Convergence and Approximation of Invariant Measures for Neural Field Lattice Models under Noise Perturbation

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Abstract. This paper is mainly concerned with limiting behaviors of invariant measures for neural field lattice models in random environment. First of all, we consider the convergence relation of invariant measures between the stochastic neural field lattice model and the corresponding deterministic model in weighted spaces, and prove any limit of a sequence of invariant measures of such lattice model must be an invariant measure of its limiting system as the noise intensity tends to zero. Then we are devoted to studying the numerical approximation of invariant measure of such stochastic neural lattice model. To this end, we firstly consider convergence of invariant measures between such neural lattice model and the system with neurons only interacting with its n-neighborhood, then we further prove convergence relation of invariant measures between the system with n-neighborhood and its finite dimensional truncated system. By this procedure, the invariant measure of the stochastic neural lattice models can be approximated by the numerical invariant measure of finite dimensional truncated system based on the Backward Euler-Maruyama scheme. Therefore, the invariant measure of deterministic neural field lattice model can be observed by the invariant measure of BEM scheme when the noise is not negligible.

Keywords and phrases: Stochastic neural field lattice model; Weighted space; Nonlinear white noise; Invariant measure; Numerical invariant measure

1 Introduction

Lattice systems have wide applications in many areas, such as physics, biology sciences, pattern formation, etc. (see, for instance, [?, ?, ?] and the references therein). A system in reality is usually affected by uncertainty due to some external "noise", stochastic lattice systems with linear and nonlinear noises thus were studied in [?, ?, ?, ?] for the unweighted spaces and [?, ?] for the weighted ones.

Neural networks are receiving very much attention due to their importance in several interesting applications, such as image processing, optimization problems, associative memory and pattern recognition [?, ?, ?, ?]. For neural networks system with time delay, convergence properties of the equilibrium point have been extensively investigated, see, e.g. [?, ?]. Recently, an integral model was proposed to take into account a finite transmission speed as a space-dependent retardation [?], which was well established in computational neuroscience and known as the neural field model. Continuous neural field models may be also used to describe the average activity of neural populations by nonlinear integro-differential equations [?]. In order to emphasize the discrete characters of neural networks, a neural field lattice model was considered by Faye [?] and the existence and uniqueness of traveling front solutions were investigated. Such neural lattice model may not only be regarded as space discretization of a continuous neural field model,

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but also extends the famous Hopfield neural networks with finite neurons in [?]. In reference [?], the neural field lattice system with switching effects was formulated as a differential inclusion on a weighted space of infinite sequences. Recently, the authors investigated the existence of invariant measures in a weighted space for the following neural field lattice model driven by nonlinear white noise in [?]:

$$\begin{cases} du_i(t) = \Big(f_i\left(u_i\right) + \sum_{j\in\mathbb{Z}^d} k_{i,j}\phi\left(u_j\right) + g_i\Big)dt + \varepsilon\big(\lambda_i\left(u_i\right) + h_i\big)dW_i(t), \quad t > \tau, \\ u_i(\tau) = u_{\tau,i}, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, $u_\tau := (u_{\tau,i})_{i\in\mathbb{Z}^d}$ is the initial data. Here u_i represents the neural activity such as neural synapse of the ith node, $\varepsilon \in (0, 1]$ is a parameter representing the noise intensity, $f_i : \mathbb{R} \to \mathbb{R}$ describes the attenuation of neural activity of the ith node, $\phi : \mathbb{R} \to \mathbb{R}$ is the activation function, $k_{i,j}$ describes the connection strength from the jth to the ith node, and the time independent functions g_i and h_i describe the external forcing at the ith location for the drift and diffusion. We also refer the reader to [?, ?, ?, ?, ?, ?, ?, ?, ?, ?] for invariant measures of stochastic dynamical systems including lattice ones.

When the noise intensity $\varepsilon = 0$, (1.1) becomes the following deterministic neural field lattice model:

$$\begin{cases} du_i(t) = \Big(f_i\left(u_i\right) + \sum_{j\in\mathbb{Z}^d} k_{i,j}\phi\left(u_j\right) + g_i\Big)dt, \quad t > \tau, \\ u_i(\tau) = u_{\tau,i}. \end{cases} \tag{1.2}$$

We refer the reader to [?, ?, ?] for more results on deterministic neural lattice models and [?, ?] for invariant measures of deterministic or random dynamical systems. In this paper, we would like to observe numerically the invariant measure of (1.2) when the real world is regarded as intrinsically a little noisy. To this end, we will investigate the limiting behavior and numerical approximation of invariant measures of (1.1) from the following two aspects.

The first goal is to establish the convergence relation of invariant measures for the stochastic neural field lattice system (1.1) in a weighed space as the noise intensity $\varepsilon \to \varepsilon_0 \in [0, 1]$, which is called the zero-noise limits problem in the references [?] for $\varepsilon_0 = 0$. Such problem goes back to Kolmogorov [?], and is also referred as stochastic stability in monographs [?, ?]. The limiting behavior of invariant measures of stochastic equations has been discussed, e.g., see [?, ?, ?, ?], where invariant measures in [?, ?] were considered in the Hilbert space l^2 consisting of real-valued square summable bi-infinite sequences. We extend some results to a weighed space l^2_ρ. Such space satisfies $l^2 \subset l^\infty \subset l^2_\rho$ and hence contains many infinite sequences whose components are bounded and traveling wave solutions. By carrying out a careful analysis, two results are obtained as follows: we are first concerned with the tightness of the set of all invariant measures of (1.1) in l^2_ρ which is proved by uniform tail-estimates of solutions in l^2_ρ and the technique of stopping times as stated in [?], and then we prove any limit of a sequence of invariant measures of (1.1) must be the invariant measure of the limit system. According to [?], if one accepts that the world is intrinsically a little noisy, then such zero-noise limits are the observable invariant measures, which represent idealizations of what we see.

In order to make such observability computable, our second aim is to study the numerical invariant measure of (1.1). In [?], some computer aided estimates were used to approximate the stationary measure of a chaotic chemical reaction model with additive noise, and the estimation of numerical error was obtained by the method in [?]. Different from [?, ?], the Fourier approximation of invariant measures was investigated in [?]. One can also see [?, ?, ?] for numerical solutions and approximation of the invariant measures of finite dimensional stochastic differential equations. However, as far as we are aware, there is no result available for the numerical invariant measure of (1.1). Since the dimension of system (1.1) is infinite, we cannot discretize it directly to simulate by computer. To overcome this issue, we try to adopt the finite dimensional approximation method to deal with the numerical invariant measure of such infinite dimensional system, which is different from the method in [?]. More precisely, we will investigate numerical approximation of invariant measures of (1.1) from the following three steps.

Firstly, we consider the following case in which each neuron is only interacting with the neurons within its n-neighborhood:

$$\begin{cases} du_i(t) = \Big(f_i\left(u_i\right) + \sum_{j=i-n}^{i+n} k_{i,j}\phi\left(u_j\right) + g_i\Big)dt + \varepsilon\left(\lambda_i\left(u_i\right) + h_i\right)dW_i(t),\ t > \tau, \\ u_i(\tau) = u_{\tau,i}, \end{cases} \tag{1.3}$$

where $i \pm n := (i_1 \pm n, \dots, i_d \pm n) \in \mathbb{Z}^d$. It is worth mentioning that [?] is devoted to investigating the existence and the upper semi-continuity of random attractors for Hopfield-type neural lattice model with local n-neighborhood interconnections among neurons. Different from [?], we are concerned with the convergence of invariant measures of (1.3) as $n \to +\infty$. To this end, we first show the tightness of the set of all invariant measures of (1.3) for all $n \in \mathbb{Z}^+$ (see Lemma 5.2). Then we are going to prove the uniform convergence of solutions in probability. Due to different number of neurons and the weighted parameter ρ, the arguments in references [?, ?] cannot be used to verify it directly. In order to address this problem, we utilize some properties of ρ and the idea contained in the proof of [?, Lemma 4.2] to obtain the desired result (see Lemma 5.3). At last, we obtain any limit of a sequence of invariant measures of (1.3) must be an invariant measure of (1.1) as $n \to +\infty$ (see Theorem 3.2).

Secondly, we further consider the case in which the size of the neural network is finite. Noticing the total number of neurons we considered above is still infinitely large, then by truncating (1.3) directly, we obtain the following finite dimensional system

$$\begin{cases} du_i(t) = \Big(f_i\left(u_i\right) + \sum_{j=i-n}^{i+n} k_{i,j}\phi\left(u_j\right) + g_i\Big)dt + \varepsilon\left(\lambda_i\left(u_i\right) + h_i\right)dW_i(t),\ t > \tau, \\ u_i(\tau) = u_{\tau,i}, \end{cases} \tag{1.4}$$

where $i \in \mathbb{Z}_N^d := \{(i_1, \cdots, i_d)\,|\,i_1, \cdots, i_d \in \{-N, \cdots, 0, \cdots, N-1, N\}\}$ and $N \geq n$. It is worth mentioning that the idea of finite-dimensional approximations of equilibrium measures was firstly introduced in [?] for coupled map lattices. We apply such idea to consider finite-dimensional approximations of (1.3), and investigate the limiting behavior of invariant measures for (1.4) with respect to the number N of nodes. Similar to the above argument, we further prove the sequence of invariant measures of (1.4) must converge to an invariant measure of (1.3) as $N \to +\infty$ by the different proof from [?]. We would also like to point out that the limiting behavior of random attractors for (1.4) can be studied according to [?].

Finally, we investigate the numerical invariant measure of (1.4). Notice that, the Euler-Maruyama (EM) method was applied to investigate numerical solutions and approximation of the invariant measures of stochastic differential equations in [?, ?], where both the drift coefficients and the diffusion coefficients are required to be globally Lipschitz continuous. However in the locally Lipschitz case, EM numerical solutions to stochastic differential equations fail to be ergodic (see [?] for more details). Therefore, the Backward Euler-Maruyama (BEM) method was used to approximate the invariant measure in [?, ?, ?] where the drift coefficients do not need to satisfy a globally Lipschitz condition. Following this approach, we construct numerical approximations of the invariant measure of (1.4) in l_ρ^2. More precisely, one first needs to establish the existence and uniqueness of the invariant measure of the BEM scheme. To achieve it, the asymptotically attractive property of the solution of the BEM scheme in l_ρ^2 is proved under some additional conditions on λ_i, β_i and ρ_i, which play key roles in the proof of Lemma 5.8. Then we show that the invariant measure of the BEM scheme converges to the invariant measure of (1.4) in the sense of Wasserstein distance (see Theorem 3.4). As a consequence, the invariant measure of the original neural lattice model (1.1) can be approximated by the invariant measure of BEM scheme (3.1) (see Theorem 3.5).

In conclusion, the above convergence analysis shows that the invariant measure of zero-noise limit of (1.1) is numerically observable. These convergence relations between numerical invariant measure and invariant measures are given in Figure 1 below.

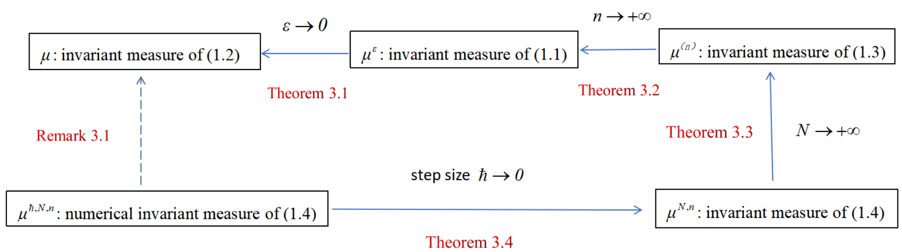

Figure 1: Convergence paths of invariant measures.

The structure of the paper is as follows. In Section 2, we first introduce a weighted Hilbert space and some necessary assumptions, as well as we prove the existence and uniqueness of solutions and the existence of invariant measures of the underlying system. Then we present some main results in Section 3. Section 4 is concerned with the convergence of invariant measures for system (1.1) as the noise intensity $\varepsilon \to \varepsilon_0 \in [0,1]$. Section 5 is devoted to establishing the numerical approximation of thet invariant measure of (1.1). we first show that invariant measures of system (1.3) converge weakly to invariant measures of system (1.1) as $n \to +\infty$ in subsection 5.1. Then we prove that invariant measures of system (1.4) converge weakly to invariant measures of system (1.3) as $N \to +\infty$ in subsection 5.2. Finally, we present that the invariant measure of BEM scheme converges weakly to that of system (1.4) in subsection 5.3. Therefore, the invariant measure of (1.2) can be approximated by the invariant measure of BEM scheme (3.1).

2 Preliminaries

In this section, we first present some assumptions, and then introduce the well-posedness of solutions as well as the existence of invariant measures of systems (1.1), (1.3) and (1.4).

2.1 Assumptions

First, we introduce some preliminaries and necessary assumptions.

(H1). Let $\rho = (\rho_i)_{i\in\mathbb{Z}^d}$ satisfy $\rho_{\mathrm{i}} > 0$ for all $i \in \mathbb{Z}^d$ and $\rho_\Sigma := \sum_{i\in\mathbb{Z}^d} \rho_i < +\infty$.

Consider the weighted space $l^2_\rho := \left\{ u = (u_i)_{i\in\mathbb{Z}^d} : \sum_{i\in\mathbb{Z}^d} \rho_i u_{\mathrm{i}}^2 < +\infty \right\}$ with the inner product $\langle u, v\rangle := \sum_{i\in\mathbb{Z}^d} \rho_i u_i v_i$ for $u = (u_i)_{i\in\mathbb{Z}^d}$, $v = (v_i)_{i\in\mathbb{Z}^d} \in l^2_\rho$ and norm $\|u\|_\rho := \sqrt{\sum_{i\in\mathbb{Z}^d} \rho_i u_i^2}$. It is easy to show l^2_ρ is a separable Hilbert space. Next, we introduce some assumptions which have been presented in [?].

(H2). There exists a constant $\kappa > 0$ such that $\sum_{j\in\mathbb{Z}^d} \frac{k^2_{i,j}}{\rho_j} \le \kappa$, $\forall i \in \mathbb{Z}^d$.

(H3). For each $i \in \mathbb{Z}^d$, $f_{\mathrm{i}} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $f_i(0) = 0$ and locally bounded derivatives, i.e., there exists a non-decreasing function $L_f(\cdot) \in \mathcal{C}\left(\mathbb{R}^+, \mathbb{R}^+\right)$ such that for any $r \in \mathbb{R}^+$ and $i \in \mathbb{Z}^d$, $\max_{\rho_i x\in[-r,r]} |f_i'(x)| \le L_f(r)$.

(H4). For each $i \in \mathbb{Z}^d$, the state dependent nonlinear diffusion term $\lambda_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and there exists a non-decreasing function $L_\lambda(\cdot) \in \mathcal{C}\left(\mathbb{R}^+, \mathbb{R}^+\right)$ such that for any $r \in \mathbb{R}^+$

and $i \in \mathbb{Z}^d$, $\max\limits_{\rho_i x\in[-r,r]} |\lambda_i'(x)| \le L_\lambda(r)$.
In addition, there exist $a = (a_i)_{i\in\mathbb{Z}^d} \in l^\infty$ and $b = (b_i)_{i\in\mathbb{Z}^d} \in l^2_\rho$ such that for any $x \in \mathbb{R}$, $|\lambda_i(x)| \le a_i|x|+b_i$.

(H5). The activation function ϕ is globally Lipschitz continuous with Lipschitz constant L_ϕ, and there exists $b_\phi > 0$ such that for any $x \in \mathbb{R}$, $|\phi(x)| \le L_\phi|x| + b_\phi$.

(H6). There exist $\alpha > 0$ and $\beta = (\beta_i)_{i\in\mathbb{Z}^d} \in l^2_\rho$ such that for any $x, y \in \mathbb{R}$ and $i \in \mathbb{Z}^d$, $(x-y)(f_i(x) - f_i(y)) \le -\alpha|x-y|^2 + \beta_i^2$.

For convenience, we define the operators F, $\mathcal{K}$ and Λ by $F(u) = (f_i(u_i))_{i\in\mathbb{Z}^d}$, $\Lambda(u) = (\lambda_i(u_i))_{i\in\mathbb{Z}^d}$ and $\mathcal{K}(u) = (K_i(u_i))_{i\in\mathbb{Z}^d}$ with $K_i(u_i) := \sum\limits_{j\in\mathbb{Z}^d} k_{i,j}\phi(u_j)$. Then $\mathcal{K}(u)$ is globally Lipschitz continuous by **(H5)**, and it follows from **(H4)** and **(H6)** that $F(u)$ and $\Lambda(u)$ satisfy locally Lipschitz condition, that is, for any $u, v \in l^2_\rho$ with $\|u\|^2_\rho \le R$, $\|v\|^2_\rho \le R$ and $R > 0$,

$$\|F(u) - F(v)\|^2_\rho \le L_f^2(2R\sqrt{\rho_\Sigma})\|u-v\|^2_\rho, \qquad \|\Lambda(u) - \Lambda(v)\|^2_\rho \le L_\lambda^2(2R\sqrt{\rho_\Sigma})\|u-v\|^2_\rho.$$

Similarly, define $\mathcal{K}^{(n)}(u) = \left(K_i^{(n)}(u_i)\right)_{i\in\mathbb{Z}^d}$ with $K_i^{(n)}(u_i) := \sum\limits_{j=i-n}^{i+n} k_{i,j}^{(n)}\phi(u_j)$, then $\|\mathcal{K}^{(n)}(u) - \mathcal{K}^{(n)}(v)\|^2_\rho \le \rho_\Sigma\kappa L_\phi^2\|u-v\|^2_\rho$. In particular, denote $F^N(u) := (f_i(u_i))_{i\in\mathbb{Z}^d_N}$, $G^N := (g_{\rm i})_{{\rm i}\in\mathbb{Z}^d_N}$ and $\mathcal{K}^N(u) := \left(K_i^{(n)}(u_i)\right)_{i\in\mathbb{Z}^d_N}$ with $K_i^{(n)}(u_i) := \sum\limits_{j=i-n}^{i+n} k_{i,j}^{(n)}\phi(u_j)$.

In order to rewrite the term $(\lambda_i(u_i) + h_i)\,dW_i(t)$ $(i \in \mathbb{Z}^d_N)$ as a vector in l^2_ρ, we define $\Lambda_i(u) = (\lambda_i(u_i))\,e_i$ and $H_i = (h_i)\,e_i$, where e_i represents the infinite sequence with 1 at position i and 0 elsewhere. Then $\Lambda(u) = \sum\limits_{i\in\mathbb{Z}^d}\Lambda_i(u)$ and $H = \sum\limits_{i\in\mathbb{Z}^d} H_i$ for every $u \in l^2_\rho$. Moreover, for all $u, v \in l^2_\rho$, there hold

$$\|\Lambda(u)\|^2_\rho = \sum_{i\in\mathbb{Z}^d}\|\Lambda_i(u)\|^2_\rho \quad \text{and} \quad \|\Lambda(u) - \Lambda(v)\|^2_\rho = \sum_{i\in\mathbb{Z}^d}\|\Lambda_i(u) - \Lambda_i(v)\|^2_\rho.$$

2.2 Well-posedness of solutions and the existence of invariant measures

Following the above procedures, (1.1), (1.3) and (1.4) can be rewritten respectively as:

$$\begin{cases} du(t) = (F(u(t)) + \mathcal{K}(u(t)) + G)\,dt + \varepsilon\sum\limits_{i\in\mathbb{Z}^d}(\Lambda_i(u) + H_i)\,dW_i(t), \\ u(\tau) = u_\tau = (u_{\tau,i})_{i\in\mathbb{Z}^d}, \end{cases} \tag{2.1}$$

$$\begin{cases} du(t) = \left(F(u(t)) + \mathcal{K}^{(n)}(u(t)) + G\right)dt + \varepsilon\sum\limits_{i\in\mathbb{Z}^d}(\Lambda_i(u) + H_i)\,dW_i(t), \\ u(\tau) = u_\tau = (u_{\tau,i})_{i\in\mathbb{Z}^d} \end{cases} \tag{2.2}$$

and

$$\begin{cases} du(t) = \left(F^N(u(t)) + \mathcal{K}^N(u(t)) + G^N\right)dt + \varepsilon\sum\limits_{i\in\mathbb{Z}^d_N}(\Lambda_i(u) + H_i)\,dW_i(t), \\ u(\tau) = u_\tau = (u_{\tau,i})_{i\in\mathbb{Z}^d_N}. \end{cases} \tag{2.3}$$

With these assumptions as well as the discussion of Theorem 2.3 in [**?**], we have

Theorem 2.1. *Let* **(H1)**-**(H6)** *hold. Then, for any $\tau \in \mathbb{R}$ and $\mathcal{F}_\tau$-measurable initial data $u_\tau \in \mathcal{L}^2(\Omega, l^2_\rho)$, the stochastic system (2.1) possesses a unique solution $u \in \mathcal{L}^2\left(\Omega, \mathcal{C}\left([\tau, \tau+T], l^2_\rho\right)\right)$ and satisfies, for all $t \ge \tau$ and almost $\omega \in \Omega$,*

$$u(t) = u_\tau + \int_\tau^t (F(u(s)) + \mathcal{K}(u(s)) + G(s))\mathrm{d}s + \varepsilon\sum_{i\in\mathbb{Z}^d}\int_\tau^t (\Lambda_i(u(s)) + H_i(s))\,\mathrm{d}W_i(s).$$

Remark 2.1. *As special cases of Theorem 2.1, for any $\tau \in \mathbb{R}$ and initial data $u_\tau \in \mathcal{L}^2\left(\Omega, l_\rho^2\right)$, systems (2.2) and (2.3) possess a unique solution $u^{(n)}, u^{N,n} \in \mathcal{L}^2\left(\Omega, \mathcal{C}\left([\tau, \tau+T], l_\rho^2\right)\right)$, respectively.*

Next, we introduce the existence of invariant measures for stochastic systems (2.1), (2.2) and (2.3). More details on the concept of invariant measure, one can see [?], so we omit it here.

(H7). $2L_\phi\sqrt{2\kappa\rho_\Sigma} + 4\|a\|_\infty^2 < \alpha$.

Theorem 2.2 ([?] Theorem 4.6)**.** *Let* **(H1)**-**(H7)** *hold. Then the stochastic system (2.1) has an invariant measure on l_ρ^2, that is, there exists a probability measure μ^ε on l_ρ^2 such that for any bounded and continuous function $\varphi : l_\rho^2 \to \mathbb{R}$, $\int_{l_\rho^2}\left(\int_{l_\rho^2}\varphi(v)p(\tau, u; t, \mathrm{d}v)\right)\mathrm{d}\mu^\varepsilon(u) = \int_{l_\rho^2}\varphi(u)\mathrm{d}\mu^\varepsilon(u)$ for $t \geq \tau$.*

Remark 2.2. *As an immediate consequence of Theorem 2.2, we obtain the stochastic systems (2.2) and (2.3) have probability measures $\mu^{(n)}, \mu^{N,n}$ on l_ρ^2, respectively.*

3 Main results

In this section, we will state the main results in this paper. We begin this section with the following theorem which shows the limiting behavior of invariant probability measures of system (2.1) as the noise intensity $\varepsilon \to \varepsilon_0 \in [0, 1]$.

Theorem 3.1. *Let* **(H1)**-**(H7)** *hold. If $\varepsilon_n \to \varepsilon_0 \in [0,1]$ and $\mu^{\varepsilon_n} \in \mathcal{S}^{\varepsilon_n}$, then there exist a subsequence ε_{n_k} and an invariant measure $\mu^{\varepsilon_0} \in \mathcal{S}^{\varepsilon_0}$ such that $\mu^{\varepsilon_{n_k}} \longrightarrow \mu^{\varepsilon_0}$ weakly.*

This proof is contained in Section 4.

Noting that the dimension of system (2.1) is infinite, it is natural to consider adopting the finite dimensional approximation method to deal with the numerical invariant measure of such infinite dimensional system. Firstly, we investigate the limiting behavior of invariant measures of system (2.2) as the interconnection parameter $n \to +\infty$. For that, we need extra assumptions on the connection strength $k_{i,j}$ and activation function ϕ:

(H8). $k_{i,j}^{(n)} \to k_{i,j}$ as $n \to +\infty$ in the sense that for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that for any $n \geq N(\epsilon)$ and $\in \mathbb{Z}^d$, $\sum_{j\in\mathbb{Z}^d}\frac{\left(k_{i,j}^{(n)} - k_{i,j}\right)^2}{\rho_j} \leq \epsilon$.

(H9). ϕ can be bounded in the sense that there exists b_ϕ such that for any $x \in \mathbb{R}$, $|\phi(x)| \leq b_\phi$.

Theorem 3.2 is concerned with the limiting behavior of invariant measure of (2.2) as $n \to +\infty$, which is different from [?, Theorem 6.1] where the authors deal with the case $\varepsilon \to \varepsilon_0$.

Theorem 3.2. *Let* **(H1)**-**(H9)** *hold, and $\mu^{(n)} \in \mathcal{S}^{(n)}$, $n \in \mathbb{Z}^+$. Then there exist a subsequence $\{n_k\}_{k=1}^{+\infty}$ and an invariant probability measure μ to (2.1) such that $\mu^{(n_k)} \longrightarrow \mu$ weakly as $k \to +\infty$.*

This proof is contained in subsection 5.1. We will find, by Theorem 3.2, the invariant measure of (2.1) with infinite neighborhoods can be approximated by that of stochastic neural field lattice system with finite neighborhoods.

Next, we further investigate whether invariant probability measures of (2.3) converge to invariant probability measures of (2.2) as the size N tends to infinity, which is important for numerical approximations of invariant measures to (2.2).

Theorem 3.3. *Let* **(H1)**-**(H9)** *hold, and $\mu^{N,n} \in \mathcal{S}^{N,n}$, $N \in \mathbb{Z}^+$. Then there exist a subsequence $\{N_k\}_{k=1}^{+\infty}$ and a probability measure μ^* such that $\mu^{N_k,n} \longrightarrow \mu^*$ weakly as $k \to +\infty$. Furthermore, Lemma 5.5 implies that μ^* must be an invariant probability measure of (2.2).*

This proof is contained in subsection 5.2.

Let $\mathcal{P}(\mathbb{R}^{2N+1})$ and $\mathcal{P}(l_\rho^2)$ denote the family of all probability measures on $\mathbb{R}^{2N+1}$ and l^2_ρ, respectively. The Wasserstein distance between ν and $\tilde{\nu} \in \mathcal{P}\left(l^2_\rho\right)$ can be defined by

$$W_2(\nu,\tilde{\nu}) := \Big[\inf_{\pi\in C(\nu,\tilde{\nu})}\int_{l^2_\rho\times l^2_\rho}\|\nu_1-\nu_2\|^2_\rho \pi(d\nu_1,d\nu_2)\Big]^{\frac{1}{2}},$$

where $C(\nu,\tilde{\nu})$ denotes the set of all couplings of ν and $\tilde{\nu}$. In addition, any Borel probability measure ν on $\mathbb{R}^{2N+1}$ can be naturally extended to a Borel probability measure ν^* on l^2_ρ. Then for any ν and $\tilde{\nu}\in\mathcal{P}\left(\mathbb{R}^{2N+1}\right)$, the Wasserstein distance between ν^* and $\tilde{\nu}^*\in\mathcal{P}\left(l^2_\rho\right)$ can be defined by $W_2(\nu^*,\tilde{\nu}^*) := W_2(\nu,\tilde{\nu})$.

Define the BEM scheme

$$\begin{cases} X_{k+1} = X_k + \left(F^N\left(X_{k+1}\right) + \mathcal{K}^N\left(X_{k+1}\right) + G^N\right)\hbar + \varepsilon \sum_{i\in\mathbb{Z}^d_N}\left(\Lambda_i(X_k)+H_i\right)\Delta W_{ik}, \\ X_0 = x, \end{cases} \tag{3.1}$$

where $k\geq 0$, $\hbar>0$ is step size, $X_k := X_{t_k} = X_{k\hbar}$, $x = u^{N,n}_\tau = (u^{N,n}_{\tau,i})_{i\in\mathbb{Z}^d_N}$ and $\Delta W_{ik} = W_{it_{k+1}} - W_{it_k}$.

Now, we are going to establish the existence and uniqueness of the invariant measure of the BEM scheme and approximation of such invariant measure to that of (2.3) in the Wasserstein metric. To this end, we further have the following assumptions.

(H10). λ_i is globally Lipschitz continuous with Lipschitz constant L_λ.

(H11). $\varepsilon^2 L^2_\lambda + 2\sqrt{\rho_\Sigma\kappa}L_\phi - 2\alpha < 0$ and $2\sqrt{2\rho_\Sigma\kappa}a_\phi - \alpha < -\dfrac{1}{8}$.

Theorem 3.4. *Let* **(H1)**-**(H11)** *hold, then we have*

$$\lim_{\hbar\to 0} W_2(\mu^{N,n},\mu^{\hbar,N,n}) = 0.$$

This proof is contained in subsection 5.3. Together with Theorems 3.2-3.4, we can obtain the following result.

Theorem 3.5. *Let* **(H1)**-**(H11)** *hold and $\beta_i = 0$ for $i\in\mathbb{Z}^d$. Then the original neural lattice model* (1.1) *has a unique invariant measure μ, and $\lim_{n\to+\infty}\lim_{N\to+\infty}\lim_{\hbar\to 0}\mu^{\hbar,N,n} = \mu$ weakly.*

This proof is contained in subsection 5.3.

Remark 3.1. *Under Assumptions* **(H1)**-**(H11)** *and $\beta_i = 0$ for $i\in\mathbb{Z}^d$, we can know the invariant measures of* (1.2)-(1.4) *and* (3.1) *are unique, respectively, which mean not only these invariant measures are ergodic, but also every sequence $\mu^{\varepsilon_n}\to\mu^{\varepsilon_0}$ weakly as $n\to+\infty$ in Theorem 3.1, $\mu^{(n)}\to\mu$ weakly as $n\to+\infty$ in Theorem 3.2, and $\mu^{N,n}\to\mu^*$ weakly as $N\to+\infty$ in Theorem 3.3. In this sense, the unique invariant measure of* (1.2) *can be approximated by the invariant measure of BEM scheme* (3.1) *from Theorem 3.1 and Theorem 3.5.*

*In addition, the unique physical measure was investigated for globally coupled Anosov diffeomorphisms in [**?**] based on the Lasota-Yorke inequalities. According to [**?**, Definition 2.1 and Remark 2.2] together with the ergodicity of μ, we need prove the absolute continuity of μ with respect to a Lebesgue measure to show μ is a physical measure, which will be one of our future works.*

Remark 3.2. *As the dimension of the finite dimensional reduction goes to infinity, the problem of computing the related measure might become numerically impossible to solve. To this end, we will try to estimate the convergent rate of invariant measures in every step approximation by referring to [**?**] in the following work, by which the problem of computing the related invariant measures might become numerically possible to solve.*

4 Proof of Theorem 3.1

This section starts from the weighted tail estimate of solutions of system (2.1) below.

Lemma 4.1. *Let Assumptions* **(H1)**-**(H7)** *hold. Then, for every $R>0$ and $\epsilon>0$, there exist $T=T(R,\epsilon)>\tau$ and $N=N(\epsilon)\geq 1$ such that the solution u satisfies, for all $t\geq T, n\geq N$ and $\varepsilon\in(0,1]$,*

$$\mathbb{E}\Big(\sum_{|i|\geq n}\rho_i\left|u_i\left(t,u_\tau\right)\right|^2\Big)<\epsilon,$$

where $u_\tau\in L^2\left(\Omega,\mathcal{F}_\tau;l^2_\rho\right)$ with $\mathbb{E}\left(\|u_\tau\|_\rho^2\right)\leq R$.

Proof. Let $\varsigma:\mathbb{R}\to[0,1]$ be a smooth function such that $\varsigma(s)=\begin{cases}0, & |s|\leq 1;\\ 1, & |s|\geq 2.\end{cases}$ Given $n\in\mathbb{N}$, define ς_n by $\varsigma_n u=\left(\varsigma\left(\frac{|i|}{n}\right)u_i\right)_{i\in\mathbb{Z}^d}$. By Itô's formula, we deduce that for any $t\geq\tau$,

$$\begin{aligned}\|\varsigma_n u(t)\|_\rho^2=&\|\varsigma_n u_\tau\|_\rho^2+2\int_\tau^t\left(\varsigma_n u(s),\varsigma_n F(u(s))\right)ds+2\int_\tau^t\left(\varsigma_n u(s),\varsigma_n\mathcal{K}(u(s))\right)ds\\&+2\int_\tau^t\left(\varsigma_n u(s),\varsigma_n G\right)ds+\varepsilon^2\int_\tau^t\|\varsigma_n\Lambda(u(s))+\varsigma_n H\|_\rho^2\,ds\\&+2\varepsilon\sum_{i\in\mathbb{Z}^d}\int_\tau^t\left(\varsigma_n u(s),\left(\varsigma_n\Lambda_i(u)+\varsigma_n H_i\right)\right)\mathrm{d}W_i(s).\end{aligned}$$

Then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\|\varsigma_n u(t)\|_\rho^2\right)=&2\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n F(u(t))\right)\right)+2\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n\mathcal{K}(u(t))\right)\right)\\&+2\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n G\right)\right)+\varepsilon^2\mathbb{E}\left(\|\varsigma_n\Lambda(u(t))+\varsigma_n H\|_\rho^2\right).\end{aligned}\tag{4.1}$$

By **(H6)**, the first term on the right-hand side of (4.1) can be bounded as

$$\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n F(u(t))\right)\right)\leq-\alpha\mathbb{E}\left(\|\varsigma_n u\|_\rho^2\right)+\sum_{|i|\geq n}\rho_i\beta_i^2.\tag{4.2}$$

For the second term on the right-hand side of (4.1), we derive by **(H1)**, **(H2)** and **(H5)** that

$$\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n\mathcal{K}(u(t))\right)\right)\leq\frac{\alpha}{8}\mathbb{E}\left(\|\varsigma_n u(t)\|_\rho^2\right)+\frac{4}{\alpha}\kappa\left(L_\phi^2\mathbb{E}\left(\|u(t)\|_\rho^2\right)+b_\phi^2\rho_\Sigma\right)\sum_{|i|\geq n}\rho_i.\tag{4.3}$$

As for the third term on the right-hand side of (4.1),

$$\mathbb{E}\left(\left(\varsigma_n u(t),\varsigma_n G\right)\right)\leq\frac{\alpha}{8}\mathbb{E}\left(\|\varsigma_n(u(t))\|_\rho^2\right)+\frac{2}{\alpha}\mathbb{E}\Big(\sum_{|i|\geq n}\rho_i g_i^2\Big).\tag{4.4}$$

For the last term on the right-hand side of (4.1), by **(H4)**, we obtain

$$\varepsilon^2\mathbb{E}\left(\|\varsigma_n\Lambda(u(t))+\varsigma_n H\|_\rho^2\right)\leq 4\|a\|_\infty^2\mathbb{E}\left(\|\varsigma_n u(t)\|_\rho^2\right)+4\varepsilon^2\sum_{|i|\geq n}\rho_i b_i^2+2\varepsilon^2\sum_{|i|\geq n}\rho_i h_i^2.\tag{4.5}$$

Combining (4.2)-(4.5) with (4.1) and then using Gronwall's inequality, implies that

$$\begin{aligned}\mathbb{E}\left(\|\varsigma_n u(t)\|_\rho^2\right)\leq&e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left(\|\varsigma_n u_\tau\|_\rho^2\right)+\frac{8}{\alpha}\kappa L_\phi^2\sum_{|i|\geq n}\rho_i\int_\tau^t e^{\frac{\alpha}{2}(s-t)}\mathbb{E}\left(\|u(s)\|_\rho^2\right)ds\\&+\frac{4}{\alpha}\sum_{|i|\geq n}\rho_i\Big(\beta_i^2+\frac{2}{\alpha}g_i^2+2\varepsilon^2 b_i^2+\varepsilon^2 h_i^2+\frac{4}{\alpha}\kappa b_\phi^2\rho_\Sigma\Big).\end{aligned}\tag{4.6}$$

Since $\mathbb{E}\big(\|u_\tau\|_\rho^2 \big) \le R$, we have for every $\epsilon > 0$, there exists $T_1 = T_1(R, \epsilon) > \tau$ such that, for all $t \ge T_1$,

$$e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\big(\|\varsigma_n u_\tau\|_\rho^2 \big) < \frac{\epsilon}{3}. \tag{4.7}$$

Applying Itô's formula to (2.1) and taking expectation, we obtain that there exists $T_2 = T_2(R) > \tau$ such that for all $t \ge T_2$,

$$\begin{aligned}\mathbb{E}\big(\|u(t)\|_\rho^2 \big) \le & e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\big(\|u_\tau\|_\rho^2\big) + \frac{4}{\alpha}\big(\|\beta\|_\rho^2 + \frac{\rho_\Sigma b_\phi^2}{L_\phi}\sqrt{2\kappa\rho_\Sigma} + 2\varepsilon^2\|b\|_\rho^2\big) \\ & + C\left(\|G\|_\rho^2 + \varepsilon^2\|H\|_\rho^2\right)\int_\tau^t e^{\frac{\alpha}{2}(s-t)}ds,\end{aligned}$$

from which there exists $N_2 = N_2(\epsilon) \ge 1$ such that for all $t \ge T_2$ and $n \ge N_2$,

$$\sum_{|i|\ge n}\rho_i\int_\tau^t e^{\frac{\alpha}{2}(s-t)}\mathbb{E}\left(\|u(s)\|_\rho^2\right)ds \le \sup_{s\ge\tau}\mathbb{E}\left(\|u(s)\|_\rho^2\right)\sum_{|i|\ge n}\rho_i\int_\tau^t e^{\frac{\alpha}{2}(s-t)}ds \le \frac{\epsilon}{3}. \tag{4.8}$$

On the other hand, since $\beta, b, H, G \in l_\rho^2$, it follows from **(H1)** that there exists $N_3 = N_3(\epsilon) \ge 1$ such that for all $n \ge N_3$,

$$\sum_{|i|\ge n}\rho_i\big(\beta_i^2 + \frac{2}{\alpha}g_i^2 + 2\varepsilon^2 b_i^2 + \varepsilon^2 h_i^2 + \frac{4}{\alpha}\kappa b_\phi^2\rho_\Sigma\big) < \frac{\epsilon}{3}. \tag{4.9}$$

From (4.6)-(4.9), it follows that for every $\epsilon > 0$, there exist $N = \max\{N_1, N_2, N_3\}$ and $T = \max\{T_1, T_2\}$ such that $\mathbb{E}\big(\sum_{|i|\ge 2n}\rho_i\,|u_i(t, u_\tau)|^2 \big) \le \mathbb{E}\big(\|\varsigma_n u(t)\|_\rho^2\big) < \epsilon$ for all $n \ge N$, $t \ge T$ and $\varepsilon \in (0, 1]$. ☐

Let $\mathcal{S}^\varepsilon$ be the collection of all invariant measures of (2.1) with $\varepsilon \in (0, 1]$. By Theorem 2.2 we see that $\mathcal{S}^\varepsilon$ is nonempty. We now prove the union $\bigcup_{\varepsilon\in(0,1]}\mathcal{S}^\varepsilon$ is tight.

Lemma 4.2. *Let* **(H1)**-**(H7)** *hold. Then* $\bigcup_{\varepsilon\in(0,1]}\mathcal{S}^\varepsilon$ *is tight.*

Proof. Given $\varphi \in l_\rho^2$, denote $\tilde{u}^{\varepsilon,n}(t, \varphi) = \big(1_{[-n,n]}(k)u_k^\varepsilon(t, \varphi)\big)_{k\in\mathbb{Z}}$ and $\hat{u}^{\varepsilon,n}(t, \varphi) = \big(\big(1 - 1_{[-n,n]}(k)\big)\,u_k^\varepsilon(t, \varphi)\big)_{k\in\mathbb{Z}}$, where $n \in \mathbb{N}$, $1_{[-n,n]}$ is the characteristic function of $[-n, n]$. By Lemma 4.1, we find that for every $\epsilon \in (0, 1)$, $k \in \mathbb{N}$ and $\varphi \in l_\rho^2$, there exist $T_k = T_k(\epsilon', k, \varphi) > \tau$ and $n_k = n_k(\epsilon', k) \ge 1$ such that for all $t \ge T_k$ and $\varepsilon \in (0, 1]$, $\mathbb{E}\left(\|\hat{u}^{\varepsilon,n_k}(t, \varphi)\|_\rho^2\right) \le \frac{\epsilon'}{2^{4k}}$.
On the other hand, by the estimates of solutions to (2.1), we obtain that

$$\mathbb{E}\left(\|u^\varepsilon(t)\|_\rho^2\right) \le e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left(\|\varphi\|_\rho^2\right) + \frac{2(t-\tau)}{\alpha}\Big(2\|\beta\|_\rho^2 + \frac{2\rho_\Sigma b_\phi^2}{a_\phi}\sqrt{2\kappa\rho_\Sigma} + \frac{2}{\alpha}\|G\|_\rho^2\Big) + 4\varepsilon^2\|b\|_\rho^2 + 2\varepsilon^2\|H\|_\rho^2.$$

Then, we see that there exist $T_1 = T_1(\varphi) > \tau$ and M independent of φ and ε, such that for all $t \ge T_1$ and $\varepsilon \in (0, 1]$, $\mathbb{E}\left(\|u^\varepsilon(t, \varphi)\|_\rho^2\right) \le M$. Following the procedure as stated in [?], we obtain the desired result. ☐

The next result is concerned with the convergence of solutions to (2.1) with respect to ε.

Lemma 4.3. *Let* **(H1)**-**(H7)** *hold. Then, for every bounded subset E in l_ρ^2, $T > 0$, $\sigma > 0$ and $\varepsilon_0 \in [0, 1]$,*

$$\lim_{\varepsilon\to\varepsilon_0}\sup_{\varphi\in E}\mathbb{P}\Big(\Big\{\omega \in \Omega \mid \sup_{\tau\le t\le\tau+T}\|u^\varepsilon(t, \varphi) - u^{\varepsilon_0}(t, \varphi)\|_\rho \ge \sigma\Big\}\Big) = 0.$$

Proof. Following the stopping time idea in [?], we need only to prove

$$\lim_{\varepsilon\to\varepsilon_0}\sup_{\varphi\in E}\mathbb{P}\Big(\Big\{\omega \in \Omega \mid \sup_{\tau\le t\le\tau+T}\|u^\varepsilon(t\wedge\tau_R^\varepsilon, \varphi) - u^{\varepsilon_0}(t\wedge\tau_R^\varepsilon, \varphi)\|_\rho \ge \sigma\Big\}\Big) = 0,$$

where $\tau_R^\varepsilon = \inf_{t\geq\tau}\{\|u^\varepsilon(t,\varphi)\|_\rho \vee \|u^{\varepsilon_0}(t,\varphi)\|_\rho > R\}$, $\tau_R^\varepsilon=\infty$ if $\{t\geq\tau : \|u^\varepsilon(t,\varphi)\|_\rho \vee \|u^{\varepsilon_0}(t,\varphi)\|_\rho > R\}=\emptyset$.

By applying Itô's formula to $u^\varepsilon(t,\varphi)-u^{\varepsilon_0}(t,\varphi)$ and then taking expectation, we derive

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big) \\
\leq& 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^\varepsilon}(F(u^\varepsilon)-F(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0})ds\Big|\Big)+2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^\varepsilon}(\mathcal{K}(u^\varepsilon)-\mathcal{K}(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0})ds\Big|\Big) \\
&+\sum_{i\in\mathbb{Z}^d}\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^\varepsilon}\|(\varepsilon-\varepsilon_0)(\Lambda_i(u^{\varepsilon_0})+H_i)+\varepsilon(\Lambda_i(u^\varepsilon)-\Lambda_i(u^{\varepsilon_0})\|_\rho^2ds\Big) \\
&+2|\varepsilon-\varepsilon_0|\mathbb{E}\Big(\sup_{\tau\leq r\leq t}|\sum_{i\in\mathbb{Z}^d}\int_\tau^{r\wedge\tau_R^\varepsilon}(\Lambda_i(u^{\varepsilon_0})+H_i,u^\varepsilon-u^{\varepsilon_0})dW_i(s)|\Big) \\
&+2\varepsilon\mathbb{E}\Big(\sup_{\tau\leq r\leq t}|\sum_{i\in\mathbb{Z}^d}\int_\tau^{r\wedge\tau_R^\varepsilon}(\Lambda_i(u^\varepsilon)-\Lambda_i(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0})dW_i(s)|\Big).
\end{aligned}
\tag{4.10}
$$

For the first two terms on the right-hand side of (4.10), we have

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^\varepsilon}(F(u^\varepsilon)-F(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0})ds\Big|\Big) \\
\leq& L_f(2R\sqrt{\rho_\Sigma})\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big)ds
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^\varepsilon}\left(\mathcal{K}(u^\varepsilon)-\mathcal{K}(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0}\right)ds\Big|\Big) \\
\leq& \sqrt{\rho_\Sigma\kappa}L_\phi\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big)ds.
\end{aligned}
\tag{4.12}
$$

For the third term on the right-hand side of (4.10), we find

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}^d}\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^\varepsilon}\|(\varepsilon-\varepsilon_0)(\Lambda_i(u^{\varepsilon_0})+H_i)+\varepsilon(\Lambda_i(u^\varepsilon)-\Lambda_i(u^{\varepsilon_0}))\|_\rho^2ds\Big) \\
\leq& 4(\varepsilon-\varepsilon_0)^2\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^\varepsilon}(2R\|a\|_\infty+2\|b\|_\rho^2+\|H\|_\rho^2)ds\Big) \\
&+2\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big)ds.
\end{aligned}
\tag{4.13}
$$

By the Burkholder-Davis-Gundy inequality, we obtain the fourth term on the right-hand side of (4.10)

$$
\begin{aligned}
&|\varepsilon-\varepsilon_0|\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\Big|\sum_{i\in\mathbb{Z}^d}\int_\tau^{r\wedge\tau_R^\varepsilon}(\Lambda_i(u^{\varepsilon_0})+H_i,u^\varepsilon-u^{\varepsilon_0})dW_i(s)\Big|\Big) \\
\leq& \frac{1}{8}\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big)+C_1^2|\varepsilon-\varepsilon_0|^2\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^\varepsilon}(2R\|a\|_\infty+2\|b\|_\rho^2+\|H\|_\rho^2)ds\Big).
\end{aligned}
\tag{4.14}
$$

Similarly, the last term on the right-hand side of (4.10) can be bounded by

$$
\begin{aligned}
&2\varepsilon\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\Big|\sum_{i\in\mathbb{Z}^d}\int_\tau^{r\wedge\tau_R^\varepsilon}(\Lambda_i(u^\varepsilon)-\Lambda_i(u^{\varepsilon_0}),u^\varepsilon-u^{\varepsilon_0})dW_i(s)\Big|\Big) \\
\leq& \frac{1}{4}\mathbb{E}\Big(\sup_{\tau\leq r\leq t}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big) \\
&+\varepsilon^2C_3\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leq r\leq s}\|u^\varepsilon(r\wedge\tau_R^\varepsilon,\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^\varepsilon,\varphi)\|_\rho^2\Big)ds.
\end{aligned}
\tag{4.15}
$$

By (4.10) and (4.11)-(4.15), we have that for all $t \in [\tau, \tau+T]$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{\tau\le r\le t}\|u^{\varepsilon}(r\wedge\tau_R^{\varepsilon},\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^{\varepsilon},\varphi)\|_\rho^2\right) \\
\le& 2\left[2L_f(2R\sqrt{\rho_\Sigma})+2\sqrt{\rho_\Sigma\kappa}L_\phi+\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})+\varepsilon^2C_3\right] \\
&\cdot\int_\tau^t\mathbb{E}\left(\sup_{\tau\le r\le s}\|u^{\varepsilon}(r\wedge\tau_R^{\varepsilon},\varphi)-u^{\varepsilon_0}(r\wedge\tau_R^{\varepsilon},\varphi)\|_\rho^2\right)ds \\
&+4(\varepsilon-\varepsilon_0)^2(2+C_1^2)(2R\|a\|_\infty+2\|b\|_\rho^2+\|H\|_\rho^2)T.
\end{aligned} \tag{4.16}$$

Then, by (4.16) and Gronwall's inequality, we have

$$\begin{aligned}
&\sup_{\varphi\in E}\mathbb{P}\Big(\Big\{\omega\in\Omega \mid \sup_{\tau\le t\le\tau+T}\|u^{\varepsilon}(t\wedge\tau_R^{\varepsilon},\varphi)-u^{\varepsilon_0}(t\wedge\tau_R^{\varepsilon},\varphi)\|_\rho\ge\sigma\Big\}\Big) \\
\le& 4(\varepsilon-\varepsilon_0)^2(2+C_1^2)(2R\|a\|_\infty+2\|b\|_\rho^2+\|H\|_\rho^2)Te^{\left[4L_f(2R\sqrt{\rho_\Sigma})+4\sqrt{\rho_\Sigma\kappa}L_\phi+2\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})+2\varepsilon^2C_3\right]T}\to 0,
\end{aligned}$$

as $\varepsilon\to\varepsilon_0$, as desired. □

Now we present a proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 4.2, $\bigcup_{\varepsilon\in(0,1]}\mathcal{S}^{\varepsilon}$ is tight, which together with the fact $\mu^{\varepsilon_n}\in\bigcup_{n=1}^{+\infty}\mathcal{S}^{\varepsilon_n}$ implies that there exist a subsequence ε_{n_k} and a probability measure μ^{ε_0} such that $\mu^{\varepsilon_{n_k}}\longrightarrow\mu^{\varepsilon_0}$ weakly. It follows from Lemma 4.3 and [?, Theorem 6.1] that μ^{ε_0} is an invariant measure such that $\mu^{\varepsilon_0}\in\mathcal{S}^{\varepsilon_0}$. □

5 Numerical approximation of invariant measures for (2.1)

This section mainly aims to obtain the numerical approximation of invariant measures for (2.1) by proving Theorems 3.2-3.4. More precisely, our analysis is divided into the following three subsections.

5.1 Proof of Theorem 3.2

To prove Theorem 3.2, we first present some results that are crucial to prove the convergence of the sequence of invariant measures.

By (4.3) and (4.6) in the proof of Lemma 4.1, we conclude the following lemma.

Lemma 5.1. *Let* **(H1)**-**(H7)** *hold. Then for given $\varepsilon\in(0,1]$, the solution $u^{(n)}$ satisfies that for every $R>0$ and $\epsilon>0$, there exist $T=T(R,\epsilon)>\tau$ and $K=K(\epsilon)\ge1$ such that for all $t\ge T$, $k\ge K$ and $n\in\mathbb{Z}^+$,*

$$\mathbb{E}\Big(\sum_{|i|\ge k}\rho_i|u_i^{(n)}(t,u_\tau)|^2\Big)<\epsilon,$$

where $u_\tau\in L^2(\Omega,\mathscr{F}_\tau;l_\rho^2)$ with $\mathbb{E}\big(\|u_\tau\|_\rho^2\big)\le R$.

Let $\mathcal{S}^{(n)}$ be the collection of all invariant probability measures of (2.2) with n-neighborhood. By Remark 2.2 we deduce that $\mathcal{S}^{(n)}$ is nonempty. Moreover, from Lemma 4.2 and Lemma 5.1, the following result on weak compactness holds.

Lemma 5.2. *Let* **(H1)**-**(H7)** *hold. Then $\bigcup_{n\in\mathbb{Z}^+}\mathcal{S}^{(n)}$ is tight.*

Next, we show the following result which is concerned with the convergence of solutions to (2.2) as $n\to+\infty$.

Lemma 5.3. *Let* **(H1)**-**(H6)**, **(H8)** *and* **(H9)** *hold. Then for every bounded subset* E *in* l^2_ρ, $T > \tau$ *and* $\sigma > 0$,

$$\lim_{n\to+\infty} \sup_{\varphi\in E} \mathbb{P}\Big(\Big\{\omega\in\Omega \mid \sup_{\tau\le t\le\tau+T} \|u^{(n)}(t,\varphi) - u(t,\varphi)\|_\rho \ge \sigma\Big\}\Big) = 0.$$

Proof. Let $\tau^n_R = \inf_{t\ge\tau}\Big\{\|u^{(n)}(t,\varphi)\|_\rho \vee \|u(t,\varphi)\|_\rho > R\Big\}$. Applying Itô's formula to $u^{(n)}(t,\varphi) - u(t,\varphi)$ and then taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u^{(n)}(r\wedge\tau^n_R,\varphi) - u(r\wedge\tau^n_R,\varphi)\|^2_\rho\Big) \\
\le& 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^n_R} (F(u^{(n)}) - F(u), u^{(n)} - u)ds\Big|\Big) + 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^n_R} (\mathcal{K}^{(n)}(u^{(n)}) - \mathcal{K}(u), u^{(n)} - u)ds\Big|\Big) \\
&+ \varepsilon^2 \sum_{i\in\mathbb{Z}^d} \mathbb{E}\Big(\int_\tau^{t\wedge\tau^n_R} \|\Lambda_i(u^{(n)}) - \Lambda_i(u)\|^2_\rho ds\Big) \\
&+ 2\varepsilon\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big|\sum_{i\in\mathbb{Z}^d}\int_\tau^{r\wedge\tau^n_R} \Big(\Lambda_i(u^{(n)}) - \Lambda_i(u), u^{(n)} - u\Big)\, dW_i(s)\Big|\Big).
\end{aligned} \tag{5.1}$$

Similar to (4.11), the first term on the right-hand side of (5.1) can be bounded by

$$\begin{aligned}
&\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^n_R} (F(u^{(n)}) - F(u), u^{(n)} - u)ds\Big|\Big) \\
\le& L_f(2R\sqrt{\rho_\Sigma}) \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u^{(n)}(r\wedge\tau^n_R,\varphi) - u(r\wedge\tau^n_R,\varphi)\|^2_\rho\Big) ds.
\end{aligned} \tag{5.2}$$

After some calculations, we have for the second term on the right-hand side of (5.1)

$$\begin{aligned}
&\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^n_R} (\mathcal{K}^{(n)}(u^{(n)}) - \mathcal{K}(u), u^{(n)} - u)ds\Big|\Big) \\
\le& \mathbb{E}\Big(\int_\tau^{t\wedge\tau^n_R} \|u^{(n)} - u\|_\rho \Big[\Big(\sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{j=i-n}^{j=i+n} k_{i,j}(\phi(u^{(n)}_j) - \phi(u_j))\Big)^2\Big)^{\frac12} \\
&+ \Big(\sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{j=i-n}^{j=i+n}(k^{(n)}_{i,j} - k_{i,j})\phi(u^{(n)}_j)\Big)^2\Big)^{\frac12} + \Big(\sum_{i\in\mathbb{Z}^d}\rho_i(\sum_{|j-i|>n} k_{i,j}\phi(u_j))^2\Big)^{\frac12}\Big]ds\Big) \\
:=& \mathbb{E}\Big(\int_\tau^{t\wedge\tau^n_R} \|u^{(n)} - u\|_\rho \Big[{I_1}^{\frac12} + {I_2}^{\frac12} + {I_3}^{\frac12}\Big]ds\Big),
\end{aligned} \tag{5.3}$$

where $I_1 = \sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{j=i-n}^{j=i+n} k_{i,j}(\phi(u^{(n)}_j) - \phi(u_j))\Big)^2$, $I_2 = \sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{j=i-n}^{j=i+n}(k^{(n)}_{i,j} - k_{i,j})\phi(u^{(n)}_j)\Big)^2$ and $I_3 = \sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{|j-i|>n} k_{i,j}\phi(u_j)\Big)^2$.

Together with **(H1)** and **(H2)**, it follows that

$$I_1 \le \sum_{i\in\mathbb{Z}^d}\rho_i\Big[\sum_{j=i-n}^{j=i+n}\frac{k^2_{i,j}}{\rho_j}L^2_\phi \sum_{j=i-n}^{j=i+n}\rho_j(u^{(n)}_j - u_j)^2\Big] \le \rho_\Sigma\kappa L^2_\phi\|u^{(n)} - u\|^2_\rho. \tag{5.4}$$

By **(H8)**, we have that for every $\epsilon > 0$, there exists $N_1(\epsilon) > 0$ such that for all $n \ge N_1(\epsilon)$, $\sum_{j\in\mathbb{Z}^d}\frac{(k^{(n)}_{i,j} - k_{i,j})^2}{\rho_j} \le \epsilon$, which together with **(H1)** and **(H9)** implies that

$$I_2 \le \sum_{i\in\mathbb{Z}^d}\rho_i\Big(\sum_{j=i-n}^{j=i+n}\frac{(k^{(n)}_{i,j} - k_{i,j})^2}{\rho_j}\sum_{j=i-n}^{j=i+n}\rho_j\phi^2(u^{(n)}_j)\Big) \le \epsilon\rho^2_\Sigma b^2_\phi. \tag{5.5}$$

By **(H1)**, for any $\epsilon > 0$, there exists $I(\epsilon) > 0$ such that $\sum_{|i|>I(\epsilon)} \rho_i < \epsilon$. Choose $N_2(\epsilon) = 2I(\epsilon)$, then $|j| > I(\epsilon)$ if $|j-i| > N_2(\epsilon)$ and $|i| \le I(\epsilon)$, and hence $\sum_{|j-i|>n} \rho_j < \epsilon$ for $n \ge N_2(\epsilon)$. Then for any $n \ge N_2(\epsilon)$,

$$I_3 \le \sum_{i\in\mathbb{Z}^d} \rho_i \Big(\sum_{|j-i|>n} \frac{k_{i,j}^2}{\rho_j} \sum_{|j-i|>n} \rho_j \phi^2(u_j) \Big) \le \sum_{i\in\mathbb{Z}^d} \rho_i \kappa b_\phi^2 \sum_{|j-i|>n} \rho_j \le 2\rho_\Sigma \kappa b_\phi^2 \epsilon, \tag{5.6}$$

which together with (5.3) and (5.4)-(5.6) implies that for all $n > \max\{N_1(\epsilon), N_2(\epsilon)\}$,

$$\begin{aligned} &\mathbb{E}\Big(\Big| \int_\tau^{t\wedge\tau_R^n} (\mathcal{K}^{(n)}(u^{(n)}) - \mathcal{K}(u), u^{(n)} - u) ds \Big| \Big) \\ \le& (1 + \rho_\Sigma^{\frac{1}{2}} \kappa^{\frac{1}{2}} L_\phi) \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) ds + \frac{1}{2}(\epsilon\rho_\Sigma^2 b_\phi^2 + 2\rho_\Sigma \kappa b_\phi^2 \epsilon) T. \end{aligned} \tag{5.7}$$

The last two terms on the right-hand side of (5.1) can be bounded by

$$\begin{aligned} &\sum_{i\in\mathbb{Z}^d} \mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^n} \|\Lambda_i(u^{(n)}) - \Lambda_i(u)\|_\rho^2 ds \Big) \\ \le& L_\lambda^2(2R\sqrt{\rho_\Sigma}) \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) ds \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\varepsilon \mathbb{E}\Big(\sup_{\tau\le r\le t} \Big| \sum_{i\in\mathbb{Z}^d} \int_\tau^{r\wedge\tau_R^n} (\Lambda_i(u^{(n)}) - \Lambda_i(u), u^{(n)} - u) dW_i(s) \Big| \Big) \\ \le& \frac{1}{4} \mathbb{E}\Big(\sup_{\tau\le r\le t} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) \\ &+ \varepsilon^2 C_5 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) ds. \end{aligned} \tag{5.9}$$

By (5.1)-(5.9), we obtain for all $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\mathbb{E}\Big(\sup_{\tau\le r\le t} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) \\ \le& \Big[4L_f(2R\sqrt{\rho_\Sigma}) + 4(\rho_\Sigma^{\frac{1}{2}} \kappa^{\frac{1}{2}} L_\phi + 1) + 2\varepsilon^2 L_\lambda^2(2R\sqrt{\rho_\Sigma}) + 4\varepsilon^2 C_5 \Big] \\ &\cdot \int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s} \|u^{(n)}(r\wedge\tau_R^n, \varphi) - u(r\wedge\tau_R^n, \varphi)\|_\rho^2 \Big) ds + 2(\epsilon\rho_\Sigma^2 b_\phi^2 + 2\rho_\Sigma \kappa b_\phi^2 \epsilon) T. \end{aligned}$$

Thanks to this and the Gronwall inequality, we obtain for all $n > \max\{N_1(\epsilon), N_2(\epsilon)\}$,

$$\begin{aligned} &\sup_{\varphi\in E} \mathbb{P}\Big(\Big\{ \omega \in \Omega \mid \sup_{\tau\le t\le \tau+T} \|u^{(n)}(t\wedge\tau_R^n, \varphi) - u(t\wedge\tau_R^n, \varphi)\|_\rho \ge \sigma \Big\} \Big) \\ \le& \frac{2}{\sigma^2} \epsilon b_\phi^2 \left(\rho_\Sigma^2 + 2\rho_\Sigma \kappa \right) T e^{[4L_f(2R\sqrt{\rho_\Sigma}) + 4(\rho_\Sigma^{\frac{1}{2}} \kappa^{\frac{1}{2}} L_\phi + 1) + 2\varepsilon^2 L_\lambda^2(2R\sqrt{\rho_\Sigma}) + 4\varepsilon^2 C_5]T}. \end{aligned}$$

The proof is complete. □

Now we present a proof of Theorem 3.2.

Proof of Theorem 3.2. Since $\bigcup_{n\in\mathbb{Z}^+} \mathcal{S}^{(n)}$ is tight by Lemma 5.2, it follows from $\{\mu^{(n)}\}_{n=1}^{+\infty} \subseteq \bigcup_{n\in\mathbb{Z}^+} \mathcal{S}^{(n)}$ that there exist a subsequence $\{n_k\}_{k=1}^{+\infty}$ and a probability measure μ such that $\mu^{(n_k)} \to \mu$ weakly.

For every $\epsilon > 0$, we find by the tightness of $\{\mu^{(n_k)}\}_{n=1}^{+\infty}$ that there exists a compact set $E = E(\epsilon) \subset l_\rho^2$ such that for all $n_k \in \mathbb{Z}^+$, $\mu^{(n_k)}(E) \ge 1 - \epsilon$. For any $t \ge \tau$ and $\varphi \in UC_b(l_\rho^2)$, where $UC_b(l_\rho^2)$ is the Banach space of all bounded uniformly continuous functions defined on l_ρ^2, we deduce

$$\Big| \int_{l_\rho^2} \mathbb{E}\left(\varphi(u(t, u_\tau)) \right) \mu^{(n_k)}(du_\tau) - \int_{l_\rho^2} \varphi(u_\tau) \mu^{(n_k)}(du_\tau) \Big| \tag{5.10}$$

$$\le \int_E \mathbb{E}\left(|\varphi(u(t,u_\tau)) - \varphi(u^{(n_k)}(t,u_\tau))|\right)\mu^{(n_k)}(du_\tau) + 2\epsilon \sup_{x\in l^2_\rho}|\varphi(x)|.$$

Since $\varphi \in UC_b(l^2_\rho)$, for every $\epsilon > 0$, there exists $\eta > 0$ such that for all $y, z \in l^2_\rho$ with $\|y - z\|^2_\rho < \eta$, we have $|\varphi(y) - \varphi(z)| < \epsilon$. Thus

$$\begin{aligned}&\int_E \mathbb{E}\left(|\varphi(u(t,u_\tau)) - \varphi(u^{(n_k)}(t,u_\tau))|\right)\mu^{(n_k)}(du_\tau) \\ \le& 2\sup_{x\in l^2_\rho}|\varphi(x)| \sup_{u_\tau\in E}\mathbb{P}\Big(\sup_{t\in[\tau,\tau+T]}\|u^{(n_k)}(t,u_\tau) - u(t,u_\tau)\|^2_\rho \ge \eta\Big) + \epsilon.\end{aligned} \tag{5.11}$$

From Lemma 5.3 and (5.10)-(5.11), it follows that

$$\lim_{k\to+\infty}\Big|\int_{l^2_\rho}\mathbb{E}\left(\varphi(u(t,u_\tau))\right)\mu^{(n_k)}(du_\tau) - \int_{l^2_\rho}\varphi(u_\tau)\mu^{(n_k)}(du_\tau)\Big| \le 2\epsilon\sup_{x\in l^2_\rho}|\varphi(x)| + \epsilon.$$

Since $\mu^{(n_k)} \to \mu$ weakly and $\epsilon > 0$ is arbitrary, we obtain

$$\int_{l^2_\rho}\mathbb{E}\left(\varphi(u(t,u_\tau))\right)\mu(du_\tau) = \int_{l^2_\rho}\varphi(u_\tau)\mu(du_\tau),$$

which means μ is an invariant measure of (2.1). □

5.2 Proof of Theorem 3.3

In this subsection, we will prove Theorem 3.3 to show the relationship of invariant measures between (2.2) and (2.3).

Similar to Lemma 5.2, we obtain the following lemma.

Lemma 5.4. *Let* **(H1)**-**(H7)** *hold, and denote by* $\mathcal{S}^{N,n}$ *the collection of all invariant probability measures of* (2.3), *then* $\bigcup_{N\in\mathbb{Z}^+}\mathcal{S}^{N,n}$ *is tight.*

The next result is concerned with the convergence of solutions to (2.3) as $N \to +\infty$.

Lemma 5.5. *Let* **(H1)**-**(H6)**, **(H8)** *and* **(H9)** *hold. Then for any* $T > \tau$, $\sigma > 0$ *and every bounded subset* E *in* l^2_ρ,

$$\lim_{N\to+\infty}\sup_{\varphi\in E}\mathbb{P}\Big(\Big\{\omega\in\Omega \mid \sup_{\tau\le t\le\tau+T}\|u^{(n)}(t,\varphi) - u^{N,n}(t,\varphi)\|_\rho \ge \sigma\Big\}\Big) = 0.$$

Proof. Denote $\tau^N_R = \inf\Big\{t \ge \tau, \|u^{N,n}(t,\varphi)\|_\rho \vee \|u^{(n)}(t,\varphi)\|_\rho > R\Big\}$. Using Itô's formula to $u^{(n)}(t,\varphi) - u^{N,n}(t,\varphi)$ and then taking expectation, we have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u^{(n)}(t\wedge\tau^N_R,\varphi) - u^{N,n}(t\wedge\tau^N_R,\varphi)\|^2_\rho\Big) \\ \le& 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^N_R}(F(u^{(n)}) - F^N(u^{N,n}), u^{(n)} - u^{N,n})ds\Big|\Big) + 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^N_R}(\mathcal{K}^{(n)}(u^{(n)}) - \mathcal{K}^N(u^{N,n}), u^{(n)} - u^{N,n})ds\Big|\Big) \\ &+ 2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^N_R}(G - G^N, u^{(n)} - u^{N,n})ds\Big|\Big) + \varepsilon^2\sum_{i\in\mathbb{Z}^d}\mathbb{E}\Big(\sup_{\tau\le r\le t}\int_\tau^{r\wedge\tau^N_R}\|\Lambda_i(u^{(n)}) + H_i\|^2_\rho ds\Big) \\ &+ 2\varepsilon\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big|\int_\tau^{r\wedge\tau^N_R}\Big(\sum_{i\in\mathbb{Z}^d}(\Lambda_i(u^{(n)}) + H_i) - \sum_{i\in\mathbb{Z}^d_N}(\Lambda_i(u^{N,n}) + H_i), u^{(n)} - u^{N,n}\Big)dW_i(s)\Big|\Big).\end{aligned} \tag{5.12}$$

By (5.2), the first term on the right-hand side of (5.12) can be bounded by

$$2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau^N_R}(F(u^{(n)}) - F^N(u^{N,n}), u^{(n)} - u^{N,n})ds\Big|\Big) \tag{5.13}$$

$$
\begin{aligned}
&\le 2L_f(2R\sqrt{\rho_\Sigma})\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds\\
&\quad+\frac{1}{8}\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)+C_1\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i {u_i^{(n)}}^2 ds\Big).
\end{aligned}
$$

Similar to inequality (5.4), the second term on the right-hand side of (5.12) can be bounded by

$$
\begin{aligned}
&2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^N}(\mathcal{K}^{(n)}(u^{(n)})-\mathcal{K}^N(u^{N,n}),u^{(n)}-u^{N,n})ds\Big|\Big) \qquad (5.14)\\
&\le 2\sqrt{\rho_\Sigma\kappa}L_\phi\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds\\
&\quad+\frac{1}{8}\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)+C_2\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i.
\end{aligned}
$$

For the third term on the right-hand side of (5.12), we find

$$
\begin{aligned}
&2\mathbb{E}\Big(\Big|\int_\tau^{t\wedge\tau_R^N}(G-G^N,u^{(n)}-u^{N,n})ds\Big|\Big) \qquad (5.15)\\
&\le\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds+\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i g_i^2 ds\Big).
\end{aligned}
$$

For the fourth term on the right-hand side of (5.12), we have

$$
\begin{aligned}
&\varepsilon^2\sum_{i\in\mathbb{Z}^d}\int_\tau^{t\wedge\tau_R^N}\|\sum_{i\in\mathbb{Z}^d}(\Lambda_i(u^{(n)})+H_i)-\sum_{i\in\mathbb{Z}_N^d}(\Lambda_i(u^{N,n})+H_i)\|_\rho^2 ds \qquad (5.16)\\
&\le\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds\\
&\quad+\varepsilon^2\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|\Lambda_i(u^{(n)})\|_\rho^2 ds\Big)+\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|H_i\|_\rho^2 ds\Big).
\end{aligned}
$$

Similar to (5.9), by the Burkholder-Davis-Gundy inequality, the last term on the right-hand side of (5.12) can be estimated by

$$
\begin{aligned}
&2\varepsilon\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big|\int_\tau^{r\wedge\tau_R^N}\Big(\sum_{i\in\mathbb{Z}^d}(\Lambda_i(u^{(n)})+H_i)-\sum_{i\in\mathbb{Z}_N^d}(\Lambda_i(u^{N,n})+H_i),u^{(n)}-u^{N,n}\Big)dW_i(s)\Big|\Big)\\
&\le\frac{1}{8}\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)+C_4\Big(\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|\Lambda_i(u^{(n)})\|_\rho^2 ds\Big)\\
&\quad+C_3\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds+\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|H_i\|_\rho^2 ds\Big)\Big).
\end{aligned}
$$

By (5.12)-(5.17), we obtain for all $t\in[\tau,\tau+T]$,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big) \qquad (5.17)\\
&\le[4L_f(2R\sqrt{\rho_\Sigma})+4\sqrt{\rho_\Sigma\kappa}L_\phi+2\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})+2C_3+2]\\
&\quad\cdot\int_\tau^t \mathbb{E}\Big(\sup_{\tau\le r\le s}\|u^{(n)}(r\wedge\tau_R^N,\varphi)-u^{N,n}(r\wedge\tau_R^N,\varphi)\|_\rho^2\Big)ds\\
&\quad+C_1\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i {u_i^{(n)}}^2 ds\Big)+(\varepsilon^2+C_4)\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|\Lambda_i(u^{(n)})\|_\rho^2 ds\Big)
\end{aligned}
$$

$$+(\varepsilon^2+C_4)\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}\|H_i\|_\rho^2ds\Big)+\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i g_i^2ds\Big)+C_2\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i.$$

Due to the fact $u^{(n)}\in\mathcal{L}^2\left(\Omega,\mathcal{C}\left([\tau,\tau+T],l_\rho^2\right)\right)$, $G=(g_i)_{i\in\mathbb{Z}^d}\in l_\rho^2$, $H=(h_i)_{i\in\mathbb{Z}^d}\in l_\rho^2$, $\|\Lambda(u)\|_\rho^2\le 2\|a\|_\infty^2\|u\|_\rho^2+2\|b\|_\rho^2$ and **(H1)**, we deduce by Gronwall's inequality that

$$\begin{aligned}&\sup_{\varphi\in E}\mathbb{P}\Big(\Big\{\omega\in\Omega\mid\sup_{\tau\le t\le\tau+T}\|u^{(n)}\left(t\wedge\tau_R^N,\varphi\right)-u^{N,n}\left(t\wedge\tau_R^N,\varphi\right)\|_\rho\ge\sigma\Big\}\Big)\\ \le&\frac{1}{\sigma^2}\Big[C_1\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i {u_i^{(n)}}^2ds\Big)+C_2\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\|u^{(n)}\|_\rho^2ds\Big)\\ &+(\varepsilon^2+C_4)\mathbb{E}\Big(\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\int_\tau^{t\wedge\tau_R^N}(\|\Lambda_i(u^{(n)})\|_\rho^2+\|H_i\|_\rho^2)ds\Big)+\mathbb{E}\Big(\int_\tau^{t\wedge\tau_R^N}\sum_{i\in\mathbb{Z}^d\setminus\mathbb{Z}_N^d}\rho_i g_i^2ds\Big)\Big]\\ &\cdot e^{\left[4L_f(2R\sqrt{\rho_\Sigma})+4\sqrt{\rho_\Sigma\kappa}L_\phi+2\varepsilon^2L_\lambda^2(2R\sqrt{\rho_\Sigma})+2C_3+2\right]T}\to 0\end{aligned}$$

as $N\to+\infty$, hence the proof is therefore complete. □

Now we present a proof of Theorem 3.3.

Proof of Theorem 3.3. Similarly to the proof of Theorem 3.2, we can conclude by Lemmas 5.4 and 5.5 that any limit of a sequence of invariant measures of (2.3) must be an invariant measure of (2.2) as $N\to+\infty$. □

5.3 Proof of Theorems 3.4 and 3.5

Denote $\eta(t):=t_k$, as $t\in[t_k,t_{k+1})$, and $\eta_+(t):=t_{k+1}$, as $t\in[t_k,t_{k+1})$ for $k\ge0$. Then the continuous version of the BEM approximate solution satisfies

$$X(t)=X_{t_0}+\int_\tau^t\left(F^N(X_{\eta_+(s)})+\mathcal{K}^N(X_{\eta_+(s)})+G^N\right)ds+\varepsilon\sum_{i\in\mathbb{Z}_N^d}\int_\tau^t\left(\Lambda_i(X_{\eta(s)})+H_i\right)dW_i(s),$$

which will be used in the proof of Theorem 3.4.

Following [?, Lemma 3.3], we establish the existence and uniqueness of solutions to the BEM scheme (3.1). Next, we provide moment estimates of solutions.

Lemma 5.6. (Moment estimates) *Let* **(H1)**-**(H4)**, **(H6)** *and* **(H9)**-**(H11)** *hold. There exists a constant* $\hbar$ *such that the numerical solution of the BEM scheme with any initial value* $x\in\mathcal{L}^2\left(\Omega,\mathbb{R}^{2N+1}\right)$ *satisfies*

$$\sup_{k\ge0}\mathbb{E}\left(|X_k|^2\right)\le C(1+\mathbb{E}\left(|x|^2\right)).$$

Proof. By (3.1) and the properties of $F^N,\mathcal{K}^N,G^N$ in Section 2, we obtain

$$\begin{aligned}|X_{k+1}|^2=&\Big(\left(F^N\left(X_{k+1}\right)+\mathcal{K}^N\left(X_{k+1}\right)+G^N\right)\hbar,\ X_{k+1}\Big)+\Big(X_k+\varepsilon\sum_{i\in\mathbb{Z}_N^d}\left(\Lambda_i(X_k)+H_i\right)\Delta W_{ik},\ X_{k+1}\Big)\\ \le&\Big[\frac12-(\alpha-2\sqrt{2\rho_\Sigma\kappa}a_\phi)\hbar\Big]|X_{k+1}|^2+(2\kappa\rho_\Sigma^2b_\phi^2+\|\beta\|_\rho^2+C\|G^N\|_\rho^2)\hbar\\ &+\frac12|X_k+\varepsilon\sum_{i\in\mathbb{Z}_N^d}\left(\Lambda_i(X_k)+H_i\right)\Delta W_{ik}|^2,\end{aligned}$$

where C is a constant which depends on k. Then we have

$$1+|X_{k+1}|^2\le\frac{1+|X_k|^2}{1-(4\sqrt{2\rho_\Sigma\kappa}a_\phi-2\alpha)\hbar}(1+v_k),\tag{5.18}$$

where $v_k = \dfrac{\sum_{i\in\mathbb{Z}_N^d} \rho_i X_{ik}(\Lambda_i(X_k)+H_i)\Delta W_{ik} + \varepsilon^2 |\sum_{i\in\mathbb{Z}_N^d} (\Lambda_i(X_k)+H_i)\,\Delta W_{ik}|^2 + C_1\hbar}{1+|X_k|^2}$,
and $C_1 = 4\kappa\rho_\Sigma^2 b_\phi^2 + 2\|\beta\|_\rho^2 + 2C\|G^N\|_\rho^2 - 4\sqrt{2\rho_\Sigma\kappa a_\phi} + 2\alpha$.

Since ΔW_{ik} is independent of $\mathcal{F}_{t_k}$, we have $\mathbb{E}(\Delta W_{ik}|\mathcal{F}_{t_k}) = 0$ and $\mathbb{E}\left(|A\Delta W_{ik}|^2|\mathcal{F}_{t_k}\right) = |A|^2\hbar$, from which we find $\mathbb{E}(v_k|\mathcal{F}_{t_k}) = \dfrac{1}{1+|X_k|^2}\Big(\varepsilon^2 \sum_{i\in\mathbb{Z}_N^d} |\Lambda_i(X_k)+H_i|^2\hbar + C_1\hbar\Big)$.
Taking conditional expectation on (5.18), it yields that

$$\mathbb{E}\left(1+|X_{k+1}|^2|\mathcal{F}_{t_k}\right) \le \frac{1+|X_k|^2}{1-(4\sqrt{2\rho_\Sigma\kappa}a_\phi - 2\alpha)\hbar}(1+4\varepsilon^2\|a\|_\infty^2\hbar) + C_2\hbar, \tag{5.19}$$

where $C_2 = 4\varepsilon^2\|b\|_\rho^2 + 2\varepsilon^2 \sum_{i\in\mathbb{Z}_N^d} |H_i|^2 + 2C_1$.
On the other hand, for any $0<\hbar<\dfrac{-1}{8(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)}$, we obtain

$$\Big[1-(4\sqrt{2\rho_\Sigma\kappa}a_\phi - 2\alpha)\hbar\Big]^{-1} \le 1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar. \tag{5.20}$$

From (5.19) and (5.20), we have

$$\mathbb{E}\left(1+|X_{k+1}|^2|\mathcal{F}_{t_k}\right) \le \Big[1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar\Big](1+|X_k|^2) + C_2\hbar. \tag{5.21}$$

Then by induction, it follows from (5.21) that

$$\begin{aligned}&\mathbb{E}\left(1+|X_{k+1}|^2|\mathcal{F}_{t_0}\right)\\ \le& \Big[1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar\Big]^{k+1}(1+|x|^2) + C_2\hbar\sum_{l=1}^{k}\Big[1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar\Big]^l + C_2\hbar.\end{aligned} \tag{5.22}$$

Taking expectation on each side of (5.22), we deduce

$$\begin{aligned}&\mathbb{E}\left(1+|X_{k+1}|^2\right)\\ \le& \Big[1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar\Big]^{k+1}\Big(1+\mathbb{E}\left(|x|^2\right)\Big) + C_2\hbar\sum_{l=1}^{k}\Big[1+(2\sqrt{2\rho_\Sigma\kappa}a_\phi-\alpha)\hbar\Big]^l + C_2\hbar,\end{aligned}$$

which implies the desired result. □

The following theorem follows directly from Lemma 5.6 proving the tightness of the family of probability distributions.

Lemma 5.7. (Tightness) *Let* **(H1)**-**(H4)**, **(H6)** *and* **(H9)**-**(H11)** *hold. Then for every compact subset* $\mathscr{K}$ *of* $\mathbb{R}^{2N+1}$, *the family of probability distribution for solutions of BEM scheme* (3.1) *is tight.*

Proof. By the moment estimates in Lemma 5.6, there exists a constant $C>0$ such that $\mathbb{E}\left(|X_k|^2\right)\le C$. Define $\mathcal{Y} = \{X_k\in\mathbb{R}^{2N+1}|\quad |X_k|\le\sqrt{\frac{C}{\epsilon}}\}$, then $\mathcal{Y}$ is a bounded and closed subset of $\mathbb{R}^{2N+1}$. Thanks to the Chebyshev inequality we find that, for all $t>0$ and $x\in\mathscr{K}$,

$$P\left(\{\omega\in\Omega : X_k(t,x)\in\mathcal{Y}\}\right) = 1 - P\left(\left\{\omega\in\Omega : |X_k| > \sqrt{\frac{C}{\epsilon}}\right\}\right) \ge 1-\epsilon,$$

which means $\{\mathcal{P}(t,x)\}_{t\ge\tau}$ is tight. □

Next, the existence and uniqueness of the numerical invariant measure of (2.3) by BEM scheme is proved.

298 **Lemma 5.8.** *Let* **(H1)**-**(H11)** *hold and* $\beta_i = 0$ *for* $i \in \mathbb{Z}_N^d$, $N \in \mathbb{Z}^+$. *Then there is a unique invariant* 299 *probability measure* $\mu^{\hbar,N,n}$ *to the BEM scheme* (3.1) *which exponentially converges in the Wasserstein* 300 *distance as* $\hbar \to 0$.

Proof. Denote by $P^{\hbar}_{k\hbar}$ the probability distribution of X_k, by Lemma 5.7, one can extract a subsequence which converges weakly to an invariant measure denoted by $\mu^{\hbar,N,n} \in \mathcal{P}(\mathbb{R}^{2N+1})$. Now, it remains to verify the uniqueness of invariant measures. Assume $\mu_1^{\hbar,N,n}$, $\mu_2^{\hbar,N,n} \in \mathcal{P}(\mathbb{R}^{2N+1})$ are the invariant measures of (3.1), respectively, then we have $W_2^2(\mu_1^{\hbar,N,n}, \mu_2^{\hbar,N,n}) \le \int_{\mathbb{R}^{2N+1}\times\mathbb{R}^{2N+1}} W_2^2(\delta_x P^{\hbar}_{k\hbar}, \delta_y P^{\hbar}_{k\hbar})\pi(dx,dy)$. Note that

$$
\begin{aligned}
|X^x_{k+1} - X^y_{k+1}|^2 \le & [-\alpha + \sqrt{\rho_\Sigma \kappa} L_\phi]\, \hbar |X^x_{k+1} - X^y_{k+1}|^2 + \frac{1}{2}|X^x_{k+1} - X^y_{k+1}|^2 \\
& + \frac{1}{2}|X^x_k - X^y_k + \varepsilon \sum_{i\in\mathbb{Z}_N^d} (\Lambda_i(X^x_k) - \Lambda_i(X^y_k))\,\Delta W_{ik}|^2 + \|\beta\|_\rho^2 \hbar.
\end{aligned}
$$

Hence we obtain $|X^x_{k+1} - X^y_{k+1}|^2 \le \dfrac{|X^x_k - X^y_k|^2}{1 - 2(-\alpha + \sqrt{\rho_\Sigma\kappa}L_\phi)\hbar}(1 + v'_k)$, where

$$
v'_k = \begin{cases}
\dfrac{\varepsilon \sum_{j\in\mathbb{Z}^d} \rho_j (X^x_{jk} - X^y_{jk}) \sum_{i\in\mathbb{Z}_N^d} (\Lambda_i(X^x_{jk}) - \Lambda_i(X^y_{jk}))\Delta W_{ik}}{|X^x_k - X^y_k|^2} & \\
+ \dfrac{\varepsilon^2 |\sum_{i\in\mathbb{Z}_N^d} (\Lambda_i(X^x_k) - \Lambda_i(X^y_k))\,\Delta W_{ik}|^2 + 2\|\beta\|_\rho^2\hbar}{|X^x_k - X^y_k|^2}, & |X^x_k - X^y_k|^2 \ne 0; \\
-1, & |X^x_k - X^y_k|^2 = 0.
\end{cases}
$$

Similar to discussions in Lemma 5.6, for any $0 < \hbar < \dfrac{-1}{4(\sqrt{\rho_\Sigma\kappa}L_\phi - \alpha)}$, we have the following result

$$
\mathbb{E}\left(|X^x_{k+1} - X^y_{k+1}|^2 | \mathcal{F}_{t_k}\right) \le \left[1 + 2(\varepsilon^2 L_\lambda^2 + 2\sqrt{\rho_\Sigma\kappa}L_\phi - 2\alpha)\hbar\right] |X^x_k - X^y_k|^2.
$$

And hence

$$
W_2^2(\delta_x P^{\hbar}_{k\hbar}, \delta_y P^{\hbar}_{k\hbar}) \le \mathbb{E}\left(|X^x_k - X^y_k|^2\right) \le e^{2(\varepsilon^2 L_\lambda^2 + 2\sqrt{\rho_\Sigma\kappa}L_\phi - 2\alpha)k\hbar}|x - y|^2,
$$

301 which together with the **(H11)** completes the proof. ◻

302 Now we present a proof of Theorem 3.4.

Proof of Theorem 3.4. By the Kolmogorov-Chapman equation, Lemma 5.8 and Lemma 5.6, we have that for any k, $l > 0$,

$$
W_2^2(\delta_x P^{\hbar}_{k\hbar}, \delta_x P^{\hbar}_{(k+l)\hbar}) \le \int_{l^2_\rho} W_2^2(\delta_x P^{\hbar}_{k\hbar}, \delta_y P^{\hbar}_{k\hbar}) P^{\hbar}_{l\hbar}(x, dy) \le 2Ce^{(\varepsilon^2 L_\lambda^2 + 2\sqrt{\rho_\Sigma\kappa}L_\phi - 2\alpha)k\hbar}(1 + 2|x|^2). \tag{5.23}
$$

Let $l \to +\infty$ in (5.23), then we have

$$
W_2^2(\delta_x P^{\hbar}_{k\hbar}, \mu^{\hbar,N,n}) \le 2Ce^{(\varepsilon^2 L_\lambda^2 + 2\sqrt{\rho_\Sigma\kappa}L_\phi - 2\alpha)k\hbar}(1 + 2|x|^2).
$$

Let $\hbar_1 = \min\{\dfrac{-1}{8(2\sqrt{2\rho_\Sigma\kappa}a_\phi - \alpha)}, \dfrac{-1}{4(\sqrt{\rho_\Sigma\kappa}L_\phi - \alpha)}\}$, then for any $\epsilon > 0$, there exists $T_1 > 0$ such that for $\hbar \in (0, \hbar_1]$ and $k\hbar \ge T_1$,

$$
W_2(\delta_x P^{\hbar}_{k\hbar}, \mu^{\hbar,N,n}) < \frac{\epsilon}{4}. \tag{5.24}
$$

In addition, by Itô's formula for $u_x^{N,n}(t) - u_y^{N,n}(t)$ and using similar estimates in Lemma 4.3 and Lemma 5.6, we obtain

$$
\mathbb{E}\left(\|u_x^{N,n}(t) - u_y^{N,n}(t)\|_\rho^2\right) \le (\|x - y\|_\rho^2 + \|\beta\|_\rho^2)e^{(-2\alpha + 2\sqrt{\rho_\Sigma\kappa}L_\phi + \varepsilon^2 L_\lambda^2)(t-\tau)}.
$$

Let $P_{k\hbar}$ be the probability distribution of $u^{N,n}$, there is a $T_2 > 0$ such that for any $\hbar \in (0,1)$ and $k\hbar \geq T_2$,

$$W_2(\delta_x P_{k\hbar}, \mu^{N,n}) < \frac{\epsilon}{4}. \tag{5.25}$$

Let $T = \max\{T_1, T_2\} + \tau$ and $k = [\frac{T+1}{\hbar}]$ for any $\hbar \in (0,1)$, then $\tau < T < k\hbar \leq T+1$. Following [?, Theorem 5.3], for any given $\epsilon > 0$, there exists a constant $\hbar^* > 0$ such that for any $\hbar \in (0,\hbar^*)$,

$$W_2(\delta_x P_{k\hbar}, \delta_x P^{\hbar}_{k\hbar}) \leq \mathbb{E}(\|X(k\hbar) - u^{N,n}(k\hbar)\|^2_\rho) < \frac{\epsilon}{2}. \tag{5.26}$$

Combining with (5.24)-(5.26), the result is proved. □

Finally, we present a proof of Theorem 3.5.

Proof of Theorem 3.5. By Assumptions **(H1)-(H11)** and $\beta_i = 0$ for $i \in \mathbb{Z}^d$, we have $E(\|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|^2_\rho) \leq E(\|u^1_\tau - u^2_\tau\|^2_\rho) e^{(-2\alpha + 2\sqrt{\rho_\Sigma \kappa} L_\phi + \varepsilon^2 L^2_\lambda)(t-\tau)}$, from which we obtain the uniqueness of the invariant measure of (1.1). Similarly, we can prove the uniqueness of the invariant measure of (1.3) and (1.4). Then by Theorem 3.2, it follows that $\mu^{(n)} \to \mu$ weakly, which implies that for any $\epsilon > 0$, there exists a $n_0 = n_0(\epsilon) \in \mathbb{Z}^+$ such that for any $n \geq n_0$ and for any bounded and continuous function $\varphi : l^2_\rho \to \mathbb{R}$, $| \int_{l^2_\rho} \varphi(u) d\mu^{(n)}(u) - \int_{l^2_\rho} \varphi(u) d\mu(u) | < \frac{\epsilon}{3}$. Fix n, by the uniqueness of the invariant measure of (1.4) and Theorem 3.3, we find there exists a $N^* = N^*(n,\epsilon) \geq n$ such that for any $N \geq N^*$, $| \int_{l^2_\rho} \varphi(u) d\mu^{N,n}(u) - \int_{l^2_\rho} \varphi(u) d\mu^{(n)}(u) | < \frac{\epsilon}{3}$. For every n and N, we infer from Theorem 3.4 that $\mu^{\hbar,N,n} \to \mu^{N,n}$ weakly, so there exists a constant $\hbar^* = \hbar^*(N,n,\epsilon) > 0$ such that for any $0 < \hbar < \hbar^*$, $| \int_{l^2_\rho} \varphi(u) d\mu^{\hbar,N,n}(u) - \int_{l^2_\rho} \varphi(u) d\mu^{N,n}(u) | < \frac{\epsilon}{3}$. Then $| \int_{l^2_\rho} \varphi(u) d\mu^{\hbar,N,n}(u) - \int_{l^2_\rho} \varphi(u) d\mu(u) | < \epsilon$ for any $n \geq n_0$, $N \geq N^*$ and $\hbar \in (0,\hbar^*)$. Therefore, $\lim_{n\to+\infty} \lim_{N\to+\infty} \lim_{\hbar\to 0} \mu^{\hbar,N,n} = \mu$ weakly. □

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