Effects of real random perturbations on Monod and Haldane consumption functions in the chemostat model

Tomás Caraballo[a,c], Javier López-de-la-Cruz[b,*] and Verónica Caraballo-Romero[a]

[a] Dpto. de Ecuaciones Diferenciales y Análisis Numérico, C/ Tarfia s/n, Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain.
[b] Dpto. de Matemática Aplicada a las TIC, Escuela Técnica Superior de Ingenieros Informáticos, Campus de Montegancedo, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain.
[c] Department of Mathematics, Wenzhou University, Wenzhou, Zhejiang Province, 325035, P. R. China.

In this paper, we investigate the classical chemostat model where the consumption function of the species, in both cases Monod and Haldane, is perturbed by real random fluctuations. Once the existence and uniqueness of non-negative global solution of the corresponding random systems is ensured, we prove the existence of a deterministic compact attracting set, whence we are able to find conditions to guarantee either the extinction or the persistence of the species, the most important aim in real applications. In addition, we depict several numerical simulations to illustrate the theoretical framework, standing out our contributions, providing the biological interpretation of every result and comparing with similar works in the literature.

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1. Introduction

The chemostat is a type of bioreactor commonly used for the growth of microorganisms in culture environments. It was invented by Monod (see [1]) and Novick and Szilard (see [2]) in the 1950s and possesses many applications in real life and industry (see, for instance, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein). Moreover, it is worth mentioning that the chemostat is also very interesting as a mathematical object, in fact, there is a recent framework called *the theory of the chemostat* (see [15, 16]) where many researchers from different areas of knowledge are involved since some decades ago.

The chemostat consists of three tanks, the *feed bottle*, the *culture vessel* and the *collection vessel*. A nutrient (or substrate) is stored in the feed bootle and (at least) one population of microorganisms (or species or microbial biomass) reside in the culture vessel. Thus, the nutrient is pumped from the first tank to the second one, where it is consumed by the species and the biological process starts. Finally, another flow of material is removed

*corresponding author: javier.lopez.delacruz@upm.es

from the culture vessel to the collection vessel to keep the volume of the culture vessel constant during the experiment.

Hence, the main goal is to understand how the concentrations of nutrient and microorganisms evolve in the culture vessel. To this end, we use the chemostat model (see [15]), which is given by

$$\frac{dn}{dt} = D(n_{\text{in}} - n) - c(n)m, \tag{1}$$

$$\frac{dm}{dt} = -Dm + c(n)m, \tag{2}$$

where $n = n(t)$ and $m = m(t)$ denote the concentrations of nutrient and microorganisms at time t, respectively, $n_{\text{in}} > 0$ represents the concentration of nutrient in the feed bottle, $D > 0$ is the dilution rate and c is the function that describes how the nutrient is consumed by the microorganisms.

Among the different reasons why the chemostat device possesses such a high interest, we point out that it allows us to describe real phenomena with a very good accuracy (see [15] and references therein).

The classical consumption function considered by many authors when dealing with chemostat models is the *Monod* one

$$c(n) = \frac{c_0 n}{d + n}, \tag{3}$$

where $c_0 > 0$ denotes the maximum specific growth rate of the microorganisms and $d > 0$ is the half-saturation constant. However, it is very well known from practitioners that certain populations of microorganisms reduce their consumption of substrate if it is at large concentration in the culture vessel. In such a case, another consumption function must be considered, which is called *Haldane* (see [15, 17, 18]) and is given by

$$c(n) = \frac{c_0 n}{d + n + \dfrac{n^2}{d_i}}, \tag{4}$$

where $d_i > 0$ represents the inhibition constant and the rest of the parameters are defined as in the case of the Monod function.

The chemostat model (1)-(2) has been widely investigated in the literature, specially the case when considering the Monod consumption function. Nevertheless, it is a deterministic system and then it assumes restrictions that are very strong, particularly when taking into consideration that real life is often subject to suffer random disturbances (see, for instance, [17, 18, 19, 20, 21, 22, 23]).

In order to set up more realistic chemostat models, we propose in this paper to introduce real noise on the consumption function of the species (in both cases, Monod and Haldane) in the deterministic chemostat model (1)-(2). More precisely, the idea is to replace the maximum specific growth rate of the species c_0 by the random term

$c_0 + \phi(\xi^*(\theta_t\omega))$, where $\phi(\xi^*(\theta_t\omega))$ denotes a bounded noise (see details in Section 2) which has been proved to be a good tool when modeling real random fluctuations not only in the chemostat model but also in other models arising in population dynamics (see [17, 18, 19, 20, 21, 22, 23, 24]).

For instance, in our last paper [24] we studied the influence of the environmental noise in the classical chemostat model (1)-(2) with both Monod and Haldane consumption function, i.e., we added the noisy terms $\phi(\xi^*(\theta_t\omega))n$ and $\phi(\xi^*(\theta_t\omega))m$ at the end of the equations (1) and (2), respectively, but we kept the consumption function (in both cases, Monod and Haldane) as the purely deterministic expressions (3) and (4).

However, it is worth mentioning that typically the smooth deterministic consumption functions (3) and (4) are derived as approximations of dotted lines obtained from experimental data in the laboratory thanks to measurements in real devices. This means that the value $c(n)$ cannot be totally determined in real life (in a deterministic way) for every value n. Then, the idea of perturbing randomly such consumption functions, which is the main innovation in this paper, makes sense completely from the practical point of view, since it allows us to obtain much more realistic models. In fact, this is the main reason that encouraged us to work on this paper and the key difference between this paper and our last work [24].

Having a look at the existing literature, the authors in [25] investigated the chemostat model (1)-(2) with Monod consumption function and they replace c_0 by $c_0 + \alpha\dot{W}(t)$, where $\dot{W}(t)$ denotes the white noise and $\alpha > 0$ is the amount of noise. Let us summarize in the following paragraphs the main differences between the work in [25] and our paper.

On the one hand, the extinction of the microorganisms is obtained in [25] as long as $D > c(n_{\text{in}})$ is fulfilled, whereas the condition to have extinction in our paper is $D > c_0$, as we prove in Section 3. Notice that our condition is less restrictive, since $c(n_{\text{in}}) < c_0$.

On the other hand, the authors in [25] provide conditions to ensure the persistence of the microorganisms in the mean, i.e.,

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t m(\tau)d\tau \geq \kappa > 0,$$

whilst we prove the persistence of the microorganisms in the stronger sense

$$\lim_{t\to+\infty} m(t) \geq \kappa > 0.$$

In addition, we would like to highlight that the authors in [25] use the standard Wiener process to model random disturbances. Nevertheless, this way of modeling randomness can produce important drawbacks from the biological point of view, as we proved in [19, 26]. That is the reason why we consider in this paper another way which has been proved to be much more realistic (see [17, 18, 19, 20, 21, 22, 23, 24]).

Our paper is organized as follows. In Section 2 we introduce some preliminaries regarding the real noise. After that, in Section 3, we investigate the chemostat model (1)-(2)

with Monod consumption function, where the maximum specific growth rate of the microorganisms c_0 is subject to random bounded fluctuations. Later, in Section 4, we study the chemostat model (1)-(2) with Haldane consumption function, where c_0 is subject to random bounded fluctuations as well. In both cases we ensure the well-posedness of the corresponding random system, we prove the existence of deterministic compact attracting sets and we provide conditions to guarantee the extinction and persistence of the microorganisms. In addition, we depict numerical simulations to illustrate the theoretical work. Finally, in Section 5 we sum up our contribution in this paper and we provide detailed biological interpretations of the results proved along the paper.

2. Preliminaries

We include in this section some basic preliminaries needed to understand this paper. Details and proofs related to the results in this section can be found in [19, 27, 28] and references therein.

Let W be a two sided standard Wiener process. From Kolmogorov's theorem, W has a continuous version, that we usually denote by ω, whose canonical interpretation is as follows. Define Ω as the set of every continuous function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(0) = 0$. Then, consider $\mathcal{F}$ the Borel $\sigma-$algebra of measurable subsets of Ω generated by the compact open topology (see [29]) and $\mathbb{P}$ the Wiener measure on $\mathcal{F}$.

Define now the family $\{\theta_t\}_{t\in\mathbb{R}}$, where θ_t is a mapping from Ω to itself (see [28]) defined as $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$. Thus, the Ornstein-Uhlenbeck process is defined as

$$(t, \omega) \mapsto \xi(t, \omega) := \xi^*(\theta_t\omega) = -\int_{-\infty}^{0} e^s \theta_t\omega(r)dr, \tag{5}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$.

The following proposition collects the most importante properties needed in this paper concerning the Ornstein-Uhlenbeck process.

Proposition 2.1 (See [30])**.** *There exists a θ_t-invariant set $\Omega_0 \subset \mathcal{F}$ of Ω of full measure such that, for almost every $\omega \in \Omega_0$, the mapping* (5) *is a stationary solution of the Langevin equation $d\xi + \xi dt = d\omega$ with continuous trajectories and*

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \xi^*(\theta_r\omega)dr = 0. \tag{6}$$

is also fulfilled.

Let us now introduce the mapping $\phi : \mathbb{R} \to \left[-\frac{\varepsilon\pi}{2}, \frac{\varepsilon\pi}{2}\right]$, where $\varepsilon > 0$ is fixed, given by

$$\phi(\xi) = \varepsilon \arctan(\xi). \tag{7}$$

Hence, we consider the stochastic process $\phi(\xi^*(\theta_t\omega))$, known as *real noise*, which is bounded, i.e.,

$$-\frac{\varepsilon\pi}{2} < \phi(\xi^*(\theta_t\omega)) < \frac{\varepsilon\pi}{2} \tag{8}$$

for any $t \in \mathbb{R}$ and $\omega \in \Omega$.

In addition, the ergodic property (6), which is very useful when making calculations, as we will see later, is also satisfied by the stochastic process $\phi(\xi^*(\theta_t\omega))$, i.e.,

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \phi(\xi^*(\theta_r\omega))dr = 0 \tag{9}$$

holds true almost surely in Ω (see [17] for the proof).

3. The random chemostat model with Monod consumption function

As explained in the introduction of the paper, in this section we are interested in investigating the chemostat model (1)-(2) with Monod consumption function, where the maximum specific growth rate of the species c_0 is subject to random bounded fluctuations modeled by the real noise. The resulting random chemostat is given by

$$\frac{dn}{dt} = D(n_{\text{in}} - n) - \frac{(c_0 + \phi(\xi^*(\theta_t\omega)))n}{d+n}m, \tag{10}$$

$$\frac{dm}{dt} = -Dm + \frac{(c_0 + \phi(\xi^*(\theta_t\omega)))n}{d+n}m, \tag{11}$$

where we recall that $n = n(t)$ and $m = m(t)$ denote the nutrient and the microorganisms concentrations, respectively, $n_{\text{in}} > 0$ represents the concentration of nutrient in the feed bottle, $D > 0$ is the dilution rate, $c_0 > 0$ denotes the maximum specific growth rate of the species and $d > 0$ is the half-saturation constant.

3.1. Well-posedness of the random chemostat

From now on, we define $\mathbb{R}^2_+ := \{(n,m) \in \mathbb{R}^2 : n, m \geq 0\}$.

Theorem 3.1. *For any given initial condition $u_0 = (n_0, m_0) \in \mathbb{R}^2_+$ and $\omega \in \Omega$, the random chemostat* (10)-(11) *possesses a unique solution*

$$u(t; 0, \omega, u_0) := (n(t; 0, \omega, u_0), m(t; 0, \omega, u_0)) \in \mathcal{C}^1([0, +\infty); \mathbb{R}^2_+),$$

where $u(t; 0, \omega, u_0)$ denotes the value at time t of the solution that starts with initial condition u_0 at time 0 and depends on ω. Both $n(t; 0, \omega, u_0)$ and $m(t; 0, \omega, u_0)$ are defined equivalently and we recall that $n_0 = n(0; 0, \omega, u_0)$ and $m_0 = m(0; 0, \omega, u_0)$.

Proof. From the second statement in Proposition 2.1, we have that $\xi^*(\theta_t\omega)$ is continuous with respect to t. Since ϕ is also continuous, the vector field of (10)-(11) is continuous with respect to t. In addition, the vector field of (10)-(11) is differentiable with respect to both n and m, hence it is locally Lipschitz with respect to n and m. Thus, we obtain the existence and uniqueness of local solution of the random chemostat (10)-(11).

Next, we prove that the unique local solution of (10)-(11) remains in $\mathbb{R}^2_+$ for every initial condition in $\mathbb{R}^2_+$. To this end, it enough to notice that $m \equiv 0$ solves (11) and

$$\left.\frac{dn}{dt}\right|_{n=0} = Dn_{\text{in}} > 0.$$

Now, we prove that the unique local solution of (10)-(11) is defined for every $t \geq 0$. Let us introduce the new state variable $b = n + m$. It is easy to check that b satisfies

$$\frac{db}{dt} = Dn_{\text{in}} - Db, \tag{12}$$

whence we obtain that

$$b(t; 0, \omega, b_0) = b_0 e^{-Dt} + \left[1 - e^{-Dt}\right] n_{in} \tag{13}$$

for every $t \geq 0$, $\omega \in \Omega$ and $b_0 := n_0 + m_0 > 0$.

Finally, thanks to (13), it follows that n and m do not blow up at any finite time, whence the unique local solution of (10)-(11) is defined for every $t \geq 0$. □

3.2. Absorbing and attracting sets

Now, we are interested in proving that there exists a compact set that attracts the solutions of the random chemostat (10)-(11) forwards in time.

Theorem 3.2. *For any $\delta > 0$, the deterministic compact set*

$$\mathcal{D}_\delta := \left\{(n, m) \in \mathbb{R}^2_+ \,:\, n_{in} - \delta \leq n + m \leq n_{in} + \delta\right\} \tag{14}$$

is absorbing for the solutions of the random chemostat (10)-(11)*, i.e., for every bounded set $E \subset \mathbb{R}^2_+$ and $\omega \in \Omega$, there exists a time $T_E(\delta, \omega) > 0$ such that the solution $u(t; 0, \omega, u_0)$ of* (10)-(11) *is contained in $\mathcal{D}_\delta$ for every $u_0 \in E$ and $t \geq T_E(\delta, \omega)$.*

Proof. From (13), it yields that

$$n_{\text{in}}(1 - e^{-Dt}) \leq b(t; 0, \omega, b_0) \leq n_{\text{in}} + b_0 e^{-Dt} \tag{15}$$

for every $t \geq 0$, $\omega \in \Omega$ and $b_0 = n_0 + m_0 > 0$.

Hence, for every $\delta > 0$, $\omega \in \Omega$ and $E \subset \mathbb{R}^2_+$ bounded, there exists $T_E(\delta, \omega) > 0$ such that

$$n_{\text{in}} - \delta \leq b(t; 0, \omega, b_0) \leq n_{\text{in}} + \delta$$

for all $u_0 \in E$ and $t \geq T_E(\delta, \omega)$, whence the compact set $\mathcal{D}_\delta$ is absorbing for the solutions of the random chemostat (10)-(11) for every given $\delta > 0$. In addition, $\mathcal{D}_\delta$ is independent of the noise. □

The next corollary follows trivially thanks to Theorem 3.2.

Corollary 3.1. *The deterministic compact set*

$$\mathcal{D}_0 := \left\{(n, m) \in \mathbb{R}^2_+ \,:\, n + m = n_{in}\right\} \tag{16}$$

attracts the solutions of the random chemostat (10)-(11)*, i.e.,*

$$\lim_{t \to +\infty} \sup_{u_0 \in E} \inf_{d_0 \in \mathcal{D}_0} \|u(t; 0, \omega, u_0) - d_0\|_{\mathbb{R}^2_+} = 0. \tag{17}$$

Remark 3.1. *Notice that the compact absorbing set $\mathcal{D}_\delta$ and the compact attracting set $\mathcal{D}_0$ defined in* (14) *and* (16)*, respectively, are both independent of the noise, even though the chemostat model* (10)-(11) *is subject to randomness.*

3.3. Extinction and persistence of the microorganisms

In this subsection we aim to supply conditions that ensure the extinction of the microorganisms in the random chemostat (10)-(11) as well as conditions that guarantee that they survive, which is the most interesting result (at least, from the biological point of view) in the paper.

Theorem 3.3. *The singleton point*

$$\mathcal{D}_0^{ext} := \{(n_{in}, 0)\} \subset \mathcal{D}_0 \tag{18}$$

is attracting for the solutions of the random chemostat (10)-(11) *whether*

$$D > c_0 + \frac{\varepsilon\pi}{2} \tag{19}$$

fulfills.

Proof. From (11) and thanks to (8), we obtain

$$\frac{dm}{dt} \leq -Dm + \left(c_0 + \frac{\varepsilon\pi}{2}\right) m = \left(-D + c_0 + \frac{\varepsilon\pi}{2}\right) m. \tag{20}$$

Then, from (20), it follows that

$$m(t; 0, \omega, m_0) \leq m_0 e^{-\left(D - c_0 - \frac{\varepsilon\pi}{2}\right) t} \tag{21}$$

for every $t \geq 0$, $\omega \in \Omega$ and $m_0 > 0$.

Finally, assuming that (19) holds true, it yields that $\lim_{t\to+\infty} m(t; 0, \omega, m_0) = 0$ for every $\omega \in \Omega$ and $m_0 > 0$, whence $\mathcal{D}_0^{\text{ext}}$ attracts the solutions of the random chemostat (10)-(11). □

Remark 3.2. *Note that Theorem 3.3 holds also true when replacing condition* (19) *by $D > c_0$, which is quite finer. To prove this, it is enough to use the ergodic property* (9)*. Indeed, from* (11)*, we have that*

$$\frac{dm}{dt} \leq -Dm + (c_0 + \phi(\xi^*(\theta_t\omega)))\, m = (-D + c_0 + \phi(\xi^*(\theta_t\omega)))\, m, \tag{22}$$

whence

$$m(t; 0, \omega, m_0) \leq m_0 e^{-(D-c_0)t + \int_0^t \phi(\xi^*(\theta_r\omega))dr} \tag{23}$$

for every $t \geq 0$, $\omega \in \Omega$ and $m_0 > 0$.

Now, thanks to the property (9)*, provided $D > c_0$, we have that $\lim_{t\to+\infty} m(t; 0, \omega, m_0) = 0$ for any $\omega \in \Omega$ and $m_0 > 0$.*

Remark 3.3. *From Theorem 3.3 and Remark 3.2, we conclude that the microorganisms in the random chemostat* (10)-(11) *become extinct as long as condition* (19) *holds true, as we will also observe in the numerical simulations displayed in Section 3.4.*

Next theorem focuses on the case when the microorganisms survive.

Theorem 3.4. *The set*

$$\mathcal{D}_0^{pers} := \left\{(n,m) \in \mathbb{R}_+^2 \,:\, n+m = n_{in},\, m \geq m_b, n \geq n_b\right\} \subset \mathcal{D}_0, \tag{24}$$

which is compact and does not depend on the noise, attracts the solutions of the random chemostat (10)-(11) *as long as*

$$\left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{in} > D(d + n_{in}) \tag{25}$$

is satisfied, where $m_b > 0$ and $n_b > 0$ are given inside the proof.

Proof. Thanks to (11) and Theorem 3.2, we have that

$$\frac{dm}{dt} \geq \left[-D + \frac{\left(c_0 - \frac{\varepsilon\pi}{2}\right)(n_{\text{in}} - \delta - m)}{d + n_{\text{in}} + \delta - m}\right] m \tag{26}$$

for every $t > 0$ large enough, $\omega \in \Omega$ and $x_0 > 0$.

Notice that the expression in brackets in (26) is positive, negative or zero when the polynomial $p(m) = p_2 m + p_1(\delta)$ is positive, negative or zero, respectively, where

$$p_1(\delta) := n_{\text{in}}\left(c_0 - \frac{\varepsilon\pi}{2}\right) - D(d + n_{\text{in}}) - \delta\left(D + c_0 - \frac{\varepsilon\pi}{2}\right), \tag{27}$$

$$p_2 := D - c_0 + \frac{\varepsilon\pi}{2}. \tag{28}$$

Now, thanks to (25), $p_1(\delta) > 0$ for every $\delta > 0$ small enough. On the other hand, provided (25), it follows that

$$D < \frac{\left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{\text{in}}}{d + n_{\text{in}}} < c_0 - \frac{\varepsilon\pi}{2},$$

whence we obtain that $p_2 < 0$. Thus, the polynomial p vanishes at 0 and $m(\delta)$, where

$$m(\delta) := \frac{D(d + n_{\text{in}}) - n_{\text{in}}\left(c_0 - \frac{\varepsilon\pi}{2}\right) + \delta\left(D + c_0 - \frac{\varepsilon\pi}{2}\right)}{D - c_0 + \frac{\varepsilon\pi}{2}} > 0. \tag{29}$$

Then, $p(m) > 0$ on $(0, m(\delta))$. Therefore, from (26), we have that

$$\left.\frac{dm}{dt}\right|_{m=\bar{m}} > 0$$

for every $\bar{m} \in (0, m(\delta))$. This reasoning proves that the concentration of the microorganisms in the random chemostat (10)-(11) is above $m(\delta)$ for every t large enough and any $\delta > 0$ small enough.

Hence, it is enough to take limit when $\delta \to 0$ in (29) to obtain

$$m_b := \frac{D(d + n_{\text{in}}) - n_{\text{in}} \left(c_0 - \frac{\varepsilon\pi}{2} \right)}{D - c_0 + \frac{\varepsilon\pi}{2}} > 0, \tag{30}$$

which does not depend on $\delta > 0$.

Now, thanks to Theorem 3.2, it yields from (10) that

$$\frac{dn}{dt} \geq D(n_{\text{in}} - n) - \frac{\left(c_0 + \frac{\varepsilon\pi}{2} \right) n \left(n_{\text{in}} + \delta - n \right)}{d + n} \tag{31}$$

for every $t > 0$ large enough, $\omega \in \Omega$ and $n_0 > 0$.

It is not difficult to check that the right-hand term in (31) is positive, negative or zero when the polynomial $q(n) = q_2 n^2 + q_1(\delta) n + q_0$ is positive, negative or zero, respectively, where

$$q_2 := c_0 + \frac{\varepsilon\pi}{2} - D, \tag{32}$$

$$q_1(\delta) := n_{\text{in}} \left(D - c_0 - \frac{\varepsilon\pi}{2} \right) - Dd - \delta \left(c_0 + \frac{\varepsilon\pi}{2} \right), \tag{33}$$

$$q_0 := D n_{\text{in}} d. \tag{34}$$

It is trivial that $q_0 > 0$. Moreover, from (25), it follows that

$$D < c_0 - \frac{\varepsilon\pi}{2} < c_0 + \frac{\varepsilon\pi}{2},$$

whence $q_2 > 0$. In addition, $q_1(\delta) < 0$ for every $\delta > 0$, since all its addends are negative.

From the previous reasoning, it follows that q vanishes at $n_b(\delta)$ and $n_a(\delta)$, with

$$n_b(\delta) := \frac{-q_1(\delta) - \sqrt{q_1(\delta)^2 - 4q_2 q_0}}{2q_2} \quad \text{and} \quad n_a(\delta) := \frac{-q_1(\delta) + \sqrt{q_1(\delta)^2 - 4q_2 q_0}}{2q_2}.$$

Hence, $q(n) > 0$ for every $n \in (0, n_b(\delta))$ and then

$$\left. \frac{dn}{dt} \right|_{n=\bar{n}} > 0$$

for every $\bar{n} \in (0, n_b(\delta))$, whence we deduce that the concentration of the substrate in the random chemostat (10)-(11) is greater than $n_b(\delta)$ for every t large enough and any $\delta > 0$ small enough.

Thus, by taking limit when $\delta \to 0$ in the expression that defines $n_b(\delta)$, we have

$$n_b := \frac{-q_1 - \sqrt{q_1^2 - 4q_2q_0}}{2q_2} > 0, \tag{35}$$

where

$$q_1 := n_{\text{in}} \left(D - c_0 - \frac{\varepsilon\pi}{2} \right) - Dd.$$

As a consequence, the set $\mathcal{D}_0^{\text{pers}}$, which is compact and independent of the noise, attracts the solutions of the random chemostat (10)-(11). □

Remark 3.4. *Notice that Theorem 3.4 ensures that the microorganisms in the random chemostat* (10)-(11) *persist if condition* (25) *holds true, which is the most interesting result in real applications.*

3.4. Numerical simulations and comments

In this subsection we depict numerical simulations to illustrate the theoretical results involving the random chemostat (10)-(11). In every figure the discontinuous lines represent the solution of the chemostat (1)-(2) with Monod consumption function without noise, whereas the continuous lines are different realizations of the random system (10)-(11).

In Figure 1 we display two panels to illustrate the evolution on time of the concentration of the nutrient and the microorganisms in the random chemostat (10)-(11). In this case we set $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$ and $\varepsilon = 0.19$. In addition, we consider the initial condition $(n_0, m_0) = (5, 4)$. Since $D > c_0$, the microorganisms become extinct, as proved in Theorem 3.3 and Remark 3.2.

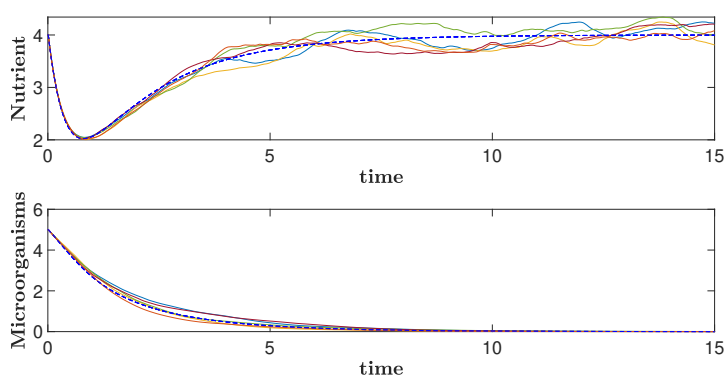

Figure 1: **Extinction of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

Now, we show in Figure 2 the phase plane associated to the numerical simulations in Figure 1, where the arrow points at the initial condition $(n_0, m_0) = (5, 4)$. As we explained above, we can observe how the concentration of the microorganisms vanishes.

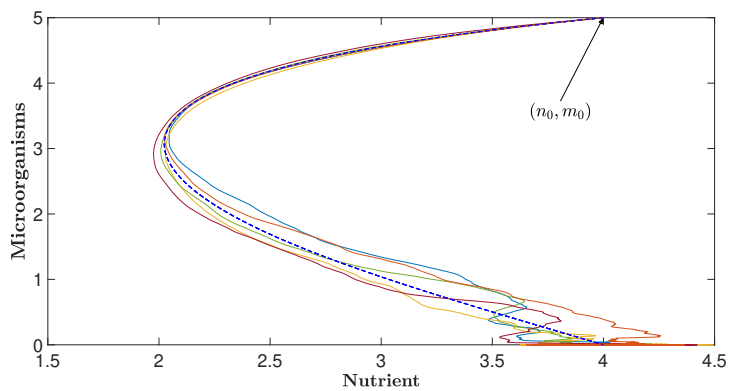

Figure 2: **Extinction of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

Next, in Figure 3 we include two panels again to illustrate the evolution on time of the concentration of the nutrient and the microorganisms in the random chemostat (10)-(11), where now we take $D = 1.1$. In this case condition (25) fulfills and then the microorganisms survive, as proved in Theorem 3.4.

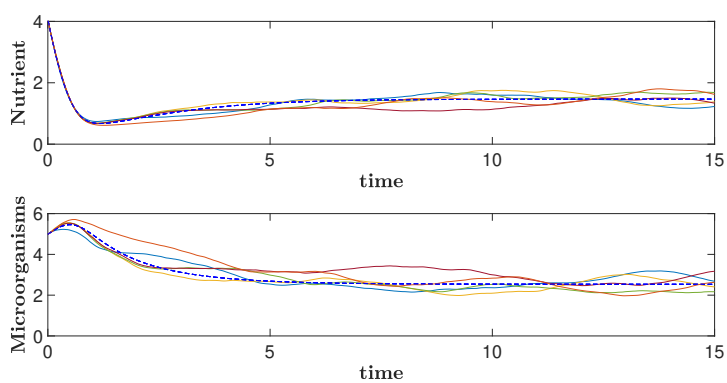

Figure 3: **Persistence of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.1$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

Finally, we depict in Figure 4 the phase plane associated to the numerical simulations in Figure 3, where the arrow points at the initial condition $(n_0, m_0) = (5, 4)$. As explained

above, where the persistence of the microorganisms is observed.

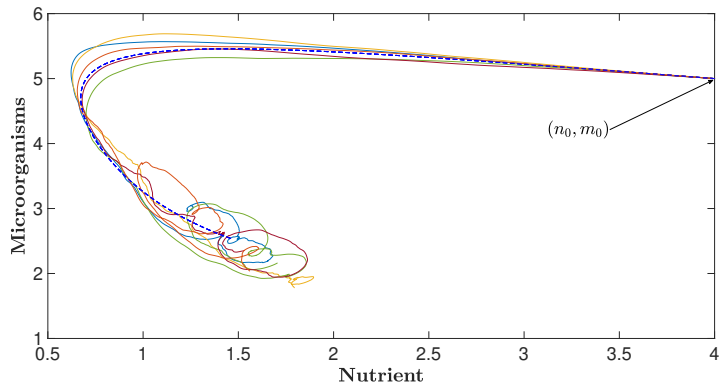

Figure 4: **Persistence of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.1$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

4. The random chemostat model with Haldane consumption function

In this section we study the chemostat model (1)-(2) with Haldane consumption function, where the maximum specific growth rate of the species c_0 is subject to random bounded fluctuations modeled by means of the real noise. The corresponding random chemostat is given by

$$\frac{dn}{dt} = D(n_{\text{in}} - n) - \frac{(c_0 + \phi(\xi^*(\theta_t\omega)))n}{d + n + \frac{n^2}{d_i}} m, \tag{36}$$

$$\frac{dm}{dt} = -Dm + \frac{(c_0 + \phi(\xi^*(\theta_t\omega)))n}{d + n + \frac{n^2}{d_i}} m, \tag{37}$$

where $n = n(t)$ and $m = m(t)$ denote the concentration of the nutrient and the microorganisms, respectively, $n_{\text{in}} > 0$ represents the concentration of nutrient in the feed bottle, $D > 0$ is the dilution rate, $c_0 > 0$ denotes the maximum specific growth rate of the species, $d > 0$ is the half-saturation constant and $d_i > 0$ describes the inhibition.

4.1. Well-posedness of the random chemostat

Again, $\mathbb{R}^2_+ = \{(n, m) \in \mathbb{R}^2 : n, m \geq 0\}$ in this section.

Theorem 4.1. *For any given initial condition $u_0 = (n_0, m_0) \in \mathbb{R}^2_+$ and $\omega \in \Omega$, the random chemostat* (36)-(37) *possesses a unique solution*

$$u(t; 0, \omega, u_0) := (n(t; 0, \omega, u_0), m(t; 0, \omega, u_0)) \in \mathcal{C}^1([0, +\infty); \mathbb{R}^2_+),$$

where $u(t;0,\omega,u_0)$ denotes the value at time t of the solution that starts with initial condition u_0 at time 0 and depends on ω. Both $n(t;0,\omega,u_0)$ and $m(t;0,\omega,u_0)$ are defined equivalently and we recall that $n_0 = n(0;0,\omega,u_0)$ and $m_0 = m(0;0,\omega,u_0)$.

The proof of Theorem 4.1 is analogous to the proof of Theorem 3.1 in the previous section, then we decided to skip it here in order not to be redundant.

4.2. Absorbing and attracting sets

In this subsection, our aim is to prove that there exists a compact set that attracts the solutions of the random chemostat (36)-(37) forwards in time.

Theorem 4.2. *For any given $\delta > 0$, the deterministic compact set*

$$\mathcal{H}_\delta := \left\{(n,m) \in \mathbb{R}^2_+ \,:\, n_{in} - \delta \leq n + m \leq n_{in} + \delta\right\} \tag{38}$$

is absorbing for the solutions of the random chemostat (36)-(37)*, i.e., for every bounded set $E \subset \mathbb{R}^2_+$ and $\omega \in \Omega$, there exists a time $T_E(\delta,\omega) > 0$ such that the solution $u(t;0,\omega,u_0)$ of* (36)-(37) *is contained in $\mathcal{H}_\delta$ for every $u_0 \in E$ and $t \geq T_E(\delta,\omega)$.*

The proof of Theorem 4.2 coincides with the proof of the Theorem 3.2 in the previous section, then it is omitted here.

Remark 4.1. *We would like to highlight that the set $\mathcal{H}_\delta$ is the same that the one obtained in Theorem 3.2 in Section 3 when considering the Monod consumption function rather than the Haldane one. This simply means that the consumption function (or, more precisely, the inhibition constant d_i) does not play any relevant role to ensure the existence of an absorbing set for the solutions of the corresponding random chemostat. Moreover, we can find in this case a set $\mathcal{H}_\delta$ which is independent of the noise, even though the corresponding model is affected by randomness.*

Now, the next corollary follows trivially from Theorem 4.2.

Corollary 4.1. *The deterministic compact set*

$$\mathcal{H}_0 := \left\{(n,m) \in \mathbb{R}^2_+ \,:\, n + m = n_{in}\right\} \tag{39}$$

attracts the solutions of the random chemostat (36)-(37)*, i.e.,*

$$\lim_{t\to+\infty} \sup_{u_0\in E} \inf_{d_0\in\mathcal{D}_0} \|u(t;0,\omega,u_0) - d_0\|_{\mathbb{R}^2_+} = 0.$$

Remark 4.2. *Note that, since the absorbing set $\mathcal{H}_\delta$ obtained in Theorem 4.2 and the absorbing set $\mathcal{D}_\delta$ obtained in Theorem 3.2 are the same, it is quite logical that the attracting sets $\mathcal{H}_0$ and $\mathcal{D}_0$ obtained in Corollary 4.1 and Corollary 3.1, respectively, coincide as well. As we explained previously, this indicates that the differences between both consumption functions Monod and Haldane are not relevant to ensure the existence of absorbing and attracting sets for the corresponding random chemostat. Nevertheless, this will not be the case when providing conditions to ensure the extinction or, what is more important, the persistence of the microorganisms, as we will see in Section 4.3.*

4.3. Extinction and persistence of the microorganisms

This subsection is dedicated to supply conditions under which either the extinction or the survival of the microorganisms in the random chemostat (36)-(37) is obtained.

Theorem 4.3. *The singleton point*

$$\mathcal{H}_0^{ext} := \{(n_{in}, 0)\} \subset \mathcal{H}_0 \tag{40}$$

attracts the solutions of the random chemostat (36)-(37) *when*

$$D > \left(c_0 + \frac{\varepsilon\pi}{2}\right) \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}} \tag{41}$$

fulfills.

Proof. Define the mapping $f : [0, +\infty) \to \mathbb{R}$ as

$$f(n) = \frac{n}{d + n + \dfrac{n^2}{d_i}},$$

which is non-monotonic and attains its maximum on $[0, +\infty)$ at $\hat{n} = \sqrt{dd_i}$, with

$$f(\hat{n}) = \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}}.$$

Thus, from (37), it yields that

$$\frac{dm}{dt} \leq -Dm + \left(c_0 + \frac{\varepsilon\pi}{2}\right) f(\hat{n})m = \left(-D + \left(c_0 + \frac{\varepsilon\pi}{2}\right) \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}}\right) m, \tag{42}$$

whence

$$m(t; 0, \omega, m_0) \leq m_0 e^{-\left(D - \left(c_0 + \frac{\varepsilon\pi}{2}\right) \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}}\right) t} \tag{43}$$

for every $t \geq 0$, $\omega \in \Omega$ and $m_0 > 0$.

Hence, given that (9) holds true, from (43) we deduce that $\lim_{t \to +\infty} m(t; 0, \omega, m_0) = 0$ for every $\omega \in \Omega$ and $m_0 > 0$, provided that (41) is fulfilled, and the proof finishes. □

Remark 4.3. *We note that Theorem 4.3 can be also proved when replacing* (41) *by*

$$D > c_0 \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}}, \tag{44}$$

which is less restrictive. The proof is as follows. From (37)*, it follows that*

$$\frac{dm}{dt} \leq \left(-D + (c_0 + \phi(\xi^*(\theta_t\omega))) \frac{\sqrt{dd_i}}{2d + \sqrt{dd_i}} \right) m. \tag{45}$$

Then, it easy to check that

$$m(t;0,\omega,m_0) \leq m_0 e^{-\left(D - c_0 \frac{\sqrt{dd_i}}{2d+\sqrt{dd_i}}\right) t + \frac{\sqrt{dd_i}}{2d+\sqrt{dd_i}} \int_0^t \phi(\xi^*(\theta_r\omega))dr} \tag{46}$$

for every $t \geq 0$*,* $\omega \in \Omega$ *and* $m_0 > 0$*.*

Now, thanks to (9)*, provided that* (44) *fulfills, we have that* $\lim_{t\to+\infty} m(t;0,\omega,m_0) = 0$ *for any* $\omega \in \Omega$ *and* $m_0 > 0$*.*

Remark 4.4. *We would like to highlight that, in spite of the fact that the attracting sets* $\mathcal{H}_0$ *and* $\mathcal{D}_0$ *obtained in Corollary 4.1 and Corollary 3.1, respectively, coincide, the conditions under which the microorganisms go to extinction are completely different. As we can observe, both* (41) *and* (44) *depend now on the inhibition constant* d_i*, whence we deduce that the differences between the consumption functions Monod and Haldane are relevant for the asymptotic dynamics of the corresponding random chemostat, even though these differences were not important to ensure the existence of the absorbing and the attracting sets.*

Remark 4.5. *Thanks to Theorem 4.3, we found conditions under which the microorganisms in the random chemostat* (36)-(37) *become extinct.*

Finally, we focus on the survival of the microorganisms, the main goal in real applications.

Theorem 4.4. *The set*

$$\mathcal{H}_0^{per} := \left\{(n,m) \in \mathbb{R}_+^2 \,:\, n+m = n_{in},\, m \geq m_b,\, n \geq n_b\right\} \subset \mathcal{H}_0, \tag{47}$$

which is compact and does not depend on the noise, attracts the solutions of the random chemostat (36)-(37) *as long as*

$$\left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{in} > D\left(d + n_{in} + \frac{n_{in}^2}{d_i}\right) \tag{48}$$

is fulfilled, where $m_b > 0$ *and* $n_b > 0$ *are given in the proof.*

Proof. Thanks to (37) and Theorem 4.2, it follows that

$$\frac{dm}{dt} \geq \left[-D + \frac{\left(c_0 - \frac{\varepsilon\pi}{2}\right)(n_{\text{in}} - \delta - m)}{d + (n_{\text{in}} + \delta - m) + \frac{(n_{\text{in}} + \delta - m)^2}{d_i}} \right] m \tag{49}$$

for every $t > 0$ large enough, $\omega \in \Omega$ and $m_0 > 0$.

Notice that the expression in brackets in (49) is positive, negative or zero when the polynomial $p(m) = p_2 m^2 + p_1(\delta)m + p_0(\delta)$ is positive, negative or zero, respectively, with

$$p_0(\delta) = \left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{\text{in}} + \delta\left(-D - \frac{2Dn_{\text{in}}}{d_i} - \frac{D\delta}{d_i} - c_0 + \frac{\varepsilon\pi}{2}\right) - D\left(d + n_{\text{in}} + \frac{n_{\text{in}}^2}{d_i}\right), \tag{50}$$

$$p_1(\delta) = D - c_0 + \frac{\varepsilon\pi}{2} + \frac{2D}{d_i}(\delta + n_{\text{in}}), \tag{51}$$

$$p_2 = -\frac{D}{d_i}. \tag{52}$$

Observe that $p_2 < 0$ and $p_0(\delta) > 0$ for every $\delta > 0$ provided that condition (48) holds true. Therefore, since p is continuous on $\mathbb{R}$, there exists $m(\delta) > 0$ such that

$$\left.\frac{dm}{dt}\right|_{m=\bar{m}} > 0$$

for every $\bar{m} \in (0, m(\delta))$ or, in other words, the concentration of microorganisms in the random chemostat (36)-(37) is greater than $m(\delta) > 0$ for every t large enough and any $\delta > 0$. Then, it suffices to consider $m_b = \lim_{\delta \to 0} m(\delta) > 0$, which is independent of δ, to finish the first part of the proof.

On the other hand, from (36) and thanks to Theorem 4.2, we have that

$$\frac{dn}{dt} \geq D(n_{\text{in}} - n) - \frac{\left(c_0 + \frac{\varepsilon\pi}{2}\right) n\,(n_{\text{in}} + \delta - n)}{d + n + \frac{n^2}{d_i}} \tag{53}$$

for every $t > 0$ large enough, $\omega \in \Omega$ and $n_0 > 0$.

It is not difficult to check that the right-hand term in (53) is positive, negative or zero when the polynomial $q(n) = q_3 n^3 + q_2 n^2 + q_1(\delta)n + q_0$ is positive, negative or zero, respectively, where

$$q_0 = Ddn_{\text{in}}, \tag{54}$$

$$q_1(\delta) = Dn_{\text{in}} - Dd - n_{\text{in}}\left(c_0 + \frac{\varepsilon\pi}{2}\right) - \delta\left(c_0 + \frac{\varepsilon\pi}{2}\right), \tag{55}$$

$$q_2 = \frac{Dn_{\text{in}}}{d_i} - D + c_0 + \frac{\varepsilon\pi}{2}, \tag{56}$$

$$q_3 = -\frac{D}{d_i}. \tag{57}$$

It is obvious that $q_0 > 0$ and $q_3 < 0$. Moreover, since q is continuous and $q(0) = q_0 > 0$, it follows that there exists $n(\delta) > 0$ such that

$$\left.\frac{dn}{dt}\right|_{n=\bar{n}} > 0$$

for every $\bar{n} \in (0, n(\delta))$. In other words, the concentration of the substrate in the random chemostat (36)-(37) is over $n(\delta) > 0$ for every t large enough and any $\delta > 0$. Thus, it suffices to consider $n_b = \lim_{\delta \to 0} n^*(\delta) > 0$, which is independent of δ, to finish the proof. □

Remark 4.6. *Notice that Theorem 4.4 provides condition* (48)*, under which the microorganisms persist in the random chemostat* (36)-(37)*. This is, no doubt at all, the most important goal in practice.*

4.4. Numerical simulations and comments

In this last subsection we present several numerical simulations to illustrate the theoretical results involving the random chemostat (36)-(37). In every figure the discontinuous lines represent the solution of the chemostat (1)-(2) with Haldane consumption function without noise, whereas the continuous lines are different realizations of the solutions of the random system (36)-(37).

In Figure 5 we display two panels to depict the evolution on time of the concentration of substrate and microorganisms in the random chemostat (36)-(37). To this end, we set $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$, $d_i = 5$, $\varepsilon = 0.19$ and the initial condition $(n_0, m_0) = (5, 4)$. In this case, condition $D > c_0$ is fulfilled, whence the microorganisms vanishes, as proved in Theorem 4.3 and Remark 4.3.

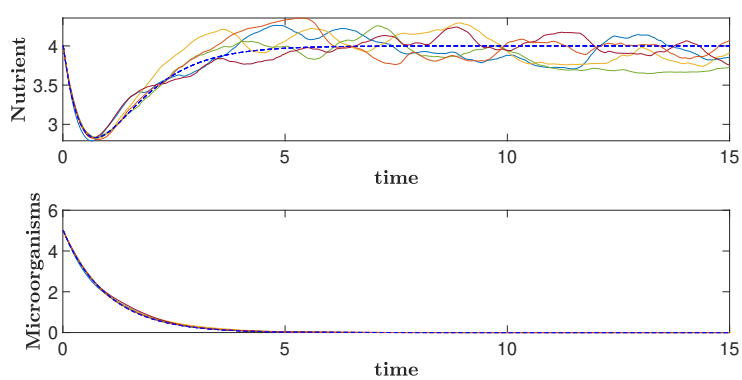

Figure 5: **Extinction of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$, $d_i = 5$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

We depict now in Figure 6 the phase plane associated to the numerical simulations

in Figure 5, where the arrow points at the initial condition $(n_0, m_0) = (5, 4)$. As we explained above, it is easy to observe how the species become extinct.

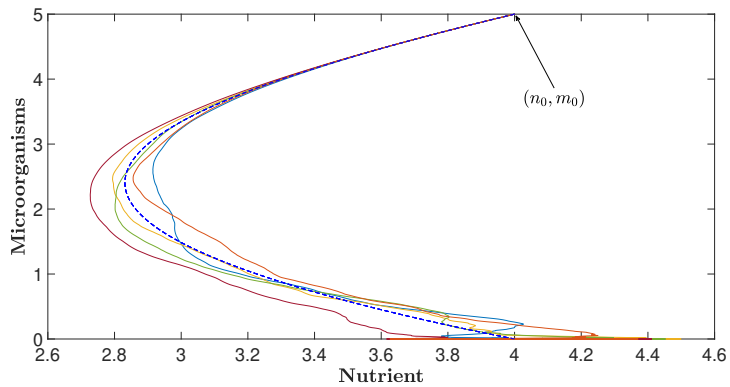

Figure 6: **Extinction of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 1.9$, $d_i = 5$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

In Figure 7 we show again two panels with the evolution on time of the concentration of substrate and microorganisms in the random chemostat (36)-(37). Nevertheless, in this case we consider $D = 0.65$. As a consequence, condition (48) holds true and then the microorganisms survive, as proved in Theorem 4.4.

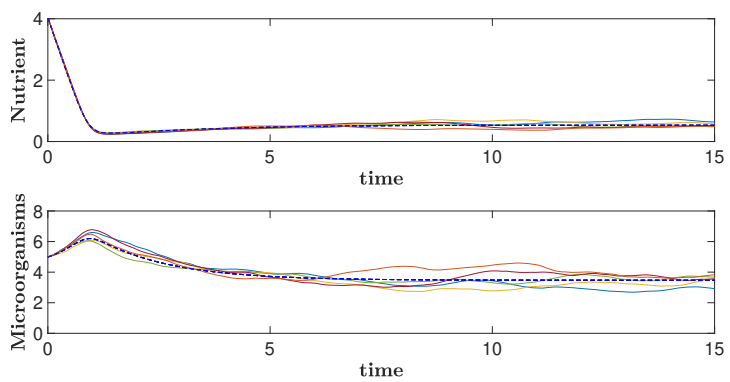

Figure 7: **Persistence of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 0.65$, $d_i = 5$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

Finally, we plot in Figure 8 the phase plane associated to the numerical simulations in Figure 7, where the arrow points at the initial condition $(n_0, m_0) = (5, 4)$ again. The

persistence of the microorganisms can be observed clearly in this figure, as we already pointed out.

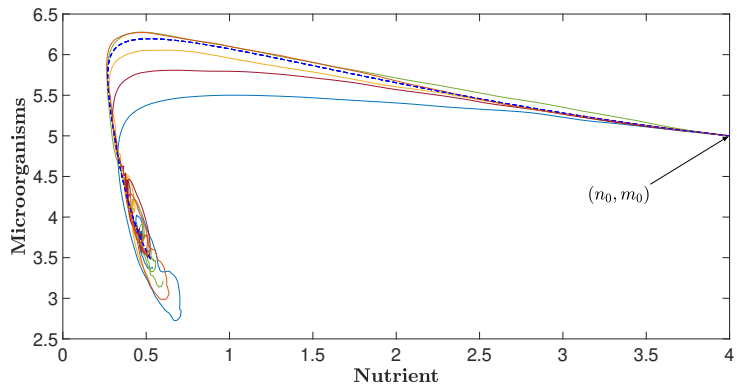

Figure 8: **Persistence of the microorganisms** where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 0.65$, $d_i = 5$ and $\varepsilon = 0.19$ with initial condition $(n_0, m_0) = (5, 4)$.

5. Conclusions

In this last section, we sum up the results proved along this work, providing biological interpretations and comparing with related works in the literature as well.

Our aim in this paper is to investigate the long-time dynamics of the chemostat model (1)-(2) with both Monod and Haldane consumption functions, where the maximum specific growth rate of the species c_0 is randomly perturbed by means of the real noise $\phi(\xi^*(\theta_t\omega))$, which is a bounded function of the well-known Ornstein-Uhlenbeck process.

Firstly, we ensured the well-posedness of both random chemostat models (10)-(11) (with Monod consumption function) and (36)-(37) (with Haldane consumption function). Moreover, we proved the existence of compact absorbing and attracting sets which, in addition, are independent of the noise, even though the original systems are subject to randomness.

Later, we provided conditions under which the microorganisms vanishes and, what is much more important from the point of view of the applications, conditions to ensure the survival of the microorganisms. Such conditions are summarized in the sequel.

5.1. Random chemostat with Monod consumption function

Concerning the random chemostat model (10)-(11), we proved in Theorem 3.3 and Remark 3.2 that the extinction of the microorganisms cannot be avoided when

$$D > c_0$$

holds true, whereas they survive if

$$\left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{\text{in}} > D(d + n_{\text{in}}) \tag{58}$$

fulfills (see Theorem 3.4).

Notice that (58) can be rewritten as $D < c_b(n_{\text{in}})$, where the function c_b is given as

$$c_b(n) = \frac{\left(c_0 - \frac{\varepsilon\pi}{2}\right) n}{d + n}. \tag{59}$$

Summing up, the microorganisms in the random chemostat (10)-(11) become extinct as long as $D > c_0$ and persists if $D < c_b(n_{\text{in}})$. Otherwise, we cannot derive more information about the extinction or persistence. This means that the dilution rate cannot be large compared with the input concentration of the species n_{in} and the consumption function c_b.

In order to help the readers, in Figure 9 we can observe a diagram where we represent the input concentration of the species n_{in} (in the $x-$axis) versus the dilution rate D (in the $y-$axis). In addition, we also include the graph of the Monod consumption function c given by (3) and the "limiting" consumption functions c_b and c_a, where c_b is defined as in (59) and c_a is given by

$$c_a(n) = \frac{\left(c_0 + \frac{\varepsilon\pi}{2}\right) n}{d + n}. \tag{60}$$

The diagram in Figure 9 must be interpreted as follows. Every point (n_{in}, D) in the red zone corresponds to the case $D > c_0$ and then the microorganisms extinguish. Nevertheless, if the point (n_{in}, D) is in the green zone, it means that $D < c_b(n_{\text{in}})$, whence the microorganisms persist. Finally, as long as the point (n_{in}, D) is in the white zone, we cannot ensure neither the persistence nor the extinction of the microorganisms.

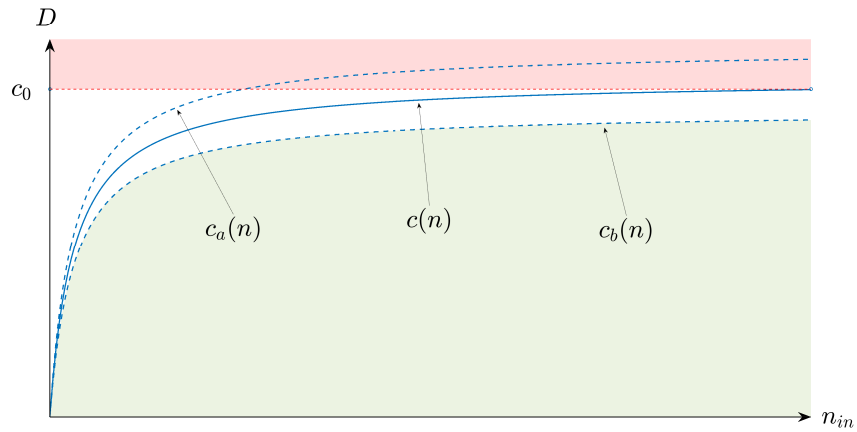

Figure 9: **Diagram in the case of Monod consumption function.**

From the previous discussion, we can notice that it is possible for practitioners to increase or decrease the input concentration of the substrate n_{in} in order to have a more or less broad branch of values for the dilution rate D such that the persistence of the microorganisms can be ensured. In addition, we can deduce the relevance of the dilution rate D when trying to keep persistence of the microorganisms in the chemostat.

As we pointed out in the introduction, in this paper we improve the results in [25], where the authors study the chemostat model (1)-(2) with Monod consumption function but they replace c_0 by $c_0 + \alpha \dot{W}(t)$, where $\dot{W}(t)$ denotes the white noise and $\alpha > 0$ is the amount of noise. In [25], the conditions for the extinction of the species is $D > c(n_{\text{in}})$ (which is more restrictive than our condition $D > c_0$) and the persistence of the species is ensured (under certain conditions) in the mean, i.e.,

$$\liminf_{t \to +\infty} \frac{1}{t} \int_0^t x(\tau) d\tau \geq \kappa > 0,$$

whilst we prove the persistence of the species in the stronger sense

$$\lim_{t \to +\infty} x(t) \geq \kappa > 0.$$

5.2. Random chemostat with Haldane consumption function

Regarding the random chemostat (36)-(37), we proved in Theorem 4.3 and Remark 4.3 that the microorganisms become extinct as long as

$$D > c(\hat{n})$$

fulfills, where we recall that $\hat{n}$ is the point at which the Haldane consumption function given by (4) attains its maximum (see Figure 11).

In addition, thanks to Theorem 4.4, it is possible to guarantee the survival of the microorganisms when

$$\left(c_0 - \frac{\varepsilon\pi}{2}\right) n_{\text{in}} > D\left(d + n_{\text{in}} + \frac{n_{\text{in}}^2}{d_i}\right) \tag{61}$$

is fulfilled.

It is easy to check that condition (61) can be rewritten equivalently as $D < c_b(n_{\text{in}})$, where the function c_b in this case is given as

$$c_b(n) = \frac{\left(c_0 - \frac{\varepsilon\pi}{2}\right) n}{d + n + \frac{n_{\text{in}}^2}{d_i}}. \tag{62}$$

In conclusion, the microorganisms extinguishes provided that condition $D > c(\hat{n})$ and persist when $D < c_b(n_{\text{in}})$ holds true. This means that the dilution rate need to be small enough compared with the input concentration of the microorganisms n_{in} and the consumption function c_b, where the parameter $\hat{n}$ plays also an important role now.

To help the reader, in Figure 9 we provide a diagram where we represent the input concentration of the microorganisms n_{in} (in the $x-$axis) and the dilution rate D (in the $y-$axis). Moreover, we also include the graph of the Haldane consumption function c given by (4) and the "limiting" consumption functions c_b and c_a, where c_b is defined as in (62) and c_a is given by

$$c_a(n) = \frac{\left(c_0 + \frac{\varepsilon\pi}{2}\right) n}{d + n + \frac{n_{\text{in}}^2}{d_i}}. \tag{63}$$

The diagram in Figure 11 must be interpreted as follows. Each point (n_{in}, D) in the red zone corresponds to the case $D > c_0$ and then the microorganisms extinguish. Nevertheless, if the point (n_{in}, D) is in the green zone, then $D < c_b(n_{\text{in}})$, whence the microorganisms persist.

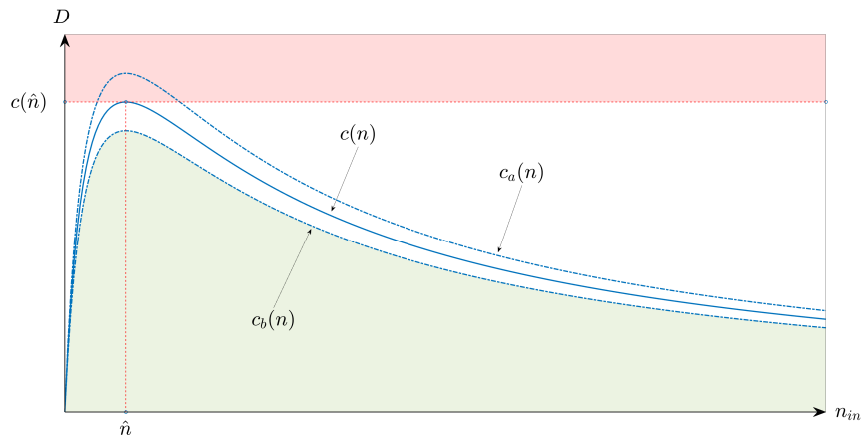

Figure 10: **Diagram in the case of Haldane consumption function.**

One could wonder what happens when $D \in [c_b(n_{\text{in}}), c(\hat{n})]$ (the white zone in Figure 11). In such a case, we can observe both extinction and persistence of the species depending on the initial condition that we consider. To illustrate this phenomenon, we depict in Figure 11 the phase plane of the random chemostat (36)-(37), where we set $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 0.91$, $d_i = 5$, $\varepsilon = 0.19$ and we consider the initial conditions $(n_0^{\text{ext}}, m_0^{\text{ext}}) = (4, 1)$ (red realization corresponding to extinction) and $(n_0^{\text{pers}}, m_0^{\text{pers}}) = (4, 1.25)$ (green realization corresponding to persistence).

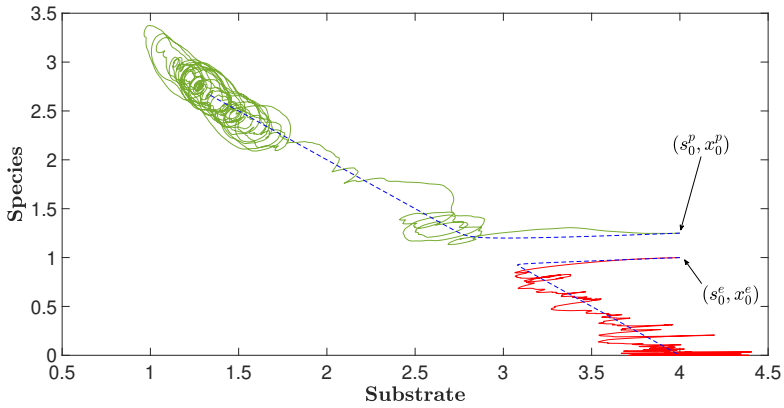

Figure 11: **Extinction and persistence in the random chemostat** (36)-(37) where $n_{\text{in}} = 4$, $d = 0.8$, $c_0 = 1.7$, $D = 0.91$, $d_i = 5$ and $\varepsilon = 0.19$ with initial conditions $(n_0^{\text{ext}}, m_0^{\text{ext}}) = (4, 1)$ (red) and $(n_0^{\text{pers}}, m_0^{\text{pers}}) = (4, 1.25)$ (green).

From the previous discussion, we deduce that practitioners can increase or decrease the

input concentration of the substrate n_{in} in order to have a more or less broad branch of values for the dilution rate D such that the persistence of the microorganisms can be ensured. In addition, it is worth pointing out the relevance of the dilution rate D when trying to keep persistence of the microorganisms in the chemostat, apart from the inhibition parameter d_i, which plays also an essential role in the dynamics of the corresponding random chemostat (36)-(37).

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