Pullback exponential attractors with explicit fractal dimensions for non-autonomous partial functional differential equations

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Abstract

The aim of this paper is to propose a new method to construct pullback exponential attractors with explicit fractal dimensions for non-autonomous infinite dimensional dynamical systems in Banach spaces. The approach is established by combining the squeezing properties and the covering of finite subspace of Banach spaces, which generalize the method established for autonomous systems in Hilbert spaces [*Eden A., Foias C., Nicolaenko B. and Temam R., Exponential Attractors for Dissipative Evolution Equations, John Wiley Sons, 1994*]. The method is especially effective for non-autonomous partial functional differential equations for which phase space decomposition based on the exponential dichotomy of the linear part or variation techniques are available for proving squeezing property. The theoretical results are illustrated by applications to several specific non-autonomous partial functional differential equations, including a retarded reaction-diffusion equation, a retarded 2D-Navier-Stokes equation and a retarded semilinear wave equation. The constructed exponential attractors possess explicit fractal dimensions which do not depend on the entropy number but only on some inner characteristics of the studied equations including the spectra of the linear part and the Lipschitz constants of the nonlinear terms and hence do not require the smooth embedding between two spaces in previous work.

Key words. *Pullback exponential attractors, non-autonomous, partial functional differential equations, squeezing property, fractal dimension, delay*

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1 Introduction

For dissipative dynamical systems generated by partial differential equations or delay differential equations, the phase space is generally not locally compact. One main technique to analyze the complex dynamics of these systems is to find the largest bounded invariant set of the system which is closed and attracts all bounded sets in the phase space, i.e., the global attractor. Moreover, if one can prove that the global attractor has finite dimension, so that, even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. Therefore, the existence and dimension estimations of global attractors for infinite dimensional dynamical systems, especially for a large class of parabolic partial differential equations and delay differential equations have drawn much attention from pure and applied mathematics community during the past decades. See, for instance, the monographs by Babin and Vishik [4], Hale [34], Ladyzhenskaya [40], Robinson [48] and Temam [52].

Nevertheless, it is pointed out in [28] and [29] that the global attractor has several drawbacks. First, it may attract the trajectories slowly and, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. A second shortcoming is that the global attractor may be sensitive to perturbations, which severely limits the application scope since the systems are only approximations of real world models. Moreover, in many situations, global attractors may not be observable in experiments or in numerical simulations because of its complicated geometric structure and may fail to capture important transient behaviors. Hence, Eden, Foias, Nicolaenko and Temam proposed in [26] the concept of exponential attractor where the theory was established based on the squeezing property in Hilbert spaces which was then extended by Babin and Nicolaenko in [1] to investigate exponential attractors of reaction-diffusion systems in an unbounded domain, and recently generalized by Zhou and Zhao in [62] and [63] to investigate the random exponential attractors for non-autonomous stochastic lattice systems with multiplicative white noise and stochastic semilinear wave equation in Hilbert spaces. Generally speaking, an exponential attractor is a compact subset with finite fractal dimension, which is positively invariant and attracts all bounded subsets at an exponential rate. It is well known that if exponential attractors exists, then they contain global attractors. Although they may be larger than the global attractors, they are more robust than global attractors under perturbations due to the exponential rate of convergence. Hence, they play significant roles in investigating asymptotic behavior of infinite dimensional nonlinear dynamical systems especially for those with fast convergence rate.

There are many evolution equations arising from real world models defined in Banach spaces, such as delayed differential equations [34] and delayed partial differential equations [57, 58, 59]. Therefore, one natural question arise, how to construct exponential attractors for systems in Banach spaces? Efendiev, Miranville and Zelik in [28] developed an alternative method and explicit construction of exponential attractors for semigroups in Banach spaces by using the so-called smoothing property of the semigroup, which originates from Ladyzhenskaya [39] for proving the finite dimensionality of global attractors. The main ingredient of their works is the following smooth property between two spaces

$$\|S(\tau^*)U_0 - S(\tau^*)V_0\|_Z \leq c\|U_0 - V_0\|_X,$$

where Z is a second Banach space which is compactly embedded into X, which has to hold for some $\tau^* > 0$ and on some bounded positively invariant subset of X. The method was then extended by Czaja, Efendiev, Miranville and Zelik [20] to construct exponential and uniform attractors for systems in Banach spaces, which has also been widely used to estimate the fractal dimension and construct exponential attractors for deterministic [12, 20, 29, 30, 35, 46] and random systems [10, 41, 50, 55].

Although, this method is effective for systems in Banach spaces, the construction cannot provide explicit bounds of the fractal dimensions since it depends on the choice of another embedding space which may vary from space to space. Furthermore, the dimension estimation depends on the entropy number between two spaces for which is generally quite difficult, if not impossible, to obtain an explicit bound. Only for some specific examples can the explicit entropy number be obtained, see, for instance [38, 53, 61]. Indeed, in the very recent works [35] and [46], the authors pointed out only for scalar equations the entropy number of the embedding $C \hookrightarrow C^1$ is explicitly known, which yields an estimate for the fractal dimension of the exponential attractors. Hence, one naturally wonders whether we can construct exponential attractors with explicit bounds of fractal dimensions for systems in Banach spaces that only depend on the inner characteristics of the system. In our recent work [36], we affirm this by extending Eden, Foias, Nicolaenko and Temam's work [26] to autonomous systems in Banach spaces with applications to functional differential equations. In this paper, we go one step further to construct pullback exponential attractors with explicit fractal dimensions of nonautonomous dynamical systems in Banach spaces.

It should be pointed out that in [24], the authors also established exponential attractors in Banach spaces with explicit bounds by the idea originated from Mané [44]. Their results are obtained under the assumption that the semiflow is C^1 and the linearized semiflow at every point inside the absorbing ball can be split into the sum of a compact operator plus a contraction, which were further generalized by [64] to relax the condition of existence of compact absorbing sets to existence of bounded absorbing sets. Actually, squeezing properties are omnipresent in systems whose linear parts admit exponential dichotomies, which plays significant roles in study invariant manifolds of nonlinear dynamical systems while the verification of C^1 smoothness may be tedious, especially for functional differential equations. Therefore, in this paper, we establish a new method by combining the squeezing property proposed in early work of [31] and the covering lemma of the finite dimensional subspace of Banach space established in [44]. We do not need the strict restriction of the boundedness of derivative of the semiflow and we also obtain explicit bounds on the dimension of the invariant set in Banach spaces. The main contributions of this work are in the following three folds.

- Unlike the works [43, 47, 51], where the functional differential equations are recast into Hilbert spaces to study the dimensions of global attractors, we directly propose a construction procedure of pullback exponential attractors in the natural spaces of functional differential equations, i.e., Banach spaces.

• Compared with the recent works [35] and [46], where Banach spaces are taken as the phase spaces, we derive explicit bounds on fractal dimensions of the constructed pullback exponential attractors that only depend on the spectrum of the linear parts and the Lipschitz constants of the nonlinear parts while not related to the entropy number.

• Different from the early works [24, 36, 64], where autonomous systems are studied, we investigated non-autonomous systems generated by different kinds of non-autonomous partial functional differential equations in this work. Specifically, we will consider both non-autonomous linear part and non-autonomous nonlinear part by semigroup approach or variational technique. Even more, the approach used here is quite different from [24, 64].

The outline of our paper is as follows. In Section 2, we recall basic notions and results from the theory of infinite dimensional dynamical systems, introduce the notion of pullback exponential attractors and propose the construction procedure of pullback exponential attractors. In Section 3, we construct pullback exponential attractors for non-autonomous retarded reaction-diffusion equations. We consider two situations, i.e., the non-autonomous effect appears in the linear part and the nonlinear part respectively. For the former scenario, we prove the squeezing property under the assumption that the process generated by the linear part admits an exponential dichotomy, while in the latter situation, we adopt a variational technique. Indeed, even the existence of pullback attractors is new for the non-autonomous retarded reaction-diffusion equations. Sections 4 and 5 are devoted to applications of the theoretical results to the retarded 2D-Navier-Stokes equation and the retarded semilinear wave equation by variational techniques, since for these two equations, it is quite difficult to show the linear parts generate semigroups in the natural phase space. At last, we summarize the paper and point out some potential directions for future research in Section 6.

2 Pullback exponential attractor

We first introduce some preliminaries for establishing our main results, including the definitions of evolution process, pullback exponential attractors as well as some hypotheses.

Definition 2.1. *A two-parameter set of mappings* $\{U(t,s) : t, s \in \mathbb{R}, t \geqslant s\}$ *acting on* X*, i.e.,* $U(t,s) : X \to X$ *for all real numbers* $t, s \in \mathbb{R}$ *with* $t \geqslant s$*, is said to be an evolution process on* X *if it satisfies*

$$\begin{aligned} U(t,\tau)U(\tau,s) &= U(t,s), \quad \forall t,\tau,s \in \mathbb{R}, \quad t \geqslant \tau \geqslant s, \\ U(s,s) &= \mathrm{Id}_X, \quad \forall s \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $\mathrm{Id}_X : X \to X$ *represents the identity map on* X*. Moreover, if*

$$\mathcal{T} \times X \ni (t,s,x) \mapsto U(t,s)x \in X$$

is continuous, where $\mathcal{T} := \{(t,s) \in \mathbb{R}\times\mathbb{R} \mid t \geq s\}$*, then it is called a continuous evolution process.*

We now give the following definition of pullback attractor and pullback exponential attractors.

Definition 2.2. *Let* $\{U(t,s) : t,s \in \mathbb{R}, t \geqslant s\}$ *be an evolution process in* X*. A family of nonautonomous sets* $\mathcal{A} = \{\mathcal{A}(t) \mid t \in \mathbb{R}\}$ *is said to be the global pullback attractor for* U *if it satisfies the following properties:*
(i) $\mathcal{A}(t) \subset X$ *is non-empty and compact for all* $t \in \mathbb{R}$*,*
(ii) $\mathcal{A}$ *is strictly invariant, i.e.*

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \quad \forall t \geq s, s \in \mathbb{R},$$

(iii) $\mathcal{A}$ *pullback attracts all bounded sets, i.e. for every bounded subset* $D \subset X$ *and* $t \in \mathbb{R}$*,*

$$\lim_{s\to\infty} \operatorname{dist}_H(U(t,t-s)D, \mathcal{A}(t)) = 0,$$

and $\mathcal{A}$ *is minimal within the families of closed subsets that pullback attract all bounded subsets of* X*.*

Definition 2.3. *Let* $\{U(t,s) : t,s \in \mathbb{R}, t \geqslant s\}$ *be an evolution process in* X*. The family of non-empty compact subsets* $\mathcal{M} = \{\mathcal{M}(t) \mid t \in \mathbb{R}\}$ *is called a pullback exponential attractor for the evolution process* $\{U(t,s) : t,s \in \mathbb{R}, t \geqslant s\}$ *if*
(i) $\mathcal{M}$ *is positively invariant, i.e.*

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t) \quad \forall t \geq s.$$

(ii) the fractal dimension of the sections $\mathcal{M}(t), t \in \mathbb{R}$*, is uniformly bounded, i.e.,*

$$\sup_{t\in\mathbb{R}} \{\dim_f(\mathcal{M}(t))\} < \infty,$$

where $\dim_f(\mathcal{M}(t)) < \infty$ *is defined as*

$$\dim_f(\mathcal{M}(t)) = \lim_{\varepsilon\to 0} \frac{\ln\left(N_\varepsilon^X(\mathcal{M}(t))\right)}{\ln\left(\frac{1}{\varepsilon}\right)},$$

and $N_\varepsilon^X(\mathcal{M}(t))$ *denotes the minimal number of* ε*-balls in* X *with centres in* $\mathcal{M}(t)$ *needed to cover* A*.*
(iii) $\mathcal{M}$ *exponentially pullback attracts all bounded sets, i.e. there exists a constant* $\omega > 0$ *such that for every bounded subset* $D \subset X$ *and every* $t \in \mathbb{R}$

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_X(U(t,t-s)D, \mathcal{M}(t)) = 0,$$

where $\operatorname{dist}_X(A,B)$ *denotes the Hausdorff semi-distance defined by between* A *and* B*, defined as*

$$\operatorname{dist}_X(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b), \qquad \text{for } A, B \subseteq X.$$

Definition 2.4. *A family of time dependent bounded subsets $\mathcal{B}(t), t \in \mathbb{R}$ is said to grow at most sub-exponentially in the past provided*

$$\lim_{t \to -\infty} R_{\mathcal{B}(t)} e^{\varsigma t} = 0 \quad \forall \varsigma > 0,$$

where $R_{\mathcal{B}(t)}$ denotes the diameter of $\mathcal{B}(t) \subset X$.

As in the autonomous case, the existence of compact absorbing sets is the crucial property in order to obtain pullback attractors. For the following result see [19].

Lemma 2.1. *Let $\{U(t,s) : t, s \in \mathbb{R}, t \geqslant s\}$ be a two-parameter process, and suppose $U(t,s) : X \to X$ is continuous for all $t \geqslant s$. If there exists a family of compact (pullback) absorbing sets $\{B(t)\}_{t \in \mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, and $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \bigcup_{\substack{D \subset X \\ bounded}} \Lambda_D(t),$$

where

$$\Lambda_D(t) = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{s \geqslant n} U(t, t-s) D}$$

For a finite dimensional subspace F of a Banach space X, denote by $B_r^F(x)$ the ball in F of center x and radius r, that is $B_r^F(x) = \{y \in F | \|y - x\| \leq r\}$. For later use, we introduce the following covering lemma of balls in finite dimensional Banach spaces which was proved in [44].

Lemma 2.2. *For every finite dimensional subspace F of a Banach space X, we have*

$$N\left(r_1, B_{r_2}^F(0)\right) \leq m 2^m \left(1 + \frac{r_1}{r_2}\right)^m, \tag{2.2}$$

for all $r_1 > 0, r_2 > 0$, where $m = \dim F$ and $N\left(r_1, B_{r_2}^F(0)\right)$ is the minimum number of balls needed to cover the ball of radius r_1 by balls $B_{r_2}^F(0)$ of radius r_2 calculated in the metric space X.

For notation simplicity, we will write $\{U(t,s) : t, s \in \mathbb{R}, t \geqslant s\}$ simply as $\{U(t,s)\}$ in the following. In order to construct the pullback exponential attractor we need to impose the following assumptions on the process $\{U(t,s)\}$.

($\mathcal{H}_1$) For the process $\{U(t,s)\}$ there exists a family of bounded sets $\mathcal{B}(t) \subset X$, $t \in \mathbb{R}$, that pullback absorbs all bounded subsets of X. That is, for all bounded subsets $D \subset X$ and all $t \in \mathbb{R}$ there exists $T_{D,t} > 0$ such that

$$U(t, t-s) D \subset \mathcal{B}(t) \quad \text{for all } s \geq T_{D,t}.$$

($\mathcal{H}_2$) The family of bounded sets $\mathcal{B}(t) \subset X, t \in \mathbb{R}$ is positively invariant, that is, $U(t,s)\mathcal{B}(s) \subseteq \mathcal{B}(t)$ for all $t \geq s$.

($\mathcal{H}_3$) There is a finite dimensional projection $P(t) : X \to P(t)X$ with finite dimension

$$\Lambda = \dim\{P(t)X\} \tag{2.3}$$

and there are three positive numbers M_1, M_2, M_3 and two constants λ_0 and λ_1 such that

$$\|P(t)U(t,s)\varphi - P(t)U(t,s)\psi\| \le M_1 e^{\lambda_0 (t-s)} \|\varphi - \psi\| \tag{2.4}$$

and

$$\|(I-P(t))U(t,s)\varphi - (I-P(t))U(t,s)\psi\| \le (M_2 e^{\lambda_1 (t-s)} + M_3 e^{\lambda_0 (t-s)}) \|\varphi - \psi\| \tag{2.5}$$

for any $t \in \mathbb{R}$ and some $s_0 \ge 0$ and φ, ψ in $\mathcal{B}(t)$.

In the following, we are devoted to the construction of exponential attractors for the discrete evolution process $\{U(n,m)\}$.

Theorem 2.1. *Let $\{U(n,m)\}$ be a discrete evolution process in X and the assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$ are satisfied for discrete times $n,m \in \mathbb{Z}$. Moreover, we assume that the diameter of the family of absorbing sets $\{\mathcal{B}(n)\}, n \in \mathbb{Z}$, grows at most sub-exponentially in the past, and there exists $0 < \alpha < M_1$ such that $\zeta := \alpha e^{\lambda_0} + M_2 e^{\lambda_1} + M_3 e^{\lambda_0} < 1$. Then, there exists a pullback exponential attractor $\{\mathcal{M}(n)\}$ for the semigroup $\{U(n,m)\}$, and the fractal dimension is bounded by*

$$\dim_f \mathcal{A} \le \frac{\ln \Lambda + \Lambda \ln(2 + \frac{2M_1}{\alpha})}{-\ln(\alpha e^{\lambda_0} + M_2 e^{\lambda_1} + M_3 e^{\lambda_0})} < \infty. \tag{2.6}$$

Proof. **1) Covering of $U(n, n-m)\mathcal{B}(n-m)$.** We construct the covering of $U(n, n-m)\mathcal{B}(n-m)$ by inductively defining a family of sets $W^m(n)$ in $m \in \mathbb{N}^+$ that depend on the time instant $n \in \mathbb{Z}$ and satisfy the following properties

$$\begin{cases} (W1) & W^m(n) \subset U(n, n-m)\mathcal{B}(n-m) \subset \mathcal{B}(n), \\ (W2) & \sharp W^m(n) \le N^m, \\ (W3) & U(n, n-m)\mathcal{B}(n-m) \subset \bigcup_{u \in W^m(n)} B_{\zeta^m R_{\mathcal{B}(n-m)}}(u), \end{cases} \tag{2.7}$$

where $\sharp W^m(n)$ represents the number of elements of $W^m(n)$.

We first consider the case $m = 1$, i.e., we construct a covering of the image when $U(n, n-1)\mathcal{B}(n-1)$. Denote by $R_{\mathcal{B}(i)} := \sup_{u \in \mathcal{B}(i)} \|u\|_X, i = n-m, n-(m-1), \cdots, n-1$, then for any $u_1 \in \mathcal{B}(n-1)$, we have $\mathcal{B}(n-1) \subset B_{R_{\mathcal{B}(n-1)}}(u_1)$. For any $u \in \mathcal{B}(n-1) \cap B\left(u_1, R_{\mathcal{B}(n-1)}\right)$, it follows from $(\mathcal{H}_3)$ that

$$\|P(n)U(n, n-1)u - PU(n, n-1)u_1\| \le M_1 e^{\lambda_0} R_{\mathcal{B}(n-1)}, \tag{2.8}$$

and

$$\|(I-P(n))U(n, n-1)u - (I-P(n))U(n, n-1)u_1\| \le (M_2 e^{\lambda_1} + M_3 e^{\lambda_0}) R_{\mathcal{B}(n-1)}. \tag{2.9}$$

By Lemma 2.2, we can find $y_1^1, \ldots, y_1^{n_1}$ such that

$$B_{P(n)X}\left(PU(n, n-1)u_n, M_1 e^{\lambda_0} R_{\mathcal{B}(n-1)}\right) \subset \bigcup_{j=1}^{n_1} B_{P(n)X}\left(y_1^j, \alpha e^{\lambda_0} R_{\mathcal{B}(n-1)}\right) \tag{2.10}$$

with

$$n_1 \le \Lambda 2^{\Lambda} \left(1 + \frac{M_1}{\alpha}\right)^{\Lambda}, \tag{2.11}$$

where Λ is the dimension of $P(n)X$ and we have denoted by $B_{P(n)X}(y,r)$ the ball in $P(n)X$ of radius r and center y. Set

$$u_1^j = y_1^j + (I - P(n))U(n,n-1)u_1 \tag{2.12}$$

for $j = 1, \dots, n_1$ and $W^1 = \{u_1^1, u_1^2, \cdots, u_1^{n_1}\}$. Then, for any $u \in \mathcal{B}(n-1) \cap B\left(u_1, R_{\mathcal{B}(n-1)}\right)$, there exists a j such that

$$\begin{aligned}\left\|U(n,n-1)u - u_1^j\right\| &\le \left\|P(n)U(n,n-1)u - y_1^j\right\| + \|(I-P(n))U(n,n-1)u - (I-P(n))U(n,n-1)u_1\| \\ &\le \left(\alpha e^{\lambda_0} M_1 + M_2 e^{\lambda_1} + M_3 e^{\lambda_0}\right) R_{\mathcal{B}(n-1)} \\ &\le \zeta R_{\mathcal{B}(n-1)}.\end{aligned} \tag{2.13}$$

indicating $(W3)$ is satisfied for $m = 1$. Furthermore, it is clear from the definition of W^1 that it satisfies $(W1)$ and $(W2)$. This completes the proof of the case $m = 1$.

Assume that the sets $W^l(n)$ satisfying (2.7) have already been constructed for all $m \le l$ and $n \in \mathbb{Z}$, i.e., there exists covering

$$U(n,n-l)\mathcal{B}(n-l) \subset \bigcup_{u \in W^l(n)} B_{\zeta^l R_{\mathcal{B}(n-l)}}. \tag{2.14}$$

We construct in the sequel the covering of $W^{l+1}(n)$ satisfying (2.7). By the process property and the induction hypothesis $(W3)$, we have

$$\begin{aligned}U(n,n-(l+1))\mathcal{B}(n-(l+1)) &= U(n,n-1)U(n-1,n-1-l)\mathcal{B}(n-1-l) \\ &\subset \bigcup_{u \in W^l(n-1)} U(n,n-1)B_{\zeta^l R_{\mathcal{B}(n-l-1)}}(u).\end{aligned} \tag{2.15}$$

In other words, $U(n,n-(l+1))\mathcal{B}(n-(l+1))$ can be covered by $\bigcup_{u \in W^l(n-1)} U(n,n-1)B_{\zeta^l R_{\mathcal{B}(n-l-1)}}(u)$. We construct in the following a covering of $\bigcup_{u \in W^l(n-1)} U(n,n-1)B_{\zeta^l R_{\mathcal{B}(n-l-1)}}(u)$.

Let $u_l \in W^l(n-1)$. It follows from induction hypothesis $(W1)$ that $u_l \in W^l(n-1) \subset U(n-1,n-1-l)\mathcal{B}(n-l-1) \subset \mathcal{B}(n-1)$. Therefore, for any $u \in \mathcal{B}(n-1) \cap B\left(u_l, \zeta^l R_{\mathcal{B}(n-l-1)}\right)$, it follows from $(\mathcal{H}_3)$ that

$$\|P(n)U(n,n-1)u - P(n)U(n,n-1)u_l\| \le M_1 e^{\lambda_0} \zeta^l R_{\mathcal{B}(n-l-1)}, \tag{2.16}$$

and

$$\|(I-P(n))U(n,n-1)u - (I-P(n))U(n,n-1)u_l\| \le (M_2 e^{\lambda_1} + M_3 e^{\lambda_0})\zeta^l R_{\mathcal{B}(n-l-1)}. \tag{2.17}$$

By Lemma 2.2, we can find $y_l^1, \dots, y_l^{n_l}$ such that

$$B_{PX}\left(P(n)U(n,n-1)u_n, M_1 e^{\lambda_0} \zeta^l R_{\mathcal{B}(n-l-1)}\right) \subset \bigcup_{j=1}^{n_l} B_{P(n)X}\left(y_l^j, \alpha e^{\lambda_0} \zeta^l R_{\mathcal{B}(n-l-1)}\right) \tag{2.18}$$

with

$$n_l \le \Lambda 2^\Lambda \left(1 + \frac{M_1}{\alpha}\right)^\Lambda, \tag{2.19}$$

where Λ is the dimension of $P(n)X$ and we have denoted by $B_{P(n)X}(y,r)$ the ball in $P(n)X$ of radius r and center y. Set

$$u_l^j = y_l^j + (I-P(n))U(n,n-1)u_l \in U(n-1,n-1-l)\mathcal{B}(n-1-l) \tag{2.20}$$

for $j=1,\ldots,n_l$. Then, it follows from (2.16)-(2.18) that, for any $u \in \mathcal{B}(n-1) \cap B\left(u_l, \zeta^l R_{\mathcal{B}(n-l-1)}\right)$, there exists a j such that

$$\begin{aligned}\left\|U(n,n-1)u-u_l^j\right\| &\le \left\|P(n)U(n,n-1)u-y_l^j\right\| + \|(I-P(n))U(n,n-1)u-(I-P(n))U(n,n-1)u_l\| \\ &\le \left(\alpha e^{\lambda_0}+M_2e^{\lambda_1}+M_3e^{\lambda_0}\right)\zeta^l R_{\mathcal{B}(n-l-1)} = \zeta^{l+1}R_{\mathcal{B}(n-l-1)}.\end{aligned} \tag{2.21}$$

This implies that $\bigcup_{u\in W^l(n-1)} U(n,n-1)B_{\zeta^l R_{\mathcal{B}(n-l-1)}}(u)$ is covered by balls with radius $\zeta^{l+1}R_{\mathcal{B}(n-l-1)}$ and centers $\{u_l^1,u_l^2,\cdots,u_l^{n_l}\}$ and hence $(W3)$ holds. Denote the new set of centres by $W^{l+1}(n)$. From the induction hypothesis, we have $\sharp W^l(n) \le [\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda]^l$, which yields $\sharp W^{l+1}(n) \le n_l \sharp W^l(n) \le [\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda]^{l+1}$ and proves $(W2)$. By construction the set of centres W^{l+1}(n), we can see $W^{l+1}(n) \subset U(n,n-(l+1))\mathcal{B}(n-(l+1))$, which concludes the proof of the properties $(W1)$.

2) Construction of random exponential attractor for $\{U(n,m)\}$. We define $E^1(n) := W^1(n)$ and set

$$E^{m+1}(n) := W^{m+1}(n) \cup U(n,n-1)E^m(n-1), \quad m \in \mathbb{N}^+. \tag{2.22}$$

Then, if follows from the definition of the sets $E^m(n)$, the properties of the sets $W^m(n)$ and the positive invariance of the absorbing set $\mathcal{B}(n)$ that the family of sets $E^m(n), m\in\mathbb{N}^+$ satisfies

(E1) $\quad U(n,n-1)E^m(n-1) \subset E^m(n), \quad E^m(n) \subset U(n,m)\mathcal{B}(m),$

(E2) $\quad E^m(n) = \bigcup_{i=0}^n U(n,n-i)W^{m-i}(n-i), \quad \sharp E^m(n) \le \sum_{i=0}^n (\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda)^i,$

(E3) $\quad U(n,m)\mathcal{B}(m) \subset \bigcup_{u\in E^m(n)} B_{\zeta^n R_{\mathcal{B}}}(u).$

Based on the family of sets $E^m(n)$, we define $\mathcal{M}(n) := \overline{\bigcup_{m\in\mathbb{N}^+} E^m(n)}$ and show that it yields an exponential attractor for the semigroup $\{U(n,m)\}$.

Positive invariance of $\mathcal{M}(n)$. It follows from property $(E1)$ that, for all $l \in \mathbb{Z}$, we have

$$U(l+n,n)\bigcup_{n\in\mathbb{Z}} E^m(n) = \bigcup_{n\in\mathbb{N}^+} U(l+n,n)E^m(n) \subset \bigcup_{n\in\mathbb{N}^+} E^{n+l} \subset \bigcup_{n\in\mathbb{N}^+} E^m(n+l). \tag{2.23}$$

Thanks to the continuity property in $(\mathcal{H}_1)$, we can take closure in both sides of (2.23), giving rise to

$$U(l+n,n)\mathcal{M}(n) := U(l+n,n)\overline{\bigcup_{m\in\mathbb{N}^+} E^m(n)} = \overline{\bigcup_{m\in\mathbb{N}^+} U(l+n,n)E^m(n)} \subset \overline{\bigcup_{m\in\mathbb{N}^+} E^m(n+l)} = \mathcal{M}(n+l). \tag{2.24}$$

Compactness and finite dimensionality of $\mathcal{M}(n)$. We prove in the sequel that the set $\mathcal{M}(n)$ is non-empty, precompact and of finite fractal dimension. It follows from $(E1)$ that, for any $l\in\mathbb{Z}$ and $m \ge l+1$,

$$E^m(n) \subset U(n,n-m)\mathcal{B}(n-m) \subset U(n,n-l)U(n-l,n-m)\mathcal{B}(n-m) \subset U(n,n-l)\mathcal{B}(n-l). \tag{2.25}$$

Thus, for any $l \in \mathbb{Z}$, we have

$$\bigcup_{m\in\mathbb{N}^+} E^m(n) = \bigcup_{m=0}^{l} E^m(n) \cup \bigcup_{m=l+1}^{\infty} E^m(n) \subset \bigcup_{m=0}^{l} E^m(n) \cup U(n,n-l)\mathcal{B}(n-l). \tag{2.26}$$

Since $\zeta < 1$, for any given $\varepsilon > 0$, there exists $l \in \mathbb{Z}$ such that

$$\zeta^{l+1} R_{\mathcal{B}(n-l)} \le \varepsilon < \zeta^l R_{\mathcal{B}(n-l+1)}, \tag{2.27}$$

which combined with the fact

$$U(n,n-l)\mathcal{B}(n-l) \subset \bigcup_{u\in W^l(n)} B_\varepsilon(u), \tag{2.28}$$

indicates that the estimate of the number of ε-balls in X needed to cover $\bigcup_{m\in\mathbb{N}^+} E^m(n)$ is

$$\begin{aligned} N_\varepsilon\left(\bigcup_{m\in\mathbb{N}^+} E^m(n)\right) &\le \sharp\left(\bigcup_{m=0}^{l} E^l\right) + \sharp W^l \le (l+1)\sharp E^l + (\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda)^l \\ &\le 2(l+1)^2[\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda]^l. \end{aligned} \tag{2.29}$$

This proves the precompactness of $\bigcup_{m\in\mathbb{N}^+} E^m(n)$ in X, which directly implies the closure $\mathcal{M}(n) := \overline{\bigcup_{m\in\mathbb{N}^+} E^m(n)}$ is compact in X since X is a Banach space.

It follows from (2.27) and (2.29) that the fractal dimension of the set $\mathcal{M}(n)$ can be estimated by

$$\begin{aligned} \dim_f \mathcal{M}(n) &= \limsup_{\varepsilon\to 0} \frac{\ln N_\varepsilon(\mathcal{M}(n))}{-\ln\varepsilon} \\ &\le \limsup_{l\to\infty} \frac{\ln[2(l+1)]^2 + \ln[\Lambda 2^\Lambda\left(1+\frac{M_1}{\alpha}\right)^\Lambda]^l}{-l\ln\zeta - \ln R_{\mathcal{B}(n-l+1)}} \\ &= \frac{\ln\Lambda + \Lambda\ln(2+\frac{2M_1}{\alpha})}{-\ln\zeta + \varsigma} < \infty, \end{aligned} \tag{2.30}$$

for any $\varsigma > 0$, where the last inequality follows from the assumption that $R_{\mathcal{B}(n-l+1)}$ grows at most sub-exponentially in the past. Due to the arbitrariness of $\varsigma > 0$, we have

$$\dim_f \mathcal{M}(n) \le \frac{\ln\Lambda + \Lambda\ln(2+\frac{2M_1}{\alpha})}{-\ln\zeta} < \infty. \tag{2.31}$$

3) Exponential attraction of $\mathcal{M}(n)$. It remains to show that the set $\mathcal{M}(n)$ exponentially attracts all bounded subsets of X at time $n \in \mathbb{Z}$. It follows from assumptions $(\mathcal{H}_1)$ that, for any bounded subset $D \subset X$, there exists an $n_{D,n} \in \mathbb{Z}$ such that $U(k,k-l)D \subset \mathcal{B}(k)$ for all $l \ge n_{D,n}$ and $k \le n$. If

$l \geq n_{D,n}+1$, that is $l = n_{D,n}+n_0$ with some $n_0 \in \mathbb{N}$, then

$$
\begin{aligned}
\operatorname{dist}_X\left(U(n,n-l)D, \mathcal{M}(n)\right) &= \operatorname{dist}_X\left(U(n,n-l)D, \overline{\bigcup_{m=0}^{\infty} E^m(n)}\right) \\
&\leq \operatorname{dist}_X\left((U(n,n-n_0)(U(n-n_0,n-n_0-n_{D,n})D, \bigcup_{m=0}^{\infty} E^m(n)\right) \\
&\leq \operatorname{dist}_X\left((U(n,n-n_0)\mathcal{B}(n-n_0), \bigcup_{m=0}^{\infty} E^m(n)\right) \\
&\leq \operatorname{dist}_X\left((U(n,n-n_0)\mathcal{B}(n-n_0), E^{n_0}\right) \\
&\leq \zeta^{n_0} R_{\mathcal{B}(n-n_0)} \leq ce^{-\omega n}
\end{aligned}
\tag{2.32}
$$

for some constants $c \geq 0$ and $\omega > 0$, since $\zeta < 1$. This completes the proof. □

By adopting the same procedure as the proof of Theorem 3.2 in [12], we have the following results about the existence of exponential attractors for evolution process in Banach spaces.

Theorem 2.2. *For the evolution process $\{U(t,s)\}$ on the Banach space X and assume that hypothesis $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold and conditions of Theorem 2.1 hold. Then, there exists a pullback exponential attractor $\mathcal{M}(t)$ for the evolution process $\{U(t,s)\}$, and the fractal dimension is bounded by*

$$
\dim_f \mathcal{A} \leq \frac{\ln \Lambda + \Lambda \ln(2+\frac{2M_1}{\alpha})}{-\ln(\alpha e^{\lambda_0} + M_2 e^{\lambda_1} + M_3 e^{\lambda_0})} < \infty. \tag{2.33}
$$

Remark 2.1. *By* (2.33)*, we can see the fractal dimension of the pullback exponential attractors constructed in theorem 2.2 depends on the parameter α. Generally, if we take $\alpha \uparrow M_1$ and assume $M_1 e^{\lambda_0} + M_2 e^{\lambda_1} + M_3 e^{\lambda_0} < 1$, then for all $\alpha \in (0, M_1)$, we have $\alpha e^{\lambda_0 t_0} + M_2 e^{\lambda_1 t_0} + M_3 e^{\lambda_0 t_0} < 1$ and hence we get an α-independent estimation*

$$
d_H \leq \frac{-\ln \Lambda - \Lambda \ln 4}{\ln(2e^{\lambda_0 t_0} + 2M_2 e^{\lambda_1 t_0} + 2M_3 e^{\lambda_0 t_0})}. \tag{2.34}
$$

3 Retarded reaction-diffusion equation

This section is devoted to the existence of pullback exponential attractors for nonautonomous retarded reaction-diffusion equations with asymptotically autonomous linear part or nonautonomous nonlinear part.

3.1 Asymptotic autonomous linear part

We first consider the following nonautonomous retarded reaction-diffusion equation on bounded domain with Dirichlet boundary condition and asymptotically autonomous linear part

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = a(t,x)\frac{\partial^2}{\partial x^2}u(t,x) - c(t,x)u(t,x) - l(t,x)u(t-r,x) + f(u(t-r,x)), 0 \le x \le \pi, t \ge 0, \\ u(t,0) = u(t,\pi) = 0, t \ge 0, \\ u(t,x) = \phi(t,x), 0 \le x \le \pi, -r \le t \le 0. \end{cases} \tag{3.1}$$

where $a, c, l \in C_b^\mu(\mathbb{R}_+, C[0,\pi]), \mu \in (0,1)$, the space of uniformly bounded, μ-Hölder continuous $C[0,\pi]$-valued functions, $a(t,x) \ge a_0 > 0$ and

$$a(t,x) \to 1, \quad c(t,x) \to c, \quad l(t,x) \to l$$

uniformly for $x \in [0,\pi]$ as $t \to \infty$. Let $H = L^2(0,\pi)$ with inner product $(\xi,\eta) = \int_0^\pi \xi(x)\eta(x)dx$, norm $\|\xi\|_H = [\int_0^\pi \xi^2(x)dx]^{1/2}$ for any $\xi, \eta \in H$ and $X = C([-r,0],H)$ the continuous function from $[-r,0]$ to H endowed with the supremum norm $\|\phi\|_X = \sup_{\theta\in[-r,0]} \|\phi(\theta)\|_H$ for any $\phi \in X$. Let $D = W^{2,2}(0,\pi) \cap W_0^{1,2}(0,\pi)$ be endowed with the usual norm. Set

$$A(t)\varphi = a(t,\cdot)\varphi'' + c(t,\cdot)\varphi \quad \text{and} \quad A\varphi = \varphi'' - c\varphi$$

for $\varphi \in D$ and $t \ge 0$. On X we further define

$$L(t)\phi = -l(t,\cdot)\phi(-r) \text{ and } L\phi = -l\phi(-r),$$

and define $A_U : X \to X$ by

$$A_U\phi = A\phi(0) + L\phi \tag{3.2}$$

for any $\phi \in X$. It follows from [57] that the characteristic values of the linear part A_U are the roots of the following characteristic equation

$$n^2 - \left(\lambda + c + le^{-\lambda r}\right) = 0, n = 1, 2, \cdots. \tag{3.3}$$

Since A_U is compact, it follows from [57, Theorem 1.2 (i)] that the spectrum of A_U is point spectrum, which we denote by $\varrho_1 > \varrho_2 > \cdots$ with multiplicity $n_1, n_2, \cdots$, where ϱ_1 is defined as

$$\varrho_1 = \max\left\{\mathrm{Re}\,\lambda : n^2 - \left(\lambda + c + le^{-\lambda r}\right) = 0\right\}, n = 1, 2, \cdots. \tag{3.4}$$

In the following, we always assume that $l - c < 1$ and it follows from [57, Lemma 1.13, p. 73] that if $c > 0$, $l > 0$ and $l - c < 1$, then $\varrho_1 < 0$. By the results in [49, p. 3541], the evolutions process $S(t,\sigma) : \mathbb{R} \times \mathbb{R} \times X \to X$ defined by $S(t,\sigma)\phi = u_t^\phi(\cdot,\sigma)$ with $u^\phi(t,\sigma)$ being the solution of the following linear part of (3.1)

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t) + L(t)u_t \\ u_\sigma = \phi \end{cases} \tag{3.5}$$

which admits an exponential dichotomy with a two dimensional stable subspace, that is, there exist a positive constants K and a negative constant $\beta < 0$, and a m dimension projection operators $P(t) : X \to X_m, s \in \mathbb{R}$ and $Q(t) = I - P(t) : X \to X_m^{\perp}, t \in \mathbb{R}$ such that

$$\|Q(t)S(t,s)\| = \|S(t,s)Q(s)\| \leq Ke^{\beta(t-s)}, \quad t \geq s. \tag{3.6}$$

Moreover, by definition of ϱ_1, there exist positive constants $-\gamma < \varrho_1$ and K_0 such that

$$\|S(t,\sigma)\phi\|_X < K_0 \mathrm{e}^{-\gamma(t-\sigma)}\|\phi\|_X \tag{3.7}$$

for all $t \geq \sigma$. We assume that f satisfies the following global Lipschitz condition.

Hypothesis A3 $\|f(\phi_1) - f(\phi_2)\| \leq L\|\phi_1 - \phi_2\|$ for any $\phi_1, \phi_2 \in X$.

It follows from Theorem 2.6 in [57] and some standard contraction techniques, one can see under assumption **Hypothesis A3**, the non-autonomous nonlinear equation (3.1) admits a solution $u^{\phi}(t,\sigma)$ for any $t \in [\sigma - r, \infty)$, which is also continuous with respect to the initial condition and can be represented as

$$u_t^{\phi}(\cdot,\sigma) = S(t,\sigma)\phi + \int_{\sigma}^{t} S(t,s)X_0 f(u_s^{\phi}(\cdot,\sigma))ds, \quad t \geq 0, \tag{3.8}$$

where $X_0 : [-r, 0] \to B(X, X)$ is given by $X_0(\theta) = 0$ if $-r \leq \theta < 0$ and $X_0(0) = Id$, where $B(X, X)$ is the family of bounded linear operators on X.

Define the non-autonomous evolution process generated by (3.1) by $\Phi(t,\sigma)\phi = u_t^{\phi}(\cdot,\sigma)$ for any $\phi \in X$, which is continuous for any $t \geq \sigma$. In the following, we construct exponential attractors for $\Phi(t,\sigma)$. By similar techniques as those in the proof of Theorem 3.1 from [37], we can see that $\Phi(t,\sigma)$ admits a family of positively invariant absorbing sets $\mathcal{B}(\sigma)$ for any $\sigma \in \mathbb{R}$, implying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold.

Theorem 3.1. *Assume that* **Hypothesis A4** *as well as* **Hypothesis A3** *hold,* $K_0 < 1$ *and* $K_0 L_f - \gamma < 0$*. Then, the dynamical system* Φ *admits an invariant absorbing set* $\mathcal{B}(\sigma)$ *defined by*

$$\mathcal{B}(\sigma) = \{\phi \in C | \|\phi\|_X \leq \frac{1}{1-K_0}[\frac{K_0 f(0)}{\gamma} + \frac{1}{\gamma - K_0 L_f}]\}. \tag{3.9}$$

Subsequently, we prove the squeezing property of Φ, i.e., $(\mathcal{H}_3)$ holds.

Theorem 3.2. *Let P be the finite dimensional projection defined by* (3.6)*,* K, β, γ *and* K_0 *being defined by* (3.6) *and* (3.5) *respectively and assumptions of Theorem 3.1 hold. Then, we have*

$$\|P(t)\Phi(t,\sigma)\varphi - P(t)\Phi(t,\sigma)\psi\|_X \leq 2e^{(K_0 L_f - \gamma)(t-\sigma)}\|\varphi - \psi\|_X \tag{3.10}$$

and

$$\|(I - P(t))\Phi(t,\sigma)\varphi - (I - P(t))\Phi(t,\sigma)\psi\|_X \leq (Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{-\gamma + L_f - \beta}e^{(L_f - \gamma)(t-\sigma)})\|\varphi - \psi\|_X \tag{3.11}$$

for any $t \geq 0$ *and* $\varphi, \psi \in \mathcal{B}$.

Proof. For any $\varphi, \psi \in X$, denote by $y = \varphi - \psi$ and $w_t(\cdot, \sigma) = \Phi(t, \sigma)\varphi - \Phi(t, \sigma)\psi = u_t^{\varphi}(\cdot, \sigma) - u_t^{\psi}(\cdot, \sigma)$. Then it follows from (3.10) that

$$w_t(\cdot, \sigma) = S(t, \sigma)y + \int_{\sigma}^{t} S(t, s)X_0[f(u_s^{\varphi}) - f(u_s^{\psi})]ds, \quad t \geq 0. \tag{3.12}$$

Taking projection $I - P(t)$ on both sides of (3.12) leads to

$$\begin{aligned} \|(I - P(t))w_t(\cdot, \sigma)\|_X =& \|(I - P(t))S(t, \sigma)y + \int_{\sigma}^{t} (I - P(t))S(t, s)X_0[f(u_s^{\varphi}) - f(u_s^{\psi})]ds\|_X \\ \leq& Ke^{\beta(t-\sigma)}\|y\|_X + L_f K_0 \int_{\sigma}^{t} e^{-\gamma(t-s)}\|(I - P(t))w_s\|_X ds. \end{aligned} \tag{3.13}$$

Multiplying both sides of (3.13) by $e^{\gamma(t-\sigma)}$ implies

$$e^{\gamma(t-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\|_X \leq Ke^{(\beta+\gamma)(t-\sigma)}\|y\|_X + L_f K_0 \int_{\sigma}^{t} e^{\gamma(s-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\|_X ds. \tag{3.14}$$

Applying the Gronwall inequality, we have

$$e^{\gamma(t-\sigma)}\|(I - P(t))w_t(\cdot, \sigma)\|_X \leq \|y\|_X[Ke^{(\beta+\gamma)(t-\sigma)} + \frac{KL_f K_0}{\beta + \gamma - L_f K_0}(e^{(\beta+\gamma)(t-\sigma)} - e^{L_f K_0})], \tag{3.15}$$

indicating that

$$\begin{aligned} \|(I - P(t))w_t(\cdot, \sigma)\|_X \leq& \|y\|_X[Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{\beta + \gamma - L_f K_0}(e^{\beta(t-\sigma)} - e^{(K_0 L_f - \gamma)(t-\sigma)})] \\ \leq& \|y\|_X[Ke^{\beta(t-\sigma)} + \frac{KL_f K_0}{-\gamma + L_f K_0 - \beta}e^{(K_0 L_f - \gamma)(t-\sigma)}]. \end{aligned} \tag{3.16}$$

Hence, the second part holds with $\lambda_0 = L_f K_0 - \gamma$, $\lambda_1 = \beta$, $M_2 = K$ and $M_3 = \frac{KL_f K_0}{-\beta - \gamma + L_f K_0}$.

Subsequently, we prove the first part. Since $S(t, \sigma)y = P(t)S(t, \sigma)y + (I - P(t))S(t, \sigma)y$, we have

$$\|P(t)S(t, \sigma)y\|_X \leq \|S(t, \sigma)y\|_X + \|(I - P(t))S(t, \sigma)y\|_X. \tag{3.17}$$

Taking projection $P(t)$ on both sides of (3.12) and on account of (3.17) yields

$$\begin{aligned} \|P(t)w_t(\cdot, \sigma)\|_X =& \|S(t, \sigma)y\|_X + \|(I - P(t))S(t, \sigma)y\|_X + \int_{\sigma}^{t} \|P(t)S(t, s)X_0[f(u_s^{\varphi}) - f(u_s^{\psi})]ds\|_X \\ \leq& (K_0 e^{-\gamma(t-\sigma)} + Ke^{\beta(t-\sigma)})\|y\|_X + L_f K_0 \int_{\sigma}^{t} e^{-\gamma(t-s)}\|Pw_t(\cdot, \sigma)\|_X ds \\ \leq& (K_0 + K)e^{-\gamma(t-\sigma)}\|y\|_X + L_f K_0 \int_{\sigma}^{t} e^{-\gamma(t-s)}\|Pw_t(\cdot, \sigma)\|_X ds. \end{aligned} \tag{3.18}$$

Multiplying both sides of (3.18) by $e^{\gamma(t-\sigma)}$ implies

$$e^{\gamma(t-\sigma)}\|Pw_t(\cdot, \sigma)\|_X \leq (K_0 + K)\|y\|_X + L_f K_0 \int_{\sigma}^{t} e^{\gamma(s-\sigma)}\|Pw_t(\cdot, \sigma)\|_X ds. \tag{3.19}$$

Applying again the Gronwall inequality,

$$e^{\gamma(t-\sigma)}\|Pw_t(\cdot,\sigma)\| \leq (K_0+K)\|y\|_X e^{L_f K_0(t-\sigma)}, \tag{3.20}$$

indicating that

$$\|Pw_t(\cdot,\sigma)\| \leq (K_0+K)\|y\|_X e^{(L_f K_0-\gamma)(t-\sigma)}. \tag{3.21}$$

Hence, the first part holds by taking $M_1 = (K_0+K)$ and $\lambda_0 = L_f K_0 - \gamma$. $\square$

By Theorem 2.2, we have the following results about existence of a pullback exponential attractor $\mathcal{M}(t)$ for the nonlinear dynamical system Φ generated by (3.1).

Theorem 3.3. *Let P be the finite dimensional projection defined by* (3.6), K, β, γ *and* K_0 *being defined by* (3.6) *and* (3.5) *respectively and assumptions of Theorem 3.1 hold. Moreover, assume there exists* $\alpha > 0$ *such that* $\zeta := \alpha e^{(L_f K_0-\gamma)} + Ke^{\beta} + \frac{KL_f K_0}{-\gamma+L_f K_0-\beta}e^{(L_f K_0-\gamma)} < 1$. *Then,* (3.1) *admits an exponential attractor* $\mathcal{M}(t)$ *whose fractal dimension satisfies*

$$\dim_f \mathcal{M}(t) \leq \frac{\ln m + m\ln(2+\frac{2(K_0+K)}{\alpha})}{-\ln(\alpha e^{(L_f K_0-\gamma)} + Ke^{\beta} + \frac{KL_f K_0}{-\gamma+L_f K_0-\beta}e^{(L_f K_0-\gamma)})} < \infty. \tag{3.22}$$

We have the following special case about the fractal dimension of global attractor $\mathcal{M}(t)$ in the case $m = 1$.

Corollary 3.1. *Let P be the finite dimensional projection defined by* (3.6), K, β, γ *and* K_0 *being defined by* (3.6) *and* (3.5) *respectively and assumptions of Theorem 3.1 hold with* $m = 1$. *Moreover, assume there exists* $\alpha > 0$ *such that* $\zeta := (\alpha+K)e^{(L_f K_0-\gamma)} + Ke^{\varrho_1} < 1$. *Then,* (3.1) *admits an exponential attractor* $\mathcal{M}(t)$ *whose fractal dimension is bounded as*

$$\dim_f \mathcal{M}(t) \leq \frac{\ln(2+\frac{2(K_0+K)}{\alpha})}{-\ln[(\alpha+K)e^{(L_f K_0+\varrho_1)} + Ke^{\varrho_1}]} < \infty. \tag{3.23}$$

3.2 Nonautonomous nonlinear term

Subsequently, we consider the situation in which the nonautonomous effect comes from the nonlinear term, i.e., the following nonautonomous initial boundary value problem with delay

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - au(x,t) - bu(x,t-r) + f(t,u(x,t-r)) + h(x), 0 \leq x \leq \pi, t \geq \tau, \\ u(0,t) = u(\pi,t) = 0, t \geq \tau, \\ u(x,t) = \phi(t)(x), 0 \leq x \leq \pi, \tau - r \leq t < \tau, \\ u(x,\tau) = u_0(x), 0 \leq x \leq \pi. \end{cases} \tag{3.24}$$

Here, a, b are positive constants, f is the nonlinear delayed forcing term and $h(x)$ is the time independent and spatial dependent external force. The uniform attractors of (3.24) on unbounded domain when the nonautonomous term does not depend on u_t has been investigate in [54]. Here, we restrict ourselves to the domain $[0,\pi]$ and pay attention to the existence as well as the topological dimensions estimation of pullback attractors.

Due to the nonautonomous effect accurses in the nonlinear term of (3.24), we do not use the semigroup approach as Section 4.1.1 but adopt the variational one developed in [9]. We first introduce more notations that will be used in the remaining of this section. Let $C_0^\infty(0,\pi)$ be the space of smooth function on $(0,\pi)$ with compact support. Set $\mathcal{V}=\{u\in C_0^\infty(0,\pi):\operatorname{div} u=0\}$ and denote by H the closure of $\mathcal{V}$ in $L^2(0,\pi)$ with inner product $(\cdot,\cdot)$ defined by $(u,v)=\int_0^\pi u(x)v(x)\mathrm{d}x$ and norm $|\cdot|$ defined by $|u|=(u,u)^{1/2}$ for any $u,v\in L^2(0,\pi)$. Let V be the closure of $\mathcal{V}$ in $H_0^1(0,\pi)$ with scalar product $((\cdot,\cdot))$ defined by $((u,v))=\int_0^\pi \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x$ and norm $\|\cdot\|$ defined by $\|u\|=((u,u))^{1/2}$ for any $u,v\in H_0^1(0,\pi)$. Denote by $L_H=L^2([-r,0],H)$. By the standard theory of Soblev spaces, one can see $V\subset H\equiv H'\subset V'$, where H' and V' are the dual spaces of V and H respectively and the injections are dense and compact. Denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V' and $X=C([-r,0],H)$ with the usual supremum norm $\|\cdot\|_X$. Set $D(A)=H^2\cap V$, then $Au=\tilde{P}\Delta u,\forall u\in D(A)$, ($\tilde{P}$ the ortho-projector from $L^2(0,\pi)$ onto H.

Let $0<\mu_1<\mu_2<\cdots<\mu_m<\cdots$ be the eigenvalues of $-A$ with eigenfunctions $e_1,e_2,\cdots,e_m,\cdots$. Denote by $V_m=\operatorname{span}\{e_1,e_2,\cdots,e_m\}$ and $V_m^\perp=\operatorname{span}\{e_{m+1},e_{m+2},\cdots\}$ the finite dimensional space spanned by $\{e_1,e_2,\cdots,e_m\}$ and its orthogonal complement respectively. Let $P_m:V\to V_m$ be the finite projection of V onto V_m defined by

$$P_m\phi=\Sigma_{i=1}^m(\phi,e_i)e_i, \tag{3.25}$$

for any $\phi\in V$. Then we have $\mu_{m+1}|(I-P_m)v|\le\|v\|=(-Av,v)$ and $\mu_1|v|\le\|v\|=(-Av,v)$ for any $v\in H$, indicating that

$$(Av,v)=-\|v\|\le-\mu_{m+1}|(I-P_m)v| \tag{3.26}$$

and

$$(Av,v)=-\|v\|\le-\mu_1|v| \tag{3.27}$$

for any $v\in V$. Let $C_V=C^0([-r,0];V)$ be the space of continuous function from $[-r,0]$ to V equipped with the usual supremum norm. Similar to [22, 16], we define the following m-dimensional projector $P=\hat{P}_m$ in C_V by

$$P\phi=(P\phi)(\theta)=\sum_{k=1}^m \mathrm{e}^{-\mu_k\theta}\left(\phi(0),\mathrm{e}_k\right)\mathrm{e}_k\equiv\mathrm{e}^{-A\theta}P_m\phi(0), \tag{3.28}$$

from C_V onto C_{V_m}, based on the projection operator P_m. It follows from (3.28) that

$$\|P\phi\|_X\le\mathrm{e}^{-\mu_1 r}|P_m\phi(0)| \tag{3.29}$$

and

$$\|(I-P)\phi\|_X\le\mathrm{e}^{-\mu_{m+1}r}|P_m\phi(0)| \tag{3.30}$$

Remark 3.1. *By [22, 16], P is the spectral projector of the infinitesimal generator $\mathcal{A}$ of the linear semigroup T_t in C_V defined by*

$$T_t[u] \equiv u_t(\theta) = \begin{cases} u(t+\theta), & \text{if } t+\theta > 0 \\ u_0(t+\theta), & \text{if } t+\theta \leqslant 0 \end{cases}$$

with $u(t)$ being the solution of (3.24) *with $a = b = 0$, $f(t, u(x, t-r)) \equiv 0$.*

For notation simplicity, denote by $g(t, \psi) = -b\psi(-r) + f(t, \psi(-r))$ for $\psi \in X$ and thus, (3.24) can be rewritten in the following abstract form on H

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) - au(t) + g(t, u_t) + h, t \geq \tau, \\ u(\tau) = u_0, u(t) = \phi(t-\tau), \quad t \in [\tau - r, \tau). \end{cases} \tag{3.31}$$

Now, we can analyze problem (3.31) for a more general functional g by imposing the following assumptions, similarly as it was done in [9]. Let us consider $g : \mathbb{R} \times X \to H$ satisfying:
Hypothesis A4 For all $\xi \in X$, $g(\cdot, \xi) : \mathbb{R} \to H$ is measurable.
Hypothesis A5 For all $t \in \mathbb{R}$, $g(t, 0) = 0$,
Hypothesis A6 There exists $L_g > 0$ such that $\forall t \in \mathbb{R}, \forall \xi, \eta \in X$

$$|g(t, \xi) - g(t, \eta)| \leqslant L_g \|\xi - \eta\|_X.$$

Hypothesis A7 There exists $m_0 \geqslant 0, C_g > 0$ such that for all $l \in [0, m_0], \tau \leqslant t, u$ and $v \in C([\tau - r, t]; H)$, the continuous function space from $[\tau - r, t]$ to H, the following inequality holds

$$\int_\tau^t e^{ls} |g(s, u_s) - g(s, v_s)|^2 \, \mathrm{d}s \leqslant C_g^2 \int_{\tau - r}^t e^{ls} |u(s) - v(s)|^2 \, \mathrm{d}s.$$

Following similar techniques as [23, Theorem 2.3] and [54, Theorem 8], we have the following results on the existence of solutions.

Lemma 3.1. *Assume that* **Hypothesis A4 − A7** *hold. Then, for each $\tau \in \mathbb{R}$*
(i) for any $\phi \in X$, there exists a solution $u(\cdot)$ to problem (3.24) *with $u \in L^2(\tau - r, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap C([\tau - r, T]; H), \forall T > \tau$,*
(ii) for any $\phi \in C([\tau - r, T]; V)$, problem (3.24) *admits a strong solution*

$$u \in L^2(\tau, T; H) \cap C([\tau - r, T]; V), \quad \forall T > \tau.$$

It follows from Lemma 3.1 and [9, Theorem 9] that the family of mappings $U(t, \tau) : X \to X$ defined by

$$U(t, \tau)\phi = u_t(\cdot; \tau, (\phi(0), \phi)) \tag{3.32}$$

is a continuous process for any $\phi \in X$ and any $\tau \leqslant t$. For later analysis, we also introduce the product space $M_H = H \times L_H$, which is a Hilbert space with associated norm

$$\|(u_0, \phi)\|_{M_H}^2 = |u_0|^2 + \int_{-r}^0 |\phi(s)|^2 \, \mathrm{d}s, \quad \text{for } (u_0, \phi) \in M_H. \tag{3.33}$$

Then, we can define the corresponding process on M_H as

$$S(t,\tau)\left(u_0,\phi\right)=\left(u\left(t;\tau,\left(u_0,\phi\right)\right),u_t\left(\cdot;\tau,\left(u_0,\phi\right)\right)\right),\quad \text{for } \left(u_0,\phi\right)\in M_H,\tau\leqslant t.$$

Following Remark 7 in [9], we also consider the family of mappings $\tilde{U}(\cdot,\cdot):M_H\to L_H$ defined as

$$\tilde{U}(t,\tau)\left(u_0,\phi\right)=u_t\left(\cdot;\tau,\left(u_0,\phi\right)\right),\quad \text{for } \left(u_0,\phi\right)\in M_H,\quad \text{and}\quad \tau\leqslant t.$$

Observe that

$$U(t,\tau)\phi=\tilde{U}(t,\tau)(\phi(0),\phi)\quad \text{for any } t\geqslant\tau, \text{ and any } \phi\in X.$$

In this way, we then have that the process $S(t,\tau)$ can be rewritten as

$$S(t,\tau)\left(u_0,\phi\right)=\Big(u\left(t;\tau,\left(u_0,\phi\right)\right),\tilde{U}(t,\tau)\left(u_0,\phi\right)\Big).$$

Moreover, define the linear mapping j by

$$j:\phi\in X\mapsto j(\phi)=(\phi(0),\phi)\in H\times X.$$

This map is obviously continuous from X into $H\times X$ and into M_H. Noticing that for all $(u_0,\phi)\in M_H$ it holds that $\tilde{U}(t,\tau)\left(u_0,\phi\right)\in X$ provided that $t\geqslant\tau+r$, we then can write

$$S(t,\tau)\left(u_0,\phi\right)=j\left(\tilde{U}(t,\tau)\left(u_0,\phi\right)\right),\quad \text{for } \left(u_0,\phi\right)\in M_H,t\geqslant\tau+r.$$

It follows from Theorem 9 in [9] that the above defined $U(t,\tau):X\to X$ and $S(t,\tau):M_H\to M_H$ are both continuous processes for $t\geq\tau$.

Using similar arguments in [9], we have the following results concerning existence of global pullback attractors of (3.1), implying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. For readers' convenience, we provide an outline of the proof here. Details can be found in [9].

Theorem 3.4. *Assume that* **Hypothesis A4 − A7** *hold for any $\tau\leqslant t$ with $m_0>0$ and $l+1+2C_g-2(\mu_1+a)<0$. Then, there exists a unique uniformly bounded pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ for the process $U(t,\tau)$.*

Proof. We first show the existence of a family of bounded absorbing sets $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ of $\tilde{U}(t,\tau)$ in X. By [9, Definition 10], it suffices to prove that for any bounded $\tilde{D}\subset M_H$ and $t\in\mathbb{R}$, there exists $T_{\tilde{D}}(t)$ such that for all $s\geqslant T_{\tilde{D}}(t)$ it holds $\tilde{U}(t,t-s)\tilde{D}(t)\subset B(t)$. Let $\tilde{D}$ be bounded in M_H, that is, there exists $d>0$ such that

$$\left|u_0\right|^2+\|\phi\|_X^2\leqslant d^2,\quad \text{for all } \left(u_0,\phi\right)\in\tilde{D}. \tag{3.34}$$

Take $(u_0,\phi)\in\tilde{D}$, $\tau\in\mathbb{R}$ and denote as usual $u(\cdot)=u\left(\cdot;\tau,\left(u_0,\phi\right)\right)$. Taking inner product on both sides of (3.31) and on account of (3.27) we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2&=(Au,u)-a(u,u)+\left(g\left(t,u_t\right),u\right)+(h,u)\\&\leqslant-(a+\mu_1)|u|^2+\frac{1}{2C_g}\left|g\left(t,u_t\right)\right|^2+\frac{C_g}{2}|u|^2+\frac{1}{2}\left|h\right|^2+\frac{1}{2}|u|^2\end{aligned} \tag{3.35}$$

implying that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 \leqslant \frac{1}{C_g}\left|g\left(t,u_t\right)\right|^2+\left(C_g+1-2(a+\mu_1)\right)|u|^2+|h|^2. \tag{3.36}$$

Choose $l\in(0,m_0)$ such that $l+2C_g-2(\mu_1+a)<0$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{lt}|u(t)|^2\right)&=le^{lt}|u(t)|^2+e^{lt}\frac{d}{dt}|u(t)|^2\\&\leqslant e^{lt}[(l+C_g+1-2(\mu_1+a))|u(t)|^2+\frac{1}{C_g}\left|g\left(t,u_t\right)\right|^2+|h|^2].\end{aligned} \tag{3.37}$$

Integrating from τ to $t(\geq\tau)$,

$$\begin{aligned}e^{lt}|u(t)|^2-e^{l\tau}\left|u_0\right|^2\leqslant&\int_\tau^t\frac{e^{ls}}{C_g}\left|g\left(s,u_s\right)\right|^2\ \mathrm{d}s+\int_\tau^t e^{ls}\left|h\right|^2\ \mathrm{d}s+\int_\tau^t e^{ls}\left(l+C_g+1-2(\mu_1+a)\right)|u(s)|^2\ \mathrm{d}s\\\leqslant&C_g\int_{\tau-r}^{\tau}e^{ls}|\phi(s-\tau)|^2\ \mathrm{d}s+\frac{e^{lt}\left|h\right|^2}{l}+\int_\tau^t e^{ms}\left(l+2C_g+1-2(\mu_1+a)\right)|u(s)|^2\ \mathrm{d}s\\\leqslant&\frac{e^{lt}\left|h\right|^2}{l}+C_g e^{l\tau}\int_{-r}^0|\phi(\theta)|^2\ \mathrm{d}\theta.\end{aligned} \tag{3.38}$$

Thus,

$$|u(t)|^2\leqslant\frac{|h|^2}{l}+d^2\left(1+C_g\right)e^{-l(t-\tau)},\ \text{for all } t\geqslant\tau. \tag{3.39}$$

Taking $t\geqslant\tau+r$, we have for $\theta\in[-r,0]$

$$\begin{aligned}|u(t+\theta)|^2&\leqslant\tilde{d}^2\left(1+C_g\right)e^{-l(t+\theta)}e^{lt}\\&\leqslant\frac{|h|^2}{l}+d^2e^{lr}\left(1+C_g\right)e^{-l(t-\tau)}.\end{aligned} \tag{3.40}$$

Setting time $t-s$ instead of τ and denoting $u(\cdot)=u\left(\cdot,t-s,\cdot\right)$ lead to

$$\|\tilde{U}(t,t-s)\left(u_0,\phi\right)\|_X=\|u_t\|_X^2\leqslant\frac{|h|^2}{l}+d^2\left(1+C_g\right)e^{l(r-s)}\ \text{for all } t,\ \text{and } s\geqslant r. \tag{3.41}$$

Therefore, there exists sufficient large $T_{\tilde{D}}(t)$ such that for all $s\geqslant T_{\tilde{D}}(t)$ it holds

$$d^2\left(1+C_g\right)e^{l(r-s)}\leq\frac{|h|^2}{l},$$

implying that the balls $\mathcal{B}(t)=B_X\left(0,2\frac{|h|^2}{l}\right)$ form an absorbing family of bounded sets for the mappings $\tilde{U}(t,\tau)$. By [9, Lemma 11], we can see $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ is a family of absorbing sets for $U(t,\tau)$ in X.

By taking the inner product of (3.31) with Δu and using similar techniques as those in the proofs of Theorem 15 and Corollary 16 in [9], there exist positive constants ρ_V,β_1,β_2 such that, for any bounded set $D\subset C_H$ and for the above absorbing time $T_{\tilde{D}}(t)$ corresponding to the set $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$, it follows

$$\|U(t,t-s)\phi\|_{C_V}^2=\|u_t(\cdot;t-s,j(\phi))\|_{C_V}^2=\max_{\theta\in[-r,0]}\|u(t+\theta;t-s,j(\phi))\|^2\leqslant\rho_V^2,$$

$$\int_{t+\theta_1}^{t+\theta_2}|Au(\sigma;t-s,j(\phi))|^2\ \mathrm{d}\sigma\leqslant\beta_1\left|\theta_2-\theta_1\right|+\beta_2,$$

for all $s \geqslant T_D+1+r, t \in \mathbb{R}, \phi \in D$, and $\theta_1, \theta_2 \in [-r, 0]$. In particular, the family $\{B_2(t)\}_{t\in\mathbb{R}}$, where $B_2(t)=B_2=B_{C_V}(0,\rho_V)$, is absorbing for the process $U(\cdot,\cdot)$. Moreover, the family $\{B_S(t)\}_{t\in\mathbb{R}}$, where $B_S(t)=B_{C_V}(0,\rho_V)\times B_{L_V^2}\left(0,h^{1/2}\rho_V\right)$, is absorbing for $S(\cdot,\cdot)$.

Apparently, the above defined $\{B_2(t)\}_{t\in\mathbb{R}}$ is a family of bounded sets in C_V, which is also (uniformly) absorbing for $\tilde{U}(\cdot,\cdot)$. Set $\tilde{B}_2=j\left(B_2\right)$, then, there exists $\tilde{T}_{\tilde{B}_2}=T_{B_2}+1+h>0$ such that $\tilde{U}(t,t-s)\tilde{B}_2\subset B_2$ for all $t\in\mathbb{R}$, and all $s\geqslant \tilde{T}_{\tilde{B}_2}$. Now, for each $t\in\mathbb{R}$, consider the set

$$B_3(t)=\bigcup_{s\geqslant \tilde{T}'_{\tilde{B}_2}} \tilde{U}(t,t-s)\tilde{B}_2\subset B_2\subset C_V.$$

Thus, $\{B_3(t)\}_{t\in\mathbb{R}}$ is a family of uniformly bounded sets in C_V which is (uniformly) absorbing for $\tilde{U}(\cdot,\cdot)$. By similar techniques as those in the proof of Theorem 17 from [9], each $B_3(t)$ is relatively compact in X, then $\left\{\overline{B_3(t)}\right\}_{t\in\mathbb{R}}$ (where the closure is taken in X) is a family of compact absorbing sets in X for $\tilde{U}(\cdot,\cdot)$. Consequently, it is also a family of compact (uniform) absorbing sets for the process $U(\cdot,\cdot)$ in X, which ensures the existence of the pullback attractors $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ for the process $U(t,\tau)$. The uniqueness of these attractors holds since they are uniformly bounded. □

Theorem 3.5. *Let P be the m dimensional projection on X defined by* (3.28) *and assume $C_g^2+1-2(\mu_1+a)>0$. Then, we have*

$$\|PU(t,\tau)\varphi-PU(t,\tau)\psi\|_X\leq 2\left(C_g^2r+1\right)e^{(C_g^2+1-2(\mu_1+a))(t-\tau)}\|\varphi-\psi\|_X \tag{3.42}$$

and

$$\|(I-P)U(t,\tau)\varphi-(I-P)U(t,\tau)\psi\|_X\leqslant \mathrm{e}^{-\mu_{m+1}r}\left(C_g^2r+1\right)^{1/2}e^{\frac{C_g^2+1-2(\mu_{m+1}+a)}{2}(t-\tau)}\|\varphi-\psi\|_X, \tag{3.43}$$

for any $t\geq\tau$, and φ,ψ in $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, where μ_1, μ_{m+1} and C_g are defined in (3.26) *and* **Hypothesis A7**, *respectively.*

Proof. For any two initial conditions $\varphi,\psi\in C_V$, denote by $u(\cdot)=u\left(\cdot;\tau,\varphi\right)$ and $v(\cdot)=u\left(\cdot;\tau,\psi\right)$ the corresponding solutions to (3.31) with initial time $\tau\in\mathbb{R}$ respectively. Moreover, denote by $y=\varphi-\psi$ and $w(t)=u^{\varphi}(t)-u^{\psi}(t)$. It follows from (3.31) that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t)=Aw(t)-aw(t)+g\left(t,u_t\right)-g\left(t,v_t\right). \tag{3.44}$$

Denoting by $w_m(t)=(I-P_m)w(t)$ and multiplying both sides of (3.31) by $w^m(t)$ lead to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w^m(t)|^2=(Aw(t),w^m(t))-a(w(t),w^m(t))+\left(g\left(t,u_t\right)-g\left(t,v_t\right),w^m(t)\right). \tag{3.45}$$

Keeping in mind that $(Aw^m,w^m)=-\|w^m\|\leq-\mu_{m+1}|(I-P_m)w|$, then we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w^m|^2&\leqslant-(\mu_{m+1}+a)|w^m|^2+|g\left(t,u_t\right)-g\left(t,v_t\right)||w^m|\\&\leqslant-(\mu_{m+1}+a)|w^m|^2+\frac{1}{2}|g\left(t,u_t\right)-g\left(t,v_t\right)|^2+\frac{1}{2}|w^m|^2.\end{aligned} \tag{3.46}$$

Integrating both sides of (3.46) from τ to t,

$$
\begin{aligned}
|w^m(t)|^2 - |w^m(\tau)|^2 &\leqslant \int_\tau^t |g(s,u_s) - g(s,v_s)|^2 \, \mathrm{d}s + \int_\tau^t (1 - 2(\mu_{m+1}+a)) |w^m(s)|^2 \, \mathrm{d}s \\
&\leqslant C_g^2 \int_{\tau-r}^t |w^m(s)|^2 \, \mathrm{d}s + \int_\tau^t \left(C_g^2 + 1 - 2(\mu_{m+1}+a)\right) |w^m(s)|^2 \, \mathrm{d}s \\
&\leqslant C_g^2 \|\varphi - \psi\|_{L_H}^2 + \int_\tau^t \left(C_g^2 + 1 - 2(\mu_{m+1}+a)\right) |w^m(s)|^2 \, \mathrm{d}s.
\end{aligned} \tag{3.47}
$$

Consequently, for any $t \geqslant \tau$, we have

$$
\begin{aligned}
|w^m(t)|^2 &\leqslant C_g^2 \|y\|_{L_H}^2 + \|u_0 - v_0\|^2 + \int_\tau^t \left(C_g^2 + \frac{1}{2} - (\mu_{m+1}+a)\right) |w^m(s)|^2 \, \mathrm{d}s \\
&\leqslant C_g^2 \|y\|_{L_H}^2 + \|u_0 - v_0\|^2 + \int_\tau^t \left(C_g^2 + 1 - 2(\mu_{m+1}+a)\right) |w^m(s)|^2 \, \mathrm{d}s.
\end{aligned} \tag{3.48}
$$

The Gronwall lemma implies now, for any $t \geqslant \tau$,

$$
|w^m(t)|^2 \leqslant \left(C_g^2 |y|_{L_H}^2 + \|u_0 - v_0\|^2\right) e^{\int_\tau^t \left(C_g^2 + 1 - 2(\mu_{m+1}+a)\right) \mathrm{d}s}. \tag{3.49}
$$

Assume that $t \geqslant \tau + r$. Then $t + \theta \geqslant \tau$ for any $\theta \in [-\tau, 0]$ and it follows from (3.26) that

$$
\|(I-P)U(t,\tau)\varphi - (I-P)U(t,\tau)\psi\|_X^2 = \|Pw_t^m\|_X^2 \quad \leqslant \mathrm{e}^{-2\mu_{m+1} r} \left(C_g^2 r + 1\right) \|\varphi - \psi\|_X^2 e^{(C_g^2 + 1 - 2(\mu_{m+1}+a))(t-\tau)} \tag{3.50}
$$

and, thus,

$$
\|(I-P)U(t,\tau)\varphi - (I-P)U(t,\tau)\psi\|_X \leqslant \mathrm{e}^{-\mu_{m+1} r} \left(C_g^2 r + 1\right)^{1/2} e^{\frac{C_g^2 + 1 - 2(\mu_{m+1}+a)}{2}(t-\tau)} \|\varphi - \psi\|_X \tag{3.51}
$$

This completes the proof of the second statement.

Now, we concentrate on the first part. Repeating the same procedure of the above proof but replacing w^m by w and $(Aw, w) = -\|w\| \leq -\mu_1 |w|$ gives rise to

$$
\|w_t\|_X \leqslant \mathrm{e}^{-\mu_1 r} \left(C_g^2 r + 1\right)^{1/2} e^{\frac{C_g^2 + 1 - 2(\mu_1 + a)}{2}(t-\tau)} \|\varphi - \psi\|_X. \tag{3.52}
$$

By state decomposition, we have

$$
\|Pw_t\|_X = \|w_t - (I - P(t))w_t\|_X \leqslant \|w_t\|_X + \|(I-P)w_t\|_X \tag{3.53}
$$

Incorporating (3.51) and (3.52) into (3.53) leads to

$$
\|PU(t,\tau)\varphi - PU(t,\tau)\psi\|_X \leqslant 2\mathrm{e}^{-\mu_1 r} \left(C_g^2 r + 1\right)^{1/2} e^{\frac{C_g^2 + 1 - 2(\mu_1 + a)}{2}(t-\tau)} \|\varphi - \psi\|_X. \tag{3.54}
$$

Therefore the first part holds, which completes the proof of the theorem. □

Theorem 3.5 implies $(\mathcal{H}_3)$ holds with $M_1 = 2\mathrm{e}^{-\mu_1 r}\left(C_g^2 r + 1\right)^{1/2}$, $M_2 = \mathrm{e}^{-\mu_{m+1} r}\left(C_g^2 r + 1\right)^{1/2}$, $M_3 = 0$, $\lambda_0 = \frac{C_g^2 + 1 - 2(\mu_1 + a)}{2}$, $\lambda_1 = \frac{C_g^2 + 1 - 2(\mu_{m+1} + a)}{2}$. Hence, by Theorem 2.2, we have the following results about existence of a pullback exponential attractor $\mathcal{M}(t)$ for the nonlinear dynamical system $U(t,\tau)$ generated by (3.24).

Theorem 3.6. *Let P be the finite dimensional projection defined by* (3.28), μ_1, μ_{m+1} *and* C_g *are defined in* (3.26) *and* **Hypothesis A7** *respectively and assumptions of Theorem 3.4 hold. Moreover, assume there exists* $\alpha > 0$ *such that* $\zeta := \alpha e^{\frac{C_g^2+1-2(\mu_1+a)}{2}} + \mathrm{e}^{-\mu_{m+1}r}\left(C_g^2 r+1\right)^{1/2} e^{\frac{C_g^2+1-2(\mu_{m+1}+a)}{2}} < 1$. *Then,* (3.24) *admits a pullback exponential attractor* $\mathcal{M}(t)$ *whose fractal dimension has an upper bound*

$$\dim_f \mathcal{M}(t) \leq \frac{\ln m + m\ln(2+\frac{4\mathrm{e}^{-\mu_1 r}\left(C_g^2 r+1\right)^{1/2}}{\alpha})}{-\ln(\alpha e^{\frac{C_g^2+1-2(\mu_1+a)}{2}} + \mathrm{e}^{-\mu_{m+1}r}\left(C_g^2 r+1\right)^{1/2} e^{\frac{C_g^2+1-2(\mu_{m+1}+a)}{2}})} < \infty. \tag{3.55}$$

4 Retarded 2D-Navier-Stokes equations

In this section, we are concerned with the existence of pullback exponential attractors for the following delayed 2D-Navier-Stokes equation on an open bounded domain $\Omega \subset \mathbb{R}^2$ with regular boundary Γ,

$$\begin{cases} \frac{\partial u}{\partial t} - \vartheta\Delta u + \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i} = f - \nabla p + g\left(t, u_t\right) \quad \text{in } (\tau,+\infty)\times\Omega, \\ \operatorname{div} u = 0 \text{ in } (\tau,+\infty)\times\Omega, \\ u = 0 \quad \text{on } (\tau,+\infty)\times\Gamma, \\ u(t,x) = \phi(t-\tau,x), t\in[\tau-r,\tau], \quad x\in\Omega. \end{cases} \tag{4.1}$$

Here, u is the velocity field of the fluid, $\vartheta > 0$ is the kinematic viscosity, f is a nondelayed external force field, p is the pressure, $\tau \in \mathbb{R}$ is the initial time, g is another external force with some hereditary characteristics and ϕ the initial datum in the interval of time $[-r, 0]$ with r being a fixed positive number, representing the time delay. In the case $t = \tau$, i.e., at the initial time, the initial velocity field is $u(\tau, x) = \phi(0, x)$. The investigation of retarded Navier-Stokes problems dates back to [7, 8], where the authors studied the existence and asymptotic behavior of solutions. In [9], they further established the existence of pullback attractors. Recently, Qin and Su [47] studied the Hursdorff dimension of Navier-Stokes-Voigt equations with a distributed delay, they recast the equation in a Hilbert space and directly adopted the approach proposed in [17] and [31]. Here, we directly deal with the problem in the natural phase space, i.e., the Banach space. Moreover, we also consider the nonautonomous case.

Notation in this section are the same as those in Section 3.2 but with a general domain Ω and the state in two dimensional space. For instance, $\mathcal{V} = \left\{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\right\}$ and H denotes the closure of $\mathcal{V}$ in $\left(L^2(\Omega)\right)^2$. Other notation can be found in Section 3.2 and [9]. Let $b : V \times V \times V \to \mathbb{R}$ be a trilinear form defined by

$$b(u,v,w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x \quad \forall u,v,w \in V.$$

Set $B : V \times V \to V'$ by $\langle B(u,v), w\rangle = b(u,v,w)$ and $B(u) = B(u,u)$ $\forall u,v,w \in V$. Denoting $D(A) = H^2 \cap V$, then $Au = -\tilde{P}\Delta u, \forall u \in D(A)$, ($\tilde{P}$ being the ortho-projector from $\left(L^2(\Omega)\right)^2$ onto H). Hence, (4.1) can be recast in the following abstract form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) + \vartheta Au(t) + B(u(t)) = f(t) + g\left(t, u_t\right) \text{ in } \mathcal{D}'\left(\tau,+\infty; V'\right), \\ u(\tau) = u_0, u(t) = \phi(t-\tau), \quad t\in(\tau-r,\tau). \end{cases} \tag{4.2}$$

Denote $X = C^0([-r, 0]; H), L_H = L^2(-r, 0; H)$ and $L_V^2 = L^2(-r, 0; V)$. By [23, Theorem 2.3] and [9, Theorem 17,Corollary 16], we have the following existence of solutions as well as pullback attractors of (4.1), indicating $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ of Theorem 2.2 hold.

Lemma 4.1. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, $g : \mathbb{R}\times X \to \left(L^2(\Omega)\right)^2$ such that hypotheses* **Hypothesis A4-Hypothesis A7** *hold. Then, the following statements hold.*
(I) For each $\tau \in \mathbb{R}$ and $\phi \in C_V$, there exists a unique solution u to (5.2) which belongs to the space $C^0([\tau - r, +\infty); V)$.
(II) Let $U(t, \tau) : C_H \to C_H$ be defined by $U(t, \tau)\phi = u_t(\cdot, \tau, \phi)$ for all $\tau \leqslant t$ and $\phi \in C^0([\tau - r, t]; H)$ and μ_1 is the first eigenvalue of the operator A. If $m_0 > 0$ and $\vartheta\mu_1 > C_g$, then there exist a positive constant ρ_V and a unique uniformly bounded pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, which is inside $B_{C_V}(0, \rho_V)$, the absorbing ball in C_V with center 0 and radius ρ_V.

Let V_m and P being defined by (3.25) and (3.28) in Section 3.2, then we have the following results concerning the squeezing property.

Theorem 4.1. *Let P be defined by (3.28), μ_1, ρ_V be given in Lemma 4.1. Assume assume that $\vartheta\mu_1 > C_g$ and there exists $c_0' > 0$ such that $C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1 > 0$. Then, we have*

$$\|PU(t, \tau)\varphi - PU(t, \tau)\psi\| \leq e^{-\mu_1 r}(C_g^2 r + 1)^{1/2} e^{\frac{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)}{2}(t-\tau)}\|\varphi - \psi\|_X \tag{4.3}$$

and

$$\|(I - P)U(t, \tau)\varphi - (I - P)U(t, \tau)\psi\| \leq e^{-\mu_{m+1} r}(C_g^2 r + 1)^{1/2} e^{\frac{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)}{2}(t-\tau)}\|\varphi - \psi\|_X, \tag{4.4}$$

for any $t \geq \tau$, and φ, ψ in $\mathcal{A}(t)$, where μ_1, μ_{m+1} and C_g are defined in (3.26), (3.27) and **Hypothesis A7** *respectively.*

Proof. For any two initial conditions $\varphi, \psi \in \mathcal{A}(t)$, denote by $u(\cdot) = u(\cdot; \tau, \varphi)$ and $v(\cdot) = u(\cdot; \tau, \psi)$ the corresponding solutions to (4.2) with initial time $\tau \in \mathbb{R}$ respectively. Moreover, denote $w(t) = u(t) - v(t)$. It follows from (4.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) + \vartheta A w(t) + B(u(t)) - B(v(t)) = g(t, u_t) - g(t, v_t). \tag{4.5}$$

Multiply both sides of (4.2) by $w(t)$ and take into account that $B(u(t)) - B(v(t)) = B(w(t), u(t)) + B(v(t), w(t))$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 + \vartheta\|w(t)\|^2 + b(w(t), u(t), w(t)) + b(v(t), w(t), w(t)) = (g(t, u_t) - g(t, v_t), w(t)). \tag{4.6}$$

Keeping in mind that $(Av, v) = -\|v\| \leq -\mu_1|v|$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 + \vartheta\|w\|^2 &\leqslant c_0|w|\|u\|\,\|w\| + |g(t, u_t) - g(t, v_t)|\,|w| \\ &\leqslant \frac{1}{2}c_0'|w|^2\|u\|^2 + \vartheta\|w\|^2 + \frac{1}{2}|g(t, u_t) - g(t, v_t)|^2 + \frac{1}{2}|w|^2, \end{aligned} \tag{4.7}$$

and therefore, noticing that $u(t) \in \mathcal{A}(t)$, we have $\|u(t)\|^2 \leq \rho_V$, and

$$
\begin{aligned}
|w(t)|^2 - |w(\tau)|^2 &\leqslant \int_\tau^t |g(s,u_s) - g(s,v_s)|^2 \, \mathrm{d}s + \int_\tau^t \left(c_0'\|u(s)\|^2 - \vartheta\mu_{m+1} + 1\right)|w(s)|^2 \, \mathrm{d}s \\
&\leqslant C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 \, \mathrm{d}s + \int_\tau^t \left(C_g^2 + c_0'\rho_V^2 - \vartheta\mu_1 + 1\right)|w(s)|^2 \, \mathrm{d}s \\
&\leqslant C_g^2 \|\varphi - \psi\|_{L_H}^2 + \int_\tau^t \left(C_g^2 + c_0'\rho_V^2 - \vartheta\mu_1 + 1\right)|w(s)|^2 \, \mathrm{d}s.
\end{aligned}
\tag{4.8}
$$

Consequently,

$$
|w(t)|^2 \leqslant C_g^2\|\varphi - \psi\|_{L_H}^2 + \|u_0 - v_0\|^2 + \int_\tau^t \left(C_g^2 + c_0'\rho_V^2 - \vartheta\mu_1 + 1\right)|w(s)|^2 \, \mathrm{d}s, \forall t \geqslant \tau. \tag{4.9}
$$

The Gronwall lemma implies now for any $t \geqslant \tau$,

$$
|w(t)|^2 \leqslant \left(C_g^2\|\varphi - \psi\|_{L_H}^2 + \|u_0 - v_0\|^2\right) e^{\int_\tau^t \left(C_g^2 + c_0'\rho_v^2 - \vartheta\mu_1 + 1\right)\mathrm{d}s}. \tag{4.10}
$$

Assume now that $t \geqslant \tau + r$, then $t + \theta \geqslant \tau$ for any $\theta \in [-\tau, 0]$ and it holds

$$
\begin{aligned}
|w(t+\theta)|^2 &\leqslant (C_g^2 r + 1) e^{\int_\tau^t \left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)\mathrm{d}s} \|\varphi - \psi\|_X^2 \\
&\leqslant (C_g^2 r + 1) e^{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)(t-\tau)} \|\varphi - \psi\|_X^2.
\end{aligned}
\tag{4.11}
$$

Thus, by (3.30), we have

$$
\|(I-P)w_t\|_X^2 \leqslant e^{-2\mu_{m+1}r}(C_g^2 r + 1) e^{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)(t-\tau)} \|\varphi - \psi\|_X^2, \tag{4.12}
$$

implying that

$$
\|(I-P)w_t\|_X \leqslant e^{-\mu_{m+1}r}(C_g^2 r + 1)^{1/2} e^{\frac{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)}{2}(t-\tau)} \|\varphi - \psi\|_X. \tag{4.13}
$$

Now, we concentrate on the first part. By (3.29) and (4.11), we have

$$
\|Pw_t\|_X = e^{-\mu_1 r}(C_g^2 r + 1)^{1/2} e^{\frac{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)}{2}(t-\tau)} \|\varphi - \psi\|_X. \tag{4.14}
$$

Therefore the first part holds, which completes the proof of the theorem. □

Theorem 3.5 implies $(\mathcal{H}_3)$ holds with $M_1 = e^{-\mu_1 r}(C_g^2 r + 1)^{1/2}, M_2 = e^{-\mu_{m+1} r}(C_g^2 r + 1)^{1/2}, M_3 = 0, \lambda_0 = \lambda_1 = \frac{\left(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1\right)}{2}$. Hence, by Theorem 2.2, we have the following results about existence of a pullback exponential attractor $\mathcal{M}(t)$ for the nonlinear evolution process $U(t,\tau)$ generated by (4.2).

Theorem 4.2. *Let P be the finite dimensional projection defined by* (3.28)*, μ_1, μ_{m+1} and C_g defined in* (3.26), (3.27) *and* **Hypothesis A7** *respectively and assumptions of Lemma 4.1 and Theorem 4.1 hold. Moreover, assume there exists $\alpha > 0$ such that $\zeta := (\alpha + e^{-\mu_{m+1}r}(C_g^2 r + 1)^{1/2}) e^{\frac{(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1)}{2}} < 1$. Then,* (4.2) *admits a pullback exponential attractor $\mathcal{M}(t)$ with fractal dimension satisfying*

$$
\dim_f \mathcal{M}(t) \leq \frac{\ln m + m \ln(2 + \frac{2e^{-\mu_1 r}(C_g^2 r + 1)^{1/2}}{\alpha})}{-\ln(\alpha + e^{-\mu_{m+1}r}(C_g^2 r + 1)^{1/2}) - \frac{(C_g^2 + c_0'\rho_V - \vartheta\mu_1 + 1)}{2}} < \infty. \tag{4.15}
$$

Remark 4.1. *In [Theorem 6.2, [47]], the authors investigated dimensions of global attractors of the following 2D Navier-Stokes-Voigt equations with a distributed delay*

$$\begin{cases} \frac{d}{dt}u - v\Delta u - \alpha^2 \frac{d}{dt}\Delta u + (u\cdot\nabla)u + \nabla p = f(x) + g(u_t), & (x,t)\in\Omega_0, \\ \operatorname{div} u = 0, & (x,t)\in\Omega_0, \\ u(x,t)|_{\partial\Omega} = \varphi, \quad \varphi\cdot n = 0, & (x,t)\in\partial\Omega_0, \\ u(x,0) = u^0(x), & x\in\Omega, \\ u(x,t) = \phi(t), & (x,t)\in\Omega_h, \end{cases} \tag{4.16}$$

by recasting (4.16) *into a Hilbert space and adopting the method established in [17]. The upper bounds of the Hausdorff and fractal dimensions of $\mathcal{A}$ they gave is*

$$\frac{C_3^{1/2}\alpha\lambda_1^{1/2}|\Omega|^{1/2}}{(2\pi\nu)^{1/2}\left(\frac{\nu}{\alpha} - \frac{2v}{\alpha^2\lambda_1} - \frac{2l_0}{\lambda_1}\right)^{1/2}} \left(\frac{\frac{\|\bar{f}\|^2}{v} + \frac{2C_2^2}{\lambda_1}C_g\|\varphi\|^2_{L^\infty(\partial\Omega)}}{v - 2C_1C_2\lambda_1^{-1}\|\varphi\|_{L^\infty(\partial\Omega)} - 3C_g\lambda_1^{-1}} + C_2\|\varphi\|^2_{L^\infty(\partial\Omega)} \right)^{1/2} + 1, \tag{4.17}$$

which do not depend on the time delay. In the case $\alpha = 0$, (4.16) *degenerates to* (4.1)*. However, in this case,* (4.17) *is not well defined since zero appears in the denominator, which means that the method in [17] may be ineffective for obtaining dimensions of* (4.1)*. Moreover, compared with* (4.17)*, our results depend on the time delay τ, which shows the characteristic of the equation.*

5 Retarded semilinear wave equations

This section is dedicated to dimension estimations of pullback attractors for the following retarded semilinear wave equation defined on an open bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$, with a smooth boundary $\partial\Omega = \Gamma$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \beta\dfrac{\partial u}{\partial t} - \Delta u - g(t, u_t) = f, \quad t > \tau \\ u_{|\Gamma} = 0, \quad t \geq \tau - r, \\ u(x,t) = \phi(x, t-\tau), \quad x\in\Omega, t\in[\tau - r, \tau] \\ \dfrac{\partial u}{\partial t}(x,t) = \psi(x, t-\tau), \quad x\in\Omega, t\in[\tau - r, \tau]. \end{cases} \tag{5.1}$$

Here, β is a positive constant, $f + g(t, u_t)$ is the source intensity which may depend on the history of the solution, ϕ is the initial datum on the interval $[\tau - r, \tau]$ where $r > 0$, and, as usual, u_t is defined for $\theta \in [-r, 0]$ as $u_t(\theta) = u(t+\theta)$ as well. The existence of global unique solutions and pullback attractors of (5.1) have been studied in [6] and [32] respectively. In the present work, we go a step further to establish explicit dimensions estimation of the pullback attractors. The notation in this subsection have the same meaning as those of Section 4.1.2 but with the domain being Ω. Thus, problem (5.1) can be written as a second order differential equation in H.

$$\begin{cases} u'' + \beta u' + Au - g(t, u_t) = f, \quad t > 0, \\ u(t) = \phi(t - \tau), \quad t \in [\tau - r, \tau], \\ u'(t) = \psi(t - \tau), \quad t \in [\tau - r, \tau]. \end{cases} \tag{5.2}$$

We first introduce more notations. Let Y be H or V, denote by C_Y the space $C^0([-\tau, 0]; Y)$ with the sup-norm, i.e., $\|\phi\|_{C_Y} = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_Y$, for $\phi \in C_Y$. Given another Banach space $(Z, \|\cdot\|_Z)$ such that the injection $Y \subset Z$ is continuous, we denote by $C_{Y,Z}$ the Banach space $C_Y \cap C^1([-\tau, 0]; Z)$ with the norm $\|\cdot\|_{C_{Y,Z}}$ defined by

$$\|\phi\|_{C_{Y,Z}}^2 = \|\phi\|_{C_Y}^2 + \|\phi'\|_{C_Z}^2, \quad \text{for } \phi \in C_{Y,Z}.$$

We will use the spaces $C_{D(A)}, C_V, X, C_{V,H}$ and $C_{D(A),V}$ in our analysis. Apart from **Hypothesis A4-Hypothesis A7**, we impose one more hypothesis on the function $g : \mathbb{R} \times X \to H$.

Hypothesis A8 $g \in C^1(\mathbb{R} \times X; H)$, and there exists $C > 0$ such that, for any $(t, \xi) \in \mathbb{R} \times X$ the Fréchet derivative $\delta g(t, \xi) \in \mathcal{L}(\mathbb{R} \times X, H)$ satisfies

$$\|\delta g(t, \xi)\|_{\mathcal{L}(\mathbb{R} \times X, H)} \leq C(1 + \|\xi\|_X)$$

By the results in [32] and [6], we have the following ones.

Lemma 5.1. *Assume that $f \in L^2_{loc}(\mathbb{R}, H), \phi \in C_{V,H}$ and g satisfies* **Hypothesis A4-Hypothesis A7**. *Then, for any $\tau \in \mathbb{R}$, there exists a unique solution $u(\cdot) = u(\cdot; \tau, \phi)$ to problem (5.1) such that $u \in C^0([\tau - r, \infty); V) \cap C^1([\tau - r, \infty); H)$. If in addition $f' \in L^2(\tau, T; H)$ for all $T > 0, \phi \in C_{D(A)}$, $\phi' \in C_V$, then*

$$u \in C^0([\tau, \infty); D(A)) \cap C^1([\tau, \infty); V).$$

In addition, suppose that **Hypothesis A8** *holds and $2\sqrt{2}\mu_1^{-1/2} C_g < \min\left\{\frac{\beta}{4}, \frac{\mu_1}{2\beta}\right\}$. Then, there exists a family $\{B(t)\}_{t \in \mathbb{R}}$ of bounded sets in $C_{V,H}$ which is uniformly pullback (and forward) absorbing for the process $U(\cdot, \cdot)$.*

Let V_m and P_m be defined in Section 3.2. Replace V and H by V_m and H_m gives the definiton of C_{V_m,H_m}, which is a finite dimensional subspace of the Banach space $C_{V,H}$. Define $\tilde{P}_m : C_{V,H} \to C_{V_m,H_m}$. Then we have the following results about the squeezing property.

Theorem 5.1. *Let P be the finite dimensional projection $\tilde{P}_m$ on $C_{V,H}$. Assume that* **Hypotheses A4 − A7** *hold for any $\tau \leqslant t$ with $m_0 > 0$,* **Hypothesis A8** *holds and $2\sqrt{2}\mu_1^{-1/2} C_g < \min\left\{\frac{\beta}{4}, \frac{\mu_1}{2\beta}\right\}$. Then there exists a constant $\gamma > 0$ such that*

$$\|PU(t, \tau)\varphi - PU(t, \tau)\psi\|_{C_{V,H}} \leq e^{-\mu_1 r}\left(1 + \lambda_1^{-1} C_g^2 r\right)^{\frac{1}{2}} e^{\frac{\gamma}{2}(t - \tau)} \|\phi - \psi\|_{C_{V,H}} \tag{5.3}$$

and

$$\|(I-P)U(t,\tau)\varphi-(I-P)U(t,\tau)\psi\|_{C_{V,H}} \le e^{-\mu_{m+1}r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}e^{\frac{\gamma}{2}(t-\tau)}\|\phi-\psi\|_{C_{V,H}}, \tag{5.4}$$

for any $t \geq \tau + r$, *and* φ, ψ *in* $\mathcal{A}(t)$, *where* μ_1, μ_{m+1} *and* C_g *are defined in* (3.26), (3.27) *and* **Hypothesis A7** *respectively.*

Proof. Let $\phi, \psi \in C_{V,H}$ be two initial data for our problem (5.2), and let $\tau \in \mathbb{R}$ be an initial time. Denote by $u(\cdot) = u(\cdot;\tau,\phi)$ and $v(\cdot) = u(\cdot;\tau,\psi)$ the corresponding solutions to (5.2). Then, it follows from [6, Lemma 3.1], there exists a constant $\gamma > 0$ which does not depend on the initial data and time, such that for all $t \geq \tau + r$

$$\|u_t-v_t\|^2_{C_{V,H}} \le \left(1+\mu_1^{-1}C_g^2r\right)e^{\gamma(t-\tau)}\|\phi-\psi\|^2_{C_{V,H}}, \tag{5.5}$$

implying that

$$\|u_t-v_t\|_{C_{V,H}} \le \left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}e^{\frac{\gamma}{2}(t-\tau)}\|\phi-\psi\|_{C_{V,H}}. \tag{5.6}$$

Thus, by (3.30) and (3.29), we have

$$\|(I-P)w_t\|^2_{C_{V,H}} \leqslant e^{-\mu_{m+1}r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}e^{\frac{\gamma}{2}(t-\tau)}\|\phi-\psi\|_{C_{V,H}}, \tag{5.7}$$

and

$$\|Pw_t\|_{C_{V,H}} = e^{-\mu_1 r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}e^{\frac{\gamma}{2}(t-\tau)}\|\phi-\psi\|_{C_{V,H}}. \tag{5.8}$$

Therefore the first part holds, which completes the proof of the theorem. □

Theorem 5.1 implies $(\mathcal{H}_3)$ holds with $M_1 = e^{-\mu_1 r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}$, $M_2 = e^{-\mu_{m+1}r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}$, $M_3 = 0, \lambda_0 = \mu_1 = \frac{\gamma}{2}$ Hence, by Theorem 2.2, we have the following results about existence of a pullback exponential attractor $\mathcal{M}(t)$ for the nonlinear evolution process $U(t,\tau)$ generated by (5.2).

Theorem 5.2. *Let* P *be the finite dimensional projection defined by* (3.28), γ *and* C_g *are defined in* (3.26), (3.27) *and* **Hypothesis A7** *respectively and assumptions of Lemma 5.1 and Theorem 5.1 hold. Moreover, assume there exists* $\alpha > 0$ *such that* $\zeta := (\alpha + e^{-\mu_{m+1}r}\left(1+\lambda_1^{-1}C_h^2r\right)^{\frac{1}{2}})e^{\frac{\gamma}{2}} < 1$. *Then,* (5.2) *admits a pullback exponential attractor* $\mathcal{M}(t)$ *whose fractal dimension has an upper bound*

$$\dim_f \mathcal{M}(t) \le \frac{\ln m + m\ln(2+\frac{2e^{-\mu_1 r}\left(1+\mu_1^{-1}C_g^2r\right)^{\frac{1}{2}}}{\alpha})}{-\ln(\alpha+e^{-\mu_{m+1}r}\left(1+\lambda_1^{-1}C_h^2r\right)^{\frac{1}{2}})-\frac{\gamma}{2}} < \infty. \tag{5.9}$$

6 Summary

In this paper, we established a new framework to construct exponential attractors for infinite dimensional nonautonomous dynamical systems in Banach spaces with explicit fractal dimension. To our best knowledge, there are two kinds of methods can be used to investigate exponential attractors of infinite dimensional dynamical systems, the first one is due to Eden, Foias, Nicolaenko and Temam [26], which depends on squeezing property and phase space decomposition and is established for PDEs in Hilbert spaces. Here, we extended this method to nonautonomous case in Banach spaces. The other well known method, which is also effective for constructing exponential attractors in Banach spaces was firstly proposed by Efendiev, Miranville and Zelik which depends on smoothing property and compact embedding of the systems between two spaces [28, 30]. Compared with their woks, the method here does not need the smoothing property and the entropy number of the embedding between two spaces but requires to some extend appropriate conditions on the spectrum gap and the Lipschitz constant of the nonlinear term. As pointed out in Constantin and Foias in [17], the dimensions estimation by squeezing property method may not be optimal and the more accurate method should be the Lyapunov exponents method which depends on computing traces of some linear operators generated by the linearization of the equations, requiring quasi-differentials of the underlying systems. Nevertheless, the method requires the smooth inner product of the Hilbert space geometric structure. How to extend this method to Banach spaces deserves much more effort since the Lyapunov exponents for evolution equations in Banach spaces may be not easy to obtain.

The constructed exponential attractors possess explicit fractal dimensions which do not depend on the entropy number but only on inner characteristics of the studied equations. The method shows a wide applicability to infinite dimensional dynamical systems generated by partial functional differential equations, including the retarded reaction diffusion equations, the retarded 2D-Navier-Stokes equations and the retarded semilinear wave equations. They maybe also available for investigating topological dimensions of attractors for neutral partial functional differential equations, the infinite delay case as well as some other evolution equations with certain squeeze properties in Banach spaces.

In the applications, we only consider partial functional differential equations on bounded domain. Actually, there are many real world process evolution on infinite domain, such as the mature population of species living in an infinite habitat. In such a scenario, the Laplace operator has a continuous spectrum, $H^1(\mathbb{R}^n)$ is not compactly embedded in $L^2(\mathbb{R}^n)$ and the solution semiflow do not have absorbing sets that are compact in the original topology, causing the method developed here no longer effective and new techniques should be established. This will be studied in an upcoming paper.

The definition of pullback attractors generally means which states in the infinite past will evolute to the given present state. Indeed, from causality perspective, we are more interested about what present states will evolute to in the future. The effective tools for describing the future evolution of nonautonomous system are the uniform attractor and uniform exponential attractors, which will also be studied in the near future.

Generally, random effects are omnipresent in mathematical modelings.Therefore, one anther question is, whether there are exponential attractors with explicit fractal dimension for partial functional differential equations perturbed by random effect, i.e. the stochastic partial functional differential equations(SPFDEs)? Indeed, even under what conditions do SPFDEs generate random dynamical systems have not been perfectly tackled needless to say the state decomposition and exponential dichotomy. This problem also deserves much effort in the future.

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