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Homomorphisms between graphs embedded in surfaces

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ABSTRACT

We extend the notion of graph homomorphism to cellularly embedded graphs (maps) by designing operations on vertices and edges that respect the surface topology; we thus obtain the first definition of map homomorphism that preserves both the combinatorial structure (as a graph homomorphism) and the topological structure of the surface (in particular, orientability and genus). Notions such as the core of a graph and the homomorphism order on cores are then extended to maps. We also develop a purely combinatorial framework for various topological features of a map such as the contractibility of closed walks, which in particular allows us to characterize map cores. We then show that the poset of map cores ordered by the existence of a homomorphism is connected and, in contrast to graph homomorphisms, does not contain any dense interval (so it is not universal for countable posets). Finally, we give examples of a pair of cores with an infinite number of cores between them, an infinite chain of gaps, and arbitrarily large antichains with a common homomorphic image.

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1. Introduction

Homomorphisms between sets with added structure are mappings that preserve this structure. For example, homomorphisms between graphs are mappings between their vertex sets that preserve adjacency. For multigraphs (loops and parallel edges allowed), a homomorphism is defined

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as a pair of mappings, one on vertices and the other on edges, which together preserve vertex-edge incidences. For a sense of the richness of the theory of graph homomorphisms, the reader is referred to [11].

Key to the structural understanding of graph homomorphisms is the notion of a core [10], and in particular the poset of cores, known as the homomorphism order. The enumeration of graph homomorphisms connects their theory to applications in statistical physics: the partition function of the q -state Potts model on a graph, and more generally partition functions of vertex-colouring models, are instances of homomorphism functions [6] (given by a weighted enumeration of graph homomorphisms from the given graph to a fixed edge-weighted graph). These partition functions are in turn intimately related to the Tutte polynomial of a graph, whose evaluations include enumerations of colourings and flows. In fact, any evaluation of the Tutte polynomial of a graph expressible as a homomorphism function is the partition function of a Potts model on the graph [7]. For graphs cellularly embedded in a surface (*maps*), the analogous notions of colourings (or tensions, rather) and flows (taking non-identity values in a finite group) are counted by evaluations of the surface Tutte polynomial [8,9], and have been expressed as partition functions of edge-colouring models [15]. This leaves the question of whether these enumerations can be expressed in terms of map homomorphisms as partition functions of vertex-colouring models on the map. To even begin to answer this question we need to formulate a definition of map homomorphism that extends that of graph homomorphism while respecting the topology of the graph embedding. We will then be able to adapt the fruitful analysis of graph homomorphisms via the poset of cores to the case of maps.

As we shall relate shortly, how to define homomorphisms between maps, however, is not so clear, one difficulty consisting in the preservation of both the combinatorial structure of the graph and the topological structure of the embedding.

Maps have several representations, some emphasizing their topological structure (cellular embeddings of graphs, ribbon graphs), some emphasizing their combinatorial structure (vertex-edge-face flags, rotation systems, graph-encoded maps, Tutte's permutation axiomatization). What counts as a map homomorphism may thus depend on which representation is chosen: what structure is to be preserved exactly? To start on firm ground, isomorphism of maps has just one candidate for its definition no matter what representation is chosen.

We use a version of Tutte's permutation axiomatization of maps [19, Chapter X] in which three involutions defined on a set of "crosses" (obtained by quartering edges of the map) describe how to get from one vertex, edge or face to an adjacent one whilst respecting the topology of the map (a formal definition is given in Definition 1 below). An isomorphism between maps in this representation is a bijection between their cross sets that commutes with the involutions defining the maps. By dropping the bijective condition, we obtain the definition of map homomorphism given both by Malnič, Nedela, and Škovič [16] (restricted to the case of orientable surfaces) and by Litjens and Sevenster [15] (restricted to locally bijective mappings, the context being universal covers of graphs). Map homomorphisms defined in this way preserve local combinatorial and topological structure (such as vertex-edge-face incidences and vertex rotations) but, in general, do not preserve global topological parameters such as orientability or genus.

We formulate a new definition of map homomorphism (Definition 22) that ensures that homomorphisms preserve the surface topology; our definition is mainly based on a vertex identification operation that we call *vertex gluing*. Vertex gluing, defined in terms of cross permutations (Definition 8), has the following interpretation in the representation of a map as a cellular embedding of a graph in a surface: two vertices can be identified under this operation if they can be moved continuously towards each other until they coincide without crossing any edge, or if they lie in different connected components; vertex gluing can be realized as the insertion of an edge (joining the vertices to be identified in a way that preserves genus and orientability) followed by its contraction. Any other rule of identifying vertices defined for maps generally is either a special case of vertex gluing or fails to preserve genus and orientability in some instances. We then define a map homomorphism as a sequence of such vertex gluings followed by a sequence of *duplicate edge gluings* (the analogue of suppressing parallel edges in graphs). Map homomorphisms in our sense preserve not only orientability and genus but also other key topological features (such as

the contractibility of facial walks) and the combinatorial structure of the map (when forgetting the embedding, a homomorphism between maps gives a homomorphism between their underlying graphs).

Using our definition of map homomorphism, we define a *core* of a map (Definition 26) as a minimal submap that is a homomorphic image of the whole map, analogously to how the core of a graph is defined. We establish several properties of map cores shared with graph cores, and characterize map cores in terms of a certain type of contractible closed walks, which, roughly speaking, separate off a disc from the remainder of the map's surface. This characterization exploits the “locality” of map homomorphisms, in the sense that any map homomorphism can be decomposed into a sequence of map homomorphisms that fix all but a plane submap contained within a contractible curve on the surface in which the map is embedded. After giving applications of the characterization, we show that, unlike graph homomorphisms, the map homomorphism order has no dense intervals, and is thus not universal. Finally, we produce examples of maps with an infinite number of cores between them, an example of an infinite chain of gaps, and arbitrarily large antichains of cores sharing a common homomorphic image.

We would also like to highlight that another contribution of our work is a purely combinatorial formulation for various topological features of a map in its representation as a cellular embedding of a graph in a closed surface, such as the contractibility of closed walks. (A more general notion of contractibility is needed than that given by Mohar and Thomassen [17], which does not include closed walks that revisit edges within its scope, while Cabello and Mohar [2] give a topological definition of contractibility not straightforward to translate into combinatorial terms.)

The paper is organized as follows. In Section 2 we use a permutation axiomatization for maps to develop the formalism required for later constructions and proofs. Section 2.1 focuses on the operations of deletion and contraction of edges in maps. By considering the effect of these operations on the various map parameters such as connectivity, genus and orientability, we arrive at eleven types of map edges (compared to the three edge types of ordinary, bridge and loop for multigraphs). These types serve to indicate both the combinatorial and topological role an edge plays in the map.

Section 3 is devoted to developing the notion of a homomorphism between maps. In Section 3.1 we give a formal account of the operation of vertex gluing, and in Section 3.2 we define the operation of duplicate edge gluing. In Section 3.3 we use vertex and duplicate edge gluings to define what it means to be an epimorphism from one map onto another, and the already established notion of isomorphism between maps in order to define what it means to be a monomorphism from one map into another. By composing an epimorphism and a monomorphism we finally arrive at what it means to be a homomorphism from one map to another.

Having then in hand the formal definition of a homomorphism between maps, Section 4 is devoted to the analogue of graph cores for maps. In Sections 4.2 and 4.3 topological notions such as separating and contractible curves are defined in terms of the permutation representation of maps so as to move towards the characterization of when a map is a core given in Section 4.4. Significant differences to graph cores then emerge in the structure of the partial order on the set of cores under the relation of “being homomorphic to” (Section 4.5).

2. Maps

We use the combinatorial definition of maps based on Tutte's permutation axiomatization of cellularly embedded graphs [18], [19, Ch. X] (see also [1, 12]), which is adapted so as to accommodate isolated vertices (maps without edges). Also, instead of the two involutions and basic permutation of [19, Axioms X.1 to X.4], we consider three involutions to define the map (as for generalized maps of dimension 2 [3, 13, 14], without boundary and with the artifice of empty permutation cycles to accommodate edgeless maps). We first introduce some notation.

Permutations of a finite set C are written in function notation, and the product of permutations is their composition, read from right to left: if $\alpha, \beta : C \rightarrow C$ are two permutations then $\alpha\beta$ is defined for $c \in C$ by $\alpha\beta(c) = \alpha(\beta(c))$. Parentheses are used to enclose cyclic permutations and to indicate the element being mapped; in the first case we add some spacing between the parenthesis and the

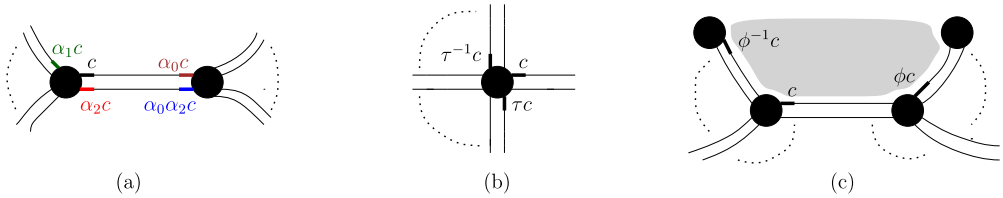


Fig. 1. (a) The permutations $\alpha_0, \alpha_1, \alpha_2$ defining a map acting on a cross c , (b) vertex permutation $\tau = \alpha_1 \alpha_2$, (c) face permutation $\phi = \alpha_1 \alpha_0$. The shaded area indicates a face, while the dots indicate that there might be some additional edges in that region.

elements of the cycle, and we may drop several in the second case (writing αc for $\alpha(c)$, note that we use Latin characters for the crosses while Greek characters for the permutations). For a sequence $\mathbf{x} = (x_1, x_2, \dots, x_k)$ of distinct elements in C , the permutation $(x_1 \ x_2 \ \dots \ x_k)$ in cycle notation is denoted by (\mathbf{x}) . For a permutation α , we let $\alpha \mathbf{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_k)$; then $(\alpha \mathbf{x}) = \alpha(\mathbf{x})\alpha^{-1}$. The reverse sequence (x_k, \dots, x_2, x_1) is denoted by $\bar{\mathbf{x}}$, and as cyclic permutations $(\bar{\mathbf{x}}) = (\mathbf{x})^{-1}$. Thus, $\overline{\alpha \mathbf{x}}$, equal to $\alpha \bar{\mathbf{x}}$, stands for the sequence $(\alpha x_k, \dots, \alpha x_2, \alpha x_1)$, and $(\alpha \bar{\mathbf{x}}) = \alpha(\mathbf{x})^{-1}\alpha^{-1}$.

Definition 1. Let V, E and F be finite sets, elements of which are called vertices, edges and faces, respectively. A map $M = (V, E, F)$ is specified by a tuple $(C, \alpha_0, \alpha_1, \alpha_2)$ in which C is a finite set, whose elements are called crosses; and

- $\alpha_0, \alpha_1, \alpha_2 : C \rightarrow C$ are involutions with no fixed points, that is, $\alpha_i^2 c = c$ and $\alpha_i c \neq c$ for each $i \in \{0, 1, 2\}$ and all $c \in C$;
- the permutations α_0 and α_2 commute, that is, $\alpha_0 \alpha_2 = \alpha_2 \alpha_0$;
- each vertex $v \in V$ is associated with a pair of cycles $(\mathbf{x})(\alpha_2 \bar{\mathbf{x}})$ in the disjoint cycle decomposition of the permutation $\alpha_1 \alpha_2$, where \mathbf{x} is a sequence of crosses (we allow \mathbf{x} to be empty, making a pair of empty cycles, which corresponds to v being an isolated vertex);
- the disjoint cycle decomposition of the involution α_1 is supplemented by pairs of empty cycles (each pair corresponding to edgeless components of the map);
- each edge $e \in E$ is associated with a pair of transpositions $(c \ \alpha_0 \alpha_2 c)(\alpha_0 c \ \alpha_2 c)$ for some cross $c \in C$;
- each face $z \in F$ is associated with a pair of cycles $(\mathbf{y})(\alpha_0 \bar{\mathbf{y}})$ in the disjoint cycle decomposition of the permutation $\alpha_1 \alpha_0$, where \mathbf{y} is a sequence of crosses (we allow \mathbf{y} to be empty, which corresponds to z being the face of an isolated vertex).

The vertex and face permutations of M are denoted respectively by

$$\tau = \alpha_1 \alpha_2 \quad \text{and} \quad \phi = \alpha_1 \alpha_0.$$

A tuple $(C, \alpha_0, \alpha_1, \alpha_2)$ may be alternatively specified by giving C, α_0, α_2 and either the vertex permutation τ or the face permutation ϕ ; this is the approach taken by Tutte [19, Chapter X]. Fig. 1 illustrates the involutions α_0, α_1 and α_2 , the vertex permutation τ and the face permutation ϕ ; in this and later figures it is convenient to use the ribbon graph representation for better visualization.

For a map $M = (V, E, F)$ specified by a tuple $(C, \alpha_0, \alpha_1, \alpha_2)$ we may write $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ when spelling out what V, E and F are is not relevant to the context. We shall also write $v \equiv (\mathbf{x})(\alpha_2 \bar{\mathbf{x}})$, $e \equiv (c \ \alpha_0 \alpha_2 c)(\alpha_0 c \ \alpha_2 c)$ and $z \equiv (\mathbf{y})(\alpha_0 \bar{\mathbf{y}})$ to indicate the cycles associated to a vertex v , an edge e , and a face z .

Let $v \equiv (\mathbf{x})(\alpha_2 \bar{\mathbf{x}})$ be a vertex such that a cross c or $\alpha_0 c$ appears in \mathbf{x} . We say that v is incident with the edge $e \equiv (c \ \alpha_0 \alpha_2 c)(\alpha_0 c \ \alpha_2 c)$ and, similarly, the edge e is incident with a face z if one cross of e appears in the corresponding pair of cycles of ϕ associated with z . With this definition of incidence, $\Gamma = (V, E)$ is the underlying graph of the map $M = (V, E, F)$ specified by $(C, \alpha_0, \alpha_1, \alpha_2)$. Isolated vertices of Γ are associated with pairs of empty cycles appended to the cycle decomposition of τ , and a corresponding pair of empty cycles is appended to the

cycle decomposition of ϕ (representing the face surrounding the isolated vertex of Γ in the given embedding).

The *degree* of a vertex v is the number of crosses in one of its associated cycles from the cycle decomposition of τ , and the *degree* of the face z is the number of crosses in one of its cycles in ϕ .

A *loop* of M is an edge incident with just one vertex, and a *link* is an edge incident with two distinct vertices. We denote the number of vertices, edges and faces of M by $\mathbf{v}(M)$, $\mathbf{e}(M)$ and $\mathbf{f}(M)$, respectively. Each orbit of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ acting on C is a *connected component* of M . Each pair of empty cycles of τ , associated with an isolated vertex, gives its own connected component. The number of connected components is denoted by $\mathbf{k}(M)$. A map is *connected* if it has just one connected component (the edgeless map on one vertex is connected).

In a connected map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$, the number $\mathbf{o}(M)$ of orbits of $\langle \alpha_0\alpha_2, \alpha_1\alpha_2 \rangle$ acting on C is either 1 or 2 [19, Theorem X.11]; M is *non-orientable* if $\mathbf{o}(M) = 1$, and *orientable* if $\mathbf{o}(M) = 2$. Generally, a (not necessarily connected) map M is said to be orientable if all its components are orientable, and non-orientable otherwise. The parameter $\mathbf{o}(M)$ is extended from connected maps to all maps by setting $\mathbf{o}(M) = \min_i \mathbf{o}(M_i)$ when M has connected components $\{M_i\}$.

The *Euler characteristic* of a map M is defined as

$$\chi(M) = \mathbf{v}(M) - \mathbf{e}(M) + \mathbf{f}(M),$$

which is used to define the *Euler genus* by

$$\mathbf{eg}(M) = 2\mathbf{k}(M) - \chi(M),$$

the *genus* by

$$\mathbf{g}(M) = \mathbf{eg}(M)/\mathbf{o}(M),$$

and the *signed genus* by

$$\mathbf{sg}(M) = [2\mathbf{o}(M) - 3]\mathbf{g}(M) = \frac{2\mathbf{o}(M) - 3}{\mathbf{o}(M)}\mathbf{eg}(M) = \begin{cases} \frac{1}{2}\mathbf{eg}(M) & \text{if } \mathbf{o}(M) = 2 \\ -\mathbf{eg}(M) & \text{if } \mathbf{o}(M) = 1. \end{cases} \tag{1}$$

The parameters \mathbf{e} , \mathbf{f} , \mathbf{v} , \mathbf{k} , χ , \mathbf{g} and \mathbf{eg} are additive over connected components: their value on M is the sum of their values over each connected component. (The parameters of orientability \mathbf{o} and signed genus \mathbf{sg} are not additive over connected components in this way.)

Two maps $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ and $M' \equiv (C', \alpha'_0, \alpha'_1, \alpha'_2)$ are *isomorphic* [19, X.4] if they have the same number of isolated vertices and there exists a bijection $\beta : C \rightarrow C'$ such that $\beta\alpha_i c = \alpha'_i \beta c$ for any $c \in C$ and $i \in \{0, 1, 2\}$. Note that β determines naturally a bijection between the vertices, edges and faces of the two maps and their respective vertex and face permutation cycles.

The (geometric) *dual* of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ is the map $M^* \equiv (C, \alpha_2, \alpha_1, \alpha_0)$. The effect of swapping α_0 and α_2 to form the dual gives $\phi = \tau^*$ and $\tau = \phi^*$: the rotational order of edges around a face of M becomes the rotational order of edges around a vertex in M^* and, dually, the rotational order of edges around a vertex of M becomes the rotational order of edges around a face of M^* . The graph underlying M^* is given by $\Gamma^* = (F, E)$ and the edge-face incidence relation.

The permutation representation of a map described in this section is better suited to our combinatorial approach than the following equivalent topological definition:

A map is a graph Γ embedded on a surface $\Sigma = \Sigma_1 \sqcup \dots \sqcup \Sigma_k$ (each Σ_i is a compact, connected 2-manifold without boundary): vertices are points and edges are closed curves connecting pairs of vertices that do not intersect, except possibly at their endpoints (informally, the graph is drawn on the surface in such a way that two edges do not cross).

The idea behind [Definition 1](#) is to describe an embedding of a graph in a surface by explaining how to go from a point next to a given vertex, edge or face to a point next to the “adjacent” vertex,

edge or face. In order to do this, edges are cut across and lengthwise into four parts or crosses¹ and the involutions describing adjacency act on this set of crosses (see Fig. 1).

From a connected combinatorial map as given by Definition 1, we can obtain a surface Σ by gluing open discs along the faces (following one of the permutations of the crosses of the face); this leads to an embedding of the underlying graph Γ of the map in Σ with the property that removing the vertices and edges of Γ from Σ leaves a set of connected components each homeomorphic to a disc. That is to say, a 2-cell embedding of Γ . Therefore, a correspondence holds between connected maps and 2-cell embeddings of connected graphs in a surface. In particular, each connected component of the graph is 2-cell embedded in its own connected surface. The genus and orientability of a connected surface Σ coincide with the genus and orientability of a connected map whose underlying graph has been 2-cell-embedded in Σ .

2.1. Deletion and contraction

Deletion and contraction of an edge e of a map M are formally defined in [9], and denoted by $M \setminus e$ and M/e , respectively.² In the case of a link, these operations have the same effect as deletion and contraction in the underlying graph of M . Deleting a loop of M likewise corresponds in the underlying graph to deleting the loop. However, the result of contracting a loop depends on its type, and is not the same as deleting it in the underlying graph: a loop is *twisted* if a cross c of the edge and $\alpha_0 c$ belong to the same permutation cycle of τ , otherwise, the loop is *non-twisted* (a cross c of the edge and $\alpha_2 \alpha_0 c$ belong to the same permutation cycle of τ). Contraction of a twisted loop has the effect of “flipping over” one side of it, and contraction of a non-twisted loop does not correspond at all to contracting the loop in the underlying graph: indeed, contracting a non-twisted loop splits its incident vertex into two vertices.

Just as deletion and contraction of edges in a graph are dual operations (with respect to matroid duality, the relevant matroid here being the cycle matroid of the graph), so deletion and contraction of edges in maps are dual operations (with respect to geometric duality). Thus [9, Proposition A.5],

$$(M/e)^* = M^* \setminus e \quad \text{and} \quad (M \setminus e)^* = M^*/e. \tag{2}$$

Similarly to graphs, the order in which deletion and contraction of the edges is performed is immaterial, which allows us to define deletion and contraction of a subset of edges. We let $M \setminus A$ denote the result of deleting the edges in A (in any order), and M/B the result of contracting the edges in B (in any order). As shown in [9, Lemma A.6], for a map M and disjoint subsets A, B of edges of M it follows that $(M \setminus A)/B = (M/B) \setminus A$.

For the purposes of this paper we only need to know the effect of these operations on the vertex and face permutations, as described in Tables 1 and 2 (actually, the definition of deletion and contraction can be retrieved from these tables). For the vertex permutation τ , the effect depends on three types of edges that have already been defined: a link, a non-twisted loop, and a twisted loop. For the face permutation ϕ , we need three further edge types: a *dual link*, *dual twisted loop* or *dual non-twisted loop* of M , defined respectively as a link, twisted loop or non-twisted loop of the dual map M^* . Combinations of these give nine edge types; see Fig. 2 for some examples.

While these nine edge types are sufficient to determine the effect of deletion and contraction on the map parameters $\mathbf{v}(M)$, $\mathbf{e}(M)$ and $\mathbf{f}(M)$, in order to determine the effect on the parameters $\mathbf{k}(M)$ and $\mathbf{o}(M)$, and hence, via Euler’s relation, on $\mathbf{g}(M)$, two further edge types are needed: a *bridge* and a *dual bridge*. The defining property of a bridge e is that

$$\mathbf{k}(M \setminus e) = \begin{cases} \mathbf{k}(M) + 1 & \text{if } e \text{ is a bridge,} \\ \mathbf{k}(M) & \text{otherwise.} \end{cases} \tag{3}$$

A dual bridge e is defined with the same property just replacing $\mathbf{k}(M \setminus e)$ by $\mathbf{k}(M/e)$.

¹ As Tutte explains [19, X.10.2], the term “cross” was suggested by their representation in diagrams as two crossed arrows, one along an edge and the other across it, these arrows being reversed by two commuting involutions. Bryant and Singerman [1] call crosses *blades*. As mentioned before, the maps given by Definition 1 can be seen as generalized maps [3,13,14], in which context these crosses are called *darts*. For Tutte, the term “dart” is used for a directed edge, corresponding to a pair of crosses related by the product of the two commuting involutions.

² The effect of deletion and contraction of an edge on the involution α_1 is described in the proof of Lemma 9.

Table 1

The effect of deletion and contraction of an edge on the vertex permutation τ , in which \mathbf{x} and \mathbf{y} are (possibly empty) sequences of crosses.

Edge type in M of e ($c \alpha_0 \alpha_2 c$) ($\alpha_0 c \alpha_2 c$)	Link	Non-twisted loop	Twisted loop
cycle pair(s) in τ	$(c \mathbf{x})(\alpha_2 c \alpha_2 \bar{\mathbf{x}})$ $(\alpha_0 \alpha_2 c \mathbf{y})(\alpha_0 c \alpha_2 \bar{\mathbf{y}})$	$(c \mathbf{x} \alpha_0 \alpha_2 c \mathbf{y})$ $(\alpha_2 \bar{\mathbf{x}} \alpha_2 c \alpha_2 \bar{\mathbf{y}} \alpha_0 c)$	$(c \mathbf{x} \alpha_0 c \mathbf{y})$ $(\alpha_2 \bar{\mathbf{x}} \alpha_2 c \alpha_2 \bar{\mathbf{y}} \alpha_0 \alpha_2 c)$
cycle pair(s) in τ after deleting e	$(\mathbf{x})(\alpha_2 \bar{\mathbf{x}})$ $(\mathbf{y})(\alpha_2 \bar{\mathbf{y}})$	$(\mathbf{x} \mathbf{y})(\alpha_2 \bar{\mathbf{x}} \alpha_2 \bar{\mathbf{y}})$	$(\mathbf{x} \mathbf{y})(\alpha_2 \bar{\mathbf{x}} \alpha_2 \bar{\mathbf{y}})$
$\mathbf{v}(M \setminus e) =$	$\mathbf{v}(M)$	$\mathbf{v}(M)$	$\mathbf{v}(M)$
cycle pair(s) in τ after contracting e	$(\mathbf{x} \mathbf{y})(\alpha_2 \bar{\mathbf{x}} \alpha_2 \bar{\mathbf{y}})$	$(\mathbf{x})(\alpha_2 \bar{\mathbf{x}})$ $(\mathbf{y})(\alpha_2 \bar{\mathbf{y}})$	$(\mathbf{x} \alpha_2 \bar{\mathbf{y}})(\alpha_2 \bar{\mathbf{x}} \mathbf{y})$
$\mathbf{v}(M/e) =$	$\mathbf{v}(M) - 1$	$\mathbf{v}(M) + 1$	$\mathbf{v}(M)$

Table 2

The effect of contraction and deletion of an edge on the face permutation ϕ , in which \mathbf{x} and \mathbf{y} are (possibly empty) sequences of crosses.

Edge type in M of e ($c \alpha_0 \alpha_2 c$) ($\alpha_0 c \alpha_2 c$)	Dual link	Dual non-twisted loop	Dual twisted loop
cycle pair(s) in ϕ	$(c \mathbf{x})(\alpha_0 c \alpha_0 \bar{\mathbf{x}})$ $(\alpha_0 \alpha_2 c \mathbf{y})(\alpha_2 c \alpha_0 \bar{\mathbf{y}})$	$(c \mathbf{x} \alpha_0 \alpha_2 c \mathbf{y})$ $(\alpha_0 \bar{\mathbf{x}} \alpha_0 c \alpha_0 \bar{\mathbf{y}} \alpha_2 c)$	$(c \mathbf{x} \alpha_2 c \mathbf{y})$ $(\alpha_0 \bar{\mathbf{x}} \alpha_0 c \alpha_0 \bar{\mathbf{y}} \alpha_0 \alpha_2 c)$
cycle pair(s) in ϕ after contracting e	$(\mathbf{x})(\alpha_0 \bar{\mathbf{x}})$ $(\mathbf{y})(\alpha_0 \bar{\mathbf{y}})$	$(\mathbf{x} \mathbf{y})(\alpha_0 \bar{\mathbf{x}} \alpha_0 \bar{\mathbf{y}})$	$(\mathbf{x} \mathbf{y})(\alpha_0 \bar{\mathbf{x}} \alpha_0 \bar{\mathbf{y}})$
$\mathbf{f}(M/e) =$	$\mathbf{f}(M)$	$\mathbf{f}(M)$	$\mathbf{f}(M)$
cycle pair(s) in ϕ after deleting e	$(\mathbf{x} \mathbf{y})(\alpha_0 \bar{\mathbf{x}} \alpha_0 \bar{\mathbf{y}})$	$(\mathbf{x})(\alpha_0 \bar{\mathbf{x}})$ $(\mathbf{y})(\alpha_0 \bar{\mathbf{y}})$	$(\mathbf{x} \alpha_0 \bar{\mathbf{y}})(\alpha_0 \bar{\mathbf{x}} \mathbf{y})$
$\mathbf{f}(M \setminus e) =$	$\mathbf{f}(M) - 1$	$\mathbf{f}(M) + 1$	$\mathbf{f}(M)$

Table 3

Effect of deletion and contraction on the Euler characteristic: the difference $\chi(M \setminus e) - \chi(M)$ (in the column headed $\setminus e$) and $\chi(M/e) - \chi(M)$ (in the column headed $/e$) according to the nine possible edge types for e (without distinguishing bridges).

Edge type in M of e	Dual link		Dual non-twisted loop		Dual twisted loop	
	$\setminus e$	$/e$	$\setminus e$	$/e$	$\setminus e$	$/e$
link	0	0	+2	0	+1	0
non-twisted loop	0	+2	+2	+2	+1	+2
twisted loop	0	+1	+2	+1	+1	+1

A bridge is a special type of link and dual non-twisted loop (it is a link as deleting a loop on a vertex v does not affect the connection of v to other vertices, and it is a dual non-twisted loop as the deletion of an edge e in disconnecting M increases the number of faces). Similarly, a dual bridge is a special type of non-twisted loop and dual link. A (dual) link that is not a bridge is called a (dual) *ordinary* link.

Fig. 2 shows maps that between them contain examples of all eleven edge types.

Euler characteristic and Euler genus. The effect of deletion and contraction of an edge e of M on the Euler characteristic $\chi(M)$ is recorded in Table 3 using the information given in Tables 1 and 2. As can be seen from the table – and as shown by Tutte [19, Theorem X.26] – contracting a link does not change the Euler characteristic (dually, deleting a dual link does not change it either).

Table 3 together with the distinction of bridges and dual bridges yield Table 4, which shows the effect of deletion and contraction on the Euler genus $\mathbf{eg}(M)$ according to the eleven edge types. The Euler genus is unchanged when deleting a dual link or a bridge; otherwise it decreases.

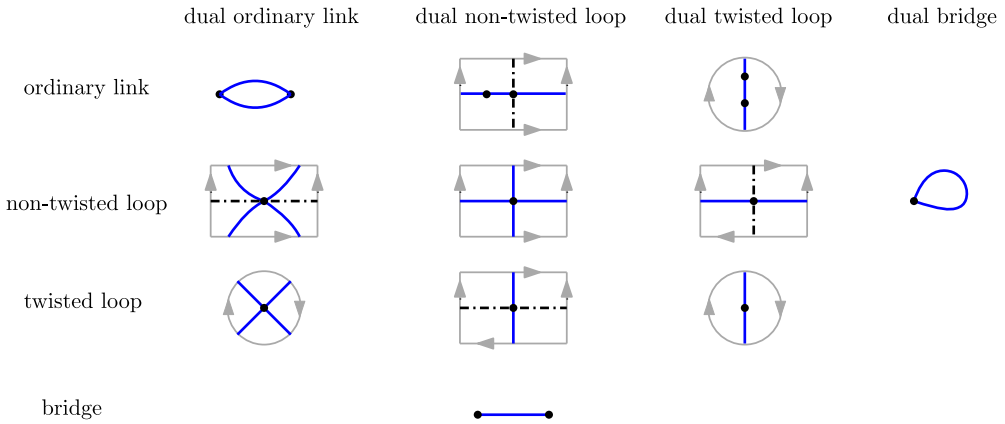


Fig. 2. Examples of maps that between them contain all eleven edge types; solid edges (in blue) are of the given type. Transpose entries in the table are dual maps (those on the diagonal self-dual). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 4

Effect of deletion and contraction on the Euler genus: the difference $\mathbf{eg}(M \setminus e) - \mathbf{eg}(M)$ (in the column headed $\setminus e$) and $\mathbf{eg}(M/e) - \mathbf{eg}(M)$ (in the column headed $/e$) according to the eleven possible edge types for e (non-twisted loops that are dual links are divided into those where the dual link is ordinary or a dual bridge, analogously for links that are dual non-twisted loops.)

Edge type in M of e	Dual link			Dual non-tw loop		Dual twisted loop	
	Dual ord. link	Dual ord. link	Dual bridge	$\setminus e$	$/e$	$\setminus e$	$/e$
	$\setminus e$	$/e$					
link ordinary	0	0		-2	0	-1	0
link bridge				0			
non-twisted loop	0	-2	0	-2	-2	-1	-2
twisted loop	0	-1		-2	-1	-1	-1

Orientability. Tutte showed [19, Theorems X.26 and X.28] that the contraction of a link does not change orientability (dually, deleting a dual link does not change orientability either); we include this result as statement (ii) in the following lemma (its proof is omitted).

Lemma 2. Let e be an edge of a connected map M .

- (i) If $\mathbf{o}(M) = 2$ then $\mathbf{o}(M \setminus e) = \mathbf{o}(M) = \mathbf{o}(M/e)$.
- (ii) If e is a dual link then $\mathbf{o}(M \setminus e) = \mathbf{o}(M)$; if e is a link then $\mathbf{o}(M/e) = \mathbf{o}(M)$.
- (iii) If e is a bridge or a dual bridge then $\mathbf{o}(M \setminus e) = \mathbf{o}(M) = \mathbf{o}(M/e)$.

Proof. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$, and $e \equiv (c \ \alpha_0 \alpha_2 c) (\alpha_0 c \ \alpha_2 c)$. If $\mathbf{o}(M) = 2$, then $\mathbf{o}(M \setminus e) = 2$, as deletion simply removes the crosses of e from the cycles of the vertex permutation τ of M , and so it cannot merge the two orientation classes (orbits under the action of $\langle \tau, \alpha_0 \alpha_2 \rangle$). By the same argument applied to the face permutation ϕ (or by duality, $\mathbf{o}(M^*) = \mathbf{o}(M)$) we have $\mathbf{o}(M/e) = 2$. This proves statement (i).

When $\mathbf{o}(M) = 2$ statement (iii) follows from statement (i). Suppose now that the edge e is a bridge and $\mathbf{o}(M) = 1$. The group $\langle \tau, \alpha_0 \alpha_2 \rangle$ acting on C has one orbit: the graph \mathcal{G}_τ on cycles of τ , in which two cycles are joined by an edge if there is a cross a in one such that $\alpha_0 \alpha_2 a$ lies in the other, is connected (as the number of connected components of \mathcal{G}_τ corresponds to the number of orbits

of $\langle \tau, \alpha_0\alpha_2 \rangle$ acting on C). The cycles

$$(c \ x)(\alpha_2c \ \alpha_2\bar{x}) \quad \text{and} \quad (\alpha_0\alpha_2c \ y)(\alpha_0c \ \alpha_2\bar{y})$$

of τ containing crosses of e have an edge in \mathcal{G}_τ connecting $(c \ x)$ and $(\alpha_0\alpha_2c \ y)$, and an edge connecting $(\alpha_2c \ \alpha_2\bar{x})$ and $(\alpha_0c \ \alpha_2\bar{y})$. After deleting e , these cycles of τ become the cycles $(x)(\alpha_2\bar{x})$ and $(y)(\alpha_2\bar{y})$ of the vertex permutation τ' of $C \setminus \{c, \alpha_0\alpha_2c, \alpha_0c, \alpha_2c\}$, and the other cycles of τ remain the same. The edge previously joining $(c \ x)$ to $(\alpha_0\alpha_2c \ y)$ and that joining $(\alpha_2c \ \alpha_2\bar{x})$ to $(\alpha_0c \ \alpha_2\bar{y})$ both disappear in the new graph $\mathcal{G}_{\tau'}$. (Since e is a bridge, the crosses in x are not in the same orbit as crosses of y under the action of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ when restricted to $C \setminus \{c, \alpha_0\alpha_2c, \alpha_0c, \alpha_2c\}$.) However, all other edges in \mathcal{G}_τ remain. Therefore, the graph $\mathcal{G}_{\tau'}$ cannot have four connected components, that is, the group $\langle \tau', \alpha_0\alpha_2 \rangle$ acting on $C \setminus \{c, \alpha_0\alpha_2c, \alpha_0c, \alpha_2c\}$ cannot have four orbits (there is either a single orbit of the action of $\langle \tau', \alpha_0\alpha_2 \rangle$ that contains the crosses of x or a single orbit that contains the crosses of y). This implies that both connected components of $M \setminus e$ cannot be orientable, and so $\mathbf{o}(M \setminus e) = \mathbf{o}(M) = 1$.

Finally, we prove that M/e is non-orientable as the single orbit of $\langle \tau, \alpha_0\alpha_2 \rangle$ acting on C remains a single orbit in M/e . After the contraction of e , the cycles of τ containing crosses of e become $(x \ y)(\alpha_2\bar{x} \ \alpha_2\bar{y})$. Then, the orbit containing the crosses of x coincides with that containing crosses of y under the action of $\langle \tau, \alpha_0\alpha_2 \rangle$ on $C \setminus \{c, \alpha_0\alpha_2c, \alpha_0c, \alpha_2c\}$. Hence, $\mathbf{o}(M/e) = 1$.

The case of a dual bridge in statement (iii) then follows by $\mathbf{o}(M^*) = \mathbf{o}(M)$, $(M \setminus e)^* = M/e$, and $(M/e)^* = M \setminus e$. \square

Genus and signed genus. In the following two lemmas we highlight those properties of the (signed) genus with respect to deletion and contraction of an edge that we shall have need for later.

Lemma 3. *Let e be an edge of a map M . Then,*

- (i) $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M)$ if and only if e is a dual link or a bridge.
- (ii) $\mathbf{sg}(M/e) = \mathbf{sg}(M)$ if and only if e is a link or a dual bridge.
- (iii) $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M) = \mathbf{sg}(M/e)$ if and only if e is a link and dual link, a bridge, or a dual bridge.

Proof. It suffices to prove statement (i), as (ii) follows by duality and (iii) is the conjunction of (i) and (ii). Suppose first that e is a dual link or a bridge. From Table 4 it follows that $\mathbf{eg}(M \setminus e) = \mathbf{eg}(M)$, and by Lemma 2 we have $\mathbf{o}(M \setminus e) = \mathbf{o}(M)$. This implies $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M)$ by Eq. (1).

Assume now that $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M)$ and that e is not a dual link. Then, the edge e is a dual loop. If $\mathbf{o}(M) = 2$, by Lemma 2, maps M and $M \setminus e$ have the same orientability. Hence $\mathbf{eg}(M \setminus e) = \mathbf{eg}(M)$, and by Table 4, this equality holds if and only if e is a bridge.

Suppose now that $\mathbf{o}(M) = 1$ and $\mathbf{o}(M \setminus e) = 2$ (otherwise we could argue as above). Eq. (1) gives $\mathbf{sg}(M) = -\mathbf{eg}(M)$ and $\mathbf{sg}(M \setminus e) = \frac{1}{2}\mathbf{eg}(M \setminus e)$. Further, by Table 4 it follows that $\mathbf{eg}(M \setminus e) \in \{\mathbf{eg}(M) - 1, \mathbf{eg}(M) - 2\}$, which gives a contradiction as the Euler genus is an integer number. Therefore, M and $M \setminus e$ must have the same orientability, and we can conclude that e is a bridge with the same argument as for the case $\mathbf{o}(M) = 2$. \square

By Lemma 3 and Table 4, we have $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M)$ if and only if $\mathbf{eg}(M \setminus e) = \mathbf{eg}(M)$. The effect of edge deletion on the signed genus, $\mathbf{sg}(M)$, is otherwise to decrease its absolute value, $\mathbf{g}(M)$:

Lemma 4. *Let e be an edge of a map M .*

- (i) If M is orientable then $\mathbf{g}(M \setminus e) < \mathbf{g}(M)$ unless e is a dual link or a bridge, in which case $\mathbf{g}(M \setminus e) = \mathbf{g}(M)$.
- (ii) If M and $M \setminus e$ are non-orientable then $\mathbf{g}(M \setminus e) < \mathbf{g}(M)$ unless e is a dual link or a bridge, in which case $\mathbf{g}(M \setminus e) = \mathbf{g}(M)$.
- (iii) If M is non-orientable and $M \setminus e$ is orientable then $2\mathbf{g}(M \setminus e) < \mathbf{g}(M)$.

Proof. If M is orientable, by Lemma 2(i), $M \setminus e$ is orientable. Statement (i) then follows by Table 4 and the fact that, under the orientability assumptions, $\mathbf{eg}(M) = 2\mathbf{g}(M)$ and $\mathbf{eg}(M \setminus e) = 2\mathbf{g}(M \setminus e)$.

Similarly, statement (ii) follows by Table 4 and the fact that $\mathbf{eg}(M) = \mathbf{g}(M)$ and $\mathbf{eg}(M \setminus e) = \mathbf{g}(M \setminus e)$ (under the orientability assumptions). In both statements, by Lemma 3, we have $\mathbf{g}(M \setminus e) = \mathbf{g}(M)$ when e is a dual link or a bridge.

By Lemma 2(i) we cannot have $\mathbf{o}(M) = 2$ and $\mathbf{o}(M \setminus e) = 1$, so the remaining case to consider is $\mathbf{o}(M) = 1$ and $\mathbf{o}(M \setminus e) = 2$, which is the hypothesis of statement (iii). By Lemma 2(ii), in this case the edge e must be a dual loop. Referring to Table 4, this implies $\mathbf{eg}(M \setminus e) < \mathbf{eg}(M)$ unless e is a bridge, in which case by Lemma 2(iii) we have $\mathbf{o}(M \setminus e) = \mathbf{o}(M)$, contrary to the assumption. Under the given orientability assumptions, $\mathbf{eg}(M) = \mathbf{g}(M)$ and $\mathbf{eg}(M \setminus e) = 2\mathbf{g}(M \setminus e)$. \square

Euler genus and signed genus of submaps. Let v be a vertex of a map $M = (V, E, F)$. The map $M - v$ obtained by deleting v is the map that results by first deleting all edges incident with v , and then removing the empty pair of cycles associated with the now isolated vertex v . A map N is an induced submap of M if $N = M - U$ for some $U \subseteq V$; similarly, N is a spanning submap of M if $N = M \setminus A$ for some $A \subseteq E$. An induced submap of a spanning submap of M is called a submap of M . All types of submaps are said to be proper when they are distinct from the map.

If M' is a submap of M then $\mathbf{v}(M') \leq \mathbf{v}(M)$ and $\mathbf{e}(M') \leq \mathbf{e}(M)$, and both the number of components and the number of faces may change in both directions.³ However, we next show that the Euler genus is monotonous.

Lemma 5. *Let M' be a submap of a map M . Then,*

- (i) $\mathbf{eg}(M') \leq \mathbf{eg}(M)$, and
- (ii) $\mathbf{eg}(M') = \mathbf{eg}(M)$ if and only if $\mathbf{sg}(M') = \mathbf{sg}(M)$.

Proof. Part (i) follows from Table 4: deleting edges does not increase the Euler genus, and neither does deleting isolated vertices. Part (ii) follows from Table 4 and Lemma 3. \square

3. Map homomorphisms

The image of a graph homomorphism can be realized as a sequence of identifications of pairs of distinct vertices, followed by the suppression of parallel edges (i.e., edges incident with the same pair of vertices, or same vertex in the case of parallel loops). In order to define the image of a homomorphism from a map, so that restricted to the underlying graph gives a graph homomorphism, we need to define how to identify a pair of vertices on a map so as to produce another map unambiguously, and to define what it means for edges of a map to be parallel. When identifying vertices in maps we also need to take into account their incident faces along with the vertex and face rotations: this is where the permutation axiomatization of maps becomes essential in order to formulate a well-defined operation on maps analogous to vertex identification in graphs.

3.1. Vertex gluing

In this section we define the operation of vertex gluing in terms of cross permutations and draw on properties of edge deletion and contraction in order to establish that vertex gluing is the only general way to identify vertices in a map while preserving genus and orientability (this property is needed, in particular, to enable restriction and composition of homomorphisms to be generally defined). First, we introduce an operation on maps represented by cross involutions, and go on to explain how vertex gluing can be formalized using this operation.

Definition 6. *Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a map and $a, b \in C$. The map obtained by riffing a and b is $M^{(ab)} \equiv (C, \alpha_0, \tilde{\alpha}_1, \alpha_2)$ in which $\tilde{\alpha}_1 a = \alpha_1 b$, $\tilde{\alpha}_1 b = \alpha_1 a$, and $\tilde{\alpha}_1 = \alpha_1$ on $C \setminus \{a, b, \alpha_1 a, \alpha_1 b\}$.*

³ For example, two loops embedded in the torus are incident with one common face, while deleting one of the loops leaves the other loop incident with two faces; deleting this single loop gives an isolated vertex and just one face again.

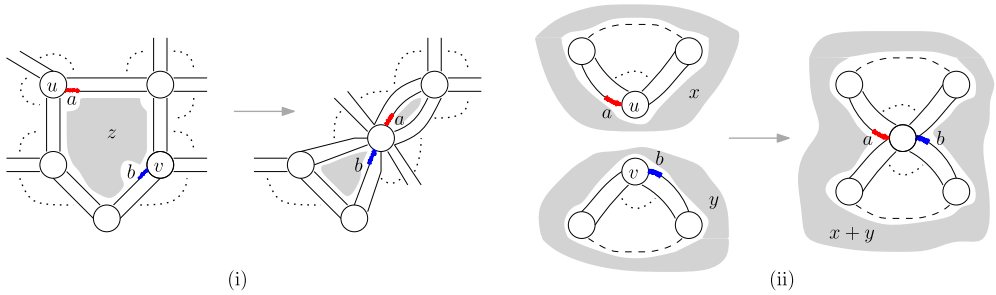


Fig. 3. Riffing the crosses a and b to obtain the map $M^{(ab)}$; this is the map that results from gluing distinct vertices u and v , whose incident crosses a and b are: (i) coincident with a face z , (ii) incident with different faces x, y in different connected components.

Fig. 3 illustrates examples of the riffing operation. With $\tilde{\alpha}_1 a = \alpha_1 b$ and $\tilde{\alpha}_1 b = \alpha_1 a$, we have $\tilde{\alpha}_1(\alpha_1 a) = \tilde{\alpha}_1^2 b = b$, and similarly $\tilde{\alpha}_1(\alpha_1 b) = a$. The involution $\tilde{\alpha}_1$ on C obtained by riffing a and b is the result of conjugating the involution α_1 by the transposition $(a b)$, or by the transposition $(\alpha_1 a \alpha_1 b)$; we thus have $M^{(ab)} = M^{(\alpha_1 a \alpha_1 b)}$. The pair of transpositions $(a \alpha_1 a) (b \alpha_1 b)$ in the disjoint cycle decomposition of α_1 is replaced by the pair $(a \alpha_1 b) (b \alpha_1 a)$.

While any pair of distinct crosses a, b of M can be riffled to produce another map, only under certain conditions are genus and orientability preserved. We next introduce some terminology to help describe what these conditions are. A cross c is *incident* with a vertex v (or a face z) if the pair of permutation cycles associated with v (respectively z) contains c . A pair of crosses are *coincident* with the vertex v (face z) if they belong to the same orbit of τ (respectively ϕ), i.e. they belong to the same permutation cycle in the pair of cycles associated with v (respectively z). If a, b are coincident with a common face z , the crosses $\alpha_1 a$ and $\alpha_1 b$ appear in the conjugate inverse cycle to $(a \dots b \dots)$; thus a, b are coincident with a common face z if and only if $\alpha_1 a, \alpha_1 b$ are coincident with a common face z .

Lemma 7. *The operation of riffing crosses a and b preserves genus and orientability when one of the following conditions holds:*

- (1) (i) crosses a and b in M are incident with distinct vertices and coincident with a common face, or (ii) these crosses are incident with vertices and faces in different connected components;
- (2) (i) crosses a and b in M are coincident with a common vertex and incident with distinct faces, or (ii) these crosses are coincident with a common vertex and face and $\mathbf{k}(M^{(ab)}) > \mathbf{k}(M)$.

In other words, in these cases, $\mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M)$.

Proof. First assume that crosses a, b satisfy condition (1). We define $M^+ \equiv (C^+, \alpha_0^+, \alpha_1^+, \alpha_2^+)$ as the map obtained from M by adding an edge $e \equiv (c \alpha_0^+ \alpha_2^+ c) (\alpha_0^+ c \alpha_2^+ c)$ so that $M^+ \setminus e = M$ and $M^+ / e = M^{(ab)}$. Explicitly, one can check that $M^+ / e = M^{(ab)}$ setting

- $C^+ = C \sqcup \{c, \alpha_0^+ c, \alpha_2^+ c, \alpha_0^+ \alpha_2^+ c\}$;
- $\alpha_0^+ = \alpha_0$ on C ;
- $\alpha_1^+ = \alpha_1$ on $C \setminus \{a, b, \alpha_1 a, \alpha_1 b\}$, and $\alpha_1^+ a = c, \alpha_1^+ b = \alpha_0^+ \alpha_2^+ c, \alpha_1^+(\alpha_1 a) = \alpha_2^+ c, \alpha_1^+(\alpha_1 b) = \alpha_0^+ c$;
- $\alpha_2^+ = \alpha_2$ on C .

When a and b belong to the same permutation cycle of ϕ , the edge e is a link and dual link in M^+ ; when they belong to different connected components of M , the edge e is a bridge of M^+ . By Lemma 3, $\mathbf{sg}(M) = \mathbf{sg}(M^+)$ and $\mathbf{sg}(M^+) = \mathbf{sg}(M^+ / e)$, whence $\mathbf{sg}(M) = \mathbf{sg}(M^{(ab)})$.

To see that riffing crosses satisfying condition (2) also preserves genus and orientability, it suffices to show that if in M crosses a and b are coincident with a common vertex and incident with

distinct faces (condition (2)(i)), then in $M^{(ab)}$ crosses a and b are incident with distinct vertices and coincident with a common face (condition (1)(i)); likewise, if in M crosses a and b are coincident with a common vertex and common face and $\mathbf{k}(M^{(ab)}) > \mathbf{k}(M)$ (condition (2)(ii)), then in $M^{(ab)}$ crosses a and b are incident with vertices and faces in different connected components (condition (1)(ii)). For then we have, $\mathbf{sg}(M^{(ab)}) = \mathbf{sg}((M^{(ab)})^{(ab)})$, and the result follows as $(M^{(ab)})^{(ab)} = M$.

It remains then to establish the effect of riffling crosses a, b on the vertex and face permutations of M under condition (2); as riffling a, b returns $M^{(ab)}$ to the original map M , it suffices to establish the effect of riffling on τ and ϕ under condition (1). (We choose this direction as it will be useful for clarifying what is involved in the key operation of vertex gluing, defined in Definition 8 and illustrated in Fig. 3 with the same notation as here.)

Suppose first condition (1)(i) that the crosses a and b are incident with distinct vertices u and v , and coincident with a common face z in M . The pair of face permutation cycles associated with z takes the form

$$\phi_z = (a \ \mathbf{x} \ b \ \mathbf{y}) (\alpha_0 \bar{\mathbf{x}} \ \alpha_0 a \ \alpha_0 \bar{\mathbf{y}} \ \alpha_0 b),$$

and the pairs of vertex permutation cycles associated with u and v take the form

$$\tau_u = (a \ \mathbf{u}) (\alpha_2 \bar{\mathbf{u}} \ \alpha_2 a) \quad \text{and} \quad \tau_v = (b \ \mathbf{v}) (\alpha_2 \bar{\mathbf{v}} \ \alpha_2 b),$$

for (possibly empty) cross sequences $\mathbf{x}, \mathbf{y}, \mathbf{u}$ and \mathbf{v} .

After riffling a and b , the face z of M is split into two faces x, y and vertices u and v are merged into a single vertex w ; in terms of the vertex permutation $\tilde{\tau} = \tilde{\alpha}_1 \alpha_2$ and face permutation $\tilde{\phi} = \tilde{\alpha}_1 \alpha_0$ of the map $M^{(ab)} \equiv (C, \alpha_0, \tilde{\alpha}_1, \alpha_2)$ obtained upon merging u and v through z ,

$$\tilde{\phi}_x = (a \ \mathbf{x}) (\alpha_0 \bar{\mathbf{x}} \ \alpha_0 a) \quad \text{and} \quad \tilde{\phi}_y = (b \ \mathbf{y}) (\alpha_0 \bar{\mathbf{y}} \ \alpha_0 b),$$

and

$$\tilde{\tau}_w = (a \ \mathbf{u} \ b \ \mathbf{v}) (\alpha_2 \bar{\mathbf{u}} \ \alpha_2 a \ \alpha_2 \bar{\mathbf{v}} \ \alpha_2 b).$$

On other crosses we have $\tilde{\tau} = \tau$ and $\tilde{\phi} = \phi$. Thus, riffling a, b consists in merging the vertices u and v into a single vertex w while splitting the common face z into faces x and y . The crosses a, b now satisfy condition (i) of part (2). Riffling a, b again returns us to condition (i) of part (1).

Suppose now condition (1)(ii) that crosses a, b are incident with vertices u, v and faces x, y that are in different connected components of M . Here we have

$$\tau_u = (a \ \mathbf{u}) (\alpha_2 \bar{\mathbf{u}} \ \alpha_2 a) \quad \text{and} \quad \tau_v = (b \ \mathbf{v}) (\alpha_2 \bar{\mathbf{v}} \ \alpha_2 b),$$

and

$$\phi_x = (a \ \mathbf{x}) (\alpha_0 \bar{\mathbf{x}} \ \alpha_0 a) \quad \text{and} \quad \phi_y = (b \ \mathbf{y}) (\alpha_0 \bar{\mathbf{y}} \ \alpha_0 b),$$

for (possibly empty) cross sequences $\mathbf{x}, \mathbf{y}, \mathbf{u}$ and \mathbf{v} . Then, the result of merging u and v into a single vertex w , along with x and y into a single face z , is to produce the map $M^{(ab)} \equiv (C, \alpha_0, \tilde{\alpha}_1, \alpha_2)$ in which

$$\tilde{\tau}_w = (\mathbf{u} \ a \ \mathbf{v} \ b) (\alpha_2 a \ \alpha_2 \bar{\mathbf{u}} \ \alpha_2 b \ \alpha_2 \bar{\mathbf{v}}).$$

$$\tilde{\phi}_z = (a \ \mathbf{x} \ b \ \mathbf{y}) (\alpha_0 \bar{\mathbf{x}} \ \alpha_0 a \ \alpha_0 \bar{\mathbf{y}} \ \alpha_0 b).$$

The crosses a, b now satisfy condition (ii) of part (2). Riffling a, b again returns us to condition (ii) of part (1). \square

Condition (1) in Lemma 7 under which genus and orientability are preserved by riffling crosses features in our definition of vertex gluing; see Fig. 3. (The inverse operation to vertex gluing of riffling crosses under condition (2), which splits a vertex while preserving genus and orientability, will feature later in Section 4.2.)

Definition 8. Let a, b be crosses of a map M such that either

- (i) a and b are coincident with a face z while a is incident with a vertex u and b is incident with a different vertex v ; or
- (ii) a, b are incident with vertices u, v and faces x, y belonging to different connected components.

The map obtained from M by gluing vertices u and v , by (i) splitting face z or by (ii) merging faces x and y , is the map $M^{(a,b)}$ obtained from M by ruffling a and b .

In case either u or v is an isolated vertex, gluing vertices u and v is simply the deletion of the isolated vertex (if both are isolated, then either is chosen arbitrarily to be deleted).

The effect of vertex gluing on the vertex and face permutations of M is given explicitly at the end of the proof of Lemma 7 above.

Having now arrived at a formal definition of vertex gluing – an operation of vertex identification in maps that preserves genus and orientability – we finish this section by establishing some of its properties relevant to the sequel. From Lemmas 9 to 11, we establish that vertex gluing commutes with deletion of a dual link or bridge. Lemma 14 tells us that permuting the order of a sequence of vertex gluings gives another sequence of vertex gluings, which by Lemma 13 results in the same map.

Lemma 9. If $e \equiv (c \ \alpha_0\alpha_2c) (\alpha_0c \ \alpha_2c)$ is an edge of $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ and a, b are crosses of M such that $\{a, b, \alpha_1a, \alpha_1b\} \cap \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\} = \emptyset$, then

$$(M \setminus e)^{(a,b)} = M^{(a,b)} \setminus e, \quad \text{and} \quad (M/e)^{(a,b)} = M^{(a,b)}/e.$$

Proof. We prove the identity for deletion of e ; the proof for its contraction is similar. The map $M^{(a,b)} \equiv (C, \alpha_0, \tilde{\alpha}_1, \alpha_2)$ has involution $\tilde{\alpha}_1$ equal to α_1 conjugated by $(a \ b)$. The map $M \setminus e \equiv (C', \alpha'_0, \alpha'_1, \alpha'_2)$ has $C' = C \setminus \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\}$, and $\alpha'_0 = \alpha_0$ and $\alpha'_2 = \alpha_2$ on C' . To obtain α'_1 from α_1 in terms of its disjoint cycle decomposition (product of transpositions) consider the (at most four) transpositions of α_1 containing a cross from edge e . We first describe how the transpositions of α'_1 are obtained from those of α_1 ; after this we describe the conditions under which pairs of empty cycles are added to α'_1 (representing new isolated vertices).

If the crosses of e are in four distinct transpositions of α_1 , after removing them, merge together into a single transposition the crosses that were paired with c and α_2c , and likewise those that were paired with α_0c and $\alpha_0\alpha_2c$. If there are three transpositions containing crosses from e , then there is one transposition containing two crosses of e , and two containing just one cross from e ; after removing the crosses of e , merge together the remaining non-empty cycles to make a single transposition in place of the original two. Finally, if there are just two transpositions containing crosses from e , then the transpositions of α'_1 are simply obtained by removing these transpositions.

The conditions under which pairs of empty cycles are added to α'_1 are as follows. If $\alpha_1c = \alpha_2c$, then a pair of empty cycles is added to represent the isolated vertex that results; likewise, if $\alpha_1\alpha_0c = \alpha_0\alpha_2c$ then a pair of empty cycles is added to represent the isolated vertex that results. (This corresponds to deleting a link with endpoint(s) of degree one.) If $\alpha_1c = \alpha_0c$ and $\alpha_1\alpha_2c = \alpha_2\alpha_0c$, then a single pair of empty cycles is added. (This corresponds to deleting a non-twisted loop, and a single new isolated vertex is obtained.) If $\alpha_1c = \alpha_2\alpha_0c$ and $\alpha_1\alpha_2c = \alpha_0c$, then a single pair of empty cycles is added. (This corresponds to deleting a twisted loop, and a single new isolated vertex is obtained.)

We thus see that the involution α'_1 of $M \setminus e$ differs from α_1 only on $\{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\} \cup \{\alpha_1c, \alpha_1\alpha_0c, \alpha_1\alpha_2c, \alpha_1\alpha_0\alpha_2c\}$.

Having now seen how deletion of an edge is defined in terms of the involution α_1 ,⁴ we observe that the hypothesis that $\{a, b, \alpha_1a, \alpha_1b\} \cap \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\} = \emptyset$ implies further that $\{a, b, \alpha_1a, \alpha_1b\} \cap \{\alpha_1c, \alpha_1\alpha_0c, \alpha_1\alpha_2c, \alpha_1\alpha_0\alpha_2c\} = \emptyset$ (by applying the involution α_1 to the two sets in the intersection). As deleting e and ruffling a, b thus change α_1 on disjoint sets of crosses, and they both preserve α_0 and α_2 on C' , these operations commute. \square

⁴ For contraction, the effect on α_1 is the same as described for deletion with the roles of α_0 and α_2 switched. (Contraction produces isolated vertices for non-twisted loops, where $\alpha_1c = \alpha_0c$ or $\alpha_1\alpha_2c = \alpha_0\alpha_2c$.)

Corollary 10. *If $e \equiv (c \ \alpha_0\alpha_2c) (\ \alpha_0c \ \alpha_2c)$ is a dual link or bridge of $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ and a, b are crosses of M such that $\{a, b, \alpha_1a, \alpha_1b\} \cap \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\} = \emptyset$, then the vertex gluing of M represented by riffling a, b is also a vertex gluing of $M \setminus e$, again represented by riffling a, b .*

Proof. As $(M \setminus e)^{(ab)} = M^{(ab)} \setminus e$ by Lemma 9, we only need to check that crosses a, b incident with distinct vertices and either coincident with a common face in M or in different connected components of M are also coincident with a common face in $M \setminus e$ or lie in different connected components of $M \setminus e$.

The effect of deleting a dual link e incident with faces x' and y' in M is to merge x' and y' into a single face z' . If a, b are coincident with a common face z of M , then the same is true in $M \setminus e$ (the same face z if $z \notin \{x', y'\}$, the merging of x' and y' if $z \in \{x', y'\}$). If a, b belong to different connected components of M and are incident with faces x, y in M , then they belong to different connected components in $M \setminus e$ and are incident with the same faces x, y in $M \setminus e$ as they are in M .

Deleting a bridge e incident with face z' in M splits z' into two faces x', y' belonging to different connected components. If a, b are coincident with a common face z of M , then the same is true in $M \setminus e$ when $z \neq z'$; if $z = z'$, crosses a, b are either coincident with common face x' or with common face y' in $M \setminus e$, or in different connected components incident with faces x' and y' . If a, b belong to different connected components of M and are incident with faces x, y in M , then they belong to different connected components in $M \setminus e$ and are incident with the same faces x, y in $M \setminus e$ as they are in M unless $\{x, y\} \cap \{z\} \neq \emptyset$, in which case the face in the component of M containing e , say $x = z$ containing a , is split into two faces, one of which contains a (however, the statement regarding the fact that a, b belong to different connected components of $M \setminus e$ remains valid). \square

Corollary 10 can be extended to Lemma 11 below to accommodate deletion of any dual link or bridge, which will allow us in Corollary 12 to induce, from a vertex gluing of M , a corresponding vertex gluing of a submap of the same genus and orientability.

Lemma 11. *Let e be either a dual link or a bridge with no endpoint of degree one of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$. Let a, b be crosses of M incident with distinct vertices and either coincident with a common face or in different connected components of M . Then there are uniquely defined crosses $a', b' \in C$ such that $(M^{(ab)}) \setminus e = (M \setminus e)^{(a'b')}$. In particular, to the vertex gluing of M represented by riffling a, b corresponds a vertex gluing of $M \setminus e$ represented by riffling a', b' .*

Proof. Let $e \equiv (c \ \alpha_0\alpha_2c) (\ \alpha_0c \ \alpha_2c)$. When $\{a, b, \alpha_1a, \alpha_1b\} \cap \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\} = \emptyset$, by Corollary 10, we can take $a' = a, b' = b$.

If e is a dual link, then a and b cannot both belong to $\{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\}$ because they are coincident with a common face (preventing $b = \alpha_0a$) and, as e is a dual link, c, α_0c belong to a different face to $\alpha_2c, \alpha_0\alpha_2c$ (preventing b from being one of these crosses). Suppose then, without loss of generality, that $a = c$ is incident with face x of M and $b \notin \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\}$ is either a cross coincident with a on face x or a cross belonging to a different connected component of M . If we set $a' = \alpha_1\alpha_2a = \tau a$, then $(M^{(ab)}) \setminus e = (M \setminus e)^{(a'b')}$.

Now suppose that e is a bridge incident with face z and that $a = c$. Possibly $b = \alpha_0\alpha_2c$, a cross coincident with a on face z , but in any event $b \notin \{\alpha_0c, \alpha_2c\}$. When $b \neq \alpha_0\alpha_2c$, the same choice of $a' = \alpha_1\alpha_2a$ gives $(M^{(ab)}) \setminus e = (M \setminus e)^{(a'b')}$, unless a is incident with an endpoint of e of degree one. When $b = \alpha_0\alpha_2c$, crosses $a' = \alpha_1\alpha_2a$ and $b' = \alpha_1\alpha_2b$ belong to different connected components of $M \setminus e$, the edge e is a loop in $M^{(ab)}$, and we have $(M^{(ab)}) \setminus e = (M \setminus e)^{(a'b')} = M/e$.

In all cases, the cross a' is equal to a when $a \notin \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\}$ and to τa when $a \in \{c, \alpha_0c, \alpha_2c, \alpha_0\alpha_2c\}$; and likewise b' is set equal to b or τb according as it does not or does belong to the crosses of e . \square

The following corollary gives a way to restrict a sequence of vertex gluings to a spanning submap, and will be used in Section 3.3 to define the restriction of a map homomorphism to a submap.

Corollary 12. *Let $M \setminus A$ be a spanning submap of M such that $\mathbf{sg}(M \setminus A) = \mathbf{sg}(M)$, and let a, b be crosses of M incident with distinct vertices u, v and either coincident with a common face or in different*

connected components. If neither u nor v are isolated vertices in $M \setminus A$, then there are crosses a', b' of $M \setminus A$ such that $M^{(ab)} \setminus A = (M \setminus A)^{(a'b')}$.

Proof. Lemma 11 and induction yield the result as any spanning submap of the same signed genus as M is, by Lemma 3(i), obtained by a sequence of deletions of dual links or bridges. \square

The following lemma records the effect of switching the order in which two riffles are composed.

Lemma 13. For two pairs of distinct crosses a, b and a', b' of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$, the following statements hold:

- (i) If a, b, a', b' are distinct, then $(M^{(ab)})^{(a'b')} = (M^{(a'b')})^{(ab)}$, and the same identity holds with $(\alpha_1 a \ \alpha_1 b)$ in place of $(a \ b)$ or with $(\alpha_1 a' \ \alpha_1 b')$ in place of $(a' \ b')$.
- (ii) If $a = a'$ and $b \neq b'$, then $(M^{(ab)})^{(a'b')} = (M^{(a'b')})^{(\alpha_1 a \ \alpha_1 b)}$, and the same identity holds with $(\alpha_1 a \ \alpha_1 b)$ in place of $(a \ b)$ or with $(\alpha_1 a' \ \alpha_1 b')$ in place of $(a' \ b')$.
- (iii) If $a = a'$ and $b = b'$, then $(M^{(ab)})^{(a'b')} = M$, and the same identity holds with either transposition $(a \ b)$ (possibly both) replaced by $(\alpha_1 a \ \alpha_1 b)$.

Proof. Suppose first that no cross among $a', b', \alpha_1 a', \alpha_1 b'$ is a cross among $a, b, \alpha_1 a, \alpha_1 b$. Then, $(a \ b) (a' \ b') = (a' \ b') (a \ b)$, and thus conjugating α_1 by $(a \ b)$ and then by $(a' \ b')$ gives the same permutation as when conjugating first by $(a' \ b')$ and then by $(a \ b)$. Hence, $(M^{(ab)})^{(a'b')} = (M^{(a'b')})^{(ab)}$, which proves (i). The same argument shows that the identity holds with $\alpha_1 a, \alpha_1 b$ replacing a, b or with $\alpha_1 a', \alpha_1 b'$ replacing a', b' .

Suppose now that $a = a'$ and $b \neq b'$. The involution

$$\alpha_1 = (a \ \alpha_1 a) (b \ \alpha_1 b) (b' \ \alpha_1 b') \dots \tag{4}$$

after conjugating by $(a \ b')$ or by $(\alpha_1 a \ \alpha_1 b')$ becomes $(a \ \alpha_1 b') (b \ \alpha_1 b) (b' \ \alpha_1 a) \dots$. Then, conjugating by $(\alpha_1 a \ \alpha_1 b)$ gives

$$(a \ \alpha_1 b') (b \ \alpha_1 a) (b' \ \alpha_1 b) \dots \tag{5}$$

Conjugating now the involution α_1 in (4) by $(a \ b)$ or by $(\alpha_1 a \ \alpha_1 b)$ gives

$$(b \ \alpha_1 a) (a \ \alpha_1 b) (b' \ \alpha_1 b') \dots,$$

and then conjugating by $(a \ b')$ gives $(b \ \alpha_1 a) (b' \ \alpha_1 b) (a \ \alpha_1 b') \dots$, which is the same involution as (5). This establishes (ii).

Finally, when $a = a'$ and $b = b'$, conjugating α_1 by $(a \ b)$ and then again by $(a \ b)$ is to conjugate by the product of transpositions $(a \ b) (a \ b)$, equal to the identity permutation. This yields (iii). \square

Before stating the next lemma, we introduce a notion needed for its proof and used further in Section 4.2 below. Distinct crosses a, b, a', b' are *interlacing* in a cycle of crosses γ if $\gamma = (a \ \mathbf{x} \ a' \ \mathbf{y} \ b \ \mathbf{z} \ b' \ \mathbf{w})$ or $\gamma = (a \ \mathbf{x} \ b' \ \mathbf{y} \ b \ \mathbf{z} \ a' \ \mathbf{w})$, where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and \mathbf{w} are (possibly empty) sequences of crosses.

Lemma 14. Let a, b and a', b' be two pairs of distinct crosses of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$, and let $M^{(ab)}$ be a vertex gluing of M and $(M^{(ab)})^{(a'b')}$ a vertex gluing of $M^{(ab)}$. Then, either

- (i) $M^{(a'b')}$ is a vertex gluing of M and $(M^{(a'b')})^{(ab)}$ is a vertex gluing of $M^{(a'b')}$, and $(M^{(ab)})^{(a'b')} = (M^{(a'b')})^{(ab)}$, or
- (ii) $M^{(\alpha_1 a' \ \alpha_1 b')}$ is a vertex gluing of M and $(M^{(\alpha_1 a' \ \alpha_1 b')})^{(ab)}$ is a vertex gluing of $M^{(\alpha_1 a' \ \alpha_1 b')}$, and $(M^{(ab)})^{(a'b')} = (M^{(\alpha_1 a' \ \alpha_1 b')})^{(ab)}$, or
- (iii) $M^{(a'b')}$ is a vertex gluing of M and $(M^{(a'b')})^{(\alpha_1 a \ \alpha_1 b)}$ is a vertex gluing of $M^{(a'b')}$, and $(M^{(ab)})^{(a'b')} = (M^{(a'b')})^{(\alpha_1 a \ \alpha_1 b)}$.

Proof. The final equalities of statements (i), (ii) and (iii) follow from Lemma 13. To establish the remainder of the lemma we shall show that if a, b are either

- (h1) incident with distinct vertices and coincident with a common face of M , or
- (h2) incident with vertices and faces in different connected components of M ,

and a', b' are either

- (h1') incident with distinct vertices and coincident with a common face of $M^{(ab)}$, or
- (h2') incident with vertices and faces in different connected components of $M^{(ab)}$,

then a', b' are either

- (c1) incident with distinct vertices and coincident with a common face of M , or
- (c2) incident with vertices and faces in different connected components of M ,

and a, b are either

- (c1') incident with distinct vertices and coincident with a common face of $M^{(a'b')}$, or
- (c2') incident with vertices and faces in different connected components of $M^{(a'b')}$,

with the appropriate changes for cases (ii) and (iii) of the statement: For (ii) we replace a' by $\alpha_1 a'$ and b' by $\alpha_1 b'$ in (c1), (c2), (c1') and (c2'); for (iii) we replace a by $\alpha_1 a$ and b by $\alpha_1 b$ in (c1), (c2), (c1') and (c2').

We take in turn the four cases according to the position of crosses a, b in M and the position of a', b' in $M^{(ab)}$. Notice that cases (ii) and (iii) only occur in the first case.

Case (h1) and (h1'): Riffing a, b in M splits the face z in M containing a, b into two faces x, y , the first containing a and the second b . By (h1'), there is a face z' in $M^{(ab)}$ containing a', b' .

If $z' \neq x$ and $z' \neq y$, then the face z' is also a face in M , and we obtain (c1) (for if the vertices of a' and b' are distinct in $M^{(ab)}$, then they are also distinct in M as, since we are performing a vertex gluing, there is a surjective mapping from the vertices of M onto the vertices of $M^{(ab)}$). Now, a and b are also coincident with the same face in $M^{(a'b')}$, by the assumption that $z' \neq x$ and $z' \neq y$ and (h1'); furthermore, they are also incident with distinct vertices in $M^{(a'b')}$ as we have just glued the vertex incident with a' and the vertex incident with b' , and if the two distinct vertices incident with a and b in M now become the same in $M^{(a'b')}$, this means that they were also glued together in $M^{(ab)}$, thus contradicting the assumption (h1'). This shows (c1') as stated.

If $z' = x$ (the case $z' = y$ is argued similarly), then we conclude that $\{a, b, a', b'\}$ are all incident with the same face z in M . Now, the condition (h1') implies that $\{a, b\} \neq \{a', b'\}$, for otherwise, a' and b' would lie in two different faces, contradicting (h1'). Consider the cycle of crosses $\gamma = (a \ \alpha_0 a \ \phi a \ \alpha_0 \phi a \ \dots \ \alpha_1 a)$ interleaving the two permutation cycles of the face z (one of them being in its order, and the other in reverse). Assume first that $|\{a, b, a', b'\}| = 4$, then the four crosses cannot be interlacing in γ for otherwise (h1') would not hold. Then we obtain (c1) and (c1'). Now assume that $|\{a, b, a', b'\}| = 3$, say without loss of generality that $a' = a$. Then there are two cases for γ , either $\gamma_1 = (a = a' \ \mathbf{x} \ b \ \mathbf{y} \ b' \ \mathbf{z})$ or $\gamma_2 = (a = a' \ \mathbf{x} \ b' \ \mathbf{y} \ b \ \mathbf{z})$ where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are non-empty sequences of crosses.⁵ In the first case, writing $\mathbf{y} = (\mathbf{y}', \alpha_1 b')$ and $\mathbf{z} = (\mathbf{z}', \alpha_1 a')$, we have $\gamma_1 = (\alpha_1 a' \ a \ \mathbf{x} \ b \ \mathbf{y}' \ \alpha_1 b' \ b' \ \mathbf{z}')$, so $\{\alpha_1 a', \alpha_1 b'\}$ and $\{a, b\}$ are not interlacing, and we conclude (c1) and (c1') hold with $\alpha_1 a'$ and $\alpha_1 b'$ (thus obtaining part (ii) of the statement). In the second case, writing $\mathbf{y} = (\mathbf{y}', \alpha_1 b)$ and $\mathbf{z} = (\mathbf{z}', \alpha_1 a)$, we have $\gamma_2 = (\alpha_1 a \ a' \ \mathbf{x} \ b' \ \mathbf{y}' \ \alpha_1 b \ b \ \mathbf{z}')$, so $\{\alpha_1 a, \alpha_1 b\}$ and $\{a', b'\}$ are not interlacing, and we have (c1) and (c1') with $\alpha_1 a$ and $\alpha_1 b$ (thus obtaining part (iii) of the statement).

Case (h1) and (h2'): As riffing a, b in M preserves connected components, crosses a', b' must also belong to different connected components of M . Thus (c2) follows. We now show that (c1') holds. When riffing a', b' in M , the only alteration to the connected component of M containing a and b may be the addition of some crosses to one of its vertices. Thus, if a and b are incident with distinct vertices in M , they are also incident with distinct vertices in $M^{(a'b')}$. The effect on the faces of this connected component is the addition of crosses to one of its faces. Thus, if a and b are coincident with a single face in M , they are also coincident with a face in $M^{(a'b')}$. This shows (c1').

⁵ The cyclicity of γ and the fact that $\{a', b'\}$ and $\{a, b\}$ are coincident with the face makes the relative position of $\{a', \alpha_1 a\}$ and $\{b', \alpha_1 b\}$ the same within γ_1 and γ_2 .

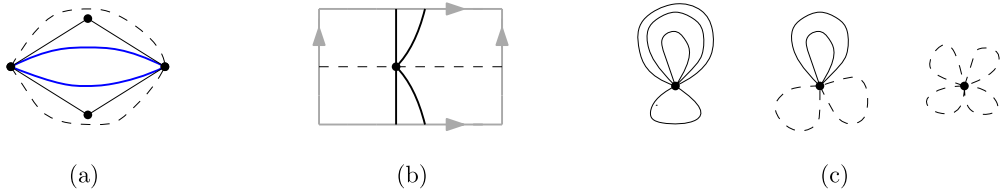


Fig. 4. (a) Two pairs (thicker blue and dashed) of duplicate edges in a plane map, (b) two duplicate loops in the torus (thicker edges), and a loop that is not duplicate with the others (dashed), (c) three plane embeddings of four loops on a vertex, the ones dashed are not duplicate with any other loop, the non-dashed ones are duplicate loops. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Case (h2) and (h1'): The riffing of a, b in M merges their respective faces x, y into a face z , with which a, b are coincident in $M^{(a,b)}$. If the face z' of $M^{(a,b)}$ containing a' and b' is not equal to z , then a', b' remain coincident with this face in M , and we have (c1) and (c2'). If $z' = z$, then there are two possibilities. The first is that a', b' are coincident with a common face of M (either x or y), so (c1) holds; also (c2') holds since gluing the vertices of a' and b' in one connected component does not affect the other connected component. The second possibility is that a' is incident with the same face of M as a or b , say the face x (but a and a' are not necessarily coincident), and b' is incident with the face y . In this case (c2) holds. Now, in $M^{(a,b)}$, crosses a' and b' are incident with the same face as a and b , and (h1') implies that a' and b' are in fact coincident with this face. Crosses a and b are in the same face permutation cycle as a' and b' if both pairs $\{a, a'\}$ and $\{b, b'\}$ are coincident on their faces of M , and different ones if both these pairs are incident but not coincident on their faces of M . (Hypothesis (h1')) prevents just one of these pairs being coincident and the other not.) Furthermore, hypothesis (h1') also implies that a' and b' are incident with distinct vertices in $M^{(a,b)}$. This shows (c1').

Case (h2) and (h2'): Riffing a and b in M merges the two connected components of a and b into one, while the connected components of a' and b' are not merged. Thus, the crosses $\{a, b, a', b'\}$ span at least three connected components in M . This shows that (c2) and (c2') hold. □

3.2. Duplicate edges

If e and f are distinct parallel edges of Γ , then the graphs obtained after deleting e and f , respectively, are isomorphic via the mapping fixing vertices and fixing all edges except e and f , which are swapped. Parallel edges are interchangeable when it comes to the existence of homomorphisms: there is a homomorphism from $\Gamma \setminus e$ to a graph Γ' if and only if there is a homomorphism from $\Gamma \setminus f$ to Γ' , and a redundancy of one of the parallel edges in that there is a homomorphism from Γ to Γ' if and only if there is a homomorphism from $\Gamma \setminus e$ to Γ' . The same statements hold for homomorphisms from the graph Γ' rather than to it.

There are, however, non-isomorphic maps with the same underlying graph which differ only in the placement of parallel edges. For example, a plane 4-cycle with a chord and an edge added in parallel to the chord, in one way bounding a face of degree two and a face of degree three, and in the other bounding two faces of degree three.

In order to carry over to maps the property of parallel edges in graphs being indistinguishable when it comes to homomorphisms, we need to add a topological constraint.

Definition 15. Two edges of a map M are duplicate if they are incident with a common face of degree two.

Fig. 4 illustrates different examples of duplicate and non-duplicate edges. One can see that duplicate edges can be merged into one while remaining in the surface of the map (much as pairs of vertices on a common face can be glued while remaining in the surface of the map).

Definition 16. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a map containing crosses a and $b = \phi a$ such that $e \equiv (a \ \alpha_0\alpha_2a) (\alpha_0a \ \alpha_2a)$ and $f \equiv (b \ \alpha_0\alpha_2b) (\alpha_0b \ \alpha_2b)$ are duplicate edges, bounding the face $(a \ b) (\alpha_0a \ \alpha_0b)$ of degree two. The map obtained by gluing e and f is $M^{[ab]} \equiv (\bar{C}, \bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2)$, in which

- $\bar{C} = C \setminus \{a, \alpha_0a, b, \alpha_0b\}$;
 - $\bar{\alpha}_0 = \alpha_0$ and $\bar{\alpha}_1 = \alpha_1$ on \bar{C} ;
 - $\bar{\alpha}_2 = \alpha_2$ on $\bar{C} \setminus \{\alpha_2a, \alpha_0\alpha_2a, \alpha_2b, \alpha_0\alpha_2b\}$, and
- $$\bar{\alpha}_2(\alpha_2a) = \alpha_0\alpha_2b, \quad \bar{\alpha}_2(\alpha_0\alpha_2a) = \alpha_2b.$$

Equivalently, we obtain $M^{[ab]}$ from M by preserving the three involutions and setting $\alpha_0\alpha_2b \leftarrow a$ and $\alpha_0\alpha_2a \leftarrow b$ (so that $\alpha_2b \leftarrow \alpha_0a$ and $\alpha_2a \leftarrow \alpha_0b$).

Observation 17. The map $M^{[ab]}$ given by Definition 16 is isomorphic to $M \setminus f$ upon setting $\alpha_0\alpha_2b \leftarrow a$, $\alpha_2b \leftarrow \alpha_0a$, and to $M \setminus e$ upon setting $\alpha_0\alpha_2a \leftarrow b$, $\alpha_2a \leftarrow \alpha_0b$.

Definition 16 extends to pairs of edges e, f that lie on a common face of larger degree, or that belong to different connected components, but we shall only apply it to pairs of edges that are duplicate: the more general operation of gluing e and f can be defined as a composition of two vertex gluings (if e and f share no endpoint) or one vertex gluing (if e and f share a single endpoint) and a duplicate edge gluing.

Edges that are duplicate in M are incident with a common vertex of degree two in M^* . The maximal induced paths in M^* have edge sets equal to the equivalence classes of the relation defined on M by taking the transitive closure of the relation of being duplicate: this equivalence relation refines that of parallel edges in the underlying graph of M .

Lemma 18. If e, f are duplicate edges in M , then $\mathbf{sg}(M \setminus e) = \mathbf{sg}(M \setminus f) = \mathbf{sg}(M)$.

Proof. The first equality follows by Observation 17, and the second equality by Lemma 3(i) as duplicate edges are dual links. \square

The following lemma tells us that duplicate edges remain duplicate under vertex gluing (except possibly when the vertex gluing involves the face bounded by the duplicate edges), and that gluing duplicates can be done before or after a vertex gluing without changing the resulting map.

Lemma 19. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a map containing crosses a and $b = \phi a$ such that $e \equiv (a \ \alpha_0\alpha_2a) (\alpha_0a \ \alpha_2a)$ and $f \equiv (b \ \alpha_0\alpha_2b) (\alpha_0b \ \alpha_2b)$ are duplicate edges. Suppose that a', b' are crosses of M such that $\{a', b', \alpha_1a', \alpha_1b'\}$ is disjoint from $\{a, \alpha_0a, b, \alpha_0b\}$. Then e and f are duplicate in $M^{(a'b')}$ and

$$(M^{[ab]})^{(a'b')} = (M^{(a'b')})^{[ab]}.$$

Proof. By the disjointedness condition on crosses $a', b', \alpha_1a', \alpha_1b'$, riffling a', b' fixes $a, b, \alpha_1a = \alpha_0b$ and $\alpha_1b = \alpha_0a$, and therefore the face of degree two $(a \ b) (\alpha_0a \ \alpha_0b)$ is unchanged, i.e. the edges e and f remain duplicate in $M^{(a'b')}$.

Riffling a', b' fixes α_0 and α_2 , and only changes α_1 on the four crosses $a', b', \alpha_1a', \alpha_1b'$, which by assumption all belong to $\bar{C} = C \setminus \{a, \alpha_0a, b, \alpha_0b\}$. The map $M^{[ab]}$ is obtained by restricting α_0, α_1 to \bar{C} , and restricting α_2 to \bar{C} while changing its values on $\alpha_2a, \alpha_0\alpha_2a, \alpha_2b, \alpha_0\alpha_2b$ from $a, \alpha_0a, b, \alpha_0b$ to $\alpha_0\alpha_2b, \alpha_2b, \alpha_0\alpha_2a, \alpha_2a$, respectively. The involutions $\alpha_0, \alpha_1, \alpha_2$ are thus by riffling/duplicate edge gluing affected as follows:

	Riffling	Duplicate edge gluing
α_0	fix	restrict to \bar{C}
α_1	change on $\{a', \alpha_1a', b', \alpha_1b'\}$	restrict to \bar{C}
α_2	fix	restrict to \bar{C} and change on $\{\alpha_2a, \alpha_0\alpha_2a, \alpha_2b, \alpha_0\alpha_2b\}$.

For each involution, duplicate edge gluing followed by riffing has the same effect as riffing followed by duplicate edge gluing. (For α_1 , the condition that the four crosses $a', b', \alpha_1 a', \alpha_1 b'$ all belong to $\bar{C} = C \setminus \{a, \alpha_0 a, b, \alpha_0 b\}$ is needed so that the restriction to \bar{C} can be followed by the changes on $\{a', b', \alpha_1 a', \alpha_1 b'\}$.) \square

In a similar way to how we “remove parallel edges” of a graph while maintaining at least one edge in each parallel class, we can glue pairs of duplicate edges in a map iteratively until no further duplicates remain. Duplicate edges e, f of M are interchangeable in the sense that there is a isomorphism from $M \setminus e$ to $M \setminus f$ that fixes all vertices and all edges apart from e or f .

3.3. Homomorphisms

We have seen how a sequence of vertex gluings followed by a sequence of duplicate edge gluings takes one map onto another map of the same signed genus: the mapping induced on the underlying graphs of the maps is an epimorphism (surjective graph homomorphism). This motivates, in particular, the following definition of an epimorphism between maps.

Definition 20. *Let M and N be maps of the same signed genus.*

An epimorphism $n : M \rightarrow N$ from M onto N is an isomorphism of N to a map obtained from M by performing a sequence of vertex gluings and a sequence of duplicate edge gluings.

A monomorphism from M into N is an isomorphism from M to a submap of N .

In an epimorphism either sequence – of vertex gluings or of duplicate edge gluings – may be empty. Notice that [Lemmas 13, 14](#) and [19](#) together imply that vertex gluings can be done first, followed by duplicate edge gluings, and the order within each sequence does not matter (that edge gluings commute with each other easily follows by interpreting it as edge deletion.)

Before giving our definition of map homomorphism, we record the following proposition, which will be used in [Section 4](#).

Proposition 21. *Let M, M' be two maps such that there exist epimorphisms $M \rightarrow M'$ and $M' \rightarrow M$ (resp., monomorphisms $M \rightarrow M'$ and $M' \rightarrow M$). Then, M and M' are isomorphic.*

Proof. If there are epimorphisms both ways between M and M' , then there can be neither vertex gluings (which reduce the number of vertices) nor duplicate edge gluings (which reduce the number of edges). Since M' is thus obtained from M by empty sequences of vertex gluings and duplicate edge gluings, there is an isomorphism between M' and M .

If there are monomorphisms both ways between M and M' , there are isomorphisms between M and a submap of M' and between M' and a submap of M . As the maps are finite there is thus an isomorphism between M and M' . \square

Definition 22. *Let M, M' be maps of the same signed genus. A map homomorphism from M to M' is a composition of an epimorphism from M onto a submap N of M' and a monomorphism from N into M' . In symbols, a map homomorphism $h : M \rightarrow M'$ is the composition $m \circ n$ of an epimorphism $n : M \rightarrow N$ and a monomorphism $m : N \rightarrow M'$.*

A homomorphism from one map to another gives a graph homomorphism between their underlying graphs: a map homomorphism can thus be regarded as a graph homomorphism that satisfies further topological constraints specified by an embedding of the graph as a map.

A homomorphism $h : M \rightarrow M'$ given by the composition $h = m \circ n$, has *image*, denoted by $h(M)$, equal to the image of its constituent epimorphism n ; i.e., $h(M) = N$. Abusing notation a little further, a function on crosses is associated with h , and we let $h(c)$ denote the cross of M' to which cross c of M is sent by h . When the map homomorphism h is given solely by a sequence of vertex gluings, as represented by riffles, h fixes each cross (as a map on crosses); it is the involution α_1 that is altered. When the map homomorphism h consists solely of a single duplicate edge gluing, then it sends four crosses to four other crosses, as described in [Definition 16](#). Further, the isomorphism

defining the monomorphism from $h(M)$ into M' consists of a bijection between crosses of $h(M)$ and a subset of crosses of M' .

Associated with a homomorphism $h : M \rightarrow M'$, there is a function from vertices of M to vertices of M' : the crosses incident with a given vertex v of M are sent by h to crosses incident with a unique vertex v' in M' . Likewise, a function is defined by $h : M \rightarrow M'$ from edges of M to edges of M' : the four crosses incident with an edge e of M are sent by h to four crosses incident with an edge e' in M' . These two functions on vertices and on edges define a graph homomorphism between the underlying graphs of M and M' .

Restriction of a homomorphism. Lemmas 11 and 12 yield a procedure for converting a sequence of vertex gluings applied to M into a sequence of vertex gluings applied to a spanning submap $M \setminus A$ of the same genus and orientability as M .

Definition 23. Let M be a map with vertex permutation τ , and $M \setminus A$ a spanning submap of M of the same signed genus. Let $h = m \circ n$ be a map homomorphism from M to M' . The restriction of h to $M \setminus A$, denoted by $h_{|M \setminus A} : M \setminus A \rightarrow M'$, is given by restricting n and m as follows:

Restriction of n : first, modify its sequence of vertex gluings to vertex gluings of $M \setminus A$ by replacing a vertex gluing represented by a riffle of crosses a, b in M incident with vertices u, v by

- a vertex gluing of $M \setminus A$ represented by a riffle of $\tau^i a, \tau^j b$ in $M \setminus A$ (where $i, j \geq 0$ are minimal with respect to $\tau^i a, \tau^j b$ being crosses of $M \setminus A$, if such i, j exist);
- deleting isolated vertex u (if $\tau^i a$ is not a cross of $M \setminus A$ for any i , but $\tau^j b$ is a cross of $M \setminus A$ for some j), or isolated vertex v (if $\tau^j b$ is not a cross of $M \setminus A$ for any j , but $\tau^i a$ is a cross of $M \setminus A$ for some i);
- deleting one of the isolated vertices u and v (if $\tau^i a, \tau^j b$ are not crosses of $M \setminus A$ for any i, j).

Second, modify its sequence of duplicate edge gluings by removing any gluings of duplicate edges, one or both of which belong to A .

Restriction of m : we restrict the isomorphism that defines m to the image of the submap $M \setminus A$ under the restricted epimorphism n .

The restriction $h_{|M \setminus A}$ involves gluing the same vertex pairs as in the sequence of vertex gluings defining the constituent epimorphism of h (what may differ are the faces merged or split by the vertex gluings): the graph homomorphism corresponding to h from the underlying graph of M to that of M' is restricted to the underlying graph of $M \setminus A$.

Having introduced the restriction to a spanning submap $M \setminus A$, we next define the restriction of a homomorphism $h : M \rightarrow M'$ to a submap $N = (M \setminus A) - U$, where U is a subset of isolated vertices of $M \setminus A$. To obtain $h_{|N}$ we just perform the restriction $h_{|M \setminus A}$ with the modification that in the vertex gluing sequence, when there is a choice between deleting a vertex in U and one outside U , the vertex in U is deleted, and after which any remaining vertices in U are deleted. If N has the same signed genus as M , then so does $M \setminus A$, so we have the following result.

Proposition 24. Let N be a submap of a map M of the same signed genus as M , and let $h : M \rightarrow M'$ be a homomorphism from M to a map M' . Then, the restriction $h_{|N} : N \rightarrow M'$ is a homomorphism.

In light of the preceding proposition, one may ask if it is possible to define a restriction $h_{|N}$ for any submap N even if N is a submap with a different signed genus as M ; this is, in general, not possible, as Fig. 5 shows; if N is a plane submap, then we are not allowed to follow the sequence of vertex gluings that h demands. This is the reason for the condition that $\mathbf{sg}(N) = \mathbf{sg}(M)$.

Composition of homomorphisms. Let M, M', M'' be maps of the same signed genus, and let $h : M \rightarrow M'$ and $h' : M' \rightarrow M''$ be map homomorphisms. The image $h(M)$ is a submap of M' of the same signed genus which implies, by Proposition 24, that the restriction $h'_{|h(M)} : h(M) \rightarrow M''$ is a homomorphism. The composition of $h'_{|h(M)}$ with h defines the composition of h and h' , denoted by $h' \circ h : M \rightarrow M''$.

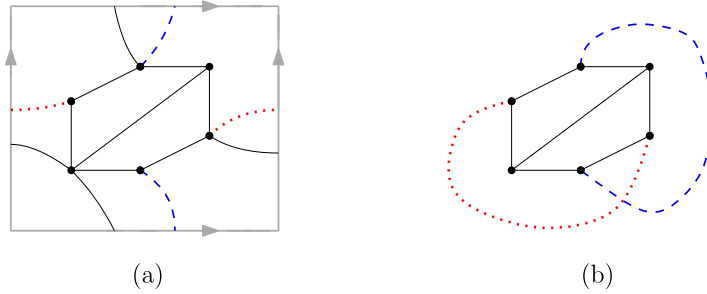


Fig. 5. (a) A map M embedded in the torus, the dotted red and dashed blue lines indicate gluings of the corresponding vertices, (b) a plane submap N of M in which the vertex gluings of (a) cannot be performed. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proposition 25. *The composition $h' \circ h$ of map homomorphisms h and h' is a map homomorphism.*

Decomposing $h = m \circ n$ and $h' = m' \circ n'$ into their constituent epimorphism and monomorphism, the composition $h' \circ h$ has epimorphism $n'_{|h(M)} \circ n$, whose vertex gluing sequence is the concatenation of the sequence of vertex gluings defining n and those defining the restriction of n' to $h(M)$. The sequence of duplicate edge gluings is formed from the concatenation of the sequence of duplicate edge gluings defining n and the inverse image under $n'_{|h(M)}$ of the edges in the sequence of duplicate edge gluings defining the restriction of n' to $h(M)$, where edges in the inverse image are taken in an arbitrary order.

Since the composition of map homomorphisms is again a map homomorphism, the existence of a homomorphism between maps defines a transitive relation; as the identity mapping from M to itself is a homomorphism, this relation defines a quasi-order on maps. In Section 4.5 a partial order is derived from this quasi-order by defining the analogue of graph cores for maps.

4. Cores

A core of a graph Γ [10] is a subgraph Δ with the property that, for any graph Γ' , there exists a homomorphism from Γ to Γ' if and only if there exists a homomorphism from Δ to Γ' , and no proper subgraph of Δ has this property. Equivalently, Δ is a core of Γ if it is a minimal subgraph of Γ (with respect to containment) for which there exists a graph homomorphism from Γ to Δ . A core of a graph is, up to isomorphism, unique.

We define cores for maps analogously to how cores are defined for graphs.

Definition 26. *A submap N of a map M is a core of M if there is a homomorphism $M \rightarrow N$ but no homomorphism $M \rightarrow N'$ for any proper submap N' of N .*

As for graphs, we say that a map is a core when it is its own core. A core of a map M , as a homomorphic image of M , has the same signed genus as M , and has no duplicate edges. If a graph is not connected, then its core may also not be connected (consider, for instance, the disjoint union of K_3 with the graph obtained from K_4 by twice subdividing each edge of a perfect matching, thus having girth four, which is a core as K_3 is not a homomorphic image of the subdivided K_4 , and both connected components are themselves graph cores); the case is similar for maps (a plane embedding of the previous graph is a map core, as any embedding of a graph core is a map core). For this reason, we focus on the quasi-ordered set of connected maps.

4.1. Basic properties

For a map M with core N , a homomorphism from M to N must be surjective (otherwise M would be mapped to a strict submap of N , contradicting that N is a core). As for graphs, the core of a map is unique.

Proposition 27. *If N and N' are cores of a map M , then N is isomorphic to N' .*

Proof. We have $\text{sg}(N) = \text{sg}(M) = \text{sg}(N')$, and there are homomorphisms $h : M \rightarrow N$ and $h' : M \rightarrow N'$. By Proposition 24, the restrictions $h_{|N'} : N' \rightarrow N$ and $h'_{|N} : N \rightarrow N'$ are homomorphisms. In addition, they are surjective since otherwise the compositions $h_{|N'} \circ h'$ and $h'_{|N} \circ h$ would be homomorphisms from M to proper submaps of N and N' , respectively, contradicting the fact that N and N' are cores. The result now follows from Proposition 21. \square

In the following proposition we establish that some basic properties of graph cores [10] have their counterparts for map cores.

Proposition 28. *The following statements hold for the core N of a map M :*

- (i) N is a core (i.e., N is its own core).
- (ii) There is a homomorphism from M to N whose restriction to N is the identity.
- (iii) N is an induced submap of M after gluing its duplicate edges.
- (iv) Given another map M' with core N' , there is a homomorphism from M to M' if and only if there is a homomorphism from N to N' .

Proof. (i) Suppose that there is a homomorphism h' from N to a proper submap N' of N . We can compose h' with a homomorphism $h : M \rightarrow N$ to obtain a homomorphism $h' \circ h : M \rightarrow N'$, which contradicts the fact that N is a core of M . Hence, N is a core.

(ii) Let $h : M \rightarrow N$, and consider the restriction $h_{|N} : N \rightarrow N$ which, by Proposition 24, is a homomorphism as $\text{sg}(N) = \text{sg}(M)$. In addition, it is surjective (since N is a core), and so $h_{|N}$ is an isomorphism (by Proposition 21 using the identity map as the other epimorphism). Thus, we can compose h with the inverse of $h_{|N}$ (also an isomorphism) to obtain a homomorphism from M to N , which is the identity on N .

(iii) Duplicate edges in any map can be glued to obtain a proper submap as homomorphic image and so N cannot contain duplicate edges.

Let N' be the submap induced in M by the vertices of N (so N is a spanning submap of N'). As $\text{sg}(N) = \text{sg}(M)$, by Lemma 5, we have $\text{sg}(N') = \text{sg}(M)$. Let $h : M \rightarrow N$ be a homomorphism that is the identity when restricted to N , which we may assume by (ii). Consider the restriction $h_{|N'} : N' \rightarrow N$, which is an epimorphism by Proposition 24 and the facts that N is a core of M and h is the identity on N . Definition 20 then establishes that N is isomorphic to a map obtained from N' after a sequence of vertex gluings followed by a sequence of duplicate edge gluings. Since $V(N) = V(N')$, the sequence of vertex gluings is empty, leaving only a sequence of duplicate edge gluings. Thus N is the induced submap N' with duplicate edges glued until none remain; see Fig. 6 for an example. Note that the glued edges are duplicated in N' , but not in M .

(iv) Let $h : M \rightarrow N$ be a homomorphism from M to its core N . Suppose first that there is a homomorphism $k : N \rightarrow N'$. Then the composition $k \circ h : M \rightarrow N'$ is a homomorphism, and N' is a submap of M' with $\text{sg}(M') = \text{sg}(N')$. Hence $k \circ h : M \rightarrow M'$ is a homomorphism.

Suppose conversely that there is a homomorphism $\ell : M \rightarrow M'$, and let $h' : M' \rightarrow N'$ be a homomorphism from M' to its core N' . Since N is a submap of M with its same signed genus, there is the monomorphism $\iota : N \rightarrow M$ just seeing N as a submap in M ; then the composition $h' \circ \ell \circ \iota$ is a homomorphism between N and N' as all the maps have the same signed genus. \square

The statement of Proposition 28(ii) holds for any submap N of M with the same signed genus as M (not necessarily a core), as we establish in the following lemma. This fact will be useful in Section 4.3 as it ensures that there is an endomorphism of M with image N if and only if there is such an endomorphism fixing N .

Lemma 29. *Let M be a map, and let N be a submap of M of the same signed genus. If there is an epimorphism $n : M \rightarrow N$, then there is an epimorphism from M onto N whose restriction to N is the identity.*

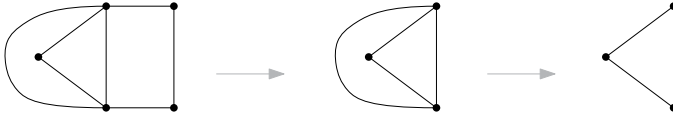


Fig. 6. A map (left) whose core is a triangle, which is not an induced submap: the duplicate edge of the submap induced on the three vertices of the triangle needs to be removed. (Notice that this edge becomes duplicate only after removing the rightmost vertices.)

Proof. If N is the core of M and there is an epimorphism $n : M \twoheadrightarrow N$, the result follows analogously to Proposition 28(ii). Suppose then that N is not a core, and let $N_0 \subset N$ be the core of N , and so the core of M (the epimorphism $n : M \twoheadrightarrow N$ can be composed with an epimorphism $N \twoheadrightarrow N_0$). By Proposition 28(ii), there exists an epimorphism $n_0 : M \twoheadrightarrow N_0$ whose restriction to N_0 is the identity. This epimorphism is defined by a sequence of vertex gluings that gives a map $N_0 \cup \{e_1, \dots, e_s\}$, where the e_i 's are duplicate edges that must be glued to obtain N_0 . Each time that a vertex gluing is performed, we identify two distinct vertices generating a map that has one less vertex. In this identification we always try to keep the vertices in N_0 or in $N \setminus N_0$; note that the two identified vertices cannot both belong to $V(N_0)$ as n_0 is the identity on N_0 . The sequence of vertex gluings and the duplicate edge gluings can be viewed in the reverse order: we construct a map N_1 from N_0 by not performing the last vertex gluing of the sequence that removes a vertex from $N \setminus V(N_0)$ and not gluing those edges e_i that are no longer duplicate with another edge in N_1 . This creates an epimorphism $n_1 : M \twoheadrightarrow N_1$ whose restriction to N_1 is the identity.

Iterating this procedure, we construct a sequence of maps $N_0 \subset N_1 \subset \dots \subset N_k \subseteq N$, where N_i is obtained from N_{i-1} by not performing the last vertex gluing (in the sequence to obtain N_{i-1}) that removes a vertex from $V(N) \setminus V(N_{i-1})$, and by not gluing those edges e_i that are no longer duplicate in N_i . We have $V(N_k) = V(N)$; then N is obtained from N_k by not gluing duplicate edges of N (which may have been glued to make N_k as they had duplicates). Associated with each N_i there is an epimorphism $n_i : M \twoheadrightarrow N_i$ whose restriction to N_i is the identity. Thus, the desired epimorphism from M onto N whose restriction to N is the identity is easily obtained from $n_k : M \twoheadrightarrow N_k$ when go from N_k to N . \square

4.2. Separating and contractible cross-circuits

The aim of this section is to translate the topological notions of a simple (i.e. non-self-intersecting) closed curve, a separating curve and a contractible curve (or at least contractible curves of a certain type) into the language of cross permutations defining a map.

In topology, a closed curve in a surface is said to be *contractible* if it can be homotopically mapped to a point; equivalently, the curve is contained within a region of the surface homeomorphic to an open disc. There have been several translations of this notion into the context of maps, where closed curves are described by walks, which are defined combinatorially rather than topologically.

Viewing a map M as an embedding of a graph Γ into a surface Σ , Mohar and Thomassen [17] define a cycle of Γ (a simple circuit, entering and leaving each vertex at most once) to be contractible in M if the closed curve that it defines in Σ is contractible. One shortcoming of this definition for our purposes is that it only applies to cycles in Γ and not to closed walks of Γ that revisit vertices or edges. This shortcoming is circumvented by the definition offered by Cabello and Mohar [2], who define a closed walk of Γ to be contractible in M if the closed curve that it defines in Σ can be ϵ -perturbed into a contractible closed curve in Σ . Such a perturbation ensures that there are only finitely many self-intersections, and edges that are traversed more than once or vertices that are visited more than once by the walk are represented by disjoint curves or points and are thus distinguished from each other. Problematic from our perspective is that both these definitions of contractibility rely on topological properties of curves in Σ determined by closed walks of Γ rather than properties of the map M as a combinatorial object.

In order to remain within the realm of maps as defined by cross involutions, we begin by introducing an analogous object to a *circuit* in a graph; in a graph a circuit is a closed walk that

repeats no edges, in a map it is a closed walk that repeats no crosses (Definition 30). In this way, we extend the scope of the term contractibility from the graph cycles of Mohar and Thomassen to cross-circuits; facial walks are then all contractible, even if they traverse an edge more than once, corresponding to the defining property of faces as enclosing topological discs.

Definition 30. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a map with vertex permutation τ and face permutation ϕ . A cross-circuit κ of M is a pair of cycles,

$$\kappa = (c_0 \ c_1 \ \dots \ c_{\ell-1}) (\alpha_0 c_{\ell-1} \ \dots \ \alpha_0 c_1 \ \alpha_0 c_0),$$

with the property that the 2ℓ crosses among $\{c_i\} \cup \{\alpha_0 c_i\}$ are pairwise distinct and for each i there is a j such that $c_i = \tau^j \phi c_{i-1}$ (indices i taken modulo ℓ).

The two cycles of the cross-circuit κ represent its traversal in opposite directions. Its length is denoted by $\ell(\kappa)$. A facial walk traversing face z is represented by the cross-circuit ϕ_z .

Recall that a closed walk in a graph is a sequence of edges which joins a sequence of vertices, starting and ending at the same vertex; for a circuit, edges are not allowed to be repeated. For orientable maps, we can find a combinatorial representation using half-edges (considering the set of half-edges to be the set of crosses in the equivalence classes modulo $\langle \alpha_0 \alpha_2, \phi \rangle$ thus having two half-edges per each edge in the graph, instead of four crosses). Thus, in a cross-circuit along an orientable map, we would traverse each half-edge at most once, so each edge at most twice, once in each direction. If a map is non-orientable, we add an extra pair of half-edges per edge in the combinatorial representation to make a total of four crosses; then a cross-circuit only uses each cross from $\{c, \alpha_0 c\}$ at most once, so each edge is used at most twice.

The choice of cross for a vertex-edge incidence reflects, not only the direction of travel (we go from c “to $\alpha_0 c$ ” and then “to $\alpha_1 \alpha_0 c$ ”), but also which “side” of the edge the walk is on, c being on one side and $\alpha_2 c$ on the other. For instance, which direction a loop is traversed is determined by which of $c, \alpha_2 c$ or $\alpha_0 c, \alpha_0 \alpha_2 c$ is chosen to represent the vertex-edge incidence, and hence the side of the edge as well. A closed walk in a map is thus given not only by the sequence of vertices and edges, but by the side along which the edges are traversed – this is determined by how the face permutation ϕ instructs the walk to take a step along an edge, which explains its role in Definition 30.

A cross-circuit $(c_0 \ c_1 \ \dots \ c_{\ell-1}) (\alpha_0 c_{\ell-1} \ \dots \ \alpha_0 c_1 \ \alpha_0 c_0)$ is said to be *non-self-intersecting* if there are no pair of distinct indices (i, j) such that the pairs of crosses $\alpha_0 c_i, c_{i+1}$ and $\alpha_0 c_j, c_{j+1}$ interlace in a cycle $(c \ \alpha_1 c \ \tau c \ \tau \alpha_1 c \ \dots \ \tau^{-1} c \ \tau^{-1} \alpha_1 c)$ consisting of all the crosses incident with a given vertex.

We next present a procedure of cutting a map around a non-self-intersecting cross-circuit, which leans on two operations that are first defined: edge splitting and vertex splitting.

Definition 31. Let $e \equiv (c \ \alpha_0 \alpha_2 c) (\alpha_0 c \ \alpha_2 c)$ be an edge of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$. The map $M' \equiv (C', \alpha'_0, \alpha'_1, \alpha'_2)$ obtained from M by splitting the edge e into two distinct edges bounding a new face of degree two is defined as follows:

- $C' = C \sqcup \{\alpha'_2 c, \alpha'_2 \alpha_0 c, \alpha'_2 \alpha_2 c, \alpha'_2 \alpha_0 \alpha_2 c\}$;
- $\alpha'_0 = \alpha_0$ on C , and

$$\begin{aligned} \alpha'_0(\alpha'_2 c) &= \alpha'_2 \alpha_0 c, & \alpha'_0(\alpha'_2 \alpha_0 c) &= \alpha'_2 c, \\ \alpha'_0(\alpha'_2 \alpha_0 \alpha_2 c) &= \alpha'_2 \alpha_2 c, & \alpha'_0(\alpha'_2 \alpha_2 c) &= \alpha'_2 \alpha_0 \alpha_2 c; \end{aligned}$$

- $\alpha'_2 = \alpha_2$ on $C \setminus \{c, \alpha_0 c, \alpha_2 c, \alpha_0 \alpha_2 c\}$, and

$$\begin{aligned} \alpha'_2(\alpha'_2 c) &= c, & \alpha'_2(\alpha'_2 \alpha_0 c) &= \alpha_0 c, \\ \alpha'_2(\alpha'_2 \alpha_0 \alpha_2 c) &= \alpha_0 \alpha_2 c, & \alpha'_2(\alpha'_2 \alpha_2 c) &= \alpha_2 c; \end{aligned}$$

- $\alpha'_1 = \alpha_1$ on C , and

$$\begin{aligned} \alpha'_1(\alpha'_2 c) &= \alpha'_2 \alpha_2 c, & \alpha'_1(\alpha'_2 \alpha_2 c) &= \alpha'_2 c, \\ \alpha'_1(\alpha'_2 \alpha_0 c) &= \alpha'_2 \alpha_0 \alpha_2 c, & \alpha'_1(\alpha'_2 \alpha_0 \alpha_2 c) &= \alpha'_2 \alpha_0 c. \end{aligned}$$

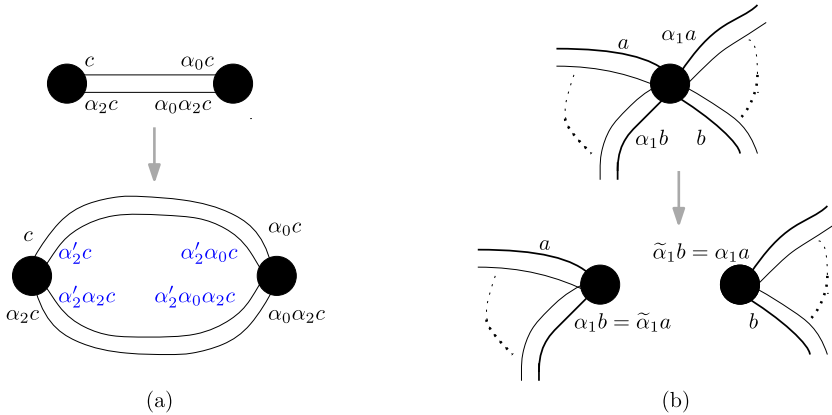


Fig. 7. (a) Edge splitting: the new crosses are labelled in blue, (b) Splitting a vertex of a map M through crosses a, b to make the map $M^{(a,b)}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Edge splitting is illustrated in Fig. 7(a), where it can be easily seen that this operation is the inverse to edge gluing (setting $a = \alpha_2'c$ and $b = \alpha_0'(\alpha_0\alpha_2c)$ in Definition 16).

Definition 32. Let a, b be a pair of distinct crosses that are coincident with a vertex v of a map $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$. The map $M^{(a,b)}$ obtained by riffling a, b is the result of splitting the vertex v through a, b .

Vertex splitting is depicted in Fig. 7(b). This operation is inverse to vertex gluing when the pair of crosses a, b satisfy condition (2) of Lemma 7 (see Fig. 3). Tutte [19, Ch. X] defines a related operation of vertex splitting, which follows the operation in Definition 32 by the insertion of a link joining the two vertices produced by splitting in our sense, in a way that preserves genus an orientability (exactly how this edge is inserted is described in the proof of Lemma 7 above). In the proof of Lemma 44 the reader can find how Tutte’s vertex splitting operation can be recovered using Definition 32, vertex-gluing, duplicate edge-gluing, and edge addition.

Informally, the procedure of cutting around a non-self-intersecting cross-circuit κ , which we define next, is performed as follows. While traversing κ we split each edge as we pass through it (in total, we split each edge as many times as it is traversed). Then, crosses of the split edges are associated with new faces of degree two (generated by the edge splitting). The property of non-self-intersecting allows us to merge these new degree-two faces when splitting vertices: as we proceed, these faces merge together to make a face of larger even degree until the last step, when the circuit closes, and the merged face is joined back on itself and the last vertex-splitting produces two new faces x and y . If this last vertex-splitting disconnects the map, x and y are in different connected components (in which case the last vertex splitting, as all the previous ones, is inverse to vertex gluing, and preserves genus and orientability). Otherwise, the two vertices produced by this last splitting are not incident with a common face (neither do they belong to different connected components); one is incident with face x and the other with face y .

Definition 33. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a map, and let

$$\kappa = (c_0 \ c_1 \ \dots \ c_{\ell-1}) (\alpha_0 c_{\ell-1} \ \dots \ \alpha_0 c_1 \ \alpha_0 c_0)$$

be a non-self-intersecting cross-circuit of M . The map $M \times \kappa$ obtained from M by cutting around κ is the result of: (i) splitting the edges of M containing $c_0, c_1, \dots, c_{\ell-1}$ in turn, producing a map $M' \equiv (C', \alpha_0', \alpha_1', \alpha_2')$, (ii) applying the sequence of vertex-splittings in M' realized by riffling crosses $\alpha_2'\alpha_0c_i$ and $\alpha_1'\alpha_2'c_{i+1}$ for $i = 0, \dots, \ell - 1$ (indices modulo ℓ).

Note that $M \times \kappa$ has the same set of crosses as M' in Definition 33. Figs. 8 and 9 illustrate examples of the procedure of cutting around a non-self-intersecting cross-circuit κ . The new facial walks produced by cutting around κ are next recorded explicitly.

Lemma 34. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a connected map, and let

$$\kappa = (c_0 \ c_1 \ \dots \ c_{\ell-1}) (\alpha_0 c_{\ell-1} \ \dots \ \alpha_0 c_1 \ \alpha_0 c_0)$$

be a non-self-intersecting cross-circuit of M . If $M' \equiv (C', \alpha'_0, \alpha'_1, \alpha'_2)$ is obtained from M by splitting the edges containing $c_0, c_1, \dots, c_{\ell-1}$ in turn as in Definition 33, then the two facial walks of $M \times \kappa$ that are not facial walks of M are

$$\begin{cases} \kappa_{\times} = (\alpha'_2 c_0 \ \alpha'_2 c_1 \ \dots \ \alpha'_2 c_{\ell-1}) (\alpha'_0 \alpha'_2 c_{\ell-1} \ \dots \ \alpha'_0 \alpha'_2 c_1 \ \alpha'_0 \alpha'_2 c_0), \\ \kappa^{\times} = (\alpha'_1 \alpha'_2 c_0 \ \alpha'_1 \alpha'_2 c_1 \ \dots \ \alpha'_1 \alpha'_2 c_{\ell-1}) (\alpha'_0 \alpha'_1 \alpha'_2 c_{\ell-1} \ \dots \ \alpha'_0 \alpha'_1 \alpha'_2 c_1 \ \alpha'_0 \alpha'_1 \alpha'_2 c_0). \end{cases}$$

Proof. This follows by recording the effect of splitting edges to form the map M' , and splitting vertices visited by κ . If the vertex being split is not the last one, the two faces incident with the two riffled crosses are different; for the last vertex we have crosses coincident with a common vertex and with a common face. See Fig. 7. Vertex splitting after edge splitting means that we slice between $\alpha'_2 c$ and $\alpha'_2 \alpha_2 c$, and between $\alpha'_2 \alpha_0 c$ and $\alpha'_2 \alpha_0 \alpha_2 c$ in Fig. 7(a) all along the edges of κ . \square

The effect of vertex splitting on the signed genus in the cases that arise when we cut around a non-self-intersecting cross-circuit is presented in the following lemma.

Lemma 35. Let $M \equiv (C, \alpha_0, \alpha_1, \alpha_2)$ be a connected map, and let a, b be crosses of M that are coincident with a vertex w . After vertex splitting w through a, b to make the map $M^{(ab)}$:

- (i) If a, b are incident with distinct faces, then $\mathbf{k}(M^{(ab)}) = \mathbf{k}(M)$ and $\mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M)$.
- (ii) If a, b are coincident with a common face and $\mathbf{k}(M^{(ab)}) > \mathbf{k}(M)$, then $\mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M)$.
- (iii) If a, b are coincident with a common face and $\mathbf{k}(M^{(ab)}) = \mathbf{k}(M)$, then $\mathbf{sg}(M^{(ab)}) \neq \mathbf{sg}(M)$.

Proof. (i) In this case, crosses a, b are in $M^{(ab)}$ incident with different vertices and coincident with a common face, so, by Lemma 7 and since $[M^{(ab)}]^{(ab)} = M$, we have $\mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M)$. That $\mathbf{k}(M^{(ab)}) = \mathbf{k}(M)$ follows from the fact that the Euler characteristic and Euler genus are unchanged.

(ii) The assumption on the number of connected components implies that splitting w into u and v disconnects the component of $M^{(ab)}$ containing u from the one containing v . Therefore, crosses a, b are in $M^{(ab)}$ incident with vertices and faces in different connected components. By Lemma 7, the result follows.

(iii) From $\mathbf{v}(M^{(ab)}) = \mathbf{v}(M) + 1$, $\mathbf{e}(M^{(ab)}) = \mathbf{e}(M)$ and $\mathbf{f}(M^{(ab)}) = \mathbf{f}(M) + 1$, it follows that $\chi(M^{(ab)}) = \chi(M) + 2$. The latter, together with the assumption $\mathbf{k}(M^{(ab)}) = \mathbf{k}(M)$, implies that $\mathbf{eg}(M^{(ab)}) = \mathbf{eg}(M) + 2$. Hence,

$$\text{if } \begin{cases} \mathbf{o}(M^{(ab)}) = \mathbf{o}(M) = 2 \\ \mathbf{o}(M^{(ab)}) = \mathbf{o}(M) = 1 \\ \mathbf{o}(M^{(ab)}) = 2, \mathbf{o}(M) = 1 \\ \mathbf{o}(M^{(ab)}) = 1, \mathbf{o}(M) = 2 \end{cases} \text{ then } \begin{cases} \mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M) + 1 \\ \mathbf{sg}(M^{(ab)}) = \mathbf{sg}(M) - 2 \\ \mathbf{sg}(M^{(ab)}) = 1 - \frac{\mathbf{sg}(M)}{2} \\ \mathbf{sg}(M^{(ab)}) = -2\mathbf{sg}(M) - 2 \end{cases} \square$$

The map $M \times \kappa$ is produced from M by first splitting (or duplicating) the edges of κ , which does not change the signed genus of M , and then by a sequence of vertex-splittings. When κ is a non-self-intersecting cross-circuit of M , by Lemma 35(i) splitting its vertices visited in turn does not change the number of connected components until possibly when splitting the last vertex. In the notation of Definition 33, this last vertex-splitting is realized by riffling the crosses $\alpha'_2 \alpha_0 c_{\ell}$ and $\alpha'_2 c_0$, which are coincident with a common vertex and with a common face, and so $\mathbf{k}(M) \leq \mathbf{k}(M \times \kappa) \leq \mathbf{k}(M) + 1$. When $\mathbf{k}(M \times \kappa) = \mathbf{k}(M)$, the non-self-intersecting cross-circuit κ is said to be *non-separating*, and it is *separating* if $\mathbf{k}(M \times \kappa) = \mathbf{k}(M) + 1$. Lemmas 35(iii) and 35(ii) give the effect on the signed genus for these two cases.

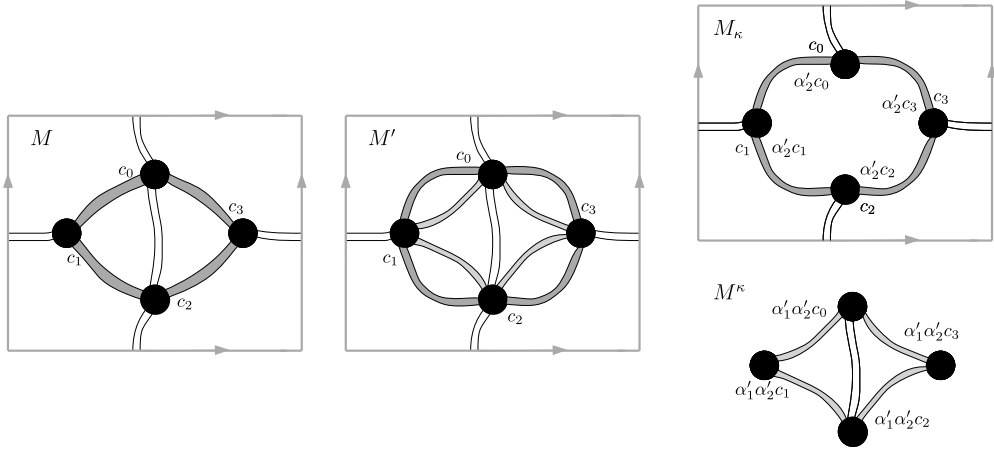


Fig. 8. A cross-circuit κ in a map M (embedded in the torus) given by $(c_0\ c_1\ c_2\ c_3)$. Splitting the edges containing crosses of κ (edges in dark grey) gives the map M' (edges in light grey come from some edge splitting). The map $M \times \kappa$ is then obtained by a sequence of vertex-splittings in M' : it is the disjoint union of M_κ and M^κ . The cross-circuit κ is contractible (M^κ is plane) and not prefacial (M_κ is not plane).

When κ is a separating cross-circuit of a connected map M , the map $M \times \kappa$ is the disjoint union of two maps:

- a connected map M_κ containing the crosses of κ and the facial walk κ_\times , and
- a connected map M^κ containing the crosses of the facial walk κ^\times .

The facial walks κ_\times and κ^\times are both described in Lemma 34; see also Figs. 8 and 9. We say that κ is *contractible* if M_κ or M^κ is a plane map. For non-connected maps, a cross-circuit is said to be contractible if it has this property in its own connected component. This notion of contractibility includes, as a special case, the contractible cycles as defined by Mohar and Thomassen [17], and is consistent with the definition given by Cabello and Mohar [2] when restricting the latter to those curves that can be ϵ -perturbed so as not to be self-intersecting. (Our term “contractible” assumes that the cross-circuit is separating and therefore non-self-intersecting.)

Definition 36. A contractible cross-circuit κ of a connected map M is said to be *prefacial* if M_κ is plane and the facial walk κ_\times traverses a cycle in the underlying graph of M_κ .

Asking a cross-circuit κ to be prefacial adds to the contractibility constraint that it is contractible in a certain direction (M_κ is plane), and the constraint on the facial walk κ_\times adds an irreducibility condition. Fig. 8 illustrates a contractible cross-circuit that is not prefacial because its direction of contraction is not towards the plane part. An example of a contractible cross-circuit κ with M_κ plane and not prefacial is given by traversing two plane loops on a vertex along the half-edges “inside” the loops. (Or take κ given by $(a\ \phi a\ b\ \phi b\ \phi^2 b)$ in the right-hand map in Fig. 3(a).) Fig. 9 illustrates a prefacial cross-circuit.

4.3. Cross-circuits and homomorphisms

In this section, we present technical tools on the interaction between map homomorphisms and different types of cross-circuits; this leads to the characterization of map cores (Theorem 42). We begin by showing that there is a correspondence between prefacial cross-circuits of a connected map and facial walks of any connected submap of the same signed genus. Prefacial cross-circuits are then used to properly partition the set of edges of a map so that homomorphisms will act independently on each subset of the partition.

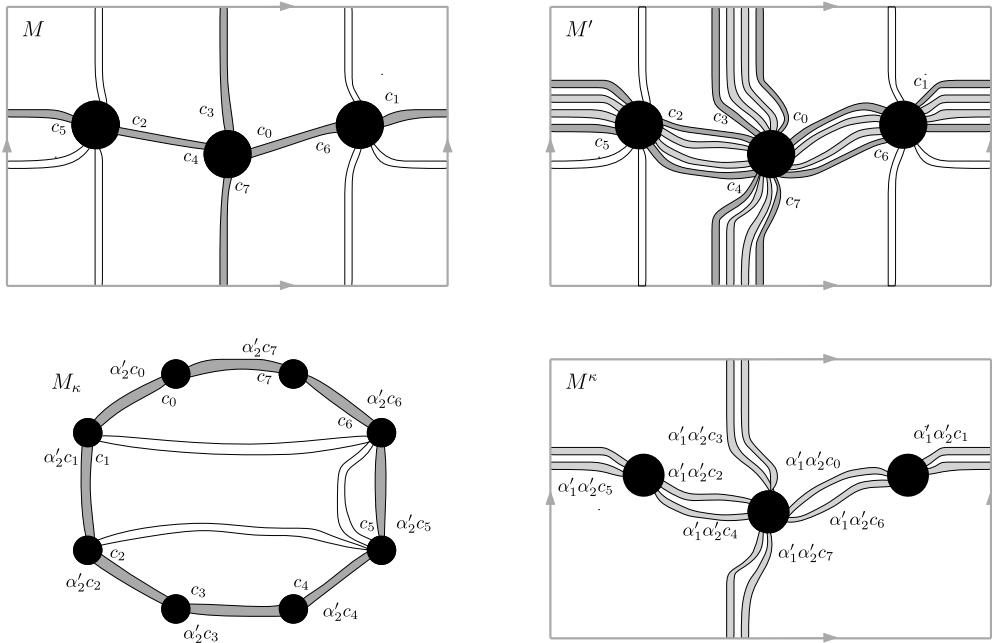


Fig. 9. A cross-circuit κ in a map M (embedded in the torus) given by $(c_0 c_1 c_2 c_3 c_4 c_5 c_6 c_7)$. The map M' is obtained by splitting the edges containing crosses of κ (edges in dark grey); each edge is split twice as it contains two crosses in the cycle defining κ (edges in light grey come from some edge splitting). The map $M \times \kappa$ is then obtained by a sequence of vertex-splittings in M' : it is the disjoint union of M_κ and M^κ . The cross-circuit κ is prefacial.

Lemma 37. Let M be a connected map, and let N be a connected submap of M with the same signed genus. Then,

- (i) Each facial walk κ in N is a prefacial cross-circuit of M .
- (ii) Conversely, for each prefacial cross-circuit κ , there exists a submap $N(\kappa)$ with the same signed genus as M such that κ is a facial walk of $N(\kappa)$.
- (iii) Each edge in $E(M) \setminus E(N)$ belongs to a unique $M_\kappa \setminus E(\kappa_\kappa)$ for some facial walk κ in N .

Proof. For (i) and (iii) it suffices to prove that if we add the edges e of $E(M) \setminus E(N)$ to N one by one (so that the resulting map is always connected), each edge e either subdivides an existing face or adds a new vertex. Then, a small inductive argument over the facial walks of the successive maps establishes the result.

As the orientability of the submaps generated by adding the edges e remains constant, so does the signed genus and the Euler genus (see Lemma 5). Then, all these submaps have the same genus, which, by Lemma 4, implies that the edge e is either a dual link (hence dividing a face into two) or a bridge (adding a new vertex), as desired.

Part (ii) follows from removing in M the edges and vertices in M_κ that do not belong to κ ; this operation creates a submap $N(\kappa)$ of M satisfying the conditions of the statement. \square

Lemma 38. Let κ be a separating cross-circuit of a connected map M .

- (i) A homomorphism $h_\kappa : M_\kappa \rightarrow M_\kappa$ fixing the crosses of κ induces a homomorphism $h : M \rightarrow M$ with the following properties:

- κ is a separating cross-circuit of $h(M)$,
- $h(M) \times \kappa$ consists of the map M^κ and the connected map $h_\kappa(M_\kappa)$,
- h is the identity on $M \setminus [E(M_\kappa) \setminus E(\kappa)]$.

(ii) The cross-circuit ϕ_z corresponding to a face z in M is a facial walk of either M_κ or M^κ . Conversely, a facial walk of M_κ (resp. M^κ) other than κ_\times (resp. κ^\times) defines a facial walk of M .

Proof. (i) Since the crosses of κ are fixed by h_κ , the riffings associated with the vertex gluings that h_κ defines in M_κ do not involve any cross from κ_\times . Further, we may assume that no cross from κ_\times is removed when gluing duplicate edges (in the notation of Definition 16 at least one of the crosses a and b does not belong to κ_\times ; if one of them belongs, say a , we can say that the edge containing a still belongs to the map). Similarly, we may consider the identity map on M^κ , and conclude that no vertex or duplicate edge gluings involve crosses from κ^\times .

As κ is separating, Lemma 35(ii)(i) give that $\text{sg}(M) = \text{sg}(M \times \kappa) = \text{sg}(M_\kappa \sqcup M^\kappa)$. Thus, we can reverse the cutting operations that form $M \times \kappa$ by gluing the appropriate vertices (in the notation of Definition 33 these vertex gluings correspond to riffing back all the crosses $(\alpha'_2\alpha_0c_i, \alpha'_1\alpha'_2c_{i-1})$, indices modulo ℓ) and then gluing the edges of κ_\times with the edges of κ^\times . This creates a homomorphism $h : M \rightarrow M$ satisfying the stated properties.

(ii) Cutting around a non-self-intersecting cross-circuit κ creates, by Lemma 34, two additional faces in $M \times \kappa$, and these precisely involve those crosses added in the process of cutting around; the remaining facial walks not involving the added crosses to $M \times \kappa$ are unchanged. As in addition additional separating hypothesis on κ , $M \times \kappa = M_\kappa \sqcup M^\kappa$ and the result follows. \square

A face z of a map M that is a face of M_κ (in the sense that the pair of cross cycle permutations of ϕ_z defines a facial walk of both M and M_κ) is said to be *contained in κ* (and κ *contains z*).

Observation 39. A prefacial cross-circuit κ of odd length is either a facial walk or contains an odd number of odd-degree faces. If κ has even length, it contains an even number of odd-degree faces. The argument to prove the odd-length statement is the following (similar for even length): M_κ is plane, and has an odd-degree face defined by κ_\times . Considering the dual map M_κ^* , there must be an odd number of other faces of odd degree in M_κ , each of which by Lemma 38(ii) is a face of M contained in κ (by the handshaking lemma, the number of odd-degree vertices is even).

Fixed a prefacial cross-circuit κ of a map M , there is a partial order \leq on contractible cross-circuits of M_κ defined by: $\lambda \leq \mu$ if λ determines a cross-circuit in M_μ and all the crosses in M_λ belong to M_μ (perhaps with the exception of λ_\times). This order has maximum element κ_\times and, for each face z of M_κ , one of its minimal elements is ϕ_z .

We next use the partial order \leq and bring together the previous results in this section to establish when a whole plane map can be sent (in a map-homomorphic way) to its surrounding prefacial cross-circuit. When no confusion may arise, we shall use $M \rightarrow \lambda$ to indicate that a map M is mapped to the map induced by the edges of a prefacial cross-circuit λ .

Lemma 40. Let κ be a prefacial cross-circuit of a connected map M . Assume that there exists a face z of M_κ such that:

- (i) z is different from the face defined by κ_\times in M_κ ;
- (ii) the degree of z is at least the degree of the face traversed by κ_\times , and both faces have degree of the same parity;
- (iii) every other face of M_κ has even degree;
- (iv) $\ell(\kappa) \leq \ell(\lambda)$ for every prefacial cross-circuit λ of M_κ such that $\phi_z \leq \lambda \leq \kappa$.

Then there is an endomorphism of M_κ that maps M_κ to the map induced by the edges of κ (equivalently κ_\times) and it is the identity over κ and κ_\times .

The following facts are used in the proof of the preceding lemma.

Observation 41. Under the hypotheses of Lemma 40 it holds that:

- M_κ and all its connected submaps are plane, and hence orientable. Thus, we can indicate a single permutation cycle associated to faces, vertices, and contractible cross-circuits.
- Any cross-circuit of M_κ that is a cycle in the underlying graph of M_κ is a contractible cross-circuit (as M_κ is plane).
- Any prefacial cross-circuit λ such that $\phi_z \leq \lambda \leq \kappa$ has the same parity as κ and ϕ_z (this follows from assumptions (ii) and (iii) by a parity argument involving the fact that if an even contractible cross-circuit contains an odd degree face, then it contains at least two of them (Observation 39)).

Proof of Lemma 40. We show the result inductively on the number of edges and vertices inside M_κ . The base case occurs when $\phi_z = \kappa$, and so M_κ is a plane cycle; the statement follows by using the identity map as a map homomorphism. Suppose now that $\phi_z \neq \kappa$; the induction argument follows a case analysis.

Case 1: There is no prefacial cross-circuit λ with $\phi_z \leq \lambda \leq \kappa$ and $\phi_z \neq \lambda \neq \kappa$. Let z' be a face different from z containing crosses of κ in M_κ (it exists as $\phi_z \neq \kappa$). We shall assume that $\phi_{z'} \neq \kappa$, otherwise κ would only contain z' and not z as well. Let $\phi_{z'} = (c_0 \cdots c_s)$ such that $c_0 \in \kappa$ and $c_1 \notin \kappa$. Note that z' must contain crosses in κ and outside κ : if all the crosses were in κ , it would be $\kappa = \phi_{z'}$ as κ_\times is a cycle in the underlying graph of the plane map; if all the crosses were outside κ , one could find a prefacial cross-circuit λ between ϕ_z and κ (using the connectedness of M and M_κ).

Case 1.1: If the vertex adjacent to c_2 belongs to κ and is the same as the vertex of c_0 , then $c_2 = c_0$; otherwise we can find a prefacial cross-circuit λ strictly shorter than κ containing z but, as there are no other prefacial circuits between them, it must be $\lambda = \phi_z$, which contradicts assumption (ii). Hence, we conclude $c_2 = c_0$ and z' is a face of length 2. By removing/gluing the edge closing z' , we obtain a map M'_κ with no new prefacial cross-circuits between ϕ_z and κ satisfying the hypotheses of Lemma 40. By induction, M'_κ can be mapped to κ by a homomorphism. Thus, the composition of first removing/gluing the edge from M_κ to M'_κ and then mapping M'_κ to κ gives us the result.

Case 1.2: If the vertex adjacent to c_2 belongs to κ and is different than the vertex of c_0 , then κ is of the form $(c_0 \ c'_1 \ \cdots \ c'_i \ c_2 \ c'_{i+2} \ \cdots \ c'_{\ell(\kappa)-1})$ and it can be partitioned into the crosses $\{c'_1, \dots, c'_i\}$ and $\{c_2, c'_{i+2}, \dots, c'_{\ell(\kappa)-1}, c_0\}$. Then, either the prefacial cross-circuit formed by $\alpha_2 \alpha_0 c_1$ and the crosses $\{c'_1, \dots, c'_i\}$, or the one formed by c_1 and $\{c_2, c'_{i+2}, \dots, c'_{\ell(\kappa)-1}, c_0\}$ contains face z . Hence, this prefacial cross-circuit must be ϕ_z as there are no prefacial cross-circuits between κ and ϕ_z (and it is not κ). Since the vertices of c_0 and c_2 are distinct, the length of ϕ_z is strictly smaller than that of κ , contradicting assumption (ii).

Case 1.3: If the vertex adjacent to c_2 does not belong to κ , then we glue the vertices of c_0 and c_2 along the face z' via $M_\kappa^{(c_0 \ c_2)}$, and remove/glue the edge of c_2 . We obtain a new map M'_κ with κ_\times in the outer face, one edge and one vertex less, and all the faces having even length (as z' had even length and we have shrink it by two units). Further, we have not created any prefacial cross-circuit between κ and ϕ_z , as it would correspond to a (perhaps several) contractible cross-circuit between κ and ϕ_z in M_κ . No other face length has been altered (in particular, the length of z). Thus, the assumptions of Lemma 40 are met, and we can apply the result by induction. The result follows by composing the vertex gluing $M_\kappa^{(c_0 \ c_2)}$ with gluing the now duplicate edge containing c_2 and the mapping from M'_κ to κ . (Note that one of the glued vertices does not belong to κ , and neither does the edge where c_2 belongs, thus the condition that the mapping is the identity over κ is preserved.)

Case 2: There exists a prefacial cross-circuit λ with $\phi_z \leq \lambda \leq \kappa$ and $\phi_z \neq \lambda \neq \kappa$. Let λ_0 be the prefacial cross-circuit that is the “closest” to ϕ_z having the same length as κ (perhaps being κ itself), that is,

- $\lambda_0 \neq \phi_z$ and $\phi_z < \lambda_0 \leq \kappa$;
- $\ell(\kappa) = \ell(\lambda_0)$ and $\ell(\lambda_0) < \ell(\lambda)$ for every λ with $\phi_z < \lambda \leq \lambda_0$.

The prefacial cross-circuit λ_0 induces a graph cycle in M_κ ; otherwise, by choosing λ'_0 to be the cross-circuit of the (minimal) graph cycle all whose crosses belong to λ_0 and that contains face z , we would have $\phi_z \leq \lambda'_0 \leq \kappa$ and $\ell(\lambda'_0) < \ell(\lambda_0)$, which contradicts assumption (iv).⁶

⁶ Note that we are not claiming that such λ'_0 is inside λ_0 ; for instance, if λ'_0 is a cycle and λ_0 is that cycle with a pendant edge inside it, then λ_0 is inside λ'_0 .

Case 2.1: If $\lambda_0 \neq \kappa$, then M_{λ_0} is a strict submap of M_κ with the same signed genus. Indeed, both maps are plane and λ_0 induces a graph cycle, and so cutting around λ_0 in M_κ is the same as removing all the edges and vertices outside λ_0 in M_κ . Thus, by induction, there is an endomorphism $M_{\lambda_0} \rightarrow \lambda_0$ that maps ϕ_z to λ_0 , and is the identity over λ_0 . Lemma 38 allows us to extend this endomorphism to a mapping $h_{\lambda_0} : M_\kappa \rightarrow M_\kappa$ (restricted to M_{λ_0} is the endomorphism $M_{\lambda_0} \rightarrow \lambda_0$, and is the identity elsewhere). Since $\lambda_0 \neq \kappa$, the image $h_{\lambda_0}(M_\kappa)$ is a strict submap of M_κ and has λ_0 as a facial walk. We can apply induction to obtain an endomorphism $h_\kappa : h_{\lambda_0}(M_\kappa) \rightarrow \kappa$ that maps λ_0 to κ , and is the identity over κ . The composition $h_{\lambda_0} \circ h_\kappa : M_\kappa \rightarrow \kappa$ has the desired properties.

Case 2.2: Assume now that $\lambda_0 = \kappa$. We perform a similar procedure of gluing a vertex and an edge (as in Case 1.3), or simply glue an edge (as in Case 1.1).

Let λ_1 be a prefacial cross-circuit with $\phi_z \prec \lambda_1 \prec \kappa$ (it exists by the assumption of Case 2) and such that there is no prefacial cross-circuit λ_2 with $\lambda_1 \prec \lambda_2 \prec \kappa$ (can be assumed as the number of prefacial walks is finite). We have $\ell(\lambda_1) > \ell(\kappa)$ (by the properties of λ_0 and that $\lambda_0 = \kappa$ in our case). A small reduction argument (as done for Case 1) shows that λ_1 is of the form $(c_0 \ c_1 \ \dots \ c_i \ c'_{i+1} \ \dots \ c'_t)$ where the c_j 's are crosses of κ (assuming we start with c_0 for convenience) and c'_{i+1}, \dots, c'_t are not. The vertex of c'_{i+1} belongs to κ , yet the vertex adjacent to c'_{i+2} does not (otherwise it would be $\ell(\lambda_1) \leq \ell(\kappa)$). The face of c_{i+1} is also the face of $\alpha_0 \alpha_2 c'_{i+1}$ (in fact, $\phi \alpha_0 \alpha_2 c'_{i+1} = c_{i+1}$), otherwise there would exist another prefacial cycle between λ_1 and κ . Moreover, the face of c_{i+1} is not z as it is not inside λ_1 . Now, we glue the vertices adjacent to c_{i+2} and c'_{i+2} by the riffing $M_\kappa^{(\alpha_0 \alpha_2 c_{i+1} \ \phi c_{i+1})}$. Observe that, for this new map:

- Condition (ii) holds, as κ and ϕ_z have not been modified.
- The parities of the prefacial cross-circuits between ϕ_z and κ are the same as κ (Observation 41).
- Condition (iii) holds since we do a gluing through a face outside z of two vertices at distance two in a face. Further, the parity of the prefacial cross-circuits between ϕ_z and κ remains the same (using Observation 41 on both the new and the old map).
- Some prefacial cross-circuits between ϕ_z and κ may have been removed or modified by reducing their size. The reduction would be of (exactly) two units, and thus the new prefacial cross-circuits between ϕ_z and κ satisfy condition (iv) with \leq (the hypothesis on Case 2 asked for a strict inequality between their lengths).

Therefore, we can apply Lemma 40 inductively to the new map $M_\kappa^{(\alpha_0 \alpha_2 c_{i+1} \ \phi c_{i+1})}$. Since the two mappings are the identity over κ (we are using here Lemma 38 for the vertex gluing that we have performed) the composition of gluing a vertex and afterwards performing the inductive step is also a mapping that is the identity over κ , thus the result follows. \square

4.4. Characterization of map cores

We bring together the notions and results of the previous sections on cores in the following theorem.

Theorem 42. *A connected map M is a core if and only if each of its prefacial cross-circuits λ either is a facial walk, or contains more than one face of odd degree, or contains no face whose degree is greater than or equal to $\ell(\lambda)$ and has the same parity as $\ell(\lambda)$.*

Proof. We argue with the contrapositive in both implications. From left to right, assume that there is a prefacial cross-circuit λ of M such that: (i) is not a facial walk, (ii) contains at most one face of odd degree, (iii) contains a face z whose degree is at least $\ell(\lambda)$ and has the same parity as $\ell(\lambda)$.

If λ has odd length, by (ii) and (iii), z is the only face of odd degree inside λ , and any prefacial cross-circuit μ with $\phi_z \preceq \mu \preceq \lambda$ has odd length (see Observation 41). If λ has even length, then all the faces inside λ have even degree (otherwise, by Observation 39, there would be at least two faces of odd degree inside λ).

Let κ be a prefacial cross-circuit such that $\phi_z \prec \kappa \preceq \lambda$ and $\ell(\kappa) = \min\{\ell(\mu) : \phi_z \preceq \mu \preceq \lambda\}$ (perhaps being λ itself). The conditions of Lemma 40 are met with face z and prefacial cross-circuit

κ . There is thus a homomorphism $h_\kappa : M_\kappa \rightarrow \kappa$ that is the identity over κ . Since κ is a prefacial cross-circuit, by Lemma 37, we can conclude that the submap M' obtained by removing the interior of κ (which is the largest submap of M where κ is a facial walk) has the same signed genus as M . Further, $\kappa \neq \phi_z$ which implies that M' is a proper submap of M . Lemma 38 (applied to h_κ) then gives a homomorphism $h : M \rightarrow M' \subsetneq M$; therefore, M is not a core.

Now, we prove the implication from right to left using the contrapositive again. Assume that M is not a core; we want to find a prefacial cross-circuit satisfying conditions (i)–(iii) stated above.

Let N be the core of M , and let $h : M \rightarrow N$ be a homomorphism, whose restriction to N is the identity (see Proposition 28(ii)). Since N is a proper submap of M with the same signed genus, Lemma 37 asserts that there is a facial walk of N that is a prefacial cross-circuit κ of M but not a facial walk of M . Consider again the submap M' of M (defined as above), and let $h' : M \rightarrow M'$ be the mapping obtained from h as follows:

- Vertices u, v are glued if h glues them and either they are in the interior of κ , or u is in the interior of κ and v is in κ .
- Edges e, e' are glued if h glues them and either they are in the interior of κ , or e is in the interior of κ and e' belongs to κ .

Once we cannot perform any of the two previous operations, we stop and obtain h' . Observe that we never glue two vertices or two edges of κ as h is the identity over N .

We now construct a layered and labelled graph T by matching the above described gluing instructions; the layers will represent the state, at that stage of the process, of the faces in the interior of κ , which are denoted by $\{f_1, \dots, f_r\}$. In the first layer of T , we place r vertices labelled f_1, \dots, f_r (one vertex per face). Now, if a vertex gluing subdivides a face (into two new faces), we place two new vertices (in the following layer) adjacent to the vertex that represents the original face (the parent); the edges are directed, always pointing towards the new vertices, which are both labelled with the label of the parent. If we glue two edges, the two faces that they bound are merged into one; this is represented by a vertex in the new layer, which is adjacent to the parent vertices by directed edges (again pointing towards the “new”). This new vertex takes the label of the face with the largest degree among the two that we are merging (if there is a tie between two faces of degree 2, we choose the label of one of them arbitrarily). For the remaining vertices (representing faces, including those that have not been subdivided), we place a new vertex in the new layer with the same label as the parent, and with a directed edge pointing from the old vertex to the new one.

The preceding procedure ensures that all the vertices of T bear one of the labels of the original faces inside κ , that is, $\{f_1, \dots, f_r\}$. Further, from one layer of T to the next, the process involve some subdivision of faces (when two vertices are glued) and some destruction of faces (when two edges are glued); at the end, only one face remains, represented by a unique vertex in the last layer of T , whose label indicates the original face that has not been destroyed/merged-into-another-one. We next discuss how the parity of the degree of the faces evolve in this process.

When a face of odd degree is subdivided, a new face of odd degree and a face of even degree appear; if we subdivide a face of even degree, two faces of odd degree or two faces of even degree appear. Then, no faces of odd degree are created when subdividing even degree faces, as there is only one face at the end of the process, and when an odd degree face is created, actually two are created, and none of them can be removed (we can only remove faces of degree 2). Further, the vertices of any path in T traversing different layers (from higher to lower layers) have the property that if their label is the same, the sequence of degrees of their associated faces is weakly decreasing. Let f_i be the label of the unique vertex in the last layer, and consider a path with all vertices labelled f_i that goes from the first to the last layer; we conclude that:

- the parity of f_i is the same as κ (it is maintained through the process, and such facial walk is modified until it becomes κ);
- the degree of the other faces in M is even, and the degree of f_i is larger or equal than $\ell(\kappa)$.

Thus, κ is a prefacial cross-circuit of M satisfying conditions (i)–(iii); this finishes the contrapositive of the implication from right to left and completes the proof of the whole statement. \square

Application of the characterization to four classes of maps. A bouquet is a map with one vertex, all edges being loops. Dually, a quasi-tree is a map with one face, also known as a unicellular map. (Bouquets and quasi-trees are necessarily connected.) Theorem 42 says that: a bouquet is a core if and only if it has no duplicate edges; a quasi-tree is a core if and only if either it is plane and consists of one edge or one isolated vertex, or it is not plane and has no vertices of degree one. (These two facts can be shown directly.) For the other two classes of connected maps, we exploit the parity conditions for κ in Theorem 42 to conclude the following. A connected map with no vertices of degree one and all faces of odd degree is a core. Any plane connected core map satisfies exactly one of the following properties: (i) consists on a single edge or a single isolated vertex (when all faces have even degree and thus the underlying graph is bipartite by Observation 41), or (ii) is a cycle of odd length (thus having precisely two faces, both of odd degree), or (iii) has at least four odd degree faces (four instead of three using Observation 39).

Graph cores vs. map cores. Since a map homomorphism is a graph homomorphism, any embedding of a graph core is a map core. However, there are map cores for which the underlying graph is not a graph core, for example, the quasi-tree embedding of a vertex with two loops on a torus. Further, the characterization of quasi-trees as map cores (stated in the previous paragraph) gives a sharp contrast between map cores and graph cores. Indeed, the fact that any connected graph has a quasi-tree embedding in some surface, possibly non-orientable [4, Page 123, Third paragraph], yields that any connected graph without vertices of degree one can be the underlying graph of a map core by embedding it in an appropriate surface.

On the other side, identifying two vertices by a graph homomorphism can be seen as gluing two vertices by a map homomorphism, using an appropriate embedding (placing a face between the two vertices to be glued); to deal with parallel edges, the graph is embedded so that they are duplicated. Therefore, any graph homomorphism can be viewed as the composition of several map homomorphisms (perhaps by changing the embeddings of the corresponding underlying graphs).

This discussion prompts a natural question on whether map cores can be used to identify graph cores: for every connected graph Γ that is not a core, is it possible to find a map M with underlying graph Γ such that M is not a (map) core?

We answer this question in the negative. Let Γ be the graph resulting from identifying two copies of the complete graph K_5 by a vertex (it has 9 vertices and no additional edges). The core of Γ is K_5 . In fact, all graph homomorphisms from Γ leading to a subgraph (not necessarily a core) are obtained by mapping one of the copies of K_5 into the other. Now, suppose that there exists a map M with underlying graph Γ that is not a map core. We can apply the arguments used in the proof of Theorem 42 to conclude that there exist a prefacial cross-circuit κ (which is not a facial walk) and a homomorphism from M to a strict submap of M that maps the interior of κ to κ . In particular, this homomorphism is a graph homomorphism from Γ to a subgraph that must be K_5 . Therefore, the underlying graph of the interior of κ should contain K_5 , which is a contradiction as the interior of κ is a plane map.

4.5. The poset of map cores

Having now described how to tell whether a map is a core, we introduce and study the poset of map cores, comparing its properties with those of the poset of graph cores.

For fixed $g \in \mathbb{Z}$, let $\mathcal{M}(g)$ and $\mathcal{M}_c(g)$ be the sets of, respectively, maps and cores of signed genus g . For $M, M' \in \mathcal{M}(g)$, we say that $M \leq M'$ if there exists a map homomorphism from M to M' , that is, if $M \rightarrow M'$. The associated strict order relation, $M < M'$, holds if $M \rightarrow M'$ and $M' \not\rightarrow M$. The relation \leq is defined analogously on $\mathcal{M}_c(g)$; indeed, it is obtained from the preorder $(\mathcal{M}(g), \leq)$ upon quotienting out by the equivalence relation \cong defined by $M \cong M'$ if $M \leq M'$ and $M' \leq M$.

Lemma 43. *The relation \leq defines a preorder on $\mathcal{M}(g)$ and a partial order on $\mathcal{M}_c(g)$.*

Proof. The preorder part follows by Proposition 21 and the fact that the composition of homomorphisms between maps in $\mathcal{M}(g)$ is again a homomorphism. For cores, Proposition 27 ensures that the preorder is in fact a partial order, since we are considering maps up to isomorphism. \square

The first property that we establish is that the poset $(\mathcal{M}_c(g), \leq)$ is connected, by first showing that the preorder $(\mathcal{M}(g), \leq)$ is connected. Recall that a preorder (P, \leq) is connected if for every pair of elements a, b there is a sequence $a = c_0, c_1, \dots, c_{k-1}, c_k = b$ such that $c_i \leq c_{i+1}$ or $c_{i+1} \leq c_i$ for each $i = 0, 1, \dots, k - 1$.

Lemma 44. *Given connected maps M and M' of the same signed genus and with at least one edge, M' can be obtained from M by a sequence of vertex gluings, vertex splittings (Definition 32 with the extra condition of Lemma 7(2)(i)), edge splittings, and duplicate edge gluings.*

Proof. Vertex gluing and vertex splitting under the conditions indicated are inverse operations, as are duplicate edge gluing and edge splitting, and all these operations preserve signed genus and connectivity. A connected map with more than one face has some edge that is a dual link; dually, a connected map with more than one vertex has some edge that is a link. For a dual link $e \equiv (c \ \alpha_2\alpha_0c) (\ \alpha_2c \ \alpha_0\alpha_2c)$, crosses c and α_2c belong to different faces. Splitting the vertex containing crosses c and α_2c by riffling c and τc has the effect of adding a vertex and removing a face (so that all the crosses of e now belong to a single face), while the signed genus and the connectivity are preserved, so the number of faces is reduced. This procedure can be repeated until the map becomes a quasi-tree. If this quasi-tree has more than one vertex, by contracting links we can reduce the number of vertices, while preserving signed genus (Lemma 4(ii)) and connectivity, thus producing no new faces; the quasi-tree can thus be made simultaneously into a bouquet by a sequence of such link contractions. Such a map with one vertex and one face is called by Tutte a *unitary map*. If the map has at least two edges, contracting a link $e \equiv (c \ \alpha_2\alpha_0c) (\ \alpha_2c \ \alpha_0\alpha_2c)$ can be realized by a sequence of vertex gluings/vertex splittings (Definition 32 with the extra condition of Lemma 7(2)(i)) and duplicate edge gluings/edge splittings as follows: (1) glue the two vertices incident with e by riffling crosses c and ϕc , making e become a loop; then (2) split the vertex incident with this loop by riffling c and $\alpha_2\alpha_0c$, thereby creating a leaf vertex; then (3) glue this leaf vertex to the vertex containing $\phi^{-1}c$ by riffling $\alpha_0\alpha_2c$ and $\phi^{-1}c$, thereby creating a pair of duplicate edges; and finally (4) glue these duplicate edges.

We have just seen how any connected map with at least one edge can be taken to a unitary map by a sequence of vertex gluings/vertex splittings and duplicate edge gluings/edge splittings. The *orientable canonical map* of genus g is the unitary map whose vertex permutation has a cycle of the form⁷ $(c_1 \ c_2 \ \alpha_0\alpha_2c_2 \ \alpha_0\alpha_2c_1 \ \dots \ c_{2g-1} \ c_{2g} \ \alpha_0\alpha_2c_{2g} \ \alpha_0\alpha_2c_{2g-1})$, and the *non-orientable canonical map* of genus g has vertex permutation with a cycle of the form

$$(c_1 \ \alpha_0c_1 \ c_2 \ \alpha_0c_2 \ \dots \ c_g \ \alpha_0c_g).$$

As remarked after Definition 32, Tutte defines an operation of vertex splitting [19, Figs. X.7.1 and X.7.2] related to ours in following our vertex splitting operation by the insertion of a link; this makes his operation inverse to that of contracting a link. Thus, to realize Tutte's vertex splitting operation we take the inverse sequence of four operations above that realized the operation of link contraction. In other words, start with the inverse of step (4) above (split an edge to make a duplicate edge), follow this by the inverse of step (3) (split an endpoint of the duplicate edge to make a leaf vertex), then carry out the inverse of step (2) (glue the leaf vertex to the vertex incident to it), finishing with the inverse of step (1) (split the endpoint of the loop). Tutte showed [19, Theorem X.37] that, using vertex splittings in his sense and link contractions, one can transform any unitary map into a canonical map while maintaining the signed genus.⁸ Hence the same is true when using vertex gluings and splittings in our sense, along with edge splittings and duplicate edge

⁷ When $g = 0$ the canonical map is an isolated vertex and the vertex permutation consists of a pair of empty cycles; as Tutte does not allow edgeless maps, he defines the canonical map of genus zero to be the map consisting of a single link.

⁸ The crosses belonging to a *cross cap* (twisted loop appearing as a pair c, α_0c in a cycle of the vertex permutation) or to a *handle* (two interlaced non-twisted loops $a, b, \alpha_0\alpha_2a, \alpha_0\alpha_2b$ in a cycle of the vertex permutation) appear consecutively in the canonical map, or, to use Tutte's term, they are *assembled*: Tutte in this way gives a combinatorial proof of the classification theorem for compact surfaces.

gluings, as these operations can simulate the two operations used by Tutte in the way described above.

Finally, then, we compose the sequence of operations that reduces M to the canonical map of its signed genus with the inverse of the sequence of operations reducing M' to the same canonical map, which takes the canonical map to M' . In this way we move from M to M' via the canonical map of their signed genus by a sequence of operations of the type described in the lemma statement. \square

Theorem 45. *The poset $(\mathcal{M}_c(g), \leq)$ is connected.*

Proof. By Lemma 44, for each pair of maps $M, M' \in \mathcal{M}(g)$ with at least one edge there is a sequence of maps $M_0 = M, M_1, M_2, \dots, M_k = M'$ such that there is either a homomorphism from M_i to M_{i+1} (when M_{i+1} is obtained from M_i by vertex gluing or duplicate edge gluing) or from M_{i+1} to M_i (when M_{i+1} is obtained from M_i by vertex splitting or edge splitting); all the operations are the ones described in Lemma 44. With the additional observation that the single vertex can be mapped everywhere in the case of $g = 0$, this shows that the preorder $(\mathcal{M}(g), \leq)$ is connected; then, by considering the corresponding homomorphisms between their cores, we obtain that the poset $(\mathcal{M}_c(g), \leq)$ is also connected. \square

We recall some further definitions from poset theory. Let a, b be elements of a poset $(P, <)$ such that $a < b$. The poset P is *dense* between a and b if for any $c, d \in P$ with $a \leq c < d \leq b$, there exists an element e such that $c < e < d$; the pair a, b forms a *gap* in P if there is no element c with $a < c < b$ (alternatively, b covers a , i.e., b is an immediate successor of a). In the poset of (cores of) graphs with order $G \leq H$ defined by the existence of a graph homomorphism $G \rightarrow H$, each comparable pair of cores (with the exception of the one between the vertex and K_2) defines an interval that is a dense total order (actually, such interval is universal, in the sense that every countable partial order can be seen as one of its suborders [5]). The following theorem shows that the partial order of map cores is drastically different from that of graph cores.

Theorem 46. *In the partial order $(\mathcal{M}_c(g), \leq)$, there is no pair of cores N_1, N_2 with $N_1 < N_2$ and a dense total order between them.*

Proof. Suppose for a contradiction that there is a dense total order between two cores $N_1, N_2 \in \mathcal{M}_c(g)$, and assume first that both are connected. Thus, there are infinitely many cores in between N_1 and N_2 , and there must exist two cores $N_3, N_4 \in \mathcal{M}_c(g)$ with $N_1 < N_3 < N_4 < N_2$ such that every core between them (including them) have the same number of odd-degree faces,⁹ and there is a dense total order between them. Let $\mathcal{N}_{3,4}$ be the set of cores between N_3 and N_4 .

Claim. There exist two cores $N_5, N_6 \in \mathcal{N}_{3,4}$ with $N_5 < N_6$ such that:

- there is a dense total order of cores from $\mathcal{N}_{3,4}$ in between N_5 and N_6 , and
- for each pair of cores $N' < N''$ in between N_5 and N_6 (including them) in the total order, there is a homomorphism $N' \rightarrow N''$ that induces a one-to-one correspondence between faces of odd degree and preserves their degree.

Proof of the claim. Let $N, \tilde{N} \in \mathcal{N}_{3,4}$ with $N < \tilde{N}$, and let $h : N \rightarrow \tilde{N}$. The image under h of each odd-degree face of N is a face that induces a prefacial cross-circuit κ in \tilde{N} of length smaller or equal than its degree (here we use that $h(N)$ is a submap of \tilde{N} and Lemma 37; note that the image of a face does not induce a prefacial cross-circuit, but rather a union of prefacial cross-circuits, we select one of them). Each of these κ 's contains exactly one face z_κ of odd degree in \tilde{N} as, by Observation 39, it should contain at least one, and it cannot contain more since the number of odd faces in \tilde{N} would be strictly larger than that in N , a contradiction with $N, \tilde{N} \in \mathcal{N}_{3,4}$. Since \tilde{N} is a core, by Theorem 42,

⁹ N_2 has a finite number of odd-degree faces, and the total number of odd-degree faces is non-decreasing when taking homomorphic images: removal of parallel edges preserves their degree, while vertex gluing always produces one odd-degree face from an odd-degree face, and either zero or two odd-degree faces from an even-degree face.

either $\phi_{z_\kappa} = \kappa$, or the degree of z_κ is strictly smaller than $\ell(\kappa)$. As there are infinitely many cores in $\mathcal{N}_{3,4}$, totally ordered, these degrees stabilize between two cores, which are the desired N_5, N_6 . These cores N_5, N_6 can be further assumed to contain a dense total order between them, as there is a dense total order between N_3 and N_4 . \square

Let $\mathcal{N}_{5,6}$ be the set of cores totally ordered in between N_5 and N_6 . We next prove that there are two cores in $\mathcal{N}_{5,6}$ with finitely many cores in between them, which contradicts the assumption that there is a dense total order between N_1 and N_2 . Let $h : N \rightarrow N'$ where $N < N'$ are two cores in $\mathcal{N}_{5,6}$. Each even-length facial walk λ of N turns into several even-length facial walks in $h(N) \subseteq N'$ whose total length is at most $\ell(\lambda)$ (it may happen that the face completely disappears); this follows from the fact that no even-degree face can be subdivided (applying the sequence of vertex and duplicate edge gluings) to create two odd-degree faces. These even-degree faces in $h(N)$ become prefacial cross-circuits κ in the core N' . If κ is not a facial walk, by Theorem 42, the faces in the interior of κ have even degree strictly smaller than $\ell(\kappa)$. Therefore, when we consider a homomorphism from N_5 into a core in $\mathcal{N}_{5,6}$, its image is obtained by either reducing the number of faces (gluing duplicated edges) or subdividing even-degree faces into smaller even-degree faces (gluing vertices); one can place within these faces a plane submap with all the faces of strictly smaller even degree (we use that the codomain is another core, and Theorem 42 where the prefacial cross-circuit κ is a part of a facial walk from N_5). Since there is an absolute minimum on the degree of a face, this process cannot be performed infinitely many times.¹⁰ Hence, at some point, the number and the degree of the even-degree faces is stabilized. Since we have also stabilized the number of odd-degree faces and their respective degree, and there is only a finite number of maps with such parameters, we conclude that the interval between N_5 and N_6 is not dense. This finishes the proof when the cores N_1, N_2 are connected.

Suppose now that the cores N_1, N_2 are not connected; we argue similarly on the connected components as they are cores and a map homomorphism can only merge them. Since the signed genus is fixed and the Euler genus is additive along connected components, up to plane connected components, we can find two cores N_3, N_4 with $N_1 < N_3 < N_4 < N_2$ such that the profile of connected components with different signed genera (different from 0) is the same, and it is the same with all the cores in-between. Regarding the plane connected components, we next use an argument on the number of odd-degree faces.

As mentioned in the applications of Theorem 42, each of the plane connected components either has at least three odd-degree faces or it is a plane odd cycle or it is bipartite (all faces of even length). In the latter case, the core is an edge, which can be mapped anywhere else. We also know that there cannot be more than one plane odd cycle among the connected components since otherwise one could be mapped into another and the whole map would not be a core. Furthermore, a homomorphism between cores of $\mathcal{N}_{3,4}$ cannot merge one plane connected component into a non-plane one, since that would strictly increase the number of odd-degree faces of the non-plane component (each plane component besides, possibly, the odd cycle, has at least three odd-degree faces, while merging two connected components can reduce the total number of odd-degree faces by at most two). Thus, as the overall number of odd-degree faces is bounded in N_4 , the number of plane connected components is also bounded for each of the cores in $\mathcal{N}_{3,4}$. Therefore, there exist $N_5, N_6 \in \mathcal{N}_{3,4}$ with $N_5 < N_6$ that have the same profile of connected components and odd-degree faces (the genera of the components, and the number of odd-degree faces on each of the connected components), and with a dense total order between them. Applying the same reasoning as in the connected case, we find two cores in $\mathcal{N}_{5,6}$ with finitely many cores in between them. \square

Corollary 47. *In the preorder $(\mathcal{M}(g), \leq)$, there is no pair of maps M_1, M_2 with $M_1 < M_2$ and a dense total order between them.*

¹⁰ When fixing the image of a homomorphism, the number of faces themselves is finite, even though we could fill a given face with arbitrarily many configurations that are cores using faces of strictly smaller degree; this arbitrary number will be fixed again, and all those inner faces will have strictly smaller degree. Thus, this procedure cannot be repeated infinitely many times once that arbitrary number is fixed and a particular collection of cores is chosen.

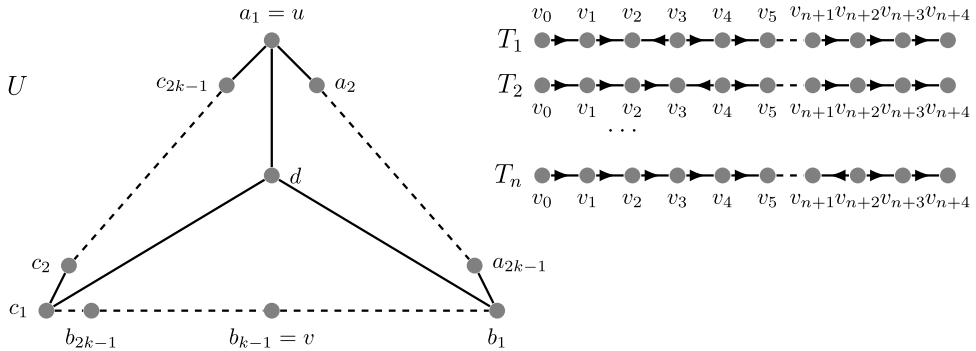


Fig. 10. Copy of the graph U , and the dimaps T_i .

Proof. This follows from the fact that a dense total order between two maps would imply, by Proposition 28 part (iv), a dense total order between their cores; here we also use that if $M_1 < M_2$, then their cores are different. \square

Let us now exhibit an infinite chain of map cores. For the remainder of this section we use C_i to denote the plane map of a graph cycle of length i . Let $T_{i,j}$ be the plane map obtained by adding to C_4 a path of length i between the first and the third vertex, and a path of length j between the second and the fourth vertex (each of these paths subdivides one of the faces induced by the C_4 on the plane). Thus, $T_{i,j}$ has four faces, two of length $i + 2$ and two of length $j + 2$. Since $T_{2k+1, 2s+1}$ has all faces of odd degree and no vertices of degree one, then $T_{2k+1, 2s+1}$ is a core for all $k, s \geq 0$. By folding one edge appropriately, we have $T_{2k+1, 2s+1} \rightarrow T_{2k+1, 2(s-1)+1}$ for $s \geq 1$ and $k \geq 0$; symmetrically, $T_{2k+1, 2s+1} \rightarrow T_{2(k-1)+1, 2s+1}$ for $s \geq 0$ and $k \geq 1$. As $T_{i,3}$ has a face of length 5, we conclude that $C_5 \rightarrow \dots \rightarrow T_{2s+1,3} \rightarrow T_{2(s-1)+1,3} \rightarrow \dots \rightarrow T_{3,3} \rightarrow T_{1,3}$ is an infinite (ordered) chain of cores.

We can also find an infinite chain of gaps by using that a plane connected map that is a core is bipartite (an edge or an isolated vertex) or a cycle of odd length or has at least four faces of odd degree, and that the codomain of a map homomorphism must have at least as many odd-degree faces as the domain.

Theorem 48. Let C_i denote the plane map of the cycle graph of length i . For each $k \geq 1$, $C_{2k+3} \rightarrow C_{2k+1}$ and there is a gap between C_{2k+3} and C_{2k+1} .

For each signed genus g , we obtain an infinite antichain in $\mathcal{M}_c(g)$ and $\mathcal{M}(g)$ by considering maps with an increasing number of odd-degree faces, and an increasing degree for each of these faces. In [10] the authors consider, for each graph H , the family $\mathcal{G}(H) = \{G \mid G \text{ is a graph and } G \rightarrow H\}$ and show the existence of a graph H_0 for which $\mathcal{G}(H_0)$ contains an infinite antichain [10, Theorem 6] (the argument uses only planar graphs); we next present a similar construction to find arbitrarily large antichains in the posets given by $\mathcal{F}(H) = \{M \mid M \text{ is a map and } M \rightarrow H\}$.

Theorem 49. For every even n , and odd $k \geq 3$, there exist plane maps B and A_1, \dots, A_n such that

- the underlying graphs of A_1, \dots, A_n are graph cores, and B is a map core,
- each of B and A_1, \dots, A_n have underlying graph of odd girth $2k + 1$,
- $A_i \rightarrow B$ for each $i \in [n]$, and $A_i \not\rightarrow A_j$ for distinct $i, j \in [n]$.

Proof. We slightly adapt the arguments from [10, Theorem 6]; see Figs. 10 and 11 for illustrations of the constructions.

Let U be a graph with vertices $\{a_1, a_2, \dots, a_{2k-1}, b_1, b_2, \dots, b_{2k-1}, c_1, c_2, \dots, c_{2k-1}, d\}$ where there is a cycle $(a_1, a_2, \dots, a_{2k-1}, b_1, b_2, \dots, b_{2k-1}, c_1, c_2, \dots, c_{2k-1}, a_1)$ and d is adjacent to b_1, c_1, a_1 . Now use the label u for a_1 and v for b_{k-1} . A plane embedding of U is a map core since all the

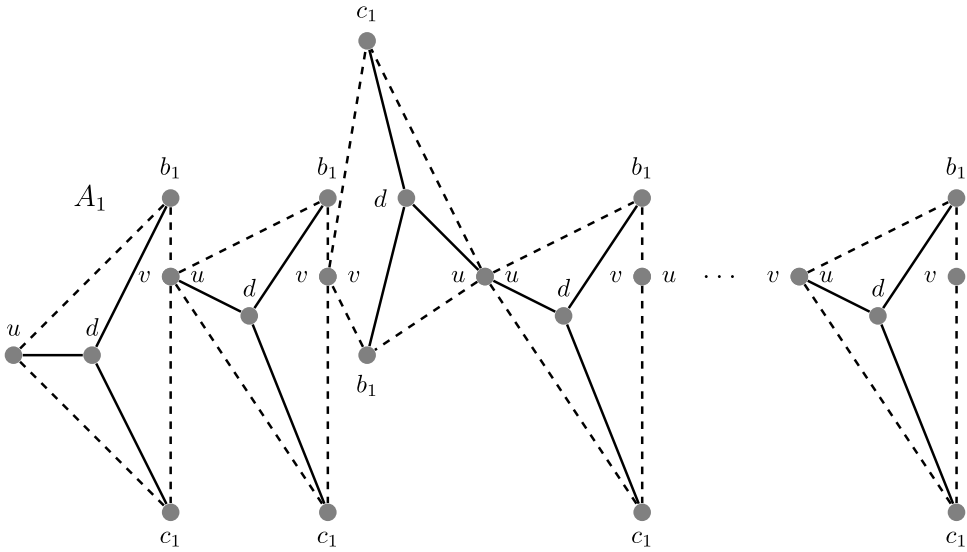


Fig. 11. Exchanging edges of T_1 by copies of U to create A_1 . The map B is obtained by gluing the first u and the last v , along the exterior face.

faces have odd degree. It is also a graph core as each odd cycle should be mapped to an odd cycle, and a cycle of length the odd girth should be mapped to a cycle of length the odd girth; two of the cycles cannot be mapped to a single one as otherwise the third odd-girth-length cycle would be shortened.

Let T_i , $1 \leq i \leq n$, be a path with $n + 4$ directed edges, all of them oriented in one direction but the $i + 2$ -th edge that is oriented in the opposite direction. Let G_i be the graph obtained by replacing each edge of T_i by a copy of U , placing vertex u as the tail of the edge and v as the head. Now, consider the plane embeddings of the graphs G_i , denoted by A_i , where all the vertices b_1 of the respective triangles are “facing up” except the b_1 corresponding to the copy of U given by the backward edge, which is “facing down”. The plane map B is formed by gluing the first vertex u with the last vertex v . Thus, B consists of $n + 4$ copies of U with all but one vertex b_1 to the outside of the circle, and one of the vertices b_1 to the inside. It is clear that $A_i \rightarrow B$, but by the argument in [10, Theorem 6] all the graphs G_i are graph cores and $A_i \not\rightarrow A_j$ (indeed, assume not, since U is a core, each copy of U in A_i should be mapped to a copy of U , vertices u in different copies should be mapped among them as well as vertices v , since v is located slightly to one side along an odd-girth-length cycle; we would thus have a graph homomorphism between T_i and T_j , a contradiction) and thus A_i are map cores. Further, for n even, each face of B has odd degree, and thus it is a map core. This follows from the fact that each face of a plane embedding of U has odd degree, and so the path uv has odd length on one side and even length on the other; in particular, by joining an even number of U 's together, and merging the first u and the last v as in A_i , we obtain two faces, both of odd degree (the facial walk of one face is obtained as the addition of an odd number of odd-length paths plus an even-length path, and the facial walk of the other face is obtained as the addition of an odd number of even-length paths plus an odd-length path). \square

In [10, Theorem 6], the graph codomain of the antichain is the odd cycle of length $2k + 1$, which is a (graph) core and does not depend on n , but only on k . However, since the number of odd faces of A_1, \dots, A_n increases with n , when using the same construction the map core B , codomain of the antichain, also depends on n . Thus, we can find arbitrarily large antichains, but not an infinite one. This raises the following question:

Is there a map B for which $\mathcal{F}(B)$ contains an infinite antichain?

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References

- [1] R.P. Bryant, D. Singerman, Foundations of the theory of maps on surfaces with boundary, *Quart. J. Math. Oxford Ser. (2)* 36 (141) (1985) 17–41.
- [2] S. Cabello, B. Mohar, Finding shortest non-separating and non-contractible cycles for topologically embedded graphs, *Discrete Comput. Geom.* 37 (2007) 213–235.
- [3] G. Damiand, P. Lienhardt, Removal and contraction for n -dimensional generalized maps, in: I. Nyström, G. Sanniti di Baja, S. Svensson (Eds.), *Discrete Geometry for Computer Imagery*, in: LNCS, vol. 2886, 2003, pp. 408–419.
- [4] J. Edmonds, On the surface duality of linear graphs, *J. Res. Nat. Bur. Standards Sect. B* 69 (5) (1965) 121–123.
- [5] J. Fiala, J. Hubička, Y. Long, J. Nešetřil, Fractal property of the graph homomorphism order, *European J. Combin.* 66 (2017) 101–109, Selected papers of EuroComb15.
- [6] M. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphism of graphs, *J. Amer. Math. Soc.* 20 (1) (2007) 37–51.
- [7] D. Garijo, A. Goodall, J. Nešetřil, Distinguishing graphs by their left and right homomorphism profiles, *European J. Combin.* 32 (7) (2011) 1025–1053.
- [8] A. Goodall, T. Krajewski, G. Regts, L. Vena, A Tutte polynomial for maps, *Combin. Probab. Comput.* 27 (6) (2018) 913–945.
- [9] A. Goodall, B. Litjens, G. Regts, L. Vena, A Tutte polynomial for maps II: the non-orientable case, *European J. Combin.* 86 (2020) 103095, 32.
- [10] P. Hell, J. Nešetřil, The core of a graph, *Discrete Math.* 109 (1–3) (1992) 117–126.
- [11] P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, in: Oxford Lecture Series in Mathematics and Its Applications, Oxford Univ. Press, 2004.
- [12] G.A. Jones, Graph embeddings, groups and Riemann surfaces, in: *Algebraic Methods in Graph Theory I, II*, Szeged 1978, in: Colloquium of Mathematical Society János Bolyai, vol. 25, North-Holland Publishing Co., 1981.
- [13] P. Lienhardt, Topological models for boundary representation: a comparison with n -dimensional generalized maps, *Comput. Aided Des.* 23 (1) (1991) 59–82.
- [14] P. Lienhardt, n -dimensional generalized combinatorial maps and cellular quasi-manifolds, *Int. J. Comput. Geom. Appl.* 4 (3) (1994) 275–324.
- [15] B. Litjens, B. Sevenster, Partition functions and a generalized coloring-flow duality for embedded graphs, *J. Graph Theory* 88 (2) (2018) 271–283.
- [16] A. Malnič, R. Nedela, M. Škoviera, Regular homomorphisms and regular maps, *European J. Combin.* 23 (4) (2002) 449–461.
- [17] B. Mohar, C. Thomassen, *Graphs on Surfaces*, J. Hopkins Univ. Press, 2001.
- [18] W.T. Tutte, What is a map? in: *New Directions in the Theory of Graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, 1973, pp. 309–325.
- [19] W.T. Tutte, *Graph Theory*, in: *Encyclopedia of Mathematics and its Applications*, vol. 21, Cambridge Univ. Press, 2001, p. xxii+335.