Stability of regular attractors for non-autonomous random dynamical systems and applications to stochastic Newton-Boussinesq equations with delays☆

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Abstract

In this paper, we establish theoretical results on the stability of random regular attractors. First, we introduce a backward regular attractor, which is a new type of attractor defined by a minimal backward pullback attracting set. We then establish an existence theorem for such an attractor, and prove it is long time stable. Eventually, we prove the long time stability of regular pullback random attractors. As an application, we consider stochastic non-autonomous Newton-Boussinesq equations with variable and distributed delays. Since solutions of the equations have no higher regularity, we prove their regular asymptotic compactness via the spectrum decomposition technique.

Keywords: Backward regular attractor; pullback random attractor; stability; delay; Newton-Boussinesq equations.
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1. Introduction

As we know, a random dynamical system can be generated by an evolution equation with stochastic perturbations (see [2, 5, 15] and the references therein). When such an equation has a time-dependent external force, we usually investigate its asymptotic behavior by non-autonomous random dynamical systems (NRDSs). A pullback random attractor (PRA) $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ plays a crucial role in dealing with dynamics of NRDSs. Hence, the existence of PRA for NRDSs has been extensively studied, see, e.g., [1, 8, 12, 18, 21, 22, 25, 26] and the references therein. Note that a PRA is a bi-parametric set depending on time and sample parameters. Then it is natural that we study the time-dependent or sample-dependent properties of PRA.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, then we claim that $(\Omega, \mathfrak{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system if $\theta_{\cdot}(\cdot) : \mathbb{R} \times \Omega \to \Omega$ is a $(\mathcal{B}(\mathbb{R}) \times \mathfrak{F}, \mathfrak{F})$-measurable mapping, $\theta_0(\cdot)$ is the identity

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on Ω, $\theta_{t+s}(\cdot) = \theta_t\theta_s(\cdot)$ for all $t, s \in \mathbb{R}$ and $P(\theta_t(\cdot)) = P$ for all $t \in \mathbb{R}$. Very recently, the authors in [9] studied the finite time stability of the PRA $\mathcal{A}$:

$$\lim_{\tau\to\tau_0} \mathrm{dist}_X(\mathcal{A}(\tau,\omega), \mathcal{A}(\tau_0,\omega)) = 0, \quad \text{for all } \omega \in \Omega,$$

where X is a Banach space with norm $\|\cdot\|_X$ and $\mathrm{dist}_X(\cdot,\cdot)$ denotes the Hausdorff semi-distance under the topology of X. In [28], Wang and Li established the long time stability of the PRA $\mathcal{A}$:

$$\lim_{\tau\to-\infty} \mathrm{dist}_X(\mathcal{A}(\tau,\omega), A(\omega)) = 0, \quad \text{for all } \omega \in \Omega,$$

where $A(\omega)$ is a nonempty compact set. Besides, they proved in [29] the probabilistic continuity of the PRA $\mathcal{A}$:

$$\lim_{s\to s_0} P\{\omega \in \Omega : \mathrm{dist}^h_X(\mathcal{A}(\tau,\theta_s\omega), \mathcal{A}(\tau,\theta_{s_0}\omega)) \geq \delta\} = 0, \quad \text{for all } \tau \in \mathbb{R},\ \delta > 0,$$

and

$$\lim_{\tau\to\tau_0} P\{\omega \in \Omega : \mathrm{dist}^h_X(\mathcal{A}(\tau,\omega), \mathcal{A}(\tau_0,\omega)) \geq \delta\} = 0, \quad \text{for all } \delta > 0,$$

where $\mathrm{dist}^h_X(\cdot,\cdot)$ denotes the Hausdorff distance under the topology of X.

Let Y be a Banach space with norm $\|\cdot\|_Y$ and $Y \hookrightarrow X$. Assume that $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is an (X,Y)-regular PRA (see Definition 2.4). To our knowledge, its long time stability has not been considered. In this paper, we will study that via an (X,Y)-backward regular attractor $\mathcal{E} = \{\mathcal{E}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in the sense of Definition 2.6. Such an attractor is defined by a compact set in X and Y, **dividedly invariant** (see Definition 2.5) and backward pullback attracting under the topology of Y.

Then, we establish an existence theorem of the (X,Y)-backward regular attractors $\mathcal{E}$ by regular backward pullback asymptotic compactness of NRDSs defined in Definition 2.8 (see Theorem 2.9). In addition, we also prove its long time stability, that is, there exists a nonempty compact set $B(\omega)$ such that

$$\lim_{\tau\to-\infty} \mathrm{dist}_Y(\mathcal{E}(\tau,\omega), B(\omega)) = 0, \quad \text{for all } \omega \in \Omega, \tag{1.1}$$

where $B(\omega)$ is minimal with respect to satisfying (1.1). Using the definition of $\mathcal{E}$, (1.1) is deduced easily (see Theorem 2.10). Finally, we use the existence of backward regular attractor to establish the long time stability of the (X,Y)-regular PRA $\mathcal{A}$:

$$\lim_{\tau\to-\infty} \mathrm{dist}_Y(\mathcal{A}(\tau,\omega), B_1(\omega)) = 0, \quad \text{for all } \omega \in \Omega, \tag{1.2}$$

where $B_1(\omega)$ is the minimal nonempty compact set satisfying (1.2). Furthermore, we also prove $B(\omega) = B_1(\omega)$ for all $\omega \in \Omega$ (see Theorem 2.11).

It is worth mentioning that the study of delay partial differential equations (PDEs) has drawn much attention on account of the importance of delay effects in physics, chemistry, engineering, biology, economics and in other real world applications. In addition, due to the presence of uncertainty and noise almost everywhere in our real life, it is sensible to consider a model with stochasticity or randomness to solve problems. The authors in references [16, 19, 20, 30, 31, 32, 33] have widely investigated the asymptotic behavior of stochastic delay PDEs. As an application, we consider the following stochastic non-autonomous retarded Newton-Boussinesq equations:

$$\begin{cases} d\tilde{\xi} - (\Delta\tilde{\xi} - u\frac{\partial\tilde{\xi}}{\partial x} - v\frac{\partial\tilde{\xi}}{\partial y} - \frac{R_a}{P_a}\frac{\partial\tilde{\vartheta}}{\partial x})dt = (h_1(\tilde{\xi}(t-\rho(t)), x, y) + f(t,x,y))dt + \tilde{\xi} \circ dW, \\ \Delta\Psi = \tilde{\xi}, u = \frac{\partial\Psi}{\partial y}, v = -\frac{\partial\Psi}{\partial x}, \\ d\tilde{\vartheta} - (\frac{1}{P_a}\Delta\tilde{\vartheta} - u\frac{\partial\tilde{\vartheta}}{\partial x} - v\frac{\partial\tilde{\vartheta}}{\partial y})dt = (\int_{-\varrho}^{0} h_2(\eta, \tilde{\vartheta}(t+\eta))d\eta + g(t,x,y))dt + \tilde{\vartheta} \circ dW, \\ \tilde{\xi}|_{\partial\mathcal{O}} = 0, \tilde{\vartheta}|_{\partial\mathcal{O}} = 0, \Psi|_{\partial\mathcal{O}} = 0, \\ \tilde{\xi}(\tau+\eta, x, y) := \tilde{\phi}(\eta, x, y), \tilde{\vartheta}(\tau+\eta, x, y) := \tilde{\varphi}(\eta, x, y), \ \eta \in [-\varrho, 0], \end{cases} \tag{1.3}$$

where $t > \tau$, $\tau \in \mathbb{R}$, $(x,y) \in \mathcal{O}$, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\mathcal{O}$. The positive constants R_a and P_a stand for the Rayleigh and Prandtl numbers, respectively. (u,v), $\tilde{\xi}$, $\tilde{\vartheta}$ and Ψ are the velocity vector of the fluid, the vortex, the flow temperature and the flow function, respectively. W is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathfrak{F}, P)$, f, g are given non-autonomous forcings, $\varrho > 0$ denotes the delay time of (1.3) h_1 and h_2 stand for variable delay and distributed delay, respectively. The Newton-Boussinesq equation was introduced in [7, 10] as a model for Benard flow. The deterministic and non-delay version of (1.3), i.e. $\tilde{\xi} = \tilde{\vartheta} = \varrho = 0$, has been studied in [11, 23, 24, 27]. As for the stochastic Newton-Boussinesq equations (1.3), no results have been reported on the asymptotic behavior of solutions, this issue remains open even for the stochastic model (1.3) without delay. Hence the aim of this work is to deal with stability of backward regular attractors for NRDSs. Since solutions to such equations have no higher regularity, we use the spectrum decomposition technique to overcome the difficulty.

We organize the rest of this paper as follows. In Section 2, we introduce some concepts related to two kinds of regular attractors, including (X,Y)-regular $\mathfrak{D}$-pullback random attractors and (X,Y)-backward regular attractors, then establish their abstract stability results. In Section 3, we apply the theoretical results to the stochastic Newton-Boussinesq equations with delay (1.3).

2. Theoretical results: stability of regular attractors for non-autonomous random dynamical systems

2.1. Concept and properties of backward regular attractors

In this subsection, we introduce a concept of a new type of attractors called a backward regular attractor and discuss its properties. For this purpose, we first recall some concepts related to pullback random attractors, see [25] for more details. Suppose that X is a separable Banach space with Borel σ-algebra $\mathfrak{B}(X)$ and norm $\|\cdot\|_X$, $(\Omega, \mathfrak{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

Definition 2.1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous NRDS on X over $(\Omega, \mathfrak{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $t, s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(i) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous;
(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity operator on X;
(iii) $\Phi(t+s, \tau, \omega, \cdot) = \Phi(t, \tau+s, \theta_s\omega, \Phi(s, \tau, \omega, \cdot))$;
(iv) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathfrak{B}(\mathbb{R}^+) \times \mathfrak{F} \times \mathfrak{B}(X), \mathfrak{B}(X))$-measurable.

Let $\mathfrak{D} = \{\mathcal{D} | \mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\}$, where $\mathcal{D}(\tau,\omega)$ is a nonempty subset of X. Let $\mathfrak{D}$ be inclusion-closed (resp. backward-closed) if $\hat{\mathcal{D}} \in \mathfrak{D}$(resp. $\tilde{\mathcal{D}} \in \mathfrak{D}$) whenever $\hat{\mathcal{D}}(\tau,\omega) \subset \mathcal{D}(\tau,\omega)$(resp. $\tilde{\mathcal{D}}(\tau,\omega) = \overline{\bigcup_{s\le\tau} \mathcal{D}(s,\omega)}$) and $\mathcal{D} \in \mathfrak{D}$.

Definition 2.2. A family $\mathcal{K} = \{\mathcal{K}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called a closed measurable $\mathfrak{D}$-pullback absorbing set for Φ if $\mathcal{K}$ is measurable in ω w.r.t. $\mathfrak{F}$ and for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, $\mathcal{K}(\tau, \omega)$ is a closed nonempty subset of X and there exists a $T := T(\tau, \omega, \mathcal{D})$ such that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)) \subseteq \mathcal{K}(\tau, \omega).$$

Assume that Y is a separable Banach space with norm $\|\cdot\|_Y$, and $Y \hookrightarrow X$. Then we recall the following definitions related to the (X, Y)-regular $\mathfrak{D}$-pullback random attractor. Inspired by [17], we introduce the following two definitions.

Definition 2.3. An NRDS Φ is said to be regular $\mathfrak{D}$-pullback asymptotically compact if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n\in\mathbb{N}}$ has a convergence subsequence in Y whenever $t_n \to +\infty$ and $x_n \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$.

Definition 2.4. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called an (X, Y)-regular $\mathfrak{D}$-pullback random attractor for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(i) $\mathcal{A}(\cdot, \cdot)$ is a random set in X;
(ii) $\mathcal{A}(\cdot, \cdot)$ is compact in X and Y;
(iii) $\mathcal{A}(\cdot, \cdot)$ is invariant, that is, $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(t + \tau, \theta_t\omega)$;
(iv) $\mathcal{A}(\cdot, \cdot)$ is $\mathfrak{D}$-pullback attracting under the topology of Y, that is, for every $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_Y(\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where $\mathrm{dist}_Y(\cdot, \cdot)$ denotes the Hausdorff semi-distance on Y.

Next, we introduce the definition of an (X, Y)-**backward regular attractor**.

Definition 2.5. A family $\mathcal{B} = \{\mathcal{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called **dividedly invariant** if there exists an invariant set $E = \{E(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\mathcal{B}(\tau, \omega) = \overline{\bigcup_{s\leq\tau} E(s, \omega)}$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.6. A family $\mathcal{E} = \{\mathcal{E}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is said to be an (X, Y)-**backward regular attractor** if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(i) $\mathcal{E}(\cdot, \cdot)$ is compact in X and Y;
(ii) $\mathcal{E}(\cdot, \cdot)$ is dividedly invariant;
(iii) $\mathcal{E}(\cdot, \cdot)$ is backward $\mathfrak{D}$-pullback attracting under the topology of Y, that is, for every $\mathcal{D} \in \mathfrak{D}$ and $s \leq \tau$,

$$\lim_{t\to+\infty} \mathrm{dist}_Y(\Phi(t, s - t, \theta_{-t}\omega, \mathcal{D}(s - t, \theta_{-t}\omega)), \mathcal{E}(\tau, \omega)) = 0.$$

Finally, we discuss the minimality of (X, Y)-backward regular attractors.

Theorem 2.7. *An (X, Y)-backward regular attractor $\mathcal{E} = \{\mathcal{E}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is a minimal backward $\mathfrak{D}$-pullback attracting set.*

Proof. Suppose that $\mathcal{B} = \{\mathcal{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is compact and backward $\mathfrak{D}$-pullback attracting. We now show that $\mathcal{B}(\tau, \omega) \supseteq \mathcal{E}(\tau, \omega)$. It follows from the divided invariance of $\mathcal{E}(\cdot, \cdot)$ that there exists an invariant set $E = \{E(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ such that $\mathcal{E}(\tau, \omega) = \overline{\bigcup_{s\leq\tau} E(s, \omega)}$. By the invariance of $E(\cdot, \cdot)$ and the backward $\mathfrak{D}$-pullback attractiveness of $\mathcal{B}(\cdot, \cdot)$, we obtain that

$$\mathrm{dist}_Y(E(s, \omega), \mathcal{B}(\tau, \omega)) = \mathrm{dist}_Y(\Phi(t, s - t, \theta_{-t}\omega, E(s - t, \theta_{-t}\omega)), \mathcal{B}(\tau, \omega)) \to 0,$$

as $t \to +\infty$. It follows from the compactness of $\mathcal{B}(\cdot, \cdot)$ that $E(s, \omega) \subseteq \mathcal{B}(\tau, \omega)$ for all $s \leq \tau$, and therefore we have $\mathcal{E}(\cdot, \cdot) \subseteq \mathcal{B}(\cdot, \cdot)$. This completes the proof. $\square$

2.2. Existence and stability of (X, Y)-backward regular attractors

This subsection is concerned with the existence and stability of (X, Y)-backward regular attractors. We start with the following definition with respect to regular backward $\mathfrak{D}$-pullback asymptotically compact for Φ.

Definition 2.8. An NRDS Φ is said to be regular backward $\mathfrak{D}$-pullback asymptotically compact if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, x_n)\}_{n\in\mathbb{N}}$ has a convergence subsequence in Y whenever $s_n \leq \tau$, $t_n \to +\infty$ and $x_n \in \mathcal{D}(s_n - t_n, \theta_{-t_n}\omega)$.

Let us now present an important result on an existence theorem of backward regular attractors.

Theorem 2.9. *Suppose that an NRDS Φ is regular backward $\mathfrak{D}$-pullback asymptotically compact and has a closed measurable $\mathfrak{D}$-pullback absorbing set $\mathcal{B} \in \mathfrak{D}$ in X, where $\mathfrak{D}$ is backward closed. Then Φ has an (X, Y)-backward regular attractor $\mathcal{E} \in \mathfrak{D}$.*

Proof. Since Φ is regular backward $\mathfrak{D}$-pullback asymptotically compact, by Definition 2.8 we obtain Φ is $\mathfrak{D}$-pullback asymptotically compact. Since $\mathfrak{D}$ is backward closed, then $\mathfrak{D}$ is inclusion closed, which combined with the fact that Φ has a closed measurable $\mathfrak{D}$-pullback absorbing set $\mathcal{B} \in \mathfrak{D}$ in X implies all conditions of [25, Lemma 2.21] are satisfied. Hence we obtain Φ has a $\mathfrak{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$.

We now show that $\mathcal{A}$ is an (X, Y)-regular $\mathfrak{D}$-pullback random attractor. Based on Definition 2.4, we first prove $\mathcal{A}$ is compact in Y. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(\tau, \omega)$. By the invariance of $\mathcal{A}$, there exists a $v_n \in \mathcal{A}(\tau - t_n, \theta_{-t_n}\omega)$ such that $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)v_n = u_n$. Since Φ is regular backward $\mathfrak{D}$-pullback asymptotically compact, we obtain $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence (not relabeled) in Y, and so there is a $u \in Y$ such that $\|u_n - u\|_Y \to 0$ as $n \to \infty$. To prove $\mathcal{A}$ is compact in Y, we need to show that $u \in \mathcal{A}(\tau, \omega)$. Since $\mathcal{A}(\tau, \omega)$ is compact in X, then there exists a $\tilde{u} \in \mathcal{A}(\tau, \omega)$ such that $\|u_n - \tilde{u}\|_X \to 0$ as $n \to \infty$. Since $Y \hookrightarrow X$, we have

$$\|u - \tilde{u}\|_X \leq \|u_n - u\|_X + \|u_n - \tilde{u}\|_X \leq \|u_n - u\|_Y + \|u_n - \tilde{u}\|_X \to 0, \text{ as } n \to \infty, \tag{2.1}$$

which implies $u = \tilde{u}$, and so $u \in \mathcal{A}(\tau, \omega)$. Then we have $\mathcal{A}$ is compact in Y. Next we prove $\mathcal{A}$ is $\mathfrak{D}$-pullback attracting under the topology of Y, that is, for each $\mathcal{D} \in \mathfrak{D}$, we show that

$$\lim_{t\to+\infty} \text{dist}_Y(\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0. \tag{2.2}$$

If (2.2) is not right, then there are $\delta > 0$, $t_n \to +\infty$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$ satisfying

$$\text{dist}_Y(\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n), \mathcal{A}(\tau, \omega)) \geq 3\delta. \tag{2.3}$$

Note that $\mathcal{A}$ is a $\mathfrak{D}$-pullback random attractor in X, it follows

$$\lim_{n\to\infty} \text{dist}_X(\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n), \mathcal{A}(\tau, \omega)) = 0, \tag{2.4}$$

for every $\mathcal{D} \in \mathfrak{D}$. By (2.4) and the compactness of $\mathcal{A}$ in X, we can find a $\bar{u} \in \mathcal{A}(\tau, \omega)$ such that $\|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n) - \bar{u}\|_X = 0$ as $n \to \infty$. Since Φ is regular backward $\mathfrak{D}$-pullback asymptotically compact, there is $\hat{u} \in Y$ such that $\|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n) - \hat{u}\|_Y = 0$. By the same method as in (2.1), we obtain $\hat{u} = \bar{u} \in \mathcal{A}(\tau, \omega)$. Then we have

$$\begin{aligned} &\text{dist}_Y(\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n), \mathcal{A}(\tau, \omega)) \\ &\quad \leq \|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_n) - \hat{u}\|_Y + \text{dist}_Y(\hat{u}, \mathcal{A}(\tau, \omega)) \to 0, \text{ as } n \to \infty, \end{aligned}$$

which contradicts (2.3). Hence (2.2) holds. Then we obtain $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is an (X, Y)-regular $\mathfrak{D}$-pullback random attractor.

Let $\mathcal{E}(\tau, \omega) = \overline{\bigcup_{s\le\tau} \mathcal{A}(s, \omega)}$. By the invariance of $\mathcal{A}$, we obtain $\mathcal{E}$ is dividedly invariant. To prove that $\mathcal{E}$ is an (X, Y)-backward regular attractor, we first show that $\mathcal{E}$ is compact in X and Y. Since $Y \hookrightarrow X$, we only need to prove $\mathcal{E}$ is compact in Y. Let $\{u_n\}_{n\in\mathbb{N}} \subset \bigcup_{s\le\tau} \mathcal{A}(s, \omega)$, then for each u_n, there exists a $s_n \le \tau$ such that $u_n \in \mathcal{A}(s_n, \omega)$. By the invariance of $\mathcal{A}$, there is a $v_n \in \mathcal{A}(s_n - t_n, \theta_{-t_n}\omega)$ such that $\Phi(t_n, s_n - t_n, \theta_{-t_n\omega}, v_n) = u_n$. It follows from the regular backward $\mathfrak{D}$-pullback asymptotic compactness of Φ that $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence we obtain $\bigcup_{s\le\tau} \mathcal{A}(s, \omega)$ is pre-compact in Y, and therefore $\mathcal{E}$ is compact in Y. Next, we show that $\mathcal{E}$ is backward $\mathfrak{D}$-pullback attracting under the topology of Y. For each $\mathcal{D} \in \mathfrak{D}$ and $s \le \tau$, we have

$$\begin{aligned}&\mathrm{dist}_Y(\Phi(t, s-t, \theta_{-t}\omega, \mathcal{D}(s-t, \theta_{-t}\omega)), \mathcal{E}(\tau, \omega))\\ &\quad \le \mathrm{dist}_Y(\Phi(t, s-t, \theta_{-t}\omega, \mathcal{D}(s-t, \theta_{-t}\omega)), \mathcal{A}(s, \omega)).\end{aligned}$$

Then, since $\mathcal{A}$ is $\mathfrak{D}$-pullback attracting under the topology of Y, we have

$$\lim_{t\to+\infty} \mathrm{dist}_Y(\Phi(t, s-t, \theta_{-t}\omega, \mathcal{D}(s-t, \theta_{-t}\omega)), \mathcal{E}(\tau, \omega)) = 0.$$

Hence we obtain $\mathcal{E}$ is backward $\mathfrak{D}$-pullback attracting under the topology of Y. Since $\mathfrak{D}$ is backward closed, we have $\mathcal{E} \in \mathfrak{D}$. Then Φ has an (X, Y)-backward regular attractor $\mathcal{E} \in \mathfrak{D}$. This proof is concluded. □

The following theorem is concerned with the stability of (X, Y)-backward regular attractors, which is deduced easily.

Theorem 2.10. *Suppose that Φ has an (X, Y)-backward regular attractor $\mathcal{E} \in \mathfrak{D}$, then $\mathcal{E}$ is long time stable, that is, there is a non-empty compact set $B(\omega)$ such that*

$$\lim_{\tau\to-\infty} \mathrm{dist}_Y(\mathcal{E}(\tau, \omega), B(\omega)) = 0, \quad \textit{for each} \ \ \omega \in \Omega, \tag{2.5}$$

where $B(\omega)$ is minimal satisfying (2.5)*.*

Proof. Let $B(\omega) = \bigcap_{\tau\le 0} \overline{\bigcup_{s\le\tau} \mathcal{E}(s, \omega)}$. By the divide invariance of $\mathcal{E}(\cdot, \cdot)$, we obtain $\tau \mapsto \mathcal{E}(\tau, \omega)$ is increasing, which along with the compactness of $\mathcal{E}(\tau, \omega)$ implies $B(\omega) = \bigcap_{\tau\le 0} \mathcal{E}(\tau, \omega)$. Hence, by the theorem of nested compact sets, we obtain $B(\omega)$ is non-empty compact in Y. We now prove that (2.5) holds. If (2.5) is false, then there are $\delta > 0$ and $x_n \in \mathcal{E}(\tau_n, \omega)$ with $\tau_n \to -\infty$ such that for all $n \in \mathbb{N}$,

$$\mathrm{dist}_Y(x_n, B(\omega)) \ge \delta. \tag{2.6}$$

Without loss of generality, we assume that $\tau_n \le 0$ for all $n \in \mathbb{N}$. Using the monotonicity of $\tau \mapsto \mathcal{E}(\tau, \omega)$, we have $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{E}(0, \omega)$. Then there is an $x \in \mathcal{E}(0, \omega) \subseteq B(\omega)$ such that $x_n \to x$ in Y. Consequently,

$$\mathrm{dist}_Y(x_n, B(\omega)) \le \|x_n - x\|_Y + \mathrm{dist}_Y(x, B(\omega)) \to 0, \quad \text{as} \ \ n \to \infty,$$

which contradicts (2.6).

Next, we show the minimality of $B(\omega)$. If there is a non-empty compact set $A(\omega)$ such that (2.5) holds, then we deduce

$$\lim_{\tau\to-\infty} \mathrm{dist}_Y(\mathcal{E}(\tau,\omega), A(\omega)) = 0, \quad \text{for each} \ \ \omega \in \Omega. \tag{2.7}$$

Let $x \in B(\omega)$. Note that $B(\omega)$ is an α-limit set of $\mathcal{E}(\cdot,\cdot)$. Then there are $x_n \in \mathcal{E}(\tau_n,\omega)$ and $\tau_n \to -\infty$ such that $x_n \to x$ in Y. By the definition of Hausdorff semi-distance and (2.7), we can find $y_n \in A(\omega)$ such that

$$\|x_n - y_n\|_Y = \mathrm{dist}_Y(x_n, A(\omega)) \le \mathrm{dist}_Y(\mathcal{E}(\tau_n,\omega), A(\omega)) \to 0, \quad \text{as} \ \ n \to \infty. \tag{2.8}$$

Since $A(\omega)$ is compact, there exists $y \in A(\omega)$ such that $y_n \to y$ in Y, which together with $x_n \to x$ in Y and (2.8) implies $x = y$. Thus we obtain $B(\omega) \subset A(\omega)$. The proof is complete. □

2.3. Relationship of (X,Y)-backward regular attractors and (X,Y)-regular pullback random attractors

This subsection is devoted to the stability of (X,Y)-backward regular attractors, which implies that of (X,Y)-regular pullback random attractors.

Theorem 2.11. *Suppose that Φ possesses an (X,Y)-backward regular attractor $\mathcal{E} \in \mathfrak{D}$ and an (X,Y)-regular $\mathfrak{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. Then $\mathcal{A}$ is long time stable, that is, there is a non-empty compact set $A(\omega)$ such that*

$$\lim_{\tau\to-\infty} \mathrm{dist}_Y(\mathcal{A}(\tau,\omega), A(\omega)) = 0, \quad \textit{for each} \ \ \omega \in \Omega, \tag{2.9}$$

where $A(\omega)$ is minimum that satisfying (2.9)*. In addition, $A(\omega) = B(\omega)$, where $B(\omega)$ is given in Theorem 2.10.*

Proof. It follows from the definition of (X,Y)-backward regular attractors, Theorems 2.7 and 2.9 that $\mathcal{A}(\tau,\omega) \subseteq \mathcal{E}(\tau,\omega)$. Let $A(\omega) = \bigcap_{\tau\le 0} \overline{\bigcup_{s\le\tau} \mathcal{A}(s,\omega)}$. Since $\mathcal{A}(\tau,\omega) \subseteq \mathcal{E}(\tau,\omega)$, we have $\overline{\bigcup_{s\le\tau} \mathcal{A}(s,\omega)} \subseteq \overline{\bigcup_{s\le\tau} \mathcal{E}(s,\omega)}$. By the monotonicity of $\tau \mapsto \mathcal{E}(\tau,\omega)$, we derive $\overline{\bigcup_{s\le\tau} \mathcal{A}(\tau,\omega)} \subseteq \mathcal{E}(\tau,\omega)$, and, consequently, $\overline{\bigcup_{s\le\tau} \mathcal{A}(\tau,\omega)}$ is compact in Y. Applying the theorem of nested compact sets we have $A(\omega)$ is non-empty compact in Y. We now check if (2.9) holds. If it does not hold, then there are $\delta > 0$ and $x_n \in \mathcal{A}(\tau_n,\omega)$ with $\tau_n \to -\infty$ such that

$$\mathrm{dist}_Y(x_n, A(\omega)) \ge \delta. \tag{2.10}$$

Note that $\{x_n\}_{n\in\mathbb{N}} \subset \overline{\bigcup_{s\le 0} \mathcal{A}(s,\omega)} \subseteq \mathcal{E}(0,\omega)$. By the compactness of $\mathcal{E}$, there exists $x \in Y$ such that $x_n \to x$ in Y. Since $A(\omega)$ is an α-limit set of $\mathcal{E}(\cdot,\cdot)$, we have $x \in A(\omega)$. Then we obtain

$$\mathrm{dist}_Y(x_n, A(\omega)) \le \|x_n - x\|_Y + \mathrm{dist}_Y(x, A(\omega)) \to 0, \quad \text{as} \ \ n \to \infty,$$

which contradicts (2.10). Hence (2.9) holds. As for the minimality of $A(\omega)$, it is similar to Theorem 2.10.

Next, we prove $A(\omega) = B(\omega)$. Using the definition of (X,Y)-backward regular attractors, Theorems 2.7 and 2.9, we obtain $\mathcal{E}(\tau,\omega) = \overline{\bigcup_{s\le\tau} \mathcal{A}(s,\omega)}$. By the monotonicity of $\tau \mapsto \mathcal{E}(\tau,\omega)$, we have

$$B(\omega) = \bigcap_{\tau\le 0} \overline{\bigcup_{s\le\tau} \mathcal{E}(s,\omega)} = \bigcap_{\tau\le 0} \mathcal{E}(\tau,\omega) = \bigcap_{\tau\le 0} \overline{\bigcup_{s\le\tau} \mathcal{A}(s,\omega)} = A(\omega).$$

This completes the proof. □

3. Application: Stochastic Newton-Boussinesq equations with delays

In the section, we apply our theoretical results to (1.3). To this end, we need to define a continuous NRDS for (1.3) and derive the backward uniform estimates of the solution for (1.3). In this paper, we use $\|\cdot\|$ and $(\cdot,\cdot)$ stand for the norm and the inner product of $L^2(\mathcal{O}) := H$, respectively. $\|\cdot\|_V$ and $\|\cdot\|_p$ $(p > 2)$ represent the norm of $H_0^1(\mathcal{O}) := V$ and $L^p(\mathcal{O})$, respectively.

3.1. Continuous non-autonomous random dynamical systems

We first identify the Wiener process $W(\cdot)$ with $\omega(\cdot)$ on a probability space $(\Omega, \mathfrak{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\}$, $\mathfrak{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and P is the Wiener measure. Define a group of measure-preserving transformation $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}$, then $(\Omega, \mathfrak{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. Next, we transform the stochastic equation (1.3) into the corresponding random one. For this purpose, let

$$\xi(t,\tau,\omega,\phi) = e^{-z(\theta_t\omega)}\tilde{\xi}(t,\tau,\omega,\tilde{\phi}), \quad \vartheta(t,\tau,\omega,\varphi) = e^{-z(\theta_t\omega)}\tilde{\vartheta}(t,\tau,\omega,\tilde{\varphi}), \quad t \geq \tau, \ \tau \in \mathbb{R}, \tag{3.1}$$

where $z(\theta_t\omega) = -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)ds$ is the one-dimensional Ornstein-Uhlenbeck process, which is a stationary solution of the Langevin equation: $dz + zdt = dW(t)$. As usual [2, 3, 4], there is a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega_0 \subset \Omega$ of full measure such that $t \mapsto z(\theta_t\omega)$ is pathwise continuous on Ω_0 and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0, \quad \forall \omega \in \Omega_0, \tag{3.2}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z(\theta_s\omega)|ds = \mathbb{E}(|z|) = \frac{1}{\sqrt{\pi}}, \quad \forall \omega \in \Omega_0. \tag{3.3}$$

For convenience, we do not distinguish the spaces Ω_0 and Ω in this paper. In addition, we define a function $J(\cdot,\cdot)$ by

$$J(u,v) = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}, \tag{3.4}$$

which satisfies:

$$\begin{cases} (J(u,v),v) = 0, \quad \text{for all } u \in H^1(\Omega), \ v \in H^2(\Omega) \cap H_0^1(\Omega), \\ \|J(u,v)\| \leq c\|u\|_{H^2}\|v\|_{H^2}, \quad \text{for all } u \in H^2(\Omega), \ v \in H^2(\Omega). \end{cases} \tag{3.5}$$

It follows from (3.1) and (3.5) that (1.3) can be rewritten as the following random equation:

$$\begin{cases} \frac{\partial \xi}{\partial t} - \Delta\xi + e^{-z(\theta_t\omega)}J(\Psi, e^{z(\theta_t\omega)}\xi) + \frac{R_a}{P_a}\frac{\partial\vartheta}{\partial x} \\ = e^{-z(\theta_t\omega)}(h_1(e^{z(\theta_{t-\rho(t)}\omega)}\xi(t-\rho(t)), x, y) + f(t,x,y)) + z(\theta_t\omega)\xi, \\ \Delta\Psi = e^{z(\theta_t\omega)}\xi, \\ \frac{\partial\vartheta}{\partial t} - \frac{1}{P_r}\Delta\vartheta + e^{-z(\theta_t\omega)}J(\Psi, e^{z(\theta_t\omega)}\vartheta) \\ = e^{-z(\theta_t\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{-z(\theta_{t+\eta}\omega)}\vartheta(t+\eta))d\eta + g(t,x,y)) + z(\theta_t\omega)\vartheta, \\ \xi|_{\partial\mathcal{O}} = 0, \vartheta|_{\partial\mathcal{O}} = 0, \Psi|_{\partial\mathcal{O}} = 0, \\ \xi(\tau+\eta, x, y) = e^{-z(\theta_{\tau+\eta}\omega)}\tilde{\xi}(\tau+\eta, \tau, \omega, \tilde{\phi}) := \phi(\eta, x, y), \\ \vartheta(\tau+\eta, x, y) = e^{-z(\theta_{\tau+\eta}\omega)}\tilde{\vartheta}(\tau+\eta, \tau, \omega, \tilde{\varphi}) := \varphi(\theta, x, y), \ \eta \in [-\varrho, 0]. \end{cases} \tag{3.6}$$

We now prove the existence of a continuous NRDS for (1.3) over $(\Omega, \mathfrak{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. To this end, we need to impose the following assumptions.

(D) $\rho(\cdot) \in C^1(\mathbb{R})$ satisfies

$$0 \le \rho(t) \le \varrho,\ \forall t \in \mathbb{R} \text{ and } \varrho_* = \sup_{t\in\mathbb{R}} \rho'(t) < 1. \tag{3.7}$$

(H1) h_1: $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies $h_1(0, \cdot, \cdot) = 0$ and there exists a positive constant L_{h_1} such that

$$|h_1(s_1, x, y) - h_1(s_2, x, y)| \le L_{h_1}|s_1 - s_2|,\ \forall s_1, s_2 \in \mathbb{R}, (x, y) \in \mathbb{R}^2. \tag{3.8}$$

(H2) h_2: $[-\varrho, 0] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies $h_2(\cdot, 0) = 0$ and there exists a positive function $L_{h_2}(\cdot)$ such that

$$|h_2(\eta, s_1) - h_2(\eta, s_2)| \le L_{h_2}(\eta)|s_1 - s_2|,\ \forall \eta \in [-\varrho, 0], s_1, s_2 \in \mathbb{R}, \tag{3.9}$$

where $L_{h_2}(\cdot) : [-\varrho, 0] \to \mathbb{R}^+$ satisfies $L_{h_2}(\cdot) \in L^2(-\varrho, 0)$.

We denote by $C_H := C([-\varrho, 0]; H)$ and $C_V := C([-\varrho, 0]; V)$ equipped with norms:

$$\|u\|_{C_H} = \sup_{\eta\in[-\varrho,0]} \|u(\eta)\|, \qquad \|u\|_{C_V} = \sup_{\eta\in[-\varrho,0]} \|u(\eta)\|_V.$$

Since (3.6) can be regarded as a deterministic equation parameterized by $\omega \in \Omega$, by the same method as in [13, 14] we derive the well-posedness of (3.6):

Lemma 3.1. *Suppose that* **(D)**, **(H1)** *and* **(H2)** *hold and* $f, g \in L^2_{loc}(\mathbb{R}, H)$. *Then, for all* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(\phi, \varphi) \in C_H \times C_H$, (3.6) *has a unique weak solution*

$$(\xi, \vartheta) \in C([\tau - \varrho, +\infty); H \times H) \cap L^2_{loc}(\tau, +\infty; V \times V),$$

such that $(\xi(\tau + \eta, \tau, \omega, \phi), \vartheta(\tau + \eta, \tau, \omega, \varphi)) = (\phi(\eta), \varphi(\eta))$ *for all* $\eta \in [-\varrho, 0]$.

Consider a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times (C_H \times C_H) \to (C_H \times C_H)$ by

$$\Phi(t, \tau, \omega, (\phi, \varphi)) = (\xi_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \phi), \vartheta_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \varphi)),\ t \in \mathbb{R}^+,\ \tau \in \mathbb{R},\ \omega \in \Omega, \tag{3.10}$$

for all $(\phi, \varphi) \in C_H \times C_H$, where (ξ, ϑ) is the solution of (3.6). Then, by Lemma 3.1 it is easy to check that Φ is a continuous NRDS for (3.6) in the sense of [25]. By (3.10), we obtain (1.3) has a continuous NRDS $\tilde{\Phi}$. Notice that Φ and $\tilde{\Phi}$ are equivalent (see [6, Theorem 3.4]), so we only consider Φ in this paper.

3.2. Backward uniform estimates of solutions

To derive backward uniform estimates of solutions, we need to assume that the non-autonomous terms f, g are **backward tempered**:

(G) $f, g \in L^2_{loc}(\mathbb{R}, H)$ are **backward limitable**:

$$\lim_{\beta\to+\infty} \sup_{s\le\tau} \int_{-\infty}^{s} e^{\beta(r-s)}(\|f(r)\|^2 + \|g(r)\|^2)dr = 0,\ \forall \tau \in \mathbb{R}, \tag{3.11}$$

which implies $f, g \in L^2_{loc}(\mathbb{R}, H)$ are **backward tempered**:

$$\sup_{s\le\tau} \int_{-\infty}^{s} e^{\beta(r-s)}(\|f(r)\|^2 + \|g(r)\|^2)dr < +\infty,\ \forall \tau \in \mathbb{R}, \beta > 0. \tag{3.12}$$

By (3.3) we define a positive constant $\bar{\gamma}$:

$$\bar{\gamma} := \frac{\lambda}{2(1+P_a)} - 2\mathbb{E}|z| - \left(\frac{L_{h_1}}{(1-\varrho_*)^{\frac{1}{2}}} + \varrho^{\frac{1}{2}}\|L_{h_2}(\cdot)\|_{L^2(-\varrho,0)}\right) e^{\frac{\lambda\varrho}{4(1+P_a)}}(\mathbb{E}(e^{-2z}) + \mathbb{E}(e^{2z})), \quad (3.13)$$

where λ denotes the Poincaré constant. Define a random variable $\gamma(\omega)$:

$$\gamma(\omega) = \frac{\lambda}{2(1+P_a)} - 2|z(\omega)| - \left(\frac{L_{h_1}}{(1-\varrho_*)^{\frac{1}{2}}} + \varrho^{\frac{1}{2}}\|L_{h_2}(\cdot)\|_{L^2(-\varrho,0)}\right) e^{\frac{\lambda\varrho}{4(1+P_a)}}(e^{-2z(\omega)} + e^{2z(\omega)}), \quad (3.14)$$

for all $\omega \in \Omega$. By the ergodic theorem [4, Theorem 2.1] we have

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \gamma(\theta_l\omega)dl = \mathbb{E}(\gamma) = \bar{\gamma}. \quad (3.15)$$

For the backward compactness and measurability of attractors for (3.6), we need to consider two attraction universes. Let $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $C_H \times C_H$, we claim that $\mathcal{D}$ is tempered if

$$\lim_{t\to+\infty} e^{-\beta t}\|\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2_{C_H\times C_H} = 0, \ \forall \tau \in \mathbb{R}, \omega \in \Omega, \beta > 0, \quad (3.16)$$

and $\mathcal{D}$ is **backward tempered** if

$$\lim_{t\to+\infty} e^{-\beta t}\sup_{s\le\tau}\|\mathcal{D}(s-t,\theta_{-t}\omega)\|^2_{C_H\times C_H} = 0, \ \forall \tau \in \mathbb{R}, \omega \in \Omega, \beta > 0. \quad (3.17)$$

Denote by $\mathfrak{D}$ the universe formed by the collection of all tempered families in $C_H \times C_H$:

$$\mathfrak{D} = \{\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \mathcal{D} \text{ satisfies (3.16)}\}.$$

Then $\mathfrak{D}$ is inclusion-closed. Another universe $\mathfrak{B}$ is composed of all **backward tempered** families in $C_H \times C_H$:

$$\mathfrak{B} = \{\mathcal{B} = \{\mathcal{B}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \mathcal{B} \text{ satisfies (3.17)}\}.$$

Then $\mathfrak{B}$ is **backward-closed**: $\mathcal{B} \in \mathfrak{B}$, then $\widetilde{\mathcal{B}} \in \mathfrak{B}$ with $\widetilde{\mathcal{B}}(\tau,\omega) = \bigcup_{s\le\tau}\mathcal{B}(s,\omega)$. It is simple to deduce that $\mathfrak{B} \subset \mathfrak{D}$ and $\mathfrak{B}$ is inclusion-closed.

3.2.1. Pullback absorbing sets

From now on, the c and $C(\omega)$ denote a positive constant and an intrinsic positive random variable, respectively.

Lemma 3.2. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. The following conclusions are true:*

(i) For each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, there exists $T_d := T_d(\mathcal{D},\tau,\omega) \ge 2\varrho + 2$ such that

$$\sup_{\tilde{\eta}\in[-2\varrho-2,0]}(\|\xi(\tau+\tilde{\eta},\tau-t,\theta_{-\tau}\omega,\phi)\|^2 + \|\vartheta(\tau+\tilde{\eta},\tau-t,\theta_{-\tau}\omega,\varphi)\|^2) \le c(1+R_d(\tau,\omega)), \quad (3.18)$$

for all $t \geq T_d$ and $(\phi, \varphi) \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$, where

$$R_d(\tau, \omega) = \int_{-\infty}^{0} e^{\int_0^r \gamma(\theta_l \omega) dl} e^{-2z(\theta_r \omega)} (\|f(r+\tau)\|^2 + \|g(r+\tau)\|^2) dr. \tag{3.19}$$

(ii) For each $\mathcal{B} \in \mathfrak{B}$, there exists $T_b := T_b(\mathcal{B}, \tau, \omega) \geq 2\varrho + 2$ such that

$$\sup_{s \leq \tau} \sup_{\tilde{\eta} \in [-2\varrho-2, 0]} (\|\xi(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \phi)\|^2 + \|\vartheta(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \varphi)\|^2) \leq c(1 + R_b(\tau, \omega)), \tag{3.20}$$

for all $t \geq T_b$ and $(\phi, \varphi) \in \mathcal{B}(s-t, \theta_{-t}\omega)$ with $s \leq \tau$, where

$$R_b(\tau, \omega) = \sup_{s \leq \tau} R_d(s, \omega). \tag{3.21}$$

Proof. Taking the inner product of the third equation of (3.6) with $\vartheta(r, s-t, \theta_{-s}\omega, \varphi)$ in H yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dr}\|\vartheta(r)\|^2 + \frac{1}{P_a}\|\nabla\vartheta(r)\|^2 + e^{-z(\theta_{r-s}\omega)}(J(\Psi, e^{z(\theta_{r-s}\omega)}\vartheta(r)), \vartheta(r)) \\
&= e^{-z(\theta_{r-s}\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta, \vartheta(r)) + e^{-z(\theta_{r-s}\omega)}(g(r), \theta(r)) + z(\theta_{r-s}\omega)\|\vartheta(r)\|^2.
\end{aligned}$$

Using the Poincaré inequality we have

$$\|\nabla\vartheta(r)\|^2 \geq \frac{\lambda}{2}\|\vartheta(r)\|^2 + \frac{1}{2}\|\nabla\vartheta(r)\|^2.$$

By the first equality of (3.5) we obtain

$$(J(\Psi, e^{z(\theta_{r-s}\omega)}\vartheta(r)), \vartheta(r)) = 0.$$

The Young inequality implies

$$e^{-z(\theta_{r-s}\omega)}(g(r), \vartheta(r)) \leq \frac{\lambda}{4P_a}\|\vartheta(r)\|^2 + \frac{P_a}{\lambda}e^{-2z(\theta_{r-s}\omega)}\|g(r)\|^2.$$

Hence, we deduce

$$\begin{aligned}
\frac{d}{dr}\|\vartheta(r)\|^2 + \frac{1}{P_a}\|\nabla\vartheta(r)\|^2 &\leq 2e^{-z(\theta_{r-s}\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta, \vartheta(r)) \\
&\quad + \frac{2P_a}{\lambda}e^{-2z(\theta_{r-s}\omega)}\|g(r)\|^2 + (2z(\theta_{r-s}\omega) - \frac{\lambda}{2P_a})\|\vartheta(r)\|^2.
\end{aligned} \tag{3.22}$$

Multiplying (3.22) by $e^{\int_{s-t}^{r} \gamma(\theta_{l-s}\omega)dl}$, and then integrating this result over $[s-t, s+\tilde{\eta}]$ with $\tilde{\eta} \in [-2\varrho-2, 0]$ and $t \geq 2\varrho+2$ we have

$$\begin{aligned}
&e^{\int_{s-t}^{s+\tilde{\eta}} \gamma(\theta_{l-s}\omega)dl}\|\vartheta(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \varphi)\|^2 + \frac{1}{P_a}\int_{s-t}^{s+\tilde{\eta}} e^{\int_{s-t}^{r} \gamma(\theta_{l-s}\omega)dl}\|\nabla\vartheta(r)\|^2 dr \\
&\leq \|\varphi\|_{C_H}^2 + c\int_{s-t}^{s+\tilde{\eta}} e^{\int_{s-t}^{r} \gamma(\theta_{l-s}\omega)dl} e^{-2z(\theta_{r-s}\omega)}\|g(r)\|^2 dr \\
&\quad + \int_{s-t}^{s+\tilde{\eta}} (2z(\theta_{r-s}\omega) + \gamma(\theta_{r-s}\omega) - \frac{\lambda}{2P_a}) e^{\int_{s-t}^{r} \gamma(\theta_{l-s}\omega)dl}\|\vartheta(r)\|^2 dr \\
&\quad + 2\int_{s-t}^{s+\tilde{\eta}} e^{\int_{s-t}^{r} \gamma(\theta_{l-s}\omega)dl} e^{-z(\theta_{r-s}\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta, \vartheta(r))dr.
\end{aligned} \tag{3.23}$$

The Young inequality implies

$$2\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{-z(\theta_{r-s}\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta, \vartheta(r))dr$$
$$\leq \varrho^{\frac{1}{2}}e^{\frac{\lambda\varrho}{4(1+P_a)}}\|L_{h_2}(\cdot)\|_{L^2(-\varrho,0)}\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{-2z(\theta_{r-s}\omega)}\|\vartheta(r)\|^2dr$$
$$+\frac{\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}\|\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta\|^2dr}{\varrho^{\frac{1}{2}}e^{\frac{\lambda\varrho}{4(1+P_a)}}\|L_{h_2}(\cdot)\|_{L^2(-\varrho,0)}}.$$

By (3.9), $h_2(\cdot,0)=0$ and the Hölder inequality we derive

$$\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}\|\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta\|^2dr$$
$$\leq \|L_{h_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{-\varrho}^{0}\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{2z(\theta_{r+\eta-s}\omega)}\|\vartheta(r+\eta)\|^2drd\eta$$
$$\leq e^{\frac{\lambda\varrho}{2(1+P_a)}}\|L_{h_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{-\varrho}^{0}\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r+\eta}\gamma(\theta_{l-s}\omega)dl}e^{2z(\theta_{r+\eta-s}\omega)}\|\vartheta(r+\eta)\|^2drd\eta$$
$$\leq \varrho e^{\frac{\lambda\varrho}{2(1+P_a)}}\|L_{h_2}(\cdot)\|^2_{L^2(-\varrho,0)}\|\varphi\|^2_{C_H}\int_{s-t-\varrho}^{s-t} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{2z(\theta_{r-s}\omega)}dr$$
$$+\varrho e^{\frac{\lambda\varrho}{2(1+P_a)}}\|L_{h_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{2z(\theta_{r-s}\omega)}\|\vartheta(r)\|^2dr,$$

where we use $\gamma(\theta_l\omega)\leq\frac{\lambda}{2(1+P_a)}$ in (3.14). Then, we have

$$2\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{-z(\theta_{r-s}\omega)}(\int_{-\varrho}^{0} h_2(\eta, e^{z(\theta_{r+\eta-s}\omega)}\vartheta(r+\eta))d\eta, \vartheta(r))dr$$
$$\leq \varrho^{\frac{1}{2}}e^{\frac{\lambda\varrho}{4(1+P_a)}}\|L_{h_2}(\cdot)\|_{L^2(-\varrho,0)}\int_{s-t}^{s+\tilde\eta} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}(e^{-2z(\theta_{r-s}\omega)}+e^{2z(\theta_{r-s}\omega)})\|\vartheta(r)\|^2dr$$
$$+\varrho e^{\frac{\lambda\varrho}{2(1+P_a)}}\|L_{h_2}(\cdot)\|^2_{L^2(-\varrho,0)}\|\varphi\|^2_{C_H}\int_{s-t-\varrho}^{s-t} e^{\int_{s-t}^{r}\gamma(\theta_{l-s}\omega)dl}e^{2z(\theta_{r-s}\omega)}dr. \tag{3.24}$$

Inserting (3.24) into (3.23), by (3.14) we deduce

$$\|\vartheta(s+\tilde\eta, s-t, \theta_{-s}\omega, \varphi)\|^2+\frac{1}{P_a}\int_{s-t}^{s+\tilde\eta} e^{\int_{s+\tilde\eta}^{r}\gamma(\theta_{l-s}\omega)dl}\|\nabla\vartheta(r)\|^2dr$$
$$\leq ce^{\frac{\lambda(\varrho+1)}{1+P_a}}\|\varphi\|^2_{C_H}(e^{\int_0^{-t}\gamma(\theta_l\omega)dl}+\int_{-t-\varrho}^{-t} e^{\int_0^{r}\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)$$
$$+ce^{\frac{\lambda(\varrho+1)}{1+P_a}}\int_{-t}^{0} e^{\int_0^{r}\gamma(\theta_l\omega)dl}e^{-2z(\theta_r\omega)}\|g(r+s)\|^2dr. \tag{3.25}$$

Taking the inner product of the first equation of (3.6) with $\xi(r, s-t, \theta_{-s}\omega, \phi)$ in H yields

$$\frac{1}{2}\frac{d}{dr}\|\xi(r)\|^2+\|\nabla\xi(r)\|^2+e^{-z(\theta_{r-s}\omega)}(J(\Psi, e^{z(\theta_{r-s}\omega)}\xi(r)), \xi(r))+\frac{R_a}{P_a}(\frac{\partial\vartheta}{\partial x}, \xi(r))$$
$$=e^{-z(\theta_{r-s}\omega)}(h_1(e^{z(\theta_{r-\rho(r)-s}\omega)}\xi(r-\rho(r)))+f(r), \xi(r))+z(\theta_{r-s}\omega)\|\xi(r)\|^2. \tag{3.26}$$

Similarly to (3.25), we have

$$
\begin{aligned}
&\|\xi(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \phi)\|^2 \\
&\quad \le c\int_{s-t}^{s+\tilde{\eta}} e^{\int_{s+\tilde{\eta}}^{r}\gamma(\theta_{l-s}\omega)dl}\|\nabla\vartheta(r)\|^2 dr + ce^{\frac{\lambda(\varrho+1)}{1+P_a}}\|\phi\|_{C_H}^2(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr) \\
&\qquad + ce^{\frac{\lambda(\varrho+1)}{1+P_a}}\int_{-t}^{0} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{-2z(\theta_r\omega)}\|f(r+s)\|^2 dr,
\end{aligned}
$$

which along with (3.25) implies

$$
\begin{aligned}
\|\xi(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \phi)\|^2 \le{}& ce^{\frac{\lambda(\varrho+1)}{1+P_a}}(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \\
&+ ce^{\frac{\lambda(\varrho+1)}{1+P_a}}\int_{-t}^{0} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{-2z(\theta_r\omega)}(\|f(r+s)\|^2 + \|g(r+s)\|^2)dr.
\end{aligned}
\tag{3.27}
$$

Combining (3.25) and (3.27), for all $\tilde{\eta} \in [-2\varrho-2, 0]$,

$$
\begin{aligned}
&\|\xi(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \phi)\|^2 + \|\vartheta(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \varphi)\|^2 \\
&\quad \le c(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \\
&\qquad + c\int_{-t}^{0} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{-2z(\theta_r\omega)}(\|f(r+s)\|^2 + \|g(r+s)\|^2)dr.
\end{aligned}
\tag{3.28}
$$

It follows from (3.2) and (3.15) that there exists $T := T(\bar{\gamma}, \omega) \ge 2\varrho + 2$ such that

$$
|z(\theta_r\omega)| \le \frac{\bar{\gamma}}{8}|r|, \qquad \left|\int_0^r (\gamma(\theta_l\omega) - \bar{\gamma})dl\right| \le \frac{\bar{\gamma}}{2}|r|, \text{ for all } |r| \ge T. \tag{3.29}
$$

(i) Let $s = \tau$ in (3.28). If $(\phi, \varphi) \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$ in (3.28). By (3.16) and (3.29) we have

$$
\begin{aligned}
&(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \\
&\quad \le c(e^{-\frac{\bar{\gamma}}{2}t} + e^{-\frac{\bar{\gamma}}{4}t})\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|_{C_H\times C_H}^2 \to 0, \quad \text{as } t \to +\infty.
\end{aligned}
$$

Hence, there exists $T_d = T_d(\mathcal{D}, \tau, \omega) > T$ such that for all $t \ge T_d$,

$$
(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \le 1,
$$

which implies (3.18) holds.

(ii) If $\phi \in \mathcal{B}(s - t, \theta_{-t}\omega)$ with $s \le \tau$ in (3.28). Thanks to (3.17) and (3.29) we obtain

$$
\begin{aligned}
&(e^{\int_0^{-t}\gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r\gamma(\theta_l\omega)dl}e^{2z(\theta_r\omega)}dr)\sup_{s\le\tau}(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \\
&\quad \le (e^{-\frac{\bar{\gamma}}{2}t} + e^{-\frac{\bar{\gamma}}{4}t})\sup_{s\le\tau}\|\mathcal{B}(s - t, \theta_{-t}\omega)\|_{C_H\times C_H}^2 \to 0, \quad \text{as } t \to +\infty.
\end{aligned}
$$

Then, there exists a $T_b := T_b(\mathcal{B}, \tau, \omega) \geq T$ such that for all $t \geq T_b$,

$$(e^{\int_0^{-t} \gamma(\theta_l\omega)dl} + \int_{-t-\varrho}^{-t} e^{\int_0^r \gamma(\theta_l\omega)dl} e^{2z(\theta_r\omega)}dr) \sup_{s\leq\tau}(\|\phi\|_{C_H}^2 + \|\varphi\|_{C_H}^2) \leq 1.$$

Taking the supremum of (3.5) over $s \in [-\infty, \tau)$ yields (3.3) as desired. $\square$

Based on Lemma 3.2, Φ possesses two pullback absorbing sets $\mathcal{K}_d \in \mathfrak{D}$ and $\mathcal{K}_b \in \mathfrak{B}$.

Lemma 3.3. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. The following results hold true:*

(i) Φ *has a closed measurable* $\mathfrak{D}$*-pullback absorbing set* $\mathcal{K}_d = \{\mathcal{K}_d(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$*, defined by*

$$\mathcal{K}_d(\tau, \omega) = \{(\phi, \varphi) \in C_H \times C_H : \|(\phi, \varphi)\|_{C_H\times C_H}^2 \leq c(1 + R_d(\tau, \omega))\}. \tag{3.30}$$

(ii) Φ *has a closed* $\mathfrak{B}$*-pullback absorbing set* $\mathcal{K}_b = \{\mathcal{K}_b(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{B}$*, given by*

$$\mathcal{K}_b(\tau, \omega) = \{(\phi, \varphi) \in C_H \times C_H : \|(\phi, \varphi)\|_{C_H\times C_H}^2 \leq c(1 + R_b(\tau, \omega))\} = \overline{\bigcup_{s\leq\tau} \mathcal{K}_d(s, \omega)}, \tag{3.31}$$

where $R_d(\tau, \omega)$ *and* $R_b(\tau, \omega)$ *are given in* (3.19) *and* (3.21) *respectively.*

Proof. By (3.18) and (3.20), we obtain $\mathcal{K}_d$ is a $\mathfrak{D}$-pullback absorbing set and $\mathcal{K}_b$ is a $\mathfrak{B}$-pullback absorbing set. In addition, it is straightforward to derive that $\omega \to R_d(\tau, \omega)$ is measurable since it is the integral of some random variables. Hence, $\mathcal{K}_d$ is a measurable set.

Now, we prove $\mathcal{K}_d \in \mathfrak{D}$ and $\mathcal{K}_b \in \mathfrak{B}$. Given $\beta > 0$ and let $\delta = \min\{\frac{\bar{\gamma}}{3}, \frac{\beta}{5}\}$. Then, by (3.2) and (3.15), there exists $T_1 := T_1(\delta, \omega) > 0$ such that

$$e^{2|z(\theta_t\omega)|} \leq e^{\delta|t|}, \quad \left|\int_0^t (\gamma(\theta_r\omega) - \bar{\gamma})dr\right| \leq \delta|t|, \text{ for all } |t| \geq T_1. \tag{3.32}$$

Hence, thanks to (3.12) and (3.32) we obtain, for all $t \geq T_1$ and $r \leq 0$,

$$\begin{aligned}
e^{\int_0^r \gamma(\theta_{l-t}\omega)dl} &= e^{\int_{-t}^{r-t} \gamma(\theta_l\omega)dl} \\
&\leq e^{\int_0^{r-t}(\gamma(\theta_l\omega)-\bar{\gamma})dl+\bar{\gamma}(r-t)-\int_0^{-t}(\gamma(\theta_l\omega)-\bar{\gamma})dl+\bar{\gamma}t} \\
&\leq e^{\left|\int_0^{r-t}(\gamma(\theta_l\omega)-\bar{\gamma})dl\right|+\left|\int_0^{-t}(\gamma(\theta_l\omega)-\bar{\gamma})dl\right|+\bar{\gamma}r} \\
&\leq e^{2\delta t}e^{(\bar{\gamma}-\delta)r} \\
&\leq e^{2\delta t}e^{2\delta r}.
\end{aligned}$$

Note that $\mathcal{K}_b$ is increasing ($\mathcal{K}_b(\tau_1, \omega) \subset \mathcal{K}_b(\tau_2, \omega)$ if $\tau_1 \leq \tau_2$). Then, we have, for all $t \geq T_1$ and $r \leq 0$,

$$\begin{aligned}
&e^{-\beta t} \sup_{s\leq\tau} \|\mathcal{K}_b(s-t, \theta_{-t}\omega)\|_{C_H\times C_H}^2 \\
&= e^{-\beta t}\|\mathcal{K}_b(\tau-t, \theta_{-t}\omega)\|_{C_H\times C_H}^2 \\
&\leq\ ce^{-\beta t}(1 + R_b(\tau-t, \theta_{-t}\omega)) = ce^{-\beta t}(1 + \sup_{s\leq\tau} R_d(s-t, \theta_{-t}\omega)) \\
&= ce^{-\beta t} + ce^{-\beta t} \sup_{s\leq\tau} \int_{-\infty}^0 e^{\int_0^r \gamma(\theta_{l-t}\omega)dl} e^{-2z(\theta_{r-t}\omega)}(\|f(r+s-t)\|^2 + \|g(r+s-t)\|^2)dr \\
&\leq ce^{-\beta t} + ce^{-(\beta-3\delta)t} \sup_{s\leq\tau} \int_{-\infty}^0 e^{\delta r}(\|f(r+s-t)\|^2 + \|g(r+s-t)\|^2)dr \\
&\leq ce^{-\beta t} + ce^{-(\beta-4\delta)t} \sup_{s\leq\tau} \int_{-\infty}^0 e^{\delta r}(\|f(r+s)\|^2 + \|g(r+s)\|^2)dr \to 0, \text{ as } t \to +\infty.
\end{aligned}$$

Then by (3.17) we obtain that $\mathcal{K}_b \in \mathfrak{B}$. By $R_d(\tau - t, \theta_{-t}\omega) \le R_b(\tau - t, \theta_{-t}\omega)$ and (3.16) we imply that $\mathcal{K}_d \in \mathfrak{D}$. The proof is concluded. $\square$

3.2.2. Backward pullback asymptotic compactness

Lemma 3.4. *Suppose that* **(D)**, **(H1)**-**(H2)** *and* **(G)** *hold. For each* $\mathcal{B} \in \mathfrak{B}$, $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$ *we have*

$$\sup_{s\le\tau} \sup_{\tilde{\eta}\in[-\varrho-1,0]} (\|\nabla\xi(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \phi)\|^2 + \|\nabla\vartheta(s+\tilde{\eta}, s-t, \theta_{-s}\omega, \varphi)\|^2)$$
$$\le C(\omega)(1 + R_b(\tau,\omega))^2(1 + R_b(\tau,\omega) + F(\tau)), \tag{3.33}$$

for all $t \ge T_b$ *(the same number as in Lemma 3.2) and* $(\phi, \varphi) \in \mathcal{B}(s-t, \theta_{-t}\omega)$ *with* $s \le \tau$*, where*

$$F(\tau) := \sup_{s\le\tau} \int_{-\infty}^{s} e^{\beta(r-s)}(\|F(r)\|^2 + \|g(r)\|^2)dr.$$

Proof. Integrating (3.22) over $[s-\varrho-2, s]$, by the Young inequality, (3.9) and (3.20) we deduce

$$\begin{aligned}\sup_{s\le\tau} \int_{s-\varrho-2}^{s} \|\nabla\vartheta(r)\|^2 dr &\le C(\omega)(1 + R_b(\tau,\omega)) + C(\omega) \sup_{s\le\tau} \int_{s-\varrho-2}^{s} \|g(r)\|^2 dr \\ &\le C(\omega)(1 + R_b(\tau,\omega)) + C(\omega)e^{\beta(\varrho+2)} \sup_{s\le\tau} \int_{s-\varrho-2}^{s} e^{\beta(r-s)}\|g(r)\|^2 dr \\ &\le C(\omega)(1 + R_b(\tau,\omega) + F(\tau)).\end{aligned} \tag{3.34}$$

Integrating (3.26) over $[s-\varrho-2, s]$, by the Young inequality, (3.8), (3.20) and (3.34) we obtain

$$\sup_{s\le\tau} \int_{s-\varrho-2}^{s} \|\nabla\xi(r)\|^2 dr \le C(\omega)(1 + R_b(\tau,\omega) + F(\tau)). \tag{3.35}$$

Multiplying the first equation of (3.6) by $-\Delta\xi(r, s-t, \theta_{-s}\omega, \phi)$ we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dr}\|\nabla\xi(r)\|^2 + \|\Delta\xi(r)\|^2 + e^{-z(\theta_{r-s}\omega)}(J(\Psi, e^{z(\theta_{r-s}\omega)}\xi), -\Delta\xi(r)) + \frac{R_a}{P_a}(\frac{\partial\vartheta}{\partial x}, -\Delta\xi(r)) \\ &\quad = e^{-z(\theta_{r-s}\omega)}(h_1(e^{z(\theta_{r-\rho(r)-s}\omega)}\xi(r-\rho(r))), -\Delta\xi(r)) + e^{-z(\theta_{r-s}\omega)}(f(r), -\Delta\xi(r)) + z(\theta_{r-s}\omega)\|\nabla\xi(r)\|^2.\end{aligned}$$

It follows from (3.4), $\Delta\Psi = e^{z(\theta_{r-s}\omega)}\xi$ and the Ladyzhenskaya inequality that

$$\begin{aligned}&-e^{-z(\theta_{r-s}\omega)}(J(\Psi, e^{z(\theta_{r-s}\omega)}\xi), -\Delta\xi(r)) \\ &\quad = \int_{\mathcal{O}} \frac{\partial\Psi}{\partial y}\frac{\partial\xi}{\partial x}\Delta\xi dxdy - \int_{\mathcal{O}} \frac{\partial\Psi}{\partial x}\frac{\partial\xi}{\partial y}\Delta\xi dxdy \\ &\quad \le \|\frac{\partial\Psi}{\partial y}\|_4\|\frac{\partial\xi}{\partial x}\|_4\|\Delta\xi\| + \|\frac{\partial\Psi}{\partial x}\|_4\|\frac{\partial\xi}{\partial y}\|_4\|\Delta\xi\| \\ &\quad \le \frac{1}{2^{1/4}}\|\frac{\partial\Psi}{\partial y}\|^{\frac{1}{2}}\|\frac{\partial\Psi}{\partial y}\|_V^{\frac{1}{2}}\|\frac{\partial\xi}{\partial x}\|^{\frac{1}{2}}\|\frac{\partial\xi}{\partial x}\|_V^{\frac{1}{2}}\|\Delta\xi\| + \frac{1}{2^{1/4}}\|\frac{\partial\Psi}{\partial x}\|^{\frac{1}{2}}\|\frac{\partial\Psi}{\partial x}\|_V^{\frac{1}{2}}\|\frac{\partial\xi}{\partial y}\|^{\frac{1}{2}}\|\frac{\partial\xi}{\partial y}\|_V^{\frac{1}{2}}\|\Delta\xi\| \\ &\quad \le c\|\Delta\Psi\|\|\nabla\xi\|^{\frac{1}{2}}\|\Delta\xi\|^{\frac{3}{2}} \le \frac{1}{6}\|\Delta\xi(r)\|^2 + ce^{4z(\theta_{r-s}\omega)}\|\xi(r)\|^4\|\nabla\xi(r)\|^2.\end{aligned}$$

The Young inequality implies

$$\begin{aligned}&-\frac{R_a}{P_a}(\frac{\partial\vartheta}{\partial x}, -\Delta\xi(r)) + e^{-z(\theta_{r-s}\omega)}(f(r), -\Delta\xi(r)) \\ &\quad \le \frac{1}{6}\|\Delta\xi(r)\|^2 + c\|\nabla\vartheta(r)\|^2 + ce^{-2z(\theta_{r-s}\omega)}\|f(r)\|^2.\end{aligned}$$

By (3.8), $h_1(0,\cdot,\cdot)=0$ and the Young inequality we derive

$$\begin{aligned}
&e^{-z(\theta_{r-s}\omega)}(h_1(e^{z(\theta_{r-\rho(r)-s}\omega)}\xi(r-\rho(r))),-\Delta\xi(r))\\
&\quad\leq \frac{1}{6}\|\Delta\xi(r)\|^2+ce^{-2z(\theta_{r-s}\omega)}\|h_1(e^{z(\theta_{r-\rho(r)-s}\omega)}\xi(r-\rho(r)))\|^2\\
&\quad\leq \frac{1}{6}\|\Delta\xi(r)\|^2+ce^{-2z(\theta_{r-s}\omega)}e^{2z(\theta_{r-\rho(r)-s}\omega)}\|\xi(r-\rho(r))\|^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\frac{d}{dr}\|\nabla\xi(r)\|^2+\|\Delta\xi(r)\|^2 &\leq ce^{4z(\theta_{r-s}\omega)}\|\xi(r)\|^4\|\nabla\xi(r)\|^2+c\|\nabla\vartheta(r)\|^2+ce^{-2z(\theta_{r-s}\omega)}\|f(r)\|^2\\
&\quad+ce^{-2z(\theta_{r-s}\omega)}e^{2z(\theta_{r-\rho(r)-s}\omega)}\|\xi(r-\rho(r))\|^2+z(\theta_{r-s}\omega)\|\nabla\xi(r)\|^2. \qquad (3.36)
\end{aligned}$$

Integrating (3.36) on $[\zeta,s+\tilde{\eta}]$ with $\zeta\in[s+\tilde{\eta}-1,s+\tilde{\eta}]$ and $\tilde{\eta}\in[-\varrho-1,0]$, and then integrating the result on $[s+\tilde{\eta}-1,s+\tilde{\eta}]$ w.r.t. ζ, we obtain for all $\tilde{\eta}\in[-\varrho-1,0]$,

$$\begin{aligned}
\|\nabla\xi(s+\tilde{\eta})\|^2\leq\ & C(\omega)\int_{s-\varrho-2}^{s}(\|\xi(r)\|^4\|\nabla\xi(r)\|^2+\|\nabla\vartheta(r)\|^2+\|f(r)\|^2)dr\\
&+C(\omega)\int_{s-\varrho-2}^{s}(\|\xi(r-\rho(r))\|^2+\|\nabla\xi(r)\|^2)dr. \qquad (3.37)
\end{aligned}$$

It follows from (3.20), (3.34) and (3.35) that

$$\begin{aligned}
&\sup_{s\leq\tau}\int_{s-\varrho-2}^{s}(\|\xi(r)\|^4\|\nabla\xi(r)\|^2+\|\nabla\vartheta(r)\|^2+\|\xi(r-\rho(r))\|^2+\|\nabla\xi(r)\|^2)dr\\
&\quad\leq c\sup_{s\leq\tau}\sup_{\tilde{\eta}\in[-2\varrho-2,0]}(1+\|\xi(s+\tilde{\eta})\|^4)\int_{s-\varrho-2}^{s}(\|\nabla\xi(r)\|^2+\|\nabla\vartheta(r)\|^2)dr\\
&\quad\leq C(\omega)(1+R_b(\tau,\omega))^2(1+R_b(\tau,\omega)+F(\tau)). \qquad (3.38)
\end{aligned}$$

Notice that

$$\sup_{s\leq\tau}\int_{s-\varrho-2}^{s}\|f(r)\|^2dr\leq\sup_{s\leq\tau}e^{\beta(\varrho+2)}\int_{s-\varrho-2}^{s}e^{\beta(r-s)}\|f(r)\|^2dr\leq cF(\tau). \qquad (3.39)$$

Substituting (3.38) and (3.39) into (3.37) we obtain

$$\sup_{s\leq\tau}\sup_{\tilde{\eta}\in[-2\varrho-2,0]}\|\nabla\xi(s+\tilde{\eta})\|^2\leq C(\omega)(1+R_b(\tau,\omega))^2(1+R_b(\tau,\omega)+F(\tau)). \qquad (3.40)$$

Similarly to (3.40), we have

$$\sup_{s\leq\tau}\sup_{\tilde{\eta}\in[-\varrho-1,0]}\|\nabla\vartheta(s+\tilde{\eta})\|^2\leq C(\omega)(1+R_b(\tau,\omega))^2(1+R_b(\tau,\omega)+F(\tau)),$$

which along with (3.40) implies (3.33). The proof is complete. □

Lemma 3.5. *Suppose that* **(D)**, **(H1)**-**(H2)** *and* **(G)** *hold. For each* $\mathcal{B}\in\mathfrak{B}$, $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$ *we have*

$$\begin{aligned}
&\sup_{s\leq\tau}\int_{s-\varrho}^{s}(\|\frac{\partial\xi}{\partial r}(r,s-t,\theta_{-s}\omega,\phi)\|^2+\|\frac{\partial\vartheta}{\partial r}(r,s-t,\theta_{-s}\omega,\varphi)\|^2)dr\\
&\quad\leq C(\omega)(1+R_b(\tau,\omega))^3(1+R_b(\tau,\omega)+F(\tau)), \qquad (3.41)
\end{aligned}$$

for all $t\geq T_b$ *and* $(\phi,\varphi)\in\mathcal{B}(s-t,\theta_{-t}\omega)$ *with* $s\leq\tau$.

Proof. Integrating (3.36) on $[s-\varrho-1,s]$, by (3.33) and (3.38) we obtain

$$\sup_{s\le\tau}\int_{s-\varrho-1}^{s}\|\Delta\xi(r)\|^2\le C(\omega)(1+R_b(\tau,\omega))^2(1+R_b(\tau,\omega)+F(\tau)). \tag{3.42}$$

Similarly to (3.42) we have

$$\sup_{s\le\tau}\int_{s-\varrho-1}^{s}\|\Delta\vartheta(r)\|^2\le C(\omega)(1+R_b(\tau,\omega))^2(1+R_b(\tau,\omega)+F(\tau)). \tag{3.43}$$

Multiplying the first equation of (3.6) by $\frac{\partial\xi}{\partial r}$, by the Young inequality, the second inequality of (3.5) and (3.8) we deduce

$$\begin{aligned}\|\frac{\partial\xi}{\partial r}\|^2\le{}& c\|\Delta\xi(r)\|^2+c\|\Delta\Psi(r)\|^2\|\Delta\xi(r)\|^2+c\|\nabla\vartheta(r)\|^2+c|z(\theta_{r-s}\omega)|^2\|\xi(r)\|^2\\&+ce^{-2z(\theta_{r-s}\omega)}(e^{-2z(\theta_{r-\rho(r)-s}\omega)}\|\xi(r-\rho(r))\|^2+\|f(r)\|^2).\end{aligned}$$

Integrating the above inequality over $[s-\varrho,s]$ yields

$$\begin{aligned}\int_{s-\varrho}^{s}\|\frac{\partial\xi}{\partial r}\|^2dr\le{}& C(\omega)\int_{s-\varrho}^{s}(\|\Delta\xi(r)\|^2+\|\xi(r)\|^2\|\Delta\xi(r)\|^2+\|\nabla\vartheta(r)\|^2+\|\xi(r)\|^2)dr\\&+C(\omega)\int_{s-\varrho}^{s}(\|\xi(r-\rho(r))\|^2+\|f(r)\|^2)dr.\end{aligned} \tag{3.44}$$

Inserting (3.20), (3.34) and (3.42) into (3.44) yields

$$\sup_{s\le\tau}\int_{s-\varrho}^{s}\|\frac{\partial\xi}{\partial r}\|^2dr\le C(\omega)(1+R_b(\tau,\omega))^3(1+R_b(\tau,\omega)+F(\tau)). \tag{3.45}$$

By the same method as in (3.45) we derive

$$\sup_{s\le\tau}\int_{s-\varrho}^{s}\|\frac{\partial\vartheta}{\partial r}\|^2dr\le C(\omega)(1+R_b(\tau,\omega))^3(1+R_b(\tau,\omega)+F(\tau)),$$

which together with (3.45) implies (3.41) holds. □

Lemma 3.6. *Suppose that* **(D)**, **(H1)**-**(H2)** *and* **(G)** *hold. Then,* Φ *defined by* (3.10) *is backward pullback asymptotically compact, that is, for each* $\mathcal{B}\in\mathfrak{B}$, $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$, $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n))\}_{n\in\mathbb{N}}$ *has a convergence subsequence in* $C_H\times C_H$ *whenever* $s_n\le\tau$, $t_n\to+\infty$ *and* $(\phi_n,\varphi_n)\in\mathcal{B}(s_n-t_n,\theta_{-t_n}\omega)$.

Proof. Based on the Ascoli-Arzelà theorem, we need to prove $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n))(\tilde\eta)\}_{n\in\mathbb{N}}$ is pre-compact in $C_H\times C_H$ for each $\tilde\eta\in[-\varrho,0]$ and $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n))\}_{n\in\mathbb{N}}$ is equi-continuous.

By Lemmas 3.2, 3.4 and the Sobolev embedding $V\times V\hookrightarrow H\times H$ is compact, we have $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n)(\tilde\eta)\}_{n\in\mathbb{N}}$ is pre-compact in $C_H\times C_H$ for each $\tilde\eta\in[-\varrho,0]$.

Now, we show that $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))\}_{n\in\mathbb{N}}$ is equi-continuous. For each $\tilde{\eta}_1, \tilde{\eta}_2 \in [-\varrho, 0]$ with $\tilde{\eta}_1 < \tilde{\eta}_2$, by (3.41) we have

$$
\begin{aligned}
&\|\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))(\tilde{\eta}_1) - \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))(\tilde{\eta}_2)\| \\
&\quad = \|\xi(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-t_n}\omega, \phi_n) - \xi(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-t_n}\omega, \phi_n)\| \\
&\quad\quad + \|\vartheta(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-t_n}\omega, \varphi_n) - \vartheta(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-t_n}\omega, \varphi_n)\| \\
&\quad \leq \int_{s+\tilde{\eta}_1}^{s+\tilde{\eta}_2} (\|\frac{\partial \xi}{\partial r}\| + \|\frac{\partial \theta}{\partial r}\|) dr \\
&\quad \leq \left(\left(\int_{s-\varrho}^{s} \|\frac{\partial \xi}{\partial r}\|^2 dr \right)^{\frac{1}{2}} + \left(\int_{s-\varrho}^{s} \|\frac{\partial \vartheta}{\partial r}\|^2 dr \right)^{\frac{1}{2}} \right) |\tilde{\eta}_1 - \tilde{\eta}_2|^{\frac{1}{2}} \\
&\quad \leq C(\omega)(1 + R_b(\tau, \omega))^3 (1 + R_b(\tau, \omega) + F(\tau)) |\tilde{\eta}_1 - \tilde{\eta}_2|^{\frac{1}{2}},
\end{aligned}
$$

which implies $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))\}_{n\in\mathbb{N}}$ is equi-continuous. The proof is complete. □

3.2.3. Regular backward pullback asymptotic compactness

Note that $-\Delta$ is a self-adjoint operator in H, and so $-\Delta$ has a corresponding eigenvalue sequence $\{\lambda_i\}_{i=1}^{\infty}$ and a complete orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H, where $\lambda_i > 0$ with $i = 1, 2, \cdots$, $\lambda_i \leq \lambda_j$ when $i < j$ and $\lambda_i \to +\infty$ as $i \to +\infty$. Let $P_j : H \to span\{e_1, e_2, \cdots, e_j\}$ be the projection operator, and so $v \in V$ has the following decomposition:

$$
v = P_j v + (I - P_j) v = v_{j,1} + v_{j,2}, \quad j \in \mathbb{N}. \tag{3.46}
$$

Lemma 3.7. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. For each $\mathcal{B} \in \mathfrak{B}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon, \tau, \omega) > 0$ with $\tilde{\eta}_1, \tilde{\eta}_2 \in [-\varrho, 0]$ and $|\tilde{\eta}_1 - \tilde{\eta}_2| < \delta$ such that*

$$
\begin{aligned}
&\sup_{s\leq\tau} (\|\nabla\xi_{j,1}(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \phi) - \nabla\xi_{j,1}(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \phi)\|^2 \\
&\quad + \|\nabla\vartheta_{j,1}(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \varphi) - \nabla\vartheta_{j,1}(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \varphi)\|^2) < \varepsilon,
\end{aligned} \tag{3.47}
$$

for all $t \geq T_b$ and $(\phi, \varphi) \in \mathcal{B}(s - t, \theta_{-t}\omega)$ with $s \leq \tau$.

Proof. By $\|\nabla\xi_{j,1}\| \leq \lambda_j^{\frac{1}{2}} \|\xi_{j,1}\|$ and (3.41) we derive

$$
\begin{aligned}
&\sup_{s\leq\tau} \|\nabla\xi_{j,1}(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \phi) - \nabla\xi_{j,1}(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \phi)\| \\
&\quad \leq \lambda_j^{\frac{1}{2}} \sup_{s\leq\tau} \|\xi_{j,1}(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \phi) - \xi_{j,1}(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \phi)\| \\
&\quad \leq \lambda_j^{\frac{1}{2}} \sup_{s\leq\tau} \|\xi(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \phi) - \xi(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \phi)\| \\
&\quad \leq \lambda_j^{\frac{1}{2}} \int_{s+\tilde{\eta}_1}^{s+\tilde{\eta}_2} \|\frac{\partial \xi}{\partial r}\| dr \leq \lambda_j^{\frac{1}{2}} \left(\int_{s-\varrho}^{s} \|\frac{\partial \xi}{\partial r}\|^2 dr \right)^{\frac{1}{2}} |\tilde{\eta}_1 - \tilde{\eta}_2|^{\frac{1}{2}} \\
&\quad \leq C(\omega) \lambda_j^{\frac{1}{2}} (1 + R_b(\tau, \omega))^3 (1 + R_b(\tau, \omega) + F(\tau)) |\tilde{\eta}_1 - \tilde{\eta}_2|^{\frac{1}{2}},
\end{aligned}
$$

which implies there exists $\delta := \delta(\varepsilon, \tau, \omega, j) > 0$ such that, when $|\tilde{\eta}_1 - \tilde{\eta}_2| < \delta$,

$$
\sup_{s\leq\tau} \|\nabla\xi_{j,1}(s + \tilde{\eta}_1, s - t, \theta_{-s}\omega, \phi) - \nabla\xi_{j,1}(s + \tilde{\eta}_2, s - t, \theta_{-s}\omega, \phi)\| < \frac{\varepsilon}{2}.
$$

Similarly, we have

$$\sup_{s\le\tau}\|\nabla\vartheta_{j,1}(s+\tilde\eta_1,s-t,\theta_{-s}\omega,\varphi)-\nabla\vartheta_{j,1}(s+\tilde\eta_2,s-t,\theta_{-s}\omega,\varphi)\|<\frac{\varepsilon}{2}.$$

Hence (3.47) holds. The proof is complete. $\square$

Lemma 3.8. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. For each $\mathcal{B}\in\mathfrak{B}$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and any $\varepsilon>0$, there exists a $j_0:=j_0(\varepsilon,\tau,\omega)\in\mathbb{N}>0$ such that*

$$\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}(\|\nabla\xi_{j,2}(s+\tilde\eta,s-t,\theta_{-s}\omega,\phi)\|^2+\|\nabla\vartheta_{j,2}(s+\tilde\eta,s-t,\theta_{-s}\omega,\varphi)\|^2)<\varepsilon, \tag{3.48}$$

for all $j\ge j_0$ and $\phi\in\mathcal{B}(s-t,\theta_{-t}\omega)$ with $s\le\tau$.

Proof. Taking the inner product of the first equation of (3.6) with $-\Delta\xi_{i,2}(r,s-t,\theta_{-s}\omega,\phi)$ in H, by the same method as in (3.36) we deduce

$$\begin{aligned}\frac{d}{dr}\|\nabla\xi_{j,2}(r)\|^2+\|\Delta\xi_{j,2}(r)\|^2&\le ce^{4z(\theta_{r-s}\omega)}\|\xi(r)\|^4\|\nabla\xi(r)\|^2+c\|\nabla\vartheta(r)\|^2+ce^{-2z(\theta_{r-s}\omega)}\|f(r)\|^2\\&\quad+ce^{-2z(\theta_{r-s}\omega)}e^{2z(\theta_{r-\rho(r)-s}\omega)}\|\xi(r-\rho(r))\|^2+z(\theta_{r-s}\omega)\|\nabla\xi_{j,2}(r)\|^2.\end{aligned}$$

By $\|\Delta\xi_{j,2}(r)\|^2\le\lambda_j\|\nabla\xi_{j,2}(r)\|^2$, we have

$$\begin{aligned}\frac{d}{dr}e^{\lambda_j r}\|\nabla\xi_{j,2}(r)\|^2&\le ce^{\lambda_j r}e^{4z(\theta_{r-s}\omega)}\|\xi(r)\|^4\|\nabla\xi(r)\|^2+c\|\nabla\vartheta(r)\|^2+ce^{\lambda_j r}e^{-2z(\theta_{r-s}\omega)}\|f(r)\|^2\\&\quad+ce^{\lambda_j r}e^{-2z(\theta_{r-s}\omega)}e^{2z(\theta_{r-\rho(r)-s}\omega)}\|\xi(r-\rho(r))\|^2+z(\theta_{r-s}\omega)e^{\lambda_j r}\|\nabla\xi_{j,2}(r)\|^2.\end{aligned}$$

Integrating the above inequality on $[\zeta,s+\tilde\eta]$ with $\zeta\in[s+\tilde\eta-1,s+\tilde\eta]$ and $\tilde\eta\in[-\varrho,0]$, and then integrating this result on $[s+\tilde\eta-1,s+\tilde\eta]$ with respect to ζ, we obtain

$$\begin{aligned}&e^{\lambda_j(s+\tilde\eta)}\|\nabla\xi_{j,2}(s+\tilde\eta)\|^2\\&\quad\le C(\omega)\int_{s+\tilde\eta-1}^{s+\tilde\eta}e^{\lambda_j r}(\|\xi(r)\|^4\|\nabla\xi(r)\|^2+\|\nabla\vartheta(r)\|^2+\|f(r)\|^2)dr\\&\qquad+C(\omega)\int_{s+\tilde\eta-1}^{s+\tilde\eta}e^{\lambda_j r}(\|\xi(r-\rho(r))\|^2+\|\nabla\xi_{j,2}(r)\|^2)dr\\&\quad\le C(\omega)\int_{s-1}^{s}e^{\lambda_j(r+\tilde\eta)}(\|\xi(r+\tilde\eta)\|^4\|\nabla\xi(r+\tilde\eta)\|^2+\|\nabla\vartheta(r+\tilde\eta)\|^2+\|f(r+\tilde\eta)\|^2)dr\\&\qquad+C(\omega)\int_{s-1}^{s}e^{\lambda_j(r+\tilde\eta)}(\|\xi(r+\tilde\eta-\rho(r+\tilde\eta))\|^2+\|\nabla\xi(r+\tilde\eta)\|^2)dr,\end{aligned}$$

which implies

$$\begin{aligned}&\|\nabla\xi_{j,2}(s+\tilde\eta)\|^2\\&\quad\le C(\omega)\int_{s-1}^{s}e^{\lambda_j(r-s)}(\|\xi(r+\tilde\eta)\|^4\|\nabla\xi(r+\tilde\eta)\|^2+\|\nabla\vartheta(r+\tilde\eta)\|^2+\|f(r+\tilde\eta)\|^2)dr\\&\qquad+C(\omega)\int_{s-1}^{s}e^{\lambda_j(r-s)}(\|\xi(r+\tilde\eta-\rho(r+\tilde\eta))\|^2+\|\nabla\xi(r+\tilde\eta)\|^2)dr.\end{aligned} \tag{3.49}$$

We now estimate every term of (3.49). By (3.20) and (3.33) we derive

$$
\begin{aligned}
&\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_j(r-s)}(\|\xi(r+\tilde\eta)\|^4\|\nabla\xi(r+\tilde\eta)\|^2+\|\nabla\vartheta(r+\tilde\eta)\|^2)dr\\
&\quad\le \sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho-1,0]}(\|\xi(s+\tilde\eta)\|^4\|\nabla\xi(s+\tilde\eta)\|^2+\|\nabla\vartheta(r+\tilde\eta)\|^2)\int_{s-1}^{s} e^{\lambda_j(r-s)}dr\\
&\quad\le \frac{C(\omega)}{\lambda_j}(1+R_b(\tau,\omega))^3(1+R_b(\tau,\omega)+F(\tau))\to 0\ \text{ as } k\to+\infty,
\end{aligned}
\tag{3.50}
$$

and

$$
\begin{aligned}
&\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_j(r-s)}(\|\xi(r+\tilde\eta-\rho(r+\tilde\eta))\|^2+\|\nabla\xi(r+\tilde\eta)\|^2)dr\\
&\quad\le (\sup_{s\le\tau}\sup_{\tilde\eta\in[-2\varrho-1,0]}\|\xi(s+\tilde\eta)\|^2+\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho-1,0]}\|\nabla\xi(s+\tilde\eta)\|^2)\int_{s-1}^{s} e^{\lambda_j(r-s)}dr\\
&\quad\le \frac{C(\omega)}{\lambda_j}(1+R_b(\tau,\omega))^3(1+R_b(\tau,\omega)+F(\tau))\to 0\ \text{ as } k\to+\infty.
\end{aligned}
\tag{3.51}
$$

By (3.11) we imply

$$
\begin{aligned}
\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_j(r-s)}\|f(r+\tilde\eta)\|^2dr &\le \sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\int_{s+\tilde\eta-1}^{s+\tilde\eta} e^{\lambda_j(r-(s+\tilde\eta))}\|f(r)\|^2dr\\
&= \sup_{\tilde\eta\in[-\varrho,0]}\sup_{s\le\tau+\tilde\eta}\int_{s-1}^{s} e^{\lambda_j(r-s)}\|f(r)\|^2dr\\
&\le \sup_{s\le\tau}\int_{s-1}^{s} e^{\lambda_j(r-s)}\|f(r)\|^2dr\to 0\ \text{ as } j\to+\infty.
\end{aligned}
\tag{3.52}
$$

Substituting (3.50)-(3.52) into (3.49) yields

$$
\lim_{j\to+\infty}\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\|\nabla\xi_{j,2}(s+\tilde\eta)\|^2=0. \tag{3.53}
$$

Similarly to (3.53), we have

$$
\lim_{j\to+\infty}\sup_{s\le\tau}\sup_{\tilde\eta\in[-\varrho,0]}\|\nabla\vartheta_{j,2}(s+\tilde\eta)\|^2=0,
$$

which along with (3.53) implies (3.48) holds. This completes the proof. □

Lemma 3.9. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. Then* Φ *in* (3.10) *is regular backward* $\mathfrak{B}$*-pullback asymptotically compact, more precisely, for each* $\tau\in\mathbb{R},\omega\in\Omega$ *and* $\mathcal{B}\in\mathfrak{B}$*, the sequence* $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n)\}_{n\in\mathbb{N}}$ *is pre-compact in* $C_V\times C_V$*, whenever* $s_n\le\tau$*,* $t_n\to+\infty$ *and* $(\phi_n,\varphi_n)\in\mathcal{B}(s_n-t_n,\theta_{-t_n}\omega)$*.*

Proof. Based on the Ascoli-Arzelà theorem, we prove the conclusion in the following two steps.

Step 1. We prove $\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega,(\phi_n,\varphi_n))\}_{n\in\mathbb{N}}$ in $C_V\times C_V$ is equi-continuous from $[-\varrho,0]$ to $V\times V$.

Without loss of generality, we assume that $t_n \geq T_b$ for all $n \in \mathbb{N}$ because of $t_n \to +\infty$. Assuming $\tilde{\eta}_1, \tilde{\eta}_2 \in [-\varrho, 0]$ with $\xi_2 > \xi_1$. Let $j_1 > j_0$ (j_0 is given in Lemma 3.8), by (3.47) and (3.48) we obtain

$$\begin{aligned}
&\|\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))(\tilde{\eta}_1) - \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))(\tilde{\eta}_2)\|_{V\times V} \\
&\quad \leq \|\xi(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n) - \xi(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n)\|_V \\
&\qquad + \|\vartheta(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \varphi_n) - \vartheta(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \varphi_n)\|_V \\
&\quad \leq \|\xi_{j_1,1}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n) - \xi_{j_1,1}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n)\|_V \\
&\qquad + \|\vartheta_{j_1,1}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \varphi_n) - \vartheta_{j_1,1}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \varphi_n)\|_V \\
&\qquad + \|(\xi_{j_1,2}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n), \vartheta_{j_1,2}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \varphi_n))\|_{V\times V} \\
&\qquad + \|(\xi_{j_1,2}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n), \vartheta_{j_1,2}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \varphi_n))\|_{V\times V} \\
&\quad \leq c\|\nabla\xi_{j_1,1}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n) - \nabla\xi_{j_1,1}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n)\| \\
&\qquad + c\|\nabla\vartheta_{j_1,1}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \varphi_n) - \nabla\vartheta_{j_1,1}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \varphi_n)\| \\
&\qquad + c\|(\nabla\xi_{j_1,2}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n), \nabla\vartheta_{j_1,2}(s_n + \tilde{\eta}_1, s_n - t_n, \theta_{-s_n}\omega, \varphi_n))\|_{H\times H} \\
&\qquad + c\|(\nabla\xi_{j_1,2}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n), \nabla\vartheta_{j_1,2}(s_n + \tilde{\eta}_2, s_n - t_n, \theta_{-s_n}\omega, \varphi_n))\|_{H\times H} \\
&\quad < c\sqrt{\varepsilon},
\end{aligned}$$

which implies $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))\}_{n\in\mathbb{N}}$ is equi-continuous.

Step 2. For each fixed $\tilde{\eta} \in [-\varrho, 0]$, we prove $\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (\phi_n, \varphi_n))(\tilde{\eta})$ is pre-compact in $V \times V$.

By (3.3) we obtain $\{(\xi_{j_1,1}(s_n + \tilde{\eta}, s_n - t_n, \theta_{-s_n}\omega, \phi_n), \vartheta_{j_1,1}(s_n + \tilde{\eta}, s_n - t_n, \theta_{-s_n}\omega, \varphi_n))\}$ is bounded in $V \times V$ and thus it is pre-compact in the j_1-dimensional subspace $V_{j_1} \times V_{j_1}$. Then, there is an index subsequence n^* of n such that $\{(\xi_{j_1,1}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \phi_{n^*}), \vartheta_{j_1,1}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \varphi_{n^*}))\}$ is a Cauchy sequence in $V_{j_1} \times V_{j_1}$. Let n^*, m^* enough large, by (3.48) we have

$$\begin{aligned}
&\|\xi(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \phi_{n^*}) - \xi(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \phi_{m^*})\|_V \\
&+ \|\vartheta(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \varphi_{n^*}) - \vartheta(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \varphi_{m^*})\|_V \\
&\quad \leq \|\xi_{j_1,1}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \phi_{n^*}) - \xi_{j_1,1}(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \phi_{m^*})\|_V \\
&\qquad + \|\vartheta_{j_1,1}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \varphi_{n^*}) - \vartheta_{j_1,1}(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \varphi_{m^*})\|_V \\
&\qquad + \|(\xi_{j_1,2}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \phi_{n^*}), \vartheta_{j_1,2}(s_{n^*} + \tilde{\eta}, s_{n^*} - t_{n^*}, \theta_{-s_{n^*}}\omega, \varphi_{n^*})\|_{V\times V} \\
&\qquad + \|(\xi_{j_1,2}(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \phi_{m^*}), \vartheta_{j_1,2}(s_{m^*} + \tilde{\eta}, s_{m^*} - t_{m^*}, \theta_{-s_{m^*}}\omega, \varphi_{m^*})\|_{V\times V} \\
&\quad < c\sqrt{\varepsilon}.
\end{aligned}$$

Then we obtain the sequence $\{(\xi(s_{n^*}+\tilde{\eta}, s_{n^*}-t_{n^*}, \theta_{-s_{n^*}}\omega, \phi_{n^*}), \vartheta(s_{n^*}+\tilde{\eta}, s_{n^*}-t_{n^*}, \theta_{-s_{n^*}}\omega, \varphi_{n^*}))\}$ is a Cauchy sequence in $V \times V$, and so it is convergent.

Hence, all conditions of the Ascoli-Arzelà theorem are satisfied. The proof is complete. □

3.2.4. Existence, long time stability of (X, Y)-regular pullback random attractors and (X, Y)-backward regular attractors

Theorem 3.10. *Suppose that* **(D)**, **(H1)**, **(H2)** *and* **(G)** *hold. The following results are true.*

(1) Φ in (3.10) has a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A}_d \in \mathfrak{D}$, defined by

$$\mathcal{A}_d(\tau, \omega) := \bigcap_{T>0} \overline{\bigcup_{t\geq T} \Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{K}_d(\tau - t, \theta_{-t}\omega)}. \tag{3.54}$$

(2) Φ in (3.10) *has a unique backward compact $\mathfrak{B}$-pullback bi-parametric attractor $\mathcal{A}_b \in \mathfrak{B}$, defined by*

$$\mathcal{A}_b(\tau,\omega) := \bigcap_{T>0} \overline{\bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{K}_b(\tau-t,\theta_{-t}\omega)}. \tag{3.55}$$

(3) $\mathcal{A}_d = \mathcal{A}_b$, that is, $\mathcal{A}_d$ is backward compact and $\mathcal{A}_b$ is measurable.

Proof. (1) By the same method as in Lemma 3.6, we have Φ is $\mathfrak{D}$-pullback asymptotically compact, which along with the (i) of Lemma 3.2 implies all conditions of [25, Lemma 2.21] are fulfilled, and so Φ in (3.10) has a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A}_d \in \mathfrak{D}$, defined by (3.54).

(2) It follows from the (ii) of Lemmas 3.2 and 3.6 that all conditions of [28, Theorem 3.9] are satisfied, and so (2) holds.

(3) By (3.31), (3.54) and (3.55) we obtain $\mathcal{A}_d \subset \mathcal{A}_b$. We also need to prove $\mathcal{A}_b \subset \mathcal{A}_d$. Note that $\mathfrak{B} \subset \mathfrak{D}$. Hence, we have $\mathcal{A}_b \in \mathfrak{D}$. By the invariance of $\mathcal{A}_b$ and the $\mathfrak{D}$-pullback attraction of $\mathcal{A}_d$ we have, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\text{dist}_{C_H\times C_H}(\mathcal{A}_b(\tau,\omega),\mathcal{A}_d(\tau,\omega)) = \text{dist}_{C_H\times C_H}(\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{A}_b(\tau-t,\theta_{-t}\omega),\mathcal{A}_d(\tau,\omega)) \to 0,$$

as $t \to +\infty$, which implies $\mathcal{A}_b(\tau,\omega) \subset \mathcal{A}_d(\tau,\omega)$ and so $\mathcal{A}_b \subset \mathcal{A}_d$. Then we have $\mathcal{A}_d = \mathcal{A}_b$. □

Theorem 3.11. *Suppose that* **(D)***,* **(H1)***,* **(H2)** *and* **(G)** *hold. Then, $\mathcal{A}_b$ derived in Theorem 3.10 is a backward compact regular pullback random attractor and Φ in* (3.10) *has an (X,Y)-backward regular attractor $\mathcal{A}$. Moreover, $\mathcal{A}_b$ and $\mathcal{A}$ are long time stable.*

Proof. Using the method of proof of Theorem 2.9, by Lemma 3.9 we obtain $\mathcal{A}_b$ is a backward compact regular pullback random attractor. It follows from Lemmas 3.2 and 3.9 that all conditions of Theorem 2.9 are satisfied, and so Φ in (3.10) has an (X,Y)-backward regular attractor $\mathcal{A}$. By Theorems 2.10 and 2.11, we obtain $\mathcal{A}_b$ and $\mathcal{A}$ are long time stable. The proof is complete. □

Data availability

No data was used for the research described in the article.

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