The existence and asymptotic behavior of solutions to 3D viscous primitive equations with Caputo fractional time derivatives

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Abstract

On the one hand, the primitive three-dimensional viscous equations for large-scale ocean and atmosphere dynamics are commonly used in weather and climate predictions. On the other hand, ever since the middle of the last century, it has been widely recognized that the climate variability exhibits long-time memory. In this paper, we first prove the global existence of weak solutions to the primitive equations of large-scale ocean and atmosphere dynamics with Caputo fractional time derivatives. Then we establish the existence of an absorbing set, which is positively invariant. Finally, an attractor (strictly speaking, the minimal attracting set containing all the limiting dynamics) is constructed for the time fractional primitive equations, which means that the present state of a system may have long-time influences on the states in far future. However, there was no work on the long-time behavior of the time fractional primitive equations and we fill this gap in this paper.

Keywords: Primitive equations, weak Caputo derivatives, weak solutions, absorbing sets, attractors

1. Introduction

As far as we know, the origin of primitive equations can be traced to the 90's of the last century [19]. The system, which consists of the hydrodynamic equations with the Coriolis force and thermodynamic equations, was used to study the extremely complicated atmospheric phenomena, and to predict the weather and possible climate changes. In recent decades the well-posedness and the long-time behavior of solutions for primitive equations have been extensively aroused scholastic interesting with the developments of the studies of atmosphere science, see, e.g., [2, 3, 10, 15, 16, 22, 23, 25] and the references therein.

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Ever since the middle of the last century, it has been widely recognized that the climate variability exhibits long-time memory, which means that the present state of a system may have long-time influences on the states in far future. Therefore, it is natural to study fractional Lorenz systems and complex chaotic attractors in the fractional systems [9, 18, 24]. From a macroscopic point of view, time fractional partial differential equations play a central role in the modeling of anomalous subdiffusive phenomena, which arise in a wide range of natural systems such as transport process in porous media, systems with memory, avascular tumor growth, foraging behavior of animals, fluid mechanics and viscoelasticity. In this context, it is natural that there exists an interest in the well-posedness of fractional Navier-Stokes equations, e.g., the weak solutions [20, 21, 26], and the mild solutions [4, 5]. To date, we do not know of any published work on 3D viscous primitive equations with time fractional differential operator.

The purpose of this paper is to fill this gap and to investigate the long-time behavior of the weak solutions to the following primitive equations of large-scale ocean and atmosphere dynamics with Caputo fractional time derivatives:

$$D_c^\gamma v + (v\cdot\nabla_H)v + w\partial_z v + \nabla_H p + f_0 k\times v + L_1 v = 0, \tag{1.1}$$

$$\partial_z p + T = 0, \tag{1.2}$$

$$\nabla_H\cdot v + \partial_z w = 0, \tag{1.3}$$

$$D_c^\gamma T + (v\cdot\nabla_H)T + w\partial_z T + L_2 T = Q. \tag{1.4}$$

Here $D_c^\gamma v$ and $D_c^\gamma T$ are, respectively, the Caputo fractional derivatives of v and T with $\gamma\in(0,1)$. The horizontal velocity $v=(v^1,v^2)$, the vertical velocity w, the temperature T, the pressure p are unknowns. f_0 is the Coriolis parameter, Q is a given heat source, the notation $k\times v$ is the vector product of $k=(0,0,1)$ and $v=(v^1,v^2,0)$. Set $\nabla_H=(\partial_x,\partial_y)$, $\nabla_H\cdot=\partial_x+\partial_y$ and $\triangle_H=\partial_x^2+\partial_y^2$. Define $L_1=-\frac{1}{R_1}\triangle_H-\frac{1}{R_2}\partial_z^2$, $L_2=-\frac{1}{R_3}\triangle_H-\frac{1}{R_4}\partial_z^2$, where R_i $(i=1,2,3,4)$ are positive Reynolds numbers. We aim to consider problem (1.1)-(1.4) in a cylindrical domain Ω=M$\times(-h,0)$ with M being a sufficiently smooth and bounded domain in $\mathbb{R}^2$. We divide the boundary of Ω into three parts:

$$\Gamma_u=\{(x,y,z)\in\overline{\Omega}: z=0\}, \tag{1.5}$$

$$\Gamma_b=\{(x,y,z)\in\overline{\Omega}: z=-h\}, \tag{1.6}$$

$$\Gamma_s=\{(x,y,z)\in\overline{\Omega}: (x,y)\in\partial M, -h\leqslant z\leqslant 0\}, \tag{1.7}$$

where Γ_u, Γ_b and Γ_s describe the top, bottom and lateral of ocean, respectively. The boundary and initial conditions are given by

$$\partial_z v=\eta, w=0, \partial_z T=-\alpha(T-\varrho)\ on\ \Gamma_u, \tag{1.8}$$

$$\partial_z v=0, w=0, \partial_z T=0\ on\ \Gamma_b, \tag{1.9}$$

$$v\cdot\overrightarrow{n}=0, \partial_{\overrightarrow{n}}v\times\overrightarrow{n}=0, \partial_{\overrightarrow{n}}T=0\ on\ \Gamma_s, \tag{1.10}$$

$$(v(0),T(0))=(v_0,T_0), \tag{1.11}$$

where $\alpha > 0$, $\overrightarrow{n}$ means the unit outward normal vector to ∂M, $\eta(x, y)$ and $\varrho(x, y)$ stand for the wind stress and the typical temperature distribution on the top surface of the ocean, respectively.

Suppose that p_s is an unknown function on Γ_b. By (1.2)-(1.3) we have

$$w(t; x, y, z) = W(v)(t; x, y, z) = -\int_{-h}^{z} \nabla_H \cdot v(t; x, y, \xi) d\xi, \tag{1.12}$$

$$p(t; x, y, z) = p_s(t; x, y) - \int_{-h}^{z} T(t; x, y, \xi) d\xi, \tag{1.13}$$

then one can recast equations (1.1)-(1.4) in a simple form

$$D_c^\gamma v + (v \cdot \nabla_H)v + W(v)\partial_z v + \nabla_H \Big(p_s - \int_{-h}^{z} T(t; x, y, \xi) d\xi\Big) + f_0 k \times v + L_1 v = 0, \tag{1.14}$$

$$D_c^\gamma T + (v \cdot \nabla_H)T + W(v)\partial_z T + L_2 T = Q, \tag{1.15}$$

$$\int_{-h}^{0} \nabla_H \cdot v(t; x, y, \xi) d\xi = 0. \tag{1.16}$$

Notice that the boundary conditions (1.8)-(1.10) can be converted into the homogeneous case (see [3, 10, 11]). For simplicity and without loss of generality we will assume that $\eta = 0$, $\varrho = 0$. Therefore, system (1.14)-(1.16) can be supplemented with the following boundary and initial conditions

$$\partial_z v = 0, w = 0, \partial_z T = -\alpha T \ on \ \Gamma_u, \tag{1.17}$$

$$\partial_z v = 0, w = 0, \partial_z T = 0 \ on \ \Gamma_b, \tag{1.18}$$

$$v \cdot \overrightarrow{n} = 0, \partial_{\overrightarrow{n}} v \times \overrightarrow{n} = 0, \partial_{\overrightarrow{n}} T = 0 \ on \ \Gamma_s, \tag{1.19}$$

$$(v(0), T(0)) = (v_0, T_0). \tag{1.20}$$

This work consists of two major parts. The first one is devoted to the existence of local and global weak solutions to problem (1.14)-(1.20). To this end, we follow the Faedo-Galerkin approximation but properly apply the compactness criteria for time fractional PDEs introduced in [13] and the uniform estimates of solutions established by integration by parts, Nirenberg-Gagliardo's inequalities and Sobolev's embeddings. The second part focuses on the asymptotic behavior of weak solutions to problem (1.14)-(1.20). Based on the elementary inequality for Caputo fractional differential equations [12], we first establish the existence of an absorbing set, which is positively invariant. Since the solution mapping of a general Caputo fractional differential equation does not, in general, generate a semi-group [6] (see [7] also for the generation of semi-dynamical systems in some sense), the omega limit set of the absorbing set cannot be called the attractor of (1.14)-(1.20). Then, the minimal attracting set containing all limiting dynamics is introduced to investigate the long-time behavior of the weak solutions to (1.14)-(1.20).

The present paper is structured as follows. In Section 2, we briefly recall an elementary inequality, fundamental notation and results about Caputo fractional derivatives. Section 3 is devoted to establishing a priori estimates for weak solutions. Meanwhile, the proof of global weak solutions to problem (1.14)-(1.20) is summarized in Section 4. In Section 5, we investigate the asymptotic behavior of the weak solutions to problem (1.14)-(1.20).

2. Preliminaries

2.1. Phase spaces and fractional Gronwall inequalities

We begin with some basic notation used in this paper.

Let B be a Banach space with B' being the dual of B, and let $T^* > 0$ be given arbitrarily. The notation $C_c^\infty((0,T^*);B)$ represents all infinitely differentiable functions from $(0,T^*)$ to B but with compact supports. $L^p((0,T^*);B)$ denotes the Banach space of all Lebesgue measurable functions satisfying $\left(\int_0^{T^*}\|f\|_B^p dt\right)^{\frac{1}{p}} < \infty$. $L^p((0,T^*);B)$ is endowed with the norm $\|f\|_{L^p((0,T^*);B)} = \left(\int_0^{T^*}\|f\|_B^p dt\right)^{\frac{1}{p}}$ if $p \in [1,\infty)$ while the essential supremum if $p=\infty$. We denote $L^p(\Omega)$ and $L^p(M)$ the usual Lebesgue spaces with the norm

$$\|f\|_{L^p} = \begin{cases} \left(\int_\Omega |f|^p dxdydz\right)^{\frac{1}{p}}, & f\in L^p(\Omega), \\ \left(\int_M |f|^p dxdy\right)^{\frac{1}{p}}, & f\in L^p(M), \end{cases}$$

where $p\in[1,\infty)$. Denote by $H^m(\Omega)$ and $H^m(M)$ the usual Sobolev spaces of functions, together with all their covariant derivatives, which are in $L^2(\Omega)$ and $L^2(M)$, respectively. For $f \in H^m(\Omega)$ or $f \in H^m(M)$, the norm is given by

$$\|f\|_{H^m} = \begin{cases} \left[\int_\Omega \left(\sum_{1\leqslant k\leqslant m}\sum_{i_j=x,y,z;j=1,\dots,k} |\nabla_{i_1}\cdots\nabla_{i_k} f|^2 + |f|^2\right)\right]^{\frac{1}{2}}, & f\in H^m(\Omega), \\ \left[\int_M \left(\sum_{1\leqslant k\leqslant m}\sum_{i_j=x,y;j=1,\dots,k} |\nabla_{i_1}\cdots\nabla_{i_k} f|^2 + |f|^2\right)\right]^{\frac{1}{2}}, & f\in H^m(M). \end{cases}$$

Let

$$\mathcal{V}_1 := \Big\{v \in (C^\infty(\Omega))^2 : \partial_z v|_{z=0} = 0, \partial_z v|_{z=-h} = 0, v\cdot\overrightarrow{n}|_{\Gamma_s} = 0, \partial_{\overrightarrow{n}} v \times \overrightarrow{n}|_{\Gamma_s} = 0,$$
$$\int_{-h}^0 \nabla_H\cdot v(t;x,y,\xi)d\xi = 0\Big\},$$

and

$$\mathcal{V}_2 := \Big\{T \in C^\infty(\Omega) : (\partial_z T + \alpha T)|_{z=0} = 0, \partial_z T|_{z=-h} = 0, \partial_{\overrightarrow{n}} T|_{\Gamma_s} = 0\Big\}.$$

Then we introduce some spaces

$$\begin{aligned} &V_1 = \textit{the closure of } \mathcal{V}_1 \textit{ under the } H^1 \textit{ topology}, \\ &V_2 = \textit{the closure of } \mathcal{V}_2 \textit{ under the } H^1 \textit{ topology}, \\ &H_1 = \textit{the closure of } \mathcal{V}_1 \textit{ under the } L^2 \textit{ topology}, \\ &H_2 = L^2(\Omega), \end{aligned}$$

and the inner products and norms are given as follows:

$$
\begin{aligned}
&\langle v,\widetilde{v}\rangle_{V_1}=\frac{1}{R_1}\int_\Omega \nabla_H v\cdot\nabla_H\widetilde{v}dxdydz+\frac{1}{R_2}\int_\Omega \partial_z v\cdot\partial_z\widetilde{v}dxdydz,\\
&\langle T,\widetilde{T}\rangle_{V_2}=\frac{1}{R_3}\int_\Omega \nabla_H T\cdot\nabla_H\widetilde{T}dxdydz+\frac{1}{R_4}\int_\Omega \partial_z T\cdot\partial_z\widetilde{T}dxdydz+\frac{\alpha}{R_4}\int_{\Gamma_u} T\widetilde{T}d\Gamma_u,\\
&\langle v,\widetilde{v}\rangle_{H_1}=\int_\Omega v\cdot\widetilde{v}dxdydz,\\
&\langle T,\widetilde{T}\rangle_{H_2}=\int_\Omega T\cdot\widetilde{T}dxdydz,\\
&\|v\|_{V_1}=\langle v,v\rangle_{V_1}^{\frac{1}{2}},\quad \|T\|_{V_2}=\langle T,T\rangle_{V_2}^{\frac{1}{2}},\\
&\|v\|_{H_1}=\langle v,v\rangle_{H_1}^{\frac{1}{2}},\quad \|T\|_{H_2}=\langle T,T\rangle_{H_2}^{\frac{1}{2}}.
\end{aligned}
$$

We shall frequently use the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined as follows:

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},\quad z\in\mathbb{C}.$$

In particular for the case $\beta=1$, we denote by $E_\alpha(z)$ the one-parameter function $E_{\alpha,1}(z)$. For later use, we collect the following uniform estimates on Mittag-Leffler functions; see [1].

Lemma 2.1. *For every $\alpha\in[0,1)$ and $x\geqslant 0$, we have*

$$\frac{1}{1+\Gamma(1-\alpha)x}\leqslant E_\alpha(-x)\leqslant\frac{1}{1+\frac{1}{\Gamma(1+\alpha)}x}.$$

Now we recall an elementary inequality of Gronwall type; see [12] for more details.

Lemma 2.2. *Let $u(t)\geqslant 0$ be continuous and satisfy*

$$D_c^\gamma u(t)\leqslant 2a-2bu(t).$$

Then

$$u(t)\leqslant u(0)E_\gamma(-2bt^\gamma)+\frac{a}{b}(1-E_\gamma(-2bt^\gamma)).$$

2.2. Fractional setting

In this subsection, let us recapitulate some definitions and preliminary facts on fractional derivatives, and the compactness criteria for time fractional PDEs; see [13, 14] for more details.

Definition 2.3. *Let B be a Banach space. For a locally integrable function $u\in L^1_{loc}((0,T^*);B)$, if there exists $u_0\in B$ such that*

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t\|u(s)-u_0\|_B ds=0,$$

we call u_0 the right limit of u at $t = 0$, denoted as $u(0+) = u_0$. Similarly, we define $u(T^*-)$ to be the constant $u_{T^*} \in B$ such that

$$\lim_{t \to T^*-} \frac{1}{T^* - t} \int_t^{T^*} \|u(s) - u_{T^*}\|_B ds = 0.$$

First, we recall the left Caputo derivatives for functions valued in $\mathbb{R}^d$. We define the modified Riemann-Liouville operators $J_\beta u(t) := g_\beta * (\theta(t)u(t))$ for $\beta > -1$, where the distributions g_β are given by

$$g_\beta(t) := \begin{cases} \frac{\theta(t)}{\Gamma(\beta)} t^{\beta-1}, & \beta > 0, \\ \delta(t), & \beta = 0, \\ \frac{1}{\Gamma(1+\beta)} D(\theta(t)t^\beta), & \beta \in (-1, 0). \end{cases}$$

Here $\theta(t)$ is the standard Heaviside step function, $\Gamma(\beta)$ is the Gamma function, and D typically denotes the distributional derivative.

Definition 2.4. *Let $0 < \gamma < 1$. Consider $u \in L^1_{loc}(0, T^*)$ possessing a right limit $u(0+) = u_0$ at $t = 0$ in the sense of Definition 2.3. The γ-th order left Caputo derivative of u is a distribution in $\mathcal{D}'(-\infty, T^*)$ with support in $[0, T^*)$, given by*

$$D_c^\gamma u := J_{-\gamma} u - u_0 g_{1-\gamma} = g_{-\gamma} * \Big((u - u_0)\theta(t)\Big).$$

If u is absolutely continuous on $(0, T^*)$, then the left Caputo derivative is reduced to the traditional definition of Caputo derivative

$$D_c^\gamma u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{u}(s)}{(t-s)^\gamma} ds, \tag{2.1}$$

where $\dot{u}$ means the time derivative of u.

Lemma 2.5. *Suppose $E(\cdot) \in L^1_{loc}([0, \infty); \mathbb{R})$ is continuous at $t = 0$. If there exists $f(t) \in L^1_{loc}([0, \infty); \mathbb{R})$ satisfying*

$$D_c^\gamma E(t) \leqslant f(t),$$

where this inequality means that $f(t) - D_c^\gamma E(t)$ is a non-negative distribution, then

$$E(t) \leqslant E(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds, \quad a.e. \tag{2.2}$$

Now we move on to the right Caputo derivatives for functions valued in $\mathbb{R}^d$. We set $\widetilde{g}_\gamma(t) := \frac{\theta(-t)}{\Gamma(\gamma)}(-t)^{\gamma-1}$ if $\gamma > 0$ and $\widetilde{g}_{-\gamma}(t) := -\frac{1}{\Gamma(1-\gamma)} D(\theta(-t)(-t)^{-\gamma})$ if $0 < \gamma < 1$. Then we have

Definition 2.6. *Let $0 < \gamma < 1$. Consider $u \in L^1_{loc}(-\infty, T^*)$ such that u has a left limit $u(T^*-)$ at $t = T^*$ in the sense of Definition 2.3. The γ-th order right Caputo derivative of u is a distribution in $\mathcal{D}'(\mathbb{R})$ with support in $(-\infty, T^*]$, given by*

$$\widetilde{D}_{c;T^*}^\gamma u := \widetilde{g}_{-\gamma} * (\theta(T^* - t)(u(t) - u(T^*-))). \tag{2.3}$$

Notice that if u is absolutely continuous on (a, T^*), $a < T^*$, then

$$\widetilde{D}_{c;T^*}^{\gamma} u := -\frac{1}{\Gamma(1-\gamma)} \int_t^{T^*} (s-t)^{-\gamma} \dot{u}(s) ds, \quad \forall t \in (a, T^*).$$

Consequently if $\varphi \in C_c^{\infty}(-\infty, T^*)$,

$$-\frac{1}{\Gamma(1-\gamma)} \int_t^{T^*} (s-t)^{-\gamma} \dot{\varphi}(s) ds = \widetilde{D}_{c;T^*}^{\gamma} \varphi = \widetilde{g}_{-\gamma} * \varphi = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^{T^*} (s-t)^{-\gamma} \varphi(s) ds,$$

thanks to the fact $\theta(T^* - t)(\varphi - \varphi(T^*)) = \varphi$ if $\varphi \in C_c^{\infty}(-\infty, T^*)$.

We now turn to the Caputo derivatives for functions valued in general Banach spaces. We fix $T^* > 0$ and define

$$\mathcal{D}' := \Big\{ v \;\Big|\; v : C_c^{\infty}((-\infty, T^*); \mathbb{R}) \to B \textit{ is a bounded linear operator} \Big\}.$$

It is clear that $\mathcal{D}'$ is the analogy of the distributions $\mathcal{D}'(\mathbb{R})$.

Definition 2.7. *Let B be a Banach space and $u \in L^1_{loc}([0, T^*); B)$. Let $u_0 \in B$. We define the weak Caputo derivative of u associated with initial data u_0 to be $D_c^{\gamma} u \in \mathcal{D}'$ such that for any test function $\varphi \in C_c^{\infty}((-\infty, T^*); \mathbb{R})$,*

$$\langle \varphi, D_c^{\gamma} u \rangle := \int_{-\infty}^{T^*} (\widetilde{D}_{c;T^*}^{\gamma} \varphi)(u - u_0) \theta(t) dt = \int_{-\infty}^{T^*} (\widetilde{D}_{c;T^*}^{\gamma} \varphi)(u - u_0) dt. \tag{2.4}$$

Proposition 2.8. *Let $\gamma \in (0, 1)$. If $u : [0, T^*) \to B$ is $C^1((0, T^*); B) \cap C^0([0, T^*); B)$, and $u \mapsto E(u) \in \mathbb{R}$ is a C^1 convex functional on B, then*

$$D_c^{\gamma} u(t) = \frac{1}{\Gamma(1-\gamma)} \Big(\frac{u(t) - u(0)}{t^{\gamma}} + \gamma \int_0^t \frac{u(t) - u(s)}{(t-s)^{\gamma+1}} ds \Big),$$

and

$$D_c^{\gamma} E(u(t)) \leqslant \langle D_c^{\gamma} u(t), D_u E(u) \rangle,$$

where $D_u E(\cdot) : B \to B'$ is the Fréchet differential and $\langle \cdot, \cdot \rangle$ is understood as the dual pairing between B and B'.

Proposition 2.9. *Suppose Y is a reflexive Banach space, $\gamma \in (0, 1)$ and $T^* > 0$. Assume $u_n \to u$ in $L^{p^*}((0, T^*); Y)$, $p^* \geqslant 1$. If there is an assignment of initial values $u_{0,n}$ for u_n such that the weak Caputo derivatives $D_c^{\gamma} u_n$ are bounded in $L^r((0, T^*); Y)$ ($r \in [1, \infty)$), then*
(i) There is a subsequence such that $u_{0,n}$ converges weakly to some value $u_0 \in Y$.
(ii) If $r > 1$, there exists a subsequence such that $D_c^{\gamma} u_{n_k}$ converges weakly to f and u_{0,n_k} converges weakly to u_0, and f is the Caputo derivative of u with initial value u_0 so that

$$u = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds. \tag{2.5}$$

Further, if $r \geqslant 1/\gamma$, then, $u(0+) = u_0$ in Y in the sense of Definition 2.3.

The following theorem could be viewed as a generalization of the Aubin-Lions lemma.

Lemma 2.10. *Let $T^*>0$, $\gamma\in(0,1)$ and $p^*\in[1,\infty)$. Let X, B, Y be Banach spaces. $X\hookrightarrow B$ compactly and $B\hookrightarrow Y$ continuously. Suppose $W\subset L^1_{loc}((0,T^*);X)$ satisfies:*

(i) There exists $r_1\in[1,\infty)$ and $\widetilde{C_1}>0$ such that $\forall u\in W$,

$$\sup_{t\in(0,T^*)} J_\gamma(\|u\|_X^{r_1}) = \sup_{t\in(0,T^*)} \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\|u\|_X^{r_1}(s)ds \leqslant \widetilde{C_1}.$$

(ii) There exists $p_1\in(p^*,\infty]$, W is bounded in $L^{p_1}((0,T^*);B)$.

(iii) There exist $r_2\in[1,\infty)$ and $\widetilde{C_2}>0$ such that $\forall u\in W$, there is an assignment of initial value u_0 for u so that the weak Caputo derivative satisfies:

$$\|D_c^\gamma u\|_{L^{r_2}((0,T^*);Y)} \leqslant \widetilde{C_2}.$$

Then, W is relatively compact in $L^{p^*}((0,T^*);B)$.

2.3. Weak solutions

We give the definition of weak solutions in the distributed sense for the problem (1.14)-(1.20).

Definition 2.11. *Let $\gamma\in(0,1)$. A couple of functions (v,T) with $v\in L^\infty((0,T^*);\,H_1)\cap L^2((0,T^*);V_1)$, $T\in L^\infty((0,T^*);H_2)\cap L^2((0,T^*);V_2)$, $D_c^\gamma v\in L^{\frac43}((0,T^*);(H^{-2}(\Omega))^2)$ and $D_c^\gamma T\in L^{\frac43}((0,T^*);H^{-2}(\Omega))$ is called a weak solution to (1.14)-(1.20) with initial datas $v_0\in H_1$ and $T_0\in H_2$ if*

$$\begin{aligned}
<v-v_0,\widetilde{D}^\gamma_{c;T^*}\varphi_1> &= \int_0^{T^*}\int_\Omega \Big((v\cdot\nabla_H)\varphi_1 + W(v)\partial_z\varphi_1\Big)\cdot v\,dxdydzdt\\
&+\int_0^{T^*}\int_\Omega \nabla_H\cdot\varphi_1\Big(p_s-\int_{-h}^z T(t;x,y,\xi)d\xi\Big)dxdydzdt\\
&-\int_0^{T^*}\int_\Omega (f_0k\times v)\cdot\varphi_1 dxdydzdt - \int_0^{T^*}\int_\Omega v\cdot L_1\varphi_1 dxdydzdt,
\end{aligned}$$

and

$$\begin{aligned}
<T-T_0,\widetilde{D}^\gamma_{c;T^*}\varphi_2> &= \int_0^{T^*}\int_\Omega \Big((v\cdot\nabla_H)\varphi_2 + W(v)\partial_z\varphi_2\Big)T\,dxdydzdt\\
&-\int_0^{T^*}\int_\Omega TL_2\varphi_2 dxdydzdt + \int_0^{T^*}\int_\Omega \varphi_2 Q\,dxdydzdt,
\end{aligned}$$

for any (φ_1,φ_2) with $\varphi_1\in C_c^\infty([0,T^*);(H^2(\Omega))^2\cap V_1)$ and $\varphi_2\in C_c^\infty([0,T^*);H^2(\Omega)\cap V_2)$. Moreover, (v,T) is called a global weak solution if (v,T) is defined on $[0,\infty)$ so that its restriction on any interval $[0,T^*)$, $T^*>0$, is a weak solution.

In the following, C is a positive constant and may be different from line to line and even if in the same line.

3. Some a priori estimates

It is well-known that L_1 and L_2 are positive self-adjoint operators with compact inverse. Therefore, the space $H_1 \times H_2$ possesses an orthonormal basis $\{e_j\}_{j=1}^{\infty} := \{(e_j^1, e_j^2)\}_{j=1}^{\infty}$ of eigenfunctions $\{\lambda_j\}_{j=1}^{\infty} := \{(\lambda_j^1, \lambda_j^2)\}_{j=1}^{\infty}$ of the operators

$$L_1 e_j^1 = \lambda_j^1 e_j^1, \quad L_2 e_j^2 = \lambda_j^2 e_j^2,$$

where $0 < \lambda_1^i \leqslant \lambda_2^i \leqslant \cdots$ and $\lim_{j\to\infty} \lambda_j^i = \infty$ for $i = 1, 2$; see [2] for more details. Then for $v_0 \in H_1$ and $T_0 \in H_2$, we have $v_0 = \sum_{j=1}^{\infty} \alpha_j^1 e_j^1$ and $T_0 = \sum_{j=1}^{\infty} \alpha_j^2 e_j^2$. It is easy to see that $\|v\|_{H_1}^2 \leqslant (\lambda_1^1)^{-1}\|v\|_{V_1}^2$ and $\|T\|_{H_2}^2 \leqslant (\lambda_1^2)^{-1}\|T\|_{V_2}^2$ for any $v \in V_1$ and $T \in V_2$. For convenience, we set $\lambda_1 := \min\{\lambda_1^1, \lambda_1^2\}$.

To prove the existence of weak solutions, we shall use the Galerkin method. Let (v_m, T_m) be an approximate solution of the problem (1.14)-(1.20), where

$$v_m = \sum_{j=1}^{m} c_{jm}^1 e_j^1, \ T_m = \sum_{j=1}^{m} c_{jm}^2 e_j^2. \tag{3.1}$$

Then we have

$$\begin{aligned} &\langle e_j^1, D_c^\gamma v_m\rangle + \langle e_j^1, (v_m \cdot \nabla_H) v_m\rangle + \langle e_j^1, W(v_m)\partial_z v_m\rangle + \langle e_j^1, f_0 k \times v_m\rangle \\ &\quad + \langle e_j^1, \nabla_H\Big(p_s - \int_{-h}^{z} T_m(t; x, y, \xi) d\xi\Big)\rangle + \langle e_j^1, L_1 v_m\rangle = 0, \end{aligned} \tag{3.2}$$

$$\langle e_j^2, D_c^\gamma T_m\rangle + \langle e_j^2, (v_m \cdot \nabla_H) T_m\rangle + \langle e_j^2, W(v_m)\partial_z T_m\rangle + \langle e_j^2, L_2 T_m\rangle = \langle e_j^2, Q\rangle. \tag{3.3}$$

According to the properties of the basis, (3.2)-(3.3) can be reduced to the following FODE system

$$D_c^\gamma(c_m) = F_m(c_m), \tag{3.4}$$

$$c_m(0) = (\alpha_1^1, \dots, \alpha_m^1, \alpha_1^2, \dots, \alpha_m^2), \tag{3.5}$$

where $c_m = (c_{1m}^1, \dots, c_{mm}^1, c_{1m}^2, \dots, c_{mm}^2)$ and F_m is a quadratic vector-valued function of c_m.

Lemma 3.1. *(i) Let $v_0 \in H_1$, $T_0 \in H_2$ and $Q \in H_2$. Then for any $m \geqslant 1$, there exists a unique solution (v_m, T_m) of the problem (3.2)-(3.3), which is defined on $[0, \infty)$ satisfying*

$$\|v_m\|_{L^\infty((0,\infty);H_1)}^2 + \|T_m\|_{L^\infty((0,\infty);H_2)}^2 \leqslant C(\|v_0\|_{H_1}^2 + \|T_0\|_{H_2}^2) + C, \tag{3.6}$$

and for any $t \geqslant 0$,

$$\begin{aligned} &\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}\Big(\|v_m(s)\|_{V_1}^2 + \|T_m(s)\|_{V_2}^2\Big) ds \\ &\leqslant \|v_0\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_0\|_{H_2}^2 + Ct^\gamma. \end{aligned} \tag{3.7}$$

(ii) There exist (v,T) with $v \in L^\infty((0,\infty);H_1) \cap L^2_{loc}([0,\infty);V_1)$ and $T \in L^\infty((0,\infty);H_2) \cap L^2_{loc}([0,\infty);V_2)$, and a subsequence $\{m_k\}_{k=1}^\infty$ such that

$$v_{m_k} \to v \text{ in } L^2_{loc}([0,\infty);H_1),$$

$$T_{m_k} \to T \text{ in } L^2_{loc}([0,\infty);H_2).$$

Further, (v,T) has weak Caputo derivatives $D_c^\gamma v \in L^{\frac{4}{3}}_{loc}([0,\infty);(H^{-2}(\Omega))^2)$ and $D_c^\gamma T \in L^{\frac{4}{3}}_{loc}([0,\infty);H^{-2}(\Omega))$.

Proof. (i) Since F_m is smooth and $\partial F_m(c_m)/\partial c_m$ is locally Lipschitz continuous in c_m, by using the same idea as in [8, 17] but for the system (3.4)-(3.5), we can conclude that the solution c_m of (3.4)-(3.5) exists on $[0,T_b^m)$ and $c_m \in C^1(0,T_b^m) \cap C^0[0,T_b^m)$. Here either $T_b^m = \infty$ or $T_b^m < \infty$ and $\limsup_{t\to T_b^m-} |c_m| = \infty$ where $|c_m| = \sqrt{\sum_{j=1}^m (c_{jm}^1)^2 + \sum_{j=1}^m (c_{jm}^2)^2}$. Therefore, by the form (3.1) of (v_m,T_m) we have

$$\begin{aligned} &v_m \in C^1((0,T_b^m);V_1) \cap C^0([0,T_b^m);V_1),\\ &T_m \in C^1((0,T_b^m);V_2) \cap C^0([0,T_b^m);V_2). \end{aligned} \tag{3.8}$$

By using Proposition 2.8, we conclude that

$$D_c^\gamma\Big(\frac{1}{2}\|v_m\|_{H_1}^2\Big) \leqslant \langle v_m, D_c^\gamma v_m\rangle, \quad D_c^\gamma\Big(\frac{1}{2}\|T_m\|_{H_2}^2\Big) \leqslant \langle T_m, D_c^\gamma T_m\rangle. \tag{3.9}$$

It follows from (3.1) and (3.3) that

$$\begin{aligned} &\langle T_m, D_c^\gamma T_m\rangle + \frac{1}{R_3}\|\nabla_H T_m\|_{L^2}^2 + \frac{1}{R_4}\|\partial_z T_m\|_{L^2}^2 + \frac{\alpha}{R_4}\|T_m|_{z=0}\|_{L^2}^2 \\ &= \int_\Omega T_m Q dxdydz - \int_\Omega \Big((v_m\cdot\nabla_H)T_m + W(v_m)\partial_z T_m\Big)T_m dxdydz. \end{aligned} \tag{3.10}$$

After using the integration by parts formula, we have

$$\int_\Omega \Big((v_m\cdot\nabla_H)T_m + W(v_m)\partial_z T_m\Big)T_m dxdydz = 0. \tag{3.11}$$

By the Hölder and Young inequalities, we find that

$$\begin{aligned} &\int_\Omega T_m Q dxdydz \leqslant \|T_m\|_{H_2}\|Q\|_{L^2} \\ &\leqslant \Big(h^2R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 + \frac{1}{4\Big(h^2R_4 + \frac{R_4h}{\alpha}\Big)}\|T_m\|_{H_2}^2 \\ &\leqslant \Big(h^2R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 + \frac{1}{2R_4}\|\partial_z T_m\|_{L^2}^2 + \frac{\alpha}{2R_4}\|T_m|_{z=0}\|_{L^2}^2. \end{aligned} \tag{3.12}$$

Inserting (3.11)-(3.12) and (3.9) into (3.10), we conclude that

$$D_c^\gamma\|T_m\|_{H_2}^2+\|T_m\|_{V_2}^2\leqslant 2\Big(h^2R_4+\frac{R_4h}{\alpha}\Big)\|Q\|_{L^2}^2. \tag{3.13}$$

Since $\int_\Omega(f_0k\times v_m)\cdot v_m dxdydz=0$, by (3.1) and (3.2) we have

$$\begin{aligned}&\langle v_m, D_c^\gamma v_m\rangle+\frac{1}{R_1}\|\nabla_H v_m\|_{L^2}^2+\frac{1}{R_2}\|\partial_z v_m\|_{L^2}^2\\ &=-\int_\Omega\Big((v_m\cdot\nabla_H)v_m+W(v_m)\partial_z v_m\Big)\cdot v_m dxdydz\\ &\quad-\int_\Omega\nabla_H\Big(p_s-\int_{-h}^z T_m(t;x,y,\xi)d\xi\Big)\cdot v_m dxdydz.\end{aligned} \tag{3.14}$$

Arguing as in (3.11), we find that

$$\int_\Omega\Big((v_m\cdot\nabla_H)v_m+W(v_m)\partial_z v_m\Big)\cdot v_m dxdydz=0. \tag{3.15}$$

Notice that

$$\int_\Omega\nabla_H p_s\cdot v_m dxdydz=-\int_M p_s\Big(\int_{-h}^0\nabla_H\cdot v_m dz\Big)dxdy=0, \tag{3.16}$$

and

$$\begin{aligned}&\int_\Omega\Big(-\nabla_H\int_{-h}^z T_m(t;x,y,\xi)d\xi\Big)\cdot v_m dxdydz\leqslant h\|v_m\|_{V_1}\|T_m\|_{H_2}\\ &\leqslant\frac{1}{2}\|v_m\|_{V_1}^2+\frac{h^2(\lambda_1^2)^{-1}}{2}\|T_m\|_{V_2}^2.\end{aligned} \tag{3.17}$$

In view of (3.9) we find that

$$D_c^\gamma\|v_m\|_{H_1}^2+\|v_m\|_{V_1}^2\leqslant h^2(\lambda_1^2)^{-1}\|T_m\|_{V_2}^2. \tag{3.18}$$

This together with (3.13) implies that

$$\begin{aligned}&D_c^\gamma\Big(\|v_m\|_{H_1}^2+(1+h^2(\lambda_1^2)^{-1})\|T_m\|_{H_2}^2\Big)+\|v_m\|_{V_1}^2+\|T_m\|_{V_2}^2\\ &\leqslant 2(1+h^2(\lambda_1^2)^{-1})\Big(h^2R_4+\frac{R_4h}{\alpha}\Big)\|Q\|_{L^2}^2.\end{aligned} \tag{3.19}$$

We deduce from Lemma 2.2 that for any $0<t<T_b^m$,

$$\begin{aligned}&\|v_m\|_{H_1}^2+(1+h^2(\lambda_1^2)^{-1})\|T_m\|_{H_2}^2\leqslant\Big(\|v_m(0)\|_{H_1}^2+(1+h^2(\lambda_1^2)^{-1})\|T_m(0)\|_{H_2}^2\Big)E_\gamma(-\lambda_1t^\gamma)\\ &\quad+\frac{2}{\lambda_1}(1+h^2(\lambda_1^2)^{-1})\Big(h^2R_4+\frac{R_4h}{\alpha}\Big)\|Q\|_{L^2}^2(1-E_\gamma(-\lambda_1t^\gamma))\\ &\leqslant C(\|v_0\|_{H_1}^2+\|T_0\|_{H_2}^2)+C,\end{aligned} \tag{3.20}$$

where we have used $\sup_{t\geqslant 0} E_\gamma(-\lambda_1 t^\gamma)\leqslant 1$ due to Lemma 2.1. This ensures that $T_b^m=\infty$. In fact, if $T_b^m<\infty$ then $\limsup_{t\to T_b^m-}(\|v_m\|_{H_1}^2+\|T_m\|_{H_2}^2)=\infty$, which contradicts (3.20). Therefore,

$$\|v_m\|^2_{L^\infty((0,\infty);H_1)}+\|T_m\|^2_{L^\infty((0,\infty);H_2)}\leqslant C(\|v_0\|^2_{H_1}+\|T_0\|^2_{H_2})+C. \tag{3.21}$$

On the other hand, it follows from (3.19) that for any $t\geqslant 0$

$$\begin{aligned}&\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Big(\|v_m(s)\|^2_{V_1}+\|T_m(s)\|^2_{V_2}\Big)ds\\ &\quad\leqslant \|v_0\|^2_{H_1}+(1+h^2(\lambda_1^2)^{-1})\|T_0\|^2_{H_2}+Ct^\gamma.\end{aligned} \tag{3.22}$$

(ii) Let $T^*>0$ be given arbitrarily. By using Lemma 2.10, now it only remains to show that

$$\|D_c^\gamma v_m\|_{L^{\frac43}((0,T^*);(H^{-2}(\Omega))^2)}\leqslant C,\ \ \|D_c^\gamma T_m\|_{L^{\frac43}((0,T^*);H^{-2}(\Omega))}\leqslant C.$$

For this end, we take test functions $\varphi_1\in L^4((0,T^*);(H^2(\Omega))^2\cap V_1)$ and $\varphi_2\in L^4((0,T^*);H^2(\Omega)\cap V_2)$ with $\|\varphi_1\|_{L^4((0,T^*);(H^2(\Omega))^2))}\leqslant 1$, $\|\varphi_2\|_{L^4((0,T^*);H^2(\Omega))}\leqslant 1$. Denote

$$\varphi_{1m}:=P_m\varphi_1,\ \ \varphi_{2m}:=P_m\varphi_2, \tag{3.23}$$

where P_m is a projection mapping as in (3.1). Then we have

$$\|\varphi_{1m}\|_{L^4((0,T^*);(H^2(\Omega))^2)}\leqslant\|P_m\|\|\varphi_1\|_{L^4((0,T^*);(H^2(\Omega))^2)}\leqslant C, \tag{3.24}$$

$$\|\varphi_{2m}\|_{L^4((0,T^*);H^2(\Omega))}\leqslant\|P_m\|\|\varphi_2\|_{L^4((0,T^*);H^2(\Omega))}\leqslant C. \tag{3.25}$$

Thanks to the integration by parts formula, we obtain that

$$\begin{aligned}&\langle D_c^\gamma v_m,\varphi_1\rangle=\langle D_c^\gamma v_m,\varphi_{1m}\rangle\\ &\ =\int_0^{T^*}\int_\Omega (v_m\cdot\nabla_H)\varphi_{1m}\cdot v_m dxdydzdz+\int_0^{T^*}\int_\Omega W(v_m)\partial_z\varphi_{1m}\cdot v_m dxdydzdt\\ &\ \ +\int_0^{T^*}\int_\Omega \nabla_H\cdot\varphi_{1m}p_s dxdydzdt-\int_0^{T^*}\int_\Omega\nabla_H\cdot\varphi_{1m}\int_{-h}^z T_m(t;x,y,\xi)d\xi dxdydzdt\\ &\ \ -\int_0^{T^*}\int_\Omega (f_0k\times v_m)\cdot\varphi_{1m}dxdydzdt-\int_0^{T^*}\int_\Omega\varphi_{1m}\cdot L_1 v_m dxdydzdt:=\sum_{i=1}^6 I_i,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}&\langle D_c^\gamma T_m,\varphi_2\rangle=\langle D_c^\gamma T_m,\varphi_{2m}\rangle\\ &\ =\int_0^{T^*}\int_\Omega (v_m\cdot\nabla_H)\varphi_{2m}T_m dxdydzdt+\int_0^{T^*}\int_\Omega W(v_m)\partial_z\varphi_{2m}T_m dxdydzdt\\ &\ \ -\int_0^{T^*}\int_\Omega\varphi_{2m}L_2T_m dxdydzdt+\int_0^{T^*}\int_\Omega\varphi_{2m}Q dxdydzdt:=\sum_{i=7}^{10} I_i.\end{aligned} \tag{3.27}$$

For the first term I_1, in view of (3.6), (3.24) and the Nirenberg-Gagliardo inequality $\|v_m\|_{L^4} \leqslant C\|v_m\|_{H_1}^{\frac{1}{4}}\|v_m\|_{V_1}^{\frac{3}{4}}$, we deduce from the Hölder inequality that

$$\begin{aligned}\Big|I_1\Big| \leqslant& \int_0^{T^*} \|\nabla_H\varphi_{1m}\|_{L^2}\|v_m\|_{L^4}^2 dt\\ \leqslant& C\int_0^{T^*} \|\varphi_{1m}\|_{H^2}\|v_m\|_{V_1}^{\frac{3}{2}} dt\\ \leqslant& C\Big(\int_0^{T^*} \|\varphi_{1m}\|_{H^2}^4 dt\Big)^{\frac{1}{4}}\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{3}{4}}\\ \leqslant& C\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{3}{4}}.\end{aligned} \tag{3.28}$$

Similarly, it follows from the Nirenberg-Gagliardo inequality $\|v_m\|_{L^3} \leqslant C\|v_m\|_{H_1}^{\frac{1}{2}}\|v_m\|_{V_1}^{\frac{1}{2}}$ and $H^1(\Omega) \subset L^6(\Omega)$ that

$$\begin{aligned}\Big|I_2\Big| \leqslant& \int_0^{T^*} \|\partial_z\varphi_{1m}\|_{L^6}\|W(v_m)\|_{L^2}\|v_m\|_{L^3} dt\\ \leqslant& C\int_0^{T^*} \|\varphi_{1m}\|_{H^2}\|v_m\|_{V_1}^{\frac{3}{2}}\|v_m\|_{H_1}^{\frac{1}{2}} dt\\ \leqslant& C\Big(\int_0^{T^*} \|\varphi_{1m}\|_{H^2}^4 dt\Big)^{\frac{1}{4}}\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{3}{4}}\\ \leqslant& C\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{3}{4}}.\end{aligned} \tag{3.29}$$

Since $\varphi_{1m} \in L^4((0,T^*);V_1)$, we have

$$\Big|I_3\Big| = \Big|\int_0^{T^*}\int_M p_s \int_{-h}^0 \nabla_H\cdot\varphi_{1m} dzdxdydt\Big| = 0. \tag{3.30}$$

Applying the Hölder inequality results in

$$\begin{aligned}&\Big|I_4\Big| + \Big|I_5\Big| + \Big|I_{10}\Big|\\ &\leqslant C\int_0^{T^*} \|\varphi_{1m}\|_{H^2}\|T_m\|_{H_2} dt + C\int_0^{T^*} \|\varphi_{1m}\|_{L^2}\|v_m\|_{H_1} dt + C\int_0^{T^*} \|Q\|_{L^2}\|\varphi_{2m}\|_{L^2} dt\\ &\leqslant C\Big(\int_0^{T^*} \|T_m\|_{V_2}^2 dt\Big)^{\frac{1}{2}} + C\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{3}{4}} + C\|Q\|_{L^2}(T^*)^{\frac{1}{2}}.\end{aligned} \tag{3.31}$$

For the terms I_6 and I_9, by the Hölder inequality we have

$$\begin{aligned}\Big|I_6\Big| + \Big|I_9\Big| \leqslant& C\int_0^{T^*} \|\varphi_{1m}\|_{H^2}\|v_m\|_{V_1} dt + C\int_0^{T^*} \|\varphi_{2m}\|_{H^2}\|T_m\|_{V_2} dt\\ \leqslant& C\Big(\int_0^{T^*} \|v_m\|_{V_1}^2 dt\Big)^{\frac{1}{2}} + C\Big(\int_0^{T^*} \|T_m\|_{V_2}^2 dt\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.32}$$

Arguing as in (3.28)-(3.29), we deduce that

$$
\begin{aligned}
\Big|I_7\Big| \leqslant & C\int_0^{T^*}\|\varphi_{2m}\|_{H^2}\|v_m\|_{L^4}\|T_m\|_{L^4}dt \\
\leqslant & C\int_0^{T^*}\|\varphi_{2m}\|_{H^2}\|v_m\|_{H_1}^{\frac{1}{4}}\|v_m\|_{V_1}^{\frac{3}{4}}\|T_m\|_{H_2}^{\frac{1}{4}}\|T_m\|_{V_2}^{\frac{3}{4}}dt \\
\leqslant & C\int_0^{T^*}\|\varphi_{2m}\|_{H^2}\|v_m\|_{V_1}^{\frac{3}{4}}\|T_m\|_{V_2}^{\frac{3}{4}}dt \\
\leqslant & C\Big(\int_0^{T^*}\|v_m\|_{V_1}^2dt\Big)^{\frac{3}{8}}\Big(\int_0^{T^*}\|T_m\|_{V_2}^2dt\Big)^{\frac{3}{8}},
\end{aligned} \tag{3.33}
$$

and

$$
\begin{aligned}
\Big|I_8\Big| \leqslant & \int_0^{T^*}\|\partial_z\varphi_{2m}\|_{L^6}\|W(v_m)\|_{L^2}\|T_m\|_{L^3}dt \\
\leqslant & C\int_0^{T^*}\|\varphi_{2m}\|_{H^2}\|v_m\|_{V_1}\|T_m\|_{H_2}^{\frac{1}{2}}\|T_m\|_{V_2}^{\frac{1}{2}}dt \\
\leqslant & C\Big(\int_0^{T^*}\|v_m\|_{V_1}^2dt\Big)^{\frac{1}{2}}\Big(\int_0^{T^*}\|T_m\|_{V_2}^2dt\Big)^{\frac{1}{4}}.
\end{aligned} \tag{3.34}
$$

By (3.7), we observe that

$$
\begin{aligned}
&\int_0^{T^*}(\|v_m\|_{V_1}^2+\|T_m\|_{V_2}^2)dt \\
&\quad\leqslant (T^*)^{1-\gamma}\int_0^{T^*}(T^*-t)^{\gamma-1}(\|v_m\|_{V_1}^2+\|T_m\|_{V_2}^2)dt \\
&\leqslant C(T^*)^{1-\gamma}+CT^*.
\end{aligned} \tag{3.35}
$$

Inserting (3.28)-(3.35) into (3.26)-(3.27) yields

$$
\|D_c^\gamma v_m\|_{L^{\frac{4}{3}}((0,T^*);(H^{-2}(\Omega))^2)} \leqslant C, \quad \|D_c^\gamma T_m\|_{L^{\frac{4}{3}}((0,T^*);H^{-2}(\Omega))} \leqslant C. \tag{3.36}
$$

Recall from (3.6)-(3.7) that

$$
\|v_m\|_{L^\infty((0,T^*);H_1)} \leqslant C, \qquad \sup_{0\leqslant t<T^*} J_\gamma(\|v_m\|_{V_1}^2) \leqslant C, \tag{3.37}
$$

and

$$
\|T_m\|_{L^\infty((0,T^*);H_2)} \leqslant C, \qquad \sup_{0\leqslant t<T^*} J_\gamma(\|T_m\|_{V_2}^2) \leqslant C. \tag{3.38}
$$

Then by Lemma 2.10, we can find a couple of subsequences $\{(v_{m_k},T_{m_k})\}_{k=1}^\infty$ such that

$$
v_{m_k}\to v \ in \ L^2((0,T^*);H_1), \quad T_{m_k}\to T \ in \ L^2((0,T^*);H_2). \tag{3.39}
$$

According to Proposition 2.9, v and T have weak Caputo derivatives with initial data v_0 and T_0 such that

$$D_c^\gamma v \in L^{\frac{4}{3}}((0,T^*);(H^{-2}(\Omega))^2), \quad D_c^\gamma T \in L^{\frac{4}{3}}((0,T^*);H^{-2}(\Omega)), \tag{3.40}$$

respectively.

By using a standard diagonal argument, v and T can be defined on $(0,\infty)$ and

$$D_c^\gamma v \in L^{\frac{4}{3}}_{loc}((0,\infty);(H^{-2}(\Omega))^2), \quad D_c^\gamma T \in L^{\frac{4}{3}}_{loc}((0,\infty);H^{-2}(\Omega)), \tag{3.41}$$

such that there is a subsequence, denoted by $\{(v_{m_k},T_{m_k})\}_{k=1}^\infty$ satisfying

$$v_{m_k} \to v \ in \ L^2_{loc}((0,\infty);H_1), \quad T_{m_k} \to T \ in \ L^2_{loc}((0,\infty);H_2). \tag{3.42}$$

By taking the further subsequence, still denoted by $\{(v_{m_k},T_{m_k})\}_{k=1}^\infty$, it follows from (3.42) that

$$v_{m_k} \to v \ a.e. \ in \ [0,\infty)\times\Omega, \quad T_{m_k} \to T \ a.e. \ in \ [0,\infty)\times\Omega. \tag{3.43}$$

For any $0<t_1<t_2<\infty$, integrating inequality (3.6) from t_1 to t_2, we have

$$\int_{t_1}^{t_2} \Big(\|v_{m_k}\|^2_{H_1} + \|T_{m_k}\|^2_{H_2}\Big)dt \leqslant \Big(C(\|v_0\|^2_{H_1} + \|T_0\|^2_{H_2}) + C\Big)(t_2-t_1).$$

This together with Lebesgue's dominated convergence theorem implies that

$$\int_{t_1}^{t_2} \Big(\|v\|^2_{H_1} + \|T\|^2_{H_2}\Big)dt \leqslant \Big(C(\|v_0\|^2_{H_1} + \|T_0\|^2_{H_2}) + C\Big)(t_2-t_1),$$

which ensures that $v \in L^\infty((0,\infty);H_1)$ and $T \in L^\infty((0,\infty);H_2)$.

From (3.35), we know that $\{v_{m_k}\}_{k=1}^\infty$ is bounded in $L^2((0,T^*);V_1)$ and $\{T_{m_k}\}_{k=1}^\infty$ is bounded in $L^2((0,T^*);V_2)$. By a standard diagonal argument again, there exists a subsequence (relabelled the same) $\{(v_{m_k},T_{m_k})\}_{k=1}^\infty$ such that v_{m_k} and T_{m_k} converge weakly to v and T in $L^2_{loc}((0,\infty);V_1)$ and $L^2_{loc}((0,\infty);V_2)$, respectively, and consequently $v\in L^2_{loc}((0,\infty);V_1)$ and $T \in L^2_{loc}((0,\infty);V_2)$. The proof of this lemma is completed. □

4. Global existence of weak solutions

In this section, we shall establish the global existence of weak solutions for (1.14)-(1.20) by using Galerkin's approximation method.

Theorem 4.1. *Suppose $v_0 \in H_1$, $T_0 \in H_2$ and $Q \in H_2$. Then, system* (1.14)-(1.20) *has a global weak solution (v,T) with $v \in L^\infty((0,\infty);H_1)\cap L^2_{loc}((0,\infty);V_1)$ and $T \in L^\infty((0,\infty);H_2)\cap L^2_{loc}((0,\infty);V_2)$ in the sense of Definition 2.11.*

Proof. We recall from Lemma 3.1 that there is a subsequence, still denoted by $\{(v_m, T_m)\}_{m=1}^{\infty}$ such that

$$v_m \rightharpoonup v \ in\ L^2_{loc}((0,\infty); V_1), \quad T_m \rightharpoonup T \ in\ L^2_{loc}((0,\infty); V_2), \tag{4.1}$$

$$v_m \to v \ in\ L^2_{loc}((0,\infty); H_1), \quad T_m \to T \ in\ L^2_{loc}((0,\infty); H_2). \tag{4.2}$$

Let T^* be given arbitrarily. Then for any $\varphi_1 \in C_c^\infty((0,T^*); (H^3(\Omega))^2 \cap V_1)$ and $\varphi_2 \in C_c^\infty((0,T^*); H^3(\Omega) \cap V_2)$, we define $\varphi_{1j} := P_j\varphi_1$ and $\varphi_{2j} := P_j\varphi_2$ as in (3.23). We first fix $j \geqslant 1$, and then we obtain that for $m \geqslant j$,

$$\begin{aligned}
&\langle \widetilde{D}^\gamma_{c;T^*}\varphi_{1j}, v_m - v_{0m}\rangle = \langle \varphi_{1j}, D^\gamma_c v_m\rangle \\
&= -\langle \varphi_{1j}, (v_m \cdot \nabla_H)v_m + W(v_m)\partial_z v_m\rangle - \langle \varphi_{1j}, f_0 k \times v_m\rangle \\
&\quad - \langle \varphi_{1j}, \nabla_H\Big(p_s - \int_{-h}^{z} T_m(t;x,y,\xi)d\xi\Big)\rangle - \langle \varphi_{1j}, L_1 v_m\rangle \\
&= \int_0^{T^*}\int_\Omega \Big((v_m\cdot\nabla_H)\varphi_{1j} + W(v_m)\partial_z\varphi_{1j}\Big)\cdot v_m dxdydzdt \\
&\quad + \int_0^{T^*}\int_\Omega \nabla_H\cdot\varphi_{1j}\Big(p_s - \int_{-h}^{z} T_m(t;x,y,\xi)d\xi\Big)dxdydzdt \\
&\quad - \int_0^{T^*}\int_\Omega (f_0 k \times v_m)\cdot\varphi_{1j}dxdydzdt - \int_0^{T^*}\int_\Omega v_m\cdot L_1\varphi_{1j}dxdydzdt,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\langle \widetilde{D}^\gamma_{c;T^*}\varphi_{2j}, T_m - T_{0m}\rangle = \langle \varphi_{2j}, D^\gamma_c T_m\rangle \\
&= -\langle \varphi_{2j}, (v_m \cdot \nabla_H)T_m + W(v_m)\partial_z T_m\rangle - \langle \varphi_{2j}, L_2 T_m\rangle + \langle \varphi_{2j}, Q\rangle \\
&= \int_0^{T^*}\int_\Omega \Big((v_m\cdot\nabla_H)\varphi_{2j} + W(v_m)\partial_z\varphi_{2j}\Big)T_m dxdydzdt \\
&\quad - \int_0^{T^*}\int_\Omega T_m L_2\varphi_{2j}dxdydzdt + \int_0^{T^*}\int_\Omega \varphi_{2j}Q dxdydzdt,
\end{aligned} \tag{4.4}$$

where $v_{0m} = P_m v_0$ and $T_{0m} = P_m T_0$.

By using the Hölder inequality, (3.24) and (4.2), we have

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega \Big((v_m\cdot\nabla_H)\varphi_{1j}\cdot v_m - (v\cdot\nabla_H)\varphi_{1j}\cdot v\Big)dxdydzdt \\
&\leqslant \int_0^{T^*} \|v_m\|_{L^4}\|\nabla_H\varphi_{1j}\|_{L^4}\|v_m - v\|_{H_1}dt \\
&\quad + \int_0^{T^*} \|v\|_{L^4}\|\nabla_H\varphi_{1j}\|_{L^4}\|v - v_m\|_{H_1}dt \\
&\leqslant C(\|v_m\|_{L^2((0,T^*);V_1)} + \|v\|_{L^2((0,T^*);V_1)})\|\varphi_1\|_{L^\infty((0,T^*);H^2)}\|v_m - v\|_{L^2((0,T^*);H_1)} \\
&\leqslant C\|v_m - v\|_{L^2((0,T^*);H_1)} \to 0 \quad as\ m \to \infty,
\end{aligned} \tag{4.5}$$

and similarly,

$$\int_0^{T^*}\int_\Omega \Big((v_m\cdot\nabla_H)\varphi_{2j}T_m-(v\cdot\nabla_H)\varphi_{2j}T\Big)dxdydzdt\to 0 \quad as\ m\to\infty. \tag{4.6}$$

Notice that by applying the Hölder inequality, the Nirenberg-Gagliardo inequality and $H^2(\Omega)\subset W^{1,6}(\Omega)$, we deduce from (3.24) and (4.2) that

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega W(v_m)\partial_z\varphi_{1j}\cdot(v_m-v)dxdydzdt\\
&\leqslant \int_0^{T^*}\|W(v_m)\|_{L^2}\|\partial_z\varphi_{1j}\|_{L^6}\|v_m-v\|_{L^3}dt\\
&\leqslant C\int_0^{T^*}\|v_m\|_{V_1}\|\varphi_{1j}\|_{H^2}\|v_m-v\|_{H_1}^{\frac12}\|v_m-v\|_{V_1}^{\frac12}dt\\
&\leqslant C\|v_m\|_{L^2((0,T^*);V_1)}\|v_m-v\|_{L^2((0,T^*);H_1)}\|v_m-v\|_{L^2((0,T^*);V_1)}\|\varphi_1\|_{L^\infty((0,T^*);H^2)}\\
&\leqslant C\|v_m-v\|_{L^2((0,T^*);H_1)}\to 0 \quad as\ m\to\infty,
\end{aligned} \tag{4.7}$$

and by using $H^3(\Omega)\subset W^{2,3}(\Omega)$ and $H^3(\Omega)\subset W^{1,\infty}(\Omega)$,

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega (W(v_m)-W(v))\partial_z\varphi_{1j}\cdot v dxdydzdt\\
&=\int_0^{T^*}\int_\Omega\int_{-h}^0 \Big(v_m(t;x,y,\xi)-v(t;x,y,\xi)\Big)d\xi\nabla_H(\partial_z\varphi_{1j}\cdot v)dxdydzdt\\
&\leqslant C\int_0^{T^*}\|v_m-v\|_{H_1}\Big(\|\nabla_H\partial_z\varphi_{1j}\|_{L^3}\|v\|_{L^6}+\|\partial_z\varphi_{1j}\|_{L^\infty}\|v\|_{V_1})dt\\
&\leqslant C\int_0^{T^*}\|v_m-v\|_{H_1}\|\varphi_{1j}\|_{H^3}\|v\|_{V_1}dt\\
&\leqslant C\|v_m-v\|_{L^2((0,T^*);H_1)}\|\varphi_1\|_{L^\infty((0,T^*);H^3)}\|v\|_{L^2((0,T^*);V_1)}\\
&\leqslant C\|v_m-v\|_{L^2((0,T^*);H_1)}\to 0 \quad as\ m\to\infty.
\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) together, we find that

$$\int_0^{T^*}\int_\Omega \Big(W(v_m)\partial_z\varphi_{1j}\cdot v_m-W(v)\partial_z\varphi_{1j}\cdot v\Big)dxdydzdt\to 0 \quad as\ m\to\infty, \tag{4.9}$$

and in a similar way, we have

$$\int_0^{T^*}\int_\Omega \Big(W(v_m)\partial_z\varphi_{2j}T_m-W(v)\partial_z\varphi_{2j}T\Big)dxdydzdt\to 0 \quad as\ m\to\infty. \tag{4.10}$$

By (4.2), it is easy to see that

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega (v_m - v)\cdot L_1\varphi_{1j}dxdydzdt\\
&\quad\leqslant \int_0^{T^*} \|v_m - v\|_{H_1}\|L_1\varphi_{1j}\|_{L^2}dt\\
&\quad\leqslant C\|v_m - v\|_{L^2((0,T^*);H_1)}\|\varphi_1\|_{L^2((0,T^*);H^2)}\\
&\quad\leqslant C\|v_m - v\|_{L^2((0,T^*);H_1)} \to 0 \quad as\ m\to\infty,
\end{aligned} \tag{4.11}$$

$$\int_0^{T^*}\int_\Omega (T_m - T)L_2\varphi_{2j}dxdydzdt \leqslant C\|T_m - T\|_{L^2((0,T^*);H_2)} \to 0 \quad as\ m\to\infty. \tag{4.12}$$

It follows from (4.2) and the Hölder inequality that

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega \nabla_H\cdot\varphi_{1j}\int_{-h}^z \Big(T_m(t;x,y,\xi) - T(t;x,y,\xi)\Big)d\xi dxdydzdt\\
&\quad\leqslant C\int_0^{T^*} \|\nabla_H\cdot\varphi_{1j}\|_{L^2}\|T_m - T\|_{H_2}dt\\
&\quad\leqslant C\|T_m - T\|_{L^2((0,T^*);H_2)} \to 0 \quad as\ m\to\infty,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\int_0^{T^*}\int_\Omega (f_0k\times v_m)\cdot\varphi_{1j}dxdydzdt - \int_0^{T^*}\int_\Omega (f_0k\times v)\cdot\varphi_{1j}dxdydzdt\\
&\quad\leqslant C\int_0^{T^*} \|\varphi_{1j}\|_{L^2}\|v_m - v\|_{H_1}dt\\
&\quad\leqslant C\|v_m - v\|_{L^2((0,T^*);H_1)} \to 0 \quad as\ m\to\infty,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\langle \widetilde{D}^\gamma_{c;T^*}\varphi_{1j}, v_m - v_{0m}\rangle - \langle \widetilde{D}^\gamma_{c;T^*}\varphi_{1j}, v - v_0\rangle\\
&\quad= \langle \widetilde{D}^\gamma_{c;T^*}\varphi_{1j}, v_m - v\rangle + \langle \widetilde{D}^\gamma_{c;T^*}\varphi_{1j}, v_0 - v_{0m}\rangle\\
&\quad\leqslant C\int_0^{T^*} \|\widetilde{D}^\gamma_{c;T^*}\varphi_{1j}\|_{L^2}\Big(\|v_m - v\|_{H_1} + \|v_0 - v_{0m}\|_{H_1}\Big)dt\\
&\quad\leqslant C\|\widetilde{D}^\gamma_{c;T^*}\varphi_1\|_{L^2((0,T^*);L^2)}\Big(\|v_m - v\|_{L^2((0,T^*);H_1)} + \|v_0 - v_{0m}\|_{H_1}\Big)\\
&\quad\leqslant C\Big(\|v_m - v\|_{L^2((0,T^*);H_1)} + \|v_0 - v_{0m}\|_{H_1}\Big) \to 0 \quad as\ m\to\infty,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\langle \widetilde{D}^\gamma_{c;T^*}\varphi_{2j}, T_m - T_{0m}\rangle - \langle \widetilde{D}^\gamma_{c;T^*}\varphi_{2j}, T - T_0\rangle\\
&\quad\leqslant C\Big(\|T_m - T\|_{L^2((0,T^*);H_2)} + \|T_0 - T_{0m}\|_{H_2}\Big) \to 0 \quad as\ m\to\infty.
\end{aligned} \tag{4.16}$$

We fix $j \geqslant 1$. Then by taking $m \to \infty$, we deduce from (4.3) and (4.4) that

$$
\begin{aligned}
&\langle \widetilde{D}^{\gamma}_{c;T^*}\varphi_{1j}, v - v_0\rangle = \langle \varphi_{1j}, D^{\gamma}_{c} v\rangle \\
&= \int_0^{T^*}\int_\Omega \Big((v\cdot\nabla_H)\varphi_{1j} + W(v)\partial_z\varphi_{1j}\Big)\cdot v dxdydzdt \\
&+ \int_0^{T^*}\int_\Omega \nabla_H\cdot\varphi_{1j}\Big(p_s - \int_{-h}^{z} T(t;x,y,\xi)d\xi\Big)dxdydzdt \\
&- \int_0^{T^*}\int_\Omega (f_0 k \times v)\cdot\varphi_{1j}dxdydzdt - \int_0^{T^*}\int_\Omega v\cdot L_1\varphi_{1j}dxdydzdt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \widetilde{D}^{\gamma}_{c;T^*}\varphi_{2j}, T - T_0\rangle = \langle \varphi_{2j}, D^{\gamma}_{c} T\rangle \\
&= \int_0^{T^*}\int_\Omega \Big((v\cdot\nabla_H)\varphi_{2j} + W(v)\partial_z\varphi_{2j}\Big) T dxdydzdt \\
&- \int_0^{T^*}\int_\Omega T L_2\varphi_{2j}dxdydzdt + \int_0^{T^*}\int_\Omega Q\varphi_{2j}dxdydzdt.
\end{aligned}
$$

Due to $\varphi_{1j} \to \varphi_1$ in $L^{p_*}((0,T^*);(H^3(\Omega))^2)$ and $\varphi_{2j} \to \varphi_2$ in $L^{p_*}((0,T^*);H^3(\Omega))$ for any $p_* \in (1,\infty)$, we obtain the weak formulations to (1.14)-(1.20) in the sense of Definition 2.11 by taking $j \to \infty$. □

5. Asymptotic behavior

In this section we will prove the existence of a minimal attracting set which plays the role of a global attractor in the theory of autonomous dynamical systems.

5.1. Existence of absorbing sets

The following result shows the existence of absorbing sets for (1.14)-(1.20), which is an important set for the long-time behavior of the solutions.

Theorem 5.1. *Let (v,T) be a weak solution to (1.14)-(1.20). Then (v,T) can be absorbed by*

$$
\mathbb{B} = \Big\{(\bar{v},\bar{T}) \in H_1 \times H_2 : \|\bar{v}\|^2_{H_1} + (1 + h^2(\lambda_1^2)^{-1})\|\bar{T}\|^2_{H_2} \leqslant 1 + \frac{2}{\lambda_1}(1 + h^2\lambda_1^2)\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|^2_{L^2}\Big\}.
$$

In other words, for any given bounded sets $B_1 \subset H_1$ and $B_2 \subset H_2$, there exists a $T^* = T^*(B_1,B_2) > 0$ such that for any $t \geqslant T^*$, $v_0 \in B_1$ and $T_0 \in B_2$ the corresponding solution $(v(t),T(t))$ to these initial values, satisfies $(v(t),T(t)) \in \mathbb{B}$. Moreover, this set is positively invariant.

Proof. From (3.20), it can be deduced for the solution corresponding to the initial values $(v_0, T_0) \in H_1 \times H_2$ that

$$\begin{aligned}
&\|v(t)\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T(t)\|_{H_2}^2 \\
&\quad \leqslant \liminf_{m\to\infty} \Big(\|v_m\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_m\|_{H_2}^2\Big) \\
&\quad \leqslant \Big(\|v_0\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_0\|_{H_2}^2\Big) E_\gamma(-\lambda_1 t^\gamma) \\
&\qquad + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 (1 - E_\gamma(-\lambda_1 t^\gamma)).
\end{aligned} \tag{5.1}$$

For any given bounded sets $B_1 \subset H_1$ and $B_2 \subset H_2$, there exists a $T^* = T^*(B_1, B_2) > 0$ such that for any $t \geqslant T^*$, $v_0 \in B_1$ and $T_0 \in B_2$, we have

$$\Big(\|v_0\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_0\|_{H_2}^2\Big) E_\gamma(-\lambda_1 t^\gamma) < 1, \quad t \geqslant T^*. \tag{5.2}$$

Thus, if $(v(\cdot), T(\cdot))$ denotes the corresponding solution to the initial values (v_0, T_0), we have

$$\|v(t)\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T(t)\|_{H_2}^2 \leqslant 1 + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2, \quad t \geqslant T^*. \tag{5.3}$$

This ensures that $\mathbb{B}$ is an absorbing set.

Moreover, for $(v_0, T_0) \in \mathbb{B}$, it follows that

$$\|v_0\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_0\|_{H_2}^2 \leqslant 1 + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2.$$

Then, from (5.1) we deduce that

$$\begin{aligned}
&\|v(t)\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T(t)\|_{H_2}^2 \\
&\quad \leqslant \Big(\|v_0\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_0\|_{H_2}^2\Big) E_\gamma(-\lambda_1 t^\gamma) \\
&\qquad + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 (1 - E_\gamma(-\lambda_1 t^\gamma)) \\
&\quad \leqslant E_\gamma(-\lambda_1 t^\gamma) + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 \\
&\quad \leqslant 1 + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2,
\end{aligned} \tag{5.4}$$

for all $t \geqslant 0$. This establishes the positive invariance of $\mathbb{B}$ and completes the proof. □

5.2. Existence of Attractors

For any bounded subset D of $H_1 \times H_2$, the omega limit set in the weak topology is defined by

$$\begin{aligned}
\Omega_w(D) := \Big\{&(v, T) \in H_1 \times H_2 : \exists\, t_n \to \infty \textit{ and a sequence of weak solutions } (v_n(\cdot), T_n(\cdot)) \textit{ of problem} \\
&(1.14) - (1.20) \textit{ with } (v_n(0), T_n(0)) = (v_{0,n}, T_{0,n}) \in D \textit{ such that} \\
&(v_n(t_n), T_n(t_n)) \rightharpoonup (v, T) \textit{ in } H_2 \times H_2\Big\}.
\end{aligned}$$

Lemma 5.2. *Let Q belong to H_2, and let D be a bounded subset of $H_1 \times H_2$. Then $\Omega_w(D)$ is nonempty, compact and attracts D in the weak topology.*

Proof. Thanks to the reflexivity of H_1 and H_2, we infer from (5.1) that $\Omega_w(D)$ is nonempty.

In order to show the weak compactness of $\Omega_w(D)$, let $(v^i, T^i) \in \Omega_w(D)$ be any given sequence. By the definition of the omega limit set in the weak topology, for each i we can find $(v^i_{0,n_i}, T^i_{0,n_i}) \in D$ and $t^i_{n_i}$ sufficiently large such that

$$dist_w((v^i_{n_i}(t^i_{n_i}), T^i_{n_i}(t^i_{n_i})), (v^i, T^i)) < \frac{1}{n_i}, \tag{5.5}$$

where $n_i \to \infty$ as $i \to \infty$, and $dist_w(\cdot, \cdot)$ denotes the distance between two points of $H_1 \times H_2$ in the weak topology. Since (5.1) ensures that the sequence $(v^i_{n_i}(t^i_{n_i}), T^i_{n_i}(t^i_{n_i}))$ is bounded in $H_1 \times H_2$, there exists a subsequence, still denoted by $(v^i_{n_i}(t^i_{n_i}), T^i_{n_i}(t^i_{n_i}))$, such that$(v^i_{n_i}(t^i_{n_i}), T^i_{n_i}(t^i_{n_i})) \rightharpoonup (v^*, T^*)$ in $H_1 \times H_2$. This together with (5.5) implies that $(v^i, T^i) \rightharpoonup (v^*, T^*)$ in $H_1 \times H_2$.

Finally, we prove that $\Omega_w(D)$ attracts D in the weak topology. Assume on the contrary that this is not the case. Then there exist $\varepsilon_0 > 0$ and sequences t_n with $t_n \to \infty$ $(n \to \infty)$, $(v_{0,n}, T_{0,n})$ with $(v_{0,n}, T_{0,n}) \in D$ and weak solutions (v_n, T_n) of (1.14)-(1.20) with $(v_n(0), T_n(0)) = (v_{0,n}, T_{0,n})$ such that

$$dist_w((v_n(t_n), T_n(t_n)), \Omega_w(D)) > \varepsilon_0, \ \forall n \in \mathbb{N}. \tag{5.6}$$

Notice from (5.1) that $(v_n(t_n), T_n(t_n))$ is bounded in $H_1 \times H_2$. Hence, $(v_n(t_n), T_n(t_n))$ is relatively compact in the weak topology of $H_1 \times H_2$, and consequently possesses at least one cluster point $(\widetilde{v}, \widetilde{T})$. This contradicts with (5.6) since $(\widetilde{v}, \widetilde{T}) \in \Omega_w(D)$ by the definition of $\Omega_w(D)$. □

According to Lemma 5.2, it is easy to know that $\Omega_w(\mathbb{B})$ attracts $\mathbb{B}$ where $\mathbb{B}$ is the absorbing set in Theorem 5.1. However, $\Omega_w(\mathbb{B})$ cannot be called the attractor of (1.14)-(1.20), since there may be additional omega limit points that are not in $\Omega_w(\mathbb{B})$. This is caused by the effects that the solution mapping of a general Caputo fractional differential equation does not, in general, generate a semi-group [6]. Hence we need to consider the set of all omega limit points as in (5.7). This is, strictly speaking, a minimal weakly closed attracting set containing all limiting dynamics of the Caputo fractional primitive equation (1.14)-(1.20) in the weak topology.

Theorem 5.3. *Let Q belong to H_2. Then the set*

$$\Omega^* = \overline{\bigcup \{\Omega_w(D) : all\ bounded\ D \subset H_1 \times H_2\}}^w \subset \mathbb{B} \tag{5.7}$$

is weakly compact in $H_1 \times H_2$, and moreover, is the minimal weakly closed set that attracts all bounded subsets of $H_1 \times H_2$ in the weak topology.

Proof. We first show that Ω^* is a subset of the absorbing set $\mathbb{B}$. For any given bounded set $D \subset H_1 \times H_2$, let $(\widehat{v}, \widehat{T}) \in \Omega_w(D)$ be given arbitrarily. By the definition of $\Omega_w(D)$, there exist $t_n \to \infty$ and a sequence of weak solutions $(v_n(\cdot), T_n(\cdot))$ of problem (1.14)-(1.20) with $(v_n(0), T_n(0)) =$

$(v_{0,n}, T_{0,n}) \in D$ such that $(v_n(t_n), T_n(t_n)) \rightharpoonup (\widehat{v}, \widehat{T})$ in $H_1 \times H_2$. In view of (5.1), we find that

$$\begin{aligned}
&\|v_n(t_n)\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_n(t_n)\|_{H_2}^2 \\
&\leqslant \Big(\|v_{0,n}\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_{0,n}\|_{H_2}^2\Big) E_\gamma(-\lambda_1 t_n^\gamma) \\
&\quad + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2 (1 - E_\gamma(-\lambda_1 t_n^\gamma)).
\end{aligned} \tag{5.8}$$

This together with Lemma 2.1 implies that

$$\begin{aligned}
&\|\widehat{v}\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|\widehat{T}\|_{H_2}^2 \\
&\leqslant \liminf_{n\to\infty} \Big(\|v_n(t_n)\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_n(t_n)\|_{H_2}^2\Big) \\
&\leqslant 1 + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2,
\end{aligned} \tag{5.9}$$

and, consequently, $(\widehat{v}, \widehat{T}) \in \mathbb{B}$. Since $(\widehat{v}, \widehat{T}) \in \Omega_w(D)$ and $D \subset H_1 \times H_2$ are arbitrary, we have $\bigcup\{\Omega_w(D) : \text{all bounded } D \subset H_1 \times H_2\} \subset \mathbb{B}$. Let $(\overline{v}, \overline{T}) \in \Omega^*$ be given arbitrarily. Then we can find a sequence $(v_n, T_n) \in \bigcup\{\Omega_w(D) : \text{all bounded } D \subset H_1 \times H_2\}$ such that $(v_n, T_n) \rightharpoonup (\overline{v}, \overline{T})$ in $H_1 \times H_2$. Therefore,

$$\begin{aligned}
&\|\overline{v}\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|\overline{T}\|_{H_2}^2 \\
&\leqslant \liminf_{n\to\infty} \Big(\|v_n\|_{H_1}^2 + (1 + h^2(\lambda_1^2)^{-1})\|T_n\|_{H_2}^2\Big) \\
&\leqslant 1 + \frac{2}{\lambda_1}(1 + h^2(\lambda_1^2)^{-1})\Big(h^2 R_4 + \frac{R_4 h}{\alpha}\Big)\|Q\|_{L^2}^2,
\end{aligned} \tag{5.10}$$

which implies $(\overline{v}, \overline{T}) \in \mathbb{B}$, and thus $\Omega^* \subset \mathbb{B}$. Then it is clear that Ω^* is weakly compact in $H_1 \times H_2$.

By Lemma 5.2 we obtain that, for each bounded subset D of $H_1 \times H_2$, $\Omega_w(D)$ attracts D in the weak topology. Then Ω^* attracts D in the weak topology since $\Omega_w(D) \subset \Omega^*$, and thus Ω^* attracts all bounded subsets of $H_1 \times H_2$ in the weak topology. Finally, we prove that Ω^* is the minimal weakly closed set attracting any bounded set $D \subset H_1 \times H_2$ in the weak topology. In fact, if there is another weakly closed set Ω' which attracts any bounded subset of $H_1 \times H_2$ in the weak topology, then by the definition of $\Omega_w(D)$, we have $\Omega_w(D) \subset \Omega'$, and consequently $\bigcup\{\Omega_w(D) : \text{all bounded } D \subset H_1 \times H_2\} \subset \Omega'$. This implies that

$$\Omega^* = \overline{\bigcup\{\Omega_w(D) \ : \ \text{all bounded } D \subset H_1 \times H_2\}}^{w} \subset \Omega'.$$

The proof is completed. □

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