# MEAN ATTRACTORS AND INVARIANT MEASURES OF LOCALLY MONOTONE AND GENERALLY COERCIVE SPDES DRIVEN BY SUPERLINEAR NOISE

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Abstract. We study the global solvability, mean attractors and invariant measures for an abstract locally monotone and generally coercive SPDEs driven by infinite-dimensional superlinear noise defined in a dual space of intersection of finitely many Banach spaces.The main feature of this abstract system is that it covers a larger class of fundamental models which are included or not included previously. Under an extended locally monotone variational setting, we establish the global well-posedness, Itô's energy equality and existence of mean random attractors in some high-order Bochner spaces. The existence, uniqueness, support, (high-order and exponential) moment estimates, ergodicity, (pointwise and Wasserstein-type) exponentially mixing and asymptotic stability of invariant measures and evolution systems of measures are discussed for autonomous and nonautonomous stochastic equations. A stopping time technique is used to prove the convergence of solutions in probability in order to overcome the difficulty caused by the local monotonicity and superlinear growth of the coefficients. Our abstract results and unified methods are expected to be applied to various types of SPDEs like 2D Navier-Stokes equations, 2D MHD equations, 2D magnetic Bénard problem, Burgers type equations, 3D Leray $\alpha$-model, convective Brinkman-Forchheimer equations, fractional $(s,p)$-Laplacian equations with monotone nonlinearities of polynomial growth of arbitrary order, and others.

## 1. Introduction

1.1. **Statement of problems.** Let $(H,\langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space identified with its dual space $H^*$ by Riesz's representation theorem. For $i=1,2,\ldots,m\in\mathbb{N}$, let $(V_i,\|\cdot\|_{V_i})$ be a reflexive Banach space continuously and densely embedded into $H$. Let $V:=\bigcap_{i=1,2,\ldots,m}V_i$ with the norm $\|v\|_V=\sum_{i=1}^m\|v\|_{V_i}$, and ${}_{V^*}\langle\cdot,\cdot\rangle_V$ and ${}_{V_i^*}\langle\cdot,\cdot\rangle_{V_i}$ be the duality products of $V,V_i$ and their dual spaces $V^*,V_i^*$. Then $V^*=\sum_{i=1}^m V_i^*$, $f\in V^*$ if and only if $f=\sum_{i=1}^m f_i$ with $f_i\in V_i^*$, and ${}_{V^*}\langle f,v\rangle_V=\sum_{i=1}^m {}_{V_i^*}\langle f_i,v\rangle_{V_i}$ for $v\in V$. The norm of $V^*$ is defined by $\|f\|_{V^*}=\inf_{f=\sum_{i=1}^m f_i}\sum_{i=1}^m\|f_i\|_{V_i^*}$. Assume that $V$ is separable. Then we get the variational triples: $V_i\subseteq H\equiv H^*\subseteq V_i^*$ and $V\subseteq H\equiv H^*\subseteq V^*$. Given a separable Hilbert space $(U,\langle\cdot,\cdot\rangle_U)$, we denote by $\mathcal{L}_2(U,H)$ the space of Hilbert-Schmidt operators from $U$ to $H$ with the norm $\|\cdot\|_{\mathcal{L}_2(U,H)}$ and inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}_2(U,H)}$. Let $W$ be a two-sided $U$-valued cylindrical Wiener process defined on the complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$.

In this work we are concerned with the global solvability and long-time dynamics of the following stochastic evolution equation defined in $V^*$:

$$\begin{cases} dX(t)=\sum_{i=1}^m A_i(t,X(t))dt+B(t,X(t))dW(t), \quad t>\tau,\\ X(\tau)=X_0,\end{cases} \tag{1.1}$$

where the progressively measurable[1] evolution operators

$$A_i:[\tau,\tau+T]\times\Omega\times V_i\to V_i^* \text{ and } B:[\tau,\tau+T]\times\Omega\times V\to\mathcal{L}_2(U,H),\quad T>0,$$

satisfy some locally monotone, generally coercive and superlinear growth conditions: there exist constants $L_0>0$, $\vartheta_i>0$, $\theta_i>0$ $\alpha_i>1$, $\beta_i\in[1,\alpha_i)$, $\kappa\geqslant 0$, $\varpi\geqslant 0$, and $\mathcal{F}_t$-adapted nonnegative processes $\phi_1,\phi_3,\phi_5\in L^1([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$, $\phi_2,\phi_4,\phi_6\in L^\infty([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ and $\eta_i\in L^{\frac{\alpha_i}{\alpha_i-\beta_i}}([\tau,\tau+T]\times\Omega;dt\times\mathbb{P};\mathbb{R}^+)$ such that $A:=\sum_{i=1}^m A_i$ and $B$ satisfy, for all $(t,\omega)\in[\tau,\tau+T]\times\Omega$ and $v,v_1,v_2\in V$,

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[1] $A_i$ restricted to $[\tau,\tau+T]\times\Omega\times V_i$ is $\mathcal{B}([\tau,\tau+T])\times\mathcal{F}_t\times\mathcal{B}(V_i)$-measurable ($\mathcal{B}$ denotes the corresponding Borel $\sigma$-algebra).

(C1) (Hemicontinuity) The map $s \to {}_{V^*}\langle A(t,\omega,v_1+sv_2),v\rangle_V$ is continuous from $\mathbb{R}$ to $\mathbb{R}$;

(C2) (Local Monotonicity)

$$2\ {}_{V^*}\langle A(t,\omega,v_1)-A(t,\omega,v_2),v_1-v_2\rangle_V+\|B(t,\omega,v_1)-B(t,\omega,v_2)\|^2_{\mathcal{L}_2(U,H)}$$
$$\leqslant -\sum_{i=1}^m 2\vartheta_i\|v_1-v_2\|^{\alpha_i}_{V_i}+\Big(\phi_1(t,\omega)+\kappa\sum_{i=1}^m\rho_i(v_2)\Big)\|v_1-v_2\|^2_H,$$

where $\rho_i: V_i\to[0,+\infty)$ is a measurable, hemicontinuous and locally bounded function satisfying

$$\sum_{i=1}^m\rho_i(v)\leqslant L_0\Big(1+\sum_{i=1}^m\|v\|^{\alpha_i}_{V_i}\Big)(1+\|v\|^{\varpi}_H); \tag{1.2}$$

(C3) (General Coercivity)

$$2\ {}_{V^*}\langle A(t,\omega,v),v\rangle_V+\|B(t,\omega,v)\|^2_{\mathcal{L}_2(U,H)}\leqslant-\sum_{i=1}^m 2\theta_i\|v\|^{\alpha_i}_{V_i}+\phi_2(t,\omega)\|v\|^2_H+\phi_3(t,\omega);$$

(C4) (Superlinear Growth)

$$\|B(t,\omega,v)\|^2_{\mathcal{L}_2(U,H)}\leqslant\sum_{i=1}^m\eta_i(t,\omega)\|v\|^{\beta_i}_{V_i}+\phi_4(t,\omega)\|v\|^2_H+\phi_5(t,\omega),$$
$$\sum_{i=1}^m\|A_i(t,\omega,v)\|^{\frac{\alpha_i}{\alpha_i-1}}_{V_i^*}\leqslant\phi_6(t,\omega)\Big(1+\sum_{i=1}^m\|v\|^{\alpha_i}_{V_i}\Big)(1+\|v\|^{\varpi}_H).$$

1.2. **Global well-posedness of** (1.1)**.** Under our settings, we demonstrate the global existence, uniqueness and Itô's formula of solutions to (1.1) in some high-order Bochner spaces.

**Definition 1.1.** *(*Probabilistically strong solutions*) We say a continuous $H$-valued $\mathcal{F}_t$-adapted stochastic process $\{X(t)\}_{t\in[\tau,\tau+T]}$ is a solution to* (1.1) *if its $dt\times\mathbb{P}$-equivalent class $\bar X$ satisfies*

$$\bar X\in\Big(\bigcap_{i=1,2,\dots,m}L^{\alpha_i}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};V_i)\Big)\bigcap L^2([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};H),$$

*and $\mathbb{P}$-a.s., the following equation holds true in $V^*$:*

$$X(t)=X_0+\int_\tau^t A(s,\bar X(s))ds+\int_\tau^t B(s,\bar X(s))dW(s),\quad t\in[\tau,\tau+T].$$

**Theorem 1.2.** *(*Well-posedness, Itô's formula, mean energy equality) *Suppose (C1)-(C4) hold true for $\phi_3,\phi_5\in L^\ell([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ and $\eta_i\in L^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ with some $\ell\geqslant\frac{\varpi}{2}+1$. Then, for any $X_0\in L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$, problem* (1.1) *has a unique solution $\{X(t)\}_{t\in[\tau,\tau+T]}$ according to Definition 1.1 such that $X\in C([\tau,\tau+T],L^2(\Omega,\mathbb{P};H))$, and satisfies Itô's formula*

$$\begin{aligned}\|X(t)\|^{2\ell}_H&=\|X_0\|^{2\ell}_H+2\ell(\ell-1)\int_\tau^t\|X(s)\|^{2\ell-4}_H\|(B(s,X(s)))^*X(s)\|^2_U ds\\&\quad+2\ell\int_\tau^t\|X(s)\|^{2\ell-2}_H\langle X(s),B(s,X(s))dW(s)\rangle_H\\&\quad+\ell\int_\tau^t\|X(s)\|^{2\ell-2}_H\big(2\ {}_{V^*}\langle A(s,X(s)),X(s)\rangle_V+\|B(s,X(s))\|^2_{\mathcal{L}_2(U,H)}\big)ds,\ t\geqslant\tau,\quad\mathbb{P}\text{-a.s.},\end{aligned}\tag{1.3}$$

*the mean energy equality*

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[\|X(t)\|^{2\ell}_H]&=2\ell(\ell-1)\mathbb{E}[\|X(t)\|^{2\ell-4}_H\|(B(t,X(t)))^*X(t)\|^2_U]\\&\quad+\ell\mathbb{E}\Big[\|X(t)\|^{2\ell-2}_H\big(2\ {}_{V^*}\langle A(t,X(t)),X(t)\rangle_V+\|B(t,X(t))\|^2_{\mathcal{L}_2(U,H)}\big)\Big],\ t\geqslant\tau,\end{aligned}\tag{1.4}$$

*and the uniform estimate*

$$\mathbb{E}\Big[\sup_{t\in[\tau,\tau+T]}\|X(t)\|^{2\ell}_H\Big]+\mathbb{E}\Big[\int_\tau^{\tau+T}\|X(s)\|^{2\ell-2}_H\sum_{i=1}^m\|X(s)\|^{\alpha_i}_{V_i}ds\Big]\leqslant C(\tau,T,X_0).\tag{1.5}$$

The global well-posedness results in Theorem 1.2 permit us to investigate the behavior of solutions to (1.1). In the literature, the behavior of solutions to stochastic systems has been discussed in several different directions: pathwise random attractors (behavior in almost sure, see [3, 4, 6, 23, 27, 43, 44, 47, 56, 61]), mean random attractors (behavior in mean, see [25, 50, 52]), invariant measures (behavior in distribution, see [7, 17, 22, 28, 29, 32, 33]), large or moderate deviation principle (behavior in small probability, and the law of large numbers or central limit theorems [28, 29]. In the present paper we study mean random attractors and invariant measures of (1.1). To do this, we shall need the following remark.

**Remark 1.3.** *Assume that there exists* $i_1 \in [1,m]\cap\mathbb{N}$ *such that* $\alpha_{i_1} \geqslant 2$. *Then by (C2)-(C3) we deduce*

$$
\begin{aligned}
&2\ {}_{V^*}\langle A(t,\omega,v_1)-A(t,\omega,v_2),v_1-v_2\rangle_V+\|B(t,\omega,v_1)-B(t,\omega,v_2)\|^2_{\mathcal{L}_2(U,H)}\\
&\leqslant-\sum_{i=1}^m\vartheta_i\|v_1-v_2\|^{\alpha_i}_{V_i}+\Big(-\lambda_{i_1}\vartheta_{i_1}+\phi_1(t,\omega)+\kappa\sum_{i=1}^m\rho_i(v_2)\Big)\|v_1-v_2\|^2_H+\vartheta_{i_1}C_{\alpha_{i_1}},
\end{aligned}
$$

*and*

$$
2\ {}_{V^*}\langle A(t,\omega,v),v\rangle_V+\|B(t,\omega,v)\|^2_{\mathcal{L}_2(U,H)}\leqslant-\sum_{i=1}^m\theta_i\|v\|^{\alpha_i}_{V_i}+(\phi_2(t,\omega)-\lambda_{i_1}\theta_{i_1})\|v\|^2_H+\phi_3(t,\omega)+\theta_{i_1}C_{\alpha_{i_1}},
$$

*where* $C_{\alpha_{i_1}}=0$ *if* $\alpha_{i_1}=2$ *and* $C_{\alpha_{i_1}}=\alpha_{i_1}^{-1}(\alpha_{i_1}-2)(\alpha_{i_1}/2)^{2/(2-\alpha_{i_1})}$ *if* $\alpha_{i_1}>2$, *and* $\lambda_{i_1}>0$ *is the best embedding constant such that* $\sqrt{\lambda_{i_1}}\|v\|_H\leqslant\|v\|_{V_{i_1}}$.

1.3. **Mean random attractors of** (1.1)**.** A basic but very restrictive condition to investigate almost sure behavior of solutions to SPDEs by *pathwise* random attractors is that SPDEs should be converted into pathwise systems via an Ornstein-Uhlenbeck process, see [6, 8, 9, 10, 11, 12, 14, 15, 23, 31, 42, 48, 49, 54, 55, 58, 62]. In general, such a transformation can be achieved for SPDEs driven by additive or linear multiplicative noise. For SPDEs like (1.1) with nonlinear noise, it seems that there are no methods available in the literature to achieve such a conversion. Then, we alternatively study the mean (not pathwise) random dynamics of (1.1), and prove that the mean random dynamical system (RDS) generated by the solution operators has a unique mean random attractor in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ in the sense of Wang [50, 52], see Theorem 3.2. If $\ell=1$, then similar results can be found in [25, 50, 51] for stochastic parabolic equations. If $\ell=2$, the reader is referred to Wang [52] for the existence of mean random attractors of stochastic Navier-Stokes equations. We notice that a new concept of mean random invariant manifolds for mean RDSs was recently proposed by Wang [53].

1.4. **Invariant measures of** (1.1)**: autonomous case.** An important and universal conclusion in the theory of *pathwise* RDSs is that all invariant measures are supported by pathwise random attractors. Since the existence of a pathwise RDS for SPDEs like (1.1) with nonlinear noise is still unknown, we are currently unable to discuss the relationship between pathwise random attractors and invariant measures of (1.1). Although we can establish the existence of mean random attractors of (1.1), we still do not know the relationship between mean random attractors and invariant measures of (1.1). In this paper we also discuss the existence and some properties of invariant probability measures of (1.1) with nonlinear noise. Owing to the local monotonicity and superlinear growth of $A_i$ and $B$, it is difficult to prove the Feller property of the transition operators which will be used to prove the existence of invariant probability measures of (1.1). A stopping time technique is used to overcome this difficult by proving the continuous dependence on initial data of the solutions in probability, see Lemma 4.2. For a global monotonous and linear growth case, the reader is referred to [51, 57].

For the autonomous case, we prove the existence and regularity of invariant measures of (1.1) when $A_i$ and $B$ are independent of sample and time.

**Theorem 1.4.** *(Existence and support) Let assumptions in Theorem 1.2 hold true.*

*(i) If there exist $i_0, i_1 \in [1,m] \cap \mathbb{N}$ such that $V_{i_0} \hookrightarrow H$ is compact, $\alpha_{i_1} \geqslant 2$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then the transition semigroup $(P_{0,t})_{t\geqslant 0}$ for (1.1) has an invariant probability measure on $H$.*

*(ii) If there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} \geqslant 2$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then each invariant probability measure of $(P_{0,t})_{t\geqslant 0}$ on $H$ is supported by $V = \bigcap_{i=1,2,\dots,m} V_i$.*

Next, we take a stationary solution of (1.1) to look at the moment estimates of invariant measures of $(P_{0,t})_{t\geqslant 0}$, which are useful to discuss the uniqueness, ergodicity and mixing of invariant measures of (1.1).

**Theorem 1.5.** (High-order and exponential moment estimates) *Let assumptions in Theorem 1.2 hold.*

*(i) If there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} = 2$ and $\theta_{i_1} > 2\lambda_{i_1}^{-1}\aleph_\ell^{-1}[\phi_2 + 2(\ell-1)\phi_4]$, where $\aleph_\ell = 2$ if $\ell = 1$ and $\aleph_\ell = 1$ if $\ell > 1$, then every invariant measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ satisfies*

$$\int_H \|x\|_H^{2\ell-2} \sum_{i=1}^m \|x\|_{V_i}^{\alpha_i} \eta(dx) < \infty, \quad \forall\, \ell \geqslant 1. \tag{1.6}$$

*(ii) If there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} > 2$, then every invariant measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ satisfies*

$$\int_H \|x\|_H^{2\ell-2} \sum_{i=1}^m \|x\|_{V_i}^{\alpha_i} \eta(dx) < \infty, \quad \forall\, \ell \geqslant 1. \tag{1.7}$$

*(iii) If $B(v) \equiv B$, and there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} \geqslant 2$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then every invariant measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ satisfies, for any $\epsilon \in \left[0, \frac{\lambda_{i_1}\theta_{i_1} - \phi_2}{2\|B\|^2_{\mathcal{L}_2(U,H)}}\right]$,*

$$\int_H e^{\epsilon\|x\|_H^2} \sum_{i=1}^m \theta_i \|x\|_{V_i}^{\alpha_i} \eta(dx) \leqslant (\phi_3 + \theta_{i_1} C_{\alpha_{i_1}}) \int_H e^{\epsilon\|x\|_H^2} \eta(dx) \leqslant 2(\phi_3 + \theta_{i_1} C_{\alpha_{i_1}}) e^{\epsilon(2\theta_{i_1}^{-1}(\phi_3+\theta_{i_1}C_{\alpha_{i_1}}))^{2/\alpha_{i_1}}}.$$

Under some specified assumptions, we then further look at the uniqueness, ergodicity, strong mixing and exponential mixing of invariant measures of $(P_{0,t})_{t\geqslant 0}$ on $H$.

**Theorem 1.6.** (Uniqueness, ergodicity and mixing) *Let assumptions in Theorem 1.2 hold.*

*(i) If $\kappa = 0$ and there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} = 2$, $\vartheta_{i_1} > \lambda_{i_1}^{-1}\phi_1$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then every invariant measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ is unique, ergodic, strongly mixing, and exponentially mixing in the sense that for any $\varphi \in Lip_b(H)$ and $X_0 \in H$,*

$$\left|(P_{0,t}\varphi)(X_0) - \int_H \varphi(x)\eta(dx)\right| \leqslant c\big(1 + \|X_0\|_H\big) e^{-\frac{1}{2}(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t}, \tag{1.8}$$

*where $c > 0$ is a constant independent of $X_0$ and $t$. Furthermore, $\eta$ is also exponentially mixing under the Wasserstein metric of $\mathcal{P}(H)$, that is, for any $\varphi \in Lip_b(H)$ and $\mu \in \mathcal{P}(H)$ satisfying $\int_H \|x\|_H^2 \mu(x) < \infty$,*

$$d_W^{\mathcal{P}}(Q_{0,t}\mu, \eta) \leqslant c\left(1 + \int_H \|x\|_H^2 \mu(dx)\right)^{1/2} e^{-\frac{1}{2}(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t}, \tag{1.9}$$

*where $c > 0$ is a constant independent of $t$, $Q_{0,t}$ is the adjoint operator of $P_{0,t}$.*

*(ii) If $\varpi = 0$, $\kappa \neq 0$, $B(v) \equiv B$ and there exists $i_1 \in [1,m] \cap \mathbb{N}$ such that $\alpha_{i_1} = 2$, $\vartheta_{i_1} > \lambda_{i_1}^{-1}[\phi_1 + \kappa L_0(1+\theta\phi_3)]$ and $\theta_{i_1} > \lambda_{i_1}^{-1}[\phi_2 + 2(1+\kappa\theta L_0)\|B\|^2_{\mathcal{L}_2(U,H)}]$, where $\theta := \max_{i=1,2,\dots,m}\{\theta_i^{-1}\}$, then every invariant measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ is unique, ergodic, strongly mixing, and exponentially mixing in the sense that for any $\varphi \in Lip_b(H)$ and $X_0 \in H$,*

$$\left|(P_{0,t}\varphi)(X_0) - \int_H \varphi(x)\eta(dx)\right| \leqslant C(X_0) e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1} + \phi_1 + \kappa L_0(1+\theta\phi_3)]t}, \tag{1.10}$$

*where $C(X_0) > 0$ is a constant independent of $t$. Furthermore, $\eta$ is also exponentially mixing under the Wasserstein metric of $\mathcal{P}(H)$, that is, for any $\varphi \in Lip_b(H)$ and $\mu \in \mathcal{P}(H)$ satisfying $\int_H e^{\|x\|_H^2} \mu(dx) < \infty$,*

$$d_W^{\mathcal{P}}(Q_{0,t}\mu, \eta) \leqslant C_\mu e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1} + \phi_1 + \kappa L_0(1+\theta\phi_3)]t}, \tag{1.11}$$

*where $C_\mu > 0$ is a constant independent of $t$.*

Let $B(X(t))$ be replaced by $\epsilon B(X(t))$ in (1.1) for $\epsilon \in [0, 1/\sqrt{2}]$. Let $\dot{\mathcal{P}}^\epsilon(H)$ be the collection of all invariant measures of (1.1) for $\epsilon \in [0, 1/\sqrt{2}]$. Then we discuss the limiting stability of invariant measures taken from $\dot{\mathcal{P}}^\epsilon(H)$. In particular, we show that the union of $\dot{\mathcal{P}}^\epsilon(H)$ over $[0, 1/\sqrt{2}]$ is tight on $H$, and the limit of every sequence of invariant measures taken from $\bigcup_{\varepsilon\in[0,1/\sqrt{2}]} \dot{\mathcal{P}}^\epsilon(H)$ must be an invariant measure of a limiting system of (1.1).

**Theorem 1.7.** (Limiting stability) *Let assumptions in Theorem 1.2 hold.*

*(i) For every $\delta > 0$, $T > 0$, $\epsilon_0 \in [0, 1/\sqrt{2}]$ and bounded set $B \subseteq H$, we have*

$$\lim_{\epsilon\to\epsilon_0} \sup_{t\in[0,T]} \sup_{X_0\in B} \mathbb{P}(\|X^\epsilon(t, 0, X_0) - X^{\epsilon_0}(t, 0, X_0)\|_H \geqslant \delta) = 0.$$

*(ii) If there exists $i_0, i_1 \in [1, m] \cap \mathbb{N}$ such that $V_{i_0} \hookrightarrow H$ is compact, $\alpha_{i_1} \geqslant 2$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then $\bigcup_{\varepsilon\in[0,1/\sqrt{2}]} \dot{\mathcal{P}}^\epsilon(H)$ is tight on $H$. If, in addition, $\eta^{\epsilon_n} \in \dot{\mathcal{P}}^{\epsilon_n}(H)$ and $\epsilon_n \to \epsilon_0$ with $\epsilon_0, \epsilon_n \in [0, 1/\sqrt{2}]$, then there exists a subsequence $\epsilon_{n_k}$ and $\eta^{\epsilon_0} \in \dot{\mathcal{P}}^{\epsilon_0}(H)$ such that $\eta^{\epsilon_{n_k}} \to \eta^{\epsilon_0}$ weakly.*

1.5. **Invariant measures of (1.1): nonautonomous case.** In many applications to physics and other fields of science, the evolution equations are often driven by stochastic and nonautonomous forcing simultaneously. In such a case the classical concept of invariant measures for time homogeneous transition semigroups does not work for nonautonomous stochastic equations like (1.1). To close the gap, a new concept called *evolution systems of probability measures* of time inhomogeneous transition operator $(P_{\tau,t})_{t\geqslant\tau}$ was introduced and studied in [18, 19, 28, 29]. By an evolution system of probability measures of $(P_{\tau,t})_{t\geqslant\tau}$ we mean a family of probability measures $\{\eta_t\}_{t\in\mathbb{R}}$ on $H$ satisfying the *invariance* $\int_H P_{\tau,t}\varphi(x)\eta_\tau(dx) = \int_H \varphi(x)\eta_t(dx)$ for any $t \geqslant \tau \in \mathbb{R}$ and continuous bounded function $\varphi$ on $H$.

As far as we know, there are not many results on the investigation of evolution system of probability measures for nonautonomous SPDEs, and the quoted results are all concerned with SPDEs with linear drift terms and additive noise. In this paper we study the existence, uniqueness, global exponentially mixing, forward strongly mixing and backward strongly mixing of evolution system of probability measures for a class of abstract SPDEs with nonlinear drift and diffusion terms.

**Theorem 1.8.** *Let assumptions in Theorem 1.2 hold. If $\kappa = 0$ and there exists $i_1 \in [1, m] \cap \mathbb{N}$ such that $\alpha_{i_1} = 2$, $\vartheta_{i_1} > \lambda_{i_1}^{-1}\phi_1$, $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, $\lambda_{i_1}(\vartheta_{i_1} - \theta_{i_1}) + \phi_2 - \phi_1 \geqslant 0$ and $\int_{-\infty}^\tau e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds < \infty$ for any $\tau \in \mathbb{R}$, then $(P_{\tau,t})_{t\geqslant\tau}$ has a unique evolution system of probability measures $\{\eta_t\}_{t\in\mathbb{R}}$ on $H$ such that*

$$\int_H \|x\|_H^2 \eta_t(dx) \leqslant \int_{-\infty}^t e^{[\lambda_{i_1}\theta_{i_1}-\phi_2](s-t)}\phi_3(s)ds, \quad \forall\, t \in \mathbb{R}. \tag{1.12}$$

*In addition, $\{\eta_t\}_{t\in\mathbb{R}}$ is exponentially mixing in the sense that for any $\varphi \in Lip_b(H)$, $X_0 \in H$ and $t \geqslant \tau \in \mathbb{R}$,*

$$|(P_{\tau,t}\varphi)(X_0) - \int_H \varphi(x)\eta_t(dx)| \leqslant 2\|\varphi\|_{Lip}\bigg(e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\|X_0\|_H + e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1}+\phi_1]t}e^{\frac{1}{2}[\lambda_{i_1}(\vartheta_{i_1}-\theta_{i_1})+\phi_2-\phi_1]\tau}\bigg(\int_{-\infty}^\tau e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds\bigg)^{1/2}\bigg). \tag{1.13}$$

*Furthermore, $\{\eta_t\}_{t\in\mathbb{R}}$ is also exponentially mixing under the Wasserstein metric of $\mathcal{P}(H)$, that is, for any $\varphi \in Lip_b(H)$, $t \geqslant \tau \in \mathbb{R}$ and $\{\mu_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ satisfying $\int_H \|x\|_H^2\mu_t(dx) \leqslant \int_{-\infty}^t e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds$,*

$$d_W^{\mathcal{P}}(Q_{\tau,t}\mu_\tau, \eta_t) \leqslant 2e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1}+\phi_1]t}e^{\frac{1}{2}[\lambda_{i_1}(\vartheta_{i_1}-\theta_{i_1})+\phi_2-\phi_1]\tau}\bigg(\int_{-\infty}^\tau e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds\bigg)^{1/2}. \tag{1.14}$$

**Remark 1.9.** *(i) Unlike the autonomous case, we prove the existence of evolution system of probability measures without using compact Sobolev embeddings. (ii) Every evolution system of probability measures* *satisfying* (1.12) *must be unique and exponentially mixing in the sense of* (1.13)-(1.14). *(iii) By* (1.13) *we find that for every* $\varphi \in C_b(H)$ *and* $X_0 \in H$,

$$\lim_{t\to+\infty} |(P_{\tau,t}\varphi)(X_0) - \int_H \varphi(x)\eta_t(dx)| = 0, \ \tau \in \mathbb{R}$$
$$\lim_{\tau\to-\infty} |(P_{\tau,t}\varphi)(X_0) - \int_H \varphi(x)\eta_t(dx)| = 0, \ t \in \mathbb{R}.$$

*(iv) The restrictive condition* $\kappa = 0$ *means that operators* $A$ *and* $B$ *have to be global monotone. We plan a future work to prove the existence of evolution systems of probability measures of* (1.1) *in the case* $\kappa \neq 0$.

1.6. **Remarks on models.** If $m = 1$, $\vartheta = \kappa = \varpi = 0$, $\eta_i \equiv 0$ and $\phi_1 = \phi_2 = \phi_4 = \phi_6 \equiv c$, then above framework reduces to the standard or locally monotone variational framework [13, 26, 34, 40]. A significant advantage of the present variational framework is $\kappa \neq 0$, $\varpi \neq 0$, $m \neq 1$, $\vartheta \neq 0$ and $\eta_i \neq 0$, which permits us to possibly study the global well-posedness and long-term dynamics for a wide class of SPDEs covering several important models included or not included before. A typical example within the present framework but not the clcial setting of is the *tamed* Navier-Stokes equation in dimension $N = 2, 3$: $\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \beta|\mathbf{u}|^{r-1}\mathbf{u} + \nabla p = 0$, where $\mu, \beta > 0$ and $r \geqslant 1$. It is known that the 2D Navier-Stokes equation ($\beta = 0$) satisfies the framework of (1.1) with $m = 1$. However, we can show that the tamed Navier-Stokes equation satisfies the present setting for $N = 2, 3$. Let $\mathcal{P}$ be the Helmholtz-Hodge projection, and consider $A_1(\mathbf{u}) = \mathcal{P}\big(\mu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u}\big)$ and $A_2(\mathbf{u}) = -\beta\mathcal{P}\big(|\mathbf{u}|^{r-1}\mathbf{u}\big)$. It can be proved that $A_1$ satisfies (C2)-(C4) only when $N = 2$. While, currently, it is impossible to prove that $A_1$ satisfies (C2) and (C4) simultaneously if $N = 3$ due to the term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Nevertheless, by using the *dissipative effect* of $\beta|\mathbf{u}|^{r-1}\mathbf{u}$ to carefully control $(\mathbf{u}\cdot\nabla)\mathbf{u}$, we can prove that $A = A_1 + A_2$ satisfies (C2) and (C4) together.

TABLE 1. $A$ satisfies (C2) and (C4) (see Proposition 7.11)

| | $2\ {}_{V^*}\langle A(\mathbf{v}_1) - A(\mathbf{v}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V \leqslant$ | $\sum_{i=1}^m \Big[\|A_i(t,\omega,v)\|_{V_i^*}^{\alpha_i/(\alpha_i-1)} + \rho_i(v)\Big] \leqslant$ |
|---|---|---|
| $N = 2, 3$<br>$r > 3$ | $-\mu\|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 - \beta\gamma\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{L}^{r+1}}^{r+1}$<br>$+\Big[2^{\frac{4}{r-3}}\frac{r-3}{r-1}\mu^{\frac{1-r}{r-3}}(\beta(r-1))^{\frac{2}{3-r}}\Big]\|\mathbf{v}_1 - \mathbf{v}_2\|_H^2$ | $c(1 + \|\mathbf{f}(t)\|_{\mathbb{V}^*}^2)\Big(1 + \|\mathbf{v}\|_{\mathbb{V}}^2 + \|\mathbf{v}\|_{\mathbb{L}^{r+1}}^{r+1}\Big)\Big(1 + \|\mathbf{v}\|_H^{\frac{2(r-3)}{r-1}}\Big)$ |
| $N = 2$,<br>$r \geqslant 1$ | $-\mu\|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 - 2\beta\gamma\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{L}^{r+1}}^{r+1} +$<br>$\frac{27}{64\mu^3}\|\mathbf{v}_2\|_H^2\|\mathbf{v}_2\|_{\mathbb{V}}^2\|\mathbf{v}_1 - \mathbf{v}_2\|_H^2$ | $c(1 + \|\mathbf{f}(t)\|_{\mathbb{V}^*}^2)\Big(1 + \|\mathbf{v}\|_{\mathbb{V}}^2 + \|\mathbf{v}\|_{\mathbb{L}^{r+1}}^{r+1}\Big)\Big(1 + \|\mathbf{v}\|_H^2\Big)$ |
| $N = r =$<br>$3$, $\beta\mu \geqslant 1$ | $-\mu\|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 - 2\beta\gamma\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{L}^{r+1}}^{r+1}$ | $c(1 + \|\mathbf{f}(t)\|_{\mathbb{V}^*}^2)\Big(1 + \|\mathbf{v}\|_{\mathbb{V}}^2 + \|\mathbf{v}\|_{\mathbb{L}^4}^4\Big)$ |

1.7. **Applications of abstract results.** As applications of our abstract results for problem (1.1), we show that the stochastic *tamed* Navier-Stokes equation (see Example (7.2)) and a stochastic fractional $(s, p)$-Laplacian equation with polynomial growth nonlinearities of arbitrary order for all $s \in (0, 1)$ and $p \in [2, \infty)$ (see Example (7.9)), do fall within our abstract variational setting. Then by directly applying our abstract results in Theorems 1.2-1.8, we obtain the global well-posedness and long-time dynamics results for the two typical examples under some specified conditions. Our abstract results are expected to be applied to many different types of SPDEs.

1.8. **Outline of paper.** In the next section we establish the well-posedness and Itô's formula of (1.1). In Section 3 we prove the existence a uniqueness of a mean random attractor for (1.1). In Section 4 we discuss the existence and properties of invariant measures and evolution systems of measures of (1.1) in autonomous and nonautonomous cases. In the last section we illustrate our abstract results with two typical models.

## 2. Well-posedness of (1.1) in high-order Bochner spaces: Existence, uniquness and Itô's formula

In this section we prove our first main result on the global well-posedness of problem (1.1) under assumptions (C1)-(C4). The proofs are based on the Galerkin approximation and the theory of monotone operators, see e.g., [1, 5, 38, 39, 45, 46, 59, 60] for deterministic case, and [26, 34, 40] for stochastic case.

2.1. **Approximate systems and functional spaces.** Note that if $f \in H$ and $v \in V$, then ${}_{V^*}\langle f, v\rangle_V = {}_{V_i^*}\langle f_i, v\rangle_{V_i} = \langle f, v\rangle_H$. Since the separable Banach space $V$ is continuously and densely embedded into $H$, there exists an orthonormal basis $\{e_i, i \in \mathbb{N}\}$ of $H$ such that $\{e_i, i \in \mathbb{N}\} \subseteq V$, and the $span\{e_i, i \in \mathbb{N}\}$ is dense in $V$. For $n \in \mathbb{N}$, let $P_n : V^* \to H_n := \text{span}\{e_1, e_2 \cdots, e_n\}$ be the projection given by $P_n f = \sum_{i=1}^n {}_{V^*}\langle f, e_i\rangle_V e_i$ for $f \in V^*$. This is crucial to treat the operator $A = \sum_{i=1}^m A_i$. Note that $P_n|_H$ defines an orthogonal projection onto $H_n$ in $H$, and hence ${}_{V^*}\langle P_n A(t, \omega, u), v\rangle_V = \langle P_n A(t, \omega, u), v\rangle_H = {}_{V^*}\langle A(t, \omega, u), v\rangle_V$ for all $u \in V$ and $v \in H_n$. Let $\{g_i, i \in \mathbb{N}\}$ be an orthonormal basis of $U$. Let $\widetilde{P}_n : U \to U_n := \text{span}\{g_1, g_2 \cdots, g_n\}$ be an orthogonal projection, and write $W_n(t) := \widetilde{P}_n W(t) = \sum_{i=1}^n \langle W(t), g_i\rangle_U g_i$. Then we consider a finite-dimensional stochastic system on $H_n$:

$$\begin{cases} dX_n(t) = P_n A(t, X_n(t))dt + P_n B(t, X_n(t))dW_n(t), \quad t > \tau \in \mathbb{R}, \\ X_n(\tau) = P_n X_0. \end{cases} \tag{2.1}$$

According to the classical results for the solvability of finite-dimensional SDEs, see e.g., [26], we can prove that problem (2.1) has a unique continuous strong solution. In what follows, we will derive a priori estimates of solutions to (2.1) in the spaces $K_i = L^{\alpha_i}([\tau, \tau+T] \times \Omega, dt \times \mathbb{P}; V_i)$, $K_i^* = L^{\frac{\alpha_i}{\alpha_i - 1}}([\tau, \tau + T] \times \Omega, dt \times \mathbb{P}; V_i^*)$, $i = 1, 2, \dots, m$, $J = L^2([\tau, \tau + T] \times \Omega, dt \times \mathbb{P}; \mathcal{L}_2(U, H))$ and $S = L^\infty([\tau, \tau + T], dt; L^2(\Omega, \mathbb{P}; H))$. In addition, the letter $c > 0$ denotes a generic constant which may change its value in different places.

2.2. **A priori estimates.** The main difficulty of deriving a priori estimates of solutions to (2.1) in the spaces above is how to estimate the nonlinear diffusion term $B(t, \omega, v)$ with a superlinear growth rate in $v$. This difficulty can be surmounted by using the dissipativeness of the operator $A$ to carefully control the superlinear growth of $B(t, \omega, v)$. To simplify calculations, we derive the following two formulations which are just direct consequences of (C4).

**Proposition 2.1.** *Let (C4) be valid. If $\beta_i < \alpha_i$, then for any $\varepsilon, \varepsilon_1, \varepsilon_2, r_1, K > 0$ and $r_2 > r_1$,*

$$K\|B(t, \omega, v)\|^2_{\mathcal{L}_2(U,H)} \leqslant \varepsilon \sum_{i=1}^m \theta_i \|v\|^{\alpha_i}_{V_i} + K\phi_4(t, \omega)\|v\|^2_H + c\sum_{i=1}^m \eta_i^{\frac{\alpha_i}{\alpha_i - \beta_i}}(t, \omega) + K\phi_5(t, \omega), \tag{2.2a}$$

$$\begin{aligned} K\|v\|^{r_1}_H \|B(t, \omega, v)\|^2_{\mathcal{L}_2(U,H)} \leqslant{}& \varepsilon_1 \|v\|^{r_1}_H \sum_{i=1}^m \theta_i \|v\|^{\alpha_i}_{V_i} + \varepsilon_2 \|v\|^{r_2}_H + K\phi_4(t, \omega)\|v\|^{2+r_1}_H \\ &+ c\phi_5^{\frac{r_2}{r_2 - r_1}}(t, \omega) + c\sum_{i=1}^m \eta_i^{\frac{\alpha_i r_2}{(\alpha_i - \beta_i)(r_2 - r_1)}}(t, \omega). \end{aligned} \tag{2.2b}$$

**Lemma 2.2.** *Let (C1)-(C4) hold. Then for any $X_0 \in L^{2\ell}(\Omega, \mathcal{F}_\tau, \mathbb{P}; H)$, we have the following conclusions.*

*(i) If $\phi_3,\phi_5\in L^{\ell}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ and $\eta_i\in L^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ hold for any $\ell\geqslant 1$, then there exists a constant $C(\tau,T,X_0)>0$ independent of $n$ such that*

$$\mathbb{E}\left[\sup_{t\in[\tau,\tau+T]}\|X_n(t)\|_H^{2\ell}\right]+\mathbb{E}\left[\int_\tau^{\tau+T}\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\|X_n(s)\|_{V_i}^{\alpha_i}ds\right]\leqslant C(\tau,T,X_0),\quad\forall\ \ell\geqslant 1.$$

*(ii) If $\phi_3,\phi_5\in L^{\ell}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ and $\eta_i\in L^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ hold for all $\ell\geqslant\varpi/2+1$, then for every $n\in\mathbb{N}$, there exists a constant $C(\tau,T,X_0)>0$ independent of $n$ such that*

$$\|X_n\|^{2\ell}_{L^{2\ell}(\Omega,\mathbb{P};L^\infty([\tau,\tau+T,dt;H]))}+\sum_{i=1}^m\left[\|X_n\|_{K_i}^{\alpha_i}+\|A_i(\cdot,X_n)\|_{K_i^*}^{\frac{\alpha_i}{\alpha_i-1}}\right]+\|B(\cdot,X_n)\|_J^2\leqslant C(\tau,T,X_0).$$

*Proof.* (i) Applying the finite-dimensional Itô formula to (2.1) we find, $\mathbb{P}$-a.s.,

$$\begin{aligned}\|X_n(t)\|_H^2=\|P_nX_0\|_H^2&+\int_\tau^t\big(2\ {}_{V^*}\langle P_nA(s,X_n(s)),X_n(s)\rangle_V\\&+\|P_nB(s,X_n(s))\widetilde{P}_n\|^2_{\mathcal{L}_2(U,H)}\big)ds+2\int_\tau^t\langle X_n(s),P_nB(s,X_n(s))dW_n(s)\rangle_Hds.\end{aligned}\tag{2.3}$$

Note that by Parseval's identity we have $\|(P_nB(s,X_n(s))\widetilde{P}_n)^*X_n(s)\|_{\mathcal{L}_2(U,\mathbb{R})}=\|(P_nB(s,X_n(s))\widetilde{P}_n)^*X_n(s)\|_U\leqslant\|P_nB(s,X_n(s))\widetilde{P}_n\|_{\mathcal{L}_2(U,H)}\|X_n(s)\|_H\leqslant\|B(s,X_n(s))\|_{\mathcal{L}_2(U,H)}\|X_n(s)\|_H$. For any $\ell\geqslant 1$, applying the finite-dimensional Itô formula to (2.3) again, we infer from (2.2b) with $K=\ell(2\ell-1)$, $\varepsilon_1=\varepsilon_2=\ell/2$, $r_1=2\ell-2$ and $r_2=2\ell$ that $\mathbb{P}$-a.s.,

$$\begin{aligned}\|X_n(t)\|_H^{2\ell}&=\|P_nX_0\|_H^{2\ell}+2\ell(\ell-1)\int_\tau^t\|X_n(s)\|_H^{2\ell-4}\|(P_nB(s,X_n(s))\widetilde{P}_n)^*X_n(s)\|^2_{\mathcal{L}_2(U,\mathbb{R})}ds\\&\quad+\ell\int_\tau^t\|X_n(s)\|_H^{2\ell-2}\big(2\ {}_{V^*}\langle P_nA(s,X_n(s)),X_n(s)\rangle_V+\|P_nB(s,X_n(s))\widetilde{P}_n\|^2_{\mathcal{L}_2(U,H)}\big)ds+M_n(t)\\&\leqslant\|P_nX_0\|_H^{2\ell}-2\ell\int_\tau^t\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\theta_i\|X_n(s)\|_{V_i}^{\alpha_i}ds\\&\quad+\ell(2\ell-1)\int_\tau^t\|X_n(s)\|_H^{2\ell-2}\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\\&\quad+c\int_\tau^t\big((1+\phi_2(s))\|X_n(s)\|_H^{2\ell}+\phi_3^\ell(s)\big)\,ds+M_n(t)\\&\leqslant\|P_nX_0\|_H^{2\ell}-\frac{3\ell}{2}\int_\tau^t\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\theta_i\|X_n(s)\|_{V_i}^{\alpha_i}ds\\&\quad+c\int_\tau^t\bigg(\Big(1+\sum_{i=2,4}\phi_i(s)\Big)\|X_n(s)\|_H^{2\ell}+\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(s)+\sum_{i=3,5}\phi_i^\ell(s)\bigg)ds+M_n(t),\end{aligned}\tag{2.4}$$

where $M_n(t):=2\ell\int_\tau^t\|X_n(s)\|_H^{2\ell-2}\langle X_n(s),P_nB(s,X_n(s))dW_n(s)\rangle_H$. For $k\in\mathbb{N}$, we define a stopping time

$$\varsigma_n^k=\inf\left\{t\geqslant\tau:\|X_n(t)\|_H>k\ \text{ and }\ \sum_{i=1}^m\int_\tau^t\|X_n(s)\|_{V_i}^{\alpha_i}ds>k\right\},$$

where $\inf\emptyset=+\infty$. Since we can control $B(s,X_n(s)$ by Proposition 2.1, by the definition of $\varsigma_n^k$ we know the quadratic variation of $M_n(t\wedge\varsigma_n^k)$: $\langle M_n(\cdot\wedge\varsigma_n^k)\rangle_t\leqslant 4\ell^2\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{4\ell-4}\|(P_nB(s,X_n(s))\widetilde{P}_n)^*X_n(s)\|^2_{\mathcal{L}_2(U,\mathbb{R})}\leqslant 4\ell^2\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{4\ell-2}\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\leqslant C(\tau,T,k)$, $\mathbb{P}$-a.s.. Then $M_n(t)$ is a real-valued continuous local martingale. By (2.4) we obtain, for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\right]+\frac{3\ell}{2}\mathbb{E}\left[\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\theta_i\|X_n(s)\|_{V_i}^{\alpha_i}ds\right]\\&\leqslant 2\mathbb{E}[\|P_nX_0\|_H^{2\ell}]+c\int_\tau^t\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s}\bigg(\Big(1+\sum_{i=2,4}\phi_i(r\wedge\varsigma_n^k)\Big)\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\bigg)\right]ds\end{aligned}$$

$$+ c\int_\tau^{\tau+T}\mathbb{E}\left[\sum_{i=3,5}\phi_i^\ell(s)+\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(s)\right]ds+2\mathbb{E}\Big[\sup_{\tau\leqslant r\leqslant t\wedge\varsigma_n^k}|M_n(r)|\Big]. \tag{2.5}$$

By the BDG inequality for real-valued continuous local martingales, we infer from (2.2b) with $\varepsilon_1=\ell/4$, $\varepsilon_2=1/4$, $r_1=2\ell-2$ and $r_2=2\ell$ that the stochastic term in (2.5) is bounded by

$$\begin{aligned}
2\mathbb{E}\Big[\sup_{\tau\leqslant r\leqslant t\wedge\varsigma_n^k}|M_n(r)|\Big] &\leqslant 12\ell\mathbb{E}\left[\left(\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{4\ell-2}\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)^{\frac12}\right]\\
&\leqslant 12\ell\mathbb{E}\left[\sup_{\tau\leqslant s\leqslant t}\|X_n(s\wedge\varsigma_n^k)\|_H^\ell\left(\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{2\ell-2}\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)^{\frac12}\right]\\
&\leqslant \frac12\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\right)+c\mathbb{E}\left(\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|^{2\ell-2}\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\right]\\
&\leqslant \frac12\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\right]+\frac{\ell}{4}\mathbb{E}\left[\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\theta_i\|X_n(s)\|_{V_i}^{\alpha_i}ds\right]\\
&\quad+c\int_\tau^t\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s}\Big(\Big(1+\phi_4(r\wedge\varsigma_n^k)\Big)\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\Big)\right]ds+c\int_\tau^{\tau+T}\mathbb{E}\left[\phi_5^\ell(s)+\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(s)\right]ds.
\end{aligned}\tag{2.6}$$

By (2.5)-(2.6) we see, for all $\ell\geqslant1$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\leqslant r\leqslant t}\|X_n(r\wedge\varsigma_n^k)\|_H^{2\ell}\right]+\mathbb{E}\left[\int_\tau^{t\wedge\varsigma_n^k}\|X_n(s)\|_H^{2\ell-2}\sum_{i=1}^m\|X_n(s)\|_{V_i}^{\alpha_i}ds\right]\\
&\leqslant e^{c+c\sum_{i=2,4}\|\phi_i\|_{L^\infty([\tau,\tau+T]\times\Omega)}}\left(\mathbb{E}[\|X_0\|_H^{2\ell}]+\int_\tau^{\tau+T}\mathbb{E}\Big[\sum_{i=3,5}\phi_i^\ell(s)+\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(s)\Big]ds\right).
\end{aligned}\tag{2.7}$$

By Markov's inequality and (2.7) we find $\mathbb{P}\{\tau\leqslant\varsigma_n^k<\tau+\mathcal{T}\}\leqslant\mathbb{P}\big\{\sup_{\tau\leqslant r\leqslant\tau+\mathcal{T}}\|X_n(r\wedge\varsigma_n^k)\|_H\geqslant k\big\}\leqslant Ck^{-2\ell}<\infty$ for any $\mathcal{T}\in\mathbb{N}$, where $C>0$ is a constant independent of $k$. Let $\Omega_\mathcal{T}=\bigcap_{m=1}^\infty\bigcup_{k=m}^\infty\{\varsigma_n^k<\tau+\mathcal{T}\}$. By the Borel-Cantelli lemma we know $\mathbb{P}(\Omega_\mathcal{T})=\mathbb{P}\big(\bigcap_{m=1}^\infty\bigcup_{k=m}^\infty\{\tau\leqslant\varsigma_n^k<\tau+\mathcal{T}\}\big)=0$. This implies, for every $\omega\in\Omega\setminus\Omega_\mathcal{T}$, there exists $k_0=k_0(\omega)>0$ such that $\varsigma_n^k(\omega)\geqslant\tau+\mathcal{T}$ for all $k\geqslant k_0$. Taking $\Omega_0=\bigcup_{\mathcal{T}\in\mathbb{N}}\Omega_\mathcal{T}$, we have $\mathbb{P}(\Omega_0)=0$ and $\varsigma_n^k(\omega)\geqslant\tau+\mathcal{T}$ for all $\omega\in\Omega\setminus\Omega_0$, and thus $\varsigma_n^k\to\infty$ as $k\to\infty$, $\mathbb{P}$-a.s. Consequently, we complete the proof of (i) by letting $k\to\infty$ in (2.7).

(ii) Note that condition (ii) implies condition (i). Then by (C4), (i) and Proposition 2.1, there exists a constant $C(\tau,T,X_0)>0$, independent of $n$, such that for all $\ell\geqslant\varpi/2+1$,

$$\sum_{i=1}^m\mathbb{E}\left[\int_\tau^t\|A_i(s,X_n(s))\|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}}ds\right]+\mathbb{E}\left[\int_\tau^t\|B(s,X_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\right]\leqslant C(\tau,T,X_0). \tag{2.8}$$

This yields (ii). □

2.3. **Proof of Theorem 1.2.** A special attention in the proof of Theorem 1.2 is that the approximated solutions converge in both Sobolev and finitely many Banach spaces simultaneously. The idea of the proof is motivated by the classical works in [26, 34], where the SPDEs is defined in a single Banach space and the growth rate of the noise is linear. In Theorem 1.2 we consider a large class of SPDEs defined in a dual space of intersection of finitely many Banach spaces driven by superlinear noise in order to cover more models which do not fall within the variational framework. Our approaches are different from the semigroup method [17], and the regularization method [51] since the operator $A$ in there is restricted to be linear.

*Proof.* **Proof of Theorem 1.2**. By Lemma 2.2 there exist a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$X_{n_k}\to\bar X\ \text{ weakly star in }\big(L^2(\Omega,\mathbb{P};L^1([\tau,\tau+T],dt;H])\big)^*; \tag{2.9a}$$

$$X_{n_k}\to\bar X\ \text{ weakly in }L^2(\Omega,\mathbb{P};L^2([\tau,\tau+T],dt;H]),\ K_1,\ K_2,\dots,K_m; \tag{2.9b}$$

$$A_i(\cdot, X_{n_k}) \to Y_i \quad \text{weakly in } K_i^*, \quad i = 1, 2, \ldots, m; \tag{2.9c}$$

$$P_{n_k} B(\cdot, X_{n_k}) \to Z \quad \text{weakly in } J. \tag{2.9d}$$

By (2.1) we infer that for all $v \in \bigcup_{n\geqslant 1} H_n$ and $\varphi \in L^\infty([\tau, \tau+T] \times \Omega, dt \times \mathbb{P}; \mathbb{R})$,

$$\begin{aligned}\mathbb{E}\left[\int_\tau^{\tau+T} \varphi(t)\langle X_{n_k}(t), v\rangle_H dt\right] = \mathbb{E}\Bigg[\int_\tau^{\tau+T} \varphi(t)\Big\langle P_{n_k} X_0 + \int_\tau^t \sum_{i=1}^m P_{n_k} A_i(s, X_{n_k}(s))ds \\ + \int_\tau^t P_{n_k} B(s, X_{n_k}(s))dW_{n_k}(s), v\Big\rangle_H dt\Bigg]. \end{aligned} \tag{2.10}$$

Passing to the limit in (2.10) as $k \to \infty$ according to (2.9a)-(2.9d), we deduce that

$$\mathbb{E}\left[\int_\tau^{\tau+T} \varphi(t)\langle \bar{X}(t), v\rangle_H dt\right] = \mathbb{E}\left[\int_\tau^{\tau+T} \varphi(t)\Big\langle X_0 + \int_\tau^t \sum_{i=1}^m Y_i(s)ds + \int_\tau^t Z(s)dW(s), v\Big\rangle_H dt\right]. \tag{2.11}$$

Define $Y := \sum_{i=1}^m Y_i$ and $X(t) := X_0 + \int_\tau^t Y(s)ds + \int_\tau^t Z(s)dW(s)$. Then by (2.11) we find $\bar{X} = X$, $dt \times \mathbb{P}$-a.e.. Note that $\bar{X} \in \bigcap_{i=1,2,\ldots,m} K_i$, $Y \in \sum_{i=1}^m K_i^*$ and $Z \in J$, as in [35, 41], we can similarly prove that $\{X(t)\}_{t\in[\tau,\tau+T]}$ is a $H$-valued continuous $\mathcal{F}_t$-adapted stochastic process satisfying $\mathbb{E}\Big[\sup_{t\in[\tau,\tau+T]} \|X(t)\|_H^2\Big] < \infty$ and the following Itô energy equation, $\mathbb{P}$-a.s.,

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_\tau^t \big(2\ {}_{V^*}\langle Y(s), X(s)\rangle_V + \|Z(s)\|_{\mathcal{L}_2(U,H)}^2\big)ds + 2\int_\tau^t \langle X(s), Z(s)dW(s)\rangle_H. \tag{2.12}$$

Applying the finite-dimensional Itô formula to (2.12) we find the Itô energy equation for any $\ell \geqslant 1$:

$$\begin{aligned}\|X(t)\|_H^{2\ell} = \|X_0\|_H^{2\ell} + 2\ell(\ell-1)\int_\tau^t \|X(s)\|_H^{2\ell-4}\|(Z(s))^*X(s)\|_U^2 ds \\ + \ell\int_\tau^t \|X(s)\|_H^{2\ell-2}\big(2\ {}_{V^*}\langle Y(s), X(s)\rangle_V + \|Z(s)\|_{\mathcal{L}_2(U,H)}^2\big)ds + M(t), \ \mathbb{P}\text{-a.s}, \end{aligned} \tag{2.13}$$

where $M(t) = 2\ell\int_\tau^t \|X(s)\|_H^{2\ell-2}\langle X(s), Z(s)dW(s)\rangle_H$. By a stoping-time argument, as in Lemma 2.2, we can show that $\mathbb{E}\Big[\sup_{t\in[\tau,\tau+T]} \|X(t)\|_H^{2\ell}\Big] < \infty$, and that $M(t)$ is a real-valued continuous local martingale. Taking the expectation in (2.13), by the property of Itô's integral (see e.g.,[17]), we then obtain the mean energy equation:

$$\begin{aligned}\frac{d}{dt}\mathbb{E}(\|X(t)\|_H^{2\ell}) = 2\ell(\ell-1)\mathbb{E}[\|X(t)\|_H^{2\ell-4}\|(Z(t))^*X(t)\|_U^2] \\ + \ell\mathbb{E}\Big[\|X(t)\|_H^{2\ell-2}\big(2\ {}_{V^*}\langle Y(t), X(t)\rangle_V + \|Z(t)\|_{\mathcal{L}_2(U,H)}^2\big)\Big]. \end{aligned} \tag{2.14}$$

Next, we use the monotonicity method to verify $A(\cdot,\cdot,\bar{X}) = Y$ and $B(\cdot,\cdot,\bar{X}) = Z$, $dt \times \mathbb{P}$-a.s.. Let $\phi$ be a $V$-valued $\mathcal{F}_t$-adapted process such that

$$\phi \in \Big(\bigcap_{i=1,2,\ldots,m} K_i\Big) \bigcap L^{2\vartheta}(\Omega, \mathbb{P}; L^\infty([\tau, \tau+T], dt; H)).$$

For $R \in \mathbb{N}$, we define a stopping time by

$$\varsigma_\phi^R := (\tau+T) \wedge \inf\Big\{t \in [\tau, \tau+T] : \|\phi(t)\|_H \vee \sum_{i=1}^m \int_\tau^t \|\phi(r)\|_{V_i}^{\alpha_i} dr > R\Big\}.$$

Note that $\varsigma_\phi^R \to \tau+T$ as $R \to \infty$, $\mathbb{P}$-a.s.. By (2.3) and the product rule we find that

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_\tau^{t\wedge\varsigma_\phi^R}\big(\phi_1(s)+\kappa\sum_{i=1}^m \rho_i(\phi(s))\big)ds}\|X_{n_k}(t\wedge\varsigma_\phi^R)\|_H^2\right] - \mathbb{E}\left[\|P_{n_k}X_0\|_H^2\right] \\ &= \mathbb{E}\Bigg[\int_\tau^{t\wedge\varsigma_\phi^R} e^{-\int_\tau^s\big(\phi_1(r)+\kappa\sum_{i=1}^m \rho_i(\phi(r))\big)dr}\Bigg(2\ {}_{V^*}\langle P_{n_k}A(s, X_{n_k}(s)), X_{n_k}(s)\rangle_V \\ &\quad + \|P_{n_k}B(s, X_{n_k}(s))\widetilde{P}_{n_k}\|_{\mathcal{L}_2(U,H)}^2 - \Big(\phi_1(s) + \kappa\sum_{i=1}^m \rho_i(\phi(s))\Big)\|X_{n_k}(s)\|_H^2\Bigg)ds\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[\int_\tau^{t\wedge\varsigma_\phi^R} e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(r))\right)dr}\Bigg(2\ {}_{V^*}\langle A(s,X_{n_k}(s))-A(s,\phi(s)),X_{n_k}(s)-\phi(s)\rangle_V\\
&\qquad+\|P_{n_k}B(s,X_{n_k}(s))\widetilde{P}_{n_k}-P_{n_k}B(s,\phi(s))\widetilde{P}_{n_k}\|^2_{\mathcal{L}_2(U,H)}\\
&\qquad-\Bigg(\phi_1(s)+\kappa\sum_{i=1}^m\big(\rho_i(\phi(s))\big)\|X_{n_k}(s)-\phi(s)\|^2_H\Bigg)ds\Bigg]\\
&\quad+\mathbb{E}\Bigg[\int_\tau^{t\wedge\varsigma_\phi^R} e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(r))\right)dr}\Bigg(2\ {}_{V^*}\langle A(s,X_{n_k}(s))-A(s,\phi(s)),\phi(s)\rangle_V\\
&\qquad+2\ {}_{V^*}\langle A(s,\phi(s)),X_{n_k}(s)\rangle_V-\|P_{n_k}B(s,\phi(s))\widetilde{P}_{n_k}\|^2_{\mathcal{L}_2(U,H)}\\
&\qquad+2\langle P_{n_k}B(s,X_{n_k}(s))\widetilde{P}_{n_k},P_{n_k}B(s,\phi(s))\widetilde{P}_{n_k}\rangle_{\mathcal{L}_2(U,H)}\\
&\qquad+\Bigg(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(\phi(s))\Bigg)\Big(\|\phi(s)\|^2_H-2\langle X_{n_k}(s),\phi(s)\rangle_H\Big)\Bigg)ds\Bigg].
\end{aligned}
\tag{2.15}
$$

This together with condition (C2) and (2.9a)-(2.9d) implies that for any nonnegative $\psi\in L^\infty([\tau,\tau+T],dt;\mathbb{R})$,

$$
\begin{aligned}
&\mathbb{E}\left[\int_\tau^{\tau+T}\psi(t)\left(e^{-\int_\tau^{t\wedge\varsigma_\phi^R}\left(\phi_1(s)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(s))\right)ds}\|X(t\wedge\varsigma_\phi^R)\|^2_H-\|X_0\|^2_H\right)dt\right]\\
&\leqslant\liminf_{k\to\infty}\mathbb{E}\left[\int_\tau^{\tau+T}\psi(t)\left(e^{-\int_\tau^{t\wedge\varsigma_\phi^R}\left(\phi_1(s)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(s))\right)ds}\|X_{n_k}(t)\|^2_H-\|P_{n_k}X_0\|^2_H\right)dt\right]\\
&\leqslant\mathbb{E}\Bigg[\int_\tau^{\tau+T}\psi(t)\int_\tau^{t\wedge\varsigma_\phi^R}e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(r))\right)dr}\Bigg(2\ {}_{V^*}\langle Y(s)-A(s,\phi(s)),\phi(s)\rangle_V\\
&\quad+2\ {}_{V^*}\langle A(s,\phi(s)),X(s)\rangle_V-\|B(s,\phi(s))\|^2_{\mathcal{L}_2(U,H)}+2\langle Z(s),B(s,\phi(s))\rangle_{\mathcal{L}_2(U,H)}\\
&\quad+\Bigg(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(\phi(s))\Bigg)\Big(\|\phi(s)\|^2_H-2\langle X(s),\phi(s)\rangle_H\Big)\Bigg)dsdt\Bigg],
\end{aligned}
\tag{2.16}
$$

By (2.14) and the product rule we can similarly find that

$$
\begin{aligned}
&\mathbb{E}\left[\int_\tau^{\tau+T}\psi(t)\left(e^{-\int_\tau^{t\wedge\varsigma_\phi^R}\left(\phi_1(s)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(s))\right)ds}\|X(t\wedge\varsigma_\phi^R)\|^2_H-\|X_0\|^2_H\right)dt\right]\\
&=\mathbb{E}\Bigg[\int_\tau^{\tau+T}\psi(t)\int_\tau^{t\wedge\varsigma_\phi^R}e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(r))\right)dr}\Bigg(2\ {}_{V^*}\langle Y(s),X(s)\rangle_V\\
&\quad+\|Z(s)\|^2_{\mathcal{L}_2(U,H)}-\Bigg(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(\phi(s))\Bigg)\|X(s)\|^2_H\Bigg)dsdt\Bigg].
\end{aligned}
\tag{2.17}
$$

Submitting (2.17) into (2.16), and rearranging the resulting terms, as in (2.15), we arrive at

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_\tau^{\tau+T}\psi(t)\int_\tau^{t\wedge\varsigma_\phi^R}e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(\phi(r))\right)dr}\Bigg(2\ {}_{V^*}\langle Y(s)-A(s,\phi(s)),X(s)-\phi(s)\rangle_V\\
&\quad+\|B(s,\phi(s))-Z(s)\|^2_{\mathcal{L}_2(U,H)}-\Bigg(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(\phi(s))\Bigg)\|X(s)-\phi(s)\|^2_H\Bigg)dsdt\Bigg]\leqslant 0.
\end{aligned}
\tag{2.18}
$$

Take $\phi=X$ in (2.18), and letting $R\to\infty$, we find that $Z=B(\cdot,X)$, $dt\times\mathbb{P}$-a.s., on $[\tau,\tau+T]\times\Omega$. Taking $\phi=X-\varepsilon\tilde{\phi}v$ with $\varepsilon>0$, $\tilde{\phi}\in L^\infty([\tau,\tau+T]\times\Omega;dt\times\mathbb{P};\mathbb{R})$ and $v\in V$ in (2.18), dividing both sides by $\varepsilon$, and letting $\varepsilon\to0$, we infer from (C1), (C4), the hemicontinuity of $\rho_i$ and Lebesgue's theorem that

$$
\mathbb{E}\Bigg[\int_\tau^{\tau+T}\psi(t)\int_\tau^{t\wedge\varsigma_\phi^R}e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum\limits_{i=1}^m\rho_i(X(r))\right)dr}\tilde{\phi}(s)\ {}_{V^*}\langle Y(s)-A(s,X(s)),v\rangle_V dsdt\Bigg]\leqslant 0.
$$

This along with the arbitrariness of $\psi$, $\tilde{\phi}$ and $v$ implies, by letting $R\to\infty$, that $Y=A(\cdot,\bar{X})$, $dt\times\mathbb{P}$-a.s., on $[\tau,\tau+T]\times\Omega$. Then $X$ is a solution to (1.1) in the sense of Def. 1.1. The energy equations (1.3)-(1.4) follow from (2.13)-(2.14). The mean uniform estimates (1.5) are derived by using (1.3) as we did in Lemma 2.2.

Finally, we prove the pathwise uniqueness of solutions to (1.1). Let $X(t)$ and $Y(t)$ be two solutions to (1.1) in the sense of Definition 1.1. For each $R>0$, we define a stopping time

$$\varsigma^R=\inf\left\{t\geqslant\tau:\left[\|X(t)\|_H\vee\|Y(t)\|_H\vee\int_\tau^t\sum_{i=1}^m\|X(s)\|_{V_i}^{\alpha_i}ds\vee\int_\tau^t\sum_{i=1}^m\|Y(s)\|_{V_i}^{\alpha_i}ds\right]>R\right\}. \tag{2.19}$$

By (C2) and the product rule, we have

$$e^{-\int_\tau^{t\wedge\varsigma^R}\left(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(Y(s))\right))ds}\|X(t)-Y(t)\|_H^2\leqslant\|X_0-Y_0\|_H^2+M(t\wedge\varsigma^R), \tag{2.20}$$

where $M(t\wedge\varsigma^R):=2\int_\tau^{t\wedge\varsigma^R}e^{-\int_\tau^s\left(\phi_1(r)+\kappa\sum_{i=1}^m\rho_i(Y(r)\right)dr}\langle X(s)-Y(s),\big(B(s,X(s))-B(s,Y(s))\big)dW(s)\rangle_H$. By (2.19) and Prop. 2.1 we know that $\mathbb{E}[M(t\wedge\varsigma^R)]=0$. Taking the expectation in (2.20), we find

$$\mathbb{E}\left[e^{-\int_\tau^{t\wedge\varsigma^R}\left(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(Y(s))\right)ds}\|X(t\wedge\varsigma^R)-Y(t\wedge\varsigma^R)\|_H^2\right]\leqslant\mathbb{E}\left[\|X_0-Y_0\|_H^2\right].$$

If $X_0=Y_0$, $\mathbb{P}$-a.s., then $\mathbb{E}[\|X_0-Y_0\|_H^2]=0$, and so $\mathbb{E}\Big[e^{-\int_\tau^{t\wedge\varsigma^R}\left(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(Y(s))\right)ds}\|X(t\wedge\varsigma^R)-Y(t\wedge\varsigma^R)\|_H^2\Big]=0$. This implies $e^{-\int_\tau^{t\wedge\varsigma^R}\left(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(Y(s))\right)ds}\|X(t)-Y(t)\|_H^2=0$, $\mathbb{P}$-a.s.. Note that $e^{\int_\tau^{t\wedge\varsigma^R}\left(\phi_1(s)+\kappa\sum_{i=1}^m\rho_i(Y(s))\right)ds}<\infty$, $\mathbb{P}$-a.s.. Therefore $X(t\wedge\varsigma^R)=Y(t\wedge\varsigma^R)$, $\mathbb{P}$-a.s.. Then we complete the proof by letting $R\to\infty$. □

In the following sections, we assume that there exists $i_1\in[1,m]\cap\mathbb{N}$ such that $\alpha_{i_1}\geqslant 2$. By Theorem 1.2, we are able to discuss mean random attractors and invariant measures for (1.1) by using the two inequalities in Remark 1.3 rather than (C2)-(C3).

## 3. Mean random attractors of (1.1): existence and uniqueness

Given $\ell\geqslant 1$, $t\in\mathbb{R}^+$ and $\tau\in\mathbb{R}$, we define a mapping $\Phi(t,\tau):L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)\to L^{2\ell}(\Omega,\mathcal{F}_{\tau+t},\mathbb{P};H)$ by $\Phi(t,\tau)X_0=X(t+\tau,\tau,X_0)$. By Theorem 1.2 we can check that $\Phi$ defines a mean RDS for (1.1) on $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ in the sense of Wang [50] such that $\Phi(t+s,\tau)=\Phi(t,s+\tau)\circ\Phi(s,\tau)$ for all $s\in\mathbb{R}^+$. If $\ell=1$, then we may call $\Phi$ is a mean-square RDS in the sense of Kloeden and Lorenz [25]. Let $\mathcal{D}=\{\mathcal{D}(\tau)\subseteq L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H):\tau\in\mathbb{R},\mathcal{D}(\tau)\neq\emptyset\text{ is bounded}\}$ be a family of attracting sets:

$$\lim_{t\to+\infty}e^{-\frac{1}{2}\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}t+\ell\int_{\tau-t}^\tau[\|\phi_2(\varsigma)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(\varsigma)\|_{L^\infty(\Omega)}]d\varsigma}\|\mathcal{D}(\tau-t)\|_{L^{2\ell}(\Omega,\mathcal{F}_{\tau-t},\mathbb{P};H)}^{2\ell}=0, \tag{3.1}$$

where $\lambda_{i_1}$ and $\theta_{i_1}$ are given in Remark 1.3, $\aleph_\ell$ is the same as in Theorem 1.5 and $\|\mathcal{D}(\tau-t)\|_{L^{2\ell}(\Omega,\mathcal{F}_{\tau-t},\mathbb{P};H)}=\sup_{u\in\mathcal{D}(\tau-t)}\|u\|_{L^{2\ell}(\Omega,\mathcal{F}_{\tau-t},\mathbb{P};H)}$. Denote by $\mathfrak{D}$ the collection of all families $\mathcal{D}$ of sets satisfying (3.1).

**Lemma 3.1.** *Let assumptions in Theorem 1.2 and Remark 1.3 hold. If for any $\ell\geqslant 1$,*

$$\boldsymbol{G}^\ell(\tau):=\int_{-\infty}^\tau e^{\frac{1}{2}\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}(r-\tau)+\ell\int_r^\tau[\|\phi_2(\varsigma)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(\varsigma)\|_{L^\infty(\Omega)}]d\varsigma}$$
$$\times\left(1+\mathbb{E}\Bigg[\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(r)+\sum_{j=3,5}\phi_j^\ell(r)\Bigg]\right)dr<\infty,\quad\forall\,\tau\in\mathbb{R}, \tag{3.2}$$

*then for each $\tau\in\mathbb{R}$, $\mathcal{D}=\{\mathcal{D}(\tau):\tau\in\mathbb{R}\}\in\mathfrak{D}$, there exist $T=T(\tau,\mathcal{D})>0$ and a constant $c>0$, independent of $\tau$ and $\mathcal{D}$, such that* $\sup_{t\geqslant T}\sup_{X_0\in\mathcal{D}(\tau-t)}\mathbb{E}\big[\|X(\tau,\tau-t,X_0)\|_H^{2\ell}\big]\leqslant c\boldsymbol{G}^\ell(\tau)$.

*Proof.* By (1.4), Remark 1.3 and (2.2b) for $K = 2\ell(\ell-1)$, $\varepsilon_1 = \ell/2$, $\varepsilon_2 = \ell\lambda_{i_1}\theta_{i_1}/4$, $r_1 = 2\ell-2$ and $r_2 = 2\ell$, we have

$$\begin{aligned}\frac{d}{dr}\mathbb{E}[\|X(r)\|_H^{2\ell}] &\leqslant \ell(\|\phi_2(r)\|_{L^\infty(\Omega)} - \lambda_{i_1}\theta_{i_1})\mathbb{E}\big[\|X(r)\|_H^{2\ell}\big] - \ell\mathbb{E}\bigg[\|X(r)\|_H^{2\ell-2}\sum_{i=1}^m \theta_i\|X(r)\|_{V_i}^{\alpha_i}\bigg]\\ &\quad + 2\ell(\ell-1)\mathbb{E}\big[\|X(r)\|_H^{2\ell-2}\|B(r,X(r))\|^2_{\mathcal{L}_2(U,H)}\big] + \ell\mathbb{E}\big[(\phi_3(r)+\theta_{i_1}C_{\alpha_{i_1}})\|X(r)\|_H^{2\ell-2}\big]\\ &\leqslant \ell\Big(-\frac{1}{2}\aleph_\ell\lambda_{i_1}\theta_{i_1} + \|\phi_2(r)\|_{L^\infty(\Omega)} + 2(\ell-1)\|\phi_4(r)\|_{L^\infty(\Omega)}\Big)\mathbb{E}\big[\|X(r)\|_H^{2\ell}\big]\\ &\quad - \frac{\ell}{2}\mathbb{E}\bigg[\|X(r)\|_H^{2\ell-2}\sum_{i=1}^m\theta_i\|X(r)\|_{V_i}^{\alpha_i}\bigg] + c + c\mathbb{E}\bigg[\sum_{i=1}^m \eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(r) + \sum_{j=3,5}\phi_j^\ell(r)\bigg]. \end{aligned} \tag{3.3}$$

Multiplying (3.3) by $e^{\frac12\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}r - \ell\int_0^r(\|\phi_2(r)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(r)\|_{L^\infty(\Omega)})dr}$ and integrating over $(\tau-t,\tau)$, and by (3.2) we complete the proof. □

**Theorem 3.2.** *If conditions in Lemma 3.1 hold, then we have the following conclusions.*

*(i) $\Phi$ has a weakly compact $\mathfrak{D}$-pullback absorbing set $\mathcal{K}=\{\mathcal{K}(\tau):\tau\in\mathbb{R}\}\in\mathfrak{D}$ in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, given by $\mathcal{K}(\tau)=\{u\in L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H):\mathbb{E}[\|u\|_H^{2\ell}]\leqslant c\boldsymbol{G}^\ell(\tau)\}$, that is, for every $\tau\in\mathbb{R}$ and $\mathcal{D}\in\mathfrak{D}$, there exists $T=T(\tau,\mathcal{D})>0$ such that $\Phi(t,\tau-t)\mathcal{D}(\tau-t)\subseteq\mathcal{K}(\tau)$ for all $t\geqslant T$.*

*(ii) $\Phi$ has a unique mean random attractor $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}\in\mathfrak{D}$ in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ with $\mathcal{A}(\tau)=\bigcap_{r\geqslant0}\overline{\bigcup_{t\geqslant r}\Phi(t,\tau-t)\mathcal{K}(\tau-t)}^w$, where the closure is taken in the weak topology of $L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$, that is, (1) $\mathcal{A}(\tau)$ is a weakly compact subset of $L^{2\ell}(\Omega,\mathcal{F}_\tau,H)$ for every $\tau\in\mathbb{R}$; (2) $\mathcal{A}$ is a $\mathfrak{D}$-pullback weakly attracting set of $\Phi$; (3) $\mathcal{A}$ is the minimal one in $\mathfrak{D}$ satisfying (1)-(2).*

*Proof.* By $\mathbf{G}^\ell(\tau)<\infty$ we know that $\mathcal{K}(\tau)$ is bounded in $L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$. It can be checked that $\mathcal{K}(\tau)$ is *convex*. Note that a convex subset is closed in the weak topology if and only if it is closed in the strong topology, and hence $\mathcal{K}(\tau)$ is weakly compact in $L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$. By Lemma 3.1 we find that $\mathcal{K}$ is a pullback absorbing set for $\Phi$. It remains to prove $\mathcal{K}\in\mathfrak{D}$. By (3.2), we have, as $t\to+\infty$,

$$\begin{aligned}&e^{-\frac12\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}t+\ell\int_{\tau-t}^\tau[\|\phi_2(\varsigma)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(\varsigma)\|_{L^\infty(\Omega)}]d\varsigma}\|\mathcal{K}(\tau-t)\|^{2\ell}_{L^{2\ell}(\Omega,\mathcal{F}_{\tau-t},\mathbb{P};H)}\\ &\leqslant c\int_{-\infty}^{-t} e^{\frac12\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}r+\ell\int_{r+\tau}^\tau[\|\phi_2(\varsigma)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(\varsigma)\|_{L^\infty(\Omega)}]d\varsigma}\\ &\qquad\qquad\times\bigg(1+\mathbb{E}\bigg[\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(r+\tau)+\sum_{j=3,5}\phi_j^\ell(r+\tau)\bigg]\bigg)dr\\ &\leqslant c\int_{-\infty}^{\tau-t} e^{\frac12\ell\aleph_\ell\lambda_{i_1}\theta_{i_1}(r-\tau)+\ell\int_r^\tau[\|\phi_2(\varsigma)\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4(\varsigma)\|_{L^\infty(\Omega)}]d\varsigma}\\ &\qquad\qquad\times\bigg(1+\mathbb{E}\bigg[\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(r)+\sum_{j=3,5}\phi_j^\ell(r)\bigg]\bigg)dr\to0,\end{aligned}$$

which completes the proof of (i). By (i) and [50, Theorem 2.13] we complete the proof of (ii). □

**Remark 3.3.** *(i) The weak attraction of such a mean attractor is defined by a weak neighborhood base of $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$ rather than the weak metric. (ii)If $\eta_i=\phi_j\equiv0$ for $j=2,3,4,5$ and $i=1,2,\ldots,m$, then the solutions are exponentially stable to the zero point in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};H)$, see e.g., Caraballo, Kloeden and Schmalfuß* [11]*. In this case, the attractor reduces to $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}$ with $\mathcal{A}(\tau)=\{0\}$.*

**Remark 3.4.** *If both $\phi_2$ and $\phi_4$ are independent of $t$, then we shall strictly assume $\theta_{i_1}>2\lambda_{i_1}^{-1}\aleph_\ell^{-1}[\|\phi_2\|_{L^\infty(\Omega)}+2(\ell-1)\|\phi_4\|_{L^\infty(\Omega)}]$. In this case, Theorem 3.2 holds true if*

$$\int_{-\infty}^\tau e^{\ell\left(\frac12\aleph_\ell\lambda_{i_1}\theta_{i_1}-\|\phi_2\|_{L^\infty(\Omega)}-2(\ell-1)\|\phi_4\|_{L^\infty(\Omega)}\right)r}\mathbb{E}\bigg[\sum_{i=1}^m\eta_i^{\frac{\ell\alpha_i}{\alpha_i-\beta_i}}(r)+\sum_{j=3,5}\phi_j^\ell(r)\bigg]dr<\infty,\quad\forall\,\tau\in\mathbb{R}.$$

*While, if both $\phi_2$ and $\phi_4$ are dependent of $t$, then $\|\phi_2(t)\|_{L^\infty(\Omega)}$ and $\|\phi_4(t)\|_{L^\infty(\Omega)}$ should behave as small numbers when $t\to+\infty$, and we do not need to assume any additional conditions on $\phi_2$ and $\phi_4$.*

## 4. Preliminaries for invariant measures and evolution systems of measures

4.1. **Notations.** Let $\mathcal{M}(H)$ and $\mathcal{P}(H)$ be the sets of all finite and probability measures defined on the Borel $\sigma$-field $\mathcal{B}(H)$, respectively. If a measure $\eta \in \mathcal{M}(H)$ satisfies $\eta(H) = 1$, then it becomes a probability measure. Denote by $B_b(H)$ ($C_b(H)$, $C_b^u(H)$, $\mathrm{Lip}_b(H)$) the space of all bounded Borel (continuous, uniformly continuous, Lipschitz continuous) real-valued functions on $H$. Then $\mathrm{Lip}_b(H) \subseteq C_b^u(H) \subseteq C_b(H) \subseteq B_b(H)$. The space $C_b(H)$ is equipped with the supremum norm $\|\varphi\|_\infty = \sup_{x\in H} |\varphi(x)|$ for $\varphi \in C_b(H)$. The spaces $C_b(H)$ and $\mathrm{Lip}_b(H)$ are endowed with the norms:

$$\|\varphi\|_\infty := \sup_{x\in H} |\varphi(x)|, \ \varphi \in C_b(H) \ \text{ and } \ \|\varphi\|_{\mathrm{Lip}_b} := \|\varphi\|_\infty + \sup_{x,y\in H, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\|x-y\|_H}, \ \ \varphi \in \mathrm{Lip}_b(H).$$

For $\varphi \in B_b(H)$ and $\eta \in \mathcal{M}(H)$, we set $\langle \varphi, \eta\rangle = \int_H \varphi(x)\eta(dx)$. The Wasserstein metric of $\mathcal{P}(H)$ is

$$d_W^{\mathcal{P}}(\mu,\eta) := \sup_{\varphi\in \mathrm{Lip}_b(H), \|\varphi\|_{\mathrm{Lip}_b}\leqslant 1} \big|\langle\varphi,\mu_1\rangle - \langle\varphi,\mu_2\rangle\big|, \ \ \mu_1,\mu_2 \in \mathcal{P}(H).$$

Next, we take an idea from [17, Theorem 7.1.4] to prove an abstract result on the *pointwise approximation* of any bounded and continuous real-valued functions on a separable Hilbert space.

**Proposition 4.1.** *Let $X$ be a separable Hilbert space. Then every function of $C_b(X)$ can be approximated pointwisely by a function of $Lip_b(X)$, that is, if $\varphi \in C_b(X)$, then there exists $\varphi_n \in Lip_b(X)$ such that* $\sup_{n\in\mathbb{N}} \sup_{x\in X} |\varphi_n(x)| < \infty$ *and $\varphi_n(x) \to \varphi(x)$ as $n\to\infty$ for each $x \in X$.*

*Proof.* Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis of $X$. Without loss of generality, we identify $\mathbb{R}^n = \mathrm{span}\{e_1, e_2 \cdot \cdot\cdot, e_n\}$. Let $P_n : X \to \mathrm{span}\{e_1, e_2 \cdot\cdot\cdot, e_n\}$ be the orthogonal projection given by $P_n x = \sum_{i=1}^n x_i e_i$ for $x = \sum_{i=1}^\infty x_i e_i \in X$, where $x_i = \langle x, e_i\rangle_X$. For any $\varphi \in C_b(X)$, we define $\varphi_n(x) = \int_{\mathbb{R}^n} \varphi(z)\phi_n(P_n x - z)dz$ for $x = \sum_{i=1}^\infty x_i e_i \in X$, where $\phi_n$ is a nonnegative smooth function on $\mathbb{R}^n$ with compact support in the ball $B(0, \frac{1}{n})$ of $\mathbb{R}^n$, such that $\int_{\mathbb{R}^n} \phi_n(z)dz = 1$. Then

$$|\varphi_n(x) - \varphi_n(y)| = \left|\int_{\mathbb{R}^n} \varphi\left(z\right)\left[\phi_n\left(P_n x - z\right) - \phi_n\left(P_n y - z\right)\right]dz\right| \leqslant C(n)\|\varphi\|_\infty \|x-y\|_X, \ \forall\, x,y \in X,$$

where $C(n) > 0$ is a constant independent of $x$ and $y$. Thus $\varphi_n \in \mathrm{Lip}_b(X)$. By $\varphi \in C_b(X)$ and $\int_{\mathbb{R}^n} \phi_n(z)dz = 1$, we find that $\sup_{n\in\mathbb{N}}\sup_{x\in X} \varphi_n(x) < \infty$.

By the continuity of $\varphi$ at the point $x$ (fixed), we find that for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that $|\varphi(z) - \varphi(x)| < \varepsilon$ for any $z \in B(x,\delta)$. Note that for any $z \in B(P_n x, \frac{1}{n})$, we have $|z - x| \leqslant |z - P_n x| + |P_n x - x| \leqslant 1/n + |P_n x - x| \to 0$ as $n \to \infty$, which implies that there exists a $N = N(x,\varepsilon) = N(x, \delta(\varepsilon, x)) \in \mathbb{N}$ (independent of $z$) such that $z \in B(x,\delta)$ for all $n \geqslant N$. This also means that $B(P_n x, \frac{1}{n}) \subseteq B(x,\delta)$ for all $n \geqslant N$. And hence, we have, for all $n \geqslant N$,

$$\begin{aligned} |\varphi_n(x) - \varphi(x)| &= \left|\int_{\mathbb{R}^n} (\varphi(z) - \varphi(x))\phi_n(P_n x - z)dz\right| \\ &\leqslant \int_{B(P_n x, \frac{1}{n})} |\varphi(z) - \varphi(x)|\phi_n(P_n x - z)dz \\ &\leqslant \varepsilon \int_{B(P_n x, \frac{1}{n})} \phi_n(P_n x - z)dz = \varepsilon \int_{\mathbb{R}^n} \phi_n(z)dz = \varepsilon. \end{aligned}$$

This completes the proof. □

From now on, we assume that $\phi_1$, $\phi_2$ and $\phi_4$ are constants, and the operators $A_i$ and $B$ as well as other functions in (C1)-(C4) are independent of $\omega$. Let $X(t,\tau,X_0)$ be the unique solution of (1.1) for $t \geqslant \tau \in \mathbb{R}$ and $X_0 \in H$. For $\varphi \in B_b(H)$, we define an operator $(P_{\tau,t})_{t\geqslant\tau}$ acting on $B_b(H)$ for (1.1) by $(P_{\tau,t}\varphi)(X_0) = \mathbb{E}[\varphi(X(t,\tau,X_0))]$. For $\Lambda \in \mathcal{B}(H)$, we set $P_{\tau,t}(X_0,\Lambda) := (P_{\tau,t}\chi_\Lambda)(X_0) = \mathbb{P}\{\omega \in \Omega : X(t,\tau,X_0) \in \Lambda\}$, where $\chi_\cdot$ is the characteristic function. Then $P_{\tau,t}(X_0,\cdot)$ is the law or transition probability function of

$X(t,\tau,X_0)$. For $\Lambda \in \mathcal{B}(H)$ and $\eta \in \mathcal{M}(H)$, we define an adjoint operator $(Q_{\tau,t})_{t\geqslant\tau}$ acting on $\mathcal{M}(H)$ for $(P_{\tau,t})_{t\geqslant\tau}$ by $Q_{\tau,t}\eta(\Lambda) = \int_H P_{\tau,t}(x,\Lambda)\eta(dx)$.

The theories and applications of invariant measures for autonomous systems have been extensively investigated in the literature, see e.g., [17, 22, 30, 51]. Following the ideas in [18, 19, 28, 29], it is natural to introduce a family of *time-dependent* evolution systems of probability measures for the time *inhomogeneous* transition operator $(P_{\tau,t})_{t\geqslant\tau}$. This notation can be regarded as a generation of invariant probability measures in nonautonomous setting.

A family of measures $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ is called *invariant* or an *evolution system of probability measures* for $(P_{\tau,t})_{t\geqslant\tau}$ if $Q_{\tau,t}\eta_\tau = \eta_t$.

A family of probability measures $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ is called $T$*-periodic* with period $T > 0$ if $\eta_{t+T} = \eta_t$.

A measure $\eta \in \mathcal{P}(H)$ is called *ergodic* for $(P_{0,t})_{t\geqslant 0}$ if for any $\varphi \in L^2(H,\eta)$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T P_{0,t}\varphi dt = \langle \varphi, \eta\rangle \quad \text{in} \quad L^2(H,\eta).$$

A family of measures $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ is called *forward strongly mixing* for $(P_{\tau,t})_{t\geqslant\tau}$ if

$$\lim_{t\to+\infty}[(P_{\tau,t}\varphi)(X_0) - \langle\varphi,\eta_t\rangle] = 0, \quad \forall\, \tau\in\mathbb{R}, X_0\in H,\ \varphi\in C_b(H).$$

A family of measures $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ is called *backward strongly mixing* for $(P_{\tau,t})_{t\geqslant\tau}$ if

$$\lim_{\tau\to-\infty}[P_{t,\tau}\varphi(X_0) - \langle\varphi,\eta_t\rangle] = 0, \quad \forall\, t\in\mathbb{R}, X_0\in H,\ \varphi\in C_b(H).$$

A family of measures $\{\eta_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ is called *global exponentially mixing* for $(P_{\tau,t})_{t\geqslant\tau}$ if for all $t \geqslant \tau$, $X_0 \in H$, $\varphi \in \mathrm{Lip}_b(H)$, there exists a constant $\delta > 0$ independent of $t$, $x$ and $\varphi$ such that

$$|(P_{\tau,t}\varphi)(X_0) - \langle\varphi,\eta_t\rangle| \leqslant C_{\varphi_{Lip}}[\Theta(X_0)e^{\delta(\tau-t)} + e^{-\delta t}\hbar(\tau)],$$

where $C_{\varphi_{Lip}} > 0$ is a constant depending on $\varphi$, $\Theta(\cdot)$ is a nonnegative bounded function on $H$ and $\hbar(\tau)$ is a real-valued function satisfying $\lim_{\tau\to-\infty}\hbar(\tau) = 0$.

4.2. **Feller properties, Markov process and process laws.**

**Lemma 4.2.** *If assumptions in Theorem 1.2 hold, then we have the following properties.*

*(i) The transition operator $(P_{\tau,t})_{t\geqslant\tau}$ is Feller.*

*(ii) $X(t,\tau,X_0)$ is an $H$-valued Markov process for all $t\geqslant\tau$.*

*(iii)The process laws $P_{\tau,t} = P_{\tau,s}P_{s,t}$ and Chapman-Kolmogorov's relation $P^\epsilon_{\tau,t}(X_0,\cdot) = \int_H P^\epsilon_{\tau,s}(X_0,dy)P^\epsilon_{s,t}(y,\cdot)$ hold for any $-\infty < \tau \leqslant s \leqslant t < +\infty$.*

*Proof.* (i) For any $t \geqslant \tau \in \mathbb{R}$ and $\varphi \in C_b(H)$, we show $P_{\tau,t}\varphi \in C_b(H)$, that is, $\mathbb{E}[\varphi(X(t,\tau,X_{n,0}))] \to \mathbb{E}[\varphi(X(t,\tau,X_0))]$ if $X_{n,0} \to X_0$ in $H$ as $n\to\infty$. Since $X_{n,0}\to X_0$ in $H$, by Theorem 1.2 for $X(t,\tau,X_{n,0})$ and $X(t,\tau,X_0)$ we find that there exists a constant $M_1 > 0$ independent of $n$ such that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{r\in[\tau,t]}\|X(r,\tau,X_{n,0})\|_H^2\Big] + \mathbb{E}\Big[\int_\tau^t\sum_{i=1}^m\|X(s,\tau,X_{n,0})\|_{V_i}^{\alpha_i}ds\Big]\\&+\mathbb{E}\Big[\sup_{r\in[\tau,t]}\|X(r,\tau,X_0)\|_H^2\Big] + \mathbb{E}\Big[\int_\tau^t\sum_{i=1}^m\|X(s,\tau,X_0)\|_{V_i}^{\alpha_i}ds\Big] \leqslant M_1.\end{aligned}$$

By Markov's inequality, for every $\epsilon > 0$, there exists a constant $R(\epsilon) > 0$ such that

$$\mathbb{P}\Big(\Big\{\omega\in\Omega : \sup_{r\in[\tau,t]}\|X(r,\tau,X_{n,0})\|_H > R(\epsilon)\Big\}\Big) \leqslant \frac{\epsilon}{2} \text{ and } \mathbb{P}\Big(\Big\{\omega\in\Omega : \sup_{r\in[\tau,t]}\|X(r,\tau,X_0)\|_H > R(\epsilon)\Big\}\Big) \leqslant \frac{\epsilon}{2}. \quad (4.1)$$

Define $\Omega_n^\epsilon = \big\{\omega\in\Omega : \sup_{r\in[\tau,t]}\|X(r,\tau,X_{n,0})\|_H \leqslant R(\epsilon) \text{ and } \sup_{r\in[\tau,t]}\|X(r,\tau,X_0)\|_H \leqslant R(\epsilon)\big\}$. Then by (4.1) we have $\mathbb{P}(\Omega\setminus\Omega_n^\epsilon) < \epsilon$. Define a stopping time

$$\varsigma^n = \inf\Big\{t\geqslant\tau : \Big[\|X(t,\tau,X_{n,0})\|_H \vee \|X(t,\tau,X_0)\|_H \vee \int_\tau^t\sum_{i=1}^m\|X(s,\tau,X_{n,0})\|_{V_i}^{\alpha_i}ds$$

$$\vee \int_\tau^t \sum_{i=1}^m \|X(s,\tau,X_0)\|_{V_i}^{\alpha_i} ds\Big] > R(\epsilon)\Big\}. \tag{4.2}$$

As before, by (1.1), (C2) and the product rule we find

$$\begin{aligned} & e^{\phi_1(\tau - t\wedge\varsigma^n) - \kappa \sum_{i=1}^m \int_\tau^{t\wedge\varsigma^n} \rho_i(X(r,\tau,X_0))dr} \|X(t\wedge\varsigma^n,\tau,X_{n,0}) - X(t\wedge\varsigma^n,\tau,X_0)\|_H^2 \\ & \leqslant \|X_{n,0} - X_0\|_H^2 + M_n(t\wedge\varsigma^n), \end{aligned} \tag{4.3}$$

where $M_n(t\wedge\varsigma^n)$ is given by

$$\begin{aligned} M_n(t\wedge\varsigma^n) := 2\int_\tau^{t\wedge\varsigma^n} & e^{\phi_1(\tau-s) - \kappa\sum_{i=1}^m \int_\tau^s \rho_i(X(r,\tau,X_0))dr} \\ & \times \langle X(s,\tau,X_{n,0}) - X(s,\tau,X_0), \big(B(s,X(s,\tau,X_{n,0})) - B(s,X(s,\tau,X_0))\big)dW(s)\rangle_H. \end{aligned}$$

By (4.2) and Proposition 2.1 we can prove that the expectation of the quadratic variation of $M(t\wedge\varsigma^n)$ is finite. Then, taking the expectation in (4.3), we obtain

$$\begin{aligned} & \mathbb{E}\Big[e^{\phi_1(\tau - t\wedge\varsigma^n) - \kappa \sum_{i=1}^m \int_\tau^{t\wedge\varsigma^n} \rho_i(X(r,\tau,X_0))dr} \\ & \qquad \times \|X(t\wedge\varsigma^n,\tau,X_{n,0}) - X(t\wedge\varsigma^n,\tau,X_0)\|_H^2\Big] \leqslant \|X_{n,0} - X_0\|_H^2. \end{aligned}$$

By (1.2) and (4.2) there exists another constant $C(\tau,T,R(\epsilon))$ to bound $\sum_{i=1}^m \int_\tau^{t\wedge\varsigma^n} \rho_i(X(r,\tau,X_0))dr \leqslant C(\tau,T,R(\epsilon))$, $\mathbb{P}$-a.s.. This implies

$$\mathbb{E}\Big[\|X(t\wedge\varsigma^n,\tau,X_{n,0}) - X(t\wedge\varsigma^n,\tau,X_0)\|_H^2\Big] \leqslant e^{\phi_1 T + C(\tau,T,R(\epsilon))}\|X_{n,0} - X_0\|_H^2. \tag{4.4}$$

Note that $\varsigma^n(\omega) \geqslant t$, for all $\omega \in \Omega_n^\epsilon$, by the definitions of $\varsigma^n$ and $\Omega_n^\epsilon$. Then by (4.4), Chebyshev's inequality and $\mathbb{P}(\Omega\setminus\Omega_n^\epsilon) < \epsilon$, we conclude that for every $\delta > 0$,

$$\begin{aligned} & \mathbb{P}\Big(\big\{\omega\in\Omega : \|X(t,\tau,X_{n,0}) - X(t,\tau,X_0)\|_H \geqslant \delta\big\}\Big) \\ & = \mathbb{P}\Big(\big\{\omega\in\Omega\setminus\Omega_n^\epsilon : \|X(t,\tau,X_{n,0}) - X(t,\tau,X_0)\|_H \geqslant \delta\big\}\Big) \\ & \quad + \mathbb{P}\Big(\big\{\omega\in\Omega_n^\epsilon : \|X(t,\tau,X_{n,0}) - X(t,\tau,X_0)\|_H \geqslant \delta\big\}\Big) \\ & \leqslant \epsilon + \delta^{-2}\mathbb{E}\Big[\|X(t\wedge\varsigma^n,\tau,X_{n,0}) - X(t\wedge\varsigma^n,\tau,X_0)\|_H^2\Big] \\ & \leqslant \epsilon + \delta^{-2} e^{\phi_1 T + C(\tau,T,R(\epsilon))}\|X_{n,0} - X_0\|_H^2. \end{aligned} \tag{4.5}$$

Since $\epsilon$ is arbitrary and $X_{n,0} \to X_0$ in $H$, we infer from (4.5) that $\lim_{n\to\infty} \mathbb{P}\Big(\big\{\omega \in \Omega : \|X(t,\tau,X_{n,0}) - X(t,\tau,X_0)\|_H \geqslant \delta\big\}\Big) = 0$. Then there exists a subsequence $\{n_k\}_{k=1}^\infty \subseteq \{n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} \|X(t,\tau,X_{n_k,0}) - X(t,\tau,X_0)\|_H = 0$, $\mathbb{P}$-a.s.. By a contradiction argument we deduce from the uniqueness of the solutions that the original sequence $X(t,\tau,X_{n,0}) \to X(t,\tau,X_0)$ in $H$ as $n\to\infty$, $\mathbb{P}$-a.s.. This along with the continuity of $\varphi$ shows that $\varphi(X(t,\tau,X_{n,0})) \to \varphi(X(t,\tau,X_0))$ in $\mathbb{R}$ as $n\to\infty$, $\mathbb{P}$-a.s.. Since $\varphi(X(t,\tau,X_{n,0})) \leqslant \|\varphi\|_\infty$ for all $n\in\mathbb{N}$, by Lebesgue's theorem we complete the proof of (i).

(ii) By the argument of proving (i), we can follow [17, Theorem 9.14] to show that $X(t,\tau,X_0)$ is an $H$-valued Markov process for $t \geqslant \tau$.

(iii) By (ii) and [17, Corollaries 9.15], we can similarly prove (iii). □

**Lemma 4.3.** *Let assumptions in Theorem 1.2 hold. If there exist $i_1 \in [1,m]\cap\mathbb{N}$ such that $\alpha_{i_1} \geqslant 2$ and $\theta_{i_1} > \lambda_{i_1}^{-1}\phi_2$, then*

$$\begin{aligned} & \mathbb{E}\left[\|X(t)\|_H^2\right] + \sum_{i=1}^m \theta_i \int_\tau^t e^{-[\lambda_{i_1}\theta_{i_1} - \phi_2](t-s)}\mathbb{E}\left[\|X(s)\|_{V_i}^{\alpha_i}\right] ds \\ & \leqslant e^{-[\lambda_{i_1}\theta_{i_1} - \phi_2](t-\tau)}\|X_0\|_H^2 + \int_\tau^t e^{-[\lambda_{i_1}\theta_{i_1} - \phi_2](t-s)}[\theta_{i_1} C_{\alpha_{i_1}} + \phi_3(s)]ds, \end{aligned}$$

*and*

$$\sum_{i=1}^{m}\theta_i\int_{\tau}^{t}\mathbb{E}\left[\|X(s)\|_{V_i}^{\alpha_i}\right]ds\leqslant\|X_0\|_H^2+\int_{\tau}^{t}[\theta_{i_1}C_{\alpha_{i_1}}+\phi_3(s)]ds.$$

*Proof.* The proof is similar to that in Lemma 3.1, and so omitted. □

## 5. Invariant measures of (1.1): autonomous case

In this subsection we discuss the existence and several important properties including integrability, uniqueness, ergodicity, mixing and stability of invariant probability measures for equation (1.1) in the autonomous case. In this part we further assume that operators and functions in (C1)-(C4) are all independent of $t$. Then we know $P_{\tau,t}=P_{0,t-\tau}$ and $P_{\tau,t}(X_0,\cdot)=P_{0,t-\tau}(X_0,\cdot)$ for any $t\geqslant\tau\geqslant 0$.

### 5.1. **Proof of Theorem 1.4.**

*Proof.* **Proof of Theorem 1.4**. (i) (Existence) By Lemma 4.3, there exist $i_0,i_1\in[1,m]\cap\mathbb{N}$ such that

$$k^{-1}\int_0^k\mathbb{E}\left[\|X(s,0,X_0)\|_{V_{i_0}}^{\alpha_{i_0}}\right]ds\leqslant C_0,\quad k\in\mathbb{N}. \tag{5.1}$$

where $C_0:=\theta_{i_0}^{-1}(\|X_0\|_H^2+\theta_{i_1}C_{\alpha_{i_1}}+\phi_3)$. For $\varepsilon>0$ and $l\in\mathbb{N}$, we define $\mathcal{Y}_l^\varepsilon=\Big\{v\in V_{i_0}:\|v\|_{V_{i_0}}\leqslant(\varepsilon^{-1}C_0 2^{2l})^{1/\alpha_{i_0}}\Big\}$ and $\mathcal{Z}_l^\varepsilon=\{u\in H:\|u-v\|_H\leqslant\frac{1}{2^l}\text{ for some }v\in\mathcal{Y}_l^\varepsilon\}$. By Markov's inequality we get

$$\begin{aligned}\mathbb{P}\Big(\{\omega\in\Omega:X(t,0,X_0)\notin\mathcal{Z}_l^\varepsilon\}\Big)&\leqslant\mathbb{P}\Big(\{\omega\in\Omega:X(t,0,X_0)\notin\mathcal{Y}_l^\varepsilon\}\Big)\\&\quad+\mathbb{P}\Big(\{\omega\in\Omega:X(t,0,X_0)\notin\mathcal{Z}_l^\varepsilon\text{ and }X(t,0,X_0)\in\mathcal{Y}_l^\varepsilon\}\Big)\\&\leqslant\mathbb{P}\bigg(\Big\{\omega\in\Omega:\|X(t,0,X_0)\|_{V_{i_0}}>(\varepsilon^{-1}C_0 2^{2l})^{\frac{1}{\alpha_{i_0}}}\Big\}\bigg)\\&\leqslant\frac{\varepsilon}{C_0 2^{2l}}\mathbb{E}(\|X(t,0,X_0)\|_{V_{i_0}}^{\alpha_{i_0}}).\end{aligned} \tag{5.2}$$

By the compact Sobolev embedding $V_{i_0}\hookrightarrow H$ we know $\mathcal{Z}_\varepsilon:=\bigcap_{l\in\mathbb{N}}\mathcal{Z}_l^\varepsilon$ is compact in $H$. Define a family of probability measures $\{\eta_k\}_{k\in\mathbb{N}}\subseteq\mathcal{P}(H)$ by $\eta_k=k^{-1}\int_0^k P_{0,t}(X_0,\cdot)dt$. By (5.1) and (5.2) we have

$$\eta_k(H\setminus\mathcal{Z}_l^\varepsilon)=k^{-1}\int_0^k\mathbb{P}\{\omega\in\Omega:X(t,0,X_0)\notin\mathcal{Z}_l^\epsilon\}dt\leqslant\frac{\varepsilon}{C_0 2^{2l}k}\int_0^k\mathbb{E}[\|X(t,0,X_0)\|_{V_{i_0}}^{\alpha_{i_0}}]dt\leqslant\frac{\varepsilon}{2^{2l}}. \tag{5.3}$$

It follows from (5.3) that $\eta_k(H\setminus\mathcal{Z}_\varepsilon)\leqslant\sum_{l=1}^\infty\eta_k(H\setminus\mathcal{Z}_l^\varepsilon)\leqslant\sum_{l=1}^\infty\frac{\varepsilon}{2^{2l}}<\varepsilon$, and so $\eta_k(\mathcal{Z}_\varepsilon)>1-\varepsilon$. Then the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is tight on $H$. By the Prokhorov's theorem, there exists a probability measure $\eta\in\mathcal{P}(H)$ such that, up to a subsequence, $\eta_k\to\eta$ weakly as $k\to\infty$. Therefore by Lemma 4.2 and the classical Krylov-Bogolyubov method (see [17]), we know, for any $t\geqslant 0$ and $\varphi\in C_b(H)$,

$$\langle\varphi,\eta\rangle=\lim_{k\to\infty}k^{-1}\int_0^k\bigg(\int_H P_{0,r}(X_0,dy)(P_{0,t}\varphi)(y)\bigg)dr=\lim_{k\to\infty}\langle P_{0,t}\varphi,\eta_k\rangle=\langle\varphi,P_{0,t}^*\eta\rangle.$$

This completes the proof of (i).

(ii) (Regularity) It is sufficient to show that each invariant probability measure $\eta$ of $(P_{0,t})_{t\geqslant 0}$ on $H$ satisfies $\eta(V)=1$. Given $R>0$, we consider the ball $B_R^V:=\{x\in V:\|x\|_V\leqslant R\}$ in $V$. Denoting $\alpha:=\min\limits_{i=1,2,\dots,m}\alpha_i$, by Lemma 4.3 and $\theta_{i_1}>\lambda_{i_1}^{-1}\phi_2$ we find that for every $X_0\in H$, there exists $k_0(X_0)\in\mathbb{N}$ such that for all $k\geqslant k_0(X_0)$,

$$k^{-1}\int_0^k\mathbb{E}\left[\|X(s,0,X_0)\|_V^\alpha\right]ds\leqslant c+ck^{-1}\sum_{i=1}^m\int_0^k\mathbb{E}\left[\|X(s,0,X_0)\|_{V_i}^{\alpha_i}\right]ds\leqslant c, \tag{5.4}$$

where $c>0$ is a constant independent of $k$ and $X_0$. By Markov's inequality and (5.4), for all $k\geqslant k_0(X_0)$,

$$k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X(s,0,X_0)\in B_R^V\Big\}\Big)ds\geqslant 1-cR^{-\alpha}. \tag{5.5}$$

By the invariance of $\eta$ and Fubini's theorem we deduce that for all $k \geqslant k_0(X_0)$,

$$\int_H \left(k^{-1}\int_0^k \mathbb{P}\Big(\Big\{\omega\in\Omega : X(s,0,X_0)\in B_R^V\Big\}\Big)ds\right)\eta(dX_0) = \eta(B_R^V).$$

This along with (5.5) and Fatou's lemma implies

$$\eta(B_R^V) \geqslant \int_H \liminf_{k\to\infty}\left(k^{-1}\int_0^k \mathbb{P}\Big(\Big\{\omega\in\Omega : X(s,0,X_0)\in B_R^V\Big\}\Big)ds\right)\eta(dX_0) \geqslant 1 - cR^{-\alpha}.$$

Letting $R\to\infty$ in the above, we derive $\eta(V)\geqslant 1$, and hence $\eta(V) = 1$. □

## 5.2. **Proof of Theorem 1.5.**

*Proof.* **Proof of Theorem 1.5** We choose an $H$-valued $\mathcal{F}_0$-measurable random variable $X_0$ with law $\nu$ such that $\nu = \eta$. Noting that $(P_{0,t}\varphi)(X_0) = \mathbb{E}[\varphi(X(t,0,X_0))] = \langle \varphi, P_{0,t}(X_0,\cdot)\rangle = \langle P_{0,t}\varphi, \nu\rangle$ for any $\varphi\in C_b(X)$, we have $P_{0,t}(X_0,\cdot) = P^*_{0,t}\nu$. By the invariance of $\eta=\nu$ we have $P_{0,t}(X_0,\cdot) = \eta$, which means that the law of $X(t,0,X_0)$ does not change for all time $t\geqslant 0$. Then by [17] we know $X(t,0,X_0)$ is a stationary solution of (1.1). Then we are able to show (i)-(iii) of Theorem 1.5.

(i) (High-order integrability for $\alpha_{i_1}=2$). By (3.3) with $\alpha_{i_1}=2$ we find

$$\begin{aligned}\mathbb{E}[\|X(t)\|_H^{2\ell}] &+ \ell\Big(\frac{1}{2}\lambda_{i_1}\theta_{i_1} - \phi_2 - 2(\ell-1)\phi_4\Big)\int_0^t \mathbb{E}\big[\|X(r)\|_H^{2\ell}\big]dr \\ &+ \frac{\ell}{2}\int_0^t \mathbb{E}\Big[\|X(r)\|_H^{2\ell-2}\sum_{i=1}^m \theta_i\|X(r)\|_{V_i}^{\alpha_i}\Big]dr \leqslant \mathbb{E}\left[\|X_0\|_H^{2\ell}\right] + ct.\end{aligned}$$

Since $\theta_{i_1} > 2\lambda_{i_1}^{-1}(\phi_2 + 2(\ell-1)\phi_4)$ and $\eta$ is the law of the stationary solution $X(t,0,X_0)$, we obtain $\int_H \|x\|_H^{2\ell}\eta(dx) + \int_H \|x\|_H^{2\ell-2}\sum_{i=1}^m \|x\|_{V_i}^{\alpha_i}\eta(dx) \leqslant c$. This implies (1.6).

(ii) (High-order integrability for $\alpha_{i_1}>2$). By (3.3) we obtain

$$\mathbb{E}[\|X(t)\|_H^{2\ell}] + \frac{\ell}{2}\int_0^t \mathbb{E}\Big[\|X(r)\|_H^{2\ell-2}\sum_{i=1}^m \theta_i\|X(r)\|_{V_i}^{\alpha_i}\Big]dr \leqslant \mathbb{E}\left[\|X_0\|_H^{2\ell}\right] + c\int_0^t \mathbb{E}\big[\|X(r)\|_H^{2\ell}\big]dr + ct. \quad (5.6)$$

By the Sobolev embedding $V_i \hookrightarrow H$ we know $\sqrt{\lambda_{i_1}}\|x\|_H \leqslant \|x\|_{V_i}$ for $i=1,2,\ldots,m$. Since $\eta$ is the law of the stationary solution $X(t,0,X_0)$, by (5.6) and Young's inequality we have

$$\int_H \|x\|_H^{2\ell}\eta(dx) \leqslant c_1 + c_2\int_H \|x\|_H^{2\ell-2}\sum_{i=1}^m \|x\|_{V_i}^{\alpha_i}\eta(dx) \leqslant c_3\int_H \|x\|_H^{2\ell}\eta(dx) + c_4, \quad (5.7)$$

where $c_2, c_3 > 0$, $0 < c_1 \leqslant c_4$. Letting $R>0$ and splitting $\int_H \|x\|_H^{2\ell}\eta(dx)$ into two parts, by (5.7) we deduce

$$\int_H \|x\|_H^{2\ell}\eta(dx) \leqslant R^{2\ell} + cR^{2-\alpha_{i_1}}\int_H \|x\|_H^{2\ell-2}\|x\|_{V_{i_1}}^{\alpha_{i_1}}\eta(dx) \leqslant R^{2\ell} + cR^{2-\alpha_{i_1}}\int_H \|x\|_H^{2\ell}\eta(dx) + cR^{2-\alpha_{i_1}}.$$

Since $\alpha_{i_1}>2$, there exists $R>0$ such that $cR^{2-\alpha_{i_1}} = \frac{1}{2}$, then we obtain the finiteness of $\int_H \|x\|_H^{2\ell}\eta(dx)$ immediately. This along with (5.7) completes the proof of (1.7).

(iii) (Exponential integrability for $\alpha_{i_1}\geqslant 2$) By Itô's formula and (1.3) with $\ell=1$, for any $\epsilon>0$,

$$\begin{aligned}de^{\epsilon\|X(t)\|_H^2} = \epsilon e^{\epsilon\|X(t)\|_H^2}\Big(&2\langle X(t), BdW(t)\rangle_H + 2\epsilon\|B^*X(t)\|_U^2 dt \\ &+ \big(2\,{}_{V^*}\langle A(X(t)), X(t)\rangle_V + \|B\|^2_{\mathcal{L}_2(U,H)}\big)dt\Big).\end{aligned}$$

Taking the expectation in the above equality, and by a stopping time argument as in Lemma 2.2, we infer from (C3) and Remark 1.3 that

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\Big[e^{\epsilon\|X(t)\|_H^2}\Big] &= \epsilon\mathbb{E}\Big[e^{\epsilon\|X(t)\|_H^2}\Big(2\epsilon\|B^*X(t)\|_U^2 + 2\,{}_{V^*}\langle A(X(t)), X(t)\rangle_V + \|B\|^2_{\mathcal{L}_2(U,H)}\Big)\Big] \\ &\leqslant \epsilon\mathbb{E}\Big[e^{\epsilon\|X(t)\|_H^2}\Big(\big(2\epsilon\|B\|^2_{\mathcal{L}_2(U,H)} + \phi_2 - \lambda_{i_1}\theta_{i_1}\big)\|X(t)\|_H^2 - \sum_{i=1}^m \theta_i\|X(t)\|_{V_i}^{\alpha_i} + \phi_3 + \theta_{i_1}C_{\alpha_{i_1}}\Big)\Big].\end{aligned}$$

This implies for all $\epsilon \in \big[0, 2^{-1}\|B\|^{-2}_{\mathcal{L}_2(U,H)}(\lambda_{i_1}\theta_{i_1} - \phi_2)\big]$,

$$\mathbb{E}\big[e^{\epsilon\|X(t)\|_H^2}\big] + \epsilon\mathbb{E}\bigg[\int_0^t e^{\epsilon\|X(s)\|_H^2}\sum_{i=1}^m \theta_i\|X(s)\|_{V_i}^{\alpha_i} ds\bigg] \leqslant \mathbb{E}\big[e^{\epsilon\|X(0)\|_H^2}\big] + \epsilon(\phi_3 + \theta_{i_1}C_{\alpha_{i_1}})\mathbb{E}\bigg[\int_0^t e^{\epsilon\|X(s)\|_H^2} ds\bigg].$$

As before, we have $\int_H e^{\epsilon\|x\|_H^2}\sum_{i=1}^m \theta_i\|x\|_{V_i}^{\alpha_i}\eta(dx) \leqslant (\phi_3 + \theta_{i_1}C_{\alpha_{i_1}})\int_H e^{\epsilon\|x\|_H^2}\eta(dx)$. Then, for all $R > 0$,

$$\int_H e^{\epsilon\|x\|_H^2}\eta(dx) \leqslant e^{\epsilon R^2} + R^{-\alpha_{i_1}}\int_H e^{\epsilon\|x\|_H^2}\|x\|_{V_{i_1}}^{\alpha_{i_1}}\eta(dx) \leqslant e^{\epsilon R^2} + \theta_{i_1}^{-1}(\phi_3 + \theta_{i_1}C_{\alpha_{i_1}})R^{-\alpha_{i_1}}\int_H e^{\epsilon\|x\|_H^2}\eta(dx).$$

Taking $R = (2\theta_{i_1}^{-1}(\phi_3 + \theta_{i_1}C_{\alpha_{i_1}}))^{1/\alpha_{i_1}}$, we then complete the proof. □

**Remark 5.1.** *The finiteness of* $\int_H \|x\|_H^{2\ell}\eta(dx)$ *is unknown if* $\alpha_i \in (1,2)$ *for any* $i \in [1,m] \cap \mathbb{N}$.

### 5.3. **Proof of Theorem 1.6.**

*Proof.* **Proof of Theorem 1.6**. (i) Let $X(t,0,X_0)$ and $X(t,0,Y_0)$ be two solutions to (1.1). Then by Remark 1.3 with $\alpha_{i_1} = 2$, we have $\mathbb{E}\left[\|X(t,0,X_0) - X(t,0,Y_0)\|_H^2\right] \leqslant e^{-(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t}\|X_0 - Y_0\|_H^2$. Let $\eta \in \mathcal{P}(H)$ be an invariant probability measure of $(P_{0,t})_{t\geqslant 0}$. By Cauchy-Schwarz's inequality we deduce, for any $\varphi \in \mathrm{Lip}_b(H)$ and $X_0 \in H$,

$$\begin{aligned}|(P_{0,t}\varphi)(X_0) - \langle\varphi,\eta\rangle|^2 &= \bigg|\mathbb{E}[\varphi(X(t,0,X_0))] - \int_H (P_{0,t}\varphi)(Y_0)\eta(dY_0)\bigg|^2 \\ &= \bigg|\int_H \mathbb{E}[\varphi(X(t,0,X_0)) - \varphi(X(t,0,Y_0))]\eta(dY_0)\bigg|^2 \\ &\leqslant 2\|\varphi\|_{\mathrm{Lip}_b}^2\big(\|X_0\|_H^2 + \int_H \|X_0\|_H^2\eta(dX_0)\big)e^{-(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t},\end{aligned} \tag{5.8}$$

where $\int_H \|X_0\|_H^2\eta(dX_0) < \infty$ due to (i) of Theorem 1.5 with $\ell = 1$. Then (1.8) follows from (5.8).

Given $\varphi \in C_b(H)$, by Proposition 4.1, there exists $\varphi_n \in \mathrm{Lip}_b(H)$ satisfying $\sup_{x\in X}\sup_{n\in\mathbb{N}}\varphi_n(x) < \infty$ such that $\varphi_n(x) \to \varphi(x)$ for any $x \in H$. By (1.8) we have $(P_{0,t}\varphi_n)(X_0) \to \langle\varphi_n,\eta\rangle$ as $t \to +\infty$. Then by Lebesgue's theorem we deduce that $(P_{0,t}\varphi)(X_0) \to \langle\varphi,\eta\rangle$ as $t \to +\infty$, and hence $\eta$ is strongly mixing on $H$ for $(P_{0,t})_{t\geqslant 0}$.

Furthermore, for $\mu \in \mathcal{P}(H)$ satisfying $\int_H \|x\|_H^2\mu(dx) < \infty$, we have

$$\begin{aligned}d_W^{\mathcal{P}}(Q_{0,t}\mu,\eta) &= \sup_{\varphi\in\mathrm{Lip}_b(H),\|\varphi\|_{\mathrm{Lip}_b}\leqslant 1}\bigg|\int_H \mathbb{E}[\varphi(X(t,\tau,X_0))]\mu(dX_0) - \int_H \mathbb{E}[\varphi(X(t,\tau,Y_0))]\eta(dY_0)\bigg| \\ &= \sup_{\varphi\in\mathrm{Lip}_b(H),\|\varphi\|_{\mathrm{Lip}_b}\leqslant 1}\bigg|\int_H\int_H \mathbb{E}[\varphi(X(t,\tau,X_0)) - \varphi(X(t,\tau,Y_0))]\mu(dX_0)\eta(dY_0)\bigg| \\ &\leqslant \sqrt{2}\bigg(\int_H \|X_0\|_H^2\mu(dX_0) + \int_H \|Y_0\|_H^2\eta(dY_0)\bigg)^{1/2} e^{-\frac{1}{2}(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t}.\end{aligned}$$

This gives (1.9).

Let $\widehat{\eta} \in \mathcal{P}(H)$ be another invariant measure for $(P_{0,t})_{t\geqslant 0}$. As before, for any $\varphi \in \mathrm{Lip}_b(H)$,

$$\begin{aligned}|\langle\varphi,\eta\rangle - \langle\varphi,\widehat{\eta}\rangle|^2 &= \bigg|\int_H (P_{0,t}\varphi)(X_0)\eta(dX_0) - \int_H (P_{0,t}\varphi)(Y_0)\widehat{\eta}(dY_0)\bigg|^2 \\ &= \bigg|\int_H\int_H \mathbb{E}[\varphi(X(t,0,X_0)) - \varphi(X(t,0,Y_0))]\eta(dX_0)\widehat{\eta}(dY_0)\bigg|^2 \\ &\leqslant 2\|\varphi\|_{\mathrm{Lip}_b}^2 e^{-(\lambda_{i_1}\vartheta_{i_1} - \phi_1)t}\bigg(\int_H \|X_0\|_H^2\eta(dX_0) + \int_H \|Y_0\|_H^2\widehat{\eta}(dY_0)\bigg) \to 0 \quad \text{as } t \to +\infty.\end{aligned}$$

Then, by Proposition 4.1 and Lebesgue's theorem, we have $\langle\varphi,\eta\rangle = \langle\varphi,\widehat{\eta}\rangle$ for all $\varphi \in C_b(H)$. This further gives the uniqueness of $\eta$. From this and [17], we know that $\eta$ is ergodic on $H$ for $(P_{0,t})_{t\geqslant 0}$.

(ii) Let $Z(t) := X(t,0,X_0) - X(t,0,Y_0)$ be the difference of two solutions to (1.1). By Remark 1.3 with $\alpha_{i_1} = 2$ we find

$$\frac{d}{dt}\|Z(t)\|_H^2 \leqslant \Big[ -\lambda_{i_1}\vartheta_{i_1} + \phi_1 + \kappa \sum_{i=1}^m \rho_i(X(t,0,Y_0))\Big] \|Z(t)\|_H^2.$$

This along with (1.2) for $\varpi = 0$ yields

$$\begin{aligned} \mathbb{E}[\|Z(t)\|_H^2] &\leqslant \|Z(0)\|_H^2 \mathbb{E}\Big[e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1]t+\kappa\int_0^t \sum_{i=1}^m \rho_i(X(r,0,Y_0))dr}\Big] \\ &\leqslant \|Z(0)\|_H^2 \mathbb{E}\Bigg[e^{(-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0)t+\kappa\theta L_0\int_0^t \sum_{i=1}^m \theta_i\|X(s,0,Y_0)\|_{V_i}^{\alpha_i}ds}\Bigg], \end{aligned} \tag{5.9}$$

where $\theta := \max\limits_{i=1,2,\ldots,m}\{\theta_i^{-1}\}$. To estimate the term $e^{\int_0^t \sum_{i=1}^m \theta_i\|X(s,0,Y_0)\|_{V_i}^{\alpha_i}ds}$ in (5.9), we let $\Upsilon(t) := \|X(t,0,Y_0)\|_H^2 + \int_0^t \sum_{i=1}^m \theta_i \|X(s,0,Y_0)\|_{V_i}^{\alpha_i} ds$. Then by (1.3) with $\ell = 1$ for $X(t,0,Y_0)$ we obtain

$$\begin{aligned} d\Upsilon(t) = &\sum_{i=1}^m \theta_i\|X(t,0,Y_0)\|_{V_i}^{\alpha_i}dt + 2\langle X(t,0,Y_0), BdW(t)\rangle_H \\ &+ 2\ {}_{V^*}\langle A(X(t,0,Y_0)), X(t,0,Y_0)\rangle_V dt + \|B\|_{\mathcal{L}_2(U,H)}^2 dt. \end{aligned}$$

Then using Itô's formula to the above equation, we find, for any $\epsilon > 0$,

$$\begin{aligned} de^{\epsilon\Upsilon(t)} = \epsilon e^{\epsilon\Upsilon(t)}\Bigg[&\sum_{i=1}^m \theta_i\|X(t,0,Y_0)\|_{V_i}^{\alpha_i}dt + 2\ {}_{V^*}\langle A(X(t,0,Y_0)), X(t,0,Y_0)\rangle_V dt + \|B\|_{\mathcal{L}_2(U,H)}^2 dt \\ &+ 2\langle X(t,0,Y_0), BdW(t)\rangle_H + 2\epsilon\|B^*(X(t,0,Y_0))\|_U^2 dt\Bigg]. \end{aligned} \tag{5.10}$$

Taking expectation in (5.10), as in Lemma 2.2, we infer from Remark 1.3 with $\alpha_{i_1} = 2$ that

$$\frac{d}{dt}\mathbb{E}\big[e^{\epsilon\Upsilon(t)}\big] \leqslant \epsilon\phi_3\mathbb{E}\big[e^{\epsilon\Upsilon(t)}\big], \quad \forall\ \epsilon \leqslant \frac{\lambda_{i_1}\theta_{i_1} - \phi_2}{2\|B\|_{\mathcal{L}_2(U,H)}^2}.$$

This implies

$$\mathbb{E}\Bigg[e^{\epsilon\int_0^t \sum_{i=1}^m \theta_i\|X(s,0,Y_0)\|_{V_i}^{\alpha_i}ds}\Bigg] \leqslant e^{\epsilon(\|Y_0\|_H^2+\phi_3 t)}, \quad \forall\ \epsilon \leqslant \frac{\lambda_{i_1}\theta_{i_1} - \phi_2}{2\|B\|_{\mathcal{L}_2(U,H)}^2}. \tag{5.11}$$

By the condition on $\theta_{i_1}$, we have $\kappa\theta L_0 \leqslant \frac{\lambda_{i_1}\theta_{i_1}-\phi_2}{2\|B\|_{\mathcal{L}_2(U,H)}^2}$. This along with (5.9) and (5.11) for $\epsilon = \kappa\theta L_0$ yields

$$\mathbb{E}[\|X(t,0,X_0) - X(t,0,Y_0)\|_H^2] \leqslant \|X_0 - Y_0\|_H^2 e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0(1+\theta\phi_3)]t+\kappa\theta L_0\|Y_0\|_H^2}.$$

Let $\eta \in \mathcal{P}(H)$ be an invariant measure of $(P_{0,t})_{t\geqslant 0}$. As before, for any $\varphi \in \mathrm{Lip}_b(H)$ and $X_0 \in H$,

$$\begin{aligned} |P_{0,t}\varphi(X_0) - \langle\varphi,\eta\rangle|^2 &= \left|\int_H \mathbb{E}[\varphi(X(t,0,X_0)) - \varphi(X(t,0,Y_0))]\eta(dY_0)\right|^2 \\ &\leqslant \|\varphi\|_{\mathrm{Lip}_b}^2 e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0(1+\theta\phi_3)]t}\int_H \|X_0 - Y_0\|_H^2 e^{\kappa\theta L_0\|Y_0\|_H^2}\eta(dY_0) \\ &\leqslant 2\|\varphi\|_{\mathrm{Lip}_b}^2 e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0(1+\theta\phi_3)]t} e^{\|X_0\|_H^2}\int_H e^{(1+\kappa\theta L_0)\|Y_0\|_H^2}\eta(dY_0). \end{aligned}$$

Since $1 + \kappa\theta L_0 \leqslant \frac{\lambda_{i_1}\theta_{i_1}-\phi_2}{2\|B\|_{\mathcal{L}_2(U,H)}^2}$, by taking $\epsilon = 1 + \kappa\theta L_0$ in the above inequality and (iii) of Theorem 1.5, we find (1.10). This also implies that $\eta$ is strongly mixing.

Furthermore, for any $\varphi \in \mathrm{Lip}_b(H)$ and $\mu \in \mathcal{P}(H)$ satisfying $\int_H e^{\|X_0\|_H^2}\mu(dX_0) < \infty$, we have

$$\begin{aligned} d_W^{\mathcal{P}}(Q_{0,t}\mu,\eta) &= \sup_{\varphi\in\mathrm{Lip}_b(H),\|\varphi\|_{\mathrm{Lip}_b}\leqslant 1}\left|\int_H\int_H \mathbb{E}[\varphi(X(t,0,X_0)) - \varphi(X(t,0,Y_0))]\mu(dX_0)\eta(dY_0)\right| \\ &\leqslant \sqrt{2}e^{\frac{1}{2}[-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0(1+\theta\phi_3)]t}\left(\int_H e^{\|X_0\|_H^2}\mu(dX_0)\right)^{1/2}\left(\int_H e^{\left(1+\kappa\theta L_0\right)\|Y_0\|_H^2}\eta(dY_0)\right)^{1/2}. \end{aligned}$$

By taking $\epsilon = 1 + \kappa\theta L_0$ in the above inequality and (iii) of Theorem 1.5, we have (1.11).

Let $\widehat{\eta}$ be another invariant measure of $(P_{0,t})_{t\geqslant 0}$, then for any $\varphi \in \mathrm{Lip}_b(H)$, as $t \to +\infty$,

$$|\langle\varphi,\eta\rangle - \langle\varphi,\widehat{\eta}\rangle|^2 = \left|\int_H\int_H \mathbb{E}[\varphi(X(t,0,X_0)) - \varphi(X(t,0,Y_0))]\eta(dX_0)\widehat{\eta}(dY_0)\right|^2$$
$$\leqslant 2\|\varphi\|^2_{\mathrm{Lip}_b} e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1+\kappa L_0(1+\theta\phi_3)]t}\int_H e^{\|X_0\|_H^2}\eta(dX_0)\int_H e^{\left(1+\kappa\theta L_0\right)\|Y_0\|^2_H}\widehat{\eta}(dY_0) \to 0,$$

This proves the uniqueness and ergodicity of $\eta$. □

### 5.4. **Proof of Theorem 1.7.**

*Proof.* **Proof of Theorem 1.7**. (i) By Theorem 1.2 we find, for any $\epsilon \in [0, 1/\sqrt{2}]$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|X^\varepsilon(t,X_0)\|_H^{2\ell}\Big] + \mathbb{E}\Big[\int_0^T \|X^\varepsilon(s,X_0)\|_H^{2\ell-2}\sum_{i=1}^m \|X^\varepsilon(s,X_0)\|^{\alpha_i}_{V_i}ds\Big]$$
$$+\,\mathbb{E}\Big[\sup_{t\in[0,T]}\|X^{\varepsilon_0}(t,X_0)\|_H^{2\ell}\Big] + \mathbb{E}\Big[\int_0^T \|X^{\varepsilon_0}(s,X_0)\|_H^{2\ell-2}\sum_{i=1}^m \|X^{\varepsilon_0}(s,X_0)\|^{\alpha_i}_{V_i}ds\Big] \leqslant M_2,$$

where $M_2$ is a constant independent of $\epsilon$. By Markov's inequality we can prove that for every $\gamma > 0$, there exists a constant $R(\gamma)$ such that for any $\epsilon \in [0, 1/\sqrt{2}]$,

$$\mathbb{P}\Big(\Big\{\omega\in\Omega : \sup_{t\in[0,T]}\|X^\epsilon(t,X_0)\|_H > R(\gamma)\Big\}\Big) \leqslant \frac{\gamma}{2} \text{ and } \mathbb{P}\Big(\Big\{\omega\in\Omega : \sup_{t\in[0,T]}\|X^{\epsilon_0}(t,X_0)\|_H > R(\gamma)\Big\}\Big) \leqslant \frac{\gamma}{2}.$$

Define $\Omega^\gamma_\epsilon = \big\{\omega\in\Omega : \sup_{t\in[0,T]}\|X^\epsilon(t,X_{n,0})\|_H \leqslant R(\gamma) \text{ and } \sup_{t\in[0,T]}\|X^{\epsilon_0}(t,X_0)\|_H \leqslant R(\gamma)\big\}$. Define

$$\varsigma^\epsilon = \inf\Big\{t\geqslant 0 : \Big[\|X^\epsilon(t,X_0)\|_H \vee \|X^{\epsilon_0}(t,X_0)\|_H \vee \int_0^t\sum_{i=1}^m\|X^\epsilon(s,X_0)\|^{\alpha_i}_{V_i}ds \vee \int_0^t\sum_{i=1}^m\|X^{\epsilon_0}(s,X_0)\|^{\alpha_i}_{V_i}ds\Big] > R(\gamma)\Big\}.$$

By (1.1) we have

$$d(X^\epsilon(t) - X^{\epsilon_0}(t)) = A(X^\epsilon(t)) - A(X^{\epsilon_0}(t)) + \big[(\epsilon-\epsilon_0)B(X^\epsilon(t)) + \epsilon_0[B(X^\epsilon(t)) - B(X^{\epsilon_0}(t))]\big]dW(t).$$

As before, by (C2) and $\epsilon_0 \in [0, 1/\sqrt{2}]$ we have

$$\begin{aligned}\|X^\epsilon(t) - X^{\epsilon_0}(t)\|_H^2 &= \int_0^t \Big[2\,{}_{V^*}\langle A(X^\epsilon(s)) - A(X^{\epsilon_0}(s)), X^\epsilon(s) - X^{\epsilon_0}(s)\rangle_V \\ &\quad + \|(\epsilon-\epsilon_0)B(X^\epsilon(s)) + \epsilon_0[B(X^\epsilon(s)) - B(X^{\epsilon_0}(s))]\|^2_{\mathcal{L}_2(U,H)}\Big]ds \\ &\quad + 2\int_0^t \big\langle X^\epsilon(s) - X^{\epsilon_0}(s), \big[(\epsilon-\epsilon_0)B(X^\epsilon(s)) + \epsilon_0[B(X^\epsilon(s)) - B(X^{\epsilon_0}(s))]\big]dW(s)\big\rangle_H \\ &\leqslant \int_0^t \Big[\phi_1 + \kappa\sum_{i=1}^m \rho_i(X^{\epsilon_0}(s))\Big]\|X^\epsilon(s) - X^{\epsilon_0}(s)\|^2_H ds \\ &\quad + 2\int_0^t \big\langle X^\epsilon(s) - X^{\epsilon_0}(s), \big[(\epsilon-\epsilon_0)B(X^\epsilon(s)) + \epsilon_0[B(X^\epsilon(s)) - B(X^{\epsilon_0}(s))]\big]dW(s)\big\rangle_H \\ &\quad + 2|\epsilon-\epsilon_0|^2\int_0^t \|B(X^\epsilon(s))\|^2_{\mathcal{L}_2(U,H)}ds.\end{aligned}$$

By the product rule we deduce

$$\begin{aligned}&e^{-\int_0^t\left(\phi_1+\kappa\sum_{i=1}^m\rho_i(X^{\epsilon_0}(r))\right)dr}\|X^\epsilon(t) - X^{\epsilon_0}(t)\|^2_H \\ &\leqslant 2|\epsilon-\epsilon_0|^2\int_0^t e^{-\int_0^s\left(\phi_1+\kappa\sum_{i=1}^m\rho_i(X^{\epsilon_0}(r))\right)dr}\|B(X^\epsilon(s))\|^2_{\mathcal{L}_2(U,H)}ds + M_\varepsilon(t),\end{aligned} \tag{5.12}$$

where $M_\epsilon(t)$ is a continuous real-valued local martingale given by

$$\begin{aligned}M_\varepsilon(t) := 2\int_0^t & e^{-\int_0^s\left(\phi_1+\kappa\sum_{i=1}^m\rho_i(X^{\epsilon_0}(r))\right)dr} \\ &\times\langle X^\epsilon(s) - X^{\epsilon_0}(s), \big[(\epsilon-\epsilon_0)B(X^\epsilon(s)) + \epsilon_0[B(X^\epsilon(s)) - B(X^{\epsilon_0}(s))]\big]dW(s)\big\rangle_H.\end{aligned}$$

By Proposition 2.1 we can prove that $\mathbb{E}[M_\epsilon(t\wedge\varsigma^\epsilon)]=0$. Replacing $t$ by $t\wedge\varsigma^\epsilon$ and taking the expectation in (5.12), we obtain

$$\begin{aligned}&\mathbb{E}\bigg[e^{-\int_0^{t\wedge\varsigma^\epsilon}\left(\phi_1+\kappa\sum_{i=1}^m\rho_i(X^{\epsilon_0}(r))\right)dr}\|X^\epsilon(t\wedge\varsigma^\epsilon)-X^{\epsilon_0}(t\wedge\varsigma^\epsilon)\|_H^2\bigg]\\&\leqslant 2|\epsilon-\epsilon_0|^2\mathbb{E}\bigg[\int_0^{t\wedge\varsigma^\epsilon}e^{-\int_0^s\left(\phi_1+\kappa\sum_{i=1}^m\rho_i(X^{\epsilon_0}(r))\right)dr}\|B(s,X^\epsilon(s))\|^2_{\mathcal{L}_2(U,H)}ds\bigg]\\&\leqslant 2|\epsilon-\epsilon_0|^2\mathbb{E}\bigg[\int_0^{t\wedge\varsigma^\epsilon}\|B(s,X^\epsilon(s))\|^2_{\mathcal{L}_2(U,H)}ds\bigg]\leqslant C(T,R(\gamma))|\epsilon-\epsilon_0|^2\to 0\text{ as }\epsilon\to\epsilon_0,\end{aligned}\tag{5.13}$$

where $C(T,R(\gamma))>0$ is a constant independent of $\epsilon$. Note that there exists another constant $C(T,R(\gamma))$ independent of $\varepsilon$ such that $\sum_{i=1}^m\int_0^t\rho_i(X^{\epsilon_0}(r))dr\leqslant C(T,R(\gamma))$, $\mathbb{P}$-a.s.. This along with (5.13) implies $\lim_{\varepsilon\to\varepsilon_0}\mathbb{E}\big[\|X^\epsilon(t\wedge\varsigma^\epsilon)-X^{\epsilon_0}(t\wedge\varsigma^\epsilon)\|_H^2\big]=0$. Then by the arguments of (4.5) we know that $X^\epsilon(t,X_0)\to X^{\epsilon_0}(t,X_0)$ in $H$ in probability as $\epsilon\to\epsilon_0$, and thus we complete the proof of (i).

(ii) We prove that the union $\bigcup_{\epsilon\in[0,1/\sqrt{2}]}\dot{\mathcal{P}}^\epsilon(H)$ is tight on $H$, that is, for any $\delta>0$, $\epsilon\in[0,1/\sqrt{2}]$ and $\eta^\epsilon\in\dot{\mathcal{P}}^\epsilon(H)$, there exists a compact set $\mathcal{Z}^\delta$ independent of $\epsilon$ such that $\eta^\epsilon(\mathcal{Z}^\delta)>1-\delta$. By $\epsilon\in[0,1/\sqrt{2}]$, $\theta_{i_1}>\lambda_{i_1}^{-1}\phi_2$ and Lemma 4.3, we find that for every $X_0\in H$, there exists $k_0=k_0(X_0)\in\mathbb{N}$ such that for all $k\geqslant k_0$,

$$k^{-1}\int_0^k\mathbb{E}\left[\|X^\epsilon(s,X_0)\|_{V_{i_0}}^{\alpha_{i_0}}\right]ds\leqslant C_0,\tag{5.14}$$

where $C_0>0$ is a constant independent of $X_0$, $k$ and $\epsilon$. For $l\in\mathbb{N}$, we define $\mathcal{Y}_l^\delta=\big\{v\in V_{i_0}:\|v\|_{V_{i_0}}\leqslant(\delta^{-1}C_02^{2l})^{1/\alpha_{i_0}}\big\}$ and $\mathcal{Z}_l^\delta=\big\{u\in H:\|u-v\|_H\leqslant\frac{1}{2^l}\text{ for some }v\in\mathcal{Y}_l^\delta\big\}$ and $\mathcal{Z}^\delta:=\bigcap_{l=1}^\infty\mathcal{Z}_l^\delta$. By the compactness of $V_{i_0}\hookrightarrow H$ we know that $\mathcal{Z}^\delta$ is compact in $H$. It is sufficient to show $\eta^\epsilon(\mathcal{Z}^\delta)>1-\delta$. For $n\in\mathbb{N}$, we set $\mathcal{X}_n^\delta:=\bigcap_{l=1}^n\mathcal{Z}_l^\delta$. Then $\bigcap_{n=1}^\infty\mathcal{X}_n^\delta=\mathcal{Z}^\delta$ and $\mathcal{X}_{n+1}^\delta\subseteq\mathcal{X}_n^\delta$, and hence $\eta^\epsilon(\mathcal{Z}^\delta)=\eta^\epsilon\big(\bigcap_{n=1}^\infty\mathcal{X}_n^\delta\big)=\lim\limits_{n\to\infty}\eta^\epsilon\big(\mathcal{X}_n^\delta\big)$. Thus, there exists $N=N(\delta)\in\mathbb{N}$ such that $0\leqslant\eta^\epsilon(\mathcal{X}_n^\delta)-\eta^\epsilon(\mathcal{Z}^\delta)\leqslant\delta/3$ for all $n\geqslant N$. By the invariance of $\eta^\epsilon$ and Fubini's theorem,

$$\int_H\bigg(k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\in\mathcal{X}_N^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)=\eta^\epsilon(\mathcal{X}_N^\delta).$$

This along with the Fatou's lemma, Markov's inequality and (5.14) yields

$$\begin{aligned}\eta^\epsilon(\mathcal{X}_N^\delta)&=\liminf_{k\to\infty}\int_H\bigg(k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\in\mathcal{X}_N^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant\int_H\bigg(\liminf_{k\to\infty}k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\in\mathcal{X}_N^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant 1-\int_H\bigg(\limsup_{k\to\infty}k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\notin\mathcal{X}_N^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant 1-\sum_{l=1}^N\int_H\bigg(\limsup_{k\to\infty}k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\notin\mathcal{Z}_l^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant 1-\sum_{l=1}^N\int_H\bigg(\limsup_{k\to\infty}k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:X^\epsilon(s,X_0)\notin\mathcal{Y}_l^\delta\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant 1-\sum_{l=1}^N\int_H\bigg(\limsup_{k\to\infty}k^{-1}\int_0^k\mathbb{P}\Big(\Big\{\omega\in\Omega:\|X^\epsilon(s,X_0)\|_{V_{i_0}}>(\delta^{-1}C_02^{2l})^{\frac{1}{\alpha_{i_0}}}\Big\}\Big)ds\bigg)\eta^\epsilon(dX_0)\\&\geqslant 1-\sum_{l=1}^N\frac{\delta}{C_02^{2l}}\int_H\bigg(\limsup_{k\to\infty}k^{-1}\int_0^k\mathbb{E}[\|X(s,X_0)\|_{V_{i_0}}^{\alpha_{i_0}}]ds\bigg)\eta^\epsilon(dX_0)\geqslant 1-\sum_{l=1}^N\frac{\delta}{2^{2l}}\geqslant 1-\frac{\delta}{3}.\end{aligned}$$

Then we know $\eta^\epsilon(\mathcal{Z}^\delta)\geqslant\eta^\epsilon(\mathcal{X}_N^\delta)-\frac{\delta}{3}>1-\delta$. By (i) and [30, Theorem 6.1] we complete the proof. □

**Remark 5.2.** *Let $\eta_k^\epsilon=k^{-1}\int_0^kP_{0,t}^\varepsilon(X_0,\cdot)dt$, where $P_{0,t}^\epsilon(X_0,\cdot)$ denotes the law of $X^\epsilon(t,0,X_0)$. Then, as in Theorem 1.4, we know $\eta_k^\epsilon$ converges (up to a subsequence) weakly to $\eta^\epsilon$ as $k\to\infty$, which is an invariant* *probability measure of* $(P^{\epsilon}_{0,t})_{t\geqslant 0}$. *Set* $\ddot{\mathcal{P}}^{\epsilon}(H)=\{\eta^{\epsilon}:\eta^{\epsilon}\text{is a weak limt point of }\eta^{\epsilon}_k\text{ as }k\to\infty\}$. *Since we can prove that the estimates of Lemma 4.3 for* $X^{\epsilon}(t,0,X_0)$ *are uniform for* $\epsilon\in[0,1/\sqrt{2}]$, *following the proof of Theorem 1.4, we can prove that for every* $\delta>0$, $\epsilon\in[0,1/\sqrt{2}]$ *and* $\eta^{\epsilon}\in\ddot{\mathcal{P}}^{\epsilon}(H)$, *there exists a compact set* $\mathcal{Z}_\delta\subseteq H$ *(independent of* $\epsilon$*) such that* $\eta^{\epsilon}_k(\mathcal{Z}_\delta)>1-\delta$, *and hence* $\eta^{\epsilon}(\mathcal{Z}_\delta)\geqslant\limsup_{k\to\infty}\eta^{\epsilon}_k(\mathcal{Z}_\delta)\geqslant 1-\delta$. *Then* $\bigcup_{\varepsilon\in[0,1/\sqrt{2}]}\ddot{\mathcal{P}}^{\epsilon}(H)$ *is tight on* $H$.

## 6. Evolution systems of measures of (1.1): nonautonomous case

In this subsection we investigate the existence, uniqueness, forward strongly mixing, backward strongly mixing and global exponential mixing for evolution systems of probability measures of nonautonomous (1.1).

### 6.1. Proof of Theorem 1.8.

**Lemma 6.1.** *Let assumptions in Theorem 1.8 hold. Then, for each* $t\in\mathbb{R}$ *and* $X_0\in H$, *there exists* $\zeta_t\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};H)$, *independent of* $X_0$, *such that* $\lim_{\tau\to-\infty}X(t,\tau,X_0)=\zeta_t$ *in* $L^2(\Omega,\mathcal{F}_t,\mathbb{P};H)$ *and*

$$\mathbb{E}[\|X(t,\tau,X_0)-\zeta_t\|_H^2]\leqslant 4e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\left(\|X_0\|_H^2+\int_{-\infty}^{\tau}e^{[\lambda_{i_1}\theta_{i_1}-\phi_2](s-\tau)}\phi_3(s)ds\right),\quad \tau\leqslant t.$$

*Proof.* Given $h>0$, $t\in\mathbb{R}$ and $X_0\in H$, let $Z(t):=X(t,\tau,X_0)-X(t,\tau-h,X_0)$ for $\tau\leqslant t$. Then, by Remark 1.3 with $\alpha_{i_1}=2$, we obtain $\mathbb{E}(\|Z(t)\|_H^2)\leqslant e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\mathbb{E}(\|Z(\tau)\|_H^2)$. Thus, by Lemma 4.3, we see

$$\begin{aligned}
&\mathbb{E}(\|X(t,\tau,X_0)-X(t,\tau-h,X_0)\|_H^2)\\
&\leqslant 2e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\left(\|X_0\|_H^2+e^{[-\lambda_{i_1}\theta_{i_1}+\phi_2]h}\|X_0\|_H^2+\int_{\tau-h}^{\tau}e^{[-\lambda_{i_1}\theta_{i_1}+\phi_2](\tau-s)}\phi_3(s)ds\right)\\
&=2e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\left(\|X_0\|_H^2+e^{[-\lambda_{i_1}\theta_{i_1}+\phi_2]h}\|X_0\|_H^2\right)\\
&\quad+2e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1]t+[\lambda_{i_1}(\vartheta_{i_1}-\theta_{i_1})+\phi_2-\phi_1]\tau}\int_{\tau-h}^{\tau}e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds.
\end{aligned}\tag{6.1}$$

By the conditions on $\vartheta_{i_1}$, $\theta_{i_1}$ and $\phi_3$ we derive $\mathbb{E}[\|X(t,\tau,X_0)-X(t,\tau-h,X_0)\|_H^2]\to 0$ as $\tau\to-\infty$ and $h\to+\infty$. Then, by the completeness of $L^2(\Omega,\mathcal{F},\mathbb{P};H)$ and a contradiction argument, there exists $\zeta_t(X_0)\in L^2(\Omega,\mathcal{F},\mathbb{P};H)$ such that $\zeta_t(X_0)=\lim_{\tau\to-\infty}X(t,\tau,X_0)$ in $L^2(\Omega,\mathcal{F},\mathbb{P};H)$. Note that, for any $\tau\leqslant t\in\mathbb{R}$ and $X_0,Y_0\in H$, $\mathbb{E}[\|X(t,\tau,X_0)-X(t,\tau,Y_0)\|_H^2]\leqslant e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\|X_0-Y_0\|_H^2\to 0$ as $\tau\to-\infty$, and hence $\zeta_t(X_0)=\zeta_t(Y_0)$. Then $\zeta_t(X_0)$ is independent of $X_0$. Letting $h\to\infty$ in (6.1), we complete the proof. □

*Proof.* **Proof of Theorem 1.8**. By Lemma 6.1 we deduce that, for all $t\in\mathbb{R}$ and $X_0\in H$, there exists $\zeta_t\in L^2(\Omega,\mathcal{F},\mathbb{P};H)$, independent of $X_0$, such that $\lim_{\tau\to-\infty}X(t,\tau,X_0)=\zeta_t$ in $L^2(\Omega,\mathcal{F}_t,\mathbb{P};H)$. Let $\eta_t$ be the law of $\zeta_t$. Then, for all $\varphi\in\mathrm{Lip}_b(H)$, we have

$$\begin{aligned}
|(P_{\tau,t}\varphi)(X_0)-\langle\varphi,\eta_t\rangle|^2&=|\mathbb{E}[\varphi(X(t,\tau,X_0))-\varphi(\zeta_t)]|^2\\
&\leqslant 4\|\varphi\|^2_{\mathrm{Lip}_b}e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\|X_0\|_H^2\\
&\quad+4\|\varphi\|^2_{\mathrm{Lip}_b}e^{[-\lambda_{i_1}\vartheta_{i_1}+\phi_1]t}e^{(\lambda_{i_1}(\vartheta_{i_1}-\theta_{i_1})+\phi_2-\phi_1)\tau}\int_{-\infty}^{\tau}e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds.
\end{aligned}\tag{6.2}$$

This along with Proposition 4.1 implies (1.13) and $\lim_{\tau\to-\infty}(P_{\tau,t}\varphi)(X_0)=\int_H\varphi(y)\eta_t(dy)$ for any $\varphi\in C_b(H)$. Letting $\tau\to-\infty$ in $(P_{\tau,\sigma}\varphi)(X_0)=(P_{\tau,t}(P_{t,\sigma}\varphi))(X_0)$ for any $\sigma\geqslant t\geqslant\tau$, and by the Feller property of $(P_{\tau,t})_{t\geqslant\tau}$ in Lemma 4.2, we find $\int_H\varphi(y)\eta_\sigma(dy)=\int_H P_{t,\sigma}\varphi(y)\eta_t(dy)$. So, $\{\eta_t\}_{t\in\mathbb{R}}$ is an evolution system of probability measures of $(P_{\tau,t})_{t\geqslant\tau}$ on $H$.

Let $\varphi_n(x) := \|x\|_H^2 \wedge n$ for $n \in \mathbb{N}$ and $x \in H$. Then $\varphi_n \in C_b(H)$. By Lemma 4.3, we see

$$\begin{aligned}\langle \varphi_n, \eta_t\rangle &\leqslant |(P_{\tau,t}\varphi_n)(0) - \langle \varphi_n, \eta_t\rangle| + \mathbb{E}\big[\|\phi^\epsilon(t,\tau,0)\|_H^2\big] \\ &\leqslant |(P_{\tau,t}\varphi_n)(0) - \langle \varphi_n, \eta_t\rangle| + \int_{-\infty}^t e^{[\lambda_{i_1}\theta_{i_1}-\phi_2](s-t)}\phi_3(s)ds.\end{aligned}$$

Letting $\tau \to -\infty$ in the above inequality, we find

$$\int_H \big(\|x\|_H^2 \wedge n\big)\eta_t(dx) \leqslant \int_{-\infty}^t e^{[\lambda_{i_1}\theta_{i_1}-\phi_2](s-t)}\phi_3(s)ds. \tag{6.3}$$

Letting $n \to \infty$ in (6.3), by Fatou's lemma we find (1.12). Note that for any $\{\mu_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(H)$ satisfying $\int_H \|x\|_H^2 \mu_t(dx) \leqslant \int_{-\infty}^t e^{[\lambda_{i_1}\theta_{i_1}-\phi_2]s}\phi_3(s)ds$, we have

$$\begin{aligned}d_W^{\mathcal{P}}(Q_{\tau,t}\mu_\tau, \eta_t) &= \sup_{\varphi\in \mathrm{Lip}_b(H), \|\varphi\|_{\mathrm{Lip}_b}\leqslant 1}\left|\int_H\int_H \mathbb{E}[\varphi(X(t,\tau,X_0)) - \varphi(X(t,\tau,Y_0))]\mu_\tau(dX_0)\eta_\tau(dY_0)\right| \\ &\leqslant \Big(\int_H\int_H \mathbb{E}[\|X(t,\tau,X_0) - X(t,\tau,Y_0)\|_H^2]\mu_\tau(dX_0)\eta_\tau(dY_0)\Big)^{1/2} \\ &\leqslant \sqrt{2}e^{\frac12[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\Big(\int_H \|X_0\|_H^2\mu_\tau(dX_0) + \int_H \|Y_0\|_H^2\eta_\tau(dY_0)\Big)^{1/2} \\ &\leqslant 2e^{\frac12[-\lambda_{i_1}\vartheta_{i_1}+\phi_1](t-\tau)}\Big(\int_{-\infty}^\tau e^{[\lambda_{i_1}\theta_{i_1}-\phi_2](s-\tau)}\phi_3(s)ds\Big)^{1/2}.\end{aligned}$$

This proves (1.14). The uniqueness of $\{\eta_t\}_{t\in\mathbb{R}}$ can be proved similarly. □

## 7. Applications to models

In this section we will present two typical examples (do not fall within the previous frameworks in the literature) for our abstract results. From now on, we let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{O}$.

**Lemma 7.1.** *For $p \in [1,\infty)$ and $a, b \in \mathbb{R}$, there exists a constant $\gamma > 0$, depending only on $p$, such that*

$$(|a|^{p-1}a - |b|^{p-1}b)(a-b) \geqslant \gamma|a-b|^{p+1}, \tag{7.1a}$$

$$(a|a|^{p-1} - b|b|^{p-1})(a-b) \geqslant \frac12(|a|^{p-1} + |b|^{p-1})(a-b)^2. \tag{7.1b}$$

*Proof.* We only focus on (7.1b). Rearranging $(a|a|^{p-1} - b|b|^{p-1})(a-b)$, as in [36], to find

$$\begin{aligned}(a|a|^{p-1} - b|b|^{p-1})(a-b) &= (a|a|^{p-1} - b|a|^{p-1} + b|a|^{p-1} - a|b|^{p-1} + a|b|^{p-1} - b|b|^{p-1})(a-b) \\ &= (|a|^{p-1} + |b|^{p-1})(a-b)^2 + (b|a|^{p-1} - a|b|^{p-1})(a-b) \\ &= (|a|^{p-1} + |b|^{p-1})(a-b)^2 + (|a|^{p-1} + |b|^{p-1})ab - a^2|b|^{p-1} - b^2|a|^{p-1} \\ &= \frac12(|a|^{p-1} + |b|^{p-1})(a-b)^2 + \frac12\left(|a|^{p+1} + |b|^{p+1} - |a|^{p-1}b^2 - |b|^{p-1}a^2\right) \\ &\geqslant \frac12(|a|^{p-1} + |b|^{p-1})(a-b)^2 + \frac12\Big(|a|^{p+1} + |b|^{p+1} - \frac{p-1}{p+1}|a|^{p+1} \\ &\qquad - \frac{2}{p+1}b^{p+1} - \frac{p-1}{p+1}|b|^{p+1} - \frac{2}{p+1}a^{p+1}\Big) \\ &\geqslant \frac12(|a|^{p-1} + |b|^{p-1})(a-b)^2.\end{aligned}$$

□

7.1. **Fractional $(s,p)$-Laplacian equations.** Fractional calculus has many applications in modeling complex phenomena arriving from a wide range of fields within finance, engineering, physics, chemistry, biology, and others, see e.g., [2]. In particular, the solutions and their associated dynamical behavior of SPDEs with fractional Laplacian driven by nonlinear white noise have been studied in [51]. In contrast, both well-posedness and dynamics of SPDEs with fractional $(s,p)$-Laplacian driven by nonlinear white noise have not been studied in the literature. However, our abstract frameworks can be used to study the well-posedness and dynamics of a class of fractional $(s,p)$-Laplacian equations driven by nonlinear white noise for any $s\in(0,1)$ and $p\geqslant 2$.

For $s\in(0,1)$ and $p\geqslant 2$, we define the fractional $(s,p)$-Laplacian operator by, for $x\in\mathbb{R}^N$,

$$
\begin{aligned}
-(-\Delta)^s_p u(x) &= -C(N,p,s)\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+ps}}dy\\
&=\frac{C(N,p,s)}{2}\int_{\mathbb{R}^N}\frac{|u(x)-u(x+y)|^{p-2}\big(u(x+y)-u(x)\big)+|u(x)-u(x-y)|^{p-2}\big(u(x-y)-u(x)\big)}{|y|^{N+ps}}dy,
\end{aligned}
$$

where the normalized constant $C(N,p,s)=\frac{s4^s\Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}$ is defined by the Gamma function. If $p=2$, then the fractional $(s,p)$-Laplacian operator reduced to the standard fractional Laplacian operator. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by $W^{s,p}(\mathbb{R}^N)=\Big\{u\in L^p(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy<\infty\Big\}$. The norm of $W^{s,p}(\mathbb{R}^N)$ is defined by $\|u\|_{W^{s,p}(\mathbb{R}^N)}=\Big(\|u(x)\|^p_{L^p(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p}$. The so-called Gagliardo semi-norm of $W^{s,p}(\mathbb{R}^N)$ is defined by $\|u\|_{\dot W^{s,p}(\mathbb{R}^N)}=\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p}$ Then $\|u\|^p_{W^{s,p}(\mathbb{R}^N)}=\|u(x)\|^p_{L^p(\mathbb{R}^N)}+\|u\|^p_{\dot W^{s,p}(\mathbb{R}^N)}$. Due to the non-local nature of the fractional $(s,p)$-Laplacian, we introduce the spaces $L^r(\mathcal{O}):=\big\{u\in L^r(\mathbb{R}^N): u=0 \text{ a.e. on } \mathbb{R}^N\backslash\mathcal{O}\big\}$ and $W^{s,p}(\mathcal{O}):=\big\{u\in W^{s,p}(\mathbb{R}^N): u=0 \text{ a.e. on } \mathbb{R}^N\backslash\mathcal{O}\big\}$ for $s\in(0,1)$, $p\geqslant 2$ and $r\geqslant 1$.

Recall the *fractional* Poincaré inequality [16, Theorem 6.5]:

$$
\|u\|^p_{\dot W^{s,p}(\mathcal{O})}=\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\geqslant c\int_{\mathcal{O}}|u(x)|^p dx,\quad \forall\, u\in W^{s,p}(\mathcal{O}), \tag{7.2}
$$

where $c$ is a positive constant depending only on $p$, $s$, $N$ and $\mathcal{O}$. Then $\|\cdot\|_{\dot W^{s,p}(\mathcal{O})}$ is an equivalent norm of $\|\cdot\|_{W^{s,p}(\mathcal{O})}$ . For convenience we agree $\|\cdot\|_{W^{s,p}(\mathcal{O})}=\|\cdot\|_{\dot W^{s,p}(\mathcal{O})}$.

Given $s\in(0,1)$, $p\geqslant 2$, $\nu>0$ and $\tau\in\mathbb{R}$, we consider the fractional $(s,p)$-Laplacian equation on $\mathcal{O}$:

$$
\begin{cases}
\dfrac{\partial}{\partial t}u(t)+\nu(-\Delta)^s_p u(t)=\displaystyle\sum_{i=2}^m F_i(t,x,u(t)),\ t>\tau,\\
u(t,x)=0,\quad x\in\mathbb{R}^N\setminus\mathcal{O},\ t>\tau,\\
u(\tau,x)=u_0(x),\quad x\in\mathcal{O}.
\end{cases} \tag{7.3}
$$

Here $F_i:\mathbb{R}\times\mathcal{O}\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous nonlinear function satisfying

$$
F_i(t,x,\omega,s)s\leqslant-\frac{3}{2}\theta_i|s|^{\alpha_i}+\psi_i(t,x,\omega), \tag{7.4}
$$

$$
|F_i(t,x,\omega,s)|\leqslant\dot\psi_i(t,x,\omega)|s|^{\alpha_i-1}+\ddot\psi_i(t,x,\omega), \tag{7.5}
$$

$$
(F_i(t,x,\omega,s_1)-F_i(t,x,\omega,s_2))(s_1-s_2)\leqslant-\vartheta_i|s_1-s_2|^{\alpha_i}+\dddot\psi_i(t,x,\omega)|s_1-s_2|^2, \tag{7.6}
$$

where $\theta_i,\vartheta_i>0$, $\alpha_i\geqslant 2$, $\psi_i\in L^1([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};L^1(\mathcal{O}))$, $\dot\psi_i\in L^{\frac{\alpha_i}{\alpha_i-1}}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};L^\infty(\mathcal{O}))$, $\ddot\psi_i\in L^{\frac{\alpha_i}{\alpha_i-1}}([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};L^{\frac{\alpha_i}{\alpha_i-1}}(\mathcal{O}))$ and $\dddot\psi_i\in L^\infty([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};L^\infty(\mathcal{O}))$ are $\mathcal{F}_t$-adapted nonnegative processes.

Let $H:=L^2(\mathcal{O})$, $V_1:=W^{s,p}(\mathcal{O})$, $\alpha_1:=p$, $V_i:=L^{\alpha_i}(\mathcal{O})$, $i=2,\dots,m\in\mathbb{N}$, and $V:=\bigcap_{i=1,2,\dots,m}V_i$. Then we get $V\subseteq H\equiv H^*\subseteq V^*$ and $V_i\subseteq H\equiv H^*\subseteq V_i^*$. Define $A_1=-(-\Delta)^s_{\alpha_1}:V_1\to V_1^*$ by

$$
{}_{V_1^*}\langle A_1(v),\xi\rangle_{V_1}=\frac{-\nu C(N,\alpha_1,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{\alpha_1-2}\big(v(x)-v(y)\big)\big(\xi(x)-\xi(y)\big)}{|x-y|^{N+\alpha_1 s}}dxdy,\quad v,\xi\in V_1.
$$

Due to the non-local structure of the fractional $(s,\alpha_1)$-Laplacian, we extend $F(t,\cdot,\omega,s):\mathcal{O}\to\mathbb{R}$ to $\mathbb{R}^N$ by setting $F(t,x,\omega,s)=0$ for all $x\in\mathbb{R}^N\setminus\mathcal{O}$. Then we define $A_i(t,\omega,\cdot):V_i\to V_i^*$ by ${}_{V_i^*}\langle A_i(t,\omega,v),\xi\rangle_{V_i}=\int_{\mathbb{R}^N}F_i(t,x,\omega,v(x))\xi(x)dx,\ i=2,\dots,m\in\mathbb{N}$. Let $A=\sum_{i=1}^m A_i$.

**Example 7.2.** *Let us consider the following stochastic fractional $(s,p)$-Laplacian equation defined in $V^*$:*

$$\begin{cases} du(t)=A(t,u(t))dt+B(t,u(t))dW, & t>\tau\in\mathbb{R},\\ u(\tau)=u_0\in H, \end{cases}\tag{7.7}$$

*Here $B:\mathbb{R}\times\Omega\times H\to\mathcal{L}_2(U,H)$ satisfies $\|B(t,\omega,v_1)-B(t,\omega,v_2)\|^2_{\mathcal{L}_2(U,H)}\leqslant(1+\|v_2\|_H^2)\|v_1-v_2\|_H^2$ and $\|B(t,\omega,v)\|^2_{\mathcal{L}_2(U,H)}\leqslant\|v\|_H^2+b(t,\omega)$, where $b\in L^1([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};\mathbb{R}^+)$ is a $\mathcal{F}_t$-adapted nonnegative process.*

7.1.1. *Global well-posedness of Example 7.2.* Let us show that the Example 7.2 satisfies the abstract framework in Section 2, and hence it is an example of the abstract SPDE (1.1).

**Proposition 7.3.** *Example 7.2 satisfies conditions in Theorem 1.2. Then for any $\tau\in\mathbb{R}$, $T>0$, $\ell\geqslant1$ and $u_0\in L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};L^2(\mathcal{O}))$, Example 7.2 has a unique solution $\{u(t)\}_{t\in[\tau,\tau+T]}$ in the sense of Definition 1.1 such that $u\in C([\tau,\tau+T],L^2(\Omega,\mathbb{P};L^2(\mathcal{O})))$. In addition, $u$ satisfies Itô's formula and energy equation* (1.3)-(1.4) *and the uniform estimate*

$$\mathbb{E}\Big(\sup_{t\in[\tau,\tau+T]}\|u(t)\|^{2\ell}_{L^2(\mathcal{O})}\Big)+\mathbb{E}\left(\int_\tau^{\tau+T}\|u(s)\|^{2\ell-2}_{L^2(\mathcal{O})}\Big[\|u(s)\|^p_{W^{s,p}(\mathcal{O})}+\sum_{i=2}^m\|u(s)\|^{\alpha_i}_{L^{\alpha_i}(\mathcal{O})}\Big]ds\right)<\infty.$$

*Proof.* By Theorem 1.2, we only need to check that $A$ and $B$ satisfy conditions (C1)-(C4).

*Step 1.* It is easy to check that $A$ satisfies (C1).

*Step 2.* By (7.1a) we have

$$\begin{aligned}{}_{V_1^*}\langle A_1(v_1)-A_1(v_2),v_1-v_2\rangle_{V_1}&=\frac{-\nu C(N,\alpha_1,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big[(v_1(x)-v_2(x))-(v_1(y)-v_2(y))\big]\\ &\times\frac{\big[|v_1(x)-v_1(y)|^{\alpha_1-2}\big(v_1(x)-v_1(y)\big)-|v_2(x)-v_2(y)|^{\alpha_1-2}\big(v_2(x)-v_2(y)\big)\big]}{|x-y|^{N+\alpha_1s}}dxdy\\ &\leqslant-\frac{3}{2}\vartheta_1\|v_1-v_2\|^{\alpha_1}_{V_1},\end{aligned}\tag{7.8}$$

where $\vartheta_1=\frac{\nu\gamma C(N,\alpha_1,s)}{3}$. By (7.6), it follows that ${}_{V_i^*}\langle A_i(t,\omega,v_1)-A_i(t,\omega,v_2),v_1-v_2\rangle_{V_i}\leqslant-\vartheta_i\|v_1-v_2\|^{\alpha_i}_{V_i}+\|\dddot{\psi}_i(t,\omega)\|_{L^\infty(\mathcal{O})}\|v_1-v_2\|^2_H$ for $i=2,\dots,m\in\mathbb{N}$. Then, by (7.8), we deduce

$$\begin{aligned}&2\ {}_{V^*}\langle A(t,\omega,v_1)-A(t,\omega,v_2),v_1-v_2\rangle_V+\|B(t,\omega,v_1)-B(t,\omega,v_2)\|^2_{\mathcal{L}_2(U,H)}\\ &\leqslant-\sum_{i=1}^m2\vartheta_i\|v_1-v_2\|^{\alpha_i}_{V_i}+\left(2\sum_{i=2}^m\|\dddot{\psi}_i(t,\omega)\|_{L^\infty(\mathcal{O})}+(1+\|v_2\|^2_H)\right)\|v_1-v_2\|^2_H.\end{aligned}$$

Then (C2) holds.

*Step 3.* Let $\theta_1=\vartheta_1$. By (7.8) and (7.4), we find $2\ {}_{V^*}\langle A(t,\omega,v),v\rangle_V+\|B(t,\omega,v)\|^2_{\mathcal{L}_2(U,H)}\leqslant-\sum_{i=1}^m2\theta_i\|v\|^{\alpha_i}_{V_i}+\|v\|^2_H+2\sum_{i=2}^m\|\psi_i(t,\omega)\|_{L^1(\mathcal{O})}+b_3(t,\omega)$. Then (C3) holds.

*Step 4.* By the definition we get

$$\begin{aligned}{}_{V_1^*}\langle A_1(v_1),v_2\rangle_{V_1}&=\frac{-\nu C(N,\alpha_1,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_1(x)-v_1(y)|^{\alpha_1-2}\big(v_1(x)-v_1(y)\big)\big(v_2(x)-v_2(y)\big)}{|x-y|^{\frac{\alpha_1(N+\alpha_1s)}{\alpha_1-1}}|x-y|^{\frac{N+\alpha_1s}{\alpha_1}}}dxdy\\ &\leqslant\frac{\nu C(N,\alpha_1,s)}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_1(x)-v_1(y)|^{\alpha_1}}{|x-y|^{N+\alpha_1s}}dxdy\right)^{\frac{\alpha_1-1}{\alpha_1}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_2(x)-v_2(y)|^{\alpha_1}}{|x-y|^{N+\alpha_1s}}dxdy\right)^{\frac{1}{\alpha_1}}\\ &=\frac{\nu C(N,\alpha_1,s)}{2}\|v_1\|^{\alpha_1-1}_{V_1}\|v_2\|_{V_1}.\end{aligned}$$

So $\|A_1(v)\|_{V_1^*} \leqslant 2\theta_1\|v\|_{V_1}^{\alpha_1-1}$. By (7.5) we find $\|A_i(t,\omega,v)\|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}} \leqslant c\|\dot{\psi}_i(t,\omega)\|_{L^\infty(\mathcal{O})}^{\frac{\alpha_i}{\alpha_i-1}}\|v\|_{V_i}^{\alpha_i}+c\|\ddot{\psi}_i(t,\omega)\|_{L^{\frac{\alpha_i}{\alpha_i-1}}(\mathcal{O})}^{\frac{\alpha_i}{\alpha_i-1}}$ for $i=2,\dots,m\in\mathbb{N}$. Therefore $\sum_{i=1}^m \|A_i(t,\omega,v)\|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}} \leqslant c\phi_6(t,\omega)\left(1+\sum_{i=1}^m \|v\|_{V_i}^{\alpha_i}\right)$, where $\phi_6(t,\omega) = c\Big(1+\sum_{i=2}^m\Big(\|\dot{\psi}_i(t,\omega)\|_{L^\infty(\mathcal{O})}^{\frac{\alpha_i}{\alpha_i-1}}+\|\ddot{\psi}_i(t,\omega)\|_{L^{\frac{\alpha_i}{\alpha_i-1}}(\mathcal{O})}^{\frac{\alpha_i}{\alpha_i-1}}\Big)\Big)$.Thus (C4) holds. □

For simplification, we assume that all functions in Example 7.2 are independent of $\omega$, $\psi_i = \dddot{\psi}_i \equiv 0$, $\alpha_2 = 2$, $\vartheta_2 > 1$ and $\theta_2 > 1$. Note that $\|u\|_{V_2} = \|u\|_H$.

7.1.2. *Mean attractors of Example 7.2.* By Proposition 7.3, we define a mean RDS $\Phi$ on $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};L^2(\mathcal{O}))$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ for any $\ell \geqslant 1$.

**Proposition 7.4.** *If $\theta_2 > 2\aleph_\ell^{-1}(2\ell-1)$ and $\int_{-\infty}^{\tau} e^{\ell(\frac{1}{2}\aleph_\ell\theta_2-(2\ell-1))r}b(r)dr < \infty$ for all $\tau\in\mathbb{R}$, then $\Phi$ has a unique mean attractor in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};L^2(\mathcal{O}))$.*

7.1.3. *Invariant probability measures of Example 7.2.* We further assume that all functions in Example 7.2 are independent of $t$. Note that $V_1 \hookrightarrow H$ is compact. By Theorems 1.4- 1.6 we have

**Proposition 7.5.** *Example 7.2 has an invariant probability measure on $L^2(\mathcal{O})$, and every invariant probability measure of Example 7.2 is supported by $W^{s,p}(\mathcal{O})\cap(\bigcap_{i=1,2,\dots,m}L^{\alpha_i}(\mathcal{O}))$.*

**Proposition 7.6.** *Each invariant probability measure $\eta$ of Example 7.2 on $L^2(\mathcal{O})$ satisfies*

$$\int_{L^2(\mathcal{O}}\|x\|_{L^2(\mathcal{O})}^{2\ell}\eta(dx)+\int_{L^2(\mathcal{O}}\|x\|_{L^2(\mathcal{O})}^{2\ell-2}\Big[\|x\|_{W^{s,p}(\mathcal{O})}^p+\sum_{i=2}^m\|x\|_{L^{\alpha_i}(\mathcal{O})}^{\alpha_i}\Big]\eta(dx)<\infty,\quad \ell\geqslant 1,\ p>2.$$

*(ii)If $B(v)\equiv B$, then every invariant probability measure $\eta$ of Example 7.2 on $L^2(\mathcal{O})$ satisfies*

$$\int_{L^2(\mathcal{O})} e^{\epsilon\|x\|_{L^2(\mathcal{O})}^2}\Big[\|x\|_{W^{s,p}(\mathcal{O})}^p+\sum_{i=2}^m\|x\|_{L^{\alpha_i}(\mathcal{O})}^{\alpha_i}\Big]\eta(dx)<\infty,\quad \forall\ \epsilon\in\Big[0,\frac{\theta_2-1}{2\|B\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))}^2}\Big].$$

**Proposition 7.7.** *If $\kappa=0$, then every invariant probability measure of Example 7.2 on $L^2(\mathcal{O})$ must be unique, ergodic, strongly mixing and exponentially mixing.*

Let $B(u(t))$ be replaced by $\epsilon B(u(t))$ in Example 7.2 for $\epsilon\in[0,1/\sqrt{2}]$. Let $\dot{\mathcal{P}}^\epsilon(L^2(\mathcal{O}))$ be the collection of all invariant probability measures of Example 7.2 for $\epsilon\in[0,1/\sqrt{2}]$. By Theorem 1.7 we have

**Proposition 7.8.** *The set $\bigcup_{\varepsilon\in[0,1/\sqrt{2}]}\dot{\mathcal{P}}^\epsilon(L^2(\mathcal{O}))$ is tight. If, in addition, $\eta^{\epsilon_n}\in\dot{\mathcal{P}}^{\epsilon_n}(L^2(\mathcal{O}))$ and $\epsilon_n\to\epsilon_0$ with $\epsilon_0,\epsilon_n\in[0,1/\sqrt{2}]$, then there exists a subsequence and $\eta^{\epsilon_0}\in\dot{\mathcal{P}}^{\epsilon_0}(L^2(\mathcal{O}))$ such that $\eta^{\epsilon_{n_k}}\to\eta^{\epsilon_0}$ weakly.*

7.2. **Convective Brinkman-Forchheimer equations.** The convective Brinkman-Forchheimer (CBF) equation is sometimes referred to as the *tamed* Navier-Stokes equation in the literature, see Kinra and Mohan [24, 36, 37], which describes the motion of incompressible fluid flows in a saturated porous medium. Given an initial time $\tau\in\mathbb{R}$, the CBF equation defined on $\mathcal{O}\subseteq\mathbb{R}^N$ for $N=2,3$ reads

$$\frac{\partial\mathbf{u}}{\partial t}-\mu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\beta|\mathbf{u}|^{r-1}\mathbf{u}+\nabla p=\mathbf{f}(t)\quad\text{and}\quad\nabla\cdot\mathbf{u}(t)=0,\qquad t>\tau,\tag{7.9}$$

with the boundary-initial conditions:

$$\mathbf{u}=0\ \text{ on }\ \partial\mathcal{O}\times(\tau,\infty)\qquad\text{and}\qquad\mathbf{u}(\tau)=\mathbf{u}_0,\tag{7.10}$$

where $\mu$ and $\beta$ are positive constants representing the Brinkman (effective viscosity) and Forchheimer coefficients, respectively. The functions $\mathbf{u}(t,x)\in\mathbb{R}^N$, $p(t,x)\in\mathbb{R}$ and $\mathbf{f}(t,x)\in\mathbb{R}^N$ represent the velocity, pressure and external force, respectively. The numbers $r\geqslant 1$ and $r=3$ are called the dissipative and critical exponents for the global solvability of (7.9)-(7.10). Note that the CBF equation in the critical case has the same scaling as the classical Navier-Stokes equation, see Hajduk and Robinson [21].

Let $\mathcal{V} = \{\mathbf{u} \in C_0^\infty(\mathcal{O};\mathbb{R}^N) : \nabla\cdot\mathbf{u} = 0\}$. Denote by $\mathbb{V}$ and $\mathbb{L}^p$ the closures of $\mathcal{V}$ in the standard Sobolev spaces $H_0^1(\mathcal{O};\mathbb{R}^N)$ and $L^p(\mathcal{O};\mathbb{R}^N)$ for $p \geqslant 1$, respectively. Since the boundary $\partial\mathcal{O}$ is sufficiently smooth, one can characterize these spaces as $\mathbb{V} = \{\mathbf{u} \in H_0^1(\mathcal{O};\mathbb{R}^N) : \nabla\cdot\mathbf{u} = 0\}$ and $\mathbb{L}^p = \{\mathbf{u} \in L^p(\mathcal{O};\mathbb{R}^N) : \nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}\cdot\mathbf{n}|_{\partial\mathcal{O}} = 0\}$ with norms $\|\mathbf{u}\|_{\mathbb{V}} = \|\nabla\mathbf{u}\|_{L^2(\mathcal{O};\mathbb{R}^N)}$ and $\|\mathbf{u}\|_{\mathbb{L}^p} = \|\mathbf{u}\|_{L^p(\mathcal{O};\mathbb{R}^N)}$ respectively, where $\mathbf{n}$ is the outward normal to $\partial\mathcal{O}$.

Let $r \geqslant 1$ be the number in (7.9), and $\mathcal{P} : L^{\frac{r+1}{r}}(\mathcal{O};\mathbb{R}^N) \to \mathbb{L}^{\frac{r+1}{r}}$ be the Helmholtz-Hodge projection. If $r = 1$, then it becomes an orthogonal projection. Define the linear, bilinear and nonlinear operators:

$$\begin{aligned}
&\mathcal{A}(\cdot) : D(\mathcal{A}) := \mathbb{V}\cap H^2(\mathcal{O},\mathbb{R}^N) \to \mathbb{L}^2 \ \text{ by } \ \mathcal{A}(\mathbf{u}) = \mu\mathcal{P}\Delta\mathbf{u},\\
&\mathcal{B}(\cdot,\cdot) : \mathbb{L}^2\times\mathbb{V} \to \mathbb{L}^2 \text{ by } \ \mathcal{B}(\mathbf{u},\mathbf{v}) = -\mathcal{P}\big((\mathbf{u}\cdot\nabla)\mathbf{v}\big),\\
&\mathcal{C}(\cdot) : \mathbb{L}^{r+1} \to \mathbb{L}^{\frac{r+1}{r}} \ \text{ by } \ \mathcal{C}(\mathbf{u}) = -\beta\mathcal{P}\big(|\mathbf{u}|^{r-1}\mathbf{u}\big).
\end{aligned}$$

By the Gelfand tripe $\mathbb{V} \subseteq \mathbb{L}^2 \equiv (\mathbb{L}^2)^* \subseteq \mathbb{V}^*$ we know $\mathcal{A}(\cdot) : \mathbb{V} \to \mathbb{V}^*$ and $\mathcal{B}(\cdot,\cdot) : \mathbb{V}\times\mathbb{V} \to \mathbb{V}^*$ are well-defined. By integration by parts we find ${}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{V}} = -{}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle_{\mathbb{V}}$ and ${}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{v}\rangle_{\mathbb{V}} = 0$ for all $\mathbf{u},\mathbf{v}$. By the Gagliardo-Nirenberg inequality, we have, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}$,

$$|{}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{V}}| \leqslant c\|\mathbf{u}\|_{\mathbb{L}^2}^{\frac12}\|\mathbf{u}\|_{\mathbb{V}}^{\frac12}\|\mathbf{v}\|_{\mathbb{V}}\|\mathbf{w}\|_{\mathbb{L}^2}^{\frac12}\|\mathbf{w}\|_{\mathbb{V}}^{\frac12}. \tag{7.11}$$

For $r > 3$, by $\frac{r-1}{2(r+1)} = \frac{2/(r-1)}{r+1} + \frac{(r-3)/(r-1)}{2}$, $\frac{2}{r-1} + \frac{r-3}{r-1} = 1$ and by the interpolation inequality we find, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}\cap\mathbb{L}^{r+1}$,

$$|\ {}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{V}}| = |{}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle_{\mathbb{V}}| \leqslant \|\mathbf{u}\|_{\mathbb{L}^{r+1}}\|\mathbf{v}\|_{\mathbb{L}^{\frac{2(r+1)}{r-1}}}\|\mathbf{w}\|_{\mathbb{V}} \leqslant \|\mathbf{u}\|_{\mathbb{L}^{r+1}}\|\mathbf{v}\|_{\mathbb{L}^{r+1}}^{\frac{2}{r-1}}\|\mathbf{v}\|_{H}^{\frac{r-3}{r-1}}\|\mathbf{w}\|_{\mathbb{V}}. \tag{7.12}$$

Then $\mathcal{B}(\cdot,\cdot) : (\mathbb{V}\cap\mathbb{L}^{r+1})\times(\mathbb{V}\cap\mathbb{L}^{r+1}) \to \mathbb{V}^* + \mathbb{L}^{\frac{r+1}{r}}$ is also well-defined.

On taking the projection $\mathcal{P}$ onto (7.9), we have

$$\begin{cases}\dfrac{\partial\mathbf{u}}{\partial t} = \mathcal{A}(\mathbf{u}) + \mathcal{B}(\mathbf{u},\mathbf{u}) + \mathcal{C}(\mathbf{u}) + \mathbf{f}(t),\ t > \tau \in \mathbb{R},\\ \mathbf{u}(\tau) = \mathbf{u}_0.\end{cases} \tag{7.13}$$

**Example 7.9.** *Let $H := \mathbb{L}^2$,$\alpha_1 = 2$, $\alpha_2 = r+1$, $V_1 := \mathbb{V}$, $V_2 := \mathbb{L}^{r+1}$, $V := V_1\cap V_2$. Then we get the Gelfand trips: $V \subseteq H \equiv H^* \subseteq V^*$ and $V_i \subseteq H \equiv H^* \subseteq V_i^*$, $\ i = 1,2$. Let $A_1(t,\cdot) = \mathcal{A}(\cdot) + \mathcal{B}(\cdot,\cdot) + \boldsymbol{f}(t)$, $A_2(\cdot) = \mathcal{C}(\cdot)$ and $A(\cdot) = A_1(t,\cdot) + A_2(\cdot)$. Consider the stochastic CBF equation:*

$$\begin{cases}d\boldsymbol{u} = \big(A_1(t,\boldsymbol{u}(t)) + A_2(\boldsymbol{u}(t))\big)dt + B(\boldsymbol{u}(t))dW,\\ \boldsymbol{u}(\tau) = \boldsymbol{u}_0 \in H,\end{cases} \tag{7.14}$$

*where the nonlinear operator $B : \mathbb{V} \to \mathcal{L}_2(U,H)$ satisfies $\|B(\boldsymbol{v})\|_{\mathcal{L}_2(U,H)}^2 \leqslant L_1\|\boldsymbol{v}\|_{\mathbb{V}}^2 + L_2\|\boldsymbol{v}\|_H^2 + L_3$ and*

$$\|B(\boldsymbol{v}_1) - B(\boldsymbol{v}_2)\|_{\mathcal{L}_2(U,H)}^2 \leqslant L_1\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbb{V}}^2 + L_2\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_H^2,$$

*here $L_1 < \mu$, $L_2$ and $L_3$ are positive constants.*

7.2.1. *Global well-posedness of Example 7.9.*

**Lemma 7.10.** *(see* [20, p.53]*) For all $\boldsymbol{v} \in H_0^1(\mathcal{O};\mathbb{R}^N)$, we have, for $\varpi \geqslant 1$,*

$$\|\boldsymbol{v}\|_{L^\varpi(\mathcal{O},\mathbb{R}^N)} \leqslant \left(\frac{\max(2,\varpi(N-1)/N)}{2\sqrt{N}}\right)^\theta \|\boldsymbol{v}\|_{L^2(\mathcal{O};\mathbb{R}^N)}^{1-\theta}\|\boldsymbol{v}\|_{H_0^1(\mathcal{O};\mathbb{R}^N)}^\theta, \quad \theta = \frac{N(\varpi-2)}{2\varpi}.$$

**Proposition 7.11.** *Example 7.9 satisfies conditions in Theorem 1.2. Then for any $T > 0$ and $\boldsymbol{u}_0 \in L^{2\ell}(\Omega,\mathcal{F}_\tau,\mathbb{P};L^2(\mathcal{O}))$, where $\ell$ can be taken from the following three cases:*

*Case 1. $r > 3$, $N = 2,3$ and $\ell \geqslant \frac{2(r-2)}{r-1}$;*

*Case 2. $r > 3$, $N = 2,3$ and $\ell \geqslant 2$;*

*Case 3. $N = r = 3$, $\beta\mu \geqslant 1$ and $\ell \geqslant 1$.*

*Example 7.2 has a unique solution* $\{\boldsymbol{u}(t)\}_{t\in[\tau,\tau+T]}$ *in the sense of Def. 1.1 such that* $\boldsymbol{u} \in C([\tau,\tau+T], L^2(\Omega,\mathbb{P};\mathbb{L}^2))$. *In addition,* $\boldsymbol{u}$ *satisfies Itô's formula and energy equation* (1.3)-(1.4), *and*

$$\mathbb{E}\Big[\sup_{t\in[\tau,\tau+T]}\|\boldsymbol{u}(t)\|_{\mathbb{L}^2}^{2\ell}\Big]+\mathbb{E}\left[\int_\tau^{\tau+T}\|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^{2\ell-2}\Big[\|\boldsymbol{u}(s)\|_{\mathbb{V}}^2+\|\boldsymbol{u}(s)\|_{\mathbb{L}^{r+1}}^{r+1}\Big]ds\right]<\infty.$$

*Proof.* We will check that $A$ and $B$ satisfy all conditions in (C1)-(C4) by the following steps.

**Step 1**. Take a sequence $\mathbf{u}_n\to\mathbf{u}$ in $V=\mathbb{V}\cap\mathbb{L}^{r+1}$. By (7.11) and Hölder's inequality, for any $\mathbf{v}\in V$,

$$\begin{aligned}
&{}_{V^*}\langle A(t,\mathbf{u}_n)-A(t,\mathbf{u}),\mathbf{v}\rangle_V = {}_{\mathbb{V}^*}\langle \mathcal{A}(\mathbf{u}_n)-\mathcal{A}(\mathbf{u}),\mathbf{v}\rangle_{\mathbb{V}}\\
&\quad+ {}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)-\mathcal{B}(\mathbf{u},\mathbf{u}),\mathbf{v}\rangle_{\mathbb{V}} + {}_{\mathbb{L}^{\frac{r+1}{r}}}\langle \mathcal{C}(\mathbf{u}_n)-\mathcal{C}(\mathbf{u}),\mathbf{v}\rangle_{\mathbb{L}^{r+1}}\\
&\leqslant \mu\|\mathbf{v}\|_{\mathbb{V}}\|\mathbf{u}_n-\mathbf{u}\|_{\mathbb{V}} + {}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{u}_n-\mathbf{u},\mathbf{u}_n),\mathbf{v}\rangle_{\mathbb{V}} + {}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{u},\mathbf{u}_n-\mathbf{u}),\mathbf{v}\rangle_{\mathbb{V}}\\
&\quad+\beta 2^{r-2}\|\mathbf{v}\|_{\mathbb{L}^{r+1}}\left(\int_{\mathcal{O}}|\mathbf{u}_n-\mathbf{u}|^{\frac{r+1}{r}}\left(|\mathbf{u}_n|^{\frac{(r+1)(r-1)}{r}}+|\mathbf{u}|^{\frac{(r+1)(r-1)}{r}}\right)dx\right)^{\frac{r}{r+1}}\\
&\leqslant \mu\|\mathbf{v}\|_{\mathbb{V}}\|\mathbf{u}_n-\mathbf{u}\|_{\mathbb{V}}+c(\|\mathbf{u}_n\|_{\mathbb{V}}+\|\mathbf{u}\|_{\mathbb{V}})\|\mathbf{v}\|_{\mathbb{V}}\|\mathbf{u}_n-\mathbf{u}\|_{\mathbb{V}}\\
&\quad+\beta 2^{r-2}\|\mathbf{v}\|_{\mathbb{L}^{r+1}}\left(\|\mathbf{u}_n\|_{\mathbb{L}^{r+1}}^{r-1}+\|\mathbf{u}\|_{\mathbb{L}^{r+1}}^{r-1}\right)\|\mathbf{u}_n-\mathbf{u}\|_{\mathbb{L}^{r+1}}\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}$$

Then $A(t,\cdot)$ is demicontinuous, and hence hemicontinuous from $V$ to $V^*$, and thus satisfies (C1).

**Step 2**. For any $\mathbf{v}\in V=\mathbb{V}\cap\mathbb{L}^{r+1}$, by ${}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{v},\mathbf{v}),\mathbf{v}\rangle_{\mathbb{V}}=0$ we obtain

$$2\ {}_{V^*}\langle A(t,\mathbf{v}),\mathbf{v}\rangle_V+\|B(\mathbf{v})\|_{\mathcal{L}_2(U,H)}^2\leqslant (L_1-\mu)\|v\|_{\mathbb{V}}^2-2\beta\|\mathbf{v}\|_{\mathbb{L}^{r+1}}^{r+1}+L_2\|\mathbf{v}\|_H^2+\frac{1}{\mu}\|\mathbf{f}(t)\|_{\mathbb{V}^*}^2+L_3.$$

Then $A$ and $B$ satisfy condition (C3) with $\theta_1=\frac12(\mu-L_1)>0$, $\theta_2=\beta$, $\phi_2=L_2$ and $\phi_3=\frac1\mu\|\mathbf{f}(t)\|_{\mathbb{V}^*}^2+L_3$.

**Step 3**. We verify conditions (C2) and (C4) together by considering the following cases.

*Case 1*: $N=2,3$ and $r>3$. By (7.1a) we find that for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{L}^{r+1}$,

$${}_{\mathbb{L}^{\frac{r+1}{r}}}\langle\mathcal{C}(\mathbf{v}_1)-\mathcal{C}(\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{L}^{r+1}}\leqslant-\beta\gamma\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{L}^{r+1}}^{r+1}. \tag{7.15}$$

If $N=2$, similar to the 2D Navier-Stokes equation, such an inequality is enough to show that $A$ satisfies (C2) and (C4) for all $r\geqslant1$. The reader is referred to [34, Example 3.3] for more details. If $N=3$, then inequality (7.15) is not sufficient to verify that $A$ satisfies (C2) and (C4) due to the bilinear operator $\mathcal{B}$. In order to overcome the difficulty, we will improve inequality (7.15) by taking an advantage of the dissipative property of $\mathcal{C}(\mathbf{u})$ in order to control $\mathcal{B}$ when we verify the local monotonicity setting of $A$ in the case $N=3$. More specifically, by (7.1b) we find that for all $r\geqslant1$ and $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{L}^{r+1}$,

$${}_{\mathbb{L}^{\frac{r+1}{r}}}\langle\mathcal{C}(\mathbf{v}_1)-\mathcal{C}(\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{L}^{r+1}}\leqslant-\frac{\beta}{2}\left(\big\||\mathbf{v}_1|^{\frac{r-1}{2}}(\mathbf{v}_1-\mathbf{v}_2)\big\|_H^2+\big\||\mathbf{v}_2|^{\frac{r-1}{2}}(\mathbf{v}_1-\mathbf{v}_2)\big\|_H^2\right). \tag{7.16}$$

For any $r>3$, by the bilinear property of $\mathcal{B}$ we find that for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{V}\cap\mathbb{L}^{r+1}$,

$$\begin{aligned}
{}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{v}_1,\mathbf{v}_1)-\mathcal{B}(\mathbf{v}_2,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{V}}&\leqslant|{}_{\mathbb{V}^*}\langle\mathcal{B}(\mathbf{v}_1-\mathbf{v}_2,\mathbf{v}_1-\mathbf{v}_2),\mathbf{v}_2\rangle_{\mathbb{V}}|\\
&\leqslant\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}\|(\mathbf{v}_1-\mathbf{v}_2)\mathbf{v}_2\|_H\\
&\leqslant\frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^2+\frac{1}{2\mu}\|(\mathbf{v}_1-\mathbf{v}_2)\mathbf{v}_2\|_H^2\\
&=\frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^2+\frac{1}{2\mu}\int_{\mathcal{O}}|\mathbf{v}_2|^2|\mathbf{v}_1-\mathbf{v}_2|^{\frac{4}{r-1}}|\mathbf{v}_1-\mathbf{v}_2|^{\frac{2(r-3)}{(r-1)}}dx\\
&\leqslant\frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^2+\frac{1}{2\mu}\big\||\mathbf{v}_2|^{\frac{r-1}{2}}(\mathbf{v}_1-\mathbf{v}_2)\big\|_H^{\frac{4}{r-1}}\big\|\mathbf{v}_1-\mathbf{v}_2\big\|_H^{\frac{2(r-3)}{r-1}}\\
&\leqslant\frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^2+\frac{\beta}{4}\big\||\mathbf{v}_2|^{\frac{r-1}{2}}(\mathbf{v}_1-\mathbf{v}_2)\big\|_H^2\\
&\quad+2^{\frac{7-r}{r-3}}\frac{r-3}{r-1}\mu^{\frac{1-r}{r-3}}(\beta(r-1))^{\frac{2}{3-r}}\big\|\mathbf{v}_1-\mathbf{v}_2\big\|_H^2.
\end{aligned} \tag{7.17}$$

Then by (7.15)-(7.17) and the condition on $B$, we find that for all $r>3$, $N=2,3$ and $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{V}\cap\mathbb{L}^{r+1}$,

$$2\ {}_{V^*}\langle A(t,\mathbf{v}_1)-A(t,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_V+\|B(\mathbf{v}_1)-B(\mathbf{v}_2)\|_{\mathcal{L}_2(U,H)}^2$$

$$
\begin{aligned}
&= 2\ {}_{\mathbb{V}^*}\langle \mathcal{A}(\mathbf{v}_1)-\mathcal{A}(\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{V}} + 2\ {}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{v}_1,\mathbf{v}_1)-\mathcal{B}(\mathbf{v}_2,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{V}}\\
&\quad + 2\ {}_{\mathbb{L}^{\frac{r+1}{r}}}\langle \mathcal{C}(\mathbf{v}_1)-\mathcal{C}(\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{L}^{r+1}} + \|B(\mathbf{v}_1)-B(\mathbf{v}_2)\|^2_{\mathcal{L}_2(U,H)}\\
&\leqslant (L_1-\mu)\|\mathbf{v}_1-\mathbf{v}_2\|^2_{\mathbb{V}} - \beta\gamma\|\mathbf{v}_1-\mathbf{v}_2\|^{r+1}_{\mathbb{L}^{r+1}} + \Big[2^{\frac{4}{r-3}}\frac{r-3}{r-1}\mu^{\frac{1-r}{r-3}}(\beta(r-1))^{\frac{2}{3-r}}+L_2\Big]\big\|\mathbf{v}_1-\mathbf{v}_2\big\|^2_H. \quad (7.18)
\end{aligned}
$$

For any $\mathbf{v},\mathbf{w}\in V$, by (7.12) and Hölder's inequality we find ${}_{\mathbb{L}^{\frac{r+1}{r}}}\langle A_2(\mathbf{v}),\mathbf{w}\rangle_{\mathbb{L}^{r+1}} \leqslant \beta\|\mathbf{v}\|^r_{\mathbb{L}^{r+1}}\|\mathbf{w}\|_{\mathbb{L}^{r+1}}$ and ${}_{\mathbb{V}^*}\langle A_1(t,\mathbf{v}),\mathbf{w}\rangle_{\mathbb{V}} \leqslant \Big(\mu\|\mathbf{v}\|_{\mathbb{V}} + \|\mathbf{v}\|^{\frac{r+1}{r-1}}_{\mathbb{L}^{r+1}}\|\mathbf{v}\|_H^{\frac{r-3}{r-1}} + \|\mathbf{f}(t)\|_{\mathbb{V}^*}\Big)\|\mathbf{w}\|_{\mathbb{V}}$. Then $\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*} \leqslant 4\Big(\mu^2\|\mathbf{v}\|^2_{\mathbb{V}} + \|\mathbf{v}\|^{\frac{2(r+1)}{r-1}}_{\mathbb{L}^{r+1}}\|\mathbf{v}\|_H^{\frac{2(r-3)}{r-1}} + \|\mathbf{f}(t)\|^2_{\mathbb{V}^*}\Big)$ and $\|A_2(\mathbf{v})\|^{\frac{r+1}{r}}_{\mathbb{L}^{\frac{r+1}{r}}} \leqslant \beta^{\frac{r+1}{r}}\|\mathbf{v}\|^{r+1}_{\mathbb{L}^{r+1}}$. Since $\frac{2(r+1)}{r-1} < r+1$ for $r>3$, we find

$$
\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*} + \|A_2(\mathbf{v})\|^{\frac{r+1}{r}}_{\mathbb{L}^{\frac{r+1}{r}}} \leqslant c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})\left(1+\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{v}\|^{r+1}_{\mathbb{L}^{r+1}}\right)\left(1+\|\mathbf{v}\|_H^{\frac{2(r-3)}{r-1}}\right). \quad (7.19)
$$

By (7.18)-(7.19) we find that $A$ and $B$ satisfy (C2) and (C4) with $\vartheta_1=\frac12(\mu-L_1)$, $\vartheta_2=\frac12\beta\gamma$, $\phi_1 = L_2 + 2^{\frac{4}{r-3}}\frac{r-3}{r-1}\mu^{\frac{1-r}{r-3}}(\beta(r-1))^{\frac{2}{3-r}}$, $\phi_6(t,\omega)=c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})$, $\kappa=0$ and $\varpi=\frac{2(r-3)}{r-1}$ when $r>3$ and $N=2,3$.

*Case 2:* $N=2$ and $r\geqslant 1$. By Lemma 7.10 we find an alternative estimate of (7.17), for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{V}$,

$$
\begin{aligned}
{}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{v}_1,\mathbf{v}_1)-\mathcal{B}(\mathbf{v}_2,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{V}} &\leqslant \|\mathbf{v}_1-\mathbf{v}_2\|_{L^4(\mathcal{O},\mathbb{R}^N)}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}\|\mathbf{v}_2\|_{L^4(\mathcal{O},\mathbb{R}^N)}\\
&\leqslant 2^{-\frac12}\|\mathbf{v}_1-\mathbf{v}_2\|_H^{\frac12}\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^{\frac32}\|\mathbf{v}_2\|_H^{\frac12}\|\mathbf{v}_2\|_{\mathbb{V}}^{\frac12}\\
&\leqslant \frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|^2_{\mathbb{V}} + \frac{27}{128\mu^3}\|\mathbf{v}_2\|^2_H\|\mathbf{v}_2\|^2_{\mathbb{V}}\|\mathbf{v}_1-\mathbf{v}_2\|^2_H.
\end{aligned}
$$

This along with (7.15) implies that for all $\mathbf{v}_1,\mathbf{v}_2\in V$,

$$
\begin{aligned}
&2\ {}_{V^*}\langle A(t,\mathbf{v}_1)-A(t,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_V + \|B(\mathbf{v}_1)-B(\mathbf{v}_2)\|^2_{\mathcal{L}_2(U,H)}\\
&\leqslant (L_1-\mu)\|\mathbf{v}_1-\mathbf{v}_2\|^2_{\mathbb{V}} - 2\beta\gamma\|\mathbf{v}_1-\mathbf{v}_2\|^{r+1}_{\mathbb{L}^{r+1}} + \left(L_2+\frac{27}{64\mu^3}\|\mathbf{v}_2\|^2_H\|\mathbf{v}_2\|^2_{\mathbb{V}}\right)\|\mathbf{v}_1-\mathbf{v}_2\|^2_H. \quad (7.20)
\end{aligned}
$$

For $\mathbf{v}\in V$, we have $\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*}\leqslant c\big[\mu^2\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{v}\|^2_H\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*}\big]$, and hence

$$
\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*} + \|A_2(\mathbf{v})\|^{\frac{r+1}{r}}_{\mathbb{L}^{\frac{r+1}{r}}} \leqslant c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})\left(1+\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{v}\|^{r+1}_{\mathbb{L}^{r+1}}\right)\left(1+\|\mathbf{v}\|^2_H\right). \quad (7.21)
$$

By (7.20)-(7.21) we find that $A$ and $B$ satisfy (C2) and (C4) with $\vartheta_1=\frac12(\mu-L_1)$, $\vartheta_2=\beta\gamma$, $\phi_1=L_2$, $\kappa=\frac{27}{64\mu^3}$, $\rho_1=0$, $\rho_2(\cdot)=\|\cdot\|^2_H\|\cdot\|^2_{\mathbb{V}}$, $\varpi=2$ and $\phi_6(t,\omega)=c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})$ when $r>3$ and $N=2,3$.

*Case 3:* $N=r=3$ and $\beta\mu\geqslant 1$. By (7.17) we find ${}_{\mathbb{V}^*}\langle \mathcal{B}(\mathbf{v}_1,\mathbf{v}_1)-\mathcal{B}(\mathbf{v}_2,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{V}} \leqslant \frac{\mu}{2}\|\mathbf{v}_1-\mathbf{v}_2\|^2_{\mathbb{V}} + \frac{1}{2\mu}\|(\mathbf{v}_1-\mathbf{v}_2)\mathbf{v}_2\|^2_H$. By (7.16) we find ${}_{\mathbb{L}^{\frac43}}\langle \mathcal{C}(\mathbf{v}_1)-\mathcal{C}(\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_{\mathbb{L}^4} \leqslant -\frac{\beta}{2}\big\||\mathbf{v}_2|(\mathbf{v}_1-\mathbf{v}_2)\big\|^2_H$. Then

$$
\begin{aligned}
&2\ {}_{V^*}\langle A(t,\mathbf{v}_1)-A(t,\mathbf{v}_2),\mathbf{v}_1-\mathbf{v}_2\rangle_V + \|B(\mathbf{v}_1)-B(\mathbf{v}_2)\|^2_{\mathcal{L}_2(U,H)}\\
&\leqslant (L_1-\mu)\|\mathbf{v}_1-\mathbf{v}_2\|^2_{\mathbb{V}} - 2\beta\gamma\|\mathbf{v}_1-\mathbf{v}_2\|^{r+1}_{\mathbb{L}^{r+1}} + L_2\big\|\mathbf{v}_1-\mathbf{v}_2\big\|^2_H.
\end{aligned}
$$

By (7.12) we have $\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*} \leqslant c\big[\mu^2\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{v}\|^4_{\mathbb{L}^4}+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*}\big]$ and $\|A_2(\mathbf{v})\|^{\frac43}_{\mathbb{L}^{\frac43}} \leqslant \beta^{\frac43}\|\mathbf{v}\|^4_{\mathbb{L}^4}$. Then $\|A_1(t,\mathbf{v})\|^2_{\mathbb{V}^*} + \|A_2(\mathbf{v})\|^{\frac{r+1}{r}}_{\mathbb{L}^{\frac{r+1}{r}}} \leqslant c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})\left(1+\|\mathbf{v}\|^2_{\mathbb{V}}+\|\mathbf{v}\|^4_{\mathbb{L}^4}\right)$. Hence, $A$ and $B$ satisfy conditions (C2) and (C4) with $\vartheta_1=\frac12(\mu-L_1)$, $\vartheta_2=\beta\gamma$, $\phi_1=L_2$, $\kappa=0$, $\varpi=0$ and $\phi_6(t,\omega)=c(1+\|\mathbf{f}(t)\|^2_{\mathbb{V}^*})$ in the case $N=r=3$ and $\beta\mu\geqslant 1$. ☐

**Remark 7.12.** *Similar to the 3D Navier-Stokes equation as considered in* [34, Example 3.3]*, in the case* $N=3$ *and* $r\in[1,3)$*, we are currently unable to show that* $A$ *satisfies (C2) and (C4).*

Next, we discuss the dynamics of Example 7.9. Let $\lambda_1>0$ be the best constant such that the Poincaré inequality $\|\mathbf{u}\|_{\mathbb{V}}\geqslant\sqrt{\lambda_1}\|\mathbf{u}\|_{\mathbb{L}^2}$ holds. Let $\ell$ be the number taken form the three cases in Proposition 7.11.

7.2.2. *Mean attractors of Example 7.9.* By Proposition 7.11 we define a mean RDS $\Phi$ of (1.1) on $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};\mathbb{L}^2)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ for Example 7.9. By Remark 3.4 we have the following result.

**Proposition 7.13.** *If $\mu > L_1 + 4\lambda_1^{-1}\aleph_\ell^{-1}(2\ell-1)L_2$ and $\int_{-\infty}^{\tau} e^{\ell[\frac{1}{4}\aleph_\ell\lambda_1(\mu-L_1)-(2\ell-1)L_2]r}\|\boldsymbol{f}(r)\|_{\mathbb{L}^2}^2dr < \infty$ for all $\tau\in\mathbb{R}$, then $\Phi$ has a unique mean attractor in $L^{2\ell}(\Omega,\mathcal{F},\mathbb{P};\mathbb{L}^2)$.*

7.2.3. *Invariant probability measures of Example 7.9.* Assume that $\mathbf{f}$ is independent of $t$. The following results are direct consequences of Theorems 1.4-1.6.

**Proposition 7.14.** *If $\mu > L_1 + 2\lambda_1^{-1}L_2$, then Example 7.9 has an invariant probability measure on $\mathbb{L}^2$ which is supported by $\mathbb{V}\cap\mathbb{L}^{r+1}$.*

**Proposition 7.15.** *If $\mu > L_1+4\lambda_1^{-1}\aleph_\ell^{-1}(2\ell-1)L_2$, then each invariant probability measure $\eta$ of Example 7.2 on $\mathbb{L}^2$ satisfies $\int_{\mathbb{L}^2}\|x\|_{\mathbb{L}^2}^{2\ell-2}\big[\|x\|_{\mathbb{V}}^2+\|x\|_{\mathbb{L}^{r+1}}^{r+1}\big]\eta(dx)<\infty$, $\ell\geqslant 1$.*

*(ii) If $B(v)\equiv B$ and $\mu > 2\lambda_1^{-1}L_2+L_1$, then every invariant measure $\eta$ of Example 7.9 on $\mathbb{L}^2$ satisfies*

$$\int_{\mathbb{L}^2} e^{\epsilon\|x\|_{\mathbb{L}^2}^2}\Big[\|x\|_{\mathbb{V}}^2+\|x\|_{\mathbb{L}^{r+1}}^{r+1}\Big]\eta(dx)<\infty,\quad \forall\ \epsilon\in\left[0,\frac{\frac{1}{2}\lambda_1(\mu-L_1)-L_2}{2\|B\|_{\mathcal{L}_2(U,\mathbb{L}^2)}^2}\right].$$

**Proposition 7.16.** *Let one of the following conditions hold.*

*(i) $r>3$, $N=2,3$ and $\mu > L_1+2\lambda_1^{-1}\Big[L_2+2^{\frac{4}{r-3}}\frac{r-3}{r-1}\mu^{\frac{1-r}{r-3}}(\beta(r-1))^{\frac{2}{3-r}}\Big]$.*

*(ii) $N=r=3$, $\beta\mu\geqslant 1$ and $\mu > L_1+2\lambda_1^{-1}L_2$.*

*Then every invariant probability measure of Example 7.9 on $\mathbb{L}^2$ must be unique, ergodic, strongly mixing and exponentially mixing.*

Let $B(\mathbf{u}(t))$ be replaced by $\epsilon B(\mathbf{u}(t))$ in Example 7.9 for $\epsilon\in[0,1/\sqrt{2}]$. Let $\dot{\mathcal{P}}^{\epsilon}(\mathbb{L}^2)$ be the collection of all invariant probability measures of Example 7.9 for $\epsilon\in[0,1/\sqrt{2}]$. By Theorem 1.7 we have

**Proposition 7.17.** *If $\mu > 2\lambda_1^{-1}L_2+L_1$, then $\bigcup_{\varepsilon\in[0,1/\sqrt{2}]}\dot{\mathcal{P}}^{\epsilon}(\mathbb{L}^2)$ is tight. If $\eta^{\epsilon_n}\in\dot{\mathcal{P}}^{\epsilon_n}(\mathbb{L}^2)$ and $\epsilon_n\to\epsilon_0$ with $\epsilon_0,\epsilon_n\in[0,1/\sqrt{2}]$, then there exist a subsequence and $\eta^{\epsilon_0}\in\dot{\mathcal{P}}^{\epsilon_0}(\mathbb{L}^2)$ such that $\eta^{\epsilon_{n_k}}\to\eta^{\epsilon_0}$ weakly.*

7.2.4. *Evolution systems of probability measures of Example 7.9.* By Theorems 1.8 we have

**Proposition 7.18.** *Let one of the conditions in Proposition 7.16 hold. If $\int_{-\infty}^{\tau} e^{[\frac{1}{2}\lambda_1(\mu-L_1)-L_2]s}\|\boldsymbol{f}(s)\|_{\mathbb{L}^2}^2ds<\infty$ for any $\tau\in\mathbb{R}$, then Example 7.9 has a unique exponentially mixing evolution system of probability measures on $\mathbb{L}^2$ such that $\int_{L^2(\mathcal{O})}\|x\|_{L^2(\mathcal{O})}^2\eta_t(dx)\leqslant L_3(\frac{1}{2}\lambda_1(\mu-L_1)-L_2)^{-1}e^{[\frac{1}{2}\lambda_1(\mu-L_1)-L_2]t}+\mu^{-1}\int_{-\infty}^{t} e^{[\frac{1}{2}\lambda_1(\mu-L_1)-L_2]s}\|\boldsymbol{f}(s)\|_{\mathbb{L}^2}^2ds$.*

## Appendix: Examples of $F_i$ and $B$ in (7.7) and (7.14)

**Example of $F_i$ in** (7.7). For $i=2,\dots,m\in\mathbb{N}$, we consider $F_i:\mathbb{R}\times\mathbb{R}^N\times\Omega\times\mathbb{R}\to\mathbb{R}$ given by $F_i(t,x,\omega,s)=-\frac{3}{2}\theta_i|s|^{\alpha_i-2}s+\frac{h_i(t,x,\omega)}{1+s^2}$ where $\theta_i>0$, $\alpha_i>2$ are constants, $h_i\in L^\infty([\tau,\tau+T]\times\Omega,dt\times\mathbb{P};L^\infty(\mathcal{O}))$ is a $\mathcal{F}_t$-adapted nonnegative processes. Then by (7.1a) we know $F_i$ satisfies (7.4)-(7.6) with $\vartheta_i=\frac{3}{2}\gamma\theta_i$, $\dot{\psi}_i=\theta_i$ $\psi_i=\ddot{\psi}_i=\dddot{\psi}_i=h_i$.

**Example of $B$ in** (7.7) Let $b\in L^1([\tau,\tau+T]\times\Omega;dt\times\mathbb{P},\mathbb{R}^+)$ be a $\mathcal{F}_t$-adapted process. For $v\in H$, we define a mapping $B_0:\mathbb{R}\times\Omega\times H\to H$ by $B_0(t,\omega,v)=\frac{1}{2}(v\sin v+b(t,\omega))$. Then we find that $\|B_0(t,\omega,v_1)-B_0(t,\omega,v_2)\|_H^2\leqslant(1+\|v_2\|_H^2)\|v_1-v_2\|_H^2$ and $\|B_0(t,\omega,v)\|_H^2\leqslant\|v\|_H^2+b(t,\omega)$. Let $e_0\in H$ with $\|e_0\|_H=1$, and consider a special space $U$ by $U:=\mathrm{span}\{e_0\}$. Given $v\in H$, define $B:\mathbb{R}\times\Omega\times H\to\mathcal{L}_2(U,H)$ by $B(t,\omega,v)u=\langle u,e_0\rangle_U B_0(t,\omega,v)$, $u\in U$. Then $\|B(t,\omega,v)\|_{\mathcal{L}_2(U,H)}=\|B(t,\omega,v)e_0\|_H^2=\|B_0(t,\omega,v)\|_H^2\leqslant\|v\|_H^2+b(t,\omega)$ and $\|B(t,\omega,v_1)-B(t,\omega,v_2)\|_{\mathcal{L}_2(U,H)}^2=\|B_0(t,\omega,v_1)-B_0(t,\omega,v_2)\|_H^2\leqslant(1+\|v_2\|_H^2)\|v_1-v_2\|_H^2$. Then we find that $B$ satisfies all conditions in Example (7.2).

**Example of $B$ in** (7.14) Define a mapping $B_0 : V \to H$ by $B_0(\mathbf{v}) = \mathbf{v} + \sin(\mathbf{v}) + \varepsilon\mathcal{B}(\mathbf{v}, \mathbf{h})$ with $\mathbf{h} \in D(\mathcal{A})$. Then for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \in V$, we have $\|B_0(\mathbf{v})\|_H^2 \leqslant c + c\|\mathbf{v}\|_H^2 + \varepsilon c\|\mathcal{A}(\mathbf{h})\|_H^2\|\mathbf{v}\|_{\mathbb{V}}^2$ and

$$\|B_0(\mathbf{v}_1) - B_0(\mathbf{v}_2)\|_H^2 \leqslant c\|\mathbf{v}_1 - \mathbf{v}_2\|_H^2 + \varepsilon c\|\mathcal{A}(\mathbf{h})\|_H^2\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{V}}^2.$$

Let $\mathbf{e}_0 \in H$ with $\|\mathbf{e}_0\|_H = 1$, and $U := \text{span}\{\mathbf{e}_0\}$. Given $\mathbf{v} \in V$, define $B : V \to \mathcal{L}_2(U, H)$ by $B(\mathbf{v})\mathbf{u} = \langle \mathbf{u}, \mathbf{e}_0\rangle_U B_0(\mathbf{v})$, $\mathbf{u} \in U$. Then $\|B(\mathbf{v})\|_{\mathcal{L}_2(U,H)} = \|B_0(\mathbf{v})\|_H$ and $\|B(\mathbf{v}_1) - B(\mathbf{v}_2)\|_{\mathcal{L}_2(U,H)} = \|B_0(\mathbf{v}_1) - B_0(\mathbf{v}_2)\|_H$. By taking $\varepsilon$ small enough, we find that $B$ satisfies the conditions in Example (7.9).

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