Applied Mathematics and Optimization manuscript No.
(will be inserted by the editor)

Dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains

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Received: ** 2022

Abstract This paper is devoted to investigating mean dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes (LANS) equations driven by infinite delay on unbounded domains. We first prove the existence of a unique solution to stochastic 3D LANS equations with infinite delay when the non-delayed external force is locally integrable, the delay term is globally Lipschitz continuous and the nonlinear diffusion term is locally Lipschitz continuous. This enables us to define a mean random dynamical system. Besides, we find that such a dynamical system possesses a unique weak pullback mean random attractor, which is a minimal, weakly compact and weakly pullback attracting set. Furthermore, we prove the existence and uniqueness of stationary solutions (equilibrium solutions) to the corresponding deterministic equation via the classical Galerkin method, the Lax-Milgram and the Brouwer fixed theorems. The stability properties of stationary solutions are also considered. By a direct approach, we first show the local stability of stationary solutions when the delay term has a general form and then apply the abstract results to two kinds of infinite delays. Second, we establish the exponential stability of stationary solutions in the case of unbounded distributed delay. Third, we investigate the asymptotic stability of stationary solutions in the case of unbounded variable delay by constructing appropriate Lyapunov functionals. Eventually, we discuss the polynomial asymptotic stability in the particular case of proportional delay.

Keywords Stochastic 3D Lagrangian-averaged Navier-Stokes equations, Infinite delay, Unbounded domains, Weak pullback mean random attractors, Stationary solutions.

Mathematics Subject Classification (2010) 35Q30, 35A01, 35B35

Shuang Yang was supported by the China Scholarship Council (CSC No. 202006990054). The research of the first two authors has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00, and by Junta de Andalucía (Consejería de Economía y Conocimiento) under projects US-1254251 and P18-FR-4509. The research of the first and third authors has been partially supported by the National Natural Science Foundation of China (No. 11571283).

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1 Introduction

Over the last 70 years, a large number of physicists and mathematicians have used several turbulence models to obtain closure, which means capturing the physical phenomenon of turbulence at computably low resolution. The LANS model is the first to use the Lagrangian averaging technique to deal with the turbulence closure problem. The main reason is that such a model requires lower computational cost than a usual Navier-Stokes equation. As we know, the LANS model reaches closure without enhancing viscosity by modifying the nonlinearity of Navier-Stokes equations, instead of introducing any extra dissipation (see Holm [18] for more details).

Recently, the LANS model has attracted much attention. There are many interesting topics with respect to this system, including the well-posedness and asymptotic behavior of solutions as well as the existence, uniqueness and stability results of stationary solutions (see, for instance, [4,5,6,8] and the references therein).

In this article, we are mainly interested in mean dynamics and stability analysis of the following stochastic 3D LANS equations with infinite delay on unbounded domains:

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p=f(t)+g(t,u_t)+\sigma(t,u)\dot{W}, \text{ in } (\tau,+\infty)\times\mathcal{O}, \\ \text{div } u=0, \quad \text{in } (\tau,+\infty)\times\mathcal{O}, \\ u=0,\ Au=0 \text{ on } (\tau,+\infty)\times\partial\mathcal{O}, \\ u(\tau+s,x)=\phi(s,x), s\in(-\infty,0], x\in\mathcal{O}, \end{cases} \tag{1.1}$$

where $\tau\in\mathbb{R}$, $\mathcal{O}\subset\mathbb{R}^3$ is unbounded open set with boundary $\partial\mathcal{O}$, which is a Poincaré domain, that is, there exists a positive number λ such that

$$\lambda\int_{\mathcal{O}}|\psi|^2dx\le\int_{\mathcal{O}}|\nabla\psi|^2dx,\ \forall\ \psi\in H_0^1(\mathcal{O}), \tag{1.2}$$

A is the Stokes operator, the positive constants ν and α denote the kinematic viscosity of the fluid and the square of the spatial scale at which fluid motion is filtered respectively, the symbol $*$ denotes the transpose of a matrix, $u=(u_1,u_2,u_3)$ and p are the averaged (or large-scale) velocity and pressure of the fluid respectively, f is a non-delayed external force term which may depend on time, and the term g contains some hereditary characteristics, such as memory, unbounded variable or infinite distributed delay, etc, u_t denotes the segment of solutions up to time t, i.e. $u_t(s)=u(t+s)$ for all $s\le 0$, ϕ is an initial velocity field defined in $(-\infty,0]$, σ is a locally Lipschitz nonlinear diffusion coefficient, and $\dot{W}$ denotes the time derivative of a cylindrical Wiener process.

In the special case $\alpha=0$, system (1.1) becomes the classical 3D Navier-Stokes equation whose dynamical behavior has been widely investigated in [3,7,9,10,16,20,29]. The deterministic case without delay (i.e., $\sigma=0$ and g is independent of u) has been carried out, for instance, in [11], where it is showed the global well-posedness as well as the existence and finite dimensionality of global attractors. Marsden et al. [27] proved the global well-posedness and regularity of solutions, and they further derived the convergence to solutions of the corresponding inviscid problem. For the stochastic and non-delay version of (1.1), Caraballo et al. [8] studied the stochastic dynamics of the 3D LANS equations for the first time.

However, to describe better a realistic model, we should take into account some hereditary characteristics such as aftereffect, time lag, memory and time delay. Also, they play an important role in determining the evolution of physical models, and appear when the current behavior is influenced by its previous states, or one focuses on controlling the problem by imposing an external force which may depend on both its present state and the history of the solutions. The delay differential equations are significant tools to depict these delayed phenomena (see [13,28]).

Nowadays, delay differential equations have been extensively investigated. For example, the analysis of Navier-Stokes equations with some hereditary features was first studied by Caraballo and Real in [7] and developed in [3,4,6,14,15,16,22,23,24,25,26]. In the bounded delay case, the authors have considered several issues concerning the existence, uniqueness, asymptotic behavior and regularity of solutions, the existence, uniqueness and stability of stationary solutions, the existence, uniqueness of global or pullback (random) attractors. The case of unbounded or infinite delays has been analyzed in [15,17,22,23,24,25,26].

Notice that the LANS equations with finite delays or with memory have been discussed recently in [4,6]. In the first paper, the authors proved the existence of a unique solution to the stochastic 3D LANS equation when two delay terms are globally Lipschitz continuous. In the second one, some sufficient conditions ensuring the existence (and eventual uniqueness) of stationary solutions were established when the non-delayed external force is locally integrable, and the delay term is globally Lipschitz continuous. Furthermore, the exponential stability of stationary solutions is obtained. In this paper, we discuss the 3D LANS equation with infinite or unbounded delays. Inspired by [22,23,25,26], we may choose several phase spaces to deal with the infinite delay. The first one is the Banach space

$$C_\gamma(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \text{ exists in } H\}, \text{ where } \gamma > 0, \tag{1.3}$$

where H is the 3D Lebesgue-type Hilbert space. The second one is

$$C_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{s\to-\infty} \varphi(s) \text{ exists in } H\}. \tag{1.4}$$

Moreover, we also use $C_\gamma(V)$ and $C_{-\infty}(V)$, where V is another Sobolev-type subspace instead of H in (1.3) and (1.4).

The first aim of this article is to study mean dynamics of the stochastic 3D LANS equation (1.1) driven by infinite delay on unbounded domains. In the Banach space $C_\gamma(V)$, we first establish the well-posedness of problem (1.1) with infinite delay when the non-delayed external force $f(t)$ is locally integrable, the delay term $g(t, u_t)$ is globally Lipschitz continuous and the nonlinear diffusion term $\sigma(t, u)$ is locally Lipschitz continuous. To that end, we introduce a globally Lipschitz continuous cut-off function ξ_n (for every $n \in \mathbb{N}$) defined by (3.3) to approximate $\sigma(t, u)$. The solution operators enable us to define a mean random dynamical system instead of the usual pathwise random dynamical system, mainly because, in this case, there is no approach available to transfer the stochastic system (1.1) to a corresponding pathwise deterministic one. We then show the existence of a unique weak pullback random attractor for the mean random dynamical system.

Another purpose of this work is to prove the existence of a unique stationary solution to the corresponding deterministic equation (see Eq. (4.9)) and analyze the asymptotic behaviour of solutions for Eq. (1.1) towards to the stationary solution. Since our results no longer depend on Sobolev compact embeddings for unbounded domains, we modify some arguments in [6, Theorem 10], i.e. apply the classical Galerkin argument as well as the Lax-Milgram and the Brouwer fixed theorems to derive the existence and uniqueness of stationary solutions to system (4.9) on unbounded domains. Moreover, we investigate their stability properties via several methods which enable us to establish different stability sufficient conditions. We carry out this analysis in $C_{-\infty}(V)$ and even in $C_\gamma(V)$. By a direct approach, we first show the local stability of stationary solution when the delay term has a general form (unbounded variable delays and infinite distributed delays or memory terms) in $C_{-\infty}(V)$. In order to obtain our results in $C_\gamma(V)$ with $\gamma > 0$, the exponential stability in the case of unbounded variable delay fails to be proved in general (see [23] and [26] for more details). Fortunately, in the case of infinite distributed delay, we are able to prove not only stability of stationary solutions in $C_\gamma(V)$, but also exponential asymptotic stability. Although we may not prove the exponential stability for the unbounded variable delay in $C_\gamma(V)$, we will explore, at least, the asymptotic stability in $C_{-\infty}(V)$, by using the Lyapunov functionals construction proposed by Kolmanovskii and Shaikhet [21]. To make our analysis more complete, we eventually discuss the polynomial asymptotic stability in the particular case of proportional delay (also known as pantograph delay).

This paper is organized as follows. In Section 2, we describe some preliminaries, including basic theories on cylindrical Wiener process, some notations and linear operators, some suitable assumptions about the non-delayed external force f, delay term g and nonlinear diffusion coefficient σ. The well-posedness and mean dynamics of problem (1.1), driven by unbounded delay and locally Lipschitz nonlinear noise, are proved in Section 3. The last section is devoted to investigating the existence and uniqueness of stationary solutions and their stability results to the corresponding deterministic equation.

2 Preliminaries

2.1 The cylindrical Wiener process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete filtered probability space such that $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub σ-algebras of $\mathcal{F}$, which contains all $\mathbb{P}$-null sets. For $t \le 0$, suppose that $\mathcal{F}_t = \mathcal{F}_0$.

Let $\{\beta_t^j, t \geq 0, j = 1,2,3,\ldots\}$ be a sequence of mutually independent standard real valued $\mathcal{F}_t$-Wiener processes and K a separable Hilbert space with an orthonormal basis $\{e_j;\ j = 1,2,3,\ldots\}$. Suppose that $\{W(t); t \geq 0\}$ is a K-valued cylindrical Wiener process (with the covariance operator $Q : K \to K$) given by

$$W(t) = \sum_{j=1}^{\infty} \beta^j(t) e_j, \quad t \geq 0. \tag{2.1}$$

Given a separable Hilbert space H_0, we denote by $\mathcal{L}^2(K, H_0)$ the space of Hilbert-Schmidt operators from K into H_0 with the following norm

$$\|S\|^2_{\mathcal{L}^2(K,H_0)} = \mathrm{tr}(SQS^*),\ \forall S \in \mathcal{L}^2(K, H_0), \tag{2.2}$$

where tr denotes the trace of an operator, S^* is the adjoint operator of S.

For any separable Banach space X, interval $(a,b) \subset \mathbb{R}$ and $p \geq 1$, we denote by $I^p(a,b;X)$ the Banach space of all processes $\varphi \in L^p(\Omega \times (a,b), \mathcal{F} \otimes \mathcal{B}((a,b)), d\mathbb{P} \otimes dt; X)$ such that $\varphi(t)$ is $\mathcal{F}_t$-progressively measurable for a.e. $t \in (a,b)$, where $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra. We also denote by $L^p(\Omega, \mathcal{F}, d\mathbb{P}; C(a,b;X))$ with $p \geq 1$ the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes φ such that $\mathbb{E}(\sup_{a \leq t \leq b} \|\varphi(t)\|_X^p) < \infty$, where $C(a,b;X)$ is the Banach space of all continuous functions from $[a,b]$ into X. For convenience, we write $L^p(\Omega, \mathcal{F}, d\mathbb{P}; C(a,b;X))$ as $L^p(\Omega; C(a,b;X))$.

For any process $\Phi \in I^2(\tau, \tau+T; \mathcal{L}^2(K, H_0))$ and $t \in [\tau, \tau+T]$, the stochastic integral $\int_\tau^t \Phi(s) dW(s)$ is defined by the unique continuous H_0-valued $\mathcal{F}_t$-martingale such that

$$\Big(\int_\tau^t \Phi(s) dW(s), w \Big)_{H_0} = \sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s) e_j, w \Big)_{H_0} d\beta^j(s),\ \forall w \in H_0, \tag{2.3}$$

where the integral with respect to $\beta^j(s)$ is the Ito integral. By [12], if $\Phi \in I^2(\tau, \tau+T; \mathcal{L}^2(K, H_0))$ and $\phi \in L^2(\Omega, L^\infty(\tau, \tau+T; H_0))$ is $\mathcal{F}_t$-progressively measurable, then

$$\sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s) e_j, \phi(s) \Big)_{H_0} d\beta^j(s) =: \int_\tau^t \Big(\Phi(s) dW(s), \phi(s) \Big),\ \tau \leq t \leq \tau + T,$$

converges in $L^1(\Omega, C(\tau, \tau+T))$.

2.2 Notations and hypotheses

In this subsection, we introduce some notations and linear operators, recall some properties with respect to the nonlinear term $(u \cdot \nabla)(u - \alpha \Delta u) - \alpha \nabla u^* \cdot \Delta u$ in the problem (1.1), and impose some suitable assumptions.

Denote by $\mathbb{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3, \mathbb{H}_0^1(\mathcal{O}) := (H_0^1(\mathcal{O}))^3$, $\mathbb{C}_0^\infty(\mathcal{O}) := (C_0^\infty(\mathcal{O}))^3$ and

$$\mathcal{V} = \{u \in \mathbb{C}_0^\infty(\mathcal{O}) : \nabla \cdot u = 0 \ \text{ in } \ \mathcal{O}\}. \tag{2.4}$$

Let H and V be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$, respectively. Denote by $|\cdot|, \|\cdot\|$ and $\|\cdot\|_*$ the norms in H, V and V^*, respectively, where V^* is the dual space of V. Let $(\cdot,\cdot), ((\cdot,\cdot))$ be the scalar product in H and V, respectively. For all $u, v \in \mathbb{H}_0^1(\mathcal{O})$, we set

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v) = \sum_{j=1}^{3} \int_{\mathcal{O}} u_j(x) v_j(x) dx + \alpha \sum_{i,j=1}^{3} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx. \tag{2.5}$$

Denote by $\mathcal{V}(Q)$ the same space as $\mathcal{V}$ with $\mathcal{O}$ is replaced by an open set Q, analogously let $H(Q)$ and $V(Q)$ be the closure of $\mathcal{V}(Q)$ in $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$, respectively.

We now consider the Stokes operator A, defined by

$$Aw = -\mathcal{P}(\Delta w), \quad \forall\ w \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap V, \tag{2.6}$$

where $\mathbb{H}^2(\mathcal{O}) = (H^2(\mathcal{O}))^3$, $\mathcal{P}$ is the Leray operator from $\mathbb{L}^2(\mathcal{O})$ onto H. We deduce

$$(Au, v) = ((u, v)), \quad \|u\|_{\mathbb{H}^2(\mathcal{O})} \le C_1|Au|, \quad \forall u \in D(A), v \in V. \tag{2.7}$$

Denote by $\|\cdot\|_{(D(A))^*}$ the norm in $(D(A))^*$, where $(D(A))^*$ is the dual space of $D(A)$. Let $\langle\cdot,\cdot\rangle$ be the duality product between $(D(A))^*$ and $D(A)$. Identifying $v \in D(A)$ with the element $h_v \in V^*$, defined by

$$h_v(u) = ((v, u)), \; u \in V.$$

We then define a continuous linear operator $\widetilde{A} \in \mathcal{L}(D(A), (D(A))^*)$ defined dy

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av), \quad \forall\; u, v \in D(A) =: D(\widetilde{A}). \tag{2.8}$$

For all $k \geq 1$, let ξ_k, λ_k be the eigenvectors and eigenvalues of the Stokes operator A, respectively, by the definition of operator $\widetilde{A}$ and (2.8), we obtain

$$\langle \widetilde{A}\xi_k, v\rangle = \nu\lambda_k((\xi_k, v)), \tag{2.9}$$

which implies that, the eigenvalues of the operator $\widetilde{A}$ are given by $\widetilde{\lambda}_k := \nu\lambda_k$.

As in [6], we associate another inner product on $D(A) = D(\widetilde{A})$, defined by

$$(u, v)_{D(A)} := \langle \widetilde{A}u, v\rangle, \text{ and so } \; \widetilde{\lambda}_1\|u\|^2 \le \|u\|^2_{D(A)}, \; \forall u, v \in D(A). \tag{2.10}$$

By (2.7), the above is equivalent to the original inner product $((u, v)) + (Au, Av)$ for $u, v \in D(A)$.

Next, we consider the following trilinear operator:

$$b^{\#}(u, v, w) = \langle (u\cdot\nabla)v, w\rangle_{-1} + \langle \nabla u^* \cdot v, w\rangle_{-1}, \quad \forall\; (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}^1_0(\mathcal{O}), \tag{2.11}$$

where $\langle\cdot,\cdot\rangle_{-1}$ denotes the duality product between $\mathbb{H}^{-1}(\mathcal{O})$ and $\mathbb{H}^1_0(\mathcal{O})$ or between $H^{-1}(\mathcal{O})$ and $H^1_0(\mathcal{O})$. Thanks to [8, Proposition 2.2], we find

$$b^{\#}(u, v, w) = -b^{\#}(w, v, u), \quad \forall\; (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A), \tag{2.12}$$

which implies that $b^{\#}(u, v, u) = 0, \forall\; (u, v) \in D(A) \times \mathbb{L}^2(\mathcal{O})$.

Define a bilinear mapping $\widetilde{B} : D(A) \times D(A) \to (D(A))^*$, denoted by

$$\langle \widetilde{B}(u, v), w\rangle = b^{\#}(u, v - \alpha\Delta v, w), \quad \forall\; (u, v, w) \in D(A) \times D(A) \times D(A), \tag{2.13}$$

$\widetilde{B}(u)$ will be used to denote $\widetilde{B}(u, u)$ for all $u \in D(A)$. By the definition and properties of $b^{\#}$, it follows that there exists a positive constant $\widetilde{c} := \widetilde{c}(\mathcal{O})$ such that for all $(u, v, w) \in D(A) \times D(A) \times D(A)$,

$$\langle \widetilde{B}(u, v), u\rangle = 0, \; \langle \widetilde{B}(u), v\rangle = -\langle \widetilde{B}(v, u), u\rangle; \tag{2.14}$$

$$\|\widetilde{B}(u, v)\|_{(D(A))^*} \le \widetilde{c}\|u\|\|v\|_{D(A)}; \tag{2.15}$$

$$|\langle \widetilde{B}(u, v), w\rangle| \le \widetilde{c}\|u\|_{D(A)}\|v\|_{D(A)}\|w\|. \tag{2.16}$$

Recall the phase space

$$C_\gamma(V) = \{u \in C((-\infty, 0]; V) : \lim_{s\to-\infty} e^{\gamma s}u(s) \text{ exists in } V\}, \text{ where } \gamma > 0, \tag{2.17}$$

which is a Banach space with sup norm, that is,

$$\|u\|_{C_\gamma(V)} = \sup_{s\in(-\infty,0]} e^{\gamma s}\|u(s)\|. \tag{2.18}$$

In order to analyze our problem, we need to establish some suitable assumptions.

We first suppose that the non-delayed external force is locally integrable, that is, for all $\tau \in \mathbb{R}, T > 0$,

$$f \in I^2(\tau, \tau + T; \mathbb{H}^{-1}(\mathcal{O})). \tag{2.19}$$

Then, we need some assumptions on delay term.

Let $g : \mathbb{R} \times C_\gamma(V) \to \mathbb{H}^{-1}(\mathcal{O})$ satisfy the following conditions:

(G1) For any $\eta \in C_\gamma(V)$, $g(\cdot, \eta)$ is measurable;

(G2) $g(\cdot, 0) = 0$;

(G3) There exists a constant $L_g > 0$ such that for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|g(t, \eta) - g(t, \zeta)\|_{\mathbb{H}^{-1}(\mathcal{O})} \le L_g \|\eta - \zeta\|_{C_\gamma(V)};$$

(G4) There exists a constant $C_g > 0$ such that, for all $\tau \in \mathbb{R}, t \ge \tau$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \|g(s, u_s) - g(s, v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} ds \le C_g^2 \int_{-\infty}^t \|u(s) - v(s)\|^2 ds;$$

(G5) There exists a constant $\widetilde{C}_g > 0$ such that, for all $\tau \in \mathbb{R}, t \ge \tau$, all decreasing function $\varpi \in C^0([\tau, t])$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \varpi(s) \|g(s, u_s) - g(s, v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} ds \le \widetilde{C}_g^2 \int_\tau^t \varpi(s) \|u(s) - v(s)\|^2 ds.$$

For the diffusion coefficient, we will impose some suitable assumptions as follows.

Let $\sigma : \mathbb{R} \times V \to \mathcal{L}^2(K, \mathbb{L}^2(\mathcal{O}))$ be measurable, and locally Lipschitz continuous, that is, for every $r > 0$, there exists a positive constant L_r depending on r such that for all $t \in \mathbb{R}, w_1, w_2 \in V$ with $\|w_1\| \le r$ and $\|w_2\| \le r$,

$$\|\sigma(t, w_1) - \sigma(t, w_2)\|_{\mathcal{L}^2(K, \mathbb{L}^2(\mathcal{O}))} \le L_r \|w_1 - w_2\|. \tag{2.20}$$

Besides, suppose that there exists a constant $L_\sigma > 0$ such that, for all $(t, w) \in \mathbb{R} \times V$,

$$\|\sigma(t, w)\|_{\mathcal{L}^2(K, \mathbb{L}^2(\mathcal{O}))} \le L_\sigma (1 + \|w\|). \tag{2.21}$$

Now, let us define $\widetilde{f}(t)$ as

$$((\widetilde{f}(t), w)) = \langle f(t), w \rangle_{-1}, \ \forall \ (t, w) \in \mathbb{R} \times V. \tag{2.22}$$

Define $\widetilde{g} : \mathbb{R} \times C_\gamma(V) \to V$ as

$$((\widetilde{g}(t, \eta), w)) = \langle g(t, \eta), w \rangle_{-1}, \ \forall \ (t, \eta, w) \in \mathbb{R} \times C_\gamma(V) \times V. \tag{2.23}$$

By (2.19) and (2.22), we obtain $\widetilde{f} \in I^2(\tau, \tau + T; (D(A))^*)$ for any $\tau \in \mathbb{R}$ and $T > 0$. It follows from (G1)-(G5) and (2.23) that $\widetilde{g} : \mathbb{R} \times C_\gamma(V) \to V$ satisfies the following conditions:

(H1) For any $\eta \in C_\gamma(V)$, $\widetilde{g}(\cdot, \eta)$ is measurable;

(H2) $\widetilde{g}(\cdot, 0) = 0$;

(H3) Setting $L_{\widetilde{g}} = L_g$, we obtain, for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|\widetilde{g}(t, \eta) - \widetilde{g}(t, \zeta)\| \le L_{\widetilde{g}} \|\eta - \zeta\|_{C_\gamma(V)}.$$

It follows from (H2) and (H3) that, for all $\eta \in C_\gamma(V)$,

$$\|\widetilde{g}(t, \eta)\| \le L_{\widetilde{g}} \|\eta\|_{C_\gamma(V)}. \tag{2.24}$$

(H4) Letting $C_{\widetilde{g}} = C_g$, for all $\tau \in \mathbb{R}, t \ge \tau$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \|\widetilde{g}(s, u_s) - \widetilde{g}(s, v_s)\|^2 ds \le C_{\widetilde{g}}^2 \int_{-\infty}^t \|u(s) - v(s)\|^2 ds;$$

(H5) Taking $\widetilde{C}_{\widetilde{g}} = \widetilde{C}_g$, for all $\tau \in \mathbb{R}, t \ge \tau$, all decreasing function $\varpi \in C^0([\tau, t])$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \varpi(s) \|\widetilde{g}(s, u_s) - \widetilde{g}(s, v_s)\|^2 ds \le \widetilde{C}_{\widetilde{g}} \int_\tau^t \varpi(s) \|u(s) - v(s)\|^2 ds.$$

Finally, we define $\widetilde{\sigma} : \mathbb{R} \times V \to \mathcal{L}^2(K, V)$ as

$$\widetilde{\sigma}(t, \upsilon) = (I + \alpha A)^{-1} \circ \mathcal{P} \circ \sigma(t, \upsilon), \ \forall \ (t, \upsilon) \in \mathbb{R} \times V,$$

where I is the identity operator in H and $I + \alpha A : D(A) \to H$ is bijective. Moreover,

$$(((I + \alpha A)^{-1} u, w)) = (u, w), \ \forall \ u \in H, \ w \in V.$$

Hence, for the orthonormal basis $\{e_j\}$ of K, we have

$$(\sigma(t, \eta)e_j, w) = ((I + \alpha A)\widetilde{\sigma}(t, \eta)e_j, w) = ((\widetilde{\sigma}(t, \eta)e_j, w)),$$

for all $j \geq 1$ and $(t, \upsilon, w) \in \mathbb{R} \times V \times D(A)$, by (2.3), we further obtain that

$$\begin{aligned}
\Big(\int_\tau^t \sigma(s, \eta) dW(s), w \Big) &= \sum_{j=1}^\infty \int_\tau^t (\sigma(s, \eta)e_j, w) d\beta^j(s) \\
&= \sum_{j=1}^\infty \int_\tau^t \big((\widetilde{\sigma}(s, \eta)e_j, w) \big) d\beta^j(s) \\
&= \Big(\Big(\int_\tau^t \widetilde{\sigma}(s, \eta) dW(s), w \Big) \Big). \qquad (2.25)
\end{aligned}$$

It follows from (2.20), (2.21) and (2.25) that $\widetilde{\sigma} : \mathbb{R} \times V \to \mathcal{L}^2(K, V)$ is measurable, and locally Lipschitz continuous, that is, for every $\widetilde{r} > 0$, there exists a positive constant $L_{\widetilde{r}}$ depending on $\widetilde{r}$ such that, for all $t \in \mathbb{R}, w_1, w_2 \in V$ with $\|w_1\| \leq \widetilde{r}$ and $\|w_2\| \leq \widetilde{r}$,

$$\|\widetilde{\sigma}(t, w_1) - \widetilde{\sigma}(t, w_2)\|_{\mathcal{L}^2(K,V)} \leq L_{\widetilde{r}} \|w_1 - w_2\|. \qquad (2.26)$$

In addition, there exists a constant $L_{\widetilde{\sigma}} > 0$ such that, for all $(t, w) \in \mathbb{R} \times V$,

$$\|\widetilde{\sigma}(t, w)\|_{\mathcal{L}^2(K,V)} \leq L_{\widetilde{\sigma}}(1 + \|w\|). \qquad (2.27)$$

3 Well-posedness and mean dynamics for stochastic 3D LANS equations with infinite delay

In this section, we investigate the well-posedness and mean dynamics of the stochastic 3D LANS system (1.1) with infinite delay and locally Lipschitz nonlinear noise.

3.1 Well-posedness of stochastic 3D LANS equations

In this subsection, our main aim is to prove the well-posedness of the stochastic 3D LANS equation (1.1). Based on the previous operators and assumptions, we consider the following abstract equation:

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f}(t) + \widetilde{g}(t, u_t) + \widetilde{\sigma}(t, u)\dfrac{dW(t)}{dt}, \ \ \forall \ t > \tau, \\ u(\tau + s) = \phi(s), s \in (-\infty, 0]. \end{cases} \qquad (3.1)$$

Definition 1 Suppose that $\phi \in L^2(\Omega, C_\gamma(V))$ (which is a $\mathcal{F}_0$-progressively measurable V-valued processes) and $\tau \in \mathbb{R}$. A stochastic process u defined on $\mathbb{R}$ is called a solution to system (3.1) if

$$u \in I^2(\tau, \tau + T; D(A)) \cap L^2(\Omega, L^\infty(\tau, \tau + T; V)), \ \forall \ T > 0,$$

$u_\tau = \phi$, and the system (3.1) is satisfied in $(D(A))^*$, that is, for almost all $\omega \in \Omega$,

$$\begin{aligned}
&((u(t), w)) + \int_\tau^t \langle \widetilde{A}u(s), w \rangle ds + \int_\tau^t \langle \widetilde{B}(u(s)), w \rangle ds \\
&= ((\phi(0), w)) + \int_\tau^t \Big(\Big(\widetilde{f}(s) + \widetilde{g}(s, u_s), w \Big) \Big) ds + \Big(\Big(w, \int_\tau^t \widetilde{\sigma}(s, u) dW(s) \Big) \Big), \qquad (3.2)
\end{aligned}$$

for all $t \geq \tau$ and $w \in D(A)$.

In order to prove the well-posedness of problem (3.1), for every $n \in \mathbb{N}$, we introduce a sequence of cut-off functions $\xi_n : V \to V$ $(n \in \mathbb{N})$ defined by

$$\xi_n(\upsilon) = \begin{cases} \upsilon, & \text{for } \|\upsilon\| \leq n, \\ \frac{n\upsilon}{\|\upsilon\|}, & \text{for } \|\upsilon\| > n. \end{cases} \tag{3.3}$$

Then, $\xi_n : V \to V$ is globally Lipschitz continuous:

$$\|\xi_n(\upsilon_1) - \xi_n(\upsilon_2)\| \leq \|\upsilon_1 - \upsilon_2\|, \quad \forall\ \upsilon_1, \upsilon_2 \in V, \tag{3.4}$$

and

$$\|\xi_n(\upsilon)\| \leq n, \quad \forall\ \upsilon \in V. \tag{3.5}$$

Given $n \in \mathbb{N}$, let $\widetilde{\sigma}_n(t, \upsilon) = \widetilde{\sigma}(t, \xi_n(\upsilon))$, then we infer from (2.26), (2.27), (3.4) and (3.5) that, for every $n \in \mathbb{N}$, there exists a positive constant L_n such that for all $t \in \mathbb{R}, \upsilon_1, \upsilon_2 \in V$,

$$\|\widetilde{\sigma}_n(t, \upsilon_1) - \widetilde{\sigma}_n(t, \upsilon_2)\|_{\mathcal{L}^2(K,V)} \leq L_n\|\upsilon_1 - \upsilon_2\|, \tag{3.6}$$

and

$$\|\widetilde{\sigma}_n(t, \upsilon)\|_{\mathcal{L}^2(K,V)} \leq L_{\widetilde{\sigma}}(1 + \|\upsilon\|). \tag{3.7}$$

Next, we consider the approximating system $(n \in \mathbb{N})$:

$$\begin{cases} \dfrac{du^n}{dt} + \widetilde{A}u^n(t) + \widetilde{B}(u^n(t)) = \widetilde{f}(t) + \widetilde{g}(t, u^n_t) + \widetilde{\sigma}_n(t, u^n(t))\dfrac{dW(t)}{dt}, \quad \forall\ t > \tau,\ \tau \in \mathbb{R}, \\ u^n_\tau(s) = \phi(s), s \in (-\infty, 0]. \end{cases} \tag{3.8}$$

For every $n \in \mathbb{N}$, $\tau \in \mathbb{R}$ and $\phi \in L^2(\Omega, C_\gamma(V))$, the approximating system (3.8) has a unique solution u^n by using the same method as in [2]. Therefore, one can define *random stopping times* by

$$\iota_n = \inf\{t \geq \tau : \|u^n(t)\| > n\},\ \forall\ n \in \mathbb{N}, \tag{3.9}$$

where $\iota_n = +\infty$ if $\{t \geq \tau : \|u^n(t)\| > n\} = \emptyset$.

The following estimate is needed for proving the well-posedness of problem (3.1).

Lemma 1 *For all $u, v \in D(A)$, we have*

$$-2\langle \widetilde{A}\overline{w} + \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle \leq \frac{\widetilde{c}}{2}\|\overline{w}\|^2\|v\|^2_{D(A)}, \tag{3.10}$$

where $\overline{w} = u - v$.

Proof Note that

$$-2\langle \widetilde{A}\overline{w}, \overline{w}\rangle = -2\|\overline{w}\|^2_{D(A)}. \tag{3.11}$$

Thanks to (2.14), we have

$$\langle \widetilde{B}(u), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, u), u\rangle = -\langle \widetilde{B}(\overline{w}, u), v\rangle, \tag{3.12}$$

and similarly

$$\langle \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, v), v\rangle. \tag{3.13}$$

Subtracting (3.13) from (3.12),

$$\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}), v\rangle, \tag{3.14}$$

which, together with (2.15), implies that

$$\begin{aligned}|\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle| &= |\langle \widetilde{B}(\overline{w}), v\rangle| \\ &\le \|\widetilde{B}(\overline{w})\|_{(D(A))^*}\|v\|_{D(A)} \\ &\le \widetilde{c}\|\overline{w}\|\|\overline{w}\|_{D(A)}\|v\|_{D(A)} \\ &\le \|\overline{w}\|^2_{D(A)} + \frac{\widetilde{c}^2}{4}\|\overline{w}\|^2\|v\|^2_{D(A)}. \end{aligned} \tag{3.15}$$

Combining (3.11) and (3.15), we obtain (3.10) as desired.

Theorem 1 *Suppose that* (H1)-(H5)*,* (3.6) *and* (3.7) *hold. Let* $\phi \in L^2(\Omega, C_\gamma(V))$ *and* $\widetilde{f} \in I^2(\tau, \tau+T; V)$*, then the stochastic system* (3.1) *has a unique solution* u *in the sense of Definition 1 such that for every* $T > 0$*, there exists a positive constant* R *depending on* T*,* $\mathbb{E}\big(\|\phi\|^2_{C_\gamma(V)}\big)$ *and* $\mathbb{E}\big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2 ds\big)$*,*

$$\mathbb{E}\Big(\sup_{\tau\le r\le\tau+T}\|u_r\|^2_{C_\gamma(V)}\Big) + \mathbb{E}\bigg(\int_\tau^{\tau+T}\|u(s)\|^2_{D(A)}ds\bigg) \le R. \tag{3.16}$$

Proof We split the proof into the following four steps.

Step 1: *We prove* $\iota_{n+1} \ge \iota_n$ *almost surely.* In fact, we only need to prove the following equality holds:

$$u^{n+1}(t\wedge\iota_n) = u^n(t\wedge\iota_n), \quad \mathbb{P}\text{-a.s. } \ \forall\, t\ge\tau,\ n\in\mathbb{N}. \tag{3.17}$$

Without loss of generality, we may assume that $\mathbb{E}(\int_\tau^t\|u^n(s)\|^2_{D(A)}ds) < \infty$. Actually, this is a direct consequence of the Step 2. Set Let $\overline{u}^n(t) := u^{n+1}(t) - u^n(t), \varsigma(t) := e^{-\frac{\widetilde{c}}{2}\int_\tau^t\|u^n(s)\|^2_{D(A)}ds}$. Applying Ito's formula to the process $\varsigma(t)\|\overline{u}^n(t)\|^2$, by (3.10) in Lemma 1, we deduce, for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}\varsigma(t\wedge\iota_n)\|\overline{u}^n(t\wedge\iota_n)\|^2 &= -\frac{\widetilde{c}}{2}\int_\tau^{t\wedge\iota_n}\varsigma(s)\|u^n(s)\|^2_{D(A)}\|\overline{u}^n(s)\|^2ds \\ &\quad - 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\big\langle\widetilde{A}\overline{u}^n(s) + \widetilde{B}(u^{n+1}(s)) - \widetilde{B}(u^n(s)), \overline{u}^n(s)\big\rangle ds \\ &\quad + 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\big(\big(\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)\big)\big)ds \\ &\quad + \int_\tau^{t\wedge\iota_n}\varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\ &\quad + 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\Big(\Big(\overline{u}^n(s), \big(\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\big)dW(s)\Big)\Big) \\ &\le 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\big(\big(\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)\big)\big)ds \\ &\quad + \int_\tau^{t\wedge\iota_n}\varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\ &\quad + 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\Big(\Big(\overline{u}^n(s), \big(\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\big)dW(s)\Big)\Big). \end{aligned} \tag{3.18}$$

By (H5) and the Young inequality, we find

$$\begin{aligned}&2\int_\tau^{t\wedge\iota_n}\varsigma(s)\big(\big(\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)\big)\big)ds \\ &\quad\le \int_\tau^{t\wedge\iota_n}\varsigma(s)\|\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n)\|^2ds + \int_\tau^{t\wedge\iota_n}\varsigma(s)\|\overline{u}^n(s)\|^2ds\end{aligned}$$

$$\leq \widetilde{C}_{\widetilde{g}}^2 \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\overline{u}^n(s)\|^2 ds + \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\overline{u}^n(s)\|^2 ds$$

$$\leq (\widetilde{C}_{\widetilde{g}}^2 + 1) \int_\tau^t \sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds. \tag{3.19}$$

Thanks to (3.6), and the fact that $\widetilde{\sigma}_n(s,u^n) = \widetilde{\sigma}_{n+1}(s,u^n)$ for all $s \in [\tau,\iota_n]$, we obtain

$$\begin{aligned}
&\int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds \\
&= \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_{n+1}(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds \\
&\leq L_{n+1}^2 \int_\tau^t \sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds.
\end{aligned} \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), we find

$$\begin{aligned}
\varsigma(t\wedge\iota_n)\|\overline{u}^n(t\wedge\iota_n)\|^2 \leq{}& c_1 \int_\tau^t \sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds \\
&+ 2\int_\tau^{t\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), \big(\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\big)dW(s)\Big)\Big),
\end{aligned}$$

where $c_1 = \widetilde{C}_{\widetilde{g}}^2 + 1 + L_{n+1}^2$. Taking supremum and expectation of the above inequality, we find

$$\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\leq r\leq t} \varsigma(r\wedge\iota_n)\|\overline{u}^n(r\wedge\iota_n)\|^2\Big) \leq{}& c_1 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big) ds \\
&+ 2\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\Big|\int_\tau^{r\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), \big(\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\big)dW(s)\Big)\Big)\Big|\Big).
\end{aligned} \tag{3.21}$$

By the Burkholder-Davis-Gundy inequality and (3.20), the last line of (3.21) is bounded by

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\Big|\int_\tau^{r\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), \big(\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\big)dW(s)\Big)\Big)\Big|\Big) \\
&\leq 2c_2\mathbb{E}\bigg(\bigg(\int_\tau^{t\wedge\iota_n}\Big(\varsigma^2(s)\|\overline{u}^n(s)\|^2\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}\Big)ds\bigg)^{\frac{1}{2}}\bigg) \\
&\leq 2c_2\mathbb{E}\bigg(\sup_{\tau\leq s\leq t} \varsigma^{\frac{1}{2}}(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|\bigg(\int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds\bigg)^{\frac{1}{2}}\bigg) \\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq s\leq t} \varsigma(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|^2\Big) + 2c_2^2\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds\Big) \\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leq s\leq t} \varsigma(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|^2\Big) + c_3 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big) ds,
\end{aligned} \tag{3.22}$$

where $c_3 = 2c_2^2 L_{n+1}^2$. Combining (3.21) and (3.22), we obtain

$$\mathbb{E}\Big(\sup_{\tau\leq r\leq t} \varsigma(r\wedge\iota_n)\|\overline{u}^n(r\wedge\iota_n)\|^2\Big) \leq c_4 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\leq\vartheta\leq s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big) ds,\ \forall\ t \in [\tau,\tau+T], \tag{3.23}$$

where $c_4 = 2(c_1 + c_3)$. By the Gronwall Lemma, together with $0 < \varsigma \leq 1$, implies

$$\mathbb{E}\Big(\sup_{\tau\leq r\leq t} \|\overline{u}^n(r\wedge\iota_n)\|^2\Big) = 0,\ \forall\ t \in [\tau,\tau+T],$$

thus,

$$u^{n+1}(t\wedge\iota_n)=u^n(t\wedge\iota_n),\ \mathbb{P}\text{-a.s.}\ \forall\ t\geq\tau,\ n\in\mathbb{N},\tag{3.24}$$

which implies (3.17) as desired.

Step 2: *We prove* $\iota:=\lim_{n\to\infty}\iota_n=\sup_{n\in\mathbb{N}}\iota_n=\infty$*, almost surely.* Applying Ito's formula to (3.8), we find, for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\|u^n(t\wedge\iota_n)\|^2+2\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2_{D(A)}ds\\&\quad=\|\phi(0)\|^2+2\int_\tau^{t\wedge\iota_n}\big((\widetilde{f}(s)+\widetilde{g}(s,u^n_s),u^n(s))\big)ds\\&\qquad+\int_\tau^{t\wedge\iota_n}\|\widetilde{\sigma}_n(s,u^n(s))\|^2_{\mathcal{L}^2(K,V)}ds+2\int_\tau^{t\wedge\iota_n}\big(\big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\big)\big).\end{aligned}\tag{3.25}$$

By (2.24) and the Young inequality, we deduce

$$\begin{aligned}2\int_\tau^{t\wedge\iota_n}\big((\widetilde{f}(s)+\widetilde{g}(s,u^n_s),u^n(s))\big)ds&\leq2\int_\tau^{t\wedge\iota_n}(\|\widetilde{f}(s)\|^2+\|\widetilde{g}(s,u^n_s)\|^2)ds+\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2ds\\&\leq2\int_\tau^{t\wedge\iota_n}\Big(\|\widetilde{f}(s)\|^2+L^2_{\widetilde{g}}\int_\tau^{t\wedge\iota_n}\|u^n_s\|^2_{C_\gamma(V)}\Big)ds+\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2ds\\&\leq2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2ds+c_5\int_\tau^{t\wedge\iota_n}\|u^n_s\|^2_{C_\gamma(V)}ds,\end{aligned}\tag{3.26}$$

where $c_5=2L^2_{\widetilde{g}}+1$. Thanks to (3.7), we find

$$\begin{aligned}\int_\tau^{t\wedge\iota_n}\|\widetilde{\sigma}_n(s,u^n(s))\|^2_{\mathcal{L}^2(K,V)}ds&\leq2L^2_{\widetilde{\sigma}}\int_\tau^{t\wedge\iota_n}(1+\|u^n(s)\|^2)ds\\&\leq2L^2_{\widetilde{\sigma}}\int_\tau^{t\wedge\iota_n}(1+\|u^n_s\|^2_{C_\gamma(V)})ds,\ \forall\ t\in[\tau,\tau+T].\end{aligned}\tag{3.27}$$

By (3.26) and (3.27), we can rewrite (3.25) as

$$\begin{aligned}&\|u^n(t\wedge\iota_n)\|^2+2\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2_{D(A)}ds\\&\quad\leq\|\phi(0)\|^2+2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2ds+c_6\int_\tau^{t\wedge\iota_n}\|u^n_s\|^2_{C_\gamma(V)}ds\\&\qquad+2\left|\int_\tau^{t\wedge\iota_n}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\right|+2L^2_{\widetilde{\sigma}}T,\end{aligned}\tag{3.28}$$

where $c_6=c_5+2L^2_{\widetilde{\sigma}}$. We infer from (3.28) that

$$\begin{aligned}\|u^n_{t\wedge\iota_n}\|^2_{C_\gamma(V)}&\leq\max\Big\{\sup_{\vartheta\in(-\infty,\tau-t\wedge\iota_n]}e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2,\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]}e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2\Big\}\\&\leq\max\Big\{\sup_{\vartheta\in(-\infty,\tau-t\wedge\iota_n]}e^{2\gamma\vartheta}\|\phi(t\wedge\iota_n+\vartheta-\tau)\|^2,\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]}e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2\Big\}\\&\leq\max\Bigg\{\sup_{\vartheta\in(-\infty,0]}e^{2\gamma(\vartheta-t\wedge\iota_n+\tau)}\|\phi(\vartheta)\|^2,\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]}e^{2\gamma\vartheta}\Bigg(\|\phi(0)\|^2+2\int_\tau^{t\wedge\iota_n+\vartheta}\|\widetilde{f}(s)\|^2ds\\&\qquad+c_6\int_\tau^{t\wedge\iota_n+\vartheta}\|u^n_s\|^2_{C_\gamma(V)}ds+2\left|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\right|+2L^2_{\widetilde{\sigma}}T\Bigg)\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\le e^{-2\gamma(t\wedge\iota_n-\tau)}\|\phi\|^2_{C_\gamma(V)} + \|\phi\|^2_{C_\gamma(V)} + 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2ds + c_6\int_\tau^{t\wedge\iota_n}\|u^n_s\|^2_{C_\gamma(V)}ds \\
&\quad + 2\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\left|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\right| + 2L^2_{\widetilde{\sigma}}T \\
&\le 2\|\phi\|^2_{C_\gamma(V)} + 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2ds + c_6\int_\tau^{t}\|u^n_{s\wedge\iota_n}\|^2_{C_\gamma(V)}ds \\
&\quad + 2\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\left|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\right| + 2L^2_{\widetilde{\sigma}}T. \qquad (3.29)
\end{aligned}
$$

Taking supremum and expectation of (3.29), we obtain

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) &\le 2\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 2\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2ds\Big) \\
&\quad + c_6\int_\tau^t\mathbb{E}\Big(\sup_{r\in[\tau,s]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big)ds + 2L^2_{\widetilde{\sigma}}T \\
&\quad + 2\mathbb{E}\Big(\sup_{r\in[\tau,t]}\sup_{\vartheta\in[\tau-r\wedge\iota_n,0]} e^{2\gamma\vartheta}\Big|\int_\tau^{r\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\Big|\Big). \qquad (3.30)
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality and (3.27), the last term of (3.30) is bounded by

$$
\begin{aligned}
&2\mathbb{E}\Big(\sup_{r\in[\tau,t]}\sup_{\vartheta\in[\tau-r\wedge\iota_n,0]} e^{2\gamma\vartheta}\Big|\int_\tau^{r\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\Big|\Big) \\
&\le 2\mathbb{E}\Big(\sup_{r\wedge\iota_n+\vartheta\in[\tau,t\wedge\iota_n]}\Big|\int_\tau^{r\wedge\iota_n+\vartheta}\Big(\Big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\Big)\Big)\Big|\Big) \\
&\le 2c_7\mathbb{E}\bigg(\Big(\int_\tau^{t\wedge\iota_n}\|u^n_s\|^2_{C_\gamma(V)}\|\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac{1}{2}}\bigg) \\
&\le 2c_7\mathbb{E}\bigg(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|_{C_\gamma(V)}\Big(\int_\tau^{t\wedge\iota_n}\|\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac{1}{2}}\bigg) \\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) + 2c_7^2L^2_{\widetilde{\sigma}}\bigg(T + \int_\tau^t\mathbb{E}\Big(\sup_{r\in[\tau,s]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big)ds\bigg). \qquad (3.31)
\end{aligned}
$$

It follows from (3.30) and (3.31) that, for all $t\in[\tau,\tau+T]$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) &\le 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 4\mathbb{E}\Big(\int_\tau^t\|\widetilde{f}(s)\|^2ds\Big) \\
&\quad + c_9\int_\tau^t\mathbb{E}\Big(\sup_{r\in[\tau,s]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big)ds + c_8, \qquad (3.32)
\end{aligned}
$$

where $c_8 = 4L^2_{\widetilde{\sigma}}T(1+c_7^2), c_9 = 2(c_6 + 2c_7^2L^2_{\widetilde{\sigma}})$. Set

$$
c_{10} := 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 4\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2ds\Big) + c_8, \qquad (3.33)
$$

which is finite due to $\phi\in L^2(\Omega, C_\gamma(V))$ and $\widetilde{f}\in I^2(\tau,\tau+T;V)$. Applying the Gronwall lemma to (3.32), we find, for all $t\in[\tau,\tau+T]$,

$$
\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) \le c_{10}e^{c_9T} =: R_1. \qquad (3.34)
$$

Finally, we infer from (3.28) and (3.31) that, for all $t \in [\tau, \tau+T]$,

$$2\mathbb{E}\Big(\sup_{r\in[\tau,t]}\int_\tau^{r\wedge\iota_n}\|u^n(s)\|_{D(A)}^2 ds\Big) \le \mathbb{E}\Big(\|\phi(0)\|^2\Big) + 2\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2 ds\Big) + \frac{c_9}{2}\int_\tau^{\tau+T}\mathbb{E}\Big(\sup_{r\in[\tau,s]}\|u^n_{r\wedge\iota_n}\|_{C_\gamma(V)}^2\Big)ds + \frac{1}{2}\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|_{C_\gamma(V)}^2\Big) + \frac{c_8}{2}, \tag{3.35}$$

which, together with (3.34), implies that there exists a positive constant R_2,

$$\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n}\|u^n(s)\|_{D(A)}^2 ds\Big) \le R_2. \tag{3.36}$$

Combining (3.34) and (3.36), we show that, for $R = R_1 + R_2 > 0$, the following inequality holds:

$$\mathbb{E}\Big(\sup_{\tau\le r\le\tau+T}\|u^n_{r\wedge\iota_n}\|_{C_\gamma(V)}^2\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T}\|u^n(s\wedge\iota_n)\|_{D(A)}^2 ds\Big) \le R. \tag{3.37}$$

Next, we use (3.37) to prove Step 2 holds. It follows from (3.9) that

$$\{\iota_n < \tau+T\} \subseteq \{\sup_{\tau\le r\le\tau+T}\|u^n(r\wedge\iota_n)\| \ge n\},$$

which, together with Chebychev's inequality and (3.37), shows

$$\mathbb{P}\{\iota_n < \tau+T\} \le \mathbb{P}\{\sup_{\tau\le r\le\tau+T}\|u^n(r\wedge\iota_n)\| \ge n\} \le \frac{1}{n^2}\mathbb{E}\Big(\sup_{\tau\le r\le\tau+T}\|u^n(r\wedge\iota_n)\|^2\Big) \le \frac{R}{n^2},$$

and thus,

$$\sum_{n=1}^\infty \mathbb{P}\{\iota_n < \tau+T\} \le R\sum_{n=1}^\infty\frac{1}{n^2} < \infty, \tag{3.38}$$

which, together with the Borel-Cantelli lemma, implies from (3.38) that

$$\mathbb{P}\Big(\bigcap_{m=1}^\infty\bigcup_{n=m}^\infty\{\iota_n < \tau+T\}\Big) = 0.$$

Therefore, there exists a subset $\Omega_T = \cap_{m=1}^\infty \cup_{n=m}^\infty \{\iota_n < \tau+T\}$ of Ω with $\mathbb{P}(\Omega_T) = 0$ such that for each $\omega \in \Omega\setminus\Omega_T$, there exists $n_0 = n_0(\omega) > 0$ such that $\iota_n(\omega) \ge \tau+T$ for all $n \ge n_0$. Since ι_n is increasing in n, we obtain $\iota(\omega) \ge \tau+T$ for all $\omega \in \Omega\setminus\Omega_T$. Let $\Omega_0 = \bigcup_{T=1}^\infty \Omega_T$, then $\mathbb{P}(\Omega_0) = 0$ and

$$\iota(\omega) \ge \tau+T \quad \text{for all } \omega \in \Omega\setminus\Omega_0 \text{ and } T \in \mathbb{N}.$$

Consequently, we deduce that $\iota(\omega) = \infty$ for all $\omega \in \Omega\setminus\Omega_0$ as desired.

Step 3: *We prove the existence of solutions to system* (3.1). By Step 1 and Step 2, there exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega\setminus\Omega_1) = 0$ such that

$$u^{n+1}(t\wedge\iota_n,\omega) = u^n(t\wedge\iota_n,\omega)\ \forall\, n \in \mathbb{N},\ t \ge \tau, \tag{3.39}$$

$$\iota(\omega) = \lim_{n\to\infty}\iota_n(\omega) = \infty,\ \forall\, \omega \in \Omega_1. \tag{3.40}$$

By (3.39) and (3.40), for every $\omega \in \Omega_1$ and $t \ge \tau$, there exists an $n_1 = n_1(t,\omega) \ge 1$ such that for all $n \ge n_1$,

$$\iota_n(\omega) > t,\ \text{thus } u^n(t,\omega) = u^{n_1}(t,\omega). \tag{3.41}$$

Define a mapping $u : [\tau, \infty) \times \Omega \to V$ by

$$u(t,\omega) = \begin{cases} u^n(t,\omega), \text{ if } \omega \in \Omega_1 \text{ and } t \in [\tau, \iota_n(\omega)], \\ \phi(0,\omega), \quad \text{if } \omega \in \Omega \setminus \Omega_1 \text{ and } t \in [\tau, \infty). \end{cases} \tag{3.42}$$

Since u^n is a continuous V-valued process, it follows from (3.42) that u is also almost surely continuous with respect to t in $C_\gamma(V)$. Besides, we further infer from (3.42) that

$$\lim_{n\to\infty} u^n(t,\omega) = u(t,\omega), \forall \omega \in \Omega_1, \ t \geq \tau. \tag{3.43}$$

Note that u^n is $\mathcal{F}_t$-adapted, it follows from (3.43) that u is also $\mathcal{F}_t$-adapted. Thanks to (3.43), (3.37) and Fatou's lemma, we obtain that, for every $T > 0$,

$$\mathbb{E}\Big(\sup_{\tau \leq r \leq \tau+T} \|u_r\|^2_{C_\gamma(V)} \Big) + \mathbb{E}\bigg(\int_\tau^{\tau+T} \|u(s)\|^2_{D(A)} ds \bigg) \leq R, \tag{3.44}$$

where R is the same positive constant as in (3.37). Therefore, (3.16) holds.

We then prove that u is a solution to problem (3.1). Since u^n is the solution to (3.8), we deduce, for all $t \in [\tau, \tau+T]$,

$$\begin{aligned} &u^n(t \wedge \iota_n) + \int_\tau^{t\wedge\iota_n} \big(\widetilde{A}u^n(s) + \widetilde{B}(u^n(s))\big) ds \\ &\quad = \phi(0) + \int_\tau^{t\wedge\iota_n} (\widetilde{f}(s) + \widetilde{g}(s, u^n_s)) ds + \int_\tau^{t\wedge\iota_n} \widetilde{\sigma}_n(s, u^n(s)) dW(s). \end{aligned} \tag{3.45}$$

By (3.42), we find that, $u^n(t\wedge\iota_n) = u(t\wedge\iota_n)$ for all $t \in [\tau, \tau+T]$, $\mathbb{P}$-a.s., which together with the definition of $\widetilde{\sigma}_n$ implies $\widetilde{\sigma}_n(s, u^n(s)) = \widetilde{\sigma}(s, u(s))$ for all $s \in [\tau, \iota_n]$, $\mathbb{P}$-a.s., it follows from (3.45) that $\mathbb{P}$-a.s.,

$$u(t\wedge\iota_n) + \int_\tau^{t\wedge\iota_n} \big(\widetilde{A}u(s) + \widetilde{B}(u(s))\big) ds = \phi(0) + \int_\tau^{t\wedge\iota_n} (\widetilde{f}(s) + \widetilde{g}(s, u_s)) ds + \int_\tau^{t\wedge\iota_n} \widetilde{\sigma}(s, u(s)) dW(s), \tag{3.46}$$

By $\iota_n \uparrow \infty$ a.s., we can rewrite (3.46) as

$$u(t) + \int_\tau^t \big(\widetilde{A}u(s) + \widetilde{B}(u(s))\big) ds = \phi(0) + \int_\tau^t (\widetilde{f}(s) + \widetilde{g}(s, u_s)) ds + \int_\tau^t \widetilde{\sigma}(s, u(s)) dW(s),$$

and thus, we prove that u is a solution to (3.1) as desired.

Step 4: We show the uniqueness of solutions to (3.1). Let u and v be two solutions to system (3.1) with the same initial condition $u(s) = v(s) = \phi(s-\tau), \ s \leq \tau$. For every $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* T_n as follows:

$$T_n = \inf\{t \geq \tau : \|u(t)\| > n \text{ or } \|v(t)\| > n\} \wedge (\tau + T). \tag{3.47}$$

Let $\overline{w} = u - v$, applying Ito's formula to the process $\varsigma(t)\|\overline{w}(t)\|^2$ where $\varsigma(t) = e^{-\frac{\widetilde{c}}{2}\int_\tau^t \|v(s)\|^2_{D(A)} ds}$, we infer from (3.10) in Lemma 1 that, for all $t \in [\tau, \tau+T]$,

$$\begin{aligned} \varsigma(t\wedge T_n)\overline{w}(t\wedge T_n) = &-\frac{\widetilde{c}}{2} \int_\tau^{t\wedge t_n} \varsigma(s) \|v(s)\|^2_{D(A)} \|\overline{w}(s)\|^2 ds \\ &- 2\int_\tau^{t\wedge T_n} \varsigma(s) \big\langle \widetilde{A}\overline{w}(s) + \widetilde{B}(u(s)) - \widetilde{B}(v(s)), \overline{w}(s) \big\rangle ds \\ &+ 2\int_\tau^{t\wedge T_n} \varsigma(s) \big((\widetilde{g}(s, u_s) - \widetilde{g}(s, v_s), \overline{w}(s)) \big) ds \\ &+ \int_\tau^{t\wedge T_n} \varsigma(s) \|\widetilde{\sigma}(s, u(s)) - \widetilde{\sigma}(s, v(s))\|^2_{\mathcal{L}^2(K,V)} ds \end{aligned}$$

$$
\begin{aligned}
&+2\int_\tau^{t\wedge T_n}\widetilde{\varsigma}(s)\Big(\Big(\overline{w}(s),\big(\widetilde{\sigma}(s,u(s))-\widetilde{\sigma}(s,v(s))\big)dW(s)\Big)\Big)\\
&\le 2\int_\tau^{t\wedge T_n}\widetilde{\varsigma}(s)\big(\big(\widetilde{g}(s,u_s)-\widetilde{g}(s,v_s),\overline{w}(s)\big)\big)ds\\
&+\int_\tau^{t\wedge T_n}\widetilde{\varsigma}(s)\|\widetilde{\sigma}(s,u(s))-\widetilde{\sigma}(s,v(s))\|^2_{\mathcal{L}^2(K,V)}ds\\
&+2\int_\tau^{t\wedge T_n}\widetilde{\varsigma}(s)\Big(\Big(\overline{w}(s),\big(\widetilde{\sigma}(s,u(s))-\widetilde{\sigma}(s,v(s))\big)dW(s)\Big)\Big).
\end{aligned}
\tag{3.48}
$$

By the similar calculation as in Step 1, we deduce from (H5) and (3.47) that there exists $c_{11}>0$ such that

$$
\mathbb{E}\Big(\sup_{\tau\le s\le t}\widetilde{\varsigma}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big)\le c_{11}\int_\tau^t\mathbb{E}\Big(\sup_{\tau\le s\le r}\widetilde{\varsigma}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big)dr.
$$

By the Gronwall lemma and $0<\widetilde{\varsigma}\le 1$, we obtain

$$
\mathbb{E}\Big(\sup_{\tau\le s\le t}\|\overline{w}(s\wedge T_n)\|^2\Big)=0,
$$

which implies $\|\overline{w}(t\wedge T_n)\|=\|u(t\wedge T_n)-v(t\wedge T_n)\|=0$ for all $t\in[\tau,\tau+T]$ almost surely. Since u and v are continuous with respect to t, we show $T_n=\tau+T$ for large enough n. We then obtain that $\|\overline{w}(t)\|=0$ for all $t\in[\tau,\tau+T]$ almost surely. Therefore, for every $T>0$,

$$
\mathbb{P}\Big(\|\overline{w}(t)\|=0,\ \text{for all}\ \ t\in[\tau,\tau+T]\Big)=1.
$$

Since T is an arbitrary number, we further imply

$$
\mathbb{P}\Big(\|\overline{w}(t)\|=0,\ \text{for all}\ \ t\ge\tau\Big)=1,
$$

which proves the uniqueness of the solutions. This completes the proof.

By Theorem 1, one can define the following mapping:

$$
\begin{aligned}
&\Phi(t,\tau):L^2(\Omega,\mathcal{F}_\tau;C_\gamma(V))\to L^2(\Omega,\mathcal{F}_{\tau+t};C_\gamma(V)),\\
&\Phi(t,\tau)\phi=u_{t+\tau}(\cdot,\tau,\phi),\ \forall\, t\ge 0,\ \tau\in\mathbb{R},\phi\in L^2(\Omega,\mathcal{F}_\tau;C_\gamma(V)).
\end{aligned}
\tag{3.49}
$$

Thanks to [19,31], we deduce that Φ is a *mean random dynamical system* generated by (3.1) on $L^2(\Omega,\mathcal{F};C_\gamma(V))$.

3.2 Weak pullback mean random attractors

In this subsection, we are interested in proving the existence of a unique *weak pullback mean random attractor* for the stochastic system (3.1) in $L^2(\Omega,C_\gamma(V))$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$. To this end, we need to further assume

$$
\int_{-\infty}^\tau e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr<\infty,\quad\forall\,\tau\in\mathbb{R},\tag{3.50}
$$

$$
\lim_{t\to+\infty}e^{-\zeta t}\int_{-\infty}^0 e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r-t)\|^2)dr=0,\quad\forall\,\zeta>0.\tag{3.51}
$$

A family $\mathcal{D}=\{\mathcal{D}(\tau):\tau\in\mathbb{R}\}$ is a family of nonempty bounded sets such that $\mathcal{D}(\tau)\in L^2(\Omega,\mathcal{F}_\tau;C_\gamma(V))$ and further tempered if

$$
\lim_{t\to+\infty}e^{-\zeta t}\|\mathcal{D}(\tau-t)\|^2_{L^2(\Omega,\mathcal{F}_{\tau-t};C_\gamma(V))}=0,\quad\forall\,\zeta>0.\tag{3.52}
$$

We will use $\mathfrak{D}$ to denote the collection (or universe) of the above tempered families.

Lemma 2 *Suppose that* (H1)-(H5), (3.50) *and* (3.51) *hold. If* $2\gamma > \widetilde{\lambda}_1$, *then for each* $\tau \in \mathbb{R}$, *it follows*

$$\mathbb{E}(\|u_\tau(\cdot, \tau - t, \phi)\|^2_{C_\gamma(V)}) \leq R_0(\tau), \tag{3.53}$$

for all $\phi \in C_\gamma(V)$, *where*

$$R_0(\tau) = \widetilde{L} + \widetilde{L}e^{-\widetilde{\lambda}_1 \tau} \int_{-\infty}^{\tau} e^{\widetilde{\lambda}_1 r} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr \tag{3.54}$$

with $\widetilde{L}$ *a positive constant independent of* τ.

Proof By Ito's formula, we obtain from (3.1) that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 + 2\|u(t)\|^2_{D(A)} = 2\Big(\Big(\widetilde{f}(t) + \widetilde{g}(t, u_t), u(t)\Big)\Big) \\ + \|\widetilde{\sigma}(t, u)\|^2_{\mathcal{L}^2(K,V)} + 2\Big(\Big(u(t), \widetilde{\sigma}(t, u(t)) dW(t)\Big)\Big).\end{aligned} \tag{3.55}$$

By the Young inequality, (2.10) and (2.24), we deduce

$$\begin{aligned}2\Big(\Big(\widetilde{f}(t) + \widetilde{g}(t, u_t), u(t)\Big)\Big) &\leq 2\widetilde{\lambda}_1^{-\frac{1}{2}} \|\widetilde{f}(t) + \widetilde{g}(t, u_t)\| \|u(t)\|_{D(A)} \\ &\leq \widetilde{\lambda}_1^{-1} \|\widetilde{f}(t) + \widetilde{g}(t, u_t)\|^2 + \|u(t)\|^2_{D(A)} \\ &\leq 2\widetilde{\lambda}_1^{-1} \|\widetilde{f}(t)\|^2 + 2\widetilde{\lambda}_1^{-1} L^2_{\widetilde{g}} \|u_t\|^2_{C_\gamma(V)} + \|u(t)\|^2_{D(A)}.\end{aligned} \tag{3.56}$$

Thanks to (2.27), we find

$$\|\widetilde{\sigma}(t, u)\|^2_{\mathcal{L}^2(K,V)} \leq 2L^2_{\widetilde{\sigma}}(1 + \|u_t\|^2_{C_\gamma(V)}). \tag{3.57}$$

Substituting (3.56) and (3.57) into (3.55), then by (2.10), we obtain

$$\frac{d}{dt}\|u(t)\|^2 + \widetilde{\lambda}_1 \|u(t)\|^2 \leq 2\widetilde{\lambda}_1^{-1} \|\widetilde{f}(t)\|^2 + c_1 \|u_t\|^2_{C_\gamma(V)} + 2L^2_{\widetilde{\sigma}} + 2\Big(\Big(u(t), \widetilde{\sigma}(t, u(t)) dW(t)\Big)\Big), \tag{3.58}$$

where $c_1 = 2\widetilde{\lambda}_1^{-1} L^2_{\widetilde{g}} + 2L^2_{\widetilde{\sigma}}$. Taking the expectation of (3.58), we have

$$\frac{d}{dt}\mathbb{E}(\|u(t)\|^2) + \widetilde{\lambda}_1 \mathbb{E}(\|u(t)\|^2) \leq 2\widetilde{\lambda}_1^{-1} \mathbb{E}(\|\widetilde{f}(t)\|^2) + c_1 \mathbb{E}(\|u_t\|^2_{C_\gamma(V)}) + 2L^2_{\widetilde{\sigma}}. \tag{3.59}$$

Multiplying (3.58) by $e^{\lambda_1 t}$ and integrating over $(\tau - t, s)$, we obtain

$$\begin{aligned}\mathbb{E}(\|u(s, \tau - t, \phi)\|^2) \leq e^{\widetilde{\lambda}_1(\tau - t - s)} \mathbb{E}(\|\phi(0)\|^2) + 2\widetilde{\lambda}_1^{-1} \int_{\tau - t}^{s} e^{\widetilde{\lambda}_1(r - s)} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr \\ + c_1 \int_{\tau - t}^{s} e^{\widetilde{\lambda}_1(r - s)} \mathbb{E}(\|u_r\|^2_{C_\gamma(V)}) dr + 2L^2_{\widetilde{\sigma}} \widetilde{\lambda}_1^{-1}.\end{aligned} \tag{3.60}$$

Since the fact that $2\gamma > \widetilde{\lambda}_1$, we deduce

$$\begin{aligned}\mathbb{E}(\|u_s\|^2_{C_\gamma(V)}) &\leq \max\Big\{ \sup_{\vartheta \leq \tau - t - s} e^{2\gamma\vartheta} \mathbb{E}(\|u(s + \vartheta)\|^2), \sup_{\tau - t - s \leq \vartheta \leq 0} e^{2\gamma\vartheta} \mathbb{E}(\|u(s + \vartheta)\|^2) \Big\} \\ &\leq \max\Big\{ \sup_{\vartheta \leq \tau - t - s} e^{2\gamma\vartheta} \mathbb{E}(\|\phi(\vartheta - \tau + t + s)\|^2), \sup_{\tau - t - s \leq \vartheta \leq 0} e^{2\gamma\vartheta} \mathbb{E}(\|u(s + \vartheta)\|^2) \Big\} \\ &\leq \max\Bigg\{ \sup_{\vartheta \leq 0} e^{2\gamma(\vartheta + \tau - t - s)} \mathbb{E}(\|\phi(\vartheta)\|^2), \sup_{\tau - t - s \leq \vartheta \leq 0} e^{2\gamma\vartheta} \Big(e^{\widetilde{\lambda}_1(\tau - t - s - \vartheta)} \mathbb{E}(\|\phi(0)\|^2) \\ &\quad + 2\widetilde{\lambda}_1^{-1} \int_{\tau - t}^{s + \vartheta} e^{\widetilde{\lambda}_1(r - s - \vartheta)} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr + c_1 \int_{\tau - t}^{s + \vartheta} e^{\widetilde{\lambda}_1(r - s - \vartheta)} \mathbb{E}(\|u_r\|^2_{C_\gamma(V)}) dr + 2L^2_{\widetilde{\sigma}} \widetilde{\lambda}_1^{-1} \Big) \Bigg\}\end{aligned} \tag{3.61}$$

$$\leq 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1}\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr + c_1\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|u_r\|^2_{C_\gamma(V)})dr + 2L_\sigma^2\widetilde{\lambda}_1^{-1}.$$

Let $s=\tau$ in (3.61), we find

$$\mathbb{E}(\|u_\tau\|^2_{C_\gamma(V)}) \leq 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1}\int_{\tau-t}^{\tau} e^{\widetilde{\lambda}_1(r-\tau)}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr$$
$$+ c_1\int_{\tau-t}^{\tau} e^{\widetilde{\lambda}_1(r-\tau)}\mathbb{E}(\|u_r\|^2_{C_\gamma(V)})dr + 2L_\sigma^2\widetilde{\lambda}_1^{-1}. \tag{3.62}$$

Set

$$c_2 := 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1}e^{-\widetilde{\lambda}_1\tau}\int_{-\infty}^{\tau} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr + 2L_\sigma^2\widetilde{\lambda}_1^{-1}.$$

Applying the Gronwall lemma to (3.62), we deduce

$$\mathbb{E}(\|u_\tau\|^2_{C_\gamma(V)}) \leq c_2 e^{c_1\widetilde{\lambda}_1^{-1}}. \tag{3.63}$$

This yields (3.53) as desired.

Recall that the definition of a weak pullback mean random attractor $\mathcal{A}^{\mathrm{w}} = \{\mathcal{A}^{\mathrm{w}}(s) : s\in\mathbb{R}\} \in \mathfrak{D}$ is introduced by [31], that is, it is the minimum among all weakly compact and $\mathfrak{D}$-pullback w-attracting sets, where $\mathcal{A}^{\mathrm{w}}(\cdot)$ is called **$\mathfrak{D}$-pullback w-attracting** if for each $\mathcal{D}(\cdot)\in\mathfrak{D}$, $s\in\mathbb{R}$ and each neighborhood $N(\mathcal{A}^{\mathrm{w}}(s))$ under the weak topology of $L^2(\Omega,\mathcal{F}_s;C_\gamma(V))$, there is $T>0$ such that

$$\Phi(t,s-t)\mathcal{D}(s-t)\subset N(\mathcal{A}^{\mathrm{w}}(s)),\ \forall t\geq T.$$

Theorem 2 *Assume that the same hypotheses and notations in Lemma 2 hold. Then the mean random dynamical system Φ, induced by system (3.1), possesses a unique weak $\mathfrak{D}$-pullback mean random attractor $\mathcal{A}^w = \{\mathcal{A}^w(\tau) : \tau\in\mathbb{R}\}\in\mathfrak{D}$ in $L^2(\Omega,\mathcal{F};C_\gamma(V))$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$.*

Proof Let $\mathcal{K} = \{\mathcal{K}(\tau) : \tau\in\mathbb{R}\}$ with

$$\mathcal{K}(\tau) = \{\xi\in C_\gamma(V) : \mathbb{E}(\|\xi\|^2_{C_\gamma(V)}) \leq R_0(\tau)\},\ \forall\,\tau\in\mathbb{R}, \tag{3.64}$$

where $R_0(\tau)$ is given by (3.54). Since $\mathcal{K}(\tau)$ is a bounded closed convex set in $L^2(\Omega,\mathcal{F}_\tau;C_\gamma(V))$, we obtain it is weakly compact in $L^2(\Omega,\mathcal{F}_\tau;C_\gamma(V))$.

Finally, we prove $\mathcal{K}\in\mathfrak{D}$. For every $\zeta>0$, we infer from (3.51) that

$$\lim_{t\to+\infty} e^{-\zeta t}\|\mathcal{K}(\tau-t)\|^2_{L^2(\Omega,\mathcal{F}_{\tau-t},C_\gamma(V))} = \lim_{t\to+\infty} e^{-\zeta t}R_0(\tau-t)$$
$$= \widetilde{L}\lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L}\lim_{t\to+\infty} e^{-\zeta t}e^{-\widetilde{\lambda}_1(\tau-t)}\int_{-\infty}^{\tau-t} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr$$
$$= \widetilde{L}\lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L}\lim_{t\to+\infty} e^{-\zeta t}\int_{-\infty}^{0} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r+\tau-t)\|^2)dr$$
$$= \widetilde{L}\lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L}e^{-\zeta\tau}\lim_{t\to+\infty} e^{-\zeta t}\int_{-\infty}^{0} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r-t)\|^2)dr = 0,$$

which implies $\mathcal{K}$ is tempered, that is, $\mathcal{K}\in\mathfrak{D}$. This together with Lemma 2, shows $\mathcal{K}$ is a weakly compact $\mathfrak{D}$-pullback random absorbing set for Φ. By [31, Theorem 2.13], we prove the existence of a unique weak $\mathfrak{D}$-pullback mean random attractor $\mathcal{A}^{\mathrm{w}}$ for Φ.

4 Stationary solutions and their stability results

In this section, we are concerned with existence, uniqueness and stability properties of the stationary solutions to (1.1). For this end, we need to assume that $\widetilde{f}(t) \equiv \widetilde{f} \in (D(A))^*$, $\widetilde{\sigma}(t,\cdot) \equiv \widetilde{\sigma}(\cdot) : V \to \mathcal{L}^2(K,V)$ with $\widetilde{\sigma}(0) = 0$, i.e., they are independent of time. Besides, suppose that $\widetilde{\sigma}$ is globally Lipschitz continuous, that is, there exists a positive constant $C_{\widetilde{\sigma}}$ such that for all $\upsilon_1, \upsilon_2 \in V$, $\|\widetilde{\sigma}(\upsilon_1) - \widetilde{\sigma}(\upsilon_2)\|_{\mathcal{L}^2(K,V)} \le C_{\widetilde{\sigma}}\|\upsilon_1 - \upsilon_2\|$.

4.1 Existence and uniqueness of stationary solutions

We now consider the abstract equation associated to Eq. (1.1):

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f} + \widetilde{g}(t,u_t) + \widetilde{\sigma}(u)\dfrac{dW}{dt}, \quad \forall\, t > 0, \\ u(t) = \phi(t), t \in (-\infty, 0]. \end{cases} \tag{4.1}$$

We denote by $u(t) := u(t;\phi)$ the solution of Eq. (1.1) (or (3.1)) with $\tau = 0$, where $\phi = u_0$.

A stationary solution (an equilibrium solution) to problem (4.1) is an element $u_\infty \in D(A)$ satisfying

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}(t,u_\infty) + \widetilde{\sigma}(u_\infty)\frac{dW}{dt}, \quad \forall\, t \ge 0. \tag{4.2}$$

However, the above equation depends on t and a noisy term. Therefore, on the one hand, to get rid of the noise, we must assume that $\widetilde{\sigma}(u_\infty) = 0$. Consequently, we will focus on the existence and uniqueness of stationary solutions for the deterministic equation (i.e. $\widetilde{\sigma} = 0$ in (4.1)) which will be any $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}(t,u_\infty), \quad \forall\, t \ge 0. \tag{4.3}$$

On the other hand, we would need to assume that the delay term $\widetilde{g}$ would not depend on the time t, that is, there exists a function $\widetilde{g}_0 : V \to V$ such that

$$\widetilde{g}(t,\xi) = \widetilde{g}_0(\hat{\xi}) \text{ if } \xi(s) = \hat{\xi},\ \forall\ (s,t,\xi) \in \mathbb{R}^- \times \mathbb{R}^+ \times C_\gamma(V), \tag{4.4}$$

and it is Lipschitz continuous (with the same Lipschitz constant $L_{\widetilde{g}}$) and $\widetilde{g}_0(0) = 0$.

For example, if $\widetilde{g}$ is driven by unbounded variable delay, defined by

$$\widetilde{g}(t,\xi) = \widetilde{\mathcal{G}}(\xi(-h(t))), \tag{4.5}$$

where $h \in C^1([0,+\infty)), h(t) \ge 0$, $h^* = \sup_{t\ge 0} h'(t) < 1$, and $\widetilde{\mathcal{G}} : V \to V$ satisfies $\widetilde{\mathcal{G}}(0) = 0$, assume that there exists $L_{\widetilde{\mathcal{G}}} > 0$, for all $\eta, \zeta \in V$,

$$\|\widetilde{\mathcal{G}}(\eta) - \widetilde{\mathcal{G}}(\zeta)\| \le L_{\widetilde{\mathcal{G}}}\|\eta - \zeta\|. \tag{4.6}$$

In this case, the delay term $\widetilde{g}$ in our problem becomes $\widetilde{g}(t,u_t) = \widetilde{\mathcal{G}}(u(t-h(t)))$.

Another example is the case of infinite distributed delay, that is, the delay term $\widetilde{g}$ is defined by

$$\widetilde{g}(t,\xi) = \int_{-\infty}^0 \widetilde{\mathcal{H}}(s,\xi(s))ds, \tag{4.7}$$

where $\widetilde{\mathcal{H}} : (-\infty,0] \times V \to V$ with $\widetilde{\mathcal{H}}(s,0) = 0$ is measurable, and it is Lipschitz continuous with respect to its second variable, that is, there exists $L_{\widetilde{\mathcal{H}}}(s) \in L^2(-\infty,0)$ with $L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\theta)\cdot} \in L^2(-\infty,0)$, for certain $\theta > 0$, such that for all $s \in (-\infty,0], \eta, \zeta \in V$,

$$\|\widetilde{\mathcal{H}}(s,\eta) - \widetilde{\mathcal{H}}(s,\zeta)\| \le L_{\widetilde{\mathcal{H}}}(s)\|\eta - \zeta\|. \tag{4.8}$$

In this case, we can rewrite the delay term $\widetilde{g}$ in our problem as $\widetilde{g}(t,u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}(s,u(t+s))ds$.

Observe that the above both situations are within our framework, the conditions (H1)-(H5) are fulfilled for the infinite distributed delay in $C_\gamma(V)$ for $\gamma > 0$, but not necessarily for the unbounded variable delay. However, conditions (H1)-(H5) are satisfied for both delays in $C_{-\infty}(V)$.

Now, we are interested in studying the existence and uniqueness of solutions to Eq. (4.3), more precisely, we will prove the existence of a unique $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_0(u_\infty). \tag{4.9}$$

By non-compactness of Sobolev embeddings on unbounded domains, we need to modify some arguments of [6, Theorem 10]. More precisely, we use the idea of the classical Galerkin approximations as well as the Lax-Milgram and the Brouwer fixed theorems to establish the existence and uniqueness of a stationary solution to Eq. (4.9).

Theorem 3 *Assume that the above hypotheses and notations hold. If $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, then:*
(1) For all $\widetilde{f} \in (D(A))^*$, there exists at least one stationary solution to (4.9)*;*
(2) If $(1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$, the stationary solution to (4.9) *is unique.*

Proof (1) Let an orthonormal basis $B = \{w_j; j \in \mathbb{N}\} \subset \mathcal{V}$ of V such that linear combinations of elements of B are dense in $D(A)$. Denote $V_m = \text{span}[w_1, w_2, \dots, w_m]$ for $m \in \mathbb{N}$ with the norm $\|\cdot\|_{D(A)}$ of $D(A)$.

Step 1: *We use the Lax-Milgram Theorem to find a unique solution to Eq.* (4.10). More precisely, fixed $x^m \in V_m$, it suffices to find $u^m \in V_m$, which solves the equation

$$\langle \widetilde{A}u^m, w^m\rangle + \langle \widetilde{B}(u^m, x^m), w^m\rangle = \big((\widetilde{f} + \widetilde{g}_0(x^m), w^m)\big), \quad \forall\, w^m \in V_m. \tag{4.10}$$

Note that for each $x^m \in V_m$, the functional $(u, w) \mapsto \langle \widetilde{A}u, w\rangle + \langle \widetilde{B}(u, x^m), w\rangle$ is bilinear continuous and coercive in $V_m \times V_m$, besides, the functional $w \mapsto \big((\widetilde{f} + \widetilde{g}_0(x^m), w)\big)$ is linear continuous in V_m and thus by the Lax-Milgram Theorem, for each fixed $x^m \in V_m$, there exists a unique solution u^m to Eq. (4.10). Thus, we can define an operator $T_m : V_m \to V_m$, given by

$$T_m(x^m) = u^m. \tag{4.11}$$

Step 2: *We apply the Brouwer fixed point theorem to the mapping T_m* (restricted to a suitable subset of V_m) to ensure that there exists $u^m \in V_m$ such that

$$\langle \widetilde{A}u^m, w^m\rangle + \langle \widetilde{B}(u^m), w^m\rangle = \big((\widetilde{f} + \widetilde{g}_0(u^m), w^m)\big), \quad \forall\, w^m \in V_m. \tag{4.12}$$

Taking $w^m = u^m$ in (4.10), by (2.10), we deduce

$$\begin{aligned}\widetilde{\lambda}_1\|u^m\|^2 &\le \|u^m\|^2_{D(A)}\\ &= \big((\widetilde{f} + \widetilde{g}_0(x^m), u^m)\big)\\ &\le \|\widetilde{f}\|\|u^m\| + L_{\widetilde{g}}\|x^m\|\|u^m\|.\end{aligned} \tag{4.13}$$

Since $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, one can take $k > 0$ such that $k(\widetilde{\lambda}_1 - L_{\widetilde{g}}) \ge \|\widetilde{f}\|$, then

$$\widetilde{\lambda}_1\|u^m\| \le k\widetilde{\lambda}_1 - kL_{\widetilde{g}} + L_{\widetilde{g}}\|x^m\|. \tag{4.14}$$

Define $\mathcal{C}_m = \{x \in V_m : \|x\|_{D(A)} \le k\}$, which is a convex set of $D(A)$, and compact in $D(A)$. Note that the mapping T_m maps $\mathcal{C}_m$ into itself.

Let us now apply the Brouwer fixed point theorem to $T_m|\mathcal{C}_m$. To that end, we only need to prove T_m is continuous. Indeed, taking $x_1^m, x_2^m \in V_m$, we then denote by $u^m = T(x_1^m)$ and $v^m = T(x_2^m)$ the solutions to Eq. (4.10), next, taking the difference, we obtain

$$\langle \widetilde{A}(u^m - v^m), w^m\rangle + \langle \widetilde{B}(u^m, x_1^m), w^m\rangle - \langle \widetilde{B}(v^m, x_2^m), w^m\rangle = \big((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), w^m)\big), \tag{4.15}$$

for all $w^m \in V_m$. Setting $w^m = u^m - v^m$ in the above equality, by (2.10), (2.14), (2.16) and $u^m \in \mathcal{C}_m$, we find

$$\|u^m - v^m\|^2_{D(A)} = \langle \widetilde{B}(v^m, x_2^m), u^m - v^m\rangle - \langle \widetilde{B}(u^m, x_1^m), u^m - v^m\rangle + \big((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)\big)$$

$$
\begin{aligned}
&= -\langle \widetilde{B}(u^m - v^m, x_2^m), v^m\rangle + \langle \widetilde{B}(u^m - v^m, x_1^m), u^m\rangle + \big((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)\big) \\
&= -\langle \widetilde{B}(u^m - v^m, x_2^m), u^m\rangle + \langle \widetilde{B}(u^m - v^m, x_1^m), u^m\rangle + \big((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)\big) \\
&\le \widetilde{\lambda}_1^{-\frac{1}{2}}\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)}\|u^m\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)} \\
&\le (\widetilde{\lambda}_1^{-\frac{1}{2}}k + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}})\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)},
\end{aligned} \tag{4.16}
$$

which implies the continuity of T_m.

Step 3: *We take limit of the solutions proved in Step 2* to derive the existence of solutions to Eq. (4.9). Taking $w^m = u^m$ in (4.12), we have

$$
\begin{aligned}
\|u^m\|_{D(A)}^2 &= \big((\widetilde{f} + \widetilde{g}_0(u^m), u^m)\big) \\
&\le \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u^m\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|u^m\|_{D(A)}^2,
\end{aligned} \tag{4.17}
$$

which implies all solutions obtained in Step 2 are bounded by $\|u^m\|_{D(A)} \le \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|/(1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}})$. Thus, u^m has a weakly convergent subsequence (not relabeled) such that $u^m \rightharpoonup u$ in $D(A)$. Moreover, for any regular bounded set $Q \subset \mathcal{O}$, we derive the same uniform bounds of $u^m|_Q$, which together with the compact injection, yields $u^m|_Q \to u|_Q$ in $\mathbb{H}_0^1(Q)$.

For (4.12), we fix any $w_j \in B$, there exists a subsequence of u^m (relabeled the same) such that

$$
\langle \widetilde{A}u^m, w_j\rangle + \langle \widetilde{B}(u^m), w_j\rangle = \big((\widetilde{f} + \widetilde{g}_0(u^m), w_j)\big), \quad \forall\, m \ge j. \tag{4.18}
$$

Taking limit in (4.18), we find

$$
\langle \widetilde{A}u, w_j\rangle + \langle \widetilde{B}(u), w_j\rangle = \big((\widetilde{f} + \widetilde{g}_0(u), w_j)\big). \tag{4.19}
$$

In fact, the first term is obtained due to the weak convergence of $u^m \rightharpoonup u$ in $D(A)$. The bilinear mapping converges as long as they are defined on the support of w_j which is compact, so we denote by $Q_j \subset \mathcal{O}$ a bounded open set with smooth boundary containing it. Therefore, we not only deduce the weak convergence $u^m \rightharpoonup u$ in $D(A)$, but the strong convergence $u^m \to u$ in $\mathbb{H}_0^1(Q_j)$ (see [16, Lemma 2.4] for more details). For the last term,

$$
\begin{aligned}
\|\big((\widetilde{g}_0(u^m), w_j)\big) - \big((\widetilde{g}_0(u), w_j)\big)\| &\le \|\widetilde{g}_0(u^m) - \widetilde{g}_0(u)\|_{\mathbb{H}_0^1(Q_j)}\|w_j\| \\
&\le L_{\widetilde{g}}\|u^m - u\|_{\mathbb{H}_0^1(Q_j)}\|w_j\|,
\end{aligned} \tag{4.20}
$$

which converges to zero due to the strong convergence in $\mathbb{H}_0^1(Q_j)$. Consequently, (4.19) is satisfied for each w_j. Since the linear combinations of elements for $B = \{w_j; j \in \mathbb{N}\}$ are dense in $D(A)$, we deduce that (4.9) holds at least by $u_\infty = u$.

(2) Let $u_\infty^{(1)}$ and $u_\infty^{(2)}$ be two solutions to system (4.9). Then,

$$
\widetilde{A}(u_\infty^{(1)} - u_\infty^{(2)}) + \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}) = \widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}). \tag{4.21}
$$

Taking the inner product of (4.21) with $u_\infty^{(1)} - u_\infty^{(2)}$, we find

$$
\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2 + \langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)}\rangle = \big((\widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)})\big). \tag{4.22}
$$

Thanks to the fact that $\langle \widetilde{B}(u) - \widetilde{B}(v), u - v\rangle = -\langle \widetilde{B}(u - v), v\rangle$ for all $u, v \in D(A)$, then by (2.10) and (2.16), we obtain

$$
|\langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)}\rangle| \le \widetilde{c}\widetilde{\lambda}_1^{-\frac{1}{2}}\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2\|u_\infty^{(2)}\|_{D(A)}. \tag{4.23}
$$

Multiplying (4.9) by u_∞, we infer from (2.10) and (4.6) that

$$
\begin{aligned}
\|u_\infty\|_{D(A)}^2 &= \langle \widetilde{A}u_\infty, u_\infty\rangle \\
&= ((\widetilde{f}, u_\infty)) + ((\widetilde{g}_0(u_\infty), u_\infty))
\end{aligned}
$$

$$\leq \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u_\infty\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|u_\infty\|^2_{D(A)}, \tag{4.24}$$

which, together with $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, implies that

$$\|u_\infty\|_{D(A)} \leq \frac{\widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}}. \tag{4.25}$$

Actually, all solutions to (4.9) must satisfy the above bound. Therefore, combining (4.23) and (4.25), we have

$$|\langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)}\rangle| \leq \widetilde{c}\widetilde{\lambda}_1^{-1}\frac{\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}}\|u_\infty^{(1)} - u_\infty^{(2)}\|^2_{D(A)}. \tag{4.26}$$

By (2.10), the last term of (4.22) is bounded by

$$\begin{aligned}\big((\widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)})\big) &\leq L_{\widetilde{g}}\|u_\infty^{(1)} - u_\infty^{(2)}\|^2 \\ &\leq \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|u_\infty^{(1)} - u_\infty^{(2)}\|^2_{D(A)}.\end{aligned} \tag{4.27}$$

Substituting (4.26)-(4.27) into (4.22), we deduce

$$(1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}})^2\|u_\infty^{(1)} - u_\infty^{(2)}\|^2_{D(A)} \leq \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|\|u_\infty^{(1)} - u_\infty^{(2)}\|^2_{D(A)}, \tag{4.28}$$

which implies the uniqueness follows as long as we assume $(1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$.

Next, we mainly analyze the behavior of the solutions to (4.1) around these stationary solutions of (4.9).

4.2 Local stability of stationary solutions

In order to prove the local stability of stationary solutions to (4.9), we first prove the local stability of stationary solutions to (4.9) for general delay terms by using a direct method. Finally, we apply the abstract results to two kinds of infinite delays.

Theorem 4 *Assume the same hypotheses and notations in Theorem 1 and Theorem 3 hold. In addition,*

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + 2C_{\widetilde{g}} + C^2_{\widetilde{\sigma}} \tag{4.29}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (4.1)*, u_∞ is the unique stationary solution of Eq.* (4.9) *and $w(t) = u(t) - u_\infty$, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leq \mathbb{E}\Big(\|w(0)\|^2\Big) + C_{\widetilde{g}}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds. \tag{4.30}$$

Proof Applying the Ito formula to $\|w(t)\|^2$, we obtain

$$\begin{aligned}\|w(t)\|^2 = \|w(0)\|^2 &+ 2\int_0^t \langle -\widetilde{A}w(s) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\ &+ 2\int_0^t \Big(\Big(\widetilde{g}(s,u_s) - \widetilde{g}_0(u_\infty), w(s)\Big)\Big)ds + 2\int_0^t \Big(\Big(w(s), \big(\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\big)dW(s)\Big)\Big) \\ &+ \int_0^t \|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|^2_{\mathcal{L}^2(K,V)}ds.\end{aligned} \tag{4.31}$$

Taking expectation of (4.31), we have

$$\mathbb{E}\Big(\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds = \mathbb{E}\Big(\|w(0)\|^2\Big) - 2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds$$

$$
\begin{aligned}
&+ 2\mathbb{E}\bigg(\int_0^t \Big(\Big(\widetilde{g}(s, u_s) - \widetilde{g}_0(u_\infty), w(s) \Big) \Big) ds \bigg) \\
&+ \int_0^t \mathbb{E}\Big(\|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|^2_{\mathcal{L}^2(K,V)} \Big) ds. \qquad (4.32)
\end{aligned}
$$

Let us now estimate each term on the right-hand side of (4.32). By (2.10), (2.16), and the fact that $\langle \widetilde{B}(u) - \widetilde{B}(v), u - v\rangle = -\langle \widetilde{B}(u-v), v\rangle$ for all $u, v \in D(A)$, we deduce

$$
\begin{aligned}
-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle \Big) ds &= 2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(w(s)), u_\infty\rangle \Big) ds \\
&\leq 2\widetilde{c}\int_0^t \mathbb{E}(\|w(s)\|^2_{D(A)} \|u_\infty\|) ds \\
&\leq 2\widetilde{c}\widetilde{\lambda}_1^{-\frac{1}{2}} \int_0^t \mathbb{E}(\|w(s)\|^2_{D(A)} \|u_\infty\|_{D(A)}) ds. \qquad (4.33)
\end{aligned}
$$

By Theorem 3, we have

$$
\|u_\infty\|_{D(A)} \leq \frac{\widetilde{\lambda}_1^{-\frac{1}{2}} \|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}}}. \qquad (4.34)
$$

Substituting (4.34) into (4.33), we find

$$
-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle \Big) ds \leq \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}}} \int_0^t \mathbb{E}(\|w(s)\|^2_{D(A)}) ds. \qquad (4.35)
$$

By (H4), (2.10) and the Young inequality with $\epsilon_0 > 0$ to be specified later on, we deduce

$$
\begin{aligned}
&2\mathbb{E}\bigg(\int_0^t \Big(\Big(\widetilde{g}(s, u_s) - \widetilde{g}_0(u_\infty), w(s) \Big) \Big) ds \bigg) \\
&= 2\mathbb{E}\bigg(\int_0^t \Big(\Big(\widetilde{g}(s, u_s) - \widetilde{g}(s, u_\infty), w(s) \Big) \Big) ds \bigg) \\
&\leq 2\widetilde{\lambda}_1^{-\frac{1}{2}} \int_0^t \mathbb{E}\Big(\|\widetilde{g}(s, u_s) - \widetilde{g}(s, u_\infty)\| \|w(s)\|_{D(A)} \Big) ds \\
&\leq \frac{1}{\epsilon_0} \int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)} \Big) ds + \epsilon_0 \widetilde{\lambda}_1^{-1} C^2_{\widetilde{g}} \int_{-\infty}^t \mathbb{E}\Big(\|w(s)\|^2 \Big) ds \\
&\leq \frac{1}{\epsilon_0} \int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)} \Big) ds + \epsilon_0 \widetilde{\lambda}_1^{-1} C^2_{\widetilde{g}} \bigg(\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2 \Big) ds \\
&\quad + \widetilde{\lambda}_1^{-1} \int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)} \Big) ds \bigg). \qquad (4.36)
\end{aligned}
$$

By (2.10), the last term of (4.32) is bounded by

$$
\int_0^t \mathbb{E}\Big(\|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|^2_{\mathcal{L}^2(K,V)} \Big) ds \leq \widetilde{\lambda}_1^{-1} C^2_{\widetilde{\sigma}} \int_0^t E\Big(\|w(s)\|^2_{D(A)} \Big) ds. \qquad (4.37)
$$

It follows from the above inequalities that

$$
\begin{aligned}
\mathbb{E}\Big(\|w(t)\|^2 \Big) \leq \mathbb{E}\Big(\|w(0)\|^2 \Big) &+ \bigg(\frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}}} + \frac{1}{\epsilon_0} + \epsilon_0 \widetilde{\lambda}_1^{-2} C^2_{\widetilde{g}} + \widetilde{\lambda}_1^{-1} C^2_{\widetilde{\sigma}} - 2 \bigg) \times \\
&\times \bigg(\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)} \Big) ds \bigg) + \epsilon_0 \widetilde{\lambda}_1^{-1} C^2_{\widetilde{g}} \int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2 \Big) ds
\end{aligned}
$$

$$\leq \mathbb{E}\Big(\|w(0)\|^2\Big) + \widetilde{\lambda}_1^{-1}\Big(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + \frac{\widetilde{\lambda}_1}{\epsilon_0} + \epsilon_0\widetilde{\lambda}_1^{-1}C_{\widetilde{g}}^2 + C_{\widetilde{\sigma}}^2 - 2\widetilde{\lambda}_1\Big)\times$$
$$\times\Big(\int_0^t \mathbb{E}\Big(\|w(s)\|_{D(A)}^2\Big)ds\Big) + \epsilon_0\widetilde{\lambda}_1^{-1}C_{\widetilde{g}}^2\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{4.38}$$

Minimizing the right-hand side of (4.38), we choose $\epsilon_0 = \widetilde{\lambda}_1 C_{\widetilde{g}}^{-1}$ such that $\frac{\widetilde{\lambda}_1}{\epsilon_0} + \epsilon_0\widetilde{\lambda}_1^{-1}C_{\widetilde{g}}^2$ achieves its minimum value $2C_{\widetilde{g}}$. Then, by (4.29), we obtain (4.30) as desired.

In what follows, we will establish some sufficient conditions ensuring the local stability of stationary solutions to (4.9) when the delay term has particular forms in $C_{-\infty}(V)$.

Corollary 1 *Assume the same hypotheses and notations in Theorem 1 and Theorem 3 hold. Let the delay term* $\widetilde{g}(t,u_t) = \widetilde{\mathcal{G}}(u(t-h(t)))$ *satisfy* (4.5) *and* (4.6)*, moreover,*

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*} + C_{\widetilde{\sigma}}^2 \tag{4.39}$$

is satisfied. If $u(\cdot)$ *is any solution of Eq.* (4.1)*,* u_∞ *is the unique stationary solution of Eq.* (4.9) *and* $w(t) = u(t) - u_\infty$*, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leq \mathbb{E}\Big(\|w(0)\|^2\Big) + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{4.40}$$

Proof Taking $\widetilde{s} = s - h(s)$, we obtain $ds = 1/(1-h'(s))d\widetilde{s} \leq 1/(1-h^*)d\widetilde{s}$. Then, by (4.6), it follows

$$\begin{aligned}\int_0^t \|\widetilde{g}(s,u_s) - \widetilde{g}(s,v_s)\|^2 ds &= \int_0^t \|\widetilde{\mathcal{G}}(u(s-h(s))) - \widetilde{\mathcal{G}}(v(s-h(s)))\|^2 ds \\ &\leq L_{\widetilde{\mathcal{G}}}^2\int_0^t \|u(s-h(s)) - v(s-h(s))\|^2 ds \\ &\leq \frac{L_{\widetilde{\mathcal{G}}}^2}{1-h^*}\int_{-\infty}^t \|u(s)-v(s)\|^2 ds.\end{aligned} \tag{4.41}$$

By (H4), we obtain $C_{\widetilde{g}} = \frac{L_{\widetilde{\mathcal{G}}}}{\sqrt{1-h^*}}$. Thanks to (4.39), we find

$$\begin{aligned}2\widetilde{\lambda}_1 &\geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + \frac{2L_{\widetilde{\mathcal{G}}}}{(1-h^*)^{\frac{1}{2}}} + C_{\widetilde{\sigma}}^2 \\ &\geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + 2C_{\widetilde{g}} + C_{\widetilde{\sigma}}^2,\end{aligned} \tag{4.42}$$

which implies (4.29), together with Theorem 4 implies (4.40) as desired.

Corollary 2 *Assume the same hypotheses and notations in Theorem 1 and Theorem 3 hold. Let the delay term* $\widetilde{g}(t,u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}(s,u(t+s))ds$ *satisfy* (4.7) *and* (4.8)*, moreover,*

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + 2\|L_{\widetilde{\mathcal{H}}}\|_{L^2(-\infty,0)} + C_{\widetilde{\sigma}}^2 \tag{4.43}$$

holds. If $u(\cdot)$ *is any solution of Eq.* (4.1)*,* u_∞ *is the unique stationary solution of Eq.* (4.9) *and* $w(t) = u(t) - u_\infty$*, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leq \mathbb{E}\Big(\|w(0)\|^2\Big) + \|L_{\widetilde{\mathcal{H}}}\|_{L^2(-\infty,0)}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{4.44}$$

Proof The proof is similar to the one of Corollary 1. By (4.8) and (H4), we deduce $C_{\widetilde{g}} = \|L_{\widetilde{\mathcal{H}}}\|_{L^2(-\infty,0)}$. It follows from (4.43) and Theorem 4 that (4.44) holds as desired.

Remark 1 For the above infinite distributed delay, we can prove not only stability of stationary solutions in $C_{-\infty}(V)$ (see Corollary 2) even in $C_\gamma(V)$, but also their exponential asymptotic stability will be established in the next theorem.

4.3 Exponential convergence of stationary solutions

Under suitable assumptions, we prove that the solution $u(t)$ to problem (4.1) with infinite distributed delay converges exponentially to the unique stationary solution u_∞ of Eq. (4.9) in $C_\gamma(V)$ for $\gamma > 0$.

Theorem 5 *Assume the same hypotheses and notations in Theorem 1 and Theorem 3 hold. Let the delay term* $\widetilde{g}(t, u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}(s, u(t+s))ds$ *satisfy* (4.7) *and* (4.8), *moreover, there exists a constant* $0 < \rho < 2\gamma$ *such that for all* $t \geq 0$,

$$2\widetilde{\lambda}_1 \geq \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + 2(2\rho)^{-\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} + C_{\widetilde{\sigma}}^2 + \rho \tag{4.45}$$

is satisfied. If $u(\cdot)$ *is any solution of Eq.* (4.1), u_∞ *is the unique stationary solution of Eq.* (4.9) *and* $w(t) = u(t) - u_\infty$, *then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leq e^{-\rho t}\Big(1 + \frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big)\mathbb{E}\Big(\|\phi - u_\infty\|_{C_\gamma(V)}^2\Big), \tag{4.46}$$

and

$$\mathbb{E}\Big(\|w_t\|_{C_\gamma(V)}^2\Big) \leq e^{-\rho t}\Big(2 + \frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big)\mathbb{E}\Big(\|\phi - u_\infty\|_{C_\gamma(V)}^2\Big). \tag{4.47}$$

Proof Applying the Ito formula to $e^{\rho t}\|w(t)\|^2$ with $0 < \rho < 2\gamma$, we find, for all $t \geq 0$,

$$\begin{aligned}
e^{\rho t}\|w(t)\|^2 = {} & \|w(0)\|^2 + \rho\int_0^t e^{\rho s}\|w(s)\|^2 ds \\
& + 2\int_0^t e^{\rho s}\langle -\widetilde{A}(w(s)) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\
& + 2\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}(r, u(s+r)) - \widetilde{\mathcal{H}}(r, u_\infty)\big)dr, w(s)\Big)\Big)ds \\
& + \int_0^t e^{\rho s}\|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|_{\mathcal{L}^2(K,V)}^2 ds \\
& + 2\int_0^t e^{\rho s}\Big(\Big(w(s), \big(\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\big)dW(s)\Big)\Big).
\end{aligned} \tag{4.48}$$

Taking expectation of (4.48) and using (2.10), we obtain

$$\begin{aligned}
& \mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|_{D(A)}^2\Big)ds \\
& \leq \mathbb{E}\Big(\|\phi - u_\infty\|_{C_\gamma(V)}^2\Big) + \rho\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|_{D(A)}^2\Big)ds \\
& \quad - 2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \\
& \quad + 2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 \big(\widetilde{\mathcal{H}}(r, u(s+r)) - \widetilde{\mathcal{H}}(r, u_\infty)\big)dr, w(s)\Big)\Big)ds\Big) \\
& \quad + \mathbb{E}\Big(\int_0^t e^{\rho s}\|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|_{\mathcal{L}^2(K,V)}^2 ds\Big).
\end{aligned} \tag{4.49}$$

Thanks to (4.35), we deduce

$$-2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \leq \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|_{D(A)}^2\Big)ds. \tag{4.50}$$

By (2.10), (4.8) and the Young inequality with $\hat{\epsilon} > 0$ to be specified later on, the fourth line of (4.49) is bounded by

$$
\begin{aligned}
&2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 (\widetilde{\mathcal{H}}(r,u(s+r)) - \widetilde{\mathcal{H}}(r,u_\infty))dr, w(s)\Big)\Big)ds\Big) \\
&\le 2\widetilde{\lambda}_1^{-\frac{1}{2}}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}(r)\|w(s+r)\|dr\Big)\cdot\|w(s)\|_{D(A)}ds\Big) \\
&\le \hat{\epsilon}\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}(r)\|w(s+r)\|dr\Big)^2 ds\Big) + \frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|_{D(A)}^2 ds\Big) \\
&=: \hat{\epsilon}\widetilde{\lambda}_1^{-1} I + \frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|_{D(A)}^2 ds\Big), \qquad (4.51)
\end{aligned}
$$

where I is estimated as follows. By the Hölder inequality,

$$
\begin{aligned}
I &= \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}(r)\|w(s+r)\|dr\Big)^2 ds\Big) \\
&\le \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}(r)e^{-\gamma r}\|w_s\|_{C_\gamma(V)}dr\Big)^2 ds\Big) \\
&= \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|_{C_\gamma(V)}^2\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}(r)e^{-(\gamma+\rho)r}e^{\rho r}dr\Big)^2 ds\Big) \\
&\le \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|_{C_\gamma(V)}^2\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}}^2(r)e^{-2(\gamma+\rho)r}dr\int_{-\infty}^0 e^{2\rho r}dr\Big)ds\Big) \\
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|_{C_\gamma(V)}^2 ds\Big) \\
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\times \\
&\quad \mathbb{E}\Big(\int_0^t e^{\rho s}\max\Big\{\sup_{\theta\le -s} e^{2\gamma\theta}\|w(s+\theta)\|^2, \sup_{\theta\in[-s,0]} e^{2\gamma\theta}\|w(s+\theta)\|^2\Big\}ds\Big) \\
&\le \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\mathbb{E}\Big(\int_0^t \Big(e^{-(2\gamma-\rho)s}\|\phi-u_\infty\|_{C_\gamma(V)}^2 \\
&\quad + \widetilde{\lambda}_1^{-1}\sup_{\theta\in[-s,0]} e^{(2\gamma-\rho)\theta}e^{\rho(s+\theta)}\|w(s+\theta)\|_{D(A)}^2\Big)ds\Big). \qquad (4.52)
\end{aligned}
$$

By (2.10), the last term of (4.49) is bounded by

$$
\int_0^t \mathbb{E}\Big(e^{\rho s}\|\widetilde{\sigma}(u(s)) - \widetilde{\sigma}(u_\infty)\|_{\mathcal{L}^2(K,V)}^2\Big)ds \le \widetilde{\lambda}_1^{-1}C_{\widetilde{\sigma}}^2\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|_{D(A)}^2\Big)ds. \qquad (4.53)
$$

Substituting (4.50)-(4.53) into (4.49), then by $0<\rho<2\gamma$, we have

$$
\begin{aligned}
\mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big) &\le \mathbb{E}\Big(\|\phi-u_\infty\|_{C_\gamma(V)}^2\Big) + \widetilde{\lambda}_1^{-1}\Big(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + \frac{\hat{\epsilon}}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2 + \frac{\widetilde{\lambda}_1}{\hat{\epsilon}} \\
&\quad + C_{\widetilde{\sigma}}^2 + \rho - 2\widetilde{\lambda}_1\Big)\int_0^t \mathbb{E}\Big(\max_{r\in[0,s]}\{e^{\rho r}\|w(r)\|_{D(A)}^2\}\Big)ds \\
&\quad + \frac{1}{2\rho}\hat{\epsilon}\widetilde{\lambda}_1^{-1}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}^2\mathbb{E}\Big(\|\phi-u_\infty\|_{C_\gamma(V)}^2\Big)\int_0^t e^{-(2\gamma-\rho)s}ds. \qquad (4.54)
\end{aligned}
$$

Choosing $\hat{\epsilon} = (2\rho)^{\frac{1}{2}}\widetilde{\lambda}_1\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|^{-1}_{L^2(-\infty,0)}$, then

$$\min_{\hat{\epsilon}>0}\Big\{\frac{\hat{\epsilon}}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)} + \frac{\widetilde{\lambda}_1}{\hat{\epsilon}}\Big\} = 2(2\rho)^{-\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}. \tag{4.55}$$

We infer from (4.54), (4.55) and (4.45) that (4.46) holds.

By (4.46), and by $0<\rho<2\gamma$, and thus $e^{(2\gamma-\rho)\theta}\le 1$ when $\theta\le 0$, we find, for all $t\ge 0$,

$$\begin{aligned}
\mathbb{E}\Big(\|w_t\|^2_{C_\gamma(V)}\Big) &= \mathbb{E}\Big(\sup_{\theta\le 0} e^{2\gamma\theta}\|w(t+\theta)\|^2\Big)\\
&= \mathbb{E}\Big(\max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(t+\theta)-u_\infty\|^2,\ \sup_{\theta\in[-t,0]} e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)\\
&= \mathbb{E}\Big(\max\Big\{e^{-2\gamma t}\|\phi-u_\infty\|^2_{C_\gamma(V)},\ \sup_{\theta\in[-t,0]} e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)\\
&\le \max\Big\{e^{-\rho t}\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big),\\
&\qquad e^{-\rho t}\Big(1+\frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)\Big\}\\
&\le e^{-\rho t}\Big(2+\frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)}\|L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big).
\end{aligned} \tag{4.56}$$

Therefore, the proof is complete.

Remark 2 Although we may not prove the exponential stability for the unbounded variable delay in $C_\gamma(V)$ (see [23] and [26] for more details), we will explore at least the asymptotic stability in $C_{-\infty}(V)$ in the next subsection.

4.4 Asymptotic stability: the Lyapunov functional method

In this subsection, we mainly investigate the asymptotic stability of the trivial solution of the following abstract nonlinear stochastic partial functional differential system by constructing suitable Lyapunov functionals:

$$\begin{cases} du(t) = (\hat{A}(t,u(t)) + \hat{F}(t,u_t))dt + \hat{\sigma}(u(t))dW(t), \quad \forall\ t\in[0,T],\\ u(t)=\phi(t),\ t\in(-\infty,0], \end{cases} \tag{4.57}$$

where $\hat{A}(t,\cdot): D(A)\to(D(A))^*$ satisfies $\langle \hat{A}(t,u),u\rangle\le 0$, for all $u\in D(A)$, $\hat{F}(t,\cdot): C_{-\infty}(V)\to V$ and $\hat{\sigma}(\cdot): V\to \mathcal{L}^2(K,V)$ satisfy the following conditions: $\hat{F}(t,0)=\hat{\sigma}(0)=0$ and they are Lipschitz continuous, that is, there exist $L_{\hat{F}}, L_{\hat{\sigma}}>0$ such that for all $t\ge 0$, $\eta_1,\eta_2\in C_{-\infty}(V)$ and $\zeta_1,\zeta_2\in V$,

$$\|\hat{F}(t,\eta_1)-\hat{F}(t,\eta_2)\| \le L_{\hat{F}}\|\eta_1-\eta_2\|_{C_{-\infty}(V)}, \tag{4.58}$$

and

$$\|\hat{\sigma}(\zeta_1)-\hat{\sigma}(\zeta_2)\|_{\mathcal{L}^2(K,V)} \le L_{\hat{\sigma}}\|\zeta_1-\zeta_2\|. \tag{4.59}$$

By the similar estimates as in Section 3, the well-posedness of (4.57) can be proved. Fixed $T>0$ and given an initial value $\phi\in L^2(\Omega, C_{-\infty}(V))$, a solution to (4.57) is a stochastic process $u\in I^2(0,T;D(A))\cap L^2(\Omega, L^\infty(0,T;V))$ satisfying

$$\begin{cases} u(t) = \phi(0) + \int_0^t \hat{A}(s,u(s))ds + \int_0^t \hat{F}(s,u_s))ds\\ \qquad\qquad + \int_0^t \hat{\sigma}(u(s))dW(s),\ \mathbb{P}\text{-}a.s.,\ \forall\ t\in[0,T],\\ u(t)=\phi(t),\ t\in(-\infty,0], \end{cases} \tag{4.60}$$

where the first equation is understood in $(D(A))^*$.

We denote by $u(\cdot;\phi)$ the solution of Eq. (4.57) corresponding to the initial condition ϕ. For the reader convenience, we recall the following definition taken from [30].

Definition 2 The trivial solution of Eq. (4.57) is said to be p-stable, with $p>0$, if for any $\epsilon>0$, there exists $\delta>0$ such that $\mathbb{E}(\|u(t;\phi)\|^p)<\epsilon$, for all $t\geq 0$, provided that $\|\phi\|_1^p := \sup_{\theta\leq 0}\mathbb{E}(\|\phi(\theta)\|^p)<\delta$. If, besides, $\lim_{t\to+\infty}\mathbb{E}(\|u(t;\phi)\|^p)=0$ for every initial function ϕ, then the trivial solution of Eq. (4.57) is called asymptotically p-stable. In particular, if $p=2$, then the trivial solution of the system (4.57) is called asymptotically mean square stable.

Let the Lyapunov functional $U(t,\xi):[0,\infty)\times L^2(\Omega,C_{-\infty}(V))\to\mathbb{R}^+$ satisfy $U(t,\xi)=U(t,\xi(0),\xi(\theta)),\theta<0$, and for $\xi=u_t$, then set

$$\begin{aligned}&U_\xi(t,u)=U(t,\xi)=U(t,u_t)=U(t,u(t+\theta)),\ \theta<0,\\&u=\xi(0)=u(t),\end{aligned}\tag{4.61}$$

where $u(t)$ is a solution to system (4.57). By [30], the generator L of (4.57) has the following form:

$$\begin{aligned}LU(t,u_t)=&\frac{\partial U_\xi(t,u(t))}{\partial t}+\Big\langle\frac{\partial U_\xi(t,u(t))}{\partial u},\hat{A}(t,u(t))+\hat{F}(t,u_t)\Big\rangle\\&+\frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2U_\xi(t,u(t))}{\partial u^2}\hat{\sigma}(u(t))Q\hat{\sigma}^*(u(t))\Big),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes inner products in Hilbert spaces. Thanks to the Ito formula, we obtain

$$\mathbb{E}\Big(U(t,u_t)-U(s,u_s)\Big)=\int_s^t\mathbb{E}\Big(LU(r,u_r)\Big)dr,\quad t\geq s.\tag{4.62}$$

Notice that the following proposition generalizes the idea of Shaikhet in [30, Theorem 2.1] to the infinite delay version of stochastic partial differential equations.

Proposition 1 ([?]) *Suppose that there exists a continuous functional $U(t,\xi):[0,\infty)\times L^p(\Omega,C_{-\infty}(V))\to\mathbb{R}^+$ such that for the solution $u(t)$ of problem (4.57) and $p\geq 2$, the following inequalities hold for some positive constants μ_1,μ_2 and μ_3,*

$$\begin{aligned}&\mathbb{E}(U(t,u_t))\geq\mu_1\mathbb{E}(\|u(t)\|^p),\ \forall\ t\geq 0,\\&\mathbb{E}(U(0,\phi))\leq\mu_2\|\phi\|_1^p,\\&\mathbb{E}(U(t,u_t)-U(0,\phi))\leq-\mu_3\int_0^t\mathbb{E}(\|u(s)\|^p)ds,\ \forall\ t\geq 0.\end{aligned}\tag{4.63}$$

Then the trivial solution of equation (4.57) is asymptotically p-stable, that is,

$$\lim_{t\to+\infty}\mathbb{E}(\|u(t)\|^p)=0.\tag{4.64}$$

Based on the previous abstract results, we state our asymptotic stability result for our original problem as follows.

Theorem 6 *Assume that the same hypotheses and notations in Theorem 1 and Theorem 3 hold. In addition, let the delay term $\widetilde{g}(t,u_t)=\widetilde{\mathcal{G}}(u(t-h(t)))$ satisfy (4.5), (4.6), $\widetilde{f}=0$ and*

$$2\widetilde{\lambda}_1\geq\frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}+C_{\widetilde{\sigma}}^2.\tag{4.65}$$

Then $u_\infty=0$ is the unique stationary solution to problem (4.9). Moreover, the trivial solution of (4.1) is asymptotically mean square stable, that is,

$$\lim_{t\to\infty}\mathbb{E}(\|u(t;\phi)\|^2)=0.\tag{4.66}$$

Proof We first infer from the assumption $\widetilde{f}=0$ and Theorem 3 that $u_\infty=0$ is the unique stationary solution to Eq. (4.9). We then construct the Lyapunov functional $U(t,u_t):[0,\infty)\times L^2(\Omega,C_{-\infty}(V))\to\mathbb{R}^+$, defined by

$$U(t,u_t)=\|u(t)\|^2+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}\int_{t-h(t)}^{t}\|u(s)\|^2ds. \tag{4.67}$$

Let $\hat{A}(t,u)=-\widetilde{A}u(t)-\widetilde{B}(u(t))$, $\hat{F}(t,u_t)=\widetilde{g}(t,u_t)=\widetilde{\mathcal{G}}(u(t-h(t)))$, $\hat{\sigma}(u(t))=\widetilde{\sigma}(u(t))$ in (4.57), by (4.6), we obtain

$$\begin{aligned}L\|u(t)\|^2&=2\langle-\widetilde{A}(u)-\widetilde{B}(u),u\rangle+2((\widetilde{\mathcal{G}}(t-h(t)),u))+\|\widetilde{\sigma}(u(t))\|^2_{\mathcal{L}^2(K,V)}\\&\le-2\|u\|^2_{D(A)}+2\|\widetilde{\mathcal{G}}(u(t-h(t)))\|\|u\|+C_{\widetilde{\sigma}}^2\|u(t)\|^2\\&\le-2\widetilde{\lambda}_1\|u\|^2+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}\|u\|^2+(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}\|u(t-h(t))\|^2+C_{\widetilde{\sigma}}^2\|u(t)\|^2\\&=\Big(-2\widetilde{\lambda}_1+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}+C_{\widetilde{\sigma}}^2\Big)\|u\|^2+(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}\|u(t-h(t))\|^2,\end{aligned} \tag{4.68}$$

then

$$\begin{aligned}LU(t,u_t)&=L\Big(\|u(t)\|^2+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}\int_{t-h(t)}^{t}\|u(s)\|^2ds\Big)\\&\le L\|u(t)\|^2+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}\|u(t)\|^2-(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}\|u(t-h(t))\|^2\\&\le\Big(-2\widetilde{\lambda}_1+\frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*}+C_{\widetilde{\sigma}}^2\Big)\|u(t)\|^2,\end{aligned} \tag{4.69}$$

which, together with (4.65), shows $LU(t,u_t)\le0$. Moreover, the functional $U(t,u_t)$ defined in (4.67) fulfills the conditions in Proposition 1, and thus the trivial solution of (4.1) is asymptotically mean square stable in the sense of Definition 2.

Remark 3 By using the method of Lyapunov functionals construction, we obtain the asymptotic stability of the trivial solution to (4.1) with unbounded variable delay. Notice that condition (4.65) becomes exactly condition (4.39) when $\widetilde{f}=0$. Therefore, Theorem 6 ensures asymptotic stability under the same sufficient conditions which ensures only stability in Corollary 1, which means that the construction of Lyapunov functionals may provide better stability results. Furthermore, our analysis is also valid to study the asymptotic stability for the general case, that is, if the stationary solution is not the origin, in this case, we can shift it to the origin by a coordinate transformation.

4.5 Polynomial asymptotic stability for a particular case of unbounded variable delay

This subsection is concerned with the polynomial asymptotic behaviour of solutions to deterministic pantograph equations. In the particular case of proportional delay, we not only prove asymptotic stability but we can determine that the rate of convergence is at least polynomial. Now, let us consider the following deterministic pantograph equation:

$$\begin{cases}X'(t)=a_1X(t)+a_2X(\theta t),\ \forall\ t\ge0,\\X(0)=X_0,\end{cases} \tag{4.70}$$

where $a_1,a_2\in\mathbb{R}$, and $\theta\in(0,1)$.

Recall that the Dini derivative D^+F, where F is a continuous real-valued function of a real variable, defined by

$$D^+F=\limsup_{\delta\downarrow0}\frac{F(t+\delta)-F(t)}{\delta}.$$

Thanks to [1, Lemma 3.4], we restate some results which will be crucial in the polynomial asymptotic stability of stationary solutions to Eq. (4.70).

Lemma 3 *Let $a_1 \in \mathbb{R}, a_2 > 0$ and $\theta \in (0,1)$. Assume that X satisfies (4.70) with $X_0 > 0$. If there exists a continuous non-negative function $t \mapsto Y(t) : \mathbb{R}^+ \mapsto \mathbb{R}^+$,*

$$D^+Y(t) \le a_1 Y(t) + a_2 Y(\theta t), \quad t \ge 0 \tag{4.71}$$

with $0 < Y(0) < X_0$. Then $Y(t) \le X(t)$ for all $t \ge 0$.

Lemma 4 *Assume that X is the solution to Eq. (4.70). If $a_1 < 0$ and $a_2 \in \mathbb{R}$, there exists a constant $\varrho_0 = \varrho_0(a_1, a_2, \theta) > 0$,*

$$\limsup_{t \to +\infty} \frac{|X(t)|}{t^\beta} = \varrho_0 |X_0|, \tag{4.72}$$

where $\beta \in \mathbb{R}$ satisfies

$$a_1 + |a_2|\theta^\beta = 0. \tag{4.73}$$

Then, for some $\varrho_1 = \varrho_1(a_1, a_2, \theta) > 0$,

$$|X(t)| \le \varrho_1 |X_0|(1+t)^\beta, \quad t \ge 0. \tag{4.74}$$

Proof The proof is similar to [1, Lemma 3.5], thus the details are omitted here.

Notice that the polynomial asymptotic stability of the trivial solution to Eq. (4.70) is proved in the above Lemma when $\beta < 0$. In what follows, we apply the idea to deriving the polynomial asymptotic stability of stationary solution to problem (4.1).

Theorem 7 *Assume the same hypotheses and notations in Theorem 1 and Theorem 3 hold. In addition, let the system (4.1) satisfy $\widetilde{f} = 0$, the delay term $\widetilde{g}(t, u_t) = L_{\widetilde{g}} u(\theta t)$ with $\theta \in (0,1)$ and $2\widetilde{\lambda}_1 > 2|L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}}$, then the origin is the unique stationary solution to Eq. (4.9), moreover, any solution $u(t)$ of Eq. (4.1) converges to zero polynomially, that is, $\widetilde{\varrho} = \widetilde{\varrho}(L_{\widetilde{g}}, C_{\widetilde{\sigma}}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta < 0$,*

$$\mathbb{E}(\|u(t;\phi)\|^2) \le \widetilde{\varrho}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^\beta, \quad t \ge 0, \tag{4.75}$$

where β satisfies $-2\widetilde{\lambda}_1 + |L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}} + |L_{\widetilde{g}}|\theta^\beta = 0$.

Proof By $\widetilde{f} = 0$ and Theorem 3, we easily deduce that the origin is the unique stationary solution to Eq. (4.9). Applying the Ito formula to $\|u(t)\|^2$, then taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}(\|u(t)\|^2) - \mathbb{E}(\|\phi(0)\|^2) \\
&\le -2\mathbb{E}\Big(\int_0^t \|u(s)\|^2_{D(A)} ds\Big) + \Big(|L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}}\Big)\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + |L_{\widetilde{g}}|\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big) \\
&\le (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}})\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + |L_{\widetilde{g}}|\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big), \ \forall \ t > 0,
\end{aligned}$$

where we used (2.10). Let $v(t) = \mathbb{E}(\|u(t)\|^2)$, then

$$\frac{dv(t)}{dt} \le (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}})v(t) + |L_{\widetilde{g}}|v(\theta t). \tag{4.76}$$

By lemmas 3-4, we obtain that there exist $\widetilde{\varrho} = \widetilde{\varrho}(L_{\widetilde{g}}, C_{\widetilde{\sigma}}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta \in \mathbb{R}$,

$$v(t) \le \widetilde{\varrho} v(0)(1+t)^\beta. \tag{4.77}$$

Since $-2\widetilde{\lambda}_1 + 2|L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}} < 0$ and $-2\widetilde{\lambda}_1 + |L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}} + |L_{\widetilde{g}}|\theta^\beta = 0$, we deduce $\beta < 0$ and

$$\mathbb{E}(\|u(t)\|^2) \le \widetilde{\varrho}\mathbb{E}(\|\phi(0)\|^2)(1+t)^\beta \le \widetilde{\varrho}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^\beta.$$

The proof is complete.

Remark 4 In fact, we can take into account a more general case in the form of $\widetilde{g}(t, \xi) = \widetilde{G}(\xi(-(1-\theta)t))$, where $\widetilde{G}(\cdot)$ is Lipschitz continuous.

Statements and Declarations

No potential conflict of interest was reported by the authors.

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgements This work was done when Shuang Yang visited the Department of Differential Equations and Numerical Analysis at the University of Sevilla. She would like to express her thanks to all people there for their kind hospitality.

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