{h(y, t) + D(x - y) \ : \ y \in \mathbb{Z}^d\}$. We have $h(x, t+1) = \max\{h(y, t) + D(x - y) \ : \ y \in \mathcal{N}_x^*\}$.

In this paper we follow the version of ballistic deposition considered in [8], the nearest neighbour model, where $\mathcal{N} = \{z \in \mathbb{Z}^d \ : \ \|z\|_1 \leq 1\}$ (being $\|z\|_1 = \sum_{i=1}^{i=d} |z_i|$ for all $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$) and the displacement function D is given by $D(x) = 0$ for $x \in \mathcal{N} - \{\mathbf{0}\}$. We are considering that the

dimension of the substrate is $d=1$ and therefore

$$h(x,t+1) = \max\{h(x-1,t),\, h(x,t)+1, h(x+1,t)\}$$

3 P SYSTEMS

The chosen P systems model can be considered as a subclass of tissue-like P system, since we do not consider membranes surrounding other ones, but a sequence of cells linked by communication channels. The intuition behind this structure is that each cell represents a column of the aggregate and the pieces of information needed for encoding the growth process are encoded on the multisets of objects in the cells.

In this model we use two very powerful membrane computing tools: the cooperation and the use of polarisations of the cells. Both features allow us an efficient design of P systems in order to perform the simulation. The study of minimal resources, i.e., to know whether the deposition process can be simulated with fewer ingredients falls out of the scope of this paper.

Formally, a *deposition P system* of degree L is a tuple of the form $\Pi = (O, P, \mu, env, v_1, \ldots, v_L, v_{env}, R)$ where:

1. O is the alphabet of objects;

2. $P=\{0,+,-\}$ is the set of polarisations. Each membrane will be associated with one polarisation.

3. μ is a *cell structure* consisting of L cells bijectively labelled with $\{1,\ldots,L\}$. For every $i \in \{1,\ldots,L-1\}$ there exists an edge between the cell i and the cell $i+1$. We will also consider an edge between the cell L and the cell 1. For the sake of simplicity, we will identify the indices $L+1$ and 1; also, if a cell has polarisation 0, we will omit the symbol 0. The initial polarisation of all membranes is 0.

4. env is the environment. It represents the region surrounding the cell structure μ. Some objects can be also placed in this region.

5. $v_1,\ldots,v_L,v_{env}$ are strings over O, describing the *multisets of objects* placed in the corresponding cells of μ and in the environment, respectively.

6. R is a finite set of *rules*, of the following forms:

(a) $[u_1 \rightarrow u_2]_i^e$ where $i \in \{1, \ldots, L\}$, $e \in P$ and $u_1, u_2 \in O^*$. These are *object evolution rules* associated with cells and depending only on the label and the polarisation of the cell. The string u_1 has at least one object.

(b) $a[\,]_i^{e_1} \rightarrow [b]_i^{e_2}$ where $i \in \{1, \ldots, L\}$, $e_1, e_2 \in P$ and $a, b \in O$. These are *send-in rules*. An object of the environment is introduced in the membrane i possibly modified. The polarisation of the cell can also change.

(c) $[a]_i^{e_1} \rightarrow b[\,]_i^{e_2}$ where $i \in \{1, \ldots, L\}$, $e_1, e_2 \in P$ and $a, b \in O$. These are *send-out rules*. An object is sent out to the environment possibly modified. The polarisation of the cell i can also change.

(d) $[a]_i^{e_1}\ [\,]_{i+1} \rightarrow [\,]_i\ [b]_{i+1}^{e_2}$ and $[\,]_i\ [a]_{i+1}^{e_1} \rightarrow [b]_i^{e_2}\ [\,]_{i+1}$ where $i \in \{1, \ldots, L\}$, $e_1, e_2 \in P$ and $a, b \in O$. These are *communication rules*. An object a is sent, possibly modified, to an adjacent cell.

Rules are applied according to the following principles:

- Rules are used as usual in the framework of membrane computing, that is, in a maximally parallel way. In one step, each object in a cell can only be used for one rule (non deterministically chosen when there are several possibilities), but any object which can evolve by a rule of any form must do it, with a restriction: as usual in P systems with polarisations, only one change of polarisation can affect a membrane, i.e., if two or more rules of types (b), (c) and (d) can be applied to a membrane, then only one of them is applied, non-deterministically chosen.

- All the elements which are not involved in any of the operations to be applied remain unchanged.

- Several rules can be applied to different objects in the same cell simultaneously.

- If rules of type (a) are used in the same time with any of the type (b), (c) or (d), then all these are applied, but we will consider that the object evolution rules (a) are performed *before* the other ones. This consideration is useful because the rules that send objects across a membrane can also change its polarisation.

4 MODELLING BALLISTIC DEPOSITION

In this section we will consider a system of Ballistic Deposition with L columns and we will provide a deposition P system which simulates its dynamics. Let us consider the deposition P system of degree L, $\Pi = (O, \mu, env, v_1, \ldots, v_L, v_{env}, P, R)$ where:

- $O = \{p, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_{n3}, c_{n4}, \alpha, x, y, z\}$
- $v_i = \emptyset$, for all $i \in \{1, \ldots, L\}$
- $v_{env} = p$

Let us consider the following sets of rules, where $i \in \{1, \ldots, L\}$. As remarked before, we will identify the indices $L+1$ and 1 as being the same and similarly for 0 and L; and, if a cell has polarisation 0, we will omit the symbol 0. As usual, λ represents the empty word.

Set (A) **– Deposition rules:** $R^i_* \equiv p\,[\,]_i \to [c_0]^+_i$

In the BD model (see Section 2) , a particle is deposited on the top of a column randomly chosen. We simulate this process by these rules. A particle p in the environment is sent to one of the cells. This particle activates the cell (the polarisation of the cells turns on positive) and goes into the cell as the object c_0.

Set (B) **– Rules for cells with positive polarisation:**

$$R^i_1 \equiv [c_0 \to c_1]^+_i \qquad R^i_2 \equiv [c_1 \to c_2]^+_i \qquad R^i_3 \equiv [y \to z\,\alpha]^+_i$$
$$R^i_4 \equiv [\,]_i\,[c_2]^+_{i+1} \to [c_{n3}]^-_i\,[\,]_{i+1} \qquad R^i_5 \equiv [\,]_i\,[z]^+_{i+1} \to [y]_i\,[\,]_{i+1}$$

The object c_0 in a cell i evolves to c_2 in two waiting steps before being sent to the cell $i-1$ transformed into c_{n3}. These waiting steps check the occurrence of objects y inside the cell. If any y occur, each of then evolves to $z\,\alpha$ at the same time in which c_0 evolves to c_1 and in the following step each z is sent to the cell $i-1$ transformed into y (the polarisation of the cell i changes). If this happens, rule R^{i-1}_4 is not triggered because the cell containing c_2 has not positive charge.

Set (C) **– Rules for cells with negative polarisation:**

$$R^i_6 \equiv [c_3 \to c_4]^-_i \qquad R^i_7 \equiv [c_4 \to c_5]^-_i \qquad R^i_8 \equiv [c_5 \to c_6]^-_i$$
$$R^i_9 \equiv [c_6]^-_i \to p\,[\,]_i \qquad R^i_{10} \equiv [c_{n3} \to c_{n4}]^-_i \qquad R^i_{11} \equiv [c_{n4} \to x\,y\,c_5]^-_i$$
$$R^i_{12} \equiv [x]^-_i\,[\,]_{i+1} \to [\,]_i\,[z]_{i+1}$$

Set (D) **– Rules for cells with polarisation zero:**

$$R^i_{13} \equiv [c_{n4} \to c_5]_i \qquad R^i_{14} \equiv [c_5 \to c_6]_i \qquad R^i_{15} \equiv [c_6]_i \to p\,[\,]_i$$
$$R^i_{16} \equiv [z \to x\,\alpha]_i \qquad R^i_{17} \equiv [x\,y \to \lambda]_i \qquad R^i_{18} \equiv [\,]_i\,[c_2]_{i+1} \to [c_3]^-_i\,[\,]_{i+1}$$

The object c_2 produces the objects c_3 and c_4, or the objects c_{n3} and c_{n4}. In

both cases, the objects c_5 and c_6 are produced. The object c_6 sends to the environment an object p which will go into a cell in the next step according to the set of rules R^i_*.

As in the set of rule (B), the counter c_k goes on till it reaches c_6. The object c_6 sends to the environment an object p which will go into a cell in the next step according to the set of rules R^i_*. Notice that rules of R^i_{17} are cooperative rules. Each pair of objects x, y which occur in a cell disappears in the next step.

4.1 Informal description of the computations

For a better understanding of the computation, let us remark that the configurations at time $8t$ with $t \in \mathbb{N}$, represent the state of a BD system after the deposition of the t–th particle. Below we will formalise this idea, but before giving a description of the computation, we provide the intuitive meaning of some of the symbols of the alphabet at time $8t$:

- p represents the particle that arrives to the substrate. When it is deposited, it disappears from the environment, then the information encoded in several cells change. When the computation inside the cells finishes, a new particle is sent to the environment and the process starts again. In this way only in the steps $8t$, a particle p appears in the environment.

- The multiplicity of α in the cell i represents the height of the column i in the BD model.

Finally, the remaining objects inside the cells at time $8t$ are of types x and y. The objects x or y inside the cell i represent the difference of height between the cell i and the cell $i+1$.

- The multiplicity of x in the cell i represents the number of units the column i is higher than column $i+1$ in the BD model.

- The multiplicity of y in the cell i represents the number of units the column i is lower than column $i+1$ in the BD model.

From the previous description we have that at time $8t$, we can find inside a cell either objects x, y or none of them, but we will never find both simultaneously. Table 2 shows an example of evolution of a simple Ballistic Deposition system with four columns where four particles have been deposited sequentially on the columns 3,2,2 and 1. The configurations at times $8t$ with

Time	Rules	Env.	Configuration
T_0		p	$-\boxed{\ }_1-\boxed{\ }_2-\boxed{\ }_3-\boxed{\ }_4-$
T_1	R^3_*		$-\boxed{\ }_1-\boxed{\ }_2-\boxed{c_0}^{+}_3-\boxed{\ }_4-$
T_2	R^3_1		$-\boxed{\ }_1-\boxed{\ }_2-\boxed{c_1}^{+}_3-\boxed{\ }_4-$
T_3	R^3_2		$-\boxed{\ }_1-\boxed{\ }_2-\boxed{c_2}^{+}_3-\boxed{\ }_4-$
T_4	R^2_4		$-\boxed{\ }_1-\boxed{c_{n3}}^{-}_2-\boxed{\ }_3-\boxed{\ }_4-$
T_5	R^2_{10}		$-\boxed{\ }_1-\boxed{c_{n4}}^{-}_2-\boxed{\ }_3-\boxed{\ }_4-$
T_6	R^2_{11}		$-\boxed{\ }_1-\boxed{x\,y\,c_5}^{-}_2-\boxed{\ }_3-\boxed{\ }_4-$
T_7	R^2_8, R^2_{12}		$-\boxed{\ }_1-\boxed{y\,c_6}_2-\boxed{z}_3-\boxed{\ }_4-$
T_8	R^2_{15}, R^3_{16}	p	$-\boxed{\ }_1-\boxed{y}_2-\boxed{x\,\alpha}_3-\boxed{\ }_4-$

Figure 2
Table with the first eight steps

$t \in \{0, \ldots, 4\}$ represent the *surface* of the Ballistic Deposition model after the fall of the t–th particle (Fig. 3).

We finish this section by formally showing that this deposition P system of degree L simulates the ballistic deposition on a substrate with L columns. In this way we need the definition of a *representative configuration*. The idea behind the definition is quite intuitive. Along the computation, some of the configurations have no meaning with respect to the deposition process, there are merely auxiliary steps of the computation. Only some of the configurations represent states of the aggregate in the deposition process. Such configurations will be called *representative configurations*.

Note that all rules are applied in a deterministic way but the rule R^i_* ($i \in \{1, \ldots, L\}$). Depending on the rules R^i_* chosen we have different computations.

Let C be an arbitrary computation of the P system and $t \in \mathbb{N}$. We will denote by C_t the configuration of the P system at time t in such computation. $C_t(i)$ denotes the pair (m, e), where m is the multiset of objects of the cell i at time t in the configuration C_t, and e is the electrical charge of that cell (we

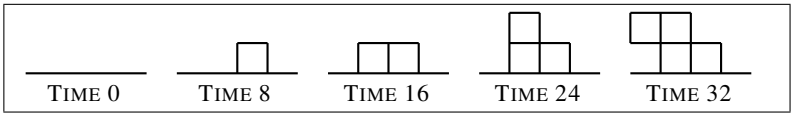

Figure 3
Example

denote it by $\{m\}^e$). Finally, $|C_t(i)|_a$ denotes the multiplicity of the object a in $C_t(i)$.

Definition *Let C_t be a configuration of a deposition P systems of degree L at time t. We will say that C_t is representative if for all $i \in \{1, \ldots, L\}$,*

- *Only objects α, x and y can occur inside the cells and all the cells have polarisation 0*
- $C_t(env) = \{p\}$
- *If $|C_t(i)|_\alpha \geq |C_t(i+1)|_\alpha$ then $|C_t(i)|_x = |C_t(i)|_\alpha - |C_t(i+1)|_\alpha$ and $|C_t(i)|_y = 0$.*
- *If $|C_t(i)|_\alpha < |C_t(i+1)|_\alpha$ then $|C_t(i)|_y = |C_t(i+1)|_\alpha - |C_t(i)|_\alpha$ and $|C_t(i)|_x = 0$*

Finally, the next theorem claims that the sequence of configurations at times $0, 8, 16, \ldots, 8t, \ldots$ represent the states of an aggregate with a ballistic deposition process.

Theorem *Let C an arbitrary computation of the P systems. For all $t \in \mathbb{N}$,the configuration C_{8t} is a representative configuration and, if $i \in \{1, \ldots, L\}$ is the chosen cell for depositing a new particle, then*

- $|C_{8t+8}(i)|_\alpha = max\{|C_{8t}(i-1)|_\alpha, |C_{8t}(i)|_\alpha + 1, |C_{8t}(i+1)|_\alpha\}$
- *For all $j \in \{1, \ldots, L\}$, $i \neq j$, we have $|C_{8t+8}(j)|_\alpha = |C_{8t}(j)|_\alpha$*

Proof See Appendix. □

5 CONCLUSIONS

Understanding how Nature works involves experimental observation and theoretical modelling. This paper is a contribution to the theoretical modelling of

a particular case of one of the most interesting process in Physics: the dynamical evolution of the frontier between two different media. In this paper, the chosen model has been Ballistic Deposition, but many other deposition processes from Physics, Chemistry and Biology can be also modelled by using similar techniques.

On the other hand, Membrane Computing techniques have been used for studying problems from many different areas, from Linguistics or Complexity Theory to Computer Graphics or Cancer Modelling[†] , but this is the first time that P systems are used to model deposition processes.

This paper can be extended in several ways. One of them is to extend the study to more dimensions, i.e., to consider the particles as 3D solids falling down onto a 2D surface. Other possible research line is to develop computer software which simulates Ballistic Depositions according with the Membrane Computing techniques presented in this paper and compare its efficiency with other approaches. A final research line is to follow this study by modelling other deposition models.

6 ACKNOWLEDGMENT

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REFERENCES

[1] E.V. Albano. (1997). Displacement of inactive phases by the reactive regime in a lattice gas model for a dimer-monomer irreversible surface reaction. *Physical Review E*, 55:7144–7152.

[2] A.-L. Barabási and H.E. Stanley. (1995). *Fractal Concepts in Surface Growth*. Cambridge University Press.

[3] E. Ben-Jacob, O. Schochet, A. Tenenbaum, I. Cohen, A. Czirok, and T. Vicsek. (1994). Generic modelling of cooperative growth patterns in bacterial colonies. *Nature*, 368:46–49.

[4] G. Ciobanu, Gh. Păun, and M.J. Pérez-Jiménez, editors. (2006). *Applications of Membrane Computing*. Springer.

[5] S. Clar, B. Drossel, and F. Schwabl. (1996). Forest fires and other examples of self-organized criticality. *Journal of Physics: Condensed Matter*, 8:6803–6824.

[6] C. Graciani-Díaz, M.A. Gutiérrez-Naranjo, and M.J. Pérez-Jiménez. (2007). A membrane computing model for ballistic depositions. In A. Romero-Jiménez M.A. Gutiérrez-Naranjo, Gh. Păun and Riscos-Núñez A, editors, *Fifth Brainstorming Week on Membrane Computing*, pages 179–197, Sevilla, Spain. Fénix Editora.

[†] See [4] for details.

[7] M.D. Penrose. (2006). Growth and roughness of the interface for ballistic deposition. *ArXiv Mathematics e-prints*.

[8] T. Seppäläinen. (2000). Strong law of large numbers for the interface in ballistic deposition. *Annales de L'Institut Henri Poincaré (B) Probabilités et Statistiques*, 36:691–735.

[9] D.N. Sutherland. (1966). Comments on vold's simulation of folc formation. *Journal of Colloid and Interface Science*, 22:300–302.

[10] M.J. Vold. (1959). A numerical approach to the problem of sediment volume. *Journal of Colloid Science*, 14:168–174.

APPENDIX

Proof [Theorem 4.1] By induction on $t \in \mathbb{N}$. The key idea of the proof is to notice that the P system is deterministic with the only exception of the set of rules R^i_* for $i \in \{1, \ldots, L\}$. This set of rules represent the non-deterministic choice of a cell in order to deposit a new particle.

$\boxed{t = 0}$

In the initial configuration C_0 all cells are empty and have polarisation 0; also, $C_0(env) = \{p\}$, then it is a representative configuration. Let us suppose that i is the chosen cell in the non-deterministic step. The first configurations are

- $C_0(i) = \{\,\} \stackrel{R^i_*}{\to} C_1(i) = \{c_0\}^+ \stackrel{R^i_1}{\to} C_2(i) = \{c_1\}^+ \stackrel{R^i_2}{\to} C_3(i) = \{c_2\}^+$
- $C_k(j) = \emptyset$, $\quad (j \in \{1, \ldots, i-1, i+1, \ldots, L, env\}$ and $k \in 1\{1, 2, 3\})$.

As the cell i has positive electrical charge in C_3, then we apply the rule R^{i-1}_4 obtaining $C_4(i-1) = \{c_{n3}\}^-$ and $C_4(j) = \emptyset$ for all $j \in \{1, \ldots, i-2, i, \ldots, L, env\}$. Hence

- $C_4(i-1) = \{c_{n3}\}^- \stackrel{R^{i-1}_{10}}{\to} C_5(i-1) = \{c_{n4}\}^- \stackrel{R^{i-1}_{11}}{\to} C_6(i-1) = \{x\, y\, c_{n5}\}$
- $C_k(j) = \emptyset$, $\quad (j \in \{1, \ldots, i-2, i, \ldots, L, env\}$ and $k \in \{5, 6\})$.

At this step rules R^{i-1}_8 and R^{i-1}_{12} are applied simultaneously given rise to

- $C_7(i-1) = \{c_6\, y\}$, $C_7(i) = \{z\}$.
- $C_7(j) = \emptyset$, $\quad (j \in \{1, \ldots, i-2, i+1, \ldots, L, env\})$.

Finally, rules R^{i-1}_9 and R^i_{16} are applied obtaining

- $C_8(i-1) = \{y\}$, $C_8(i) = \{x\, \alpha\}$, $C_8(env) = \{p\}$.

- $C_8(j) = \emptyset, \quad (j \in \{1, \ldots, i-2, i+1, \ldots, L\})$.

In configuration C_8, all the cells has polarisation 0 and only objects x, y and α occur inside them, $C_8(env) = \{p\}$. Also,
$|C_8(i-1)|_\alpha = 0 < |C_8(i)|_\alpha = 1$
$|C_8(i-1)|_y = 1 = |C_8(i)|_\alpha - |C_8(i-1)|_\alpha$
$|C_8(i-1)|_x = 0$
In addition
$|C_8(i)|_\alpha = 1 \geq |C_8(i+1)|_\alpha = 0$
$|C_8(i)|_x = 1 = |C_8(i)|_\alpha - |C_8(i+1)|_\alpha$
$|C_8(i)|_y = 0$
Finally $C_8(j) = \emptyset$ for all $j \in \{1, \ldots, i-2, i+1, \ldots, L\}$.
Hence, C_8 is a representative configuration. We also have
$|C_8(j)|_\alpha = |C_0(j)|_\alpha$ for $j \in \{1, \ldots, i-1, i+1, \ldots, L\}$
$|C_8(i)|_\alpha = max\{|C_0(i-1)|_\alpha, |C_0(i)|_\alpha + 1, |C_0(i+1)|_\alpha\} = 1$

$\boxed{t \rightarrow t+1}$

Let us suppose that C_{8t} is a representative configuration and i is the cell chosen non-deterministically. We will prove that

- C_{8t+8} is also a representative configuration.
- $|C_{8t+8}(i)|_\alpha = max\{|C_{8t}(i-1)|_\alpha, |C_{8t}(i)|_\alpha + 1, |C_{8t}(i+1)|_\alpha\}$.
- For all $j \in \{1, \ldots, L\}, i \neq j, |C_{8t+8}(j)|_\alpha = |C_{8t}(j)|_\alpha$.

We will consider five possible cases depending on the relation among $|C_{8t}(i-1)|_\alpha$, $|C_{8t}(i)|_\alpha$ and $|C_{8t}(i+1)|_\alpha$.

Case 1: $|C_{8t}(i-1)|_\alpha > |C_{8t}(i)|_\alpha \geq |C_{8t}(i+1)|_\alpha$
Case 2: $|C_{8t}(i-1)|_\alpha > |C_{8t}(i+1)|_\alpha > |C_{8t}(i)|_\alpha$
Case 3: $|C_{8t}(i+1)|_\alpha \geq |C_{8t}(i-1)|_\alpha > |C_{8t}(i)|_\alpha$
Case 4: $|C_{8t}(i)|_\alpha \geq |C_{8t}(i-1)|_\alpha$ and $|C_{8t}(i)|_\alpha \geq |C_{8t}(i+1)|_\alpha$
Case 5: $|C_{8t}(i+1)|_\alpha > |C_{8t}(i)|_\alpha \geq |C_{8t}(i-1)|_\alpha$

Proof is made by inspection of these cases. In order not to extend further we include only the first case (the rest of them are proved in a similar manner). The complete proof of each case can be found in the appendix of [6].

Case 1: Let us suppose that $|C_{8t}(i-1)|_\alpha > |C_{8t}(i)|_\alpha \geq |C_{8t}(i+1)|_\alpha$.
In this case we have

- $|C_{8t}(i-1)|_x = |C_{8t}(i-1)|_\alpha - |C_{8t}(i)|_\alpha > 0$.

- $|C_{8t}(i)|_x = |C_{8t}(i)|_\alpha - |C_{8t}(i+1)|_\alpha \geq 0$.
- $|C_{8t}(i-1)|_y = |C_{8t}(i)|_y = 0$.

Also, $max\{|C_{8t}(i-1)|_\alpha, |C_{8t}(i)|_\alpha + 1, |C_{8t}(i+1)|_\alpha\} = |C_{8t}(i-1)|_\alpha$. Having in mind that object y are present in cell i we have

- $C_{8t}(i) \stackrel{R_*^i}{\rightarrow} C_{8t+1}(i) = C_{8t}(i) \cup \{c_0\}^+ \stackrel{R_1^i}{\rightarrow} C_{8t+2}(i) = C_{8t}(i) \cup \{c_1\}^+ \stackrel{R_2^i}{\rightarrow} C_{8t+3}(i) = C_{8t}(i) \cup \{c_2\}^+$
- $C_{8t+k}(j) = C_{8t}(j), \quad (j \in \{1, \ldots, i-1, i+1, \ldots, L\})$.
- $C_{8t+k}(env) = \emptyset$ for $k \in \{1, 2, 3\}$.

As cell i has positive electrical charge in C_{8t+3}, then we apply the rule R_4^{i-1} obtaining

- $C_{8t+4}(i-1) = C_{8t}(i-1) \cup \{c_{n3}\}^-$.
- $C_{8t+4}(j) = C_{8t}(j), \quad (j \in \{1, \ldots, i-2, i, \ldots, L\})$.
- $C_{8t+4}(env) = \emptyset$.

At this step rules R_{10}^{i-1} and R_{12}^{i-1} are applied simultaneously and therefore all the copies of x in cell $i-1$ are sent into cell i transformed into copies of z. Hence

- $C_{8t+5}(i-1) = \{\alpha^{|C_{8t}(i-1)|_\alpha}\, c_{n4}\}$.
- $C_{8t+5}(i) = C_{8t}(i) \cup \{z^{|C_{8t}(i-1)|_x}\}$.
- $C_{8t+5}(j) = C_{8t}(j), \quad (j \in \{1, \ldots, i-2, i+1, \ldots, L\})$.
- $C_{8t+5}(env) = \emptyset$.

Now, rules R_{13}^{i-1} and R_{16}^i can be applied. Then

- $C_{8t+6}(i-1) = \{\alpha^{|C_{8t}(i-1)|_\alpha}\, c_5\}$.
- $C_{8t+6}(i) = \{x^{|C_{8t}(i)|_x + |C_{8t}(i-1)|_x} \alpha^{|C_{8t}(i)|_\alpha + |C_{8t}(i-1)|_x}\}$ as $|C_{8t}(i-1)|_y = 0$, $C_{8t+6}(j) = C_{8t}(j), \quad (j \in \{1, \ldots, i-2, i+1, \ldots, L\})$.
- $C_{8t+6}(env) = \emptyset$.

After that,

- $C_{8t+6}(i-1) = \{\alpha^{|C_{8t}(i-1)|_\alpha}\, c_5\} \stackrel{R_8^{i-1}}{\rightarrow} C_{8t+7}(i-1) = \{\alpha^{|C_{8t}(i-1)|_\alpha}\, c_6\}$.

- $C_{8t+7}(i) = C_{8t+6}(i)$, $C_{8t+7}(j) = C_{8t}(j)$, $\quad (j \in \{1,\ldots,i-2,i+1,\ldots,L\})$.
- $C_{8t+7}(env) = \emptyset$.

Next, the rule R_{15}^{i-1} is applied. Then,

- $C_{8t+8}(i-1) = \{\alpha^{|C_{8t}(i-1)|_\alpha}\}$.
- $C_{8t+8}(i) = C_{8t+6}(i) = \{x^{|C_{8t}(i)|_x+|C_{8t}(i-1)|_x}\ \alpha^{|C_{8t}(i)|_\alpha+|C_{8t}(i-1)|_x}\}$.
- $C_{8t+8}(j) = C_{8t}(j)$ for all $j \in \{1,\ldots,i-1,i+1,\ldots,L\}$.
- $C_{8t+8}(env) = \{p\}$.

Note that $|C_{8t+8}(i)|_\alpha = |C_{8t}(i)|_\alpha + |C_{8t}(i-1)|_x$ and by induction hypothesis $|C_{8t}(i-1)|_x = |C_{8t}(i-1)|_\alpha - |C_{8t}(i)|_\alpha$. So, $|C_{8t+8}(i)|_\alpha = |C_{8t}(i-1)|_\alpha = |C_{8t+8}(i-1)|_\alpha$, and that there is no x or y in $C_{8t+8}(i-1)$.
Finally, bearing in mind that $|C_{8t+8}(i)|_\alpha = |C_{8t}(i-1)|_\alpha > |C_{8t}(i)|_\alpha$ and $|C_{8t}(i)|_\alpha \geq |C_{8t}(i+1)|_\alpha = |C_{8t+8}(i+1)|_\alpha$, we deduce

- $|C_{8t+8}(i)|_\alpha > |C_{8t+8}(i+1)|_\alpha$.
- $$\begin{array}{rcl} |C_{8t+8}(i)|_x & \stackrel{(1)}{=} & |C_{8t}(i)|_x + |C_{8t}(i-1)|_x \\ & \stackrel{(2)}{=} & (|C_{8t}(i)|_\alpha - |C_{8t}(i+1)|_\alpha) + |C_{8t}(i-1)|_x \\ & \stackrel{(3)}{=} & (|C_{8t}(i)|_\alpha + |C_{8t}(i-1)|_x) - |C_{8t+8}(i+1)|_\alpha \\ & \stackrel{(4)}{=} & |C_{8t+8}(i)|_\alpha - |C_{8t+8}(i+1)|_\alpha \end{array}$$.

The equality (1) holds by the explicit description obtained for $C_{8t+8}(i)$. The equality (2) holds by induction hypothesis. (3) holds because for all $j \in \{1,\ldots,L\}$, $i \neq j$, $|C_{8t+8}(j)|_\alpha = |C_{8t}(j)|_\alpha$, in particular for $j = i+1$ and the last equality holds by the explicit description obtained for $C_{8t+8}(i)$.
Finally, we note that $|C_{8t+8}(i)|_y = 0$, by the explicit description obtained for $C_{8t+8}(i)$.

□