On the stochastic threshold of the COVID-19 epidemic model incorporating jump perturbations

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Abstract

This work delves into the intricate realm of epidemic modeling under the influence of unpredictable surroundings. By harnessing the power of white noise and Lévy noise, we construct a robust framework to capture the behavioral characteristics of the COVID-19 epidemic amidst erratic changes in the external environment. To enhance our comprehension of the intricate dynamics of the coronavirus, we conducted an investigation using a stochastic SIQS epidemic model that incorporates a dedicated compartment to represent populations under quarantine. Thanks to stochastic modeling techniques, we account for the inherent randomness in the transmission process and provide insights into the potential variations and uncertainties associated with the progression of the epidemic. Specifically, we show that the asymptotic behavior of our model is perfectly governed by two thresholds, $\mathcal{R}_{\sigma,J}$ and $\mathcal{R}'_{\sigma,J}$. That is to say, if $\mathcal{R}_{\sigma,J} < 1$, the disease will be removed from the population, while it will persist if $\mathcal{R}'_{\sigma,J} > 1$. Our highlight lies in obtaining the necessary and sufficient conditions for extinction in the absence of jump noise, namely $\mathcal{R}_{\sigma,0} = \mathcal{R}'_{\sigma,0}$. This means that our sufficient conditions for extinction for the jump case are also almost necessary. Finally, we present a set of computational simulations to validate our theoretical findings, supporting the results developed throughout this article. Overall, this research contributes to our understanding of the COVID-19 pandemic and its impact on the global population.

Key words: Stochastic SIQS model; Lévy jumps; Extinction; Exponentially stability; Persistence.

1 Introduction

COVID-19 is an infectious ailment caused by the novel coronavirus SARS-CoV-2, initially identified in Wuhan, China, in December 2019. In March 2020, the World Health Organization (WHO) officially declared the outbreak a global pandemic. By December 2021, this disease had affected more than 264 million individuals worldwide, leading to over 5 million in fatalities. The COVID-19 pandemic has emerged as a substantial public health crisis, profoundly impacting various facets of society, encompassing the economy, education, and healthcare systems. The primary means of COVID-19 transmission involve the inhalation of respiratory droplets expelled through coughing, sneezing, or speaking by infected individuals. Close contact with infected people also facilitates the spread of the virus, which exhibits a notably high rate of person-to-person transmission. The emergence of these communication modes has had a profound and far-reaching impact on public health. Many countries have taken crucial steps in response to the spread of COVID-19, such as imposing quarantine measures and implementing social distancing protocols. The transmission dynamics of the disease are influenced by many interconnected factors, including the behaviors exhibited by infected individuals, environmental conditions, and the effectiveness of public health interventions. These mechanisms are complex and dynamic, and researchers are employing a diverse range of tools to simulate the propagation of the disease and formulate efficient strategies for its containment. The study of the transmission of COVID-19 is a rapidly evolving field. Scientists are currently endeavoring to develop successfull treatments and vaccines for COVID-19 while also striving to gain a deeper understanding of the enduring effects that the disease may have on individuals who have gotten infected. By exploring the mechanisms of the virus, scientists aim to identify potential targets for therapies and interventions that can effectively combat the virus and minimize its impact on public health. Furthermore, studies are underway to investigate the long-term consequences of COVID-19 infection, including its potential to cause chronic health issues or exacerbate pre-existing conditions. The scientific community is working tirelessly to combat the ongoing pandemic and safeguard global health through these endeavors. Scientific investigations are underway to comprehend the societal and economic ramifications of the pandemic, encompassing the consequences of quarantine and social distancing measures on mental well-being and the economy. Mathematical biology has played a critical role in studying the COVID-19 pandemic. Recently, scientists have made significant progress in developing mathematical models aimed at understanding the underlying mechanisms that propel the transmission of the virus and forecasting the efficiency of various public health interventions. The SIQS model is a prevalent epidemic model that considers individuals who are susceptible, infected, quarantined, and those who have recovered from the disease. This model is widely utilized to assess the efficacy of quarantine and isolation measures in controlling and reducing disease transmission. By simulating different situations and studying the effect of these interventions on the propagation of the virus, researchers have been able to generate valuable insights into the potential effectiveness of these strategies in controlling the pandemic. This approach has proven to be a valuable tool for policymakers and public health officials in developing evidence-based strategies to mitigate the impact of the disease on society. Mathematical modeling was used to study the propagation of COVID-19 and inform decisions about pandemic planning, resource allocation, and interventions. Epidemic models were used to study the dynamics of the pandemic disease [20, 23, 14], describe how it spreads through contact [11, 16, 14], and assess the effectiveness of social distancing measures and other interventions [22, 8]. They have also been used to forecast the disease trend under various scenarios [10] and analyze transmission routes such as asymptomatic infectious and exposed individuals [9, 18]. This work aims to establish a SIQS epidemic model, a mathematical framework used to simulate the spread of infectious diseases and investigate the transmission dynamics of the COVID-19 pandemic. The underlying dynamics of this system are mathematically characterized by a set of differential equations, elucidating the temporal variations in the populations of susceptible, infected, and quarantined individuals. More specifically, the system can be succinctly expressed as follows:

$$\begin{cases} dS = [\Lambda - \mu S - \beta SI + \gamma I + rQ]\, dt, \\ dI = [-(\mu + d + \gamma + \lambda)I + \beta SI]\, dt, \\ dQ = [-(\mu + d + r)Q + \lambda I]\, dt. \end{cases} \tag{1}$$

The dynamics of the SIQS system is governed by a set of parameters and rates, which are defined as follows. The recruitment of susceptibles due to births and immigration is represented by the rate parameter Λ. The natural mortality rate per capita is represented by the parameter μ. The average number of appropriate contacts by an individual per time is noted by the parameter β, and this term only applies to individuals, not in quarantine. The recovery rate γ quantifies the velocity at which individuals successfully overcome the infection and return to the susceptible compartment S. In contrast, the recovery rate r captures the pace at which individuals emerge from quarantine partition Q, regain their health and rejoin the susceptible population S. The rate constant for disease-related deaths in the partitions I and Q is represented by the parameter d, and the rate at which individuals depart from the infected partition I to the quarantine partition Q is represented by the parameter λ. That mean, $\mu + d + r$ represent the rate at which individuals leave the quarantine class Q, and λ denotes the rate at which individuals enter the quarantine class Q. In [15], specific versions of the deterministic SIQS and SIQR models were introduced by Hethcote et al., which have been widely used in the modeling of infectious diseases. A comparison between the stochastic SIQS model and the deterministic model with nonlinear impact was introduced by Zhang and Huo, focusing on their long-term behaviors [26]. Wei and Chen [24] developed a stochastic epidemic SIQS model incorporating a saturation function into the incidence rate. They applied their model to the spread of hand, foot, and mouth disease in China and found that the model could better fit the data than models that assume a constant incidence rate. In [27], Zhang and Huo proposed a stochastic SIQS model and investigated the disease extinction and persistence threshold. Their study aimed to determine the critical value of the basic reproduction number below which the disease will eventually die out and above which the disease will persist in the population. But all these studies used independent standard Brownian motions variability in the mortality rates. However, in this work, we have modified the transmission coefficient by adding a stochastic term of the form $\sigma \dfrac{dB_t}{dt}$, where B_t represents a stochastic process known as Brownian motion. It is defined within a comprehensive probability space denoted as $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ denotes the sigma-algebra of events, $(\mathcal{F}_t)_{t\geq 0}$ refers to a filtration capturing the available information over time, and $\mathbb{P}$ represents the probability measure governing the system. Here, $\sigma > 0$ represents the intensity of the noise. When confronted with a situation where the population may experience rapid environmental shocks such as avian flu and SARS, the stochastic model, driven by white noise, may break continuity, affirming the significance of Lévy noise. About the works of the stochastic model driven by Lévy noise, the reader can refer to [28, 1, 12]. Here, we consider the system formulated as a stochastic SIQS model with jump perturbation coupled with white noise.

$$\begin{cases} dS = [\Lambda - \mu S - \beta SI + \gamma I + rQ]\, dt - \sigma SI dB_t - dK_t, \\ dI = [-(\mu + d + \lambda + \gamma)I + \beta SI]\, dt + \sigma SI dB_t + dK_t, \\ dQ = [-(\mu + d + r)Q + \lambda I]\, dt, \end{cases} \tag{2}$$

and $K_t = \int_0^t \int_{\mathbb{T}} J(z)S(s^-)I(s^-)\widetilde{\mathcal{N}}(dt, dz)$, where, $\widetilde{\mathcal{N}}(dt, dz) = \mathcal{N}(dt, dz) - \pi(dz)dt$ is the compensated Poisson measure, $\mathcal{N}(dt, dz)$ is a Poisson counting measure independent of $B(t)$, dt is Lebesgue measure, π is a Lévy measure defined on a measurable subset $\mathbb{T}$ of $[0, \infty)$, and the continuously differentiable function $J(.)$ represents the effects of random jumps in the population, verifying $|J(z)| \leq \frac{\mu}{\Lambda}$ for every $z \in \mathbb{T}$. Additionally, the symbols $S(s^-)$ and $I(s^-)$ denote the left-hand limits of the functions $S(s)$ and $I(s)$, respectively. For practicality, we will henceforth refer to these limits as $S(s)$ and $I(s)$. This article is structured into five distinct sections. In Section 2, we thoroughly discuss the existence and positivity of the solution, presenting a comprehensive analytical approach used to establish this result. Section 3 explores the conditions necessary for the extinction of coronavirus disease and presents analytical findings. The persistence of the disease and the conditions surrounding it are analyzed and established in Section 4. To illustrate the theoretical results, Section 5 offers several numerical examples. Lastly, in Section 6, we conclude the paper by discussing the obtained results and proposing potential avenues for future research.

2 Global existence and positivity

Let us first recall a helpful result.

Lemma 2.1 (Lipster [21]). *Consider the local martingale M_t, $t \geq t_0$, which vanishes at time t_0. Define $\varphi_{M_t} := \int_{t_0}^{t} \frac{d<M_s,M_s>}{(1+s)^2}$, where $<M_t, M_t>$ is called Meyer square process. If*

$$\mathbb{P}\left(\lim_{t\to\infty} \varphi_{M_t} < \infty\right) = 1,$$

then

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{M_t}{t} = 0\right) = 1.$$

The next theorem guarantees the solution will remain in Δ, where

$$\Delta = \left\{(x_1, x_2, x_3) \in (0,\infty)^3;\ \frac{\Lambda}{\mu + d} \leq x_1 + x_2 + x_3 \leq \frac{\Lambda}{\mu}\right\}.$$

Theorem 2.2. *Set $\Upsilon(z, S, I) = (1 + J(z)S)(1 - J(z)I)$, for $(z, S, I) \in \mathbb{T} \times (0, \frac{\Lambda}{\mu})^2$. If*

$$\sup_{0<S,I<\frac{\Lambda}{\mu}} \int_{\mathbb{T}} \ln\left(\frac{1}{\Upsilon(z,S,I)}\right)\pi(dz) < \infty. \tag{3}$$

Then, model (2) admits a unique solution $(S_t, I_t, Q_t) \in \Delta$ for any initial state $(S_0, I_0, Q_0) \in \Delta$.

Proof. Let N be the total population, we have $dN = (\Lambda - \mu N - d(I + Q))dt$, then

$$\frac{dN(s)}{ds} \geq (\Lambda - (\mu + d)N(s)) \qquad \forall s \in (0,t) \quad \text{a.s.},$$

which gives by integration between 0 and t that

$$N(t) - \frac{\Lambda}{\mu + d} \geq \left(N_0 - \frac{\Lambda}{\mu + d}\right)e^{-(\mu+d)t},$$

and hence

$$N(s) \geq \frac{\Lambda}{\mu + d} \qquad \forall s \in (0,t) \quad \text{a.s.}. \tag{4}$$

Besides, by the positivity of I and Q, we find

$$\frac{dN(t)}{dt} \leq (\Lambda - \mu N(t)),$$

and, by integrating this inequality, we have

$$N(s) \leq \frac{\Lambda}{\mu} \quad \forall s \in (0,t) \quad \text{a.s..} \tag{5}$$

For all initial conditions $(S_0, I_0, Q_0) \in \Delta$, model (2) admits a unique maximal local solution (S_t, I_t, Q_t) on the time interval $[0, \tau_e)$, where τ_e represents the timing of the explosion. This existence result follows from the locally Lipschitz continuity of the coefficients of the equations. Now, taking $\varepsilon, \varepsilon_0 > 0$ where $S_0,\ I_0,\ Q_0 > \varepsilon_0$, For $\varepsilon \leq \varepsilon_0$, we consider the stopping time

$$\tau_\varepsilon = \inf\{t \in [0, \tau_e), \quad S(t) \leq \varepsilon \quad \text{or} \quad I(t) \leq \varepsilon \quad \text{or} \quad Q(t) \leq \varepsilon\}, \tag{6}$$

and define for $(S, I, Q) \in \Delta$

$$W_1(S, I, Q) = \ln\left(\frac{1}{S}\right) + \ln\left(\frac{1}{I}\right) + \ln\left(\frac{1}{Q}\right).$$

By the Itô formula, we deduce for every $t \geq 0$ and $s \in [0, t \wedge \tau_\varepsilon]$

$$\begin{aligned} dW_1(S,I,Q) \quad = \quad & \left(3\mu + 2d + \lambda + \gamma + r - \frac{\Lambda}{S} + \beta I - \frac{\gamma I}{S} - \frac{rQ}{S} + \frac{\sigma^2 I^2}{2} - \beta S + \frac{\sigma^2 S^2}{2} - \frac{\lambda I}{Q}\right) ds \\ & - \int_{\mathbb{T}} \left(\ln\left(1 - J(z)I\right) + \ln\left(1 + J(z)S\right)\right) \pi(dz) ds + \int_{\mathbb{T}} (S - I) J(z) \pi(dz) ds \\ & + \sigma(I - S) dB - \int_{\mathbb{T}} \left(\ln\left(1 - J(z)I\right) + \ln\left(1 + J(z)S\right)\right) \widetilde{\mathcal{N}}(ds, dz). \end{aligned}$$

From (3) and (5), we deduce

$$\begin{aligned} & 3\mu + 2d + \lambda + \gamma + r - \frac{\Lambda}{S} + \beta I - \frac{\gamma I}{S} - \frac{rQ}{S} + \frac{\sigma^2 I^2}{2} - \beta S + \frac{\sigma^2 S^2}{2} - \frac{\lambda I}{Q} \\ & \quad - \int_{\mathbb{T}} \left(\ln\left(1 - J(z)I\right) + \ln\left(1 + J(z)S\right)\right) \pi(dz) + \int_{\mathbb{T}} (S - I) J(z) \pi(dz) \\ & \leq 3\mu + 2d + \lambda + \gamma + r + \beta I + \frac{\sigma^2 I^2}{2} + \frac{\sigma^2 S^2}{2} + 2\pi(\mathbb{T}) \frac{\Lambda}{\mu} \\ & \quad + \sup_{0 < S, I < \frac{\Lambda}{\mu}} \int_{\mathbb{T}} \ln\left(\frac{1}{\Upsilon(z, S, I)}\right) \pi(dz). \end{aligned} \tag{7}$$

Integrating and taking into account (7), we have for every $t \geq 0$

$$\mathbb{E}\left[W_1(S\left(t \wedge \tau_\varepsilon\right), I\left(t \wedge \tau_\varepsilon\right), Q\left(t \wedge \tau_\varepsilon\right))\right] \leq W_1\left(S_0, I_0, Q_0\right) + \chi t \leq 3 \ln\left(\frac{1}{\varepsilon_0}\right) + \chi t, \tag{8}$$

where

$$\chi = 3\mu + 2d + \lambda + \gamma + r + \beta \frac{\Lambda}{\mu} + \sigma^2 \left(\frac{\Lambda}{\mu}\right)^2 + \pi(\mathbb{T}) \frac{\Lambda}{\mu} + \sup_{0 < S, I < \frac{\Lambda}{\mu}} \int_{\mathbb{T}} \ln\left(\frac{1}{\Upsilon(z, S, I)}\right) \pi(dz).$$

Now, assuming $\tau_e < \infty$, there exists $t > 0$ where $\mathbb{P}(\tau_e < t) > 0$, and thus $\mathbb{P}(\tau_\varepsilon < t) > 0$. For $\omega \in \{\tau_e < t\}$ and (6), we have

$$\ln\left(\frac{1}{\varepsilon}\right) \leq W_1\left(S\left(t \wedge \tau_\varepsilon\right)(\omega), I\left(t \wedge \tau_\varepsilon\right)(\omega), Q\left(t \wedge \tau_\varepsilon\right)(\omega)\right),$$

one can easily obtain

$$\begin{aligned} \ln\left(\frac{1}{\varepsilon}\right) \mathbb{P}\left(\tau_\varepsilon \leq t\right) &\leq \mathbb{E}\left[W_1(S\left(\tau_\varepsilon\right), I\left(\tau_\varepsilon\right), Q\left(\tau_\varepsilon\right))\mathbf{1}_{\{\tau_\varepsilon \leq t\}}\right], \\ &\leq \mathbb{E}\left[W_1(S\left(t \wedge \tau_\varepsilon\right), I\left(t \wedge \tau_\varepsilon\right), Q\left(t \wedge \tau_\varepsilon\right))\right]. \end{aligned} \tag{9}$$

By using $\tau_\varepsilon \leq \tau_e$, (8) and (9) we have

$$\mathbb{P}\left(\tau_e \leq t\right) \leq \mathbb{P}\left(\tau_\varepsilon \leq t\right) \leq \frac{3\ln\left(\frac{1}{\varepsilon_0}\right) + \chi t}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

Letting $\varepsilon \to 0$, it follows $\mathbb{P}\left(\tau_e \leq t\right) = 0$. Consequently, $\tau_e = \infty \quad a.s.$ $\square$

3 Extinction

We refer to the works of Khasminskii [17] and Yuan and Mao [25], which provide a lemma establishing sufficient conditions to guarantee asymptotic stability in probability.

Lemma 3.1. *The trivial solution of system (2) is globally asymptotically stable in probability if there is a function $W \in \mathcal{C}^2\left(\mathbb{R}^3 \times \mathbb{S}; \mathbb{R}^+\right)$ where $\mathcal{L}W$ is definite-negative.*

In the next theorem, the global stability of model (2) will be discussed. First, consider the quadratic function

$$\Gamma(S) = -\frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right) S^2 + \beta S - (\mu + d + \lambda + \gamma), \tag{10}$$

and the quantities

$$\begin{aligned} \mathcal{R}_{\sigma,J} &= \frac{\beta\Lambda}{\mu}\left(\mu + d + \lambda + \gamma + \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2\right)^{-1}, \\ C_1 &= \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right), \\ C_2 &= -\beta + \left(\sigma^2 + \frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\frac{\Lambda}{\mu}, \\ C_3 &= \Gamma\left(\frac{\Lambda}{\mu}\right) = (\mathcal{R}_{\sigma,J} - 1)\left(\mu + d + \lambda + \gamma + \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2\right), \\ \Theta &= \begin{cases} C_3 & \text{if} \quad C_2 \leq 0, \\ \frac{C_2^2}{4C_1} + C_3 & \text{if} \quad C_2 > 0. \end{cases} \end{aligned}$$

Theorem 3.2. *Assume further that*

$$\sup_{0<S<\frac{\Lambda}{\mu}} \int_{\mathbb{T}} \ln^2(1+J(z)S)\pi(dz) < \infty. \tag{11}$$

If (3) holds, and

$$\Theta < 0, \tag{12}$$

then, for each $(S_0, I_0, Q_0) \in \Delta$, the disease-free $E_0(1,0,0)$ of model (2) is globally asymptotically stable in probability.

Proof. Let $W_2(S,I,Q) = k_1\left(\frac{\Lambda}{\mu} - S\right)^2 + k_2 I^{\frac{1}{k_2}} + k_3 Q^2$ be the Lyapunov function, where k_1, k_2 and k_3 are real positive constants that will be chosen below. We have

$$\begin{aligned}
\mathcal{L}W_2 &= -2k_1\mu\left(\frac{\Lambda}{\mu} - S\right)^2 + 2k_1\beta SI\left(\frac{\Lambda}{\mu} - S\right) - 2k_1\gamma I\left(\frac{\Lambda}{\mu} - S\right) - 2k_1 rQ\left(\frac{\Lambda}{\mu} - S\right) \\
&\quad + k_1\sigma^2 S^2 I^2 - (\mu+d+\lambda+\gamma)I^{\frac{1}{k_2}} + \beta S I^{\frac{1}{k_2}} + \frac{1}{2}\left(\frac{1}{k_2} - 1\right)\sigma^2 S^2 I^{\frac{1}{k_2}} - 2k_3(\mu+d+r)Q^2 \\
&\quad + 2k_3\lambda IQ + k_1\int_{\mathbb{T}}\left(\left(\frac{\Lambda}{\mu} - S + J(z)SI\right)^2 - \left(\frac{\Lambda}{\mu} - S\right)^2 - 2J(z)SI\left(\frac{\Lambda}{\mu} - S\right)\right)\pi(dz) \\
&\quad + k_2\int_{\mathbb{T}}\left((I+J(z)SI)^{\frac{1}{k_2}} - I^{\frac{1}{k_2}} - \frac{1}{k_2}J(z)SI^{\frac{1}{k_2}}\right)\pi(dz).
\end{aligned}$$

Since $S, I \in (0, \frac{\Lambda}{\mu})$, $I \le \frac{\Lambda}{\mu} - S$ and $Q \le \frac{\Lambda}{\mu} - S$, we deduce for all $k_2 \ge 1$

$$\begin{aligned}
\mathcal{L}W_2 &\le -2k_1\mu\left(\frac{\Lambda}{\mu} - S\right)^2 + 2k_1\beta\left(\frac{\Lambda}{\mu}\right)^2 I + k_1\sigma^2\left(\frac{\Lambda}{\mu}\right)^2 I^2 - (\mu+d+\lambda+\gamma)I^{\frac{1}{k_2}} + \beta S I^{\frac{1}{k_2}} \\
&\quad + \frac{1}{2}\left(\frac{1}{k_2} - 1\right)\sigma^2 S^2 I^{\frac{1}{k_2}} - 2k_3(\mu+d+r)Q^2 + 2(k_3\lambda - k_1(\gamma+r))IQ \\
&\quad + k_1 S^2 I^2\int_{\mathbb{T}} J^2(z)\pi(dz) + I^{\frac{1}{k_2}}\int_{\mathbb{T}}\left(k_2(I+J(z)SI)^{\frac{1}{k_2}} - k_2 - J(z)S\right)\pi(dz).
\end{aligned} \tag{13}$$

Applying Taylor-Lagrange formula and the inequality $|J(z)S| \le 1$, one can easily verify that there exists $0 < c < \frac{1}{k_2}$ such that

$$\begin{aligned}
(1+J(z)S)^{\frac{1}{k_2}} - 1 &= \frac{1}{k_2}\ln(1+J(z)S) + \frac{1}{2k_2^2}\ln^2(1+J(z)S)\exp^{c\ln(1+J(z)S)} \\
&\le \frac{1}{k_2}\ln(1+J(z)S) + \frac{1}{2k_2^2}\ln^2(1+J(z)S)\exp^{\frac{1}{k_2}\ln(2)}.
\end{aligned}$$

Applying the reverse of Fatou's lemma

$$\begin{aligned}
&\limsup_{k_2\to\infty}\int_{\mathbb{T}}\left(k_2(1+J(z)S)^{\frac{1}{k_2}} - k_2 - \frac{2^{\frac{1}{k_2}-1}}{k_2}\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}}\ln^2(1+J(z)S)\right)\pi(dz) \\
&\quad \le \int_{\mathbb{T}}\ln(1+J(z)S)\pi(dz),
\end{aligned}$$

which implies that, for all $\varepsilon > 0$, there exists k_0 such that, for all $k_2 \geq k_0$, we have

$$\begin{aligned}\int_{\mathbb{T}} \left(k_2 \left(1+J(z)S\right)^{\frac{1}{k_2}} - k_2\right)\pi(dz) &\leq \varepsilon + \int_{\mathbb{T}} \left(\ln(1+J(z)S)\right)\pi(dz) \\ &\quad + \frac{2^{\frac{1}{k_2}-1}}{k_2} \sup_{0<S<\frac{\Lambda}{\mu}} \int_{\mathbb{T}} \ln^2(1+J(z)S)\pi(dz). \end{aligned} \tag{14}$$

From (13), (14) and using the inequality

$$\ln(1+y) - y < -\frac{y^2}{4}, \quad 0 < y \leq 1, \tag{15}$$

we have

$$\begin{aligned}\mathcal{L}W_2 \leq\ & -2k_1\mu\left(\frac{\Lambda}{\mu} - S\right)^2 - 2k_3(\mu+d+r)Q^2 + 2(k_3\lambda - k_1(\gamma+r))IQ + 2k_1\beta SI\left(\frac{\Lambda}{\mu} - S\right) \\ & + k_1 S^2 I^2\left(\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right) + I^{\frac{1}{k_2}}\left(-(\mu+d+\lambda+\gamma) + \beta S - \frac{1}{2}\sigma^2 S^2 + \varepsilon\right. \\ & \left. -\frac{1}{4}S^2\int_{\mathbb{T}} J^2(z)\pi(dz) + \frac{2^{\frac{1}{k_2}-1}}{k_2}\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}} \ln^2(1+J(z)S)\pi(dz) + \frac{1}{2k_2}\sigma^2 S^2\right).\end{aligned}$$

For k_2 and k_3 such that $k_2 \geq k_0$ and $k_3 < \frac{k_1(\gamma+r)}{\lambda}$, and by using $I^j \leq (\frac{\Lambda}{\mu})^{j-\frac{1}{k_2}} I^{\frac{1}{k_2}}$ for $j \geq \frac{1}{k_2}$, we derive

$$\begin{aligned}\mathcal{L}W_2 \leq\ & I^{\frac{1}{k_2}}\left(2k_1\beta\left(\frac{\Lambda}{\mu}\right)^{3-\frac{1}{k_2}} + k_1\left(\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^{4-\frac{1}{k_2}}\right. \\ & + 2(k_3\lambda - k_1(\gamma+r))\left(\frac{\Lambda}{\mu}\right)^{2-\frac{1}{k_2}} \\ & \left. + \frac{1}{2k_2}\left(2^{\frac{1}{k_2}}\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}} \ln^2(1+J(z)S)\pi(dz) + \sigma^2\left(\frac{\Lambda}{\mu}\right)^2\right) + \Gamma(S) + \varepsilon\right) \\ & -2k_1\mu\left(\frac{\Lambda}{\mu} - S\right)^2 - 2k_3(\mu+d+r)Q^2.\end{aligned}$$

On other hand, by examining the quadratic form Γ on $(0, \frac{\Lambda}{\mu})$, we can show that $\Gamma(S) \leq \Theta$. This implies

$$\begin{aligned}\mathcal{L}W_2 \leq\ & I^{\frac{1}{k_2}}\left(2k_1\beta\left(\frac{\Lambda}{\mu}\right)^{3-\frac{1}{k_2}} + k_1\left(\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^{4-\frac{1}{k_2}}\right. \\ & + 2(k_3\lambda - k_1(\gamma+r))\left(\frac{\Lambda}{\mu}\right)^{2-\frac{1}{k_2}} \\ & \left. + \frac{1}{2k_2}\left(2^{\frac{1}{k_2}}\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}} \ln^2(1+J(z)S)\pi(dz) + \sigma^2\left(\frac{\Lambda}{\mu}\right)^2\right) + \Theta + \varepsilon\right) \\ & -2k_1\mu\left(\frac{\Lambda}{\mu} - S\right)^2 - 2k_3(\mu+d+r)Q^2.\end{aligned} \tag{16}$$

By (12), we can choose a sufficiently small ε, a sufficiently large k_2 and a sufficiently small k_1 such that

$$2k_1\beta\left(\frac{\Lambda}{\mu}\right)^{3-\frac{1}{k_2}}+k_1\left(\sigma^2+\int_{\mathbb{T}}J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^{4-\frac{1}{k_2}}+2(k_3\lambda-k_1(\gamma+r))\left(\frac{\Lambda}{\mu}\right)^{2-\frac{1}{k_2}}$$
$$+\frac{1}{2k_2}\left(2^{\frac{1}{k_2}}\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}}\ln^2(1+J(z)S)\pi(dz)+\sigma^2\left(\frac{\Lambda}{\mu}\right)^2\right)+\Theta+\varepsilon<0.$$

The proof is completed according to Lemma 3.1. □

Theorem 3.3. *Let $(S_0, I_0, Q_0)\in\Delta$, if (3) and (11) hold, then*

$$\limsup_{t\to\infty}\frac{1}{t}\ln(I(t))\leq\Theta.$$

Moreover, if $\Theta<0$, then system (2) is extinctive with exponential extinction rate.

Proof. Applying Itô's formula yields

$$\begin{aligned}\ln(I(t)) &= \ln(I_0)+\int_0^t\left(-\frac{1}{2}\sigma^2S^2(s)+\beta S(s)-(\mu+d+\lambda+\gamma)\right)ds\\ &\quad+\int_0^t\int_{\mathbb{T}}\left(\ln(1+J(z)S(s))-J(z)S(s)\right)\pi(dz)ds+\int_0^t\sigma S(s)dB_s\\ &\quad+\int_0^t\int_{\mathbb{T}}\ln(1+J(z)S(s))\widetilde{\mathcal{N}}(ds,dz).\end{aligned}$$

By (15), we obtain

$$\begin{aligned}\ln(I(t)) &\leq \ln(I_0)+\int_0^t\left(-\frac{1}{2}\left(\sigma^2+\frac{1}{2}\int_{\mathbb{T}}J^2(z)\pi(dz)\right)S^2(s)+\beta S(s)-(\mu+d+\lambda+\gamma)\right)ds\\ &\quad+\int_0^t\sigma S(s)dB_s+\int_0^t\int_{\mathbb{T}}\ln(1+J(z)S(s))\widetilde{\mathcal{N}}(ds,dz),\\ &\triangleq \ln(I_0)+\int_0^t\Gamma(S(s))ds+M_t^0+M_t^1,\end{aligned}\tag{17}$$

where $M_t^0=\int_0^t\sigma S(s)dB_s$ and $M_t^1=\int_0^t\int_{\mathbb{T}}\ln(1+J(z)S(s))\widetilde{\mathcal{N}}(ds,dz)$ are real-valued local martingales, fulfilling

$$\begin{aligned}<M_t^0,M_t^0> &= \int_0^t\sigma^2S^2(s)ds\leq\sigma^2\left(\frac{\Lambda}{\mu}\right)^2t,\\ <M_t^1,M_t^1> &= \int_0^t\int_{\mathbb{T}}\ln^2(1+J(z)S(s))\pi(dz)ds\leq\left(\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}}\ln^2(1+J(z)S)\pi(dz)\right)t.\end{aligned}$$

Then, we have from (11) and Lemma 2.1

$$\limsup_{t\to\infty}\frac{M_t^0}{t}=0\quad,\quad\limsup_{t\to\infty}\frac{M_t^1}{t}=0\quad a.s.$$

It follows from (17) that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(I(t))\leq\Theta+\limsup_{t\to\infty}\frac{M_t^0}{t}+\limsup_{t\to\infty}\frac{M_t^1}{t}.$$

Therefore, if $\Theta<0$, then $I(t)$ converges to 0 exponentially as t goes to $+\infty$. $\square$

The following theorem establishes the criteria for the moment exponential stability of the equilibrium state of model (2). Let us define

$$\Xi(\delta) = -2\left(\sigma^2+\int_{\mathbb{T}}J^2(z)\pi(dz)\right)\delta^2+2\beta\delta-\mu, \tag{18}$$

$$\Pi = \begin{cases}-\frac{1}{2}\left(\sigma^2+\frac{1}{2}\int_{\mathbb{T}}J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2+\beta\frac{\Lambda}{\mu}-\mu & \text{if } C_2\leq 0,\\ \frac{\beta^2}{2\left(\sigma^2+\frac{1}{2}\int_{\mathbb{T}}J^2(z)\pi(dz)\right)}-\mu & \text{if } C_2>0.\end{cases} \tag{19}$$

Theorem 3.4. *For $(S_0,I_0,Q_0)\in\Delta$ and $0<p\leq 1$. If (3) holds, then*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\mathbb{E}\left[X^p(t)\right]\right)\leq p\left(\Pi+\frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2\left(p\sigma^2+\int_{\mathbb{T}}J^2(z)\pi(dz)\right)\right), \tag{20}$$

where $X=\frac{\Lambda}{\mu}-S+I+Q$. Moreover, if

$$\Pi+\frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2\left(p\sigma^2+\int_{\mathbb{T}}J^2(z)\pi(dz)\right)<0,$$

then, the free-disease equilibrium state E_0 is pth moment exponentially stable in Δ.

Proof. By using Itô's formula we have

$$\begin{aligned} d\ln\left(X^p\right) &= p\left(-\mu(\frac{\Lambda}{\mu}-S)+2\beta SI-(\mu+d+2\gamma)I-(\mu+d+2r)Q\right)\frac{1}{X}dt\\ &\quad -2p\sigma^2\left(\frac{SI}{X}\right)^2dt+p\int_{\mathbb{T}}\left(\ln\left(1+2J(z)\frac{SI}{X}\right)-2J(z)\frac{SI}{X}\right)\pi(dz)dt\\ &\quad +2p\sigma\frac{SI}{X}dB+p\int_{\mathbb{T}}\ln\left(1+2J(z)\frac{SI}{X}\right)\widetilde{\mathcal{N}}(dt,dz). \end{aligned}$$

Using (15), the following inequalities

$$|J(z)|\leq\frac{\mu}{\Lambda},\quad \frac{SI}{X}\leq\frac{\Lambda}{2\mu},\quad \left(-\mu(\frac{\Lambda}{\mu}-S)-(\mu+d+2\gamma)I-(\mu+d+2r)Q\right)\frac{1}{X}\leq-\mu, \tag{21}$$

and that fact that Ξ verifies $\Xi(x)\leq\Pi$ on $x\in\left(0,\frac{\Lambda}{2\mu}\right)$, we obtain after integration the following estimation

$$\ln(X^p(t)) \leq \ln(X_0^p)+p\Pi t+pM_t^0+pM_t^1, \tag{22}$$

where

$$M_t^0=2\sigma\int_0^t\frac{S(s)I(s)}{X(s)}dB_s,\quad M_t^1=\int_0^t\int_{\mathbb{T}}\ln\left(1+2J(z)\frac{S(s)I(s)}{X(s)}\right)\widetilde{\mathcal{N}}(ds,dz),$$

are real-valued local martingales. Taking expectation yields

$$\mathbb{E}\left[\frac{X^p(t)}{X_0^p}\right] \leq \exp\left\{p\Pi t\right\} \times \mathbb{E}\left[\exp\left\{pM_t^0\right\}\right] \times \mathbb{E}\left[\exp\left\{pM_t^1\right\}\right]. \tag{23}$$

The associated exponential process of the continuous martingale pM_t^0, that is $\exp\left\{pM_t^0 - \frac{1}{2} < pM_t^0, pM_t^0 >\right\}$, is a martingale. Therefore

$$\begin{aligned} \mathbb{E}\left[\exp\left\{pM_t^0\right\}\right] &= \mathbb{E}\left[\exp\left\{\frac{1}{2} < pM_t^0, pM_t^0 >\right\}\right] = \exp\left\{\frac{1}{2}p^2\int_0^t \left(2\sigma\frac{S(s)I(s)}{X(s)}\right)^2 ds\right\} \\ &\leq \exp\left\{\frac{1}{2}p^2\sigma^2\left(\frac{\Lambda}{\mu}\right)^2 t\right\}. \end{aligned} \tag{24}$$

In contrast, the process

$$\exp\left\{pM_t^1 - \int_0^t \int_{\mathbb{T}} \left(\exp\left\{p\ln\left(1+\Psi(s,z)\right)\right\} - 1 - p\ln\left(1+\Psi(s,z)\right)\right)\pi(dz)ds\right\},$$

where $\Psi(s,z) = 2J(z)\frac{S(s)I(s)}{X(s)}$, is a martingale. Thus

$$\mathbb{E}\left[\exp\left\{pM_t^1\right\}\right] = \mathbb{E}\left[\exp\left\{\int_0^t \int_{\mathbb{T}} \left((1+\Psi(s,z))^p - 1 - p\ln\left(1+\Psi(s,z)\right)\right)\pi(dz)ds\right\}\right].$$

Using the inequalities

$$y^p < 1 + p(y-1), \qquad 0 \leq y, \quad 0 < p \leq 1,$$

and

$$\ln(1+y) - y \geq -\frac{y^2}{2}, \quad 0 \leq y, \tag{25}$$

we deduce

$$\begin{aligned} \mathbb{E}\left[\exp\left\{pM_t^1\right\}\right] &= \mathbb{E}\left[\exp\left\{\frac{p}{2}\int_0^t \int_{\mathbb{T}} \Psi^2(s,z)\pi(dz)ds\right\}\right] \\ &\leq \exp\left\{\frac{pt}{2}\left(\frac{\Lambda}{\mu}\right)^2 \int_{\mathbb{T}} J^2(z)\pi(dz)\right\}. \end{aligned} \tag{26}$$

Combining (23), (24) and (26) yields

$$\ln\left(\mathbb{E}\left[\frac{X^p(t)}{X_0^p}\right]\right) \leq pt\left(\Pi + \frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2\left(p\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right)\right).$$

We obtain the needed assertion by dividing both sides by t and letting t go to ∞. $\square$

4 Persistence

Recall that

$$\Gamma(S) \;=\; -\frac{1}{2}\left(\sigma^2+\frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right)S^2+\beta S-(\mu+d+\lambda+\gamma),$$

and

$$\mathcal{R}_{\sigma,J} \;=\; \beta\frac{\Lambda}{\mu}\left(\mu+d+\lambda+\gamma+\frac{1}{2}\left(\sigma^2+\frac{1}{2}\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2\right)^{-1}.$$

In addition, we use the quadratic function denoted as $\Gamma'(\cdot)$

$$\begin{aligned}\Gamma'(S) &= \Gamma(S)-\frac{1}{4}\int_{\mathbb{T}} J^2(z)\pi(dz)\\ &= -\frac{1}{2}\left(\sigma^2+\int_{\mathbb{T}} J^2(z)\pi(dz)\right)S^2+\beta S-(\mu+d+\lambda+\gamma),\end{aligned} \tag{27}$$

and the quantities

$$\mathcal{R}'_{\sigma,J} \;=\; \beta\frac{\Lambda}{\mu}\left(\mu+d+\lambda+\gamma+\frac{1}{2}\left(\sigma^2+\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2\right)^{-1}, \tag{28}$$

$$C'_3 \;=\; \Gamma'\left(\frac{\Lambda}{\mu}\right)=\left(\mathcal{R}'_{\sigma,J}-1\right)\left(\mu+d+\lambda+\gamma+\frac{1}{2}\left(\sigma^2+\int_{\mathbb{T}} J^2(z)\pi(dz)\right)\left(\frac{\Lambda}{\mu}\right)^2\right). \tag{29}$$

Let ν and ν' be the positive roots, on $(0,\frac{\Lambda}{\mu})$, of $\Gamma(S)=0$ and $\Gamma'(S)=0$ respectively. Let us denote

$$\xi_I=\frac{\mu+d+r}{\mu+d+r+\lambda}\left(\frac{\Lambda}{\mu}-\nu\right),\quad \xi_Q=\frac{\lambda}{\mu+d+r+\lambda}\left(\frac{\Lambda}{\mu}-\nu\right),$$
$$\xi'_I=\frac{\mu+d+r}{\mu+d+r+\lambda}\left(\frac{\Lambda}{\mu}-\nu'\right),\quad \xi'_Q=\frac{\lambda}{\mu+d+r+\lambda}\left(\frac{\Lambda}{\mu}-\nu'\right).$$

Theorem 4.1. *Let* (3) *and*

$$\sup_{0<S<\frac{\Lambda}{\mu}}\int_{\mathbb{T}}\ln\left(\frac{1}{1+J(z)S}\right)\pi(dz)<\infty, \tag{30}$$

hold. If $\mathcal{R}_{\sigma,J}>1$, then

(a) $\limsup_{t\to\infty} S(t)\geq\nu$, *a.s.,*

(b) $\liminf_{t\to\infty} I(t)\leq\xi_I$, *a.s.,*

(c) $\liminf_{t\to\infty} Q(t)\leq\xi_Q$, *a.s..*

Moreover if $\mathcal{R}'_{\sigma,J}>1$, then

(a′) $\liminf_{t\to\infty} S(t)\leq\nu'$, *a.s.,*

(b') $\limsup_{t\to\infty} I(t) \geq \xi'_I$, *a.s.*,

(c') $\limsup_{t\to\infty} Q(t) \geq \xi'_Q$, *a.s.*.

Proof. (a) From (2), we obtain

$$\begin{aligned}
\ln(I(t)) &= \ln(I_0) + \int_0^t \left(-\frac{1}{2}\sigma^2 S^2(s) + \beta S(s) - (\mu + d + \lambda + \gamma)\right) ds \\
&\quad + \int_0^t \int_{\mathbb{T}} \left(\ln\left(1 + J(z)S(s)\right) - J(z)S(s)\right) \pi(dz) \\
&\quad + \int_0^t \sigma S(s) dB_s + \int_0^t \int_{\mathbb{T}} \ln\left(1 + J(z)S(s)\right) \widetilde{\mathcal{N}}(ds, dz), \qquad (31)
\end{aligned}$$

which implies, using inequalities (15),

$$\ln(I(t)) \leq \ln(I_0) + \int_0^t \Gamma(S(s))ds + \int_0^t \sigma S(s) dB_s + \int_0^t \int_{\mathbb{T}} \ln\left(1 + J(z)S(s)\right) \widetilde{\mathcal{N}}(ds, dz). \quad (32)$$

$\Gamma(0) = -(\mu + d + \lambda + \gamma) < 0$ and $\Gamma\left(\frac{\Lambda}{\mu}\right) = C_3 > 0$. Then, equation $\Gamma(S) = 0$ has a unique root $\nu \in (0, \frac{\Lambda}{\mu})$. Furthermore, Γ is increasing on $(0, \nu)$ and for any $\varepsilon > 0$ small enough, for $0 < S \leq \nu - \varepsilon$, we have

$$\Gamma(S) \leq \Gamma(\nu - \varepsilon) < 0. \qquad (33)$$

Suppose that (a) is not true. Then, there is $\varepsilon > 0$, sufficiently small, such that

$$\mathbb{P}\left(\limsup_{t\to\infty} S(t) \leq \nu - 2\varepsilon\right) > 0.$$

Set

$$\Omega_1 = \left\{\limsup_{t\to\infty} S(t) \leq \nu - 2\varepsilon\right\}.$$

For each $\omega \in \Omega_1$, there exists $T(\omega) > 0$, where

$$S(t) \leq \nu - \varepsilon < \frac{\Lambda}{\mu} \qquad \forall\ t \geq T(\omega). \qquad (34)$$

Form (33) and (34) we have, for $T(\omega) \leq s$,

$$\Gamma(S(s)) \leq \Gamma(\nu - \varepsilon) < 0. \qquad (35)$$

Furthermore, Lemma 2.1 and (30) allow us to conclude that there is a $\Omega_2 \subset \Omega$ such that $\mathbb{P}(\Omega_2) = 1$, so that for each $\omega \in \Omega_2$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sigma S(s) dB_s = 0, \quad \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_{\mathbb{T}} \left(\ln\left(1 + J(z)S(s)\right)\right) \widetilde{\mathcal{N}}(ds, dz) = 0. \qquad (36)$$

Now, fix any $\omega \in \Omega_1 \cap \Omega_2$. Therefore, from (32) and (35) we find, for $t \geq T(\omega)$,

$$\begin{aligned} \ln(I(t)) \quad &\leq \quad \ln(I_0) + \int_0^{T(\omega)} \Gamma(S(s))ds + \int_{T(\omega)}^t \Gamma(\nu - \varepsilon)ds \\ &+ \int_0^t \sigma S(s)dB_s + \int_0^t \int_{\mathbb{T}} \ln\left(1 + J(z)S(s)\right) \widetilde{\mathcal{N}}(ds, dz). \end{aligned} \tag{37}$$

From (36) and (37), we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln(I(t)) \leq \Gamma(\nu - \varepsilon) < 0. \tag{38}$$

Thus

$$\lim_{t\to\infty} I(t) = 0. \tag{39}$$

Integrating the last equation of (2), we derive

$$Q(t) = \quad Q_0 e^{-(\mu+d+r)t} + \lambda \int_0^t I(t-s)e^{-(\mu+d+r)s}ds. \tag{40}$$

By (40) and Fatou's lemma,

$$\limsup_{t\to\infty} Q(t) \leq \frac{\lambda}{\mu + d + r} \limsup_{t\to\infty} I(t), \tag{41}$$

which in addition to (39) gives $\lim_{t\to\infty} Q(t) = 0$, and then $\lim_{t\to\infty} S(t) = \dfrac{\Lambda}{\mu}$. But this conflicts with (34). Therefore, assumption (a) must be verified.
(a') Similarly, from (31) and using (25),

$$\ln(I(t)) - \ln(I_0) \quad \geq \quad \int_0^t \Gamma'(S(s))ds + \int_0^t \sigma S(s)dB_s + \int_0^t \int_{\mathbb{T}} \ln\left(1 + J(z)S(s)\right) \widetilde{\mathcal{N}}(ds, dz). \tag{42}$$

Suppose that (a') is not true. Then, there is $\varepsilon' > 0$, sufficiently small, with $\mathbb{P}(\Omega_3) > 0$, such that

$$\Omega_3 = \left\{ \liminf_{t\to\infty} S(t) \geq \nu' + 2\varepsilon' \right\}.$$

Thus, for each $\omega \in \Omega_3$, there exists $T'(\omega) > 0$ where

$$S(t) \geq \nu' + \varepsilon' \qquad \forall\ t \geq T'(\omega). \tag{43}$$

As in (35), there is a $T'(\omega) > 0$ for each ω, such that

$$\Gamma'(S(s)) \geq \Gamma'(\nu' + \varepsilon') > 0 \qquad \forall\ s \geq T'(\omega). \tag{44}$$

By (36), (42), (44) and similarly to (37), we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln(I(t)) \geq \Gamma'(\nu' + \varepsilon') > 0.$$

Hence $\lim\limits_{t\to\infty} I(t) = \infty$, but this contradicts $I(t) < \dfrac{\Lambda}{\mu}$.
(b) By (a) and (5),

$$\liminf_{t\to\infty} I(t) + \liminf_{t\to\infty} Q(t) \le \frac{\Lambda}{\mu} - \nu, \quad a.s.. \tag{45}$$

From (40) and Fatou's lemma, we obtain

$$\liminf_{t\to\infty} I(t) \le \frac{\mu+d+r}{\lambda} \liminf_{t\to\infty} Q(t). \tag{46}$$

Combining (45) and (46) we deduce assertion (b).
(b') Similarly to (b), it easily follows from (41), (a') and (4).
(c)-(c') follow immediately from (4), (5), (a), (a'), (b) and (b'). □

5 Examples and computer simulations

The exponential variant of the Euler Maruyama method, known as exponential Euler Maruyama method, offers distinct features and advantages compared to the traditional Euler Maruyama approach (see [19, 6]). One notable advantage is its inherent ability to preserve positivity. When initialized with a positive value, the iterative process consistently generates positive solutions for all subsequent time points, thanks to the exponential form. Consequently, this numerical approach is well-suited for addressing a category of superlinearly growing Stochastic Differential Equations (SDEs) with strictly positive solutions. By using the exponential Euler Maruyama method, we have the following discrete system.

$$\begin{aligned}
S_{k+1} &= S_k \exp\Bigg[\bigg(\frac{\Lambda}{S_k} - \mu - \beta I_k + \frac{\gamma I_k}{S_k} + \frac{r Q_k}{S_k} - \frac{\sigma^2 I_k^2}{2} + \frac{1}{\widetilde{\mathcal{N}}(t_k)} \sum_{i=1}^{\widetilde{\mathcal{N}}(t_k)} \big(\ln\left(1 - J(\xi_i) I_k\right) \\
&\quad + J(\xi_i) I_k\big)\bigg) h - \sigma I_k \sqrt{h}\eta_k + \sum_{i=\widetilde{\mathcal{N}}(t_k)+1}^{\widetilde{\mathcal{N}}(t_{k+1})} \ln\left(1 - J(\xi_i) I_k\right)\Bigg], \\
I_{k+1} &= I_k \exp\Bigg[\bigg(-(\mu + d + \lambda + \gamma) + \beta S_k \frac{\sigma^2 I_k^2}{2} + \frac{1}{\widetilde{\mathcal{N}}(t_k)} \sum_{i=1}^{\widetilde{\mathcal{N}}(t_k)} \big(\ln\left(1 + J(\xi_i) S_k\right) \\
&\quad - J(\xi_i) S_k\big)\bigg) h + \sigma S_k \sqrt{h}\eta_k + \sum_{i=\widetilde{\mathcal{N}}(t_k)+1}^{\widetilde{\mathcal{N}}(t_{k+1})} \left(\ln\left(1 + J(\xi_i) S_k\right)\right)\Bigg], \\
Q_{k+1} &= Q_k \exp\left[\left(-(\mu + d + r) + \frac{\lambda I_k}{Q_k}\right) h\right].
\end{aligned}$$

where η_k, $(k = 1, 2, ...)$ are independent random variables distributed on $\mathcal{N}(0, 1)$ and ξ_i, $(i = 1, 2, ...\widetilde{\mathcal{N}}(t_k))$ is the double sequence of jump times and marks generated by the $\widetilde{\mathcal{N}}$. So, we choose the initial values $(0.45; 0.35; 0.15)$ for all examples.

Example 1. *We choose the parameter values* $\Lambda = 0.1$, $\mu = 0.098$, $d = 0.04$, $\beta = 0.2$, $\gamma = 0.01$, $\lambda = 0.03$, $r = 0.04$, $\sigma = 0.25$ *and* $J(z) = 0.35$, $z \in \mathbb{R}^+$. *This gives* $\mathcal{R}_{\sigma,J} \approx 0.8418 < 1$,

$C_1 \approx 0.0619$, $C_2 \approx -0.0737 < 0$ and $\Theta = C_3 \approx -0.0383 < 0$. Then, the first extinction condition of Theorem 3.2 is destroyed (see Figure 1(a)). To verify the second condition, we change the function J to $J(z) = 0.65$, $z \in \mathbb{R}^+$, and we keep other parameters unchanged. Then, we compute $\mathcal{R}_{\sigma,J} \approx 0.6367 < 1$, $C_1 \approx 0.1369$, $C_2 \approx 0.0793 > 0$, $C_3 \approx -0.1164$ and $\Theta = \frac{C_2^2}{4C_1} + C_3 \approx -0.1162 < 0$. Therefore, the fulfillment of the second condition for extinction, as stated in Theorem 3.2, is verified. This conclusion is further substantiated by the computer simulations presented in Figure 2 (b).

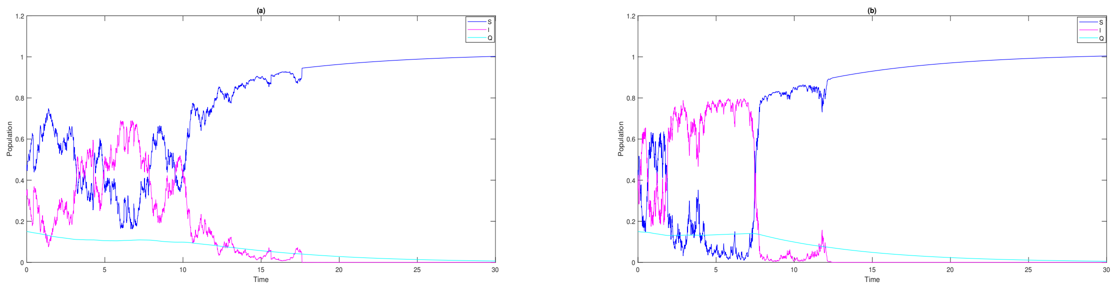

Figure 1: The single path of $(S(t); I(t); Q(t))$ for model (2) based on the values of the parameters in the example 1.

Example 2. *In Figure 2 (c) we choose $\beta = 0.15$, $\sigma = 0.1$ and $J(z) = 0.35$, $z \in \mathbb{R}^+$, and we maintain the same values for the other parameters as those specified in Example 1. So, we get $\mathcal{R}_{\sigma,J} \approx 0.7116 < 1$, $C_1 \approx 0.0356$, $C_2 \approx -0.0773 < 0$ and $\Theta = C_3 \approx -0.0620 < 0$, this satisfy the first extinction condition of Theorem 3.3. In Figure 2 (d), we change the function J to $J(z) = 0.65$, $z \in \mathbb{R}^+$. Thus, $\mathcal{R}_{\sigma,J} \approx 0.5221 < 1$, $C_1 \approx 0.1106$, $C_2 \approx 0.0758 > 0$, $C_3 \approx -0.1401$ and $\Theta = \frac{C_2^2}{4C_1} + C_3 \approx -0.1400 < 0$. Hence, the second extinction condition of Theorem 3.3 is satisfied.*

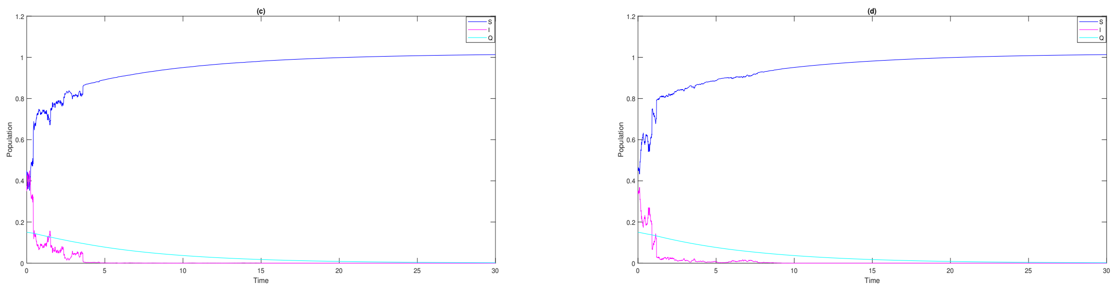

Figure 2: The single path of $(S(t); I(t); Q(t))$ for model (2) based on the parameter values of Example 2.

Example 3. *For first condition of the moment exponential stability of the equilibrium state of model (2) in Theorem 3.4, we choose $p = 0.2$, we change β to 0.09, σ to 0.1 and the function J to $J(z) = 0.1$, $z \in \mathbb{R}^+$, and we keep other parameters the same as in Example 1. We get $\mathcal{R}_{\sigma,J} \approx 0.4943 < 1$, $C_2 \approx -0.0747 < 0$ and $\Pi \approx -0.0140$. This gives*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\mathbb{E}\left[X^p(t)\right]\right) \leq p\left(\Pi + \frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2 \left(p\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right)\right) \approx -0.0015 < 0.$$

Consequently, by Theorem 3.4, the free-disease equilibrium state E_0 is pth moment exponentially stable in Δ (see Figure 3 (e)). For the second condition, we choose $p = 0.05$, and the parameter values $\beta = 0.1$, $\sigma = 0.3$ and $J(z) = 0.25$, $z \in \mathbb{R}^+$. Thus $\mathcal{R}_{\sigma,J} \approx 0.4232 < 1$, $C_2 \approx 0.0237 > 0$ and $\Pi \approx -0.0568$. This implies

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\mathbb{E}\left[X^p(t)\right]\right) \leq p\left(\Pi + \frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2 \left(p\sigma^2 + \int_{\mathbb{T}} J^2(z)\pi(dz)\right)\right) \approx -0.0011 < 0.$$

Therefore, the second condition of theorem 3.4 is satisfied. The computer simulations in Figure 3 (f) support these results.

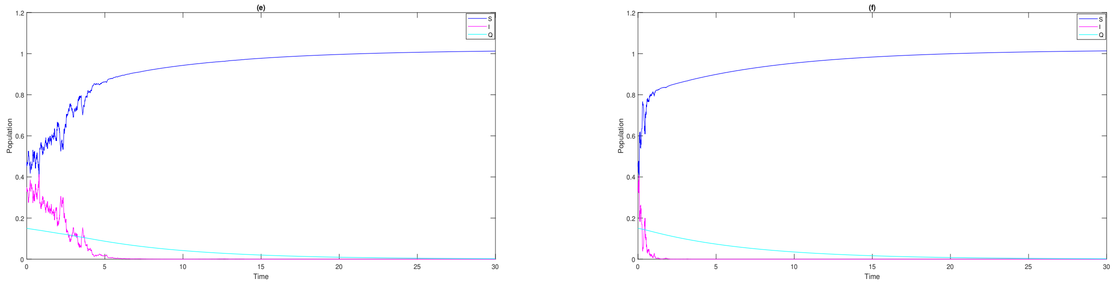

Figure 3: The single path of $(S(t); I(t); Q(t))$ for model (2) using the parameter values of Example 3.

Example 4. *Set $\Lambda = 0.07$, $\mu = 0.065$, $d = 0.01$, $\beta = 0.8$, $\gamma = 0.01$, $\lambda = 0.05$, $r = 0.02$, $\sigma = 0.25$, and $J(z) = 0.2$, $z \in \mathbb{R}^+$. We obtain $\mathcal{R}_{\sigma,J} \approx 1.8848 > 1$ and $\mathcal{R}'_{\sigma,J} \approx 1.7724 > 1$. As a result, the persistence criteria of Theorem 4.1 is verified, which is clearly illustrated by Figure 4.*

6 Conclusions

In conclusion, our study delved into a stochastic SIQS epidemic model with a quarantine compartment influenced by white and Lévy noise. We specifically focused on the perturbation of the transmission rate parameter β and its effect on epidemic dynamics. Our research has made significant contributions to the field by establishing the existence of a unique global solution to the model and by revealing the pivotal role of two thresholds in governing the asymptotic behavior of our model. The thresholds, namely $\mathcal{R}_{\sigma,J}$ and $\mathcal{R}'_{\sigma,J}$, play a crucial role in determining the behavior of the disease within the population. If $\mathcal{R}_{\sigma,J} < 1$, the disease will ultimately be eradicated, while it will persist if $\mathcal{R}'_{\sigma,J} > 1$. Our major contribution, which distinguishes our study from existing research and highlights the robustness of the identified thresholds, lies in establishing the necessary and sufficient conditions for extinction in the absence of jump noise, represented by $\mathcal{R}_{\sigma,0} = \mathcal{R}'_{\sigma,0}$. This finding illustrates the substantial impact of incorporating jump noise into our model, leading to a notable difference between the thresholds, as encapsulated by the term $\frac{1}{4}\int_{\mathbb{T}} J^2(z)\pi(dz)$. The presence of jump noise introduces an additional factor that influences the behavior of the epidemic. This highlights the potential loss of extinction that can occur in the presence of unpredictable and sudden changes in the external environment. Recognizing this potential extinction loss becomes crucial when analyzing the evolution of the epidemic in unpredictable environments. In real-world scenarios,

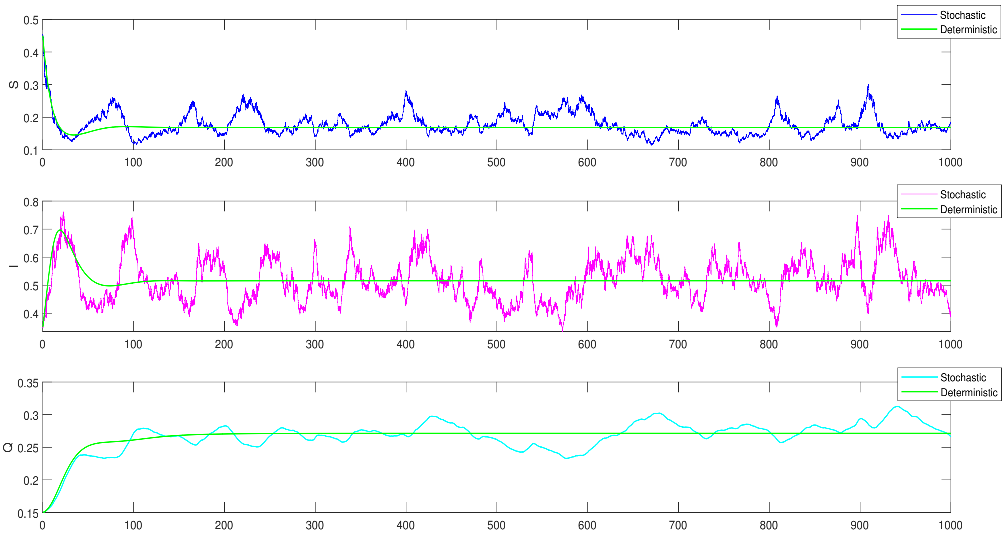

Figure 4: Results of one simulation run of SDE (2) and its corresponding deterministic model (1) for the parameter values of Example 4.

policy changes, population movements, or seasonal variations can lead to significant perturbations in the transmission dynamics. Incorporating jump noise into the model allows us to capture these sudden changes and provides a more comprehensive understanding of epidemic behavior. To validate our theoretical findings, we conducted extensive numerical simulations, further supporting our model and consistency robustness. These simulations serve as a tangible demonstration of the practical implications of our research and its ability to capture the intricate dynamics of the COVID-19 pandemic. Overall, our study offers valuable insights into the complex dynamics of the COVID-19 epidemic and its adaptation to unpredictable changes in the external environment by better understanding the critical thresholds and the influence of jump perturbations. In addition, our results have implications for policymakers designing effective intervention strategies to combat infectious diseases. The insights gained from our study can be applied to investigate the spread of various diseases and guide the implementation of public health policies. In future work, we suggest further generalizations for the model system (2), including adding a switching component to provide a more realistic formulation. Such a component would be valuable as populations may experience sudden parameter changes, such as different transmission rates in winter and summer. To address this, we propose using a continuous-time Markov chain $r(t)$ to model these sudden changes. However, the analysis of such a model would be more complex.

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