# CROSS PRODUCTS, AUTOMORPHISMS, AND GRADINGS 

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#### Abstract

The affine group schemes of automorphisms of the multilinear $r$ fold cross products on finite-dimensional vectors spaces over fields of characteristic not two are determined. Gradings by abelian groups on these structures, that correspond to morphisms from diagonalizable group schemes into these group schemes of automorphisms, are completely classified, up to isomorphism.


## 1. Introduction

Eckmann Eck43 defined a vector cross product on an $n$-dimensional real vector space $V$, endowed with a (positive definite) inner product $\mathrm{b}(u, v)$, to be a continuous map

$$
X: V^{r} \longrightarrow V \quad(1 \leq r \leq n)
$$

satisfying the following axioms:

$$
\begin{align*}
& \mathrm{b}\left(X\left(v_{1}, \ldots, v_{r}\right), v_{i}\right)=0, \quad 1 \leq i \leq r  \tag{1.1}\\
& \mathrm{~b}\left(X\left(v_{1}, \ldots, v_{r}\right), X\left(v_{1}, \ldots, v_{r}\right)\right)=\operatorname{det}\left(\mathrm{b}\left(v_{i}, v_{j}\right)\right) \tag{1.2}
\end{align*}
$$

There are very few possibilities.
Theorem 1.1 ( Eck43, Whi63]). A vector cross product exists in precisely the following cases:

- $n$ is even, $r=1$,
- $n \geq 3, r=n-1$,
- $n=7, r=2$,
- $n=8, r=3$.

Multilinear vector cross products $X$ on vector spaces $V$ over arbitrary fields of characteristic not two, relative to a nondegenerate symmetric bilinear form $\mathrm{b}(u, v)$, were classified by Brown and Gray [BG67]. These are the multilinear maps $X$ : $V^{r} \rightarrow V(1 \leq r \leq n)$ satisfying (1.1) and (1.2). The possible pairs $(n, r)$ are again those in Theorem 1.1.

The exceptional cases: $(n, r)=(7,2)$ and $(8,3)$, are intimately related to the octonion, or Cayley, algebras.

These multilinear vector cross products have become important tools in Differential Geometry or Nonassociative Algebras (see, for instance, Kar05, EKO05).

[^0]In particular, they are closely related to the exceptional classical simple Lie superalgebras: $G(3), F(4), D(2,1 ; \alpha)$ (KO03).

An elementary account on multilinear cross products, and of their connections with the exceptional basic classical simple Lie superalgebras can be found in Eld04.

The nondegenerate symmetric bilinear form $\mathrm{b}(u, v)$ is part of the definition given by Eckmann and Brown and Gray. However, this is not, in general, uniquely determined. Therefore, we will make use of the following definition.

Definition 1.2. Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$ of characteristic not two, and let $r$ be a natural number with $1 \leq r \leq n\left(n=\operatorname{dim}_{\mathbb{F}} V\right)$.

An $r$-fold cross product $X$ on $V$ is a multilinear map

$$
X: V^{r} \longrightarrow V
$$

such that there is a nondegenerate symmetric bilinear form $\mathrm{b}: V \times V \rightarrow \mathbb{F}$ satisfying conditions (1.1) and (1.2).

In this situation we will say that $X$ is a cross product on $V$ relative to b , or that b admits the $r$-fold cross product $X$.

Alternatively, we will simply say that $(V, X)$ (or $(V, X, \mathrm{~b})$, if b is fixed) is a cross product.

Two such pairs $\left(V_{1}, X_{1}\right)$ and $\left(V_{2}, X_{2}\right)$ are said to be isomorphic if the rank $r$ is the same in both cases and there is a linear isomorphism $\varphi: V_{1} \rightarrow V_{2}$ such that $\varphi\left(X_{1}\left(v_{1}, \ldots, v_{r}\right)\right)=X_{2}\left(\varphi\left(v_{1}\right), \ldots, \varphi\left(v_{r}\right)\right)$ for any $v_{1}, \ldots, v_{r} \in V_{1}$; while two such triples $\left(V_{1}, X_{1}, \mathrm{~b}_{1}\right)$ and ( $V_{2}, X_{2}, \mathrm{~b}_{2}$ ) are said to be isomorphic if there is a linear isometry $\varphi:\left(V_{1}, \mathrm{~b}_{1}\right) \rightarrow\left(V_{2}, \mathrm{~b}_{2}\right)$ such that $\varphi\left(X\left(v_{1}, \ldots, v_{r}\right)\right)=X\left(\varphi\left(v_{1}\right), \ldots, \varphi\left(v_{r}\right)\right)$ for any $v_{1}, \ldots, v_{r} \in V_{1}$.

Therefore, given a cross product $X$ on the vector space $V$ relative to the bilinear form b , we will consider two automorphism groups, depending on whether the bilinear form b is considered a part of the definition:

$$
\begin{aligned}
& \operatorname{Aut}(V, X):=\{\varphi \in \mathrm{GL}(V) \mid \\
&\left.\varphi\left(X\left(v_{1}, \ldots, v_{r}\right)\right)=X\left(\varphi\left(v_{1}\right), \ldots, \varphi\left(v_{r}\right)\right) \forall v_{1}, \ldots, v_{r} \in V\right\}, \\
& \operatorname{Aut}(V, X, \mathrm{~b}):=\operatorname{Aut}(V, X) \cap \mathrm{O}(V, \mathrm{~b})
\end{aligned}
$$

where $\mathrm{GL}(V)$ denotes the general linear group of $V$, and $\mathrm{O}(V, \mathrm{~b})$ the orthogonal group of $(V, \mathrm{~b})$. The last $\operatorname{group} \operatorname{Aut}(V, X, \mathrm{~b})$ is the one considered in BG67.

More generally, we will consider the corresponding affine group schemes, which will be treated in a functorial point of view (see Wat79] or [KMRT98, Chapter VIII]). That is, for any unital, associative, commutative $\mathbb{F}$-algebra $R$, the corresponding group of $R$-points are the following:
$\boldsymbol{\operatorname { A u t }}(V, X)(R)=\{\varphi \in \mathbf{G L}(V)(R) \mid$

$$
\left.\varphi\left(X_{R}\left(v_{1}, \ldots, v_{r}\right)\right)=X_{R}\left(\varphi\left(v_{1}\right), \ldots, \varphi\left(v_{r}\right)\right) \forall v_{1}, \ldots, v_{r} \in V_{R}\right\}
$$

$\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})(R)=\boldsymbol{\operatorname { A u t }}(V, X)(R) \cap \mathbf{O}(V, \mathrm{~b})(R)$,
where $V_{R}=V \otimes_{\mathbb{F}} R, X_{R}$ denotes the scalar extension of $X$ to $V_{R}$, and $\mathbf{G L}(V)$ and $\mathbf{O}(V, \mathrm{~b})$ denote the general linear affine group scheme and the orthogonal group scheme attached to $V$ and b .

All vector spaces considered from now on will be assumed to be finite-dimensional and defined over a ground field $\mathbb{F}$ of characteristic not two.

The paper is organized as follows. Section 2 will be devoted to reviewing the basic results and examples of cross products and will answer the question as to what extent the form b is uniquely determined by the cross product. In section 3 the affine group schemes of automorphisms of $r$-fold cross products will be determined.

In the most interesting cases, these are simply connected algebraic groups of types $G_{2}$ and $B_{3}$. In case $r=\operatorname{dim}_{\mathbb{F}} V-1$, these group schemes are close to the special orthogonal group scheme, but they may fail to be smooth. Finally, Section 4 will be devoted to classifying gradings by abelian groups, up to isomorphism, on cross products. Fine gradings, up to equivalence, will be classified too, and the associated Weyl groups will be computed. In future work, these classifications will be used to study gradings by abelian groups on the exceptional simple basic classical Lie superalgebras.

## 2. Cross products

In this section, some results on cross products will be reviewed in a way suitable for our purposes. The problem of the uniqueness of the associated bilinear form will be tackled too.

If $(V, X, \mathrm{~b})$ is an $r$-fold cross product, then it is clear that for any nonzero scalars $\alpha, \beta \in \mathbb{F}$, with $\beta^{r-1}=\alpha^{2},(V, \alpha X, \beta \mathrm{~b})$ is an $r$-fold cross product too.

Let us review the main examples of cross products. First, let us recall from BG67, §3] the notion of star operator. Let b be a nondegenerate symmetric bilinear form of discriminant 1 on the vector space $V$ of dimension $n$. Then b extends to the exterior algebra $\bigwedge V$ as follows:

$$
\mathrm{b}\left(u_{1} \wedge \cdots \wedge u_{p}, v_{1} \wedge \cdots \wedge v_{q}\right)=\left\{\begin{array}{l}
\operatorname{det}\left(\mathrm{b}\left(u_{i}, v_{j}\right)\right) \quad \text { if } p=q \\
0 \quad \text { otherwise } .
\end{array}\right.
$$

As the discriminant is 1 , there exists an element $\omega \in \bigwedge^{n} V$, unique up to sign, such that $\mathrm{b}(\omega, \omega)=1$. The star operator relative to b and $\omega$ is the linear map * $: \bigwedge V \rightarrow \bigwedge V$, such that ${ }^{*}\left(\bigwedge^{p} V\right)=\bigwedge^{n-p} V, 0 \leq p \leq n$, defined by

$$
\mathrm{b}\left({ }^{*} x, y\right)=\mathrm{b}(x \wedge y, \omega)
$$

for any $0 \leq p \leq n, x \in \bigwedge^{p} V, y \in \bigwedge^{n-p} V$.
In this case, the multilinear map

$$
\begin{aligned}
X: V^{n-1} & \longrightarrow V \\
\left(v_{1}, \ldots, v_{n-1}\right) & \mapsto{ }^{*}\left(v_{1} \wedge \cdots \wedge v_{n-1}\right)
\end{aligned}
$$

is an $(n-1)$-fold cross product on $V$ relative to b .
Let now $\mathcal{C}$ be a Cayley algebra with norm $n$. That is, $\mathcal{C}$ is an eight-dimensional unital nonassociative algebra endowed with a nondegenerate quadratic form $n$ admitting composition:

$$
\mathrm{n}(x y)=\mathrm{n}(x) \mathrm{n}(y)
$$

for all $x, y \in \mathcal{C}$. Any $x \in \mathcal{C}$ satisfies the Cayley-Hamilton equation:

$$
\begin{equation*}
x^{2}-\mathrm{n}(x, 1) x+\mathrm{n}(x) 1=0 \tag{2.1}
\end{equation*}
$$

where $\mathrm{n}(x, y)=\mathrm{n}(x+y)-\mathrm{n}(x)-\mathrm{n}(y)$ is the polar form of n . Write $\mathrm{b}_{\mathrm{n}}(x, y)=$ $\frac{1}{2} \mathrm{n}(x, y)$. The linear map $x \mapsto \bar{x}:=\mathrm{n}(x, 1) 1-x$ is the canonical involution of $\mathcal{C}$. It satisfies the equations

$$
\begin{equation*}
\mathrm{n}(x y, z)=\mathrm{n}(y, \bar{x} z)=\mathrm{n}(x, z \bar{y}) \tag{2.2}
\end{equation*}
$$

for all $x, y, z \in \mathcal{C}$. Any two elements in $\mathcal{C}$ generate an associative subalgebra.
Let $\mathcal{C}_{0}$ be the subspace orthogonal to the unity $1: \mathcal{C}_{0}=\{x \in \mathcal{C} \mid \mathrm{n}(x, 1)=0\}$. For any $x, y \in \mathcal{C}_{0}$, the product $x y$ in $\mathcal{C}$ splits as:

$$
\begin{equation*}
x y=-\mathrm{b}_{\mathrm{n}}(x, y) 1+x \times y \tag{2.3}
\end{equation*}
$$

for some $x \times y \in \mathcal{C}_{0}$. Then $\times$ is an anticommutative multiplication on $\mathcal{C}_{0}$ that satisfies:

$$
\begin{equation*}
(x \times y) \times y=\mathrm{b}_{\mathrm{n}}(x, y) y-\mathrm{b}_{\mathrm{n}}(y, y) x \tag{2.4}
\end{equation*}
$$

for all $x, y \in \mathcal{C}_{0}$ (see, e.g., KMRT98, Chapter VIII] or [EK13, Chapter 4]).
Moreover, $\left(\mathrm{C}_{0}, X^{\mathrm{C}_{0}}, \mathrm{~b}_{\mathrm{n}}\right)$ is a 2-fold cross product, with $X^{\mathrm{C}_{0}}(x, y)=x \times y$.
On the other hand, $\left(\mathcal{C}, X_{\epsilon}^{\mathcal{C}}, \mathrm{b}_{\mathrm{n}}\right)$, with $\epsilon= \pm 1$, where $X_{\epsilon}^{\mathcal{C}}$ is given by the formulas:

$$
\begin{align*}
X_{1}^{\mathrm{e}}(x, y, z) & =(x \bar{y}) z-\mathrm{b}_{\mathrm{n}}(x, y) z-\mathrm{b}_{\mathrm{n}}(y, z) x+\mathrm{b}_{\mathrm{n}}(x, z) y,  \tag{2.5}\\
X_{-1}^{\mathrm{e}}(x, y, z) & =x(\bar{y} z)-\mathrm{b}_{\mathrm{n}}(x, y) z-\mathrm{b}_{\mathrm{n}}(y, z) x+\mathrm{b}_{\mathrm{n}}(x, z) y, \tag{2.6}
\end{align*}
$$

is a 3 -fold cross product on $\mathcal{C}$ (see [BG67, Theorem 5.1]).
Moreover, the following holds for all $u_{i}, v_{i} \in \mathcal{C}, 1 \leq i \leq 3$ (see Eld96, Proposition 3]):

$$
\begin{align*}
& \mathrm{b}_{\mathrm{n}}\left(X_{\epsilon}^{\mathrm{C}}\left(u_{1}, u_{2}, u_{3}\right), X_{\epsilon}^{\mathrm{C}}\left(v_{1}, v_{2}, v_{3}\right)\right)=\operatorname{det}\left(\mathrm{b}_{\mathrm{n}}\left(u_{i}, v_{j}\right)\right) \\
& \quad+\epsilon \sum_{\sigma \text { even } \tau \text { even }} \sum_{\mathrm{b}_{\mathrm{n}}\left(u_{\sigma(1)}, v_{\tau(1)}\right) \mathrm{b}_{\mathrm{n}}\left(u_{\sigma(2)}, X_{\epsilon}^{\mathrm{C}}\left(u_{\sigma(3)}, v_{\tau(2)}, v_{\tau(3)}\right)\right)} \tag{2.7}
\end{align*}
$$

( $\epsilon$ is either 1 or -1 , and the sums are over the even permutations of $1,2,3$.)
Now we have all the ingredients to review the classification of cross products. The trace of a linear operator $X$ will be denoted by $\operatorname{tr}(X)$.

Theorem 2.1. Let $X$ be an r-fold cross product on a vector space $V$ of dimension $n, 1 \leq r \leq n$. Then one, and only one, of the following conditions holds:
(i) $n$ is even, $r=1, X^{2}=-\mathrm{id}$, and $\operatorname{tr}(X)=0$.
(ii) $n \geq 3, r=n-1$, and

$$
X\left(v_{1}, \ldots, v_{n-1}\right)={ }^{*}\left(v_{1} \wedge \cdots \wedge v_{n-1}\right)
$$

for all $v_{1}, \ldots, v_{n-1} \in V$, where $*$ is the star operator relative to a nondegenerate symmetric bilinear form b of discriminant 1 and an element $\omega \in \bigwedge^{n} V$ with $\mathrm{b}(\omega, \omega)=1$.
(iii) $n=7, r=2$, and $(V, X)$ is isomorphic to $\left(\mathcal{C}_{0}, X^{\mathrm{C}_{0}}\right)$ for a Cayley algebra C.
(iv) $n=8, r=3$, and $(V, X)$ is isomorphic to $\left(\mathcal{C}, \alpha X_{1}^{\mathcal{E}}\right)$ for a Cayley algebra $\mathcal{C}$ and $a$ nonzero scalar $\alpha \in \mathbb{F}$.
Conversely, all the pairs ( $V, X$ ) in items (i)-(iv) are cross products.
Proof. With the exception of a few details, everything follows from BG67. In particular, it is proved in BG67 that the only possible pairs $(n, r)$ are $(2 s, 1)$, $(n, n-1),(7,2)$ and $(8,3)$.

If $r=1$ and $X$ is a 1 -fold cross product relative to the bilinear form b , then $X$ satisfies $\mathrm{b}(X(u), u)=0$ and $\mathrm{b}(X(u), X(v))=\mathrm{b}(u, v)$ for all $u, v \in V$. That is, $X$ is both a skew-symmetric transformation and an isometry relative to b . If $X^{*}$ denotes the adjoint of the endomorphism $X$ relative to b , then we have $X^{*}=-X$ and $X X^{*}=\mathrm{id}$, and this implies $X^{2}=-\mathrm{id}$ and $\operatorname{tr}(X)=0$.

Conversely, let $X: V \rightarrow V$ be an endomorphism such that $X^{2}=-\mathrm{id}$ and $\operatorname{tr}(X)=0$. If $-1 \in \mathbb{F}^{2}$ and we pick $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^{2}=-1$, then $V=V_{+} \oplus V_{-}$with $V_{ \pm}=\{v \in V \mid X(v)= \pm \mathbf{i} v\}$. The condition $\operatorname{tr}(X)=0$ gives $\operatorname{dim}_{\mathbb{F}} V_{+}=\operatorname{dim}_{\mathbb{F}} V_{-}$so $n$ is even and $X$ is a 1 -fold cross product relative to any nondegenerate symmetric bilinear form $\mathrm{b}: V \times V \rightarrow \mathbb{F}$ with $\mathrm{b}\left(V_{+}, V_{+}\right)=0=\mathrm{b}\left(V_{-}, V_{-}\right)$.

If $-1 \notin \mathbb{F}^{2}$, then the subalgebra $\mathbb{K}=\mathbb{F i d} \oplus \mathbb{F} X$ of $\operatorname{End}_{\mathbb{F}}(V)$ is a field, and $V$ is a vector space over $\mathbb{K}$. For any $\mathbb{K}$-basis $\left\{v_{1}, \ldots, v_{s}\right\}$ of $V, n=2 s$ and $X$ is a 1 -fold cross product relative to the bilinear form b defined so that the $\mathbb{F}$-basis
$\left\{v_{1}, X\left(v_{1}\right), \ldots, v_{s}, X\left(v_{s}\right)\right\}$ is orthogonal and $\mathrm{b}\left(v_{i}, v_{i}\right)=\mathrm{b}\left(X\left(v_{i}\right), X\left(v_{i}\right)\right)=1$ for all $i$.

The case $n \geq 3, r=n-1$ is proved in BG67. Theorem 3.3]. For $n=7, r=2$ it follows from BG67, Theorem 4.1].

Finally, if $n=8, r=3$, and $(V, X, \mathrm{~b})$ is a 3 -fold cross product, then we have (2.7) for $\epsilon=1$ (type I) or $\epsilon=-1$ (type II). If ( $V, X, \mathrm{~b}$ ) is of type I, then $(V, X,-\mathrm{b})$ is of type II. Hence, changing b by -b if necessary, we may assume that $(V, X, \mathrm{~b})$ is of type I and (iv) follows from [BG67, Theorem 5.1] and [Eld96, Proposition 3].

Remark 2.2. The proof of Theorem 2.1 shows that the bilinear form b is not determined at all from the cross product $X$ for $r=1$. Equation (2.3) shows, on the other hand, that b is uniquely determined by $X$ in case $n=7, r=2$.

The remaining cases are dealt with in the next result.
Proposition 2.3. Let $X$ be an $r$-fold cross product on an $n$-dimensional vector space $V$ relative to the nondegenerate symmetric bilinear forms b and $\mathrm{b}^{\prime}$.

- If $n \geq 3$ and $r=n-1$, then there is a scalar $\mu \in \mathbb{F}$ with $\mu^{n-2}=1$, such that $\mathrm{b}^{\prime}=\mu \mathrm{b}$.
In particular, for $n=3, \mathrm{~b}$ is uniquely determined.
- If $n=8$ and $r=3$, then $\mathrm{b}^{\prime}$ equals either b or -b . In particular, the bilinear form b is unique if $(V, X, \mathrm{~b})$ is assumed to be of type $I$.

Proof. Let $n \geq 3, r=n-1$, and without loss of generality, assume that $\mathbb{F}$ is algebraically closed. Let $\left\{v_{1}, \ldots, v_{n}\right\}$ be an orthogonal basis relative to b , with $\mathrm{b}\left(v_{i}, v_{i}\right)=1$ for all $i$.

As $\mathrm{b}\left(X\left(v_{1}, \ldots, v_{n-1}\right), v_{i}\right)=0$ for all $i=1, \ldots, n-1$, and

$$
\mathrm{b}\left(X\left(v_{1}, \ldots, v_{n-1}\right), X\left(v_{1}, \ldots, v_{n-1}\right)\right)=\operatorname{det}\left(\mathrm{b}\left(v_{i}, v_{j}\right)\right)=1
$$

changing if necessary $v_{n}$ by $-v_{n}$, we may assume that $X\left(v_{1}, \ldots, v_{n-1}\right)=v_{n}$. From the fact that $\Phi: V^{n} \rightarrow \mathbb{F}$, given by $\Phi\left(u_{1}, \ldots, u_{n}\right)=\mathrm{b}\left(X\left(u_{1}, \ldots, u_{n-1}\right), u_{n}\right)$, is multilinear and alternating, we conclude that for any permutation $\sigma$,

$$
\begin{equation*}
X\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}\right)=(-1)^{\sigma} v_{\sigma(n)} \tag{2.8}
\end{equation*}
$$

where $(-1)^{\sigma}$ denotes the signature of $\sigma$.
But $\Phi^{\prime}: V^{n} \rightarrow \mathbb{F}$ given by $\Phi^{\prime}\left(u_{1}, \ldots, u_{n}\right)=\mathrm{b}^{\prime}\left(X\left(u_{1}, \ldots, u_{n-1}\right), u_{n}\right)$, is also alternating, and hence there is a nonzero scalar $\mu$ such that $\Phi^{\prime}=\mu \Phi$. For any permutation $\sigma$,

$$
\mathrm{b}^{\prime}\left(v_{\sigma(n)}, v_{\sigma(i)}\right)=(-1)^{\sigma} \Phi^{\prime}\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}, v_{\sigma(i)}\right)=\mu \mathrm{b}\left(v_{\sigma(n)}, v_{\sigma(i)}\right)
$$

so that we get $\mathrm{b}^{\prime}=\mu \mathrm{b}$. Equation (1.2) gives $\mu^{n-2}=1$.
In case $n=8, r=3$, assume again that $\mathbb{F}$ is algebraically closed and let $\Phi: V^{4} \rightarrow \mathbb{F}$ (respectively $\Phi^{\prime}: V^{4} \rightarrow \mathbb{F}$ ) be the alternating multilinear map given by $\Phi\left(u_{1}, u_{2}, u_{3}, u_{4}\right)=\mathrm{b}\left(X\left(u_{1}, u_{2}, u_{3}\right), u_{4}\right)$ (respectively $\Phi^{\prime}\left(u_{1}, u_{2}, u_{3}, u_{4}\right)=$ $\left.\mathrm{b}^{\prime}\left(X\left(u_{1}, u_{2}, u_{3}\right), u_{4}\right)\right)$. Using (1.2) we get:

$$
\begin{aligned}
\left|\begin{array}{rrr}
\mathrm{b}(x, x) & \mathrm{b}(x, y) & \mathrm{b}(x, v) \\
\mathrm{b}(y, x) & \mathrm{b}(y, y) & \mathrm{b}(y, v) \\
\mathrm{b}(u, x) & \mathrm{b}(u, y) & \mathrm{b}(u, v)
\end{array}\right| & =\mathrm{b}(X(x, y, u), X(x, y, v)) \\
& =\Phi(x, y, u, X(x, y, v)) \\
& =-\Phi(x, y, X(x, y, v), u) \\
& =-\mathrm{b}(X(x, y, X(x, y, v)), u)
\end{aligned}
$$

The nondegeneracy of b gives
$X(x, y, X(x, y, v))=\left|\begin{array}{ll}\mathrm{b}(x, x) & \mathrm{b}(x, v) \\ \mathrm{b}(y, x) & \mathrm{b}(y, v)\end{array}\right| y-\left|\begin{array}{cc}\mathrm{b}(x, y) & \mathrm{b}(x, v) \\ \mathrm{b}(y, y) & \mathrm{b}(y, v)\end{array}\right| x-\left|\begin{array}{ll}\mathrm{b}(x, x) & \mathrm{b}(x, y) \\ \mathrm{b}(y, x) & \mathrm{b}(y, y)\end{array}\right| v$,
and hence we get

$$
X(x, y, X(x, y, X(x, y, v)))=-\left|\begin{array}{ll}
\mathrm{b}(x, x) & \mathrm{b}(x, y) \\
\mathrm{b}(y, x) & \mathrm{b}(y, y)
\end{array}\right| X(x, y, v)
$$

that is,

$$
X(x, y, .)^{3}=-\left|\begin{array}{ll}
\mathrm{b}(x, x) & \mathrm{b}(x, y) \\
\mathrm{b}(y, x) & \mathrm{b}(y, y)
\end{array}\right| X(x, y, .)
$$

and the same happens replacing b by $\mathrm{b}^{\prime}$.
Thus, for any $x, y \in V$ with $X(x, y,) \neq$.0 we obtain:

$$
\begin{equation*}
\mathrm{b}(x, x) \mathrm{b}(y, y)-\mathrm{b}(x, y)^{2}=\mathrm{b}^{\prime}(x, x) \mathrm{b}^{\prime}(y, y)-\mathrm{b}^{\prime}(x, y)^{2} \tag{2.9}
\end{equation*}
$$

As the set of such pairs $(x, y)$ is Zariski dense in $V \times V$, we conclude that (2.9) holds for any $x, y \in V$.

For any $x \in V$ with $\mathrm{b}(x, x) \neq 0$, the subspace $S_{x}=\{v \in V \mid \mathrm{b}(x, v)=0=$ $\left.\mathrm{b}^{\prime}(x, v)\right\}$ has dimension at least 6 , so it is not isotropic for both b and $\mathrm{b}^{\prime}$. For any $v \in S_{x}$, (2.9) gives

$$
\mathrm{b}(v, v)=\frac{\mathrm{b}^{\prime}(x, x)}{\mathrm{b}(x, x)} \mathrm{b}^{\prime}(v, v)
$$

so that $\mathrm{b}=\mu \mathrm{b}^{\prime}$ on the subspace $S_{x}$ with $\mu=\frac{\mathrm{b}^{\prime}(x, x)}{\mathrm{b}(x, x)}$. But (2.9) gives

$$
\left(1-\mu^{2}\right)\left|\begin{array}{ll}
\mathrm{b}(u, u) & \mathrm{b}(u, v) \\
\mathrm{b}(v, u) & \mathrm{b}(v, v)
\end{array}\right|=0
$$

for any $u, v \in S_{x}$ and this implies $\mu^{2}=1$. Thus $\mu= \pm 1$ and hence $\mathrm{b}^{\prime}(x, x)=$ $\pm \mathrm{b}(x, x)$ for any $x \in V$ with $\mathrm{b}(x, x) \neq 0$. Therefore, the polynomial map

$$
\left(\mathrm{b}(x, x)-\mathrm{b}^{\prime}(x, x)\right)\left(\mathrm{b}(x, x)+\mathrm{b}^{\prime}(x, x)\right)
$$

is 0 on the Zariski dense set of nonisotropic vectors for $b$, so it is 0 , and hence so is one of its factors. Thus either $\mathrm{b}^{\prime}=\mathrm{b}$ or $\mathrm{b}^{\prime}=-\mathrm{b}$.

## 3. Automorphisms

This section is devoted to computing the affine group schemes of automorphisms of cross products.

## 3.1. $n$ even, $r=1$.

Let $X: V \rightarrow V$ be a 1 -fold cross product on the even-dimensional vector space $V$, relative to a nondegenerate symmetric bilinear form b. Theorem 2.1 and its proof tell us that $\mathbb{K}=\mathbb{F i d} \oplus \mathbb{F} X$ is an étale $\mathbb{F}$-algebra (either isomorphic to $\mathbb{F} \times \mathbb{F}$ if $-1 \in \mathbb{F}^{2}$, or a quadratic field of $\mathbb{F}$ otherwise). In any case, $\mathbb{K}$ is endowed with a canonical involution: $X \mapsto-X$. Recall that b is far from being determined by $X$.

Denote by $\operatorname{Cent}_{\mathbf{G L}(V)}(X)$ the group scheme of elements that commute with $X$ in the general linear group scheme. The first part of the next result is trivial.

Theorem 3.1. Let $X: V \rightarrow V$ be a 1-fold cross product on the even-dimensional vector space $V$, relative to a nondegenerate symmetric bilinear form b .

- $\boldsymbol{\operatorname { A u t }}(V, X)=\operatorname{Cent}_{\mathbf{G L}(V)}(X)$.
- $\boldsymbol{A u t}(V, X, \mathrm{~b})=\mathbf{U}(V, \mathrm{~h})$, where h is the hermitian nondegenerate form given by

$$
\begin{aligned}
& \mathrm{h}: V \times V \\
& \longrightarrow \mathbb{K} \\
& (u, v)
\end{aligned} \mapsto_{\mathrm{b}(u, v) \mathrm{id}-\mathrm{b}(X(u), v) X}
$$

for any $u, v \in V$, and $\mathbf{U}(V, \mathrm{~h})$ is the corresponding unitary group scheme, whose $R$-points are those $\varphi \in \operatorname{End}_{\mathbb{K} \otimes_{\mathbb{R}} R}\left(V_{R}\right)$ such that $\mathrm{h}_{R}(\varphi(u), \varphi(v))=$ $\mathrm{h}_{R}(u, v)$ for all $u, v \in V_{R}=V \otimes_{\mathbb{F}} R$. (The subindex $R$ denotes the natural scalar extension.)

Proof. The fact that h is hermitian (i.e., h is $\mathbb{F}$-bilinear, $\mathrm{h}(X(u), v)=X \mathrm{~h}(u, v)$, and $\mathrm{h}(u, v)=\overline{\mathrm{h}(v, u)}$, for all $u, v \in V$, where - denotes the canonical involution of $\mathbb{K})$ is clear.

Note that for $\varphi \in \operatorname{End}_{\mathbb{K} \otimes_{\mathbb{R}} R}\left(V_{R}\right)$ and $u, v \in V_{R}, \mathrm{~h}_{R}(\varphi(u), \varphi(v))=\mathrm{h}_{R}(u, v)$ if, and only if, $\mathrm{b}_{R}(\varphi(u), \varphi(v))=\mathrm{b}_{R}(u, v)$ and $\mathrm{b}_{R}\left(X_{R}(\varphi(u)), \varphi(v)\right)=\mathrm{b}_{R}\left(X_{R}(u), v\right)$. Hence $\varphi$ is an element of $\mathbf{U}(V, \mathrm{~h})(R)$ if and only if $\varphi$ is an $R$-point in the intersection of $\operatorname{Cent}_{\mathbf{G L}(V)}(X)=\boldsymbol{\operatorname { A u t }}(V, X)$ and of the orthogonal group scheme $\mathbf{O}(V, \mathrm{~b})$, whence the result.

In case $-1 \in \mathbb{F}^{2}, \mathbb{K}$ is split and Theorem 3.1 gives:
Corollary 3.2. Let $X: V \rightarrow V$ be a 1 -fold cross product on the even-dimensional vector space $V$, relative to a nondegenerate symmetric bilinear form b , over a field $\mathbb{F}$ containing a square root $\mathbf{i}$ of -1 . Then, the following conditions hold:

- $\boldsymbol{A u t}(V, X)$ is isomorphic to $\mathbf{G L}\left(V_{+}\right) \times \mathbf{G L}\left(V_{-}\right)$, where $V_{ \pm}=\{v \in V \mid$ $X(v)= \pm \mathbf{i} v\}$.
- $\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})$ is isomorphic to $\mathbf{G L}\left(V_{+}\right)$.

Proof. Any automorphism of $(V, X)$ preserves $V_{+}$and $V_{-}$, and any automorphism of $(V, X, \mathrm{~b})$ is determined by its action on $V_{+}$, because $V_{+}$and $V_{-}$are paired by b.

## 3.2. $n \geq 3, r=n-1$.

Given a vector space $V$ endowed with a nondegenerate symmetric bilinear form, we will consider the special orthogonal group scheme $\mathbf{O}^{+}(V, \mathrm{~b})$ and the group scheme $\widetilde{\mathbf{O}}(V, \mathrm{~b})$ whose $R$-points are those invertible linear automorphisms $\varphi$ of $V_{R}$ such that $\mathrm{b}_{R}(\varphi(u), \varphi(v))=\operatorname{det}(\varphi) \mathrm{b}_{R}(u, v)$ for all $u, v \in V_{R}$.

Theorem 3.3. Let $X: V^{n-1} \rightarrow V$ be an ( $n-1$ )-fold cross product on the $n$ dimensional vector space $V$, relative to a nondegenerate symmetric bilinear form b . Then the next two equations hold:

- $\boldsymbol{\operatorname { A u t }}(V, X)=\widetilde{\mathbf{O}}(V, \mathrm{~b})$.
- $\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})=\mathbf{O}^{+}(V, \mathrm{~b})$.

Proof. Let $\varphi$ be a linear automorphism of $V$. After extending scalars we may assume, as in the proof of Proposition 2.3, that there exists a basis $\left\{v_{1}, \ldots, v_{n}\right\}$ satisfying Equation (2.8). Then $\varphi \in \operatorname{Aut}(V, X)$ if, and only if, we have

$$
X\left(\varphi\left(v_{\sigma(1)}\right), \ldots, \varphi\left(v_{\sigma(n-1)}\right)\right)=\varphi\left(X\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}\right)\right)=(-1)^{\sigma} \varphi\left(v_{\sigma(n)}\right)
$$

for any permutation $\sigma$, and this happens, as b is nondegenerate, if and only if we have

$$
\begin{aligned}
\mathrm{b}\left(\varphi\left(v_{\sigma(n-1)}\right), \varphi\left(v_{\sigma(i)}\right)\right) & =(-1)^{\sigma} \Phi\left(\varphi\left(v_{\sigma(1)}\right), \ldots, \varphi\left(v_{\sigma(n-1)}\right), \varphi\left(v_{\sigma(i)}\right)\right) \\
& =(-1)^{\sigma} \operatorname{det}(\varphi) \Phi\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}, v_{\sigma(i)}\right) \\
& = \begin{cases}\operatorname{det}(\varphi) & \text { if } i=n \\
0 & \text { otherwise }\end{cases} \\
& =\operatorname{det}(\varphi) \mathrm{b}\left(v_{\sigma(n)}, v_{\sigma(i)}\right)
\end{aligned}
$$

where $\Phi$ is the alternating multilinear form considered in the proof of Proposition 2.3. We conclude that $\varphi$ is an automorphism of $(V, X)$ if and only if it satisfies

$$
\begin{equation*}
\mathrm{b}(\varphi(u), \varphi(v))=\operatorname{det}(\varphi) \mathrm{b}(u, v) \tag{3.1}
\end{equation*}
$$

for all $u, v \in V$.
The argument above is functorial, so we conclude $\boldsymbol{\operatorname { A u t }}(V, X)=\widetilde{\mathbf{O}}(V, \mathrm{~b})$. Moreover, $\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})=\boldsymbol{\operatorname { A u t }}(V, X) \cap \mathbf{O}(V, \mathrm{~b})=\mathbf{O}^{+}(V, \mathrm{~b})$.

Denote by $\boldsymbol{\mu}_{m}$ the scheme of $m$-th roots of unity. A natural short exact sequence appears:

Proposition 3.4. Let $X: V \rightarrow V$ be an ( $n-1$ )-fold cross product on the $n$ dimensional vector space $V$, relative to a nondegenerate symmetric bilinear form b . Then the determinant provides a short exact sequence:

$$
1 \longrightarrow \mathbf{O}^{+}(V, \mathrm{~b}) \longrightarrow \widetilde{\mathbf{O}}(V, \mathrm{~b}) \xrightarrow{\text { det }} \boldsymbol{\mu}_{n-2} \longrightarrow 1
$$

Proof. For any (unital, associative, commutative) $\mathbb{F}$-algebra $R$ and any element $\varphi \in \widetilde{\mathbf{O}}(V, \mathrm{~b})(R)$, Equation (3.1) shows that $\operatorname{det}(\varphi)^{2}=\operatorname{det}(\varphi)^{n}$, so that $\operatorname{det}(\varphi)$ is an $(n-2)$-th root of unity, thus showing that the exact sequence is well defined. Now, the only thing left is to prove that det gives a quotient map $\widetilde{\mathbf{O}}(V, \mathrm{~b}) \rightarrow \boldsymbol{\mu}_{n-2}$.

Given any $\mathbb{F}$-algebra $R$ and a root of unity $r \in \boldsymbol{\mu}_{n-2}(R)$, consider the degree two extension $S=R[T] /\left(T^{2}-r\right)$. Denote by $t$ the class of $T$ modulo $\left(T^{2}-r\right)$. The algebra $S$ is a free $R$-module of rank two, so $S / R$ is a faithfully flat extension. Take an orthogonal basis $\left\{v_{1}, \ldots, v_{n}\right\}$ and consider the linear automorphism $\varphi \in$ $\mathbf{G L}(V)(S)$ such that

$$
\varphi\left(v_{i}\right)=t v_{i}, 1 \leq i \leq n ; \quad \varphi\left(v_{n}\right)=t^{n-1} v_{n} .
$$

Note that $t^{2}=r=r^{n-1}=\left(t^{n-1}\right)^{2}$, so that $\varphi$ lies in $\widetilde{\mathbf{O}}(V, \mathrm{~b})(S)$, and $\operatorname{det}(\varphi)=$ $\left(t^{n-1}\right)^{2}=r$.

This shows that det : $\widetilde{\mathbf{O}}(V, \mathrm{~b}) \rightarrow \boldsymbol{\mu}_{n-2}$ is a quotient map.
In particular, for $n=3, \boldsymbol{\operatorname { A u t }}(V, X)=\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})=\mathbf{O}^{+}(V, \mathrm{~b})$.
Remark 3.5. As $\mathbf{O}(V, \mathrm{~b})$ is smooth, we get that $\widetilde{\mathbf{O}}(V, \mathrm{~b})$ is smooth if and only if so is $\boldsymbol{\mu}_{n-2}$ (see, e.g., KMRT98, (22.12)]). Therefore we obtain the following:
$\widetilde{\mathbf{O}}(V, \mathrm{~b})$ is smooth if and only if the characteristic of $\mathbb{F}$ does not divide $n-2$.
The Lie algebra of $\widetilde{\mathbf{O}}(V, \mathrm{~b})$ is

$$
\operatorname{Lie}(\widetilde{\mathbf{O}}(V, \mathrm{~b}))=\left\{f \in \operatorname{End}_{\mathbb{F}}(V) \mid \mathrm{b}(f(u), v)+\mathrm{b}(u, f(v))=\operatorname{tr}(f) \mathrm{b}(u, v) \forall u, v \in V\right\}
$$

Hence, for any $f \in \operatorname{Lie}(\widetilde{\mathbf{O}}(V, \mathrm{~b})), 2 \operatorname{tr}(f)=n \operatorname{tr}(f)$, so that $(n-2) \operatorname{tr}(f)=0$ and we get:

$$
\operatorname{Lie}(\widetilde{\mathbf{O}}(V, \mathrm{~b}))= \begin{cases}\mathfrak{s o}(V, \mathrm{~b}) & \text { if char } \mathbb{F} \text { does not divide } n-2 \text { (smooth case) }, \\ \mathfrak{s o}(V, \mathrm{~b}) \oplus \mathbb{F i d} & \text { otherwise }\end{cases}
$$

where $\mathfrak{s o}(V, \mathrm{~b})$ denotes the orthogonal Lie algebra, i.e., the Lie algebra of skewsymmetric endomorphisms of $V$ relative to b.
3.3. $n=7, r=2$.

If $(V, X)$ is a 2 -fold cross product on a seven-dimensional vector space, then ( $V, X$ ) is isomorphic to $\left(\mathcal{C}_{0}, X^{\mathcal{C}_{0}}\right)$ for a Cayley algebra $\mathcal{C}$ (Theorem 2.1). Moreover (Remark (2.2), the bilinear form is uniquely determined in this case.

The restriction map

$$
\begin{aligned}
\operatorname{Aut}(\mathcal{C}) & \longrightarrow \operatorname{Aut}\left(\mathfrak{C}_{0}, X^{\mathcal{C}_{0}}\right) \\
f & \left.\mapsto f\right|_{\mathfrak{C}_{0}}
\end{aligned}
$$

is an isomorphism of affine group schemes (see, e.g., CRE16), and this gives the next result (see SV00).

Theorem 3.6. Let ( $V, X$, b) be a 2 -fold cross product on a seven-dimensional vector space. Then $\boldsymbol{\operatorname { A u t }}(V, X)=\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})$ is a simple algebraic group of type $G_{2}$.
3.4. $n=8, r=3$.

If $(V, X)$ is a 3 -fold cross product on an eight-dimensional vector space $V$, then by Theorem[2.1, $(V, X)$ is isomorphic to $\left(\mathcal{C}, \alpha X_{1}^{\mathcal{C}}\right)$, for a Cayley algebra $\mathcal{C}$ and a nonzero scalar $\alpha \in \mathbb{F}$, where $X_{1}^{\mathcal{e}}$ is given by Equation (2.5). But $\operatorname{Aut}(V, X)=\operatorname{Aut}(V, \alpha X)$ for any nonzero $\alpha$, so it is enough to study the group scheme $\operatorname{Aut}\left(\mathcal{C}, X_{1}^{\mathrm{e}}\right)$. By Proposition 2.3, this is the same as $\operatorname{Aut}\left(\mathcal{C}, X_{1}^{\mathcal{C}}, \mathrm{b}_{\mathrm{n}}\right)$.

Consider, as in Sha90a or Eld96 the triple product on the Cayley algebra $\mathcal{C}$ given by

$$
\begin{equation*}
\{x y z\}=(x \bar{y}) z \tag{3.2}
\end{equation*}
$$

that satisfies the equation

$$
\begin{equation*}
\{x x y\}=\mathrm{n}(x) y=\{y x x\} \tag{3.3}
\end{equation*}
$$

This is called a $3 C$-product, and the pair $(\mathcal{C},\{\ldots\})$ a $3 C$-algebra.
Because of (3.3), the norm of $\mathcal{C}$ is determined by the $3 C$-product, and hence we have the equality $\operatorname{Aut}(\mathcal{C},\{\ldots\})=\boldsymbol{\operatorname { A u t }}(\mathcal{C},\{\ldots\}, \mathrm{n})$. Moreover, Equation (2.5) and Proposition 2.3 give:

$$
\begin{equation*}
\boldsymbol{\operatorname { A u t }}\left(\mathcal{C}, X_{1}^{\mathcal{C}}\right)=\boldsymbol{\operatorname { A u t }}\left(\mathcal{C}, X_{1}^{\mathcal{C}}, \mathrm{b}_{\mathrm{n}}\right)=\boldsymbol{\operatorname { A u t }}(\mathcal{C},\{\ldots\})=\boldsymbol{\operatorname { A u t }}(\mathcal{C},\{\ldots\}, \mathrm{n}) \tag{3.4}
\end{equation*}
$$

and hence it is enough to compute $\operatorname{Aut}(\mathcal{C},\{\ldots\})$. The group of rational points has been computed in Eld96, Proposition 7] and shown to be isomorphic to the spin group of $\left(\mathfrak{C}_{0},-\mathrm{n}\right)$. Here we will extend the results in [Eld96] to the group scheme setting, and will relate them to the triality phenomenon.

To begin with, let $\mathcal{C}$ be a Cayley algebra with norm n, and consider the paraCayley product $x \bullet y:=\bar{x} \bar{y}$. Let $l_{x}: y \mapsto x \bullet y$ and $r_{x}: y \mapsto y \bullet x$ be the operators of left and right multiplication by the element $x$ in the para-Cayley algebra $(\mathcal{C}, \bullet)$. Consider, following [KMRT98, §35] and EK13, §5.1], the linear map:

$$
\begin{aligned}
\mathcal{C} & \longrightarrow \operatorname{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C}) \\
x & \mapsto\left(\begin{array}{cc}
0 & l_{x} \\
r_{x} & 0
\end{array}\right) .
\end{aligned}
$$

For any $x, y \in \mathcal{C}, r_{x} l_{x}(y)=(\overline{\bar{x}} \bar{y}) \bar{x}=(y x) \bar{x}=\mathrm{n}(x) y=l_{x} r_{x}(y)$. Hence we have

$$
\left(\begin{array}{cc}
0 & l_{x} \\
r_{x} & 0
\end{array}\right)^{2}=\mathrm{n}(x) \mathrm{id}
$$

and the above linear map induces an isomorphism

$$
\begin{equation*}
\Phi: \mathfrak{C l}(\mathcal{C}, n) \longrightarrow \operatorname{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C}) \tag{3.5}
\end{equation*}
$$

which restricts to an isomorphism of the even Clifford algebra $\mathfrak{C l}(\mathcal{C}, n)_{\overline{0}}$ onto the diagonal subalgebra $\operatorname{End}_{\mathbb{F}}(\mathcal{C}) \times \operatorname{End}_{\mathbb{F}}(\mathcal{C})$. Under this isomorphism, the canonical involution $\tau$ of $\mathfrak{C l}(\mathcal{C}, \mathrm{n}): \tau(x)=x$ for all $x \in \mathcal{C}$, corresponds to the orthogonal involution $\sigma_{\mathrm{n} \perp \mathrm{n}}$ associated to the quadratic form $\mathrm{n} \perp \mathrm{n}$ on $\mathcal{C} \oplus \mathcal{C}$.

To prevent confusions between the unity of the Clifford algebra $\mathfrak{C l}(\mathcal{C}, n)$ and the unity of the Cayley algebra $\mathcal{C}$, we will denote in what follows the unity of $\mathfrak{C l}(\mathcal{C}, n)$ by 1 , and the unity of $\mathcal{C}$ by $e_{0}$. Also, the multiplication in $\mathcal{C}$ will be denoted by juxtaposition, as usual, and the one in $\mathfrak{C l}(\mathcal{C}, \mathrm{n})$ by a dot: $x \cdot y$.

The associated spin group is the group:

$$
\operatorname{Spin}(\mathcal{C}, \mathrm{n})=\left\{u \in \mathfrak{C l}(\mathcal{C}, \mathrm{n})_{\overline{0}} \mid u \cdot \tau(u)=1, u \cdot \mathcal{C} \cdot u^{-1}=\mathcal{C}\right\}
$$

The vector representation of $\operatorname{Spin}(\mathcal{C}, \mathrm{n})$ on $\mathcal{C}$ works as follows. For any $x \in \mathcal{C}$ and $u \in \operatorname{Spin}(\mathcal{C}, \mathrm{n}), \chi_{u}(x):=u \cdot x \cdot u^{-1}=u \cdot x \cdot \tau(u)$.

For any $u \in \operatorname{Spin}(\mathcal{C}, \mathrm{n}), \Phi(u)$ lies in the even part of $\operatorname{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C})$ :

$$
\Phi(u)=\left(\begin{array}{cc}
\rho_{u}^{-} & 0 \\
0 & \rho_{u}^{+}
\end{array}\right)
$$

and as $u \cdot \tau(u)=1$, it follows that $\rho_{u}^{+}$and $\rho_{u}^{-}$are orthogonal transformations: $\rho_{u}^{ \pm} \in \mathrm{O}(\mathcal{C}, \mathrm{n})$.

The equality $u \cdot x=\chi_{u}(x) \cdot u$ transfers by $\Phi$ to

$$
\left(\begin{array}{cc}
\rho_{u}^{-} & 0 \\
0 & \rho_{u}^{+}
\end{array}\right)\left(\begin{array}{cc}
0 & l_{x} \\
r_{x} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & l_{\chi_{u}(x)} \\
r_{\chi_{u}(x)} & 0
\end{array}\right)\left(\begin{array}{cc}
\rho_{u}^{-} & 0 \\
0 & \rho_{u}^{+}
\end{array}\right) .
$$

This gives,

$$
\rho_{u}^{-}(x \bullet y)=\chi_{u}(x) \bullet \rho_{u}^{+}(y), \quad \rho_{u}^{+}(y \bullet x)=\rho_{u}^{-}(y) \bullet \chi_{u}(x),
$$

for all $u \in \operatorname{Spin}(\mathcal{C}, \mathrm{n})$ and $x, y \in \mathcal{C}$. It is easy to check (EK13, Lemma 5.4]) that there is a cyclic symmetry here: if three isometries $f_{i} \in \mathrm{O}(\mathcal{C}, \mathrm{n}), 0 \leq i \leq 2$ satisfy $f_{0}(x \bullet y)=f_{1}(x) \bullet f_{2}(y)$ for all $x, y \in \mathcal{C}$, then also $f_{1}(x \bullet y)=f_{2}(x) \bullet f_{0}(y)$, and $f_{2}(x \bullet y)=f_{0}(x) \bullet f_{1}(y)$.

All the above arguments are functorial and give an isomorphism of affine group schemes (see KMRT98, §35])

$$
\begin{align*}
\operatorname{Spin}(\mathcal{C}, \mathrm{n}) & \cong\left\{\left(f_{0}, f_{1}, f_{2}\right) \in \mathbf{O}^{+}(\mathcal{C}, \mathrm{n})^{3} \mid f_{0}(x \bullet y)=f_{1}(x) \bullet f_{2}(y) \forall x, y \in \mathcal{C}\right\}  \tag{3.6}\\
u & \mapsto\left(\chi_{u}, \rho_{u}^{+}, \rho_{u}^{-}\right)
\end{align*}
$$

Recall that we denote by $e_{0}$ the unity of the Cayley algebra $\mathcal{C}$.
Proposition 3.7. The above isomorphism of affine group schemes induces an isomorphism

$$
\begin{aligned}
& \operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right) \cong\left\{\left(f_{0}, f_{1}, f_{2}\right) \in \mathbf{O}^{+}(\mathcal{C}, \mathrm{n})^{3} \mid\right. \\
& \left.\quad f_{0}(x \bullet y)=f_{1}(x) \bullet f_{2}(y) \text { and } f_{0}\left(e_{0}\right)=e_{0} \forall x, y \in \mathcal{C}\right\}
\end{aligned}
$$

Proof. Consider the linear map

$$
\begin{gathered}
\mathcal{C}_{0} \longrightarrow \mathfrak{C l}(\mathcal{C}, \mathrm{n}) \\
x \mapsto \quad e_{0} \cdot x,
\end{gathered}
$$

whose image lies in the even part $\mathfrak{C l}(\mathrm{C}, \mathrm{n})_{\overline{0}}$. Since $\left(e_{0} \cdot x\right)^{\cdot 2}=e_{0} \cdot x \cdot e_{0} \cdot x=-e_{0}^{\cdot 2} \cdot x^{\cdot 2}=$ $-\mathrm{n}\left(e_{0}\right) \mathrm{n}(x) 1=-\mathrm{n}(x) 1$, because $e_{0}$ and $x$ are orthogonal relative to n , this linear map induces an embedding

$$
\Psi: \mathfrak{C l}\left(\mathrm{C}_{0},-\mathrm{n}\right) \longrightarrow \mathfrak{C l}(\mathrm{C}, \mathrm{n})
$$

which, by dimension count, restricts to an isomorphism (also denoted by $\Psi$ ):

$$
\begin{equation*}
\Psi: \mathfrak{C l}\left(\mathrm{C}_{0},-\mathrm{n}\right) \longrightarrow \mathfrak{C l}(\mathrm{C}, \mathrm{n})_{\overline{0}} . \tag{3.7}
\end{equation*}
$$

Recall that $\tau$ denotes the canonical involution on $\mathfrak{C l}(\mathcal{C}, \mathrm{n})$. Denote by $\tau^{\prime}$ the involution on $\mathfrak{C l}\left(\mathfrak{C}_{0},-\mathrm{n}\right)$ which is -id on $\mathfrak{C}_{0}$. That is, $\tau^{\prime}$ is the composition of the canonical involution on $\mathfrak{C l}\left(\mathrm{C}_{0},-\mathrm{n}\right)$ with the parity automorphism. The restriction of $\tau^{\prime}$ to the even part equals the restriction of the canonical involution.

For all $x \in \mathcal{C}_{0}$ we get

$$
\Psi\left(\tau^{\prime}(x)\right)=-\Psi(x)=-e_{0} \cdot x=x \cdot e_{0}=\tau(x) \cdot \tau\left(e_{0}\right)=\tau\left(e_{0} \cdot x\right)=\tau(\Psi(x))
$$

so that we have

$$
\begin{equation*}
\Psi \tau^{\prime}=\tau \Psi \tag{3.8}
\end{equation*}
$$

As the elements of $\mathcal{C}_{0}$ anticommute with $e_{0}$ in $\mathfrak{C l}(\mathcal{C}, n)$ we get

$$
\begin{aligned}
& \Psi\left(\mathfrak{C l}\left(\mathrm{C}_{0},-\mathrm{n}\right)_{\overline{0}}\right)=\left\{a \in \mathfrak{C l}(\mathfrak{C}, \mathrm{n})_{\overline{0}} \mid e_{0} \cdot a=a \cdot e_{0}\right\} \\
& \Psi\left(\mathfrak{C l}\left(\mathrm{C}_{0},-\mathrm{n}\right)_{\overline{1}}\right)=\left\{a \in \mathfrak{C l}(\mathcal{C}, \mathrm{n})_{\overline{0}} \mid e_{0} \cdot a=-a \cdot e_{0}\right\}
\end{aligned}
$$

In particular, for any $u \in \operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right)$, we have $\Psi(u) \cdot e_{0}=e_{0} \cdot \Psi(u)$. Hence, for any $x \in \mathcal{C}_{0}$ we get

$$
\Psi\left(u \cdot x \cdot u^{-1}\right)=\Psi(u) \cdot e_{0} \cdot x \cdot \Psi(u)^{-1}=e_{0} \cdot \Psi(u) \cdot x \cdot \Psi(u)^{-1}
$$

so that we obtain

$$
\Psi(u) \cdot x \cdot \Psi(u)^{-1}=e_{0} \cdot \Psi\left(u \cdot x \cdot u^{-1}\right)=e_{0} \cdot e_{0} \cdot \chi_{u}(x)=\chi_{u}(x) \in \mathfrak{C}_{0}
$$

and we conclude that $\Psi(u)$ lies in $\operatorname{Spin}(\mathcal{C}, n)$.
Conversely, for any $v \in \operatorname{Spin}(\mathcal{C}, \mathrm{n})$ with $e_{0} \cdot v=v \cdot e_{0}, v$ is the image $v=\Psi(u)$ of some element $u \in \mathfrak{C l}\left(\mathfrak{C}_{0},-\mathrm{n}\right)_{\overline{0}}$. From (3.8) we get $u \cdot \tau^{\prime}(u)=1$, and for any $x \in \mathfrak{C}_{0}$,

$$
\Psi\left(u \cdot x \cdot u^{-1}\right)=v \cdot e_{0} \cdot x \cdot v^{-1}=e_{0} \cdot \chi_{v}(x)=\Psi\left(\chi_{v}(x)\right) \in \Psi\left(\complement_{0}\right),
$$

and we conclude that $u \cdot x \cdot u^{-1}$ lies in $\mathfrak{C}_{0}$, so that $u$ lies in $\operatorname{Spin}\left(\mathcal{C}_{0},-n\right)$.
Therefore $\Psi$ restricts to a group isomorphism

$$
\operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right) \cong \operatorname{Cent}_{\operatorname{Spin}\left(e^{\prime}, \mathrm{n}\right)}\left(e_{0}\right)
$$

from the spin group of $\left(\mathcal{C}_{0},-\mathrm{n}\right)$ onto the centralizer in $\operatorname{Spin}(\mathcal{C}, \mathrm{n})$ of $e_{0}$ under the vector representation.

But all these arguments are functorial, so actually $\Psi$ induces an isomorphism of affine group schemes:

$$
\operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right) \cong\left\{u \in \operatorname{Spin}(\mathcal{C}, \mathrm{n}) \mid u \cdot e_{0}=e_{0} \cdot u\right\}
$$

Finally, if we compose this isomorphism with the one in Equation (3.6) we obtain the isomorphism of affine group schemes:

$$
\begin{align*}
\operatorname{Spin}\left(\mathfrak{C}_{0},-\mathrm{n}\right) & \longrightarrow\left\{\left(f_{0}, f_{1}, f_{2}\right) \in \mathbf{O}^{+}(\mathcal{C}, \mathrm{n})^{3} \mid f_{0}(x \bullet y)=f_{1}(x) \bullet f_{2}(y)\right. \\
& \text { and } \left.f_{0}\left(e_{0}\right)=e_{0} \forall x, y \in \mathcal{C}\right\}  \tag{3.9}\\
u & \mapsto\left(\chi_{\Psi(u)}, \rho_{\Psi(u)}^{+}, \rho_{\Psi(u)}^{-}\right),
\end{align*}
$$

as required.
Remark 3.8. If $\mathcal{C}$ is a Cayley algebra, and $\left(f_{0}, f_{1}, f_{2}\right)$ is a triple of isometries satisfying $f_{0}(x \bullet y)=f_{1}(x) \bullet f_{2}(y)$ for all $x, y \in \mathcal{C}$, then we also have $f_{1}(x \bullet y)=f_{2}(x) \bullet f_{0}(y)$. If $f_{0}\left(e_{0}\right)=e_{0}$, then with $y=e_{0}$ above we get $f_{1}(\bar{x})=\overline{f_{2}(x)}$ for all $x$, so that $f_{1}(x)=\overline{f_{2}(\bar{x})}$. Then both $f_{0}$ and $f_{1}$ are determined by $f_{2}$.

Conversely, with $\left(f_{0}, f_{1}, f_{2}\right)$ as above, if $f_{1}(\bar{x})=\overline{f_{2}(x)}$ for all $x \in \mathcal{C}$, we get

$$
f_{1}(\bar{x})=f_{1}\left(x \bullet e_{0}\right)=\left\{\begin{array}{l}
f_{2}(x) \bullet f_{0}\left(e_{0}\right) \\
\overline{f_{2}(x)}=f_{2}(x) \bullet e_{0}
\end{array}\right.
$$

and we conclude that $f_{0}\left(e_{0}\right)=e_{0}$.
As both $f_{0}$ and $f_{1}$ are then determined by $f_{2}$, projecting on the third component in (3.9) gives an injective homomorphism (i.e., a closed embedding):

$$
\begin{align*}
\theta: \mathbf{S p i n}\left(\mathfrak{C}_{0},-\mathrm{n}\right) & \longrightarrow \mathbf{O}^{+}(\mathfrak{C}, \mathrm{n}) \\
u & \mapsto \rho_{\Psi(u)}^{-} . \tag{3.10}
\end{align*}
$$

This is actually the spin representation of $\operatorname{Spin}\left(\mathcal{C}_{0},-n\right)$.
Our goal now is to show that the image of the homomorphism $\theta$ in (3.10) is the automorphism group scheme $\operatorname{Aut}(\mathcal{C},\{\ldots\})=\operatorname{Aut}\left(\mathcal{C}, X_{1}^{\mathcal{C}}\right)($ see (3.4) $)$.
Remark 3.9. Composing the isomorphisms $\Phi$ in (3.5) and $\Psi$ in (3.7), for any $x \in \mathcal{C}_{0}$ we obtain:

$$
\Phi \Psi(x)=\Phi\left(e_{0} \cdot x\right)=\left(\begin{array}{cc}
0 & l_{e_{0}} \\
r_{e_{0}} & 0
\end{array}\right)\left(\begin{array}{cc}
0 & l_{x} \\
r_{x} & 0
\end{array}\right)=\left(\begin{array}{cc}
l_{e_{0}} r_{x} & 0 \\
0 & r_{e_{0}} l_{x}
\end{array}\right)=\left(\begin{array}{cc}
L_{x} & 0 \\
0 & R_{x}
\end{array}\right)
$$

where $L_{x}: y \mapsto x y, R_{x}: y \mapsto y x$, are the operators of left and right multiplication by $x$ in $\mathcal{C}$. This follows from $l_{e_{0}} r_{x}(y)=l_{e_{0}}(\bar{y} \bar{x})=\overline{\bar{y}} \bar{x}=x y$ and from $r_{e_{0}} l_{x}(y)=$ $r_{e_{0}}(\bar{x} \bar{y})=\overline{\bar{x}} \bar{y}=y x$.

Thus for $x_{1}, \ldots, x_{2 s} \in \mathcal{C}_{0}$, with $\prod_{i=1}^{2 s} \mathrm{n}\left(x_{i}\right)=1$, the image by $\theta$ in (3.10) of $u=x_{1} \cdots x_{2 s} \in \operatorname{Spin}\left(\mathrm{C}_{0}, \mathrm{n}\right)$ is $L_{x_{1}} \cdots L_{x_{2 s}}$. These elements are shown in Eld96, Theorem 10] and, in a different way, in Eld00, Corollary 2.5], to exhaust the elements in $\operatorname{Aut}(\mathcal{C},\{\ldots\})$.

We are now ready for the computation of the affine group schemes in (3.4). This result extends the results in Sha90b and Eld96, where only the groups of rational points were computed. It also gives a new perspective on the automorphisms of 3 -fold cross products.

Theorem 3.10. The group scheme of automorphisms Aut $(\mathcal{C},\{\ldots\})$ is isomorphic to $\operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right)$.
Proof. The homomorphism $\theta$ in (3.10) factors through $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ because if a triple $\left(f_{0}, f_{1}, f_{2}\right) \in \mathbf{O}^{+}(\mathcal{C}, \mathrm{n})^{3}(R)$, for an $\mathbb{F}$-algebra $R$, satisfies $f_{0}(x \bullet y)=f_{1}(x) \bullet$ $f_{2}(y)$ for all $x, y, z \in \mathcal{C}_{R}$, and with $f_{0}\left(e_{0}\right)=e_{0}$, then we have $f_{1}(\bar{x})=\overline{f_{2}(x)}$ and

$$
\begin{aligned}
f_{2}(\{x y z\}) & =f_{2}((x \bar{y}) z) \\
& =f_{2}((y \bar{x}) \bullet \bar{z})=f_{0}(y \bar{x}) \bullet f_{1}(\bar{z}) \\
& =f_{0}(\bar{y} \bullet x) \bullet \overline{f_{2}(z)} \\
& =\left(f_{1}(\bar{y}) \bullet f_{2}(x)\right) \bullet \overline{f_{2}(z)} \\
& =\left(\overline{f_{2}(y)} \bullet f_{2}(x)\right) \bullet \overline{f_{2}(z)} \\
& =\left(\overline{f_{2}(y) \overline{f_{2}(x)}}\right) f_{2}(z) \\
& =\left(f_{2}(x) \overline{f_{2}(y)}\right) f_{2}(z)=\left\{f_{2}(x) f_{2}(y) f_{2}(z)\right\}
\end{aligned}
$$

and hence we get an injective morphism $\theta: \boldsymbol{\operatorname { S p i n }}\left(\mathcal{C}_{0},-\mathrm{n}\right) \rightarrow \boldsymbol{\operatorname { A u t }}(\mathcal{C},\{\ldots\})$ (which we denote also by $\theta$ ).

Remark 3.9 and [Eld96, Theorem 10] show that $\theta_{\overline{\mathbb{F}}}$ is bijective, where $\overline{\mathbb{F}}$ denotes an algebraic closure of $\mathbb{F}$. Also, the differential $\mathrm{d} \theta$ is injective, because $\theta$ in (3.10) is injective. But the Lie algebra $\operatorname{Lie}(\operatorname{Aut}(\mathcal{C},\{\ldots\}))=\operatorname{Der}(\mathcal{C},\{\ldots\})$ is isomorphic to the orthogonal Lie algebra $\mathfrak{s o}\left(\mathcal{C}_{0},-n\right)($ Eld96, Theorem 12]), so by dimension count $\mathrm{d} \theta$ is an isomorphism $\operatorname{Lie}\left(\boldsymbol{\operatorname { S p i n }}\left(\mathfrak{C}_{0},-\mathrm{n}\right)\right) \rightarrow \operatorname{Lie}(\operatorname{Aut}(\mathcal{C},\{\ldots\}))$. We conclude from EK13, Theorem A.50] that $\theta: \boldsymbol{\operatorname { S p i n }}\left(\mathcal{C}_{0},-\mathrm{n}\right) \rightarrow \boldsymbol{\operatorname { A u t }}(\mathcal{C},\{\ldots\})$ is an isomorphism.

## 4. Gradings

Let $G$ be an abelian group and let $(V, X)$ be an $r$-fold cross product. A $G$-grading on $(V, X)$ is a vector space decomposition $\Gamma: V=\bigoplus_{g \in G} V_{g}$, such that

$$
X\left(V_{g_{1}}, \ldots, V_{g_{r}}\right) \subseteq V_{g_{1} \cdots g_{r}}
$$

for all $g_{1}, \ldots, g_{r} \in G$. The subspaces $V_{g}$ are called the homogeneous components and we write $\operatorname{deg}(v)=g\left(\right.$ or $\left.\operatorname{deg}_{\Gamma}(v)=g\right)$ for any $0 \neq v \in V_{g}$. The support of $\Gamma$ is the (finite) subset $\operatorname{Supp} \Gamma:=\left\{g \in G \mid V_{g} \neq 0\right\}$.

Two $G$-gradings $\Gamma: V=\bigoplus_{g \in G} V_{g}$ and $\Gamma^{\prime}: V=\bigoplus_{g \in G} V_{g}^{\prime}$ on $(V, X)$ are isomorphic if there is an automorphism $\varphi \in \operatorname{Aut}(V, X)$ such that $\varphi\left(V_{g}\right)=V_{g}^{\prime}$ for all $g \in G$.

For the basic facts on gradings on algebras, the reader is referred to the monograph [EK13]. Here we will review the main facts, adapted to cross products.

Any $G$-grading $\Gamma$ on $(V, X)$ determines a homomorphism of affine group schemes

$$
\rho_{\Gamma}: G_{\text {diag }} \longrightarrow \operatorname{Aut}(V, X)
$$

where $G_{\text {diag }}$ is the diagonalizable affine group scheme represented by the group algebra $\mathbb{F} G$. This is the Cartier dual of the constant group scheme $G$ (see KMRT98, §20]).

The homomorphism $\rho_{\Gamma}$ factors through the subgroup scheme $\boldsymbol{\operatorname { A u t }}(V, X, \mathrm{~b})$ if and only if $\mathrm{b}\left(V_{g}, V_{h}\right)=0$ unless $g h=e$ (the neutral element of $G$ ). In this case, $\Gamma$ will be said to be compatible with b .

Two $G$-gradings on ( $V, X$ ) are isomorphic if and only if the corresponding homomorphisms are conjugate by an element in $\operatorname{Aut}(V, X)$ [EK13, Proposition 1.36].

Conversely, any homomorphism $\rho: G_{\text {diag }} \rightarrow \boldsymbol{\operatorname { A u t }}(V, X)$ determines a grading $\Gamma$ on $(V, X)$ as follows. Take the generic element (the identity map) $\operatorname{id}_{\mathbb{F} G} \in G_{\mathrm{diag}}(G)=$ $\operatorname{Hom}(\mathbb{F} G, \mathbb{F} G)$ (algebra homomorphisms). The automorphism

$$
\rho_{\mathbb{F} G}\left(\mathrm{id}_{\mathbb{F} G}\right) \in \operatorname{Aut}\left(V_{\mathbb{F} G}, X_{\mathbb{F} G}\right)
$$

will be called the generic automorphism attached to $\rho$, and $\Gamma: V=\bigoplus_{g \in G} V_{g}$ is given by:

$$
V_{g}=\left\{v \in V \mid \rho_{\mathbb{F} G}\left(\mathrm{id}_{\mathbb{F} G}\right)(v \otimes 1)=v \otimes g\right\} .
$$

We will say that $\rho_{\mathbb{F} G}\left(\operatorname{id}_{\mathbb{F} G}\right)$ is the generic automorphism of $\Gamma$ and will be denoted by $\varphi_{\Gamma}$.

Let $\Gamma: V=\bigoplus_{g \in G} V_{g}$ be a $G$-grading on the $r$-fold cross product $(V, X)$. The diagonal group scheme $\operatorname{Diag}(\Gamma)$ is the diagonalizable group scheme whose group of $R$-points, for any $\mathbb{F}$-algebra $R$, consists of those automorphisms of the scalar extension $\left(V_{R}, X_{R}\right)$ that act by a scalar on each homogeneous component:

$$
\operatorname{Diag}(\Gamma)(R)=\left\{f \in \operatorname{Aut}\left(V_{R}, X_{R}\right)|f|_{V_{g} \otimes_{\mathbb{F}} R} \in R^{\times} \operatorname{id}_{V_{g} \otimes_{\mathbb{F}} R}\right\} .
$$

Up to isomorphism, $\operatorname{Diag}(\Gamma)$ is $U_{\text {diag }}$ for a finitely generated abelian group $U$, which is called the universal grading group of $\Gamma$. The morphism $\rho_{\Gamma}$ factors through $\operatorname{Diag}(\Gamma)$ and hence induces a natural group homomorphism $U \rightarrow G$. Moreover, $\Gamma$ may be considered, in a natural way, a grading by $U$, which is the most natural grading group for $\Gamma$.

Given a $G$-grading $\Gamma: V=\bigoplus_{g \in G} V_{g}$ and an $H$-grading $\Gamma^{\prime}: V=\bigoplus_{h \in H} V_{h}^{\prime}$ on $(V, X), \Gamma$ is said to be a refinement of $\Gamma^{\prime}$, or $\Gamma^{\prime}$ a coarsening of $\Gamma$, if for any $g \in G$ there is an element $h \in H$ such that $V_{g} \subseteq V_{h}^{\prime}$. In other words, each homogeneous component for $\Gamma^{\prime}$ is a sum of homogeneous components for $\Gamma$. The refinement is said to be proper if we have that $V_{g}$ is strictly contained in $V_{h}^{\prime}$ for at least one nonzero homogeneous component. In particular, given a $G$-grading $\Gamma: V=\bigoplus_{g \in G} V_{g}$ and a group homomorphism $\alpha: G \rightarrow H$, the grading $\Gamma^{\alpha}: V=\bigoplus_{h \in H} V_{h}^{\prime}$, where
$V_{h}^{\prime}=\bigoplus_{\alpha(g)=h} V_{g}$ for all $h \in H$, is a coarsening of $\Gamma$. Any homogeneous element for $\Gamma$ of degree $g$ is homogeneous too for $\Gamma^{\alpha}$ of degree $\alpha(g)$.

A grading $\Gamma$ is said to be fine if it admits no proper refinement. In that case, $\operatorname{Diag}(\Gamma)$ is a maximal diagonalizable subgroup scheme of $\boldsymbol{\operatorname { A u t }}(V, X)$.

Thus the classification of fine gradings up to equivalence is equivalent to the classification of maximal diagonalizable subgroup schemes of $\operatorname{Aut}(V, X)$ up to conjugation by elements in $\operatorname{Aut}(V, X)$.

Here two gradings $\Gamma: V=\bigoplus_{g \in G} V_{g}$ and $\Gamma^{\prime}: V=\bigoplus_{h \in H} V_{h}^{\prime}$ on $(V, X)$ are equivalent if there is an automorphism $\varphi \in \operatorname{Aut}(V, X)$ such that for any $g \in G$ there is $h \in H$ such that $\varphi\left(V_{g}\right)=V_{h}^{\prime}$.

Equivalence is weaker than isomorphism. For equivalence, the diagonal group schemes are imposed to be conjugate by an automorphism of $(V, X)$, while for isomorphism, we require that the morphisms of affine group schemes $\rho_{\Gamma}$ are conjugate by an automorphism.

Given a grading $\Gamma: V=\bigoplus_{g \in G} V_{g}$, the automorphism group of $\Gamma, \operatorname{Aut}(\Gamma)$ is the group of self-equivalences, that is, of automorphisms of $(V, X)$ that permute the homogeneous components. The stabilizer of $\Gamma: \operatorname{Stab}(\Gamma)$, is the kernel of the natural induced map $\operatorname{Aut}(\Gamma) \rightarrow \operatorname{Sym}(\operatorname{Supp}(\Gamma))$, where $\operatorname{Sym}(S)$ denotes the symmetric group on the set $S$. That is, $\operatorname{Stab}(\Gamma)$ is the subgroup of self-isomorphisms of $\Gamma$. The quotient $W(\Gamma):=\operatorname{Aut}(\Gamma) / \operatorname{Stab}(\Gamma)$ is called the Weyl group.

In this section, the ground field $\mathbb{F}$ will be assumed to be algebraically closed (and, as always, of characteristic not two), unless otherwise stated. All gradings by abelian groups on cross products will be classified up to isomorphism. The fine gradings will be classified up to equivalence too, thus obtaining the maximal diagonalizable subgroup schemes of the corresponding automorphism group schemes.

## 4.1. $n$ even, $r=1$.

Let $X: V \rightarrow V$ be a 1 -fold cross product on the even-dimensional vector space $V$ over an algebraically closed field $\mathbb{F}$.

Let $G$ be an abelian group, a $G$-grading $\Gamma: V=\bigoplus_{g \in G} V_{g}$ on $(V, X)$ is nothing else than a vector space decomposition of $V$ on subspaces invariant under the endomorphism $X: V \rightarrow V$.

Hence, if $V_{+}$and $V_{-}$are the subspaces defined in Corollary 3.2, a $G$-grading on ( $V, X$ ) determines $G$-gradings on the vector subspaces $V_{+}$and $V_{-}$and, conversely, any pair of $G$-gradings on $V_{+}$and $V_{-}$(as vector spaces) determines a $G$-grading on ( $V, X$ ).

In particular, up to equivalence, there is a unique fine grading $\Gamma$, with universal group $\mathbb{Z}^{n}, n=2 s$, obtained by fixing bases $\left\{v_{1}, \ldots, v_{s}\right\},\left\{w_{1}, \ldots, w_{s}\right\}$, of $V_{+}$and $V_{-}$, and assigning degrees as follows:

$$
\operatorname{deg}\left(v_{i}\right)=\epsilon_{i}, \quad \operatorname{deg}\left(w_{i}\right)=\epsilon_{s+i}, \quad i=1, \ldots, s
$$

where $\epsilon_{1}, \ldots, \epsilon_{n}$ is the canonical basis of $\mathbb{Z}^{n}$. The Weyl group $W(\Gamma)$ is $\operatorname{Sym}_{s} \times \operatorname{Sym}_{s}$, where $\operatorname{Sym}_{s}$ is the symmetric group on $s$ symbols, because any self-equivalence permutes the homogeneous components of $V_{+}$and $V_{-}$separately.

Remark 4.1. The above is valid for arbitrary fields $\mathbb{F}$ with $-1 \in \mathbb{F}^{2}$, and implies that a fine grading on $(V, X)$ is given by a pair of fine gradings on $V_{+}$and $V_{-}$as vector spaces. In particular, there is a unique fine grading up to equivalence, and the corresponding maximal diagonalizable subgroup scheme is just a maximal torus of $\mathbf{A u t}(V, X)=\mathbf{G} \mathbf{L}\left(V_{+}\right) \times \mathbf{G} \mathbf{L}\left(V_{-}\right)$. (All maximal tori are conjugate.)

If $-1 \notin \mathbb{F}^{2}$, then $\mathbb{K}=\mathbb{F i d} \oplus \mathbb{F} X$ is a field. Hence any homogeneous component of a grading $\Gamma: V=\bigoplus_{g \in G} V_{g}$ on $(V, X)$ is a vector space over $\mathbb{K}$, so $\operatorname{dim}_{\mathbb{F}} V_{g}$ is even
for any $g \in G$. In particular, $\Gamma$ is a coarsening of a grading by $\mathbb{Z}^{s}(n=2 s)$ obtained by taking a $\mathbb{K}$-basis $\left\{v_{1}, \ldots, v_{s}\right\}$ of $V$ and declaring that $\operatorname{deg}\left(v_{i}\right)=\epsilon_{i}$, where the $\epsilon_{i}$ 's are the canonical generators of $\mathbb{Z}^{s}$. Hence again there is a unique fine grading, up to equivalence, and the corresponding diagonalizable group scheme is a maximal split torus of $\boldsymbol{\operatorname { A u t }}(V, X)=\operatorname{Cent}_{\mathbf{G L}(V)}(X)$ (see Theorem 3.1).

## 4.2. $n \geq 3, r=n-1$.

Let $(V, X)$ be an $r$-fold cross product with $\operatorname{dim}_{\mathbb{F}} V=n \geq 3$ and $r=n-1$, over an algebraically closed field $\mathbb{F}$, relative to a nondegenerate symmetric bilinear form b, unique up to a scalar (Proposition (2.3). Equation (2.8) shows that $X$ is unique, up to isomorphism.

Let $G$ be an abelian group and let $\Gamma: V=\bigoplus_{g \in G} V_{g}$ be a $G$-grading on $(V, X)$. The generic automorphism $\varphi_{\Gamma}$ is in $\operatorname{Aut}(V, X)(\mathbb{F} G)=\widetilde{\mathbf{O}}(V, \mathrm{~b})(\mathbb{F} G)$.

Therefore, for any $g_{1}, g_{2} \in G$ and $v_{1} \in V_{g_{1}}, v_{2} \in V_{g_{2}}$, on the group algebra $\mathbb{F} G$ we have

$$
\mathrm{b}\left(v_{1}, v_{2}\right) g_{1} g_{2}=\mathrm{b}\left(v_{1} \otimes g_{1}, v_{2} \otimes g_{2}\right)=\operatorname{det}\left(\varphi_{\Gamma}\right) \mathrm{b}\left(v_{1}, v_{2}\right)
$$

and hence we get

$$
\begin{equation*}
\mathrm{b}\left(V_{g_{1}}, V_{g_{2}}\right)=0 \quad \text { unless } g_{1} g_{2}=\operatorname{det}\left(\varphi_{\Gamma}\right) \tag{4.1}
\end{equation*}
$$

Write $h=\operatorname{det}\left(\varphi_{\Gamma}\right)$ and consider the map

$$
\begin{aligned}
\delta: G & \longrightarrow \mathbb{Z}_{\geq 0} \\
g & \mapsto \delta(g):=\operatorname{dim}_{\mathbb{F}} V_{g}
\end{aligned}
$$

which satisfies the following restrictions:

- $\sum_{g \in G} \delta(g)=n$,
- $h=\operatorname{det}\left(\varphi_{\Gamma}\right)=\prod_{g \in G} g^{\delta(g)}$,
- $\delta(g)=\delta\left(g^{-1} h\right)$ for all $g \in G$. This is a consequence of (4.1).

By Proposition 3.4, the determinant $h=\operatorname{det}\left(\varphi_{\Gamma}\right)$ satisfies $h^{n-2}=e$. (e denotes the neutral element of $G$.)

Note that $h$ is the neutral element if and only if $\Gamma$ is compatible with b. Also, because of (4.1), for an element $g \in \operatorname{Supp} \Gamma$, the restriction $\left.\mathrm{b}\right|_{V_{g}}$ is nondegenerate if and only if $g^{2}=h$, otherwise $V_{g}$ is totally isotropic.

Conversely, given a map $\delta$ satisfying the restrictions above, we can always define a $G$-grading on $(V, X)$. For example, let $n=5, g_{1}, g_{2}, h \in G$ with $h^{3}=e, g_{1}^{2}=h$, $g_{2}^{2} \neq h$. (This implies that the elements $g_{1}, g_{2}$ and $g_{2}^{-1} h$ are different.) Assume that $\delta\left(g_{1}\right)=1, \delta\left(g_{2}\right)=2=\delta\left(g_{2}^{-1} h\right)$, and $\delta(g)=0$ for any $g \neq g_{1}, g_{2}, g_{2}^{-1} h$. Since $\mathbb{F}$ is algebraically closed we can take a basis $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$ of $V$ with $\mathrm{b}\left(v_{1}, v_{1}\right)=1=\mathrm{b}\left(v_{2}, v_{4}\right)=\mathrm{b}\left(v_{3}, v_{5}\right)$, and all the other values of $\mathrm{b}\left(v_{i}, v_{j}\right)$ equal to 0 . Then with $V_{g_{1}}=\mathbb{F} v_{1}, V_{g_{2}}=\mathbb{F} v_{2}+\mathbb{F} v_{3}, V_{g_{2}^{-1} h}=\mathbb{F} v_{4}+\mathbb{F} v_{5}$, and $V_{g}=0$ for $g \neq g_{1}, g_{2}, g_{2}^{-1} h$, we get a $G$-grading on $(V, X)$, as its generic automorphism lies in $\widetilde{\mathbf{O}}(V, \mathrm{~b})$.

Up to isomorphism, the $G$-grading associated to a map $\delta$ satisfying the restrictions above is unique, and will be denoted by $\Gamma(G, \delta)$.

We have proved our next result:
Theorem 4.2. Let $X: V^{n-1} \rightarrow V$ be an ( $n-1$ )-fold cross product on the $n$ dimensional vector space $V(n \geq 3)$ over an algebraically closed field $\mathbb{F}$, relative to a nondegenerate symmetric bilinear form b . Let $G$ be an abelian group.

Then any $G$-grading on $(V, X)$ is isomorphic to a grading $\Gamma(G, \delta)$ for a unique map $\delta: G \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the restrictions:

- $\sum_{g \in G} \delta(g)=n$,
- $\delta(g)=\delta\left(g^{-1} h\right)$ for all $g \in G$, where $h=\prod_{g \in G} g^{\delta(g)}$.

Let $\delta$ be a map satisfying the restrictions in Theorem 4.2, and let $\Gamma=\Gamma(G, \delta)$ be the associated grading. Then, joining suitable bases of the homogeneous components, we obtain a basis of $V$ consisting of homogeneous elements:

$$
\left\{u_{1}, \ldots, u_{p}, v_{1}, w_{1}, \ldots, v_{q}, w_{q}\right\} \quad(p, q \geq 0, n=p+2 q)
$$

with $\mathrm{b}\left(u_{i}, u_{i}\right)=1$ for all $i=1, \ldots, p, \mathrm{~b}\left(v_{j}, w_{j}\right)=1\left(=\mathrm{b}\left(w_{j}, v_{j}\right)\right)$ for all $j=1, \ldots, q$, and all other values of $b$ being 0 , and with

$$
\operatorname{deg}\left(u_{i}\right)=g_{i} \text { for } i=1, \ldots, p, \quad \operatorname{deg}\left(v_{j}\right)=g_{j}^{\prime}, \operatorname{deg}\left(w_{j}\right)=g_{j}^{\prime \prime} \text { for } j=1, \ldots, q
$$

for group elements $g_{1}, \ldots, g_{p}, g_{1}^{\prime}, g_{1}^{\prime \prime}, \ldots, g_{q}^{\prime}, g_{q}^{\prime \prime}$ satisfying

$$
g_{i}^{2}=h=g_{j}^{\prime} g_{j}^{\prime \prime}, \quad \text { where } h:=g_{1} \cdots g_{p} g_{1}^{\prime} g_{1}^{\prime \prime} \cdots g_{q}^{\prime} g_{q}^{\prime \prime}
$$

Note that $n=p+2 q$.
Let $U$ be the abelian group, defined by generators and relations, with generators $x_{1}, \ldots, x_{p}, y_{1}, z_{1}, \ldots, y_{q}, z_{q}$, subject to the relations:

$$
\begin{equation*}
x_{1}^{2}=\cdots=x_{p}^{2}=y_{1} z_{1}=\cdots=y_{q} z_{q}=x_{1} \cdots x_{p} y_{1} z_{1} \cdots y_{q} z_{q} \tag{4.2}
\end{equation*}
$$

(We will write $U=\left\langle x_{1}, \ldots, x_{p}, y_{1}, z_{1}, \ldots, y_{p}, z_{p}\right| x_{1}^{2}=\cdots=x_{p}^{2}=y_{1} z_{1}=\cdots=$ $\left.y_{q} z_{q}=x_{1} \cdots x_{p} y_{1} z_{1} \cdots y_{q} z_{q}\right\rangle$.)

Also, let $\delta_{U}: U \rightarrow \mathbb{Z}_{\geq 0}$ be the map defined by

$$
\delta_{U}\left(x_{i}\right)=\delta_{U}\left(y_{j}\right)=\delta_{U}\left(z_{j}\right)=1
$$

for all $i=1, \ldots, p, j=1, \ldots, q$, and $\delta_{U}(u)=0$ for any other element $u \in U$.
This is well defined and satisfies the conditions of Theorem 4.2 because the elements $x_{1}, \ldots, x_{p}, y_{1}, z_{1}, \ldots, y_{q}, z_{q}$ are all different. Actually, we get the following possibilities:

- If $p=0$, then $n=2 q$ is even with $q \geq 2$, and

$$
U=\left\langle y_{1}, z_{1}, \ldots, y_{q}, z_{q} \mid y_{1} z_{1}=\cdots=y_{q} z_{q}=y_{1} z_{1} \cdots y_{q} z_{q}\right\rangle
$$

With $u=y_{1} z_{1}=\cdots=y_{q} z_{q}=y_{1} z_{1} \cdots y_{q} z_{q}$, we have $u=u^{q}$. Then $U$ is generated by the elements $u, y_{1}, \ldots, y_{q}$ and is isomorphic to $\mathbb{Z}^{q} \times$ $\mathbb{Z} /(q-1)$ by means of the map that takes $(0, \ldots, 1, \ldots, 0 ; \overline{0})$ to $y_{j}$, and $(0, \ldots,-1, \ldots, 0 ; \overline{1})$ to $z_{j}$ ( 1 in the $j$ th position). This shows that the elements $y_{i}, z_{i}$ are all different.

- If $p=1$, then $q \geq 1, n=1+2 q$ is odd, and

$$
U=\left\langle x_{1}, y_{1}, z_{1}, \ldots, y_{q}, z_{q} \mid x_{1}^{2}=y_{1} z_{1}=\cdots=y_{q} z_{q}=x_{1} y_{1} z_{1} \cdots y_{q} z_{q}\right\rangle
$$

Thus $x_{1}^{2}=x_{1}\left(x_{1}^{2}\right)^{q}=x_{1}^{1+2 q}$, so $x_{1}^{2 q-1}=e$. If $q=1$, we get $x_{1}=e$ and $U$ is isomorphic to $\mathbb{Z}$. If $q \geq 2$, then $U$ is generated by $x_{1}, y_{1}, \ldots, y_{q}$ and it is isomorphic to $\mathbb{Z}^{q} \times \mathbb{Z} /(2 q-1)$ by means of the map that takes $(0, \ldots, 0 ; \overline{1})$ to $x_{1},(0, \ldots, 1, \ldots, 0 ; \overline{0})$ to $y_{j}$, and $(0, \ldots,-1, \ldots, 0 ; \overline{2})$ to $z_{j}( \pm 1$ in the $j$ th position).

- Finally, if $p \geq 2$, write $x_{i}=x_{1} t_{i}$, with $t_{i}^{2}=e$, for $i=2, \ldots p$. The relations (4.2) imply $x_{1}^{2}=x_{1} \cdots x_{p} y_{1} z_{1} \cdots y_{q} z_{q}=x_{1}^{p+2 q} t_{2} \cdots t_{p}$, so $t_{2} \cdots t_{p}=$ $x_{1}^{p+2 q-2}=x_{1}^{n-2}$ and $x_{1}^{2 n-4}=e$. Then $U$ is generated by the elements $x_{1}, t_{2}, \ldots, t_{p-1}, y_{1}, \ldots, y_{q}$ and it is isomorphic to $\mathbb{Z}^{q} \times \mathbb{Z} /(2 n-4) \times(\mathbb{Z} / 2)^{p-2}$ by means of the map that takes

$$
\begin{array}{rlr}
(0, \ldots, 0 ; \overline{1} ; \overline{0}, \ldots, \overline{0}) & \mapsto x_{1} & (0, \ldots, 1, \ldots, 0 ; \overline{0} ; \overline{0}, \ldots, \overline{0}) \mapsto y_{j} \\
(0, \ldots, 0 ; \overline{1} ; \overline{0}, \ldots, \overline{1}, \ldots, \overline{0}) & \mapsto x_{i} & (0, \ldots,-1, \ldots, 0 ; \overline{2} ; \overline{0}, \ldots, \overline{0}) \mapsto z_{j} \\
(0, \ldots, 0 ; \overline{n-1} ; \overline{1}, \ldots, \overline{1}) & \mapsto x_{p} & \\
\text { for } 2 \leq i \leq p-1,1 \leq j & \leq q . &
\end{array}
$$

The map $\delta_{U}$ satisfies the restrictions in Theorem 4.2, so it determines a grading $\Gamma\left(U, \delta_{U}\right)$, with one-dimensional homogeneous components, and hence fine, that refines $\Gamma(G, \delta)$. Moreover, $U$ is its universal grading group.

This gives the classification of fine gradings, up to equivalence:
Corollary 4.3. Let $X: V^{n-1} \rightarrow V$ be an $(n-1)$-fold cross product on the $n$ dimensional vector space $V(n \geq 3)$ over an algebraically closed field $\mathbb{F}$, relative to a nondegenerate symmetric bilinear form b .

Up to equivalence, the fine gradings on $(V, X)$ are the gradings $\Gamma\left(U, \delta_{U}\right)$, where $U$ is the abelian group

$$
\begin{aligned}
& U=\left\langle x_{1}, \ldots, x_{p}, y_{1}, z_{1}, \ldots, y_{p}, z_{p}\right| \\
& \left.x_{1}^{2}=\cdots=x_{p}^{2}=y_{1} z_{1}=\cdots=y_{q} z_{q}=x_{1} \cdots x_{p} y_{1} z_{1} \cdots y_{q} z_{q}\right\rangle
\end{aligned}
$$

with $p+2 q=n$, and $\delta_{U}: U \mapsto \mathbb{Z}_{\geq 0}$ is the map given by

$$
\delta_{U}\left(x_{i}\right)=\delta_{U}\left(y_{j}\right)=\delta_{U}\left(z_{j}\right)=1
$$

for all $i=1, \ldots, p, j=1, \ldots, q$, and $\delta_{U}(u)=0$ for any other element $u \in U$.
Moreover, $U$ is, up to isomorphism, the universal grading group of $\Gamma\left(U, \delta_{U}\right)$ and the following conditions hold:

- if $p=0, U$ is isomorphic to $\mathbb{Z}^{q} \times \mathbb{Z} /(q-1)$,
- if $p=1, U$ is isomorphic to $\mathbb{Z}^{q} \times \mathbb{Z} /(2 q-1)$,
- if $p>1, U$ is isomorphic to $\mathbb{Z}^{q} \times \mathbb{Z} /(2 n-4) \times(\mathbb{Z} / 2)^{p-2}$.

Any self-equivalence of a fine grading $\Gamma\left(U, \delta_{U}\right)$ in Corollary 4.3 permutes the homogeneous spaces of degrees $x_{i}, 1 \leq i \leq p$, and the pairs of homogeneous spaces of degrees $y_{i}$ and $z_{i}$. Therefore, the Weyl group is the Cartesian product of the symmetric group on $p$ symbols, and the 'signed permutation group' on $q$ symbols:

$$
W\left(\Gamma\left(U, \delta_{U}\right)\right)=\operatorname{Sym}_{p} \times\left((\mathbb{Z} / 2)^{q} \rtimes \operatorname{Sym}_{q}\right) .
$$

Example 4.4. Let $Q$ be the algebra of quaternions over the algebraically closed field $\mathbb{F}$, with norm n , which is multiplicative. Up to isomorphism $Q$ is the algebra of $2 \times 2$ matrices over $\mathbb{F}$, and the norm is given by the determinant. Let $x \mapsto \bar{x}=\mathrm{n}(x, 1) 1-x$ be the canonical involution. It satisfies $x \bar{x}=\mathrm{n}(x) 1=\bar{x} x$ for all $x \in \mathcal{Q}$. Define the multilinear map

$$
\begin{aligned}
X: Q^{3} & \longrightarrow \mathcal{Q} \\
(x, y, z) & \mapsto X(x, y, z):=x \bar{y} z-z \bar{y} x
\end{aligned}
$$

This map is alternating, because $X(x, x, z)=x \bar{x} z-z \bar{x} x=\mathrm{n}(x) z-\mathrm{n}(x) z=0$, and $X(x, y, y)=x \bar{y} y-y \bar{y} x=\mathrm{n}(y) x-\mathrm{n}(y) x=0$. Besides, for all $x, y, z \in \mathcal{Q}$, $\mathrm{n}(X(x, y, z), x)=\mathrm{n}(x \bar{y} z, x)-\mathrm{n}(z \bar{y} x, x)=\mathrm{n}(x) \mathrm{n}(\bar{y} z-z \bar{y}, 1)=0$.

It turns out that $X$ is a 3 -fold cross product on $Q$. Indeed, for any $x, y, z \in \mathcal{Q}$ and $\lambda \in \mathbb{F}$, the coefficient of $\lambda^{2}$ in both sides of the equality $\mathrm{n}((x+\lambda z) \bar{y}(x+\lambda z))=$ $\mathrm{n}(x+\lambda z)^{2} \mathrm{n}(y)$, gives

$$
\begin{equation*}
\mathrm{n}(x \bar{y} z+z \bar{y} x)+\mathrm{n}(x \bar{y} x, z \bar{y} z)=\left(\mathrm{n}(x, z)^{2}+2 \mathrm{n}(x) \mathrm{n}(z)\right) \mathrm{n}(y) . \tag{4.3}
\end{equation*}
$$

Also we have

$$
\begin{align*}
\mathrm{n}(x \bar{y} z+z \bar{y} x) & =\mathrm{n}(x \bar{y} z)+\mathrm{n}(z \bar{y} x)+\mathrm{n}(x \bar{y} z, z \bar{y} x) \\
& =2 \mathrm{n}(x) \mathrm{n}(y) \mathrm{n}(z)+\mathrm{n}(x \bar{y} z, z \bar{y} x), \tag{4.4}
\end{align*}
$$

and

$$
\begin{align*}
\mathrm{n}(x \bar{y} x, z \bar{y} z)= & \mathrm{n}(\mathrm{n}(x, y) x-\mathrm{n}(x) y, \mathrm{n}(y, z) z-\mathrm{n}(z) y) \\
= & \mathrm{n}(x, y) \mathrm{n}(y, z) \mathrm{n}(z, x)-\mathrm{n}(x) \mathrm{n}(y, z)^{2}  \tag{4.5}\\
& \quad-\mathrm{n}(z) \mathrm{n}(x, y)^{2}+2 \mathrm{n}(x) \mathrm{n}(y) \mathrm{n}(z) .
\end{align*}
$$

Putting all these together, we get:

$$
\begin{aligned}
\mathrm{n}(X(x, y, z), & X(x, y, z)) \\
& =\mathrm{n}(x \bar{y} z-z \bar{y} x, x \bar{y} z-z \bar{y} x) \\
& =\mathrm{n}(x \bar{y} z, x \bar{y} z)+\mathrm{n}(z \bar{y} x, z \bar{y} x)-2 \mathrm{n}(x \bar{y} z, z \bar{y} x) \\
& =8 \mathrm{n}(x) \mathrm{n}(y) \mathrm{n}(z)-2 \mathrm{n}(x \bar{y} z+z \bar{y} x) \quad(\text { by (4.4) }) \\
& =4 \mathrm{n}(x) \mathrm{n}(y) \mathrm{n}(z)+2 \mathrm{n}(x \bar{y} x, z \bar{y} z)-2 \mathrm{n}(x, z)^{2} \mathrm{n}(y) \quad \text { (by (4.3)) } \\
& =\left|\begin{array}{lll}
\mathrm{n}(x, x) & \mathrm{n}(x, y) & \mathrm{n}(x, z) \\
\mathrm{n}(y, x) & \mathrm{n}(y, y) & \mathrm{n}(y, z) \\
\mathrm{n}(z, x) & \mathrm{n}(z, y) & \mathrm{n}(z, z)
\end{array}\right| \quad \text { (by (4.5)), }
\end{aligned}
$$

so that $X$ turns out to be a 3 -fold cross product relative to the polar form $\mathrm{n}(.,$.$) .$
Corollary 4.3 shows that, up to equivalence, there are three different fine gradings on ( $\mathbb{Q}, X$ ) with universal groups $\mathbb{Z}^{2}(p=0, q=2), \mathbb{Z} \times \mathbb{Z} / 4(p=2, q=1)$, and $\mathbb{Z} / 4 \times(\mathbb{Z} / 2)^{2}(p=4, q=0)$.

In contrast to this, the quaternion algebra $Q$ has only two fine gradings, with universal groups $\mathbb{Z}$ and $(\mathbb{Z} / 2)^{2}$ (see, e.g., [EK13, Example 2.40]).

## 4.3. $n=7, r=2$.

Any 2 -fold cross product on a seven-dimensional vector space over an algebraically closed field $\mathbb{F}$ is isomorphic to $\left(\mathcal{C}_{0}, X^{\mathcal{C}_{0}}\right)$ for the unique Cayley algebra $\mathcal{C}$ over $\mathbb{F}$ (Theorem 2.1).

Given an abelian group $G$, any $G$-grading on ( $\complement_{0}, X^{\mathfrak{C}_{0}}$ ) extends uniquely to a grading on the Cayley algebra $\mathcal{C}$, and two such gradings on $\left(\mathcal{C}_{0}, X^{\mathcal{C}_{0}}\right)$ are isomorphic if and only if so are the extended gradings on $\mathcal{C}$ (see Eld98] or EK13, Corollary 4.25]).

Moreover, the classification of $G$-gradings, up to isomorphism, on the Cayley algebra $\mathcal{C}$ is given in EK13, Theorem 4.21], based on Eld98. There are two types of gradings, those obtained as a coarsening of the $\mathbb{Z}^{2}$-grading given by the weight space decomposition relative to a maximal torus of $\boldsymbol{\operatorname { A u t }}(\mathcal{C})$, and those equivalent to the $(\mathbb{Z} / 2)^{3}$-grading obtained by means of the Cayley-Dickson doubling process.

In particular, there exist only two fine gradings, up to equivalence, with universal group $\mathbb{Z}^{2}$ and $(\mathbb{Z} / 2)^{3}$, respectively. The respective Weyl groups are the Weyl group of the root system of type $G_{2}$ (i.e., the dihedral group of order 12) and the general linear group $\mathrm{GL}_{3}\left(\mathbb{F}_{2}\right)$ of degree 3 over the field of two elements (see EK13, Theorems 4.17 and 4.19]).

## 4.4. $n=8, r=3$.

Any 3-fold cross product on an eight-dimensional vector space over an algebraically closed field $\mathbb{F}$ is isomorphic, by Theorem 2.1 and using that $\mathbb{F}$ is algebraically closed, to $\left(\mathcal{C}, X_{1}^{\mathcal{C}}\right)$ for the unique Cayley algebra $\mathcal{C}$ over $\mathbb{F}$, where $X_{1}^{\mathcal{C}}$ is given by Equation (2.5).

Equation (3.4) shows that the group schemes $\operatorname{Aut}\left(\mathcal{C}, X_{1}^{\mathcal{C}}\right)$ and $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ coincide, where the triple product $\{\ldots\}$ is given in (3.2): $\{x y z\}:=(x \bar{y}) z$ for $x, y, z \in \mathcal{C}$. Moreover, Theorem 3.10 and its proof show that $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ is isomorphic to $\operatorname{Spin}\left(\mathfrak{C}_{0},-n\right)$ and is contained in the special orthogonal group scheme $\mathbf{O}^{+}(\mathcal{C}, n)$, where $n$ denotes the norm of the Cayley algebra $\mathcal{C}$.

Therefore, in order to study gradings on the cross product ( $\mathcal{C}, X_{1}^{\mathcal{C}}$ ), it is enough to study gradings on ( $\mathcal{C},\{\ldots\})$.

We need some previous results. Continuing the discussion in Remark 3.9, Eld96, Theorem 10] and [Eld00, Corollary 2.5] give:

$$
\begin{equation*}
\operatorname{Aut}(\mathcal{C},\{\ldots\})=\left\{\prod_{i=1}^{m} L_{x_{i}} \mid m \geq 0, x_{i} \in \mathcal{C}_{0} \forall i, \text { and } \prod_{i=1}^{m} \mathrm{n}\left(x_{i}\right)=1\right\} \tag{4.6}
\end{equation*}
$$

where $L_{x}: y \mapsto x y$ denotes the left multiplication by $x$ in $\mathcal{C}$. Take a standard basis $\left\{e_{1}, e_{2}, u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}\right\}$ of $\mathcal{C}$ (see, e.g., EK13, p. 41]), with multiplication given by:

- $e_{1}$ and $e_{2}$ are orthogonal idempotents: $e_{1}^{2}=e_{1}, e_{2}^{2}=e_{2}, e_{1} e_{2}=e_{2} e_{1}=0$,
- $e_{1} u_{i}=u_{i} e_{2}=u_{i}, e_{2} u_{i}=u_{i} e_{1}=0$, for all $i=1,2,3$,
- $e_{2} v_{i}=v_{i} e_{1}=v_{i}, e_{1} v_{i}=v_{i} e_{2}=0$, for all $i=1,2,3$,
- $u_{i} v_{i}=-e_{1}, v_{i} u_{i}=-e_{2}$, for all $i=1,2,3$,
- $u_{i} u_{i+1}=-u_{i+1} u_{i}=v_{i+2}, v_{i} v_{i+1}=-v_{i+1} v_{i}=u_{i+2}$, indices modulo 3,
- all other products between basic elements are 0 .

Lemma 4.5. Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field $\mathbb{F}$.
(i) The orbit of 1 under $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ is the 'unit sphere':

$$
\operatorname{orbit}_{A u t(\mathcal{C},\{\ldots\})}(1)=\{x \in \mathcal{C} \mid \mathrm{n}(x)=1\}
$$

(ii) The orbit of $e_{1}$ is the set of nonzero isotropic elements:

$$
\operatorname{orbit}_{\operatorname{Aut}(\mathcal{C},\{\ldots\})}\left(e_{1}\right)=\{x \in \mathcal{C} \mid \mathrm{n}(x)=0, x \neq 0\}
$$

(iii) The orbit of the pair $\left(e_{1}, e_{2}\right)$ under the diagonal action of $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ on $\mathcal{C} \times \mathcal{C}$ is the set

$$
\operatorname{orbit}_{\operatorname{Aut}(\mathcal{C},\{\ldots\})}\left(e_{1}, e_{2}\right)=\{(x, y) \in \mathcal{C} \times \mathcal{C} \mid \mathrm{n}(x)=0=\mathrm{n}(y), \mathrm{n}(x, y)=1\}
$$

Proof. As Aut $(\mathcal{C},\{\ldots\})$ is contained in the orthogonal group of the norm, it is clear that the orbit of 1 is contained in the set of norm 1 elements. Conversely, if $x \in \mathcal{C}$ and $\mathrm{n}(x)=1$, take $z \in \mathcal{C}$ orthogonal to 1 and $x$ of norm 1. Then $x=(x \bar{z}) z=L_{x \bar{z}} L_{z}(1)$ and $L_{x \bar{z}} L_{z} \in \operatorname{Aut}(\mathcal{C},\{\ldots\})$.

It is clear that the orbit of $e_{1}$ is contained in the set of nonzero elements of zero norm.

If $0 \neq x \in \mathcal{C}_{0}$ is an element with $\mathrm{n}(x)=0$, take $y \in \mathcal{C}_{0}$ with $\mathrm{n}(y)=0$ and $\mathrm{n}(x, y)=1$. Then $-x y-y x=x \bar{y}+y \bar{x}=\mathrm{n}(x, y) 1=1$ and $(x y)(x y)=(x y)(-1-$ $y x)=-x y$, as $y^{2}=0$ because of (2.1). It turns out that $f_{1}=-x y$ and $f_{2}=-y x$ are orthogonal idempotents, and thus (see, e.g. [EK13, pp. 128-129]) there is an automorphism of $\mathcal{C}$ (hence also of $(\mathcal{C},\{\ldots\})$ ) such that $\varphi\left(f_{1}\right)=e_{1}, \varphi\left(f_{2}\right)=e_{2}$.

Also, $f_{2} x=-y x^{2}=0, x f_{1}=-x^{2} y=0$, and hence $f_{1} x=x f_{2}=x$, so that $\varphi(x) \in\left\{z \in \mathcal{C}: e_{1} z=z e_{2}=z\right\}=U:=\mathbb{F} u_{1}+\mathbb{F} u_{2}+\mathbb{F} u_{3}$. But any $u \in U$ is mapped to $-u_{3}$ by a suitable automorphism of $\mathcal{C}$ that preserves $e_{1}$ and $e_{2}$. Hence we may assume that $\varphi(x)=-u_{3}$. Note that $L_{u_{2}+v_{2}} L_{u_{1}+v_{1}}\left(e_{1}\right)=-u_{3}$ and we conclude that $x$ is in the orbit of $e_{1}$.

If $\mathrm{n}(x)=0$ but $x \notin \mathcal{C}_{0}$, take $y \in \mathcal{C}_{0}$ orthogonal to $x$ and with $\mathrm{n}(y)=1$. Then $L_{y} \in \operatorname{Aut}(\mathrm{C},\{\ldots\})$ and $L_{y}(x)=y x$ satisfies $\overline{y x}=-\bar{x} y=-\mathrm{n}(x, y) 1+\bar{y} x=-y x$, so $y x \in \mathcal{C}_{0}$. By the previous paragraph, $y x$ lies in the orbit of $e_{1}$, and so does $x$.

Finally, since $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ is contained in the orthogonal group, it is clear that the orbit of $\left(e_{1}, e_{2}\right)$ is contained in $\{(x, y) \in \mathcal{C} \times \mathcal{C} \mid \mathrm{n}(x)=0=\mathrm{n}(y), \mathrm{n}(x, y)=1\}$. Conversely, let $x, y \in \mathcal{C}$ with $\mathrm{n}(x)=0=\mathrm{n}(y), \mathrm{n}(x, y)=1$, and take $z \in \mathcal{C}$ orthogonal to $1, x$, and $y$, with $\mathrm{n}(z)=1$, and take $t \in \mathcal{C}$ orthogonal to 1 and $z$ with $\mathrm{n}(t)=1$. Let $a=z t^{-1}=z \bar{t}, b=t$. Then $a, b \in \mathcal{C}_{0}, \mathrm{n}(a)=\mathrm{n}(b)=1$, so $L_{a} L_{b} \in \operatorname{Aut}(\mathcal{C},\{\ldots\})$. Besides, use (2.2) to get $\mathrm{n}\left(L_{a} L_{b}(x), 1\right)=\mathrm{n}(x, \bar{b} \bar{a})=$ $\mathrm{n}(x, \overline{a b})=\mathrm{n}(x, z)=0$, and $\mathrm{n}\left(L_{a} L_{b}(y), 1\right)=\mathrm{n}(y, z)=0$ too. Therefore, we may assume that $x$ and $y$ are elements in $\mathcal{C}_{0}$. With some of the arguments above we may
then assume that $x y=-e_{2}$ and $y x=-e_{1}, x \in V=\bigoplus_{i=1}^{3} \mathbb{F} v_{i}, y \in U=\bigoplus_{i=1}^{3} \mathbb{F} u_{i}$, and even that $x=v_{1}$ and $y=u_{1}$. The operator $\varphi=L_{u_{1}-v_{1}} L_{e_{1}-e_{2}}$ is in $\operatorname{Aut}(\mathcal{C},\{\ldots\})$ and $\varphi\left(v_{1}\right)=e_{1}, \varphi\left(u_{1}\right)=e_{2}$. The result follows.

Let us define a few natural gradings on ( $\mathcal{C},\{\ldots\}$ ).
Example 4.6 (Cartan grading). The following assignment of degrees in $\mathbb{Z}^{3}$ gives a grading on $(\mathcal{C},\{\ldots\})$, which is called the Cartan grading, as its homogeneous components are the weight spaces relative to a maximal torus $\mathbf{T}$ of $\boldsymbol{A u t}(\mathcal{C},\{\ldots\})$ :

$$
\begin{aligned}
\operatorname{deg}\left(u_{1}\right) & =(1,0,0)=-\operatorname{deg}\left(v_{1}\right), \\
\operatorname{deg}\left(u_{2}\right) & =(0,1,0)=-\operatorname{deg}\left(v_{2}\right), \\
\operatorname{deg}\left(u_{3}\right) & =(0,0,1)=-\operatorname{deg}\left(v_{3}\right), \\
\operatorname{deg}\left(e_{2}\right) & =(1,1,1)=-\operatorname{deg}\left(e_{1}\right) .
\end{aligned}
$$

This grading, denoted by $\Gamma_{\text {Cartan }}^{\mathcal{C}}$, is fine since all the homogeneous components have dimension 1, and it is straightforward to check that $\mathbb{Z}^{3}$ is its universal grading group.

The maximal torus $\mathbf{T}$ is precisely $\operatorname{Diag}\left(\Gamma_{\text {Cartan }}^{\mathcal{C}}\right)$, and we will identify the associated Weyl group $W\left(\Gamma_{\text {Cartan }}^{\mathcal{C}}\right)$ with a subgroup of $\mathrm{GL}_{3}(\mathbb{Z})=\operatorname{Aut}\left(\mathbb{Z}^{3}\right)$. Note that $\operatorname{Aut}(\Gamma)$ is the normalizer of $T=\mathbf{T}(\mathbb{F})$ in $\operatorname{Aut}(\mathcal{C},\{\ldots\})$, while $\operatorname{Stab}(\Gamma)$ is its centralizer. The quotient $W\left(\Gamma_{\text {Cartan }}^{\mathcal{C}}\right)$ is then the Weyl group of the root system of type $B_{3}$, which is the signed permutation group $(\mathbb{Z} / 2)^{3} \rtimes \mathrm{Sym}_{3}$.

Given any abelian group $G$ and a group homomorphism $\alpha: \mathbb{Z}^{3} \rightarrow G$, denote by $\Gamma^{\mathcal{C}}(G, \alpha)$ the $G$-grading on $(\mathcal{C},\{\ldots\})$ obtained as a coarsening of $\Gamma_{\text {Cartan }}^{\mathcal{C}}$ by means of $\alpha$. That is, the degree in $\Gamma^{\mathcal{C}}(G, \alpha)$ is the image under $\alpha$ of the degree in $\Gamma_{\text {Cartan }}^{\mathcal{E}}$ for any homogeneous element relative to $\Gamma_{\text {Cartan }}^{\mathcal{C}}$.

There is another interesting basis $\left\{1, w_{i} \mid 1 \leq i \leq 7\right\}$ of $\mathcal{C}$, with multiplication determined by

$$
\begin{align*}
& w_{i}^{2}=-1 \text { for all } i, \\
& w_{i} w_{i+1}=-w_{i+1} w_{i}=w_{i+3}(\text { cyclically in } i, i+1, i+3, \text { indices modulo } 7) . \tag{4.7}
\end{align*}
$$

Thus, for example, $w_{1} w_{2}=w_{4}, w_{4} w_{1}=w_{2}, \ldots$
This will be called a Cayley-Dickson basis, or CD-basis for short. Any CD-basis is an orthonormal basis relative to n . For example, one can take $w_{1}=\mathbf{i}\left(e_{1}-e_{2}\right)$, $w_{2}=u_{1}+v_{1}, w_{3}=u_{2}+v_{2}$, and then $w_{4}=w_{1} w_{2}=\mathbf{i}\left(u_{1}-v_{1}\right), w_{5}=u_{3}+v_{3}$, $w_{6}=\mathbf{i}\left(u_{3}-v_{3}\right), w_{7}=\mathbf{i}\left(u_{2}-v_{2}\right)$, where $\mathbf{i}$ is a square root of -1 .
Example 4.7. Let $H$ be an elementary 2 -subgroup of rank 3 of an abelian group $G$, so that $H$ is isomorphic to $(\mathbb{Z} / 2)^{3}$, and let $h_{1}, h_{2}, h_{3}$ be generators of $H$. Denote by $\Gamma^{\mathcal{C}}(G, H)$ the grading on the Cayley algebra $\mathcal{C}$ determined by $\operatorname{deg}\left(w_{i}\right)=h_{i}$ for $i=1,2,3$ (the elements $w_{i}, i=1,2,3$, generate the algebra $\mathcal{C}$ ). Then we have $\operatorname{deg}\left(w_{4}\right)=h_{1} h_{2}, \operatorname{deg}\left(w_{5}\right)=h_{2} h_{3}, \operatorname{deg}\left(w_{6}\right)=h_{1} h_{2} h_{3}$ and $\operatorname{deg}\left(w_{7}\right)=h_{1} h_{3}$.

This grading is, up to isomorphism, independent of the choice of generators of $H$, because of [EK13, Theorem 4.19]. Its support is, precisely, the subgroup $H$.

Any grading on $\mathcal{C}$ is a grading on $(\mathcal{C},\{\ldots\})$, so this gives a grading of $(\mathcal{C},\{\ldots\})$, also denoted by $\Gamma^{\mathcal{C}}(G, H)$.

Given a grading $\Gamma$ of $(\mathcal{C},\{\ldots\})$ by an abelian group $G$, and given an order two element $h \in G$, the shift of $\Gamma$ by $h$ is the new grading $\Gamma^{[h]}$ with $\operatorname{deg}_{\Gamma^{[h]}}(x)=$ $h \operatorname{deg}_{\Gamma}(x)$. It is clear that $\Gamma^{[h]}$ is a grading of $(\mathcal{C},\{\ldots\})$ too.
Example 4.8. Let $H$ be an elementary 2-subgroup of rank 4 of an abelian group $G$, and let $K$ be a subgroup of $H$ of index two (and hence $K$ is isomorphic to $\left.(\mathbb{Z} / 2)^{3}\right)$. Take an element $h \in H \backslash K$, and consider the shift $\left(\Gamma^{\mathcal{C}}(G, K)\right)^{[h]}$.

Item (i) of Lemma4.5shows that, up to isomorphism, this grading is independent of $h$, and will be denoted by $\Gamma^{\mathrm{C}}(G, H, K)$. Its support is $H \backslash K$.

We are ready to classify the gradings of the 3 -fold cross product ( $\mathcal{C}, X_{1}^{\mathcal{C}}$ ) or, equivalently, of the triple system $(\mathcal{C},\{\ldots\})$, up to isomorphism.

Theorem 4.9. Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field $\mathbb{F}$, let $G$ be an abelian group and let $\Gamma: \mathcal{C}=\bigoplus_{g \in G} \mathcal{C}_{g}$ be a grading of $(\mathcal{C},\{\ldots\})$. Then $\Gamma$ is isomorphic to one of the following gradings:

- $\Gamma^{\mathrm{C}}(G, \alpha)$, for a group homomorphism $\alpha: \mathbb{Z}^{3} \rightarrow G$.
- $\Gamma^{\mathcal{C}}(G, H)$ for an elementary 2-subgroup $H$ of rank 3 .
- $\Gamma^{\mathbb{C}}(G, H, K)$ for an elementary 2-subgroup $H$ of rank 4 and a subgroup $K$ of $H$ of index 2.
Gradings on different items are not isomorphic, and
- Two gradings $\Gamma^{\mathcal{C}}(G, \alpha)$ and $\Gamma^{\mathcal{C}}\left(G, \alpha^{\prime}\right)$ are isomorphic if and only if there is an element $\omega$ in the Weyl group $W\left(\Gamma_{\text {Cartan }}^{\mathcal{E}}\right)$ such that $\alpha^{\prime}=\alpha \omega$.
- Two gradings $\Gamma^{\mathcal{C}}(G, H)$ and $\Gamma^{\mathcal{C}}\left(G, H^{\prime}\right)$ are isomorphic if and only if $H=$ $H^{\prime}$.
- Two gradings $\Gamma^{\mathbb{C}}(G, H, K)$ and $\Gamma^{\mathcal{C}}\left(G, H^{\prime}, K^{\prime}\right)$ are isomorphic if and only if $H=H^{\prime}$ and $K=K^{\prime}$.

Proof. Let $\Gamma: \mathcal{C}=\bigoplus_{g \in G} \mathcal{C}_{g}$ be a grading by the abelian group $G$ of $(\mathcal{C},\{\ldots\})$.
Assume first that there is a nonzero homogeneous element $x \in \mathcal{C}_{g}$ with $\mathrm{n}(x)=0$. As $\operatorname{Aut}(\mathcal{C},\{\ldots\}) \subseteq \mathbf{O}^{+}(\mathcal{C}, \mathrm{n}), \mathrm{n}\left(\mathfrak{C}_{g_{1}}, \mathfrak{C}_{g_{2}}\right)=0$ unless $g_{1} g_{2}=e$, so there is an element $y \in \mathcal{C}_{g^{-1}}$ such that $\mathrm{n}(x, y)=1$ and $\mathrm{n}(y)=0$. By Lemma 4.5(iii), we may assume that $e_{1}$ and $e_{2}$ are homogeneous, and $\operatorname{deg}\left(e_{1}\right)=g, \operatorname{deg}\left(e_{2}\right)=g^{-1}$. But then $\left\{e_{1} \mathrm{C} e_{2}\right\}=e_{1} \mathrm{C} e_{2}=U=\mathbb{F} u_{1}+\mathbb{F} u_{2}+\mathbb{F} u_{3}$, and $\left\{e_{2} \mathrm{C} e_{1}\right\}=V=\mathbb{F} v_{1}+\mathbb{F} v_{2}+\mathbb{F} v_{3}$, are graded subspaces of $\mathcal{C}$. In particular, we can take a homogeneous basis $\left\{u_{1}^{\prime}, u_{2}^{\prime}, u_{3}^{\prime}\right\}$ of $U$ and, multiplying by a nonzero scalar $u_{3}^{\prime}$ if needed, with $\mathrm{n}\left(u_{1}^{\prime}, u_{2}^{\prime} u_{3}^{\prime}\right)=1$. There is then an automorphism of $\mathcal{C}$ that fixes $e_{1}$ and $e_{2}$ and takes $u_{i}$ to $u_{i}^{\prime}$ for $i=1,2,3$, so we may assume that the $u_{i}$ 's are homogeneous too, and so are the $v_{i}$ 's (for instance, $v_{3}=\left\{u_{1} e_{1} u_{2}\right\}$ ). Therefore, $\Gamma$ is a coarsening of the Cartan grading, and hence isomorphic to $\Gamma^{\mathcal{C}}(G, \alpha)$ for a group homomorphism $\alpha: \mathbb{Z}^{3} \rightarrow G$.

Otherwise, all homogeneous components are one-dimensional and not isotropic. As $\{x, x, x\}=\mathrm{n}(x) x$ for all $x \in \mathcal{C}$, we conclude that the support of $\Gamma$ generates a 2-elementary abelian subgroup. There are two possibilities:

- If the neutral element $e$ of $G$ is in the support, by Lemma 4.5(i) we may assume that the unity 1 of $\mathcal{C}$ is homogeneous of degree $e: 1 \in \mathcal{C}_{e}$. But then, as $x y=\{x, 1, y\}$, it follows that $\Gamma$ is a $G$-grading of $\mathcal{C}$, with one-dimensional nonisotropic homogeneous components. This gives the second possibility (see [Eld98 or [EK13, Theorem 4.21]).
- Otherwise, again by Lemma 4.5(i) we may assume that the unity 1 of $\mathcal{C}$ is homogeneous: $1 \in \mathcal{C}_{g}$, for an order 2 element $g \in G$. Then in the shift $\Gamma^{[g]}, 1$ is homogeneous of degree $g^{2}=e$, and we are in the situation of the previous item. If $K$ is the support of $\Gamma^{[g]}$, then the subgroup generated by $g$ and $K$ is 2-elementary of rank 4 , and $\Gamma=\left(\Gamma^{[g]}\right)^{[g]}$ is, up to isomorphism, the grading $\Gamma^{\mathfrak{C}}(G, H, K)$.
Now it is clear that gradings in different items are not isomorphic. The support of $\Gamma^{\mathcal{C}}(G, H)$ is $H$, and of $\Gamma^{\mathcal{C}}(G, H, K)$ is $H \backslash K$. The isomorphism condition for gradings of type $\Gamma^{\mathcal{C}}(G, \alpha)$ follows from [EK13, Proposition 4.22].

The homogeneous components of the gradings $\Gamma^{\mathcal{C}}(G, H)$ and $\Gamma^{\complement}(G, H, K)$ in Theorem 4.9 are the subspaces spanned by the elements of a CD-basis (Equation
(4.7), and hence they are all equivalent to the grading $\Gamma_{\mathrm{CD}}^{\mathrm{e}}$ over the grading group $(\mathbb{Z} / 2)^{4}$ with

$$
\begin{aligned}
& \operatorname{deg}(1)=(\overline{1}, \overline{0}, \overline{0}, \overline{0}), \quad \operatorname{deg}\left(w_{1}\right)=(\overline{1}, \overline{1}, \overline{0}, \overline{0}), \\
& \operatorname{deg}\left(w_{2}\right)=(\overline{1}, \overline{0}, \overline{1}, \overline{0}), \quad \operatorname{deg}\left(w_{3}\right)=(\overline{1}, \overline{0}, \overline{0}, \overline{1}) .
\end{aligned}
$$

(All the other homogeneous components are determined from these ones.)
Corollary 4.10. Up to equivalence, the only fine gradings of $(\mathcal{C},\{\ldots\})$ are $\Gamma_{\text {Cartan }}^{\mathcal{E}}$ and $\Gamma_{\mathrm{CD}}^{\mathrm{E}}$, with universal groups $\mathbb{Z}^{3}$ and $(\mathbb{Z} / 2)^{4}$.

Any element of the Weyl group of $\Gamma_{\mathrm{CD}}^{\mathrm{e}}$ permutes its support and gives an automorphism of the universal grading group. Consider $\mathbb{Z} / 2$ as the field $\mathbb{F}_{2}$ of elements, then $W\left(\Gamma_{\mathrm{CD}}^{\mathrm{C}}\right)$ embeds in $\left\{\gamma \in \mathrm{GL}\left(\mathbb{F}_{2}^{4}\right) \mid \varphi\left(1 \times \mathbb{F}_{2}^{3}\right) \subseteq 1 \times \mathbb{F}_{2}^{3}\right\}$, which is identified with the affine group $\operatorname{Aff}\left(3, \mathbb{F}_{2}\right)$. Lemma 4.5(i) shows that $W\left(\Gamma_{\mathrm{CD}}^{\mathcal{E}}\right)$ acts transitively on the support. Also, the grading $\Gamma_{\mathrm{CD}}^{\mathcal{C}}$ is equivalent to the fine $(\mathbb{Z} / 2)^{3}$-grading on $\mathcal{C}$, with Weyl group $\mathrm{GL}_{3}\left(\mathbb{F}_{2}\right)$. As $\operatorname{Aut}(\mathcal{C})$ is a subgroup of $\operatorname{Aut}(\mathcal{C},\{\ldots\})$, it follows that $\mathrm{GL}_{3}\left(\mathbb{F}_{2}\right)$ is contained in $W\left(\Gamma_{\mathrm{CD}}^{\mathcal{C}}\right) \leq \operatorname{Aff}\left(3, \mathbb{F}_{2}\right)$. It turns out that the Weyl group $W\left(\Gamma_{\mathrm{CD}}^{\mathcal{C}}\right)$ is the whole affine group $\operatorname{Aff}\left(3, \mathbb{F}_{2}\right)$.

The corollary above and the computation of the Weyl groups have been considered independently in AOCMpr , devoted to the classification of the fine gradings on certain Kantor systems attached to Hurwitz algebras. One such triple system corresponds to our 3 -fold cross product ( $\mathcal{C}, X_{1}^{\mathcal{C}}$ ).

Corollary 4.11. Let $\mathbf{Q}$ be a quasitorus (i.e., diagonalizable) subgroup scheme of $\operatorname{Spin}\left(\mathrm{C}_{0},-\mathrm{n}\right)$. Then either:

- $\mathbf{Q}$ is contained in a maximal torus, and hence conjugate to $\operatorname{Diag}\left(\Gamma_{\text {Cartan }}^{\mathcal{C}}\right)$, or
- $\mathbf{Q}$ is conjugate to $\operatorname{Diag}\left(\Gamma_{\mathrm{CD}}^{\mathrm{C}}\right)$, which is isomorphic to $\boldsymbol{\mu}_{2}^{4}$.

Remark 4.12. The group scheme $\operatorname{Spin}\left(\mathcal{C}_{0},-\mathrm{n}\right)$ is the simply connected group of type $B_{3}$. In contrast, the corresponding adjoint group, that is, the special orthogonal group scheme $\mathbf{O}^{+}\left(\mathfrak{C}_{0},-n\right)$, contains four maximal quasitori, up to conjugation, which are isomorphic to $\mathbf{G}_{\mathrm{m}}^{3}$ (a maximal torus), $\mathbf{G}_{\mathrm{m}}^{2} \times \boldsymbol{\mu}_{2}^{2}, \mathbf{G}_{\mathrm{m}} \times \boldsymbol{\mu}_{2}^{4}$, and $\boldsymbol{\mu}_{2}^{6}$ (see [EK13, Theorem 3.67]).

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