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EQUIVALENT QUASI–NORMS ON GENERALIZED ORLICZ SPACES

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Abstract. In this paper we show that the equivalence among the classical quasi-norms of the generalized Orlicz spaces X^{Φ} — the Orlicz quasi-norm, the Luxemburg quasi-norm and the Amemiya quasi-norm — holds under some mild conditions on the underlying quasi-Banach function space X — mainly the weak Fatou property — improving previous results of [4] for which some lattice convexity requirements for the quasi-Banach function space X were needed.

1. Introduction

The construction of the classical Orlicz spaces $L^{\Phi}(\mu)$ lies in the structure of the space of Lebesgue integrable functions $L^1(\mu)$. Indeed, given an N-function Φ, the finiteness of a given integral modular provides a criterion for determining when a given function belongs to the corresponding Orlicz space. In this case, it is well-known that the so-called Orlicz and Luxemburg norms are equivalent.

The same ideas that allow the construction of the Orlicz spaces can be applied using as underlying space any other (quasi-) Banach function space X of measurable functions instead of $L^1(\mu)$. In this way, for a Young function (or N-function) Φ we can construct the generalized Luxemburg X_L^{Φ}, Amemiya X_A^{Φ} or Orlicz X_O^{Φ} space. As we will see later we will always have the equality $X_L^{\Phi} = X_A^{\Phi}$. The inclusion $X_L^{\Phi} \subseteq X_O^{\Phi}$ always holds. However it is worth mentioning that generalized Orlicz spaces defined *a la Orlicz* and *a la Luxemburg* are in general different (see [4, Example 4.1]). In this case it is natural to ask under what conditions these spaces coincide and, when this occurs, if the equivalences among the *different* quasi-norms for the generalized Orlicz, Luxemburg or Amemiya spaces remain true.

In recent years, some authors have devoted some attention to this question. Among them, and starting from the results that were published by Jain, Persson and Upreti in [6], some of the authors of the present paper, together with other mathematicians, have proved in [4, Theorem 5.5] for an N-function Φ and a quasi-Banach function space

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X over a finite measure μ with the σ-Fatou property and having a strictly monotone quasi-renorming the equality $X_L^\Phi = X_O^\Phi$ holds, and the equivalence of the corresponding Luxemburg and Orlicz quasi-norms.

On the other hand, also in [4] it is shown that every quasi-Banach function space that satisfies a minimal requirement of convexity (the so-called L-convexity used by Kalton in [7]) allows a quasi-renorming that is strictly monotone. This result entails obtaining the equality $X_L^\Phi = X_O^\Phi$ and the equivalence of the Luxemburg and Orlicz quasi-norms for a quasi-Banach function space X over a finite measure μ with the σ-Fatou property and an N-function Φ if X is lattice r-convex for some $0<r<\infty$ (see [4, Corollary 6.6]).

In this paper, continuing the work started in [4], we are interested in showing that these results can be strongly improved. In fact, we will show that we can remove the hypothesis on the lattice convexity of the space X, and we will also replace the σ-Fatou hypothesis by a weaker request, the so-called weak Fatou property, to obtain the equivalence of the Orlicz and Luxemburg quasi-norms on the generalized Orlicz space. For this purpose, we will use as a technical resource the Amemiya quasi-norm: we will prove that it is equivalent to both the Luxemburg and the Orlicz quasi-norms.

2. Preliminaries and notation

In this article we will consider function spaces over a measure space (Ω,Σ,μ), where Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a finite positive measure defined on Σ. We will denote by $L^0(\mu)$ the space of (μ-a.e. equivalence classes of) measurable functions $f:\Omega\longrightarrow\mathbb{R}$ equipped with the topology of convergence in measure.

A *quasi-normed function space* over μ is any order ideal $X\subseteq L^0(\mu)$ which is a quasi-normed lattice with respect to the μ-a.e. order, that is, if $f\in L^0(\mu)$, $g\in X$ and $|f|\leqslant|g|$ μ-a.e., then $f\in X$ and $\|f\|_X\leqslant\|g\|_X$, where $\|\cdot\|_X$ is the quasi-norm of X. We usually denote by $C\geqslant 1$ *a quasi-triangle constant* of X, that is, $\|f+g\|_X\leqslant C(\|f\|_X+\|g\|_X)$ for all $f,g\in X$. We will always assume that characteristic function of Ω, χ_Ω, belongs to X. Note that any quasi-normed function space over μ is continuously embedded into $L^0(\mu)$, as it is proved in [10, Proposition 2.2]. A complete quasi-normed function space is called a *quasi-Banach function space*. If, in addition, the quasi-norm happens to be a norm, then X is called a *Banach function space*.

We say that a quasi-normed function space X has the σ*-Fatou property* if for any positive increasing sequence $(f_n)_n$ in X with $\sup_{n\geqslant 1}\|f_n\|_X<\infty$, we have that $\sup_{n\geqslant 1} f_n\in X$ and $\left\|\sup_{n\geqslant 1} f_n\right\|_X = \sup_{n\geqslant 1}\|f_n\|_X$. We say that a quasi-normed function space X has the *weak Fatou property* if for any positive increasing sequence $(f_n)_n$ in X, with $\sup_{n\geqslant 1}\|f_n\|_X<\infty$, we have that $\sup_{n\geqslant 1} f_n\in X$. It is known that if a quasi-normed function space has the σ-Fatou property (weak Fatou property), then it is complete and hence a quasi-Banach function space (see [10, Proposition 2.35]).

Note that the only difference among the σ-Fatou and the weak Fatou properties is that the first one requires in addition that $\left\|\sup_{n\geqslant 1} f_n\right\|_X = \sup_{n\geqslant 1}\|f_n\|_X$. Thus, each quasi-normed function space X with the σ-Fatou property obviously satisfies also the weak Fatou property. However, in general *the σ-Fatou property is not quasi-renorming invariant, while the weak Fatou property clearly is.* This fact will be relevant for us, since we will use quasi-renormings in several steps of our work.

The following result is the adaptation (regarding the proof; the statement is exactly the same) for quasi-Banach function spaces of a well-known result of Amemiya for Banach function spaces. It states that although the equality $\left\|\sup_{n\geqslant 1} f_n\right\|_X = \sup_{n\geqslant 1}\|f_n\|_X$ is not guaranteed by the weak Fatou property, it implies a weaker inequality that properly relates the norm $\left\|\sup_{n\geqslant 1} f_n\right\|_X$ with $\sup_{n\geqslant 1}\|f_n\|_X$. The proof of next result is adapted from Theorem 2 in [12, Ch. 15 § 65], that refers to an original proof of Amemiya [2].

THEOREM 1. (Amemiya) *Let X be a quasi-Banach function space with the weak Fatou property. Then there is a constant $G\geqslant 1$ (only depending on $\|\cdot\|_X$) such that $\|f\|_X \leqslant G\sup_{n\geqslant 1}\|f_n\|_X$ whenever $0\leqslant f_n\uparrow f\in X$.*

Proof. Let $C\geqslant 1$ be a quasi-triangular constant for $\|\cdot\|_X$. Suppose by contradiction that there is no such a constant $G\geqslant 1$. Then for each natural number $p\geqslant 1$ there is a sequence $(f_{pn})_n\subseteq X$ and a function $f_p\in X$ such that $0\leqslant f_{pn}\uparrow f_p$ and

$$\left\|f_p\right\|_X > p^3C^p\sup_{n\geqslant 1}\left\|f_{pn}\right\|_X, \quad p=1,2,\dots \tag{1}$$

By multiplying by appropriate constants we can further assume that

$$\sup_{n\geqslant 1}\left\|f_{pn}\right\|_X = \frac{1}{p^2C^p}, \quad p=1,2,\dots \tag{2}$$

From (1) and (2) we have that $\left\|f_p\right\|_X\geqslant p$ for all $p=1,2,\dots$ Take now the functions $g_n := f_{1n}+f_{2n}+\cdots+f_{nn}$, $n=1,2,\dots$ Then $0\leqslant g_1\leqslant g_2\leqslant\cdots$, and for every $n\geqslant 1$ we have

$$\begin{aligned}\|g_n\|_X &= \left\|\sum_{p=1}^n f_{pn}\right\|_X \leqslant \sum_{p=1}^n C^p\left\|f_{pn}\right\|_X\\ &\leqslant \sum_{p=1}^n C^p\frac{1}{p^2C^p}\\ &\leqslant \sum_{p=1}^\infty \frac{1}{p^2}<\infty.\end{aligned}$$

Thus, $\sup_{n\geqslant 1}\|g_n\|_X<\infty$ and then $g:=\sup_{n\geqslant 1} g_n\in X$, as a consequence of the weak Fatou property of X. But this gives a contradiction, since for each natural number $p\geqslant 1$, if we take $n \geqslant p$ we get

$$g \geqslant g_n = f_{1n} + f_{2n} + \cdots + f_{nn} \geqslant f_{pn}, \quad n \geqslant p$$

and so $g \geqslant f_p$ for all $p = 1, 2, \ldots$ We conclude that $\|g\|_X \geqslant \|f_p\|_X \geqslant p$ for all $p = 1, 2, \ldots,$ what gives a contradiction. $\square$

Finally, recall that a quasi-normed function space X is said to be σ*-order continuous* if for any positive increasing sequence $(f_n)_n$ in X converging μ-a.e. to a function $f \in X$, we have that $\|f - f_n\|_X \to 0$.

3. Young functions and N-functions

We collect here some results, all known, on Young's functions that we will need next.

A *Young function* is any strictly increasing convex function (and so continuous) $\Phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. From the convexity of Φ we have the following useful inequality

$$\Phi(\alpha x) \leqslant \alpha \Phi(x) \quad \text{if} \quad 0 \leqslant \alpha \leqslant 1, \quad x \geqslant 0. \tag{3}$$

A Young function Φ is called an *N-function* if Φ satisfies the two limit conditions $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$. The *complementary function* of the Young function Φ is defined as $\hat{\Phi}(y) := \sup_{x\geqslant 0}\{xy - \Phi(x)\}$, for all $y \geqslant 0$. From the definition of $\hat{\Phi}$ it is clear that Φ and $\hat{\Phi}$ satisfy the *Young inequality*

$$xy \leqslant \Phi(x) + \hat{\Phi}(y), \quad x, y \geqslant 0. \tag{4}$$

Every Young function Φ has a *right derivative*, that is, a non-decreasing, right continuous function $\varphi : [0,\infty) \longrightarrow [0,\infty)$, with $\varphi(0) = 0$, such that $\Phi(x) = \int_0^x \varphi(t)dt$ for all $x \in [0,\infty)$ (see [9, Theorem 1.1] or [11, Theorem 1.3.1]). This function φ also satisfies the following equality (see [9, (2.7)] or [11, Theorem 1.3.3]) that we will use later

$$x\varphi(x) = \Phi(x) + \hat{\Phi}(\varphi(x)), \quad x \geqslant 0. \tag{5}$$

A Young function Φ has the Δ_2*-property*, written $\Phi \in \Delta_2$, if there exists a constant $K > 1$ such that $\Phi(2x) \leqslant K\Phi(x)$ for all $x \geqslant 0$.

Next lemma is well-known and will be used later. The proof can be seen in [9, Ch. I § 1 p. 9]

LEMMA 1. *Let Φ be an N-function and take its right derivative φ. Then we have that $\lim_{x\to\infty} \varphi(x) = \infty$ and $\lim_{x\to 0} \varphi(x) = 0$.*

4. Quasi-norms on generalized Orlicz spaces

In this section we introduce the Luxemburg, Amemiya and Orlicz quasi-Banach function spaces whose relations, mainly the equivalence of their quasi-norms, will be the aim of our work. The reader can find information about in [3] and [4], in which there is an analysis of the relations among two of them, and in [6], where the same topic is studied for the Banach space case under the assumption of the σ-Fatou property.

Let Φ be a Young function and let X be a quasi-normed function space over a finite measure μ.

The (generalized) *Luxemburg space* X_L^Φ is defined as the following set:

$$X_L^\Phi := \left\{ f \in L^0(\mu) : \exists c > 0 : \Phi\left(\frac{|f|}{c}\right) \in X \right\}.$$

Given $f \in X_L^\Phi$, we define the Luxemburg lattice quasi-norm of f by

$$\|f\|_{X_L^\Phi} := \inf\left\{ c > 0 : \Phi\left(\frac{|f|}{c}\right) \in X, \text{ with } \left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X \leqslant 1 \right\}. \tag{6}$$

The (generalized) *Amemiya space* X_A^Φ is defined as the following set:

$$X_A^\Phi := \left\{ f \in L^0(\mu) : \exists k > 0 : \Phi(k|f|) \in X \right\}.$$

Given $f \in X_A^\Phi$, we define the Amemiya lattice quasi-norm of f by

$$\|f\|_{X_A^\Phi} := \inf\left\{ \frac{1}{k}\left(1 + \|\Phi(k|f|)\|_X\right), \, k > 0 \right\}. \tag{7}$$

The Luxemburg and Amemiya spaces defined above, equipped with their corresponding quasi-norms, are really quasi-normed function spaces over the finite measure μ with the same quasi-triangle constant as the one of the quasi-norm of the space X.

PROPOSITION 1. *Let $(X, \|\cdot\|_X)$ be a quasi-normed function space over the measure μ with the weak Fatou property. Let Φ be Young function. Then the Amemiya space X_A^Φ has also the weak Fatou property.*

This is a consequence of the equality $X_A^\Phi = X_L^\Phi$, the equivalence of the quasi-norms $\|\cdot\|_{X_A^\Phi}$ and $\|\cdot\|_{X_L^\Phi}$ which will be shown in Theorem 2, and of the fact that the Luxemburg space X_L^Φ has the weak Fatou property (see [3, Theorem 4]).

Now, let Φ be an N-function. The corresponding (generalized) *Orlicz space* X_O^Φ is defined as the following set:

$$X_O^\Phi := \left\{ f \in L^0(\mu) : \|f\|_{X_O^\Phi} < \infty \right\},$$

where $\|\cdot\|_{X_O^\Phi}$ is the *Orlicz quasi-norm* defined by

$$\|f\|_{X_O^\Phi} := \sup\left\{ \|fg\|_X : \hat{\Phi}(|g|) \in X, \left\|\hat{\Phi}(|g|)\right\|_X \leqslant 1 \right\}. \tag{8}$$

As we have pointed out above $X_A^\Phi = X_L^\Phi$ as sets. Nevertheless, the spaces X_L^Φ and X_O^Φ are in general different (see [4, Example 4.1]), but we proved in [4, Proposition 3.3]) that the inclusion $X_L^\Phi \subseteq X_O^\Phi$ holds and

$$\|f\|_{X_O^\Phi} \leqslant 2C\,\|f\|_{X_L^\Phi}, \quad f \in X_L^\Phi, \tag{9}$$

where $C \geqslant 1$ is a quasi-triangular constant for X.

5. Equivalence of the Amemiya and Luxemburg quasi-norms

Let us now check the inequalities that relate the Amemiya and Luxemburg quasi-norms. The next result provides the main tool in order to do it.

PROPOSITION 2. *Let X be a quasi-normed function space over a measure μ, and let Φ be a Young function. Given a function $f \in X_L^\Phi$, we have that*

$$\begin{aligned}\|f\|_{X_L^\Phi} &= \inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k>0\right\} \\ &= \inf\left\{\frac{1}{k}\max\{1, \|\Phi(k|f|)\|_X\}, k>0\right\}.\end{aligned} \tag{10}$$

Proof. Let us show first the inequality

$$\|f\|_{X_L^\Phi} \leqslant \inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k>0\right\}.$$

Let $k>0$. If $\|\Phi(k|f|)\|_X \leqslant 1$, then (by the definition of $\|f\|_{X_L^\Phi}$) we can conclude that $\|f\|_{X_L^\Phi} \leqslant \dfrac{1}{k}$. On the other hand, if we have $\|\Phi(k|f|)\|_X > 1$, using (3) we get

$$\left\|\Phi\left(\frac{|f|}{\frac{1}{k}\|\Phi(k|f|)\|_X}\right)\right\|_X = \left\|\Phi\left(\frac{k|f|}{\|\Phi(k|f|)\|_X}\right)\right\|_X \leqslant \frac{\|\Phi(k|f|)\|_X}{\|\Phi(k|f|)\|_X} = 1.$$

In this second case we have that $\|f\|_{X_L^\Phi} \leqslant \dfrac{1}{k}\|\Phi(k|f|)\|_X$. And in any case, we have that

$$\|f\|_{X_L^\Phi} \leqslant \max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}$$

for all $k>0$. Computing the infimum with respect to $k>0$, we conclude that

$$\|f\|_{X_L^\Phi} \leqslant \inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k>0\right\}.$$

Let us show now the converse inequality, that is,

$$\inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k>0\right\} \leqslant \|f\|_{X_L^\Phi}.$$

Let $c > 0$ such that $\left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X \leqslant 1$. Then

$$\inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k > 0\right\} \leqslant \max\left\{c, c\left\|\Phi\left(\frac{|f|}{c}\right)\right\|_X\right\} \leqslant c.$$

Computing the infimum with respect to $c > 0$, we get that

$$\inf\left\{\max\left\{\frac{1}{k}, \frac{1}{k}\|\Phi(k|f|)\|_X\right\}, k > 0\right\} \leqslant \|f\|_{X_L^\Phi}. \quad \square$$

The next result establishes the equivalence of the Amemiya and Luxemburg quasi-norms.

THEOREM 2. *Let X be a quasi-normed function space over a measure μ and Φ a Young function. Then the quasi-norms $\|\cdot\|_{X_A^\Phi}$ and $\|\cdot\|_{X_L^\Phi}$ are equivalent in $X_A^\Phi = X_L^\Phi$. In particular, we have that*

$$\|f\|_{X_L^\Phi} \leqslant \|f\|_{X_A^\Phi} \leqslant 2\|f\|_{X_L^\Phi}, \tag{11}$$

for all $f \in X_A^\Phi = X_L^\Phi$.

Proof. It is enough to consider Proposition 2 and the inequality

$$\max\{1, x\} \leqslant 1 + x \leqslant 2\max\{1, x\},$$

for all $x \geqslant 0$. So we get

$$\frac{1}{k}\max\{1, \|\Phi(k|f|)\|_X\} \leqslant \frac{1}{k}(1 + \|\Phi(k|f|)\|_X) \leqslant \frac{2}{k}\max\{1, \|\Phi(k|f|)\|_X\}$$

and then we obtain the equivalence among the quasi-norms just by computing the infimum on $k > 0$. $\square$

6. Stability of generalized Orlicz spaces by equivalence of Young functions and quasi-renorming of X

Later we will need to renormalize the underlying space X and change the function Φ for another one with better properties. In this section we will check that these changes will not affect the corresponding generalized Orlicz space. Something similar will be shown for the Luxemburg space and the Amemiya space. Most of these results are probably known, and, in many cases, the proofs are straightforward.

DEFINITION 1. Let Φ and Ψ Young functions. We say that Ψ is stronger than Φ if there is a constant $a > 0$ such that $\Phi(x) \leqslant \Psi(ax)$, for all $x \geqslant 0$. In this case we write $\Phi \prec \Psi$. We say that Φ and Ψ are equivalent if $\Phi \prec \Psi$ and $\Psi \prec \Phi$. In this case we write $\Phi \equiv \Psi$.

REMARK 1. It is proved in [11, Theorem 2, p. 16] that if Φ and Ψ are N-functions such that $\Phi \prec \Psi$, then $\hat{\Psi} \prec \hat{\Phi}$. In particular, if $\Phi \equiv \Psi$, then $\hat{\Phi} \equiv \hat{\Psi}$.

PROPOSITION 3. *Let Φ and Ψ be Young functions such that $\Phi \prec \Psi$ with constant $a > 0$. Then $X_L^\Psi \subseteq X_L^\Phi$, and we also have that $\|f\|_{X_L^\Phi} \leqslant a\|f\|_{X_L^\Psi}$ for all $f \in X_L^\Psi$. In particular, if $\Phi \equiv \Psi$ then $X_L^\Phi = X_L^\Psi$ and the quasi-norms $\|\cdot\|_{X_L^\Phi}$ and $\|\cdot\|_{X_L^\Psi}$ are equivalent.*

We have similar results for the Amemiya and Orlicz quasi-norms.

PROPOSITION 4. *Let Φ and Ψ be Young functions such that $\Phi \prec \Psi$ with constant $a > 0$. Then $X_A^\Psi \subseteq X_A^\Phi$ and $\|f\|_{X_A^\Phi} \leqslant a\|f\|_{X_A^\Psi}$ for all $f \in X_A^\Psi$, and if $\Phi \equiv \Psi$ then $X_A^\Phi = X_A^\Psi$ and the quasi-norms $\|\cdot\|_{X_A^\Phi}$ and $\|\cdot\|_{X_A^\Psi}$ are equivalent.*

PROPOSITION 5. *Let Φ and Ψ be N-functions such that $\Phi \prec \Psi$. Let $b > 0$ a constant associated to $\hat{\Psi} \prec \hat{\Phi}$. Then $\|f\|_{X_O^\Phi} \leqslant b\|f\|_{X_O^\Psi}$ for all $f \in X_O^\Psi$. Thus $X_O^\Psi \subseteq X_O^\Phi$. In particular, if $\Phi \equiv \Psi$ then $X_O^\Phi = X_O^\Psi$ and the quasi-norms $\|\cdot\|_{X_O^\Phi}$ and $\|\cdot\|_{X_O^\Psi}$ are equivalent.*

To conclude the first part of this section we present the following result that will allow us to assume that the Young function (N-function) Φ used for the construction of the spaces X_O^Φ, X_A^Φ and X_L^Φ can be chosen with a continuous derivative.

PROPOSITION 6. *Let Φ be a Young function (an N-function). Then there is a Young function (an N-function) Ψ with continuous derivative such that $\Phi \equiv \Psi$.*

Proof. Given the Young function (N-function) Φ, let us consider its right derivative $\varphi : [0,\infty) \longrightarrow [0,\infty)$. We know that it is non-decreasing, right continuous and $\Phi(x) = \int_0^x \varphi(t)dt$ for all $x \in [0,\infty)$. Consider now the function $\Psi : [0,\infty) \longrightarrow [0,\infty)$, given by

$$\Psi(x) := \int_0^x \frac{\Phi(u)}{u} du = \int_0^x \left[\frac{1}{u}\int_0^u \varphi(t)dt\right] du = \int_0^x \varphi(t)\log\frac{x}{t}dt. \tag{12}$$

Let us see that it satisfies the requirements of the statement of the proposition.

i) There is $a > 0$ such that $\Psi(x) \leqslant \Phi(x) \leqslant \Psi(ax)$, for all $x \geqslant 0$. Indeed, let us prove that the above inequalities are satisfied for $a = 2$. Since the function $\dfrac{\Phi(u)}{u}$ is increasing (due to the fact that Φ is convex), we have that

$$\Psi(x) = \int_0^x \frac{\Phi(u)}{u} du \leqslant \frac{\Phi(x)}{x} x = \Phi(x)$$

for all $x \geqslant 0$. On the other hand,

$$\Psi(2x) = \int_0^{2x} \frac{\Phi(u)}{u} du \geqslant \int_x^{2x} \frac{\Phi(u)}{u} du \geqslant \frac{\Phi(x)}{x} x = \Phi(x)$$

for all $x \geqslant 0$. In fact, it can be proved that $\Phi(x) \leqslant \Psi(ax)$ for all $x \geqslant 0$ and each $a \geqslant 1$.

ii) We claim that Ψ has continuous derivative. Indeed, note that the function $x \to \frac{1}{x}\int_0^x \varphi(u)du$ is continuous, and by the Fundamental Theorem of Calculus, we have that $\Psi'(x) = \frac{1}{x}\int_0^x \varphi(u)du$ for all $x \geqslant 0$.

iii) It is also clear that Ψ is increasing, since $\Psi'(x) \geqslant 0$ for all $x \geqslant 0$.

iv) We also have that Ψ is convex. Indeed, at the points where φ is continuous (except countable many points), the function Ψ' has a derivative and it is given by $\Psi''(x) = \frac{1}{x}\varphi(x) - \frac{1}{x^2}\int_0^x \varphi(u)du$, for all $x \geqslant 0$. Now it is clear that $\Psi''(x) \geqslant 0$ if and only if $x\varphi(x) \geqslant \int_0^x \varphi(u)du$ and the last inequality holds since φ is non-decreasing. In fact, any function Ψ given by formula (12) is convex.

Now, assume that Φ is an N-function. Then we have as a direct consequence of i) that $\lim_{x\to 0}\frac{\Psi(x)}{x} = 0$ and $\lim_{x\to\infty}\frac{\Psi(x)}{x} = \infty$. $\square$

Finally, we will show that, if we *renorm* the space X we get equivalent (quasi-) norms for the associated generalized Orlicz spaces. For the proof, see [4, Proposition 5.10].

PROPOSITION 7. *Let Φ be a Young function and let X an ideal of $L^0(\mu)$. Consider two equivalent lattice quasi-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X, and denote by $X_1 := (X, \|\cdot\|_1)$ and $X_2 := (X, \|\cdot\|_2)$ the corresponding quasi-normed function spaces. Then*

1) the Luxemburg quasi-norms $\|\cdot\|_{X_{1L}^\Phi}$ and $\|\cdot\|_{X_{2L}^\Phi}$ are also equivalent, and

2) the Amemiya quasi-norms $\|\cdot\|_{X_{1A}^\Phi}$ and $\|\cdot\|_{X_{2A}^\Phi}$ are equivalent too.

If moreover, Φ is an N-function, then

3) the Orlicz quasi-norms $\|\cdot\|_{X_{1O}^\Phi}$ and $\|\cdot\|_{X_{2O}^\Phi}$ are equivalent.

Recall that a quasi-norm $\|\cdot\|$ on X is not necessarily a continuous function $\|\cdot\| : x \in X \longrightarrow \|x\| \in [0, +\infty)$; which of course it is if $\|\cdot\|$ is a norm.

DEFINITION 2. Let $0 < p \leqslant 1$. A p-norm on a linear space X is a function $|||\cdot||| : X \longrightarrow [0, +\infty)$ satisfying

a) $|||x||| = 0$ if and only if $x = 0$.

b) $|||\alpha x||| = |\alpha| |||x|||$, $\alpha \in \mathbb{R}$, $x \in X$.

c) $|||x+y|||^p \leqslant |||x|||^p + |||y|||^p$, $x, y \in X$.

A p-norm on X is a quasi-norm with quasi-triangular constant $C = 2^{\frac{1}{p}-1}$, which is always continuous in X. Let us recall that the Aoki-Rolewicz Theorem guarantees that we can always find an equivalent continuous quasi-norm for the quasi-normed space X.

THEOREM 3. (Aoki-Rolewicz) *Let $(X, \|\cdot\|_X)$ be a quasi-normed space with quasi-triangular constant $C \geqslant 1$. Then there are $0 < p \leqslant 1$ and a p-norm $|||\cdot|||$ on X such that $|||x||| \leqslant \|x\|_X \leqslant 2C\,|||x|||$ for all $x \in X$, with $C = 2^{\frac{1}{p}-1}$.*

Proof. It can be found in [8, p. 7]. In fact, we can use the following definition as associated p-norm

$$|||x||| = \inf\left\{ \left[\sum_{k=1}^{n} \|x_k\|_X^p \right]^{\frac{1}{p}} : x = \sum_{k=1}^{n} x_k,\ n \geqslant 1 \right\}. \tag{13}$$

In light of (13) and the Riesz Decomposition Lemma (see [1, p. 11]), it is easy to see that if the quasi-norm $\|\cdot\|_X$ is a lattice quasi-norm, then the p-norm $|||\cdot|||$ is also a lattice p-norm. $\square$

7. Equivalences of the Amemiya, Luxemburg and Orlicz quasi-norms

In this last and main section we will establish the equality of the spaces X_A^Φ, X_L^Φ and X_O^Φ (being Φ an N-function) and the corresponding equivalence of their quasi-norms when the space X has the weak Fatou property, without any additional assumptions. In the absence of the weak Fatou property for the space X, we will prove the equivalence of the three quasi-norms in the smallest space $X_A^\Phi = X_L^\Phi$ whenever X has σ-order continuous quasi-norm. The proofs of all these results are consequence of the following proposition which is inspired by similar results given in [9] and [5].

PROPOSITION 8. *Let Φ be an N-function. For every simple function f we have that*

$$\|f\|_{X_A^\Phi} \leqslant 2\|f\|_{X_O^\Phi}. \tag{14}$$

Proof. As we have seen above, we can assume for the proof that the quasi-norm $\|\cdot\|_X$ is continuous in X and the N-function Φ has a continuous derivative φ.

Let us prove inequality (14). Consider a non-null simple function $0 \leqslant f$. Write $f = \sum_{n=1}^{N} \alpha_n \chi_{A_n}$, with $\alpha_n > 0$, where $A_n \in \Sigma$ are pairwise disjoint, $n = 1, 2, \ldots, N$. Note that $\hat{\Phi}(\varphi(kf)) \in X$, for all $k > 0$, since it is also a simple function. In fact, it is easy to see that $\hat{\Phi}(\varphi(kf)) = \sum_{n=1}^{N} \hat{\Phi}(\varphi(k\alpha_n))\chi_{A_n}$. On the other hand we have that $\beta\chi_B \leqslant f \leqslant \alpha\chi_A$, where $\alpha := \max\{\alpha_1, \ldots, \alpha_n\}$, $\beta := \min\{\alpha_1, \ldots, \alpha_n\} = \alpha_{n_0}$, $A := A_1 \cup \cdots \cup A_N$ and $B := A_{n_0}$. Since f is non-null, $\|\chi_B\|_X > 0$ and also $\|\chi_A\|_X > 0$. Therefore we have

$$\hat{\Phi}(\varphi(k\beta))\chi_B = \hat{\Phi}(\varphi(k\beta\chi_B)) \leqslant \hat{\Phi}(\varphi(kf)) \leqslant \hat{\Phi}(\varphi(k\alpha\chi_A)) = \hat{\Phi}(\varphi(k\alpha))\chi_A$$

for all $k > 0$. Computing the quasi-norm we get

$$\begin{aligned}\hat{\Phi}(\varphi(k\beta))\,\|\chi_B\|_X &= \left\|\hat{\Phi}(\varphi(k\beta\chi_B))\right\|_X \leqslant \left\|\hat{\Phi}(\varphi(kf))\right\|_X \leqslant \left\|\hat{\Phi}(\varphi(k\alpha\chi_A))\right\|_X \\ &= \hat{\Phi}(\varphi(k\alpha))\,\|\chi_A\|_X\end{aligned}$$

for all $k>0$. Since $\hat{\Phi}$ is an N-function, Lemma 1 gives that $\lim_{x\to\infty}\hat{\Phi}(\varphi(x))=\infty$, and $\lim_{x\to 0}\hat{\Phi}(\varphi(x))=0$. Thus, we also have $\lim_{k\to 0}\left\|\hat{\Phi}(\varphi(kf))\right\|_X=0$ and $\lim_{k\to\infty}\left\|\hat{\Phi}(\varphi(kf))\right\|_X=\infty$. The function

$$H:k\in[0,\infty)\longrightarrow H(k):=\left\|\hat{\Phi}(\varphi(kf))\right\|_X\in[0,\infty)$$

is continuous since the quasi-norm $\|\cdot\|_X:X\longrightarrow[0,\infty)$ is continuous and the function $k\in[0,\infty)\to\hat{\Phi}(\varphi(kf))=\sum_{n=1}^{N}\hat{\Phi}(\varphi(k\alpha_n))\chi_{A_n}\in X$ is continuous (addition of continuous functions), since $\hat{\Phi}$ and φ are continuous. Now, as a consequence of the *Darboux property* there exists $k_0>0$ such that

$$H(k_0)=\left\|\hat{\Phi}(\varphi(k_0f))\right\|_X=1. \tag{15}$$

Taking into account $\hat{\Phi}(\varphi(k_0f))\geqslant 0$ and $\Phi(k_0f)\geqslant 0$ in X, and (5), we obtain

$$\begin{aligned}\|f\|_{X_A^\Phi}&\leqslant\frac{1}{k_0}\left(1+\|\Phi(k_0f)\|_X\right)=\frac{1}{k_0}\left(\left\|\hat{\Phi}(\varphi(k_0f))\right\|_X+\|\Phi(k_0f)\|_X\right)\\&\leqslant\frac{2}{k_0}\left(\left\|\hat{\Phi}(\varphi(k_0f))+\Phi(k_0f)\right\|_X\right)=\frac{2}{k_0}\|k_0f\varphi(k_0f))\|_X\\&=2\|f\varphi(k_0f))\|_X\leqslant 2\|f\|_{X_O^\Phi}.\end{aligned}$$

The last inequality is due to the definition of quasi-norm of Orlicz and (15). This concludes the proof. □

THEOREM 4. *Let $(X,\|\cdot\|_X)$ be a quasi-Banach function space over a measure μ with the weak Fatou property, and let Φ be an N-function. Then $X_A^\Phi=X_O^\Phi$ and the Amemiya and Orlicz quasi-norms are equivalent.*

Proof. We can assume for the proof that the quasi-norm $\|\cdot\|_X$ is continuous in X and the N-function Φ has a continuous derivative φ.

Let us show first that $\|f\|_{X_O^\Phi}\leqslant C\|f\|_{X_A^\Phi}$ for all $f\in X_A^\Phi$ for a triangular constant $C\geqslant 1$ for the space X. Indeed, given $g\in L^0(\mu)$, with $\hat{\Phi}(g)\in X$ and $\left\|\hat{\Phi}(|g|)\right\|_X\leqslant 1$, and any $k>0$, by Young inequality (4) we get the following inequality

$$\|fg\|_X=\frac{1}{k}\|kfg\|_X\leqslant\frac{1}{k}\left\|\Phi(k|f|)+\hat{\Phi}(|g|)\right\|_X\leqslant\frac{C}{k}\left(\|\Phi(k|f|)\|_X+1\right).$$

Computing the supremum for g we get $\|f\|_{X_O^\Phi}\leqslant\frac{C}{k}\left(\|\Phi(k|f|)\|_X+1\right)$, for all $k>0$. Taking now the infimum for $k>0$ we can conclude that $\|f\|_{X_O^\Phi}\leqslant C\|f\|_{X_A^\Phi}$, for all $f\in X_A^\Phi$.

Let us now look at the opposite inequality. Since X has the weak Fatou property, we know that the Amemiya space X_A^Φ also has it. Therefore by Amemiya's theorem (Theorem 1), there is a constant $G>0$ such that $\|g\|_{X_A^\Phi}\leqslant G\sup_{n\geqslant 1}\|g_n\|_{X_A^\Phi}$ for every sequence $(g_n)_n \subseteq X_A^\Phi$ and all $g \in X_A^\Phi$ such that $0 \leqslant g_n \uparrow g \in X_A^\Phi$. In particular, we are going to prove that

$$\|f\|_{X_A^\Phi} \leqslant 2G\|f\|_{X_O^\Phi}, \tag{16}$$

for all $f \in X_O^\Phi$, and consequently the inclusion $X_O^\Phi \subseteq X_A^\Phi$. Indeed, recalling the inequality (14), if $f \in X_O^\Phi$, there is a sequence $(f_n)_n$ of positive simple functions such that $0 \leqslant f_n \uparrow |f|$ pointwise μ-a.e. (since f is a measurable function). In particular, $\sup\limits_{n\geqslant 1}\|f_n\|_{X_O^\Phi} \leqslant \|f\|_{X_O^\Phi}$. From (14) we get that

$$\|f_n\|_{X_A^\Phi} \leqslant 2\|f_n\|_{X_O^\Phi} \leqslant 2\|f\|_{X_O^\Phi}$$

for all $n = 1,2,\ldots$ By the weak Fatou property of X_A^Φ we get that $f \in X_A^\Phi$. Moreover, $\|f\|_{X_A^\Phi} \leqslant G\sup\limits_{n\geqslant 1}\|f_n\|_{X_A^\Phi} \leqslant 2G\sup\limits_{n\geqslant 1}\|f_n\|_{X_O^\Phi} \leqslant 2G\|f\|_{X_O^\Phi}$, for all $f \in X_O^\Phi$, which is exactly (16). $\square$

COROLLARY 1. *Let $(X,\|\cdot\|_X)$ be a quasi-Banach function space over the measure μ, with the weak Fatou property, and let Φ be an N-function. Then $X_L^\Phi = X_A^\Phi = X_O^\Phi$ and the Amemiya, Luxemburg and Orlicz quasi-norms are equivalent.*

Another consequence of the Proposition 8 is the following result.

THEOREM 5. *Let $(X,\|\cdot\|_X)$ be a quasi-normed function space over a measure μ that is σ-order continuous, and let Φ be an N-function with the Δ_2 property. Then the Luxemburg and Orlicz quasi-norms are equivalent in X_L^Φ.*

Proof. We know by (9) that $\|f\|_{X_O^\Phi} \leqslant 2C\|f\|_{X_L^\Phi}$ for every $f \in X_L^\Phi$, where $C \geqslant 1$ is a quasi-triangular constant for X. Suppose now that X is σ-order continuous and $\Phi \in \Delta_2$. Then the quasi-norm of X_L^Φ is also σ-order continuous (see [3, Theorem 5]).

Let $f \in X_L^\Phi$ and take $\varepsilon > 0$. Since f is measurable we get a simple function g such that $0 \leqslant g \leqslant |f|$ and $\||f|-g\|_{X_L^\Phi} < \varepsilon$. Then, taking into account (14), we have

$$\begin{aligned}\|f\|_{X_L^\Phi} = \||f|-g+g\|_{X_L^\Phi} &\leqslant C\left(\||f|-g\|_{X_L^\Phi} + \|g\|_{X_L^\Phi}\right) \leqslant \varepsilon C + C\|g\|_{X_L^\Phi} \\ &\leqslant \varepsilon C + C\|g\|_{X_A^\Phi} \leqslant \varepsilon C + 2C\|g\|_{X_O^\Phi} \leqslant \varepsilon C + 2C\|f\|_{X_O^\Phi}.\end{aligned}$$

Since ε is arbitrary, we obtain the result. $\square$

COROLLARY 2. *Let $(X,\|\cdot\|_X)$ be a quasi-normed function space over the measure μ that is σ-order continuous, and let Φ an N-function with the Δ_2 property. Then the Amemiya, Luxemburg and Orlicz quasi-norms are equivalent in X_L^Φ.*

The hypothesis of X being σ-order continuous in Corollary 2 is clearly not necessary for the equivalence of the quasi-norms. It is well-known that, if $\Phi \in \Delta_2$ then the Luxemburg, Amemiya and Orlicz norms are equivalent in X_L^Φ (see [4, Remark 6.8]).

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