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DYNAMICS AND LARGE DEVIATIONS FOR FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LÉVY NOISE*

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Abstract. This paper is mainly concerned with a kind of fractional stochastic evolution equations driven by Lévy noise in a bounded domain. We first state the well-posedness of the problem via iterative approximations and energy estimates. Then, the existence and uniqueness of weak pullback mean random attractors for the equations are established by defining a mean random dynamical system. Next, we prove the existence of invariant measures when the problem is autonomous by means of the fact that $H^{\gamma}(\mathcal{O})$ is compactly embedded in $L^2(\mathcal{O})$ with $\gamma \in (0,1)$. Moreover, the uniqueness of this invariant measure is presented, which ensures the ergodicity of the problem. Finally, a large deviation principle result for solutions of stochastic PDEs perturbed by small Lévy noise and Brownian motion is obtained by a variational formula for positive functionals of a Poisson random measure and Brownian motion. Additionally, the results are illustrated by the fractional stochastic Chafee–Infante equations.

Key words. Fractional Laplacian operator, Lévy noise, Brownian motion, weak mean random attractors, invariant measures, ergodicity, large deviation principle

MSC codes. 35R11, 35Q30, 65F08, 60H15, 65F10

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1. Introduction.

In this paper, we consider the following fractional stochastic PDEs driven by Lévy noise and Brownian motion:

$$
\begin{cases}
du(t) + (-\Delta)^{\gamma} u(t)dt + f(u(t))dt = g(t,u(t))dW(t) & \\
+\displaystyle\int_E h(u(t-),\xi)\tilde{N}(dt,d\xi), & \text{in } \mathcal{O}\times(\tau,\infty),\\
u(t,x) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty),\\
u(\tau,x) = u_0(x), & \text{in } \mathcal{O},
\end{cases} \tag{1.1}
$$

where $\mathcal{O} \subset \mathbb{R}^d$ $(d > 1)$ is a bounded domain with smooth boundary, $\tau \in \mathbb{R}$, the operator $(-\Delta)^{\gamma}$ with $\gamma \in (0,1)$ is the so-called fractional Laplacian, $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree with positive leading coefficient, and the functions $g(t,u)$ and $h(u,\xi)$ satisfy some conditions which will be specified later. We consider problem (1.1) with respect to a given stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}, W, N)$ and a Hilbert

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space U, where W is a two-sided U-valued cylindrical Wiener process and N is a Poisson measure induced by a stationary $\mathcal{F}_t$-Poisson point process on $(\tau,T]\times E$ with a σ-finite intensity measure $L_{T-\tau}\otimes\lambda$, $L_{T-\tau}$ is the Lebesgue measure on $(\tau,T]$ and λ is a σ-finite measure on a measurable space E, and $\tilde{N}(dt,d\xi):=N(dt,d\xi)-\lambda(d\xi)dt$ is the compensated Poisson random measure. Assume W and $\tilde{N}$ are independent.

Stochastic PDEs arise in many different fields since stochastic perturbations originated from many natural sources cannot be ignored in a realistic modeling. In recent decades, stochastic PDEs driven by Brownian motion have been extensively studied theoretically [31, 53], concerning well-posedness, existence of stationary solutions, stochastic attractors, and invariant measures. However, the fact that forcing terms may be treated stochastically does not mean that details of the stochastic treatment are arbitrary [37]. In fact, it turns out that a process not only may be Gaussian but also can exhibit skew, fat tails, and other properties usually associated with more exotic types of stochastic phenomena, such as non-Gaussian Lévy noise. For example, they have been used to develop models for neuronal activity that account for synaptic impulses occurring randomly, both in time and at different locations of a spatially extended neuron. Other applications arise in chemical reaction-diffusion systems and stochastic turbulence models [22, 33, 56].

The fractional Laplacian operator, which is written as $(-\Delta)^\gamma$ with $\gamma\in(0,1)$, has multiple equivalent characterizations [40, 42]. In the present paper, we will mainly adopt the nonlocal one (see (2.1)). Although the eigenfunctions of $(-\Delta)^\gamma$ are not smooth in the sense that they are just Hölder continuous up to the boundary of $\mathcal{O}$ but not Lipschitz continuous, it is possible to construct a continuous operator A which involves $(-\Delta)^\gamma$ (see (2.12)). By means of the fact that $W^{\gamma,2}(\mathcal{O})$ is compactly embedded in $L^2(\mathcal{O})$ and the Hilbert–Schmidt theorem, we can find the eigenfunctions e_j ($j\in\mathbb{N}$) of A which form an orthonormal basis of $L^2(\mathcal{O})$ with corresponding eigenvalues $0<\lambda_1\le\lambda_2\le\cdots\to\infty$ ($Ae_j=\lambda_j e_j$). Moreover, the domain of A^r is denoted by $D(A^r)$ which is equipped with the norm $\|u\|_{D(A^r)}=\|A^r u\|_{L^2(\mathcal{O})}$ for $u\in D(A^r)$. Notice that $\{e_j/\lambda_j^r\}$ is a complete orthonormal system of $D(A^r)$. By the Riesz representation theorem, $D(A^{-r})$ is the dual space of $D(A^r)$. In this way, we know that $D(A^r)$ is continuously embedded into $L^p(\mathcal{O})$ as long as r is large enough [46].

Nonlocal or memory effects are ubiquitous in physics and engineering [3, 44, 45]. Therefore, evolutionary equations with fractional Laplacian operator can be used to model these nonlocal effects (see [1, 12, 14, 20, 23, 49, 50, 51] and the references therein). Particularly, the solutions and their dynamics of fractional PDEs have been extensively studied by a great many researchers; see [22, 24, 32, 41, 46] and the references therein.

Consequently, it is meaningful to study the dynamics of problem (1.1). To be precise, the first goal of this paper is to analyze the well-posedness of (1.1) in $L^2(\Omega;\mathbf{D}([\tau,T];L^2(\mathbb{R}^d)))\cap L^2(\Omega;L^2(\tau,T;W^{\gamma,2}(\mathbb{R}^d)))\cap L^{p+1}(\Omega;L^{p+1}(\tau,T;L^{p+1}(\mathbb{R}^d)))$, the existence and uniqueness of weak pullback mean random attractors for the mean random dynamical systems generated by the solution operators. The second goal is to prove the existence of invariant measures and ergodicity to problem (1.1) in the autonomous case. This result holds true since $W^{\gamma,2}(\mathcal{O})$ is compactly embedded in $L^2(\mathcal{O})$, where $\mathcal{O}\subset\mathbb{R}^d$ is a bounded domain.

The third goal, which is also the main novelty of this paper, is to establish a large deviation principle to fractional stochastic PDEs (1.1) with Lévy noise by a variational representation obtained in [9] and weak convergence approach. The large deviation principle is an active and important topic in probability and statistics. Large deviation properties of stochastic PDEs driven by infinite dimensional Brownian motions and Poisson random measure have been studied in [7, 8, 9]. However, as far as the authors are aware, there are no results about the large deviation principle for fractional stochastic PDEs, and our work will fill this gap. To this end, we follow some ideas introduced by [9] which can be properly adapted to our problem. This is mainly due to the fact that the eigenfunctions of the fractional Laplacian operator $(-\Delta)^\gamma$ share properties similar to the ones of the classical Laplacian operator $-\Delta$. By carrying out a careful analysis, we need to impose some assumptions on γ, d, and p (namely, $p+1 \in (2, \frac{2d}{d-2\gamma}])$ such that $W^{\gamma,2}(\mathbb{R}^d)$ is continuously embedded in $L^{p+1}(\mathbb{R}^d)$, which allows us to accomplish the proposed study.

The paper is organized as follows. In section 2, we review the definition of fractional Laplacian operator, impose the conditions on the nonlinear terms, and introduce the concept of a large deviation principle. Then, the well-posedness of problem (1.1) is established in section 3 by an iterative method. Section 4 is devoted to the existence and uniqueness of weak pullback mean random attractors. In section 5, we study the existence of invariant measures and ergodicity to problem (1.1) when it is autonomous. In section 6, a general large deviation result to (1.1) is proved by a variational formula for positive functionals of a Poisson random measure and Brownian motion. An illustrative example concerning the Chafee–Infante model is exhibited in section 7 and an appendix with the proofs of some results concludes our paper in section 8.

2. Preliminaries.

In this section, we will introduce some basic definitions and properties of the fractional Laplacian operator, impose proper assumptions on nonlinear terms in (1.1), and recall the general criteria for a large deviation principle.

2.1. Fractional setting.

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^d$. Then the integral fractional Laplacian operator $(-\Delta)^\gamma$ with $0<\gamma<1$ is defined, for $u \in \mathcal{S}$, by

$$(-\Delta)^\gamma u(x) = -\frac{1}{2} C(d,\gamma) \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\gamma}} dy, \qquad x \in \mathbb{R}^d, \tag{2.1}$$

where $C(d,\gamma)$ is a positive constant given by

$$C(d,\gamma) = \frac{\gamma 4^\gamma \Gamma(\frac{d+2\gamma}{2})}{\pi^{\frac{d}{2}} \Gamma(1-\gamma)}. \tag{2.2}$$

The reader is referred to [36] for more details on the integral fractional operators. Moreover, for any real $0<\gamma<1$, the fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^d) := H^\gamma(\mathbb{R}^d)$ is defined by

$$H^\gamma(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\gamma}} dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^\gamma(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |u(x)|^2 dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\gamma}} dxdy \right)^{\frac{1}{2}}.$$

We denote the Gagliardo seminorm of $H^\gamma(\mathbb{R}^d)$ as $\|\cdot\|_{\dot{H}^\gamma(\mathbb{R}^d)}$, i.e.,

$$\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\gamma}} dxdy, \quad u \in H^\gamma(\mathbb{R}^d).$$

Then, for all $u\in H^\gamma(\mathbb{R}^d)$, we have $\|u\|^2_{H^\gamma(\mathbb{R}^d)}=\|u\|^2+\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^d)}$. Note that $H^\gamma(\mathbb{R}^d)$ is a Hilbert space with inner product

$$(u,v)_{H^\gamma(\mathbb{R}^d)}=\int_{\mathbb{R}^d}u(x)v(x)dx+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\gamma}}dxdy,\quad \forall u,v\in H^\gamma(\mathbb{R}^d).$$

By [36], we infer that for every fixed $\gamma\in(0,1)$ and $u\in H^\gamma(\mathbb{R}^d)$, the norm $\|u\|_{H^\gamma(\mathbb{R}^d)}$ is equivalent to $(\|u\|^2_{L^2(\mathbb{R}^d)}+\|(-\Delta)^{\frac{\gamma}{2}}u\|^2_{L^2(\mathbb{R}^d)})^{\frac12}$. More precisely, we have

$$\|u\|^2_{H^\gamma(\mathbb{R}^d)}=\|u\|^2_{L^2(\mathbb{R}^d)}+\frac{2}{C(d,\gamma)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2_{L^2(\mathbb{R}^d)},\quad \forall u\in H^\gamma(\mathbb{R}^d).$$

Since the fractional Laplacian operator $(-\Delta)^\gamma$ defined above is a nonlocal one, we here interpret the homogeneous Dirichlet boundary as $u=0$ on $\mathbb{R}^d\backslash\mathcal{O}$ instead of $u=0$ only on $\partial\mathcal{O}$. Such an interpretation is consistent with the nonlocal nature of the integral fractional Laplacian and has been used in many publications. Based on this interpretation, we recall $\mathbb{H}$ and $\mathbb{V}$ are two Hilbert spaces given by $\mathbb{H}=\{u\in L^2(\mathbb{R}^d):u=0 \text{ a.e. in } \mathbb{R}^d\backslash\mathcal{O}\}$ and $\mathbb{V}=\{u\in H^\gamma(\mathbb{R}^d):u=0 \text{ a.e. in } \mathbb{R}^d\backslash\mathcal{O}\}$, respectively. Then we have $\mathbb{V}\hookrightarrow\mathbb{H}=\mathbb{H}^*\hookrightarrow\mathbb{V}^*$, where $\mathbb{H}^*$ is identified with $\mathbb{H}$ by the Riesz representation theorem, and $\mathbb{V}^*$ and $\mathbb{H}^*$ are the dual spaces of $\mathbb{V}$ and $\mathbb{H}$, respectively.

We conclude this subsection by introducing some notation. For $p\ge1$, we denote by $L^p(\mathbb{R}^d)$ the usual L^p-space over $\mathbb{R}^d$ with the standard norm $\|\cdot\|_p$. The norms and inner products of $\mathbb{H}$ and $\mathbb{V}$ are denoted by $|\cdot|$ and $\|\cdot\|$, $(\cdot,\cdot)$ and $((\cdot,\cdot))$, respectively. Moreover, the norm of $\mathbb{V}^*$ is denoted by $\|\cdot\|_*$. For simplicity of notation, when no confusion may arise, we will use the unified notation $<\cdot,\cdot>$ to denote the dual relations between different spaces. For a Polish space E, denote by $C([0,T];E)$ and $\mathbf{D}([0,T];E)$ the spaces of continuous functions and right continuous functions with left limits from $[0,T]$ to E, respectively, endowed with the uniform topology both, if not specified.

2.2. Stochastic setting and assumptions. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all $\mathbb{P}$-null sets. The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L^2(\Omega;\mathbb{H})$, is a Banach space equipped with the norm $\|u(\cdot)\|^2_{L^2(\Omega;\mathbb{H})}=\mathbb{E}|u(\cdot,\omega)|^2$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u=\int_\Omega u(\omega)d\mathbb{P}$. Furthermore, let H and U be two separable Hilbert spaces, and let $\mathcal{L}_2(U;H)$ denote the space of Hilbert–Schmidt operators from a separable Hilbert space U to H with norm $\|\cdot\|_{\mathcal{L}_2(U;H)}$ (see [17] for more details).

Throughout this paper, we impose the following conditions on f, g, and h.

- **Assumptions on nonlinear term f.** Suppose the nonlinear term $f:\mathbb{R}\to\mathbb{R}$ has the following form:

$$f(u)=\sum_{k=1}^{2p}f_{2p-k}u^{k-1},\qquad f_0>0,\quad p\in\mathbb{N}. \tag{2.3}$$

In fact, no significant changes in the proofs of the results presented here are required if we consider, more generally, a continuously differentiable function f on $\mathbb{R}$ satisfying

$(f_1)\,|f(u)|\le l_1(1+|u|^p)$, (f_2) $u\cdot f(u)\ge -l_2+l_3|u|^{p+1}$, (f_3) $f'(u)\ge -l_4$, for some $l_j>0$, $j=1,2,3,4$.

For convenience, we fix a positive number δ, and set

$$F(u) = f(u) - \delta u, \qquad \forall u \in \mathbb{R}. \tag{2.4}$$

By (2.4) and conditions (f_1)–(f_3), after simple calculations, we find that
(F_1) $|F(u)| \le k_1(1+|u|^p)$, (F_2) $u \cdot F(u) \ge -k_2 + k_3|u|^{p+1}$, (F_3) $F'(u) \ge -k_4 := -l_4 - \delta$,
for some $k_j > 0$, $j = 1,2,3,4$.

- **Assumptions on nonlinear term g.** Suppose $g : [\tau,\infty) \times \mathbb{H} \to \mathcal{L}_2(U;\mathbb{H})$ is locally Lipschitz continuous and grows linearly in its second argument uniformly for $t \in [\tau,\infty)$, that is:
  - (g_1) For every $r > 0$, there exists a positive constant $L_g(r)$ depending on r such that for all $t \in [\tau,\infty)$, $u_1, u_2 \in \mathbb{H}$ with $|u_1| \le r$ and $|u_2| \le r$,
  $$\|g(t,u_1) - g(t,u_2)\|^2_{\mathcal{L}_2(U;\mathbb{H})} \le L_g(r)|u_1 - u_2|^2; \tag{2.5}$$
  - (g_2) There exists a positive constant C_g, such that for all $t \in [\tau,\infty)$ and $u \in \mathbb{H}$,
  $$\|g(t,u)\|^2_{\mathcal{L}_2(U;\mathbb{H})} \le C_g(1+|u|^2); \tag{2.6}$$
  - (g_3) For every fixed $u \in \mathbb{H}$, $g(\cdot,u) : [\tau,\infty) \to \mathcal{L}_2(U;\mathbb{H})$ is progressively measurable.
- **Assumptions on nonlinear term h.** Suppose $h : \mathbb{H} \times E \to \mathbb{H}$ is locally Lipschitz continuous and grows linearly in its first argument uniformly for $\xi \in E'$, where $E' \subset E$ satisfying $\lambda(E') < \infty$, precisely:
  - (h_1) For every $r > 0$, there exists a positive constant $L_h(r)$ depending on r, such that, for all $u_1, u_2 \in \mathbb{H}$ with $|u_1| \le r$ and $|u_2| \le r$,
  $$\int_{E'} |h(u_1,\xi) - h(u_2,\xi)|^2 \lambda(d\xi) \le L_h(r)|u_1 - u_2|^2; \tag{2.7}$$
  - (h_2) There exists a positive constant C_h such that, for all $u \in \mathbb{H}$,
  $$\int_{E'} |h(u,\xi)|^2 \lambda(d\xi) \le C_h(1+|u|^2); \tag{2.8}$$
  - (h_3) $h : \mathbb{H} \times E \to \mathbb{H}$ is a measurable mapping.

In light of (2.4), problem (1.1) can be put into the form when the boundary condition is replaced by $u = 0$ on $\mathbb{R}^d \backslash \mathcal{O}$:

$$\begin{aligned} &du(t) + (-\Delta)^\gamma u(t)dt + F(u(t))dt + \delta u(t)dt \\ &\quad = g(t,u(t))dW(t) + \int_E h(u(t-),\xi)\tilde{N}(dt,d\xi), \quad x \in \mathcal{O},\ t > \tau, \end{aligned} \tag{2.9}$$

with boundary and initial conditions,

$$u(t,x) = 0, \quad x \in \mathbb{R}^d \backslash \mathcal{O},\ t > \tau, \qquad \text{and} \qquad u(\tau,x) = u_0(x), \quad x \in \mathcal{O}. \tag{2.10}$$

To prove the existence and uniqueness of weak solutions to problem (2.9)–(2.10), we follow the ideas of [46]. To this end, let $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a bilinear form given by

$$a(v_1,v_2) = \delta(v_1,v_2) + \frac{1}{2}C(d,\gamma)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v_1(x)-v_1(y))(v_2(x)-v_2(y))}{|x-y|^{d+2\gamma}}dxdy, \quad \forall v_1,v_2 \in \mathbb{V}, \tag{2.11}$$

where $C(d,\gamma)$ is the constant in (2.2) and δ the one in (2.4). For convenience, we associate an operator $A:\mathbb{V}\to\mathbb{V}^*$ with a in the following way:

$$< A(v_1), v_2 >_{(\mathbb{V}^*,\mathbb{V})} = a(v_1,v_2) \qquad \forall v_1,v_2\in\mathbb{V}, \tag{2.12}$$

where $< \cdot,\cdot >_{(\mathbb{V}^*,\mathbb{V})}$ is the duality pairing of $\mathbb{V}^*$ and $\mathbb{V}$. It follows from [46] that the inverse operator $A^{-1}:\mathbb{V}^*\subset\mathbb{H}\to\mathbb{V}\subset\mathbb{H}$ is symmetric and compact. Therefore, the Hilbert–Schmidt theorem shows that A has a family of eigenfunctions $\{e_j\}_{j=1}^\infty$ such that $\{e_j\}_{j=1}^\infty$ forms an orthonormal basis of $\mathbb{H}$. Moreover, if λ_j is the eigenvalue corresponding to e_j, i.e., $Ae_j=\lambda_j e_j,\ j=1,2,\dots,$ then λ_j satisfies $0<\delta<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to\infty$ as $j\to\infty$. Actually, e_j $(j\in\mathbb{N})$ are eigenfunctions of the integral fractional Laplacian operator $(-\Delta)^\gamma$. We will consider the fractional power of the operator A. Given $u\in\mathbb{H}$, we have $u=\sum_{j=1}^\infty a_j e_j$ with $a_j=\int_{\mathbb{R}^d}u(x)e_j(x)dx$. Then, for $r>0$, define $A^r u=\sum_{j=1}^\infty a_j\lambda_j^r e_j$ provided the series is convergent for u in $\mathbb{H}$. The domain of A^r is denoted by $D(A^r)$, which is equipped with the norm $\|u\|_{D(A^r)}=|A^r u|$ for $u\in D(A^r)$. By the Riesz representation theorem, $D(A^{-r})$ is the dual space of $D(A^r)$.

2.3. Large deviation principle. For a topological space $\mathcal{E}$, denote the corresponding Borel σ-field by $\mathcal{B}(\mathcal{E})$. For a measure λ on $\mathcal{E}$ and a Hilbert space $\mathbb{H}$, let $L^2(\mathcal{E},\lambda;\mathbb{H})$ denote the space of measurable functions f from $\mathcal{E}$ to $\mathbb{H}$ such that $\int_{\mathcal{E}}|f(u)|^2\lambda(du)<\infty$. For a function $x:[0,T]\to\mathcal{E}$, we use the notation $x(t)$ to denote the evaluation of x at $t\in[0,T]$. A similar convention will be followed for stochastic processes. Eventually, we say a collection $\{u^\varepsilon\}$ of $\mathcal{E}$-valued random variables is tight if the distributions of u^ε are tight in $\mathrm{P}(\mathcal{E})$ (the space of probability measures on $\mathcal{E}$). A function $I:\mathcal{E}\to[0,\infty]$ is called a rate function on $\mathcal{E}$ if for each $M<\infty$, the level set $\{u\in\mathcal{E}:I(u)\le M\}$ is a compact subset of $\mathcal{E}$. A sequence $\{u^\varepsilon\}$ of $\mathcal{E}$-valued random variables is said to satisfy the Laplace principle upper bound (respectively, lower bound) on $\mathcal{E}$ with rate function I if for each $h\in C_b(\mathcal{E})$ (the space of real continuous bounded functions),

$$\limsup_{\varepsilon\to0}\varepsilon\log\mathbb{E}\left\{\exp\left[-\frac{1}{\varepsilon}h(u^\varepsilon)\right]\right\}\le-\inf_{u\in\mathcal{E}}\{h(u)+I(u)\}$$

and

$$\liminf_{\varepsilon\to0}\varepsilon\log\mathbb{E}\left\{\exp\left[-\frac{1}{\varepsilon}h(u^\varepsilon)\right]\right\}\ge-\inf_{u\in\mathcal{E}}\{h(u)+I(u)\},$$

respectively. The Laplace principle is said to hold for $\{u^\varepsilon\}$ with rate function I if both the Laplace upper and lower bounds hold. It is well known that when $\mathcal{E}$ is a Polish space, the family $\{u^\varepsilon\}$ satisfies the Laplace principle upper (respectively, lower) bound with a rate function I on $\mathcal{E}$ if and only if $\{u^\varepsilon\}$ satisfies the large derivation upper (respectively, lower) bound for all closed sets (respectively, open sets) with the rate function I. For more details, see [10] and the references therein.

3. Well-posedness of problem (2.9)–(2.10).

We will prove the existence and uniqueness of solutions to (2.9)–(2.10) in the following sense.

DEFINITION 3.1. *Let $u_0\in L^2(\Omega;\mathbb{H})$ be $\mathcal{F}_\tau$-measurable. An $\mathbb{H}$-valued $\mathcal{F}_t$-adapted càdlàg stochastic process u is called a solution of* (2.9)–(2.10) *if,*

$$u\in L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))\cap L^{p+1}(\Omega;L^{p+1}(\tau,T;L^{p+1}(\mathbb{R}^d))), \tag{3.1}$$

and, for all $t \geq \tau$,

$$u(t) + \int_\tau^t (-\Delta)^\gamma u(s)ds + \delta \int_\tau^t u(s)ds + \int_\tau^t F(u(s))ds = u_0 + \int_\tau^t g(s,u(s))dW(s) + \int_\tau^t \int_E h(u(s-),\xi)\tilde{N}(ds,d\xi), \tag{3.2}$$

in $(\mathbb{V} \cap L^{p+1}(\mathbb{R}^d))^*$, $\mathbb{P}$*-almost surely.*

THEOREM 3.2. *Assume conditions* (F_1)*–*(F_3)*,* (g_1)*–*(g_3) *and* (h_1)*–*(h_3) *hold. Then, for every* $\mathcal{F}_\tau$*-measurable initial value* $u_0 \in L^2(\Omega;\mathbb{H})$*, problem* (2.9)*–*(2.10) *has a unique global solution in the sense of Definition* 3.1*. Moreover, the solution* u *depends continuously on* u_0 *from* $L^2(\Omega;\mathbb{H})$ *to* $L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H})) \cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))$.

Proof. The proof of this theorem follows a standard scheme, for example, [30, Theorem 3.2] and [38, Theorem 2.1], but with particular technical difficulties caused by Lévy noise and the fractional Laplace operator. We will split the proof into five steps.

Step 1. As λ is σ-finite on the Polish space E, there exist measurable subsets $E_m \nearrow E$ satisfying $\lambda(E_m) < \infty$ for all $m \in \mathbb{N}$ and $\cup_{m=1}^\infty E_m = E$. Now, for each $m \in \mathbb{N}$ and $t > \tau$, we define the function $u_0^m \equiv u_0$ and consider recursively the equations,

$$u_n^m(t) = u_0 - \int_\tau^t (-\Delta)^\gamma u_n^m(s)ds - \delta \int_\tau^t u_n^m(s)ds - \int_\tau^t F(u_n^m(s))ds + \int_\tau^t g(s,u_{n-1}^m(s))dW(s) + \int_\tau^t \int_{E_m} h(u_{n-1}^m(s-),\xi)\tilde{N}(ds,d\xi), \quad \mathbb{P}\text{-}a.s. \tag{3.3}$$

Applying the Itô formula to $|u_n^m|^2$, we have

$$\begin{aligned} &|u_n^m(t)|^2 \\ &= |u_0|^2 - 2\int_\tau^t < (-\Delta)^\gamma u_n^m(s) + \delta u_n^m(s) + F(u_n^m(s)), u_n^m(s) > ds \\ &\quad + 2\int_\tau^t \left(u_n^m(s), g(s,u_{n-1}^m(s))dW(s)\right) + \int_\tau^t \|g(s,u_{n-1}^m(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds \\ &\quad + 2\int_\tau^t \int_{E_m} \left(u_n^m(s), h(u_{n-1}^m(s-),\xi)\right)\tilde{N}(ds,d\xi) + \int_\tau^t \int_{E_m} |h(u_{n-1}^m(s-),\xi)|^2 N(ds,d\xi). \end{aligned} \tag{3.4}$$

By definition of the fractional Laplacian operator $(-\Delta)^\gamma$ and (2.11)–(2.12), setting $\eta = \min\{\frac{C(d,\gamma)}{2}, \delta\}$, we obtain

$$\begin{aligned} -2\int_\tau^t < (-\Delta)^\gamma u_n^m(s) + \delta u_n^m(s), u_n^m(s) > ds &= -2\int_\tau^t \left(|(-\Delta)^{\frac{\gamma}{2}} u_n^m(s)|^2 + \delta |u_n^m(s)|^2\right) ds \\ &= -2\int_\tau^t \left(\frac{C(d,\gamma)}{2}\|u_n^m(s)\|^2_{\dot{H}^\gamma(\mathbb{R}^d)} + \delta|u_n^m(s)|^2\right) ds \leq -2\eta \int_\tau^t \|u_n^m(s)\|^2 ds. \end{aligned} \tag{3.5}$$

Making use of condition (F_2), we derive

$$
\begin{aligned}
&-2\int_\tau^t\int_{\mathbb{R}^d} F(u_n^m(s))u_n^m(s)dxds = -2\int_\tau^t\int_{\mathcal{O}} F(u_n^m(s))u_n^m(s)dxds\\
&\quad\le 2\int_\tau^t\int_{\mathcal{O}}\left(k_2-k_3|u_n^m(s)|^{p+1}\right)dxds \le 2k_2|\mathcal{O}|(t-\tau)-2k_3\|u_n^m\|^{p+1}_{L^{p+1}(\tau,t;L^{p+1}(\mathbb{R}^d))}.
\end{aligned}
\tag{3.6}
$$

Substituting (3.5)–(3.6) into (3.4), taking supremum with respect to $t\in[\tau,T']$ for any $\tau\le T'\le T$ and expectation, we find

$$
\begin{aligned}
&\max\left\{\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right],\ 2\eta\mathbb{E}\int_\tau^{T'}\|u_n^m(t)\|^2dt, 2k_3\mathbb{E}\int_\tau^{T'}\|u_n^m(t)\|^{p+1}_{p+1}dt\right\}\\
&\quad\le \mathbb{E}|u_0|^2+2k_2|\mathcal{O}|(T-\tau)+2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t(u_n^m(s),g(s,u_n^m(s))dW(s))\right)\\
&\qquad+\mathbb{E}\int_\tau^{T'}\|g(t,u_{n-1}^m(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt+\mathbb{E}\int_\tau^{T'}\int_{E_m}|h(u_{n-1}^m(t-),\xi)|^2N(dt,d\xi)\\
&\qquad+2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_m}(u_n^m(s),h(u_{n-1}^m(s-),\xi))\tilde{N}(ds,d\xi)\right).
\end{aligned}
\tag{3.7}
$$

By applying assumption (g_2), the Burkholder–Davis–Gundy and Young inequalities, we have

$$
\begin{aligned}
&2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\left(u_n^m(s),g(s,u_{n-1}^m(s))dW(s)\right)\right)\\
&\quad\le 2C_b\mathbb{E}\left(\int_\tau^{T'}|u_n^m(s)|^2\|g(s,u_{n-1}^m(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds\right)^{\frac12}\\
&\quad\le 2C_b\mathbb{E}\left(\sup_{t\in[\tau,T']}|u_n^m(t)|^2\int_\tau^{T'}\|g(t,u_{n-1}^m(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt\right)^{\frac12}\\
&\quad\le \frac14\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right]+4C_b^2\mathbb{E}\int_\tau^{T'}\|g(t,u_{n-1}^m(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt\\
&\quad\le \frac14\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right]+4C_b^2C_g(T-\tau)+4C_b^2C_g\mathbb{E}\int_\tau^{T'}|u_{n-1}^m(t)|^2dt,
\end{aligned}
\tag{3.8}
$$

here and in what follows, C_b is the constant obtained from the Burkholder–Davis–Gundy inequality for Brownian motion and the Poisson process. Similarly, we conclude from assumption (h_2), the Burkholder–Davis–Gundy and Young inequalities that

$$
\begin{aligned}
&2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_m}(u_n^m(s),h(u_{n-1}^m(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&\le 2C_b\mathbb{E}\left(\int_\tau^{T'}\int_{E_m}|u_n^m(s)|^2|h(u_{n-1}^m(s-),\xi)|^2\lambda(d\xi)ds\right)^{\frac12}\\
&\le 2C_b\mathbb{E}\left(\sup_{t\in[\tau,T']}|u_n^m(t)|^2\int_\tau^{T'}\int_{E_m}|h(u_{n-1}^m(s-),\xi)|^2\lambda(d\xi)ds\right)^{\frac12}
\end{aligned}
\tag{3.9}
$$

$$\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right] + 4C_b^2\mathbb{E}\int_\tau^{T'}\int_{E_m}|h(u_{n-1}^m(s-),\xi)|^2\lambda(d\xi)ds$$
$$\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right] + 4C_b^2C_h(T-\tau) + 4C_b^2C_h\mathbb{E}\int_\tau^{T'}|u_{n-1}^m(t)|^2dt.$$

By means of (g_2) and (h_2), we infer

(3.10)
$$\mathbb{E}\int_\tau^{T'}\|g(t,u_{n-1}^m(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt \leq C_g(T-\tau) + C_g\mathbb{E}\int_\tau^{T'}|u_{n-1}^m(t)|^2dt,\ \forall T'\in[\tau,T],$$

and

(3.11)
$$\mathbb{E}\int_\tau^{T'}\int_{E_m}|h(u_{n-1}^m(t-),\xi)|^2\lambda(d\xi)dt \leq C_h(T-\tau) + C_h\mathbb{E}\int_\tau^{T'}|u_{n-1}^m(t)|^2dt,\ \forall T'\in[\tau,T],$$

respectively. Thus, it follows from (3.7)–(3.11) that, for all $T'\in(\tau,T]$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right] &\leq 2\mathbb{E}|u_0|^2 + (4k_2|\mathcal{O}| \\ &\quad + 2C_1)(T-\tau) + 2C_1\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|u_{n-1}^m(s)|^2\right]dt,\end{aligned} \tag{3.12}$$

where we have denoted by $C_1 := 4C_b^2C_g + 4C_b^2C_h + C_g + C_h$. Let us define $\mathcal{U}_N^m(T') := \sup_{n\leq N}\mathbb{E}[\sup_{t\in[\tau,T']}|u_n^m(t)|^2]$ for all $N\in\mathbb{N}, T'\in[\tau,T]$. Subsequently, inequality (3.12) implies for each $T'\in[\tau,T]$,

$$\mathcal{U}_N^m(T') \leq 2\mathbb{E}|u_0|^2 + (4k_2|\mathcal{O}| + 2C_1)(T-\tau) + 2C_1\int_\tau^{T'}\mathcal{U}_N^m(t)dt.$$

The Gronwall lemma implies for each $m\in\mathbb{N}$ and for any $\tau\leq T'\leq T$ that

$$\mathbb{E}\left[\sup_{t\in[\tau,T']}|u_n^m(t)|^2\right] \leq \left(2\mathbb{E}|u_0|^2 + (4k_2|\mathcal{O}| + 2C_1)(T-\tau)\right)e^{2C_1(T-\tau)} := C_2(T),\ \forall n\in\mathbb{N},$$

where $C_2(T)$ is a positive constant depending on T. This, together with (3.7), shows

$$\mathbb{E}\int_\tau^T\|u_n^m(t)\|^2dt + \mathbb{E}\int_\tau^T\|u_n^m(t)\|_{p+1}^{p+1}dt \leq C_3(T),$$

for a positive constant $C_3(T)$. In conclusion, we proved in Step 1 that for all $m,n\in\mathbb{N}$, there exists a constant $C_4>0$ depending on T such that

$$\mathbb{E}\left[\sup_{t\in[\tau,T]}|u_n^m(t)|^2\right] + \mathbb{E}\int_\tau^T\|u_n^m(t)\|^2dt + \mathbb{E}\int_\tau^T\|u_n^m(t)\|_{p+1}^{p+1}dt \leq C_4(T). \tag{3.13}$$

Step 2. For each $m\in\mathbb{N}$ and $n_1,n_2\in\mathbb{N}$, define $\chi_{n_1,n_2}^m = u_{n_1}^m(t) - u_{n_2}^m(t)$, using similar arguments as in Step 1, by (F_3), (g_1), (h_1), the Burkholder–Davis–Gundy and Young inequalities, we obtain

$$\frac{1}{2}\mathbb{E}\left[\sup_{t\in[\tau,T']}|\chi_{n_1,n_2}^m(t)|^2\right] + 2\eta\mathbb{E}\int_\tau^{T'}\|\chi_{n_1,n_2}^m(t)\|^2dt \tag{3.14}$$

$$\leq 4k_4 \int_\tau^{T'} \mathbb{E}\left[\sup_{s\in[\tau,t]} |\chi^m_{n_1,n_2}(s)|^2\right] dt + 2C_5 \int_\tau^{T'} \mathbb{E}\left[\sup_{s\in[\tau,t]} |\chi^m_{n_1-1,n_2-1}(s)|^2\right] dt,$$
$$\forall\ \tau \leq T' \leq T,$$

where we have denoted by $C_5 = 4C_b^2 L_g + L_g + 4C_b^2 L_h + L_h$.

Let $\mathcal{V}^m(T') = \overline{\lim}_{n_1,n_2\to\infty} \mathbb{E}[\sup_{t\in[\tau,T']} |\chi^m_{n_1,n_2}(t)|^2]$. Then, by the Fatou–Lebesgue theorem and (3.14), we have $\mathcal{V}^m(T') \leq 4(2k_4 + C_5)\int_\tau^{T'} \mathcal{V}^m(t)dt$ for all $T' \in [\tau,T]$. The Gronwall lemma implies that

$$\mathcal{V}^m(T) = \overline{\lim_{n_1,n_2\to\infty}} \mathbb{E}\left[\sup_{t\in[\tau,T]} |\chi^m_{n_1,n_2}(t)|^2\right] = 0 \quad \text{and} \quad \overline{\lim_{n_1,n_2\to\infty}} \mathbb{E}\int_\tau^T \|\chi^m_{n_1,n_2}(t)\|^2 dt = 0. \tag{3.15}$$

Hence, for each $m \in \mathbb{N}$, there exists an adapted process $u^m \in L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H})) \cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))$ such that $\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[\tau,T]} |u^m_n(t) - u^m(t)|^2] = \lim_{n\to\infty} \mathbb{E}\int_\tau^T \|u^m_n(t) - u^m(t)\|^2 dt = 0$. Additionally, thanks to (3.13), we immediately derive

$$\mathbb{E}\left[\sup_{t\in[\tau,T]} |u^m(t)|^2\right] + \mathbb{E}\int_\tau^T \|u^m(t)\|^2 dt + \mathbb{E}\int_\tau^T \|u^m(t)\|^{p+1}_{p+1} dt \leq C_4(T). \tag{3.16}$$

Now, for each $m \in \mathbb{N}$, taking the limit in (3.3) as $n \to \infty$, by means of the continuity of function F, the Lipschitz condition imposed on g and h, it is easy to show that u^m is the unique solution of the equation

$$\begin{aligned} u^m(t) = u_0 &- \int_\tau^t (-\Delta)^\gamma u^m(s)ds - \delta\int_\tau^t u^m(s)ds - \int_\tau^t F(u^m(s))ds \\ &+ \int_\tau^t g(s,u^m(s))dW(s) + \int_\tau^t \int_{E_m} h(u^m(s-),\xi)\tilde{N}(ds,d\xi), \quad \tau < t \leq T,\ \mathbb{P}\text{-}a.s. \end{aligned} \tag{3.17}$$

Notice that equality (3.17) holds in $(\mathbb{V} \cap L^{p+1}(\mathbb{R}^d))^*$.

Step 3. For m_1, $m_2 \in \mathbb{N}$ with $m_2 < m_1$, we have $\lambda(E_{m_2}) < \lambda(E_{m_1})$ since E_m is increasing. Define $\mathbf{B}^{m_1,m_2}(t) = u^{m_1}(t) - u^{m_2}(t)$. Applying the Itô formula to $|\mathbf{B}^{m_1,m_2}(t)|^2$ and proceeding likewise as in Step 1, we obtain for all $T' \in [\tau,T]$ that

$$\begin{aligned} &\max\left\{\mathbb{E}\left[\sup_{t\in[\tau,T']} |\mathbf{B}^{m_1,m_2}(t)|^2\right],\ 2\eta\mathbb{E}\int_\tau^{T'} \|\mathbf{B}^{m_1,m_2}(t)\|^2 dt\right\} \\ &\quad\leq 2k_4\mathbb{E}\int_\tau^{T'} |\mathbf{B}^{m_1,m_2}(t)|^2 dt \\ &\qquad + 2\mathbb{E}\left(\sup_{t\in[\tau,T']} \int_\tau^t (\mathbf{B}^{m_1,m_2}(s), (g(s,u^{m_1}(s)) - g(s,u^{m_2}(s)))dW(s))\right) \\ &\qquad + \mathbb{E}\int_\tau^{T'} \|g(s,u^{m_1}(s)) - g(s,u^{m_2}(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})} ds \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_{m_2}}(\mathbf{B}^{m_1,m_2}(s),h(u^{m_1}(s-),\xi)-h(u^{m_2}(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&+ \mathbb{E}\int_\tau^{T'}\int_{E_{m_2}}|h(u^{m_1}(s-),\xi)-h(u^{m_2}(s-),\xi)|^2N(ds,d\xi)\\
&+ 2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_{m_1}\setminus E_{m_2}}(\mathbf{B}^{m_1,m_2}(s),h(u^{m_1}(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&+ \mathbb{E}\int_\tau^{T'}\int_{E_{m_1}\setminus E_{m_2}}|h(u^{m_1}(s-),\xi)|^2N(ds,d\xi)\\
&\le 2k_4\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt+I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

For I_1, similar to (3.8), by (g_1), the Burkholder–Davis–Gundy and Young inequalities, we have

$$
\begin{aligned}
&2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t(\mathbf{B}^{m_1,m_2}(s),(g(s,u^{m_1}(s))-g(s,u^{m_2}(s)))dW(s))\right)\\
&\le\frac14\mathbb{E}\left[\sup_{t\in[\tau,T']}|\mathbf{B}^{m_1,m_2}(t)|^2\right]+4C_b^2L_g\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt.
\end{aligned}\tag{3.19}
$$

For I_3, similar to (3.9), making use of (h_1) and the Burkholder–Davis–Gundy and Young inequalities, we obtain

$$
\begin{aligned}
&2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_{m_2}}(\mathbf{B}^{m_1,m_2}(s),h(u^{m_1}(s-),\xi)-h(u^{m_2}(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&\le\frac14\mathbb{E}\left[\sup_{t\in[\tau,T']}|\mathbf{B}^{m_1,m_2}(t)|^2\right]+4C_b^2L_h\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt.
\end{aligned}\tag{3.20}
$$

For I_2 and I_4, similar to (3.10)–(3.11), by means of (g_1) and (h_1), we derive

$$
I_2=\mathbb{E}\int_\tau^{T'}\|g(s,u^{m_1}(s))-g(s,u^{m_2}(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds\le L_g\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt,\tag{3.21}
$$

and

$$
\begin{aligned}
I_4&=\mathbb{E}\int_\tau^{T'}\int_{E_{m_2}}|h(u^{m_1}(s-),\xi)-h(u^{m_2}(s-),\xi)|^2N(ds,d\xi)\\
&\le L_h\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt,
\end{aligned}\tag{3.22}
$$

respectively. For I_5, by assumption (h_2), the Burkholder–Davis–Gundy and Young inequalities, we have

$$
\begin{aligned}
I_5&=2\mathbb{E}\left(\sup_{t\in[\tau,T']}\int_\tau^t\int_{E_{m_1}\setminus E_{m_2}}(\mathbf{B}^{m_1,m_2}(s),h(u^{m_1}(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&\le\frac14\mathbb{E}\left[\sup_{t\in[\tau,T']}|\mathbf{B}^{m_1,m_2}(t)|^2\right]+4C_b^2\mathbb{E}\int_\tau^{T'}\int_{E_{m_1}\setminus E_{m_2}}|h(u^{m_1}(t-),\xi)|^2\lambda(d\xi)dt.
\end{aligned}\tag{3.23}
$$

Consequently, substituting (3.19)–(3.23) to (3.18), we find

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\left[\sup_{t\in[\tau,T']}|\mathbf{B}^{m_1,m_2}(t)|^2\right]\\
&\le (2k_4+4C_b^2L_g+L_g+4C_b^2L_h+L_h)\int_\tau^{T'}\mathbb{E}\left[\sup_{s\in[\tau,t]}|\mathbf{B}^{m_1,m_2}(s)|^2\right]dt\\
&\quad+(4C_b^2+1)\mathbb{E}\int_\tau^{T'}\int_{E_{m_1}\backslash E_{m_2}}|h(u^{m_1}(t-),\xi)|^2\lambda(d\xi)dt.
\end{aligned}\tag{3.24}$$

It follows from the fact $E_{m_2}\subset E_{m_1}$ with $\lambda(E_{m_1}\backslash E_{m_2})\to 0$ as $m_1,m_2\to\infty$, assumption (h_2) and the property of an absolutely Lebesgue integrable function that, for any $\varepsilon>0$, there exist $M(\varepsilon)>0$ and $\delta'>0$ such that for all $m_1,m_2\ge M(\varepsilon)$,

$$\lambda(E_{m_1}\backslash E_{m_2})<\delta' \quad\text{and}\quad \int_{E_{m_1}\backslash E_{m_2}}|h(u^{m_1}(t^-),\xi)|^2\lambda(d\xi)<\varepsilon.\tag{3.25}$$

Immediately, the Gronwall lemma, together with (3.24)–(3.25) and (3.18), implies that

$$\mathbb{E}\left[\sup_{t\in[\tau,T]}|\mathbf{B}^{m_1,m_2}(t)|^2\right]\to 0,\quad\text{and}\quad \mathbb{E}\int_\tau^T\|\mathbf{B}^{m_1,m_2}(t)\|^2dt\to 0,\qquad\text{as }\ m_1,\ m_2\to\infty.$$

Therefore, there exists an $\mathcal{F}_t$-adapted process $u\in L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))$ such that $\lim_{m\to\infty}\mathbb{E}[\sup_{t\in[\tau,T]}|u^m(t)-u(t)|^2]=\lim_{m\to\infty}\mathbb{E}\int_\tau^T\|u^m(t)-u(t)\|^2dt=0$, which, combining with (3.16), yields

$$\mathbb{E}\left[\sup_{t\in[\tau,T]}|u(t)|^2\right]+\mathbb{E}\int_\tau^T\|u(t)\|^2dt+\mathbb{E}\int_\tau^T\|u(t)\|_{p+1}^{p+1}dt\le C_4(T).$$

Eventually, taking the limit in (3.17) shows that u is the unique solution of (2.9) on the interval $[\tau,T]$.

Step 4. By repeating the above arguments, we obtain the existence of the unique solution of (2.9) on the interval $[T,2T-\tau]$, which finally leads to the completion of the global existence and uniqueness of solutions to (2.9)–(2.10) by further iterations.

Step 5. *Continuity of solutions with respect to initial data.* Let $u_{0,1},\ u_{0,2}\in L^2(\Omega;\mathbb{H})$ be two $\mathcal{F}_\tau$-measurable initial data, and u_1 and u_2 are the corresponding solutions of (2.9)–(2.10) on $[\tau,T]$ for any $T>\tau$, respectively. Denote by $\bar u=u_1-u_2$ and $\bar u_0=u_{0,1}-u_{0,2}$. Then by the Itô formula, the definition of fractional Laplacian operator $(-\Delta)^\gamma$, (2.11)–(2.12) and condition (F_3), similar to (3.5), we have for every $m\in\mathbb{N}$,

$$\begin{aligned}
&|\bar u^m(t)|^2+2\eta\int_\tau^t\|\bar u^m(s)\|^2ds\\
&\le|\bar u_0|^2+2\int_\tau^t(\bar u^m(s),(g(s,u_1^m(s))-g(s,u_2^m(s)))dW(s))\\
&\quad+2k_4\int_\tau^t|\bar u^m(s)|^2ds+\int_\tau^t\|g(s,u_1^m(s))-g(s,u_2^m(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds\\
&\quad+2\int_\tau^t\int_{E_m}(\bar u^m(s),h(u_1^m(s-),\xi)-h(u_2^m(s-),\xi))\tilde N(ds,d\xi)\\
&\quad+\int_\tau^t\int_{E_m}|h(u_1^m(s-),\xi)-h(u_2^m(s-),\xi)|^2N(ds,d\xi).
\end{aligned}\tag{3.26}$$

On the one hand, by (3.26), we find that for all $\tau \le T' \le T$,

(3.27)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\le t\le T'}|\bar{u}^m(t)|^2\right]\\
&\le \mathbb{E}|\bar{u}_0|^2+2\mathbb{E}\left(\sup_{\tau\le t\le T'}\int_\tau^t(\bar{u}^m(s),(g(s,u_1^m(s))-g(s,u_2^m(s)))dW(s))\right)\\
&\quad+2k_4\int_\tau^{T'}\mathbb{E}\left[\sup_{\tau\le s\le t}|\bar{u}^m(s)|^2\right]dt+\mathbb{E}\int_\tau^{T'}\|g(s,u_1^m(s))-g(s,u_2^m(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds\\
&\quad+2\mathbb{E}\left(\sup_{\tau\le t\le T'}\int_\tau^t\int_{E_m}(\bar{u}^m(s),h(u_1^m(s-),\xi)-h(u_2^m(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&\quad+\mathbb{E}\int_\tau^{T'}\int_{E_m}|h(u_1^m(s-),\xi)-h(u_2^m(s-),\xi)|^2N(ds,d\xi).
\end{aligned}$$

Similar to (3.8)–(3.9), by the Burkholder–Davis–Gundy and Young inequalities, (g_1) and (h_1), we have

$$\begin{aligned}
&2\mathbb{E}\left(\sup_{\tau\le t\le T'}\int_\tau^t(\bar{u}^m(s),(g(s,u_1^m(s))-g(s,u_2^m(s)))dW(s))\right)\\
&\quad\le\frac{1}{4}\mathbb{E}\left[\sup_{\tau\le t\le T'}|\bar{u}^m(t)|^2\right]+4C_b^2L_g\int_\tau^{T'}\mathbb{E}\left[\sup_{\tau\le s\le t}|\bar{u}^m(s)|^2\right]dt,
\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}
&2\mathbb{E}\left(\sup_{\tau\le t\le T'}\int_\tau^t\int_{E_m}(\bar{u}^m(s),h(u_1^m(s-),\xi)-h(u_2^m(s-),\xi))\tilde{N}(ds,d\xi)\right)\\
&\quad\le\frac{1}{4}\mathbb{E}\left[\sup_{\tau\le t\le T'}|\bar{u}^m(t)|^2\right]+4C_b^2L_h\int_\tau^{T'}\mathbb{E}\left[\sup_{\tau\le s\le t}|\bar{u}^m(s)|^2\right]dt,
\end{aligned}\tag{3.29}$$

respectively. It follows from (3.27)–(3.29) and conditions (g_1) and (h_1) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\le t\le T'}|\bar{u}^m(t)|^2\right]\\
&\quad\le 2\mathbb{E}|\bar{u}_0|^2+2(2k_4+4C_b^2L_g+4C_b^2L_h+L_g+L_h)\int_\tau^{T'}\mathbb{E}\left[\sup_{\tau\le s\le t}|\bar{u}^m(s)|^2\right]dt.
\end{aligned}$$

Applying the Gronwall lemma to the above inequality, we obtain in particular for $T'=T$,

$$\mathbb{E}\left[\sup_{\tau\le t\le T}|\bar{u}^m(t)|^2\right]\le 2e^{C_6(T-\tau)}\mathbb{E}|\bar{u}_0|^2,\tag{3.30}$$

where $C_6 := 2\left(2k_4+4C_b^2L_g+4C_b^2L_h+L_g+L_h\right)$. By (3.26) and (3.30), for some $C_7 := C_7(T)$, we derive

$$\mathbb{E}\int_\tau^T\|\bar{u}^m(t)\|^2dt\le C_7\mathbb{E}|\bar{u}_0|^2.\tag{3.31}$$

In Step 3, we have proved for each initial value $u_0\in L^2(\Omega;\mathbb{H})$ and every $m\in\mathbb{N}$, the corresponding solution sequence $\{u^m\}$ is Cauchy in $L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;$

$L^2(\tau,T;\mathbb{V})$). Therefore, $u_1,\ u_2 \in L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))$ and satisfy (3.30)–(3.31). Moreover, there exists a positive constant $C(T)$ such that

$$\|u_1-u_2\|^2_{L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))}+\|u_1-u_2\|^2_{L^2(\Omega;L^2(\tau,T;\mathbb{V}))}\le C\|u_{0,1}-u_{0,2}\|^2_{L^2(\Omega;\mathbb{H})}. \tag{3.32}$$

Namely, the solution depends continuously on initial data. The proof of this theorem is complete. ∎

Remark 3.3. Notice that, under assumptions of Theorem 3.2, if u is the unique solution to problem (2.9)–(2.10) corresponding to the initial value $u_0\in L^2(\Omega;\mathbb{H})$, then there exists a sequence

$$u^m\in L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))\cap L^{p+1}(\Omega;L^{p+1}(\tau,T;L^{p+1}(\mathbb{R}^d)))(m\ge 1),$$

which converges to u in $L^2(\Omega;\mathbf{D}([\tau,T];\mathbb{H}))\cap L^2(\Omega;L^2(\tau,T;\mathbb{V}))$ and satisfies (3.17). In other words, each u^m is a solution to problem (2.9)–(2.10) but replacing E by E_m. This fact has been used in Step 5 in the previous proof and will be used repeatedly in the following sections.

4. Existence of weak mean random attractors. This section is devoted to the existence and uniqueness of weak mean random attractors for the nonautonomous fractional stochastic differential equations (2.9)–(2.10). To this end, we first define a mean random dynamical system for (2.9)–(2.10), then prove the existence and uniqueness of weak pullback mean random attractors.

Observe that it follows from Theorem 3.2 that for every $\tau\in\mathbb{R}$ and every $\mathcal{F}_\tau$-measurable initial datum $u_0\in L^2(\Omega;\mathbb{H})$, problem (2.9)–(2.10) has a unique càdlàg $\mathbb{H}$-valued $\mathcal{F}_t$-adapted solution $u(t,\tau,u_0)$ with initial condition u_0 at τ in the sense of Definition 3.1. Theorem 3.2 presented that $u(\cdot,\tau,u_0)\in L^2(\Omega;\mathbf{D}([\tau,\infty);\mathbb{H}))$, which implies that $u\in\mathbf{D}([\tau,\infty);L^2(\Omega;\mathbb{H}))$. In this way, we are able to define a cocycle generated by the problem under consideration. Given $t\in\mathbb{R}^+$ and $\tau\in\mathbb{R}$, let $\Phi(t,\tau)$ be a mapping from $L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})$ to $L^2(\Omega,\mathcal{F}_{t+\tau};\mathbb{H})$ defined by $\Phi(t,\tau)(u_0)=u(t+\tau,\tau,u_0)$, where $u_0\in L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})$. The uniqueness of solution to (2.9)–(2.10) implies that for every $t,s>0$ and $\tau\in\mathbb{R}$, $\Phi(t+s,\tau)=\Phi(t,s+\tau)\circ\Phi(s,\tau)$. This cocycle Φ is called the mean random dynamical system generated by (2.9)–(2.10) on $L^2(\Omega,\mathcal{F};\mathbb{H})$. We will study the existence and uniqueness of weak pullback random attractors for Φ.

Let $B=\{B(\tau)\subset L^2(\Omega,\mathcal{F}_\tau;\mathbb{H}):\tau\in\mathbb{R}\}$ be a family of nonempty bounded sets such that

$$\lim_{\tau\to-\infty}e^{(2\delta-L_1)\tau}\|B(\tau)\|^2_{L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})}=0, \tag{4.1}$$

where $2\delta>L_1$ with $L_1=C_g+C_h$ and $\|\mathcal{Q}\|_{L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})}=\sup_{u\in\mathcal{Q}}\|u\|_{L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})}$ for a subset $\mathcal{Q}$ in $L^2(\Omega,\mathcal{F}_\tau;\mathbb{H})$. We will use $\mathcal{D}$ to denote the collection of all families of nonempty bounded sets satisfying (4.1).

We will first derive uniform estimates on the solutions of (2.9)–(2.10), then construct a $\mathcal{D}$-pullback absorbing set for the system Φ.

Lemma 4.1. *Suppose (F_1)–(F_3), (g_1)–(g_3) and (h_1)–(h_3) hold. In addition, assume $2\delta>L_1:=C_g+C_h$. Then for every $\tau\in\mathbb{R}$ and $B\in\mathcal{D}$, there exists $T=T(\tau,B)>0$ such that for all $t\ge T$, the solution u of (2.9)–(2.10) satisfies*

$$\mathbb{E}|u(\tau,\tau-t,u_0)|^2\le\frac{2k_2|\mathcal{O}|+L_1}{2\delta-L_1}+1,\qquad\forall u_0\in B(\tau-t).$$

Proof. We will split the proof into two steps.

Step 1. As λ is σ-finite on the Polish space E, there exist measurable subsets $E_m\nearrow E$ satisfying $\lambda(E_m)<\infty$ for all $m\in\mathbb{N}$ and $\cup_{m=1}^\infty E_m=E$. Taking into account

Remark 3.3, for each $m \in \mathbb{N}$, applying Itô's formula to $e^{(2\delta-L_1)t}|u^m(r,\tau-t,u_0)|^2$ with $r \geq \tau - t$ (see, for example, [11]), we obtain

$$
\begin{aligned}
(4.2)\qquad & e^{(2\delta-L_1)r}|u^m(r,\tau-t,u_0)|^2 + 2\int_{\tau-t}^r e^{(2\delta-L_1)s}|(-\Delta)^{\frac{\gamma}{2}}u^m(s,\tau-t,u_0)|^2 ds \\
&+ 2\delta\int_{\tau-t}^r e^{(2\delta-L_1)s}|u^m(s,\tau-t,u_0)|^2 ds \\
&+ 2\int_{\tau-t}^r e^{(2\delta-L_1)s} < F(u^m(s,\tau-t,u_0)), u^m(s,\tau-t,u_0) > ds \\
&= e^{(2\delta-L_1)(\tau-t)}|u_0|^2 + (2\delta-L_1)\int_{\tau-t}^r e^{(2\delta-L_1)s}|u^m(s,\tau-t,u_0)|^2 ds \\
&+ 2\int_{\tau-t}^r e^{(2\delta-L_1)s}(u^m(s,\tau-t,u_0), g(s,u^m(s,\tau-t,u_0))dW(s)) \\
&+ \int_{\tau-t}^r e^{(2\delta-L_1)s}\|g(s,u^m(s,\tau-t,u_0))\|^2_{\mathcal{L}_2(U;\mathbb{H})} ds \\
&+ 2\int_{\tau-t}^r\int_{E_m} e^{(2\delta-L_1)s}(u^m(s,\tau-t,u_0), h(u(s-,\tau-t,u_0),\xi))\tilde{N}(ds,d\xi) \\
&+ \int_{\tau-t}^r\int_{E_m} e^{(2\delta-L_1)s}|h(u^m(s-,\tau-t,u_0),\xi)|^2 N(ds,d\xi),
\end{aligned}
$$

which implies that for all $r \geq \tau - t$,

(4.3)

$$
\begin{aligned}
& e^{(2\delta-L_1)r}\mathbb{E}|u^m(r,\tau-t,u_0)|^2 + C(d,\gamma)\int_{\tau-t}^r e^{(2\delta-L_1)s}\mathbb{E}\|u^m(s,\tau-t,u_0)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)} ds \\
&+ 2\delta\int_{\tau-t}^r e^{(2\delta-L_1)s}\mathbb{E}|u^m(s,\tau-t,u_0)|^2 ds \\
&+ 2\int_{\tau-t}^r e^{(2\delta-L_1)s}\mathbb{E} < F(u^m(s,\tau-t,u_0)), u^m(s,\tau-t,u_0) > ds \\
&= e^{(2\delta-L_1)(\tau-t)}\mathbb{E}|u_0|^2 + (2\delta-L_1)\int_{\tau-t}^r e^{(2\delta-L_1)s}\mathbb{E}|u^m(s,\tau-t,u_0)|^2 ds \\
&+ \int_{\tau-t}^r e^{(2\delta-L_1)s}\mathbb{E}\|g(s,u^m(s,\tau-t,u_0))\|^2_{\mathcal{L}_2(U;\mathbb{H})} ds \\
&+ \int_{\tau-t}^r\int_{E_m} e^{(2\delta-L_1)s}\mathbb{E}|h(u^m(s-,\tau-t,u_0),\xi)|^2 N(ds,d\xi).
\end{aligned}
$$

We now do estimates one by one for (4.3). On the one hand, by (F_2), we have

(4.4)

$$
\begin{aligned}
& 2\mathbb{E} < F(u^m(r,\tau-t,u_0)), u^m(r,\tau-t,u_0) >= 2\mathbb{E}\int_{\mathbb{R}^d} F(u^m(r,\tau-t,u_0))u^m(r,\tau-t,u_0)dx \\
&= 2\mathbb{E}\int_{\mathcal{O}} F(u^m(r,\tau-t,u_0))u^m(r,\tau-t,u_0)dx \geq 2\mathbb{E}\int_{\mathcal{O}}(-k_2+k_3|u^m(r,\tau-t,u_0)|^{p+1})dx \\
&= -2k_2|\mathcal{O}| + 2k_3\mathbb{E}\|u^m(r,\tau-t,u_0)\|^{p+1}_{p+1}.
\end{aligned}
$$

On the other hand, by means of assumptions (g_2) and (h_2), we obtain

$$
(4.5)\qquad \mathbb{E}\|g(r,u^m(r,\tau-t,u_0))\|^2_{\mathcal{L}_2(U;\mathbb{H})} \leq C_g + C_g\mathbb{E}|u^m(r,\tau-t,u_0)|^2,
$$

and

$$\int_{E_m} \mathbb{E}|h(u^m(r-,\tau-t,u_0),\xi)|^2\lambda(d\xi) \le C_h + C_h\mathbb{E}|u^m(r,\tau-t,u_0)|^2, \tag{4.6}$$

separately. Substituting (4.4)–(4.6) into (4.3), ignoring the second term of the left-hand side of (4.3) and the second term of the right-hand side of the estimate to (4.4), for every $m\in\mathbb{N}$, we find

$$e^{(2\delta-L_1)r}\mathbb{E}|u^m(r,\tau-t,u_0)|^2 \le e^{(2\delta-L_1)(\tau-t)}\mathbb{E}|u_0|^2 + e^{(2\delta-L_1)r}\frac{2k_2|\mathcal{O}|+L_1}{2\delta-L_1}.$$

Therefore, we infer on the interval $(\tau-t,\tau)$ that

$$\mathbb{E}|u^m(\tau,\tau-t,u_0)|^2 \le e^{-(2\delta-L_1)t}\mathbb{E}|u_0|^2 + \frac{2k_2|\mathcal{O}|+L_1}{2\delta-L_1}, \qquad \forall m\in\mathbb{N}. \tag{4.7}$$

Step 2. Let us proceed like in Step 3 in Theorem 3.2. For $m_1,\ m_2\in\mathbb{N}$ with $m_2<m_1$, then we have $\lambda(E_{m_2})<\lambda(E_{m_1})$. Define $\mathbf{R}_{m_1,m_2}(r,\tau-t,u_0)=u^{m_1}(r,\tau-t,u_0)-u^{m_2}(r,\tau-t,u_0)$. Similar arguments as (3.18)–(3.25) imply (replace $\mathbf{B}^{m_1,m_2}$ and $t\in[\tau,T]$ by $\mathbf{R}_{m_1,m_2}$ and $s\in[\tau-t,r]$, respectively) for every $\tau\in\mathbb{R}$, $t>\tau$ and $r>\tau-t$ that

$$\mathbb{E}\left[\sup_{s\in[\tau-t,r]}|\mathbf{R}_{m_1,m_2}(s)|^2\right]\to 0,\ \ \mathbb{E}\int_{\tau-t}^{r}\|\mathbf{R}_{m_1,m_2}(s)\|^2ds\to 0 \qquad \text{as}\ \ m_1,\ m_2\to\infty.$$

Therefore, for every $r>\tau-t$, there exists an $\mathcal{F}_t$-adapted process $u\in L^2(\Omega;\mathbf{D}([\tau-t,r];\mathbb{H}))\cap L^2(\Omega;L^2(\tau-t,r;\mathbb{V}))$ (thanks to the uniqueness of solution, this limit is denoted by the same u) such that

$$\lim_{m\to\infty}\mathbb{E}\left[\sup_{s\in[\tau-t,r]}|u^m(s)-u(s)|^2\right]=\lim_{m\to\infty}\mathbb{E}\int_{\tau-t}^{r}\|u^m(s)-u(s)\|^2ds=0,$$

which, together with (4.7), yields

$$\mathbb{E}|u(\tau,\tau-t,u_0)|^2 \le e^{-(2\delta-L_1)t}\mathbb{E}|u_0|^2 + \frac{2k_2|\mathcal{O}|+L_1}{2\delta-L_1}. \tag{4.8}$$

Since $u_0\in B(\tau-t)$ and $B\in\mathcal{D}$, one has

$$e^{-(2\delta-L_1)t}\mathbb{E}|u_0|^2 \le e^{-(2\delta-L_1)t}\|B(\tau-t)\|^2\to 0 \qquad \text{as} \quad t\to\infty,$$

which along with (4.8) concludes the proof. ∎

We will present now the existence of weakly compact $\mathcal{D}$-pullback absorbing sets to problem (2.9)–(2.10).

LEMMA 4.2. *Under assumptions of Lemma* 4.1*, the mean random dynamical system* Φ *related to* (2.9)–(2.10) *has a weakly compact* $\mathcal{D}$*-pullback absorbing set* $K=\{K(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$*, which is given, for each* $\tau\in\mathbb{R}$*, by*

$$K(\tau)=\{u\in L^2(\Omega,\mathcal{F}_\tau;\mathbb{H}):\mathbb{E}|u(\tau)|^2\le R\},\ \ \textit{where}\ R:=\frac{2k_2|\mathcal{O}|+L_1}{2\delta-L_1}+1. \tag{4.9}$$

Proof. For each $\tau \in \mathbb{R}$, it is easy to see that $K(\tau)$ given by (4.9) is a bounded closed convex subset of $L^2(\Omega, \mathcal{F}_\tau; \mathbb{H})$. Therefore, it is a weakly compact subset of $L^2(\Omega, \mathcal{F}_\tau; \mathbb{H})$. Moreover, it follows from Lemma 4.1 that, for every $\tau \in \mathbb{R}$ and $B = \{B(\tau - t)\} \in \mathcal{D}$, there exists $T = T(\tau, B) > 0$ such that $\Phi(t, \tau - t, B(\tau - t)) \subset K(\tau)$ for all $t \geq T$. On the other hand,

$$\lim_{\tau \to -\infty} e^{(2\delta - L_1)\tau} \|K(\tau)\|^2_{L^2(\Omega, \mathcal{F}_\tau; \mathbb{H})} \leq \lim_{\tau \to -\infty} e^{(2\delta - L_1)\tau} \left(\frac{2k_2|\mathcal{O}| + L_1}{2\delta - L_1} + 1 \right) = 0.$$

Hence, we have verified $K(\tau) \in \mathcal{D}$. Namely, K is a weakly compact $\mathcal{D}$-pullback absorbing set for Φ. ∎

Now, we are in a position to address the existence and uniqueness of weak $\mathcal{D}$-pullback mean random attractors to problem (2.9)–(2.10) (see [48] for the definition of this kind of attractors).

THEOREM 4.3. *Assume the conditions of Lemma* 4.1 *hold. Then the mean random dynamical system* Φ *to problem* (2.9)–(2.10) *has a unique weak* $\mathcal{D}$*-pullback mean random attractor* $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $L^2(\Omega, \mathcal{F}; \mathbb{H})$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$.

Proof. The existence and uniqueness of the weak $\mathcal{D}$-pullback mean random attractors $\mathcal{A} \in \mathcal{D}$ of Φ are immediate consequences of Lemma 4.2 and [48, Theorem 2.13]. ∎

5. Invariant measures and ergodicity.

In this section, we establish the existence of invariant measures and ergodicity to the following autonomous problem, for $x \in \mathcal{O}$ and $t > 0$,

(5.1)
$$du(t) + (-\Delta)^\gamma u(t)dt + F(u(t))dt + \delta u(t)dt = g(u(t))dW(t) + \int_E h(u(t-), \xi)\tilde{N}(dt, d\xi),$$

with boundary and initial conditions,

$$u(t, x) = 0, \quad x \in \mathbb{R}^d \backslash \mathcal{O}, \ t \geq 0, \qquad \text{and} \qquad u(0, x) = u_0, \quad x \in \mathcal{O}, \tag{5.2}$$

respectively, where δ is a positive constant as in (2.4), and W and $\tilde{N}$ are independent real-valued standard Wiener process and compensated Poisson random measure, respectively.

In the remainder of this section, we will assume the nonlinear functions $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \times E \to \mathbb{R}$ are globally Lipschitz continuous with linearly growing rate. More precisely, there exist positive constants α_1, α_2, β_1 and β_2 such that, for every $s, r \in \mathbb{R}$,

$$|g(s) - g(r)|^2 \leq \alpha_1 |s - r|^2, \qquad |g(s)|^2 \leq \beta_1 + \frac{\delta}{2}|s|^2, \tag{5.3}$$

and for $\lambda(E') < \infty$ with $E' \subset E$, suppose

$$\int_{E'} |h(s, \xi) - h(r, \xi)|^2 \lambda(d\xi) \leq \alpha_2 |s - r|^2, \qquad \int_{E'} |h(s, \xi)|^2 \lambda(d\xi) \leq \beta_2 + \frac{\delta}{2}|s|^2. \tag{5.4}$$

Under assumptions (F_1)–(F_3), (5.3)–(5.4), Theorem 3.2 shows that for every $\mathcal{F}_0$-measurable initial value u_0 in $L^2(\Omega; \mathbb{H})$, problem (5.1)–(5.2) possesses a unique càdlàg $\mathbb{H}$-valued $\mathcal{F}_t$-adapted solution $u(t, u_0)$ in the sense of Definition 3.1.

5.1. Introduction to invariant measures and ergodicity. We first provide the definitions of invariant measures and ergodicity (for more details, see [18, 34, 35] and the references therein). Let $\mathbb{X}$ be a Polish space, and let P_t and $P(t,x,\Gamma)$, $t\geq 0$, $x\in\mathbb{X}$, $\Gamma\in\mathcal{B}(\mathbb{X})$, be the corresponding transition semigroup and transition function, respectively.

DEFINITION 5.1. *A probability measure μ on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is said to be an invariant measure or a stationary measure for a given transition probability function $P(t,x,dy)$ if it satisfies*

$$\mu(A)=\int_{\mathbb{X}} P(t,x,A)\mu(dx)\ \forall A\in\mathcal{B}(\mathbb{X}),\forall t>0.$$

Equivalently, if for all $\varphi\in C_b(\mathbb{X})$ (the space of bounded and continuous Borel functions on $\mathbb{X}$) and $t\geq 0$,

$$\int_{\mathbb{X}}\varphi(x)\mu(dx)=\int_{\mathbb{X}}(P_t\varphi)(x)\mu(dx),$$

where the Markov semigroup $(P_t)_{t\geq 0}$ is defined by

$$P_t\varphi(x)=\int_{\mathbb{X}}\varphi(y)P(t,x,dy).$$

DEFINITION 5.2. *Let μ be an invariant measure for $(P_t)_{t\geq 0}$. We say that the measure μ is an ergodic measure if for all $\varphi\in L^2(\mathbb{X};\mu)$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(P_t\varphi)(x)dt=\int_{\mathbb{X}}\varphi(x)\mu(dx)\quad in\ L^2(\mathbb{X};\mu).$$

The next lemma is crucial to proving the existence of ergodicity.

LEMMA 5.3 (see [18, Theorem 3.2.6]). *If $\mu\in\mathcal{M}(\mathbb{X})$ (the set of all probability measures defined on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is the unique invariant measure for the semigroup $P_t(t>0)$, then it is ergodic.*

Now, we will introduce the transition operators to problem (5.1)–(5.2). By the construction of the solutions, we find that the transition operators are homogeneous. Therefore, let $u(t,u_0)$ be the unique solution to (5.1)–(5.2); then for any $\varphi\in C_b(\mathbb{H})$, $t\geq 0$, and $u_0\in\mathbb{H}$, the corresponding Markov transition operator P_t can be defined as [17, Theorem 9.8, p. 244]

$$(P_t\varphi)(u_0)=\mathbb{E}[\varphi(u(t,u_0))]. \tag{5.5}$$

As usual, for $\Gamma\in\mathscr{B}(\mathbb{H})$, $t\geq 0$ and $u_0\in\mathbb{H}$, we set $P(t,u_0,\Gamma)=(P_t\chi_\Gamma)(u_0)=\mathbb{P}\{\omega\in\Omega: u(t,u_0)\in\Gamma\}$, where χ_Γ denotes the characteristic (or indicator) function of Γ. Then $P(t,u_0,\cdot)$ is the probability distribution of $u(t,u_0)$. In addition, Theorem 3.2 (cf. (3.32)) proved the solution to (5.1)–(5.2) depends continuously on initial value, which implies the Feller property of P_t for $t\geq 0$. Similar arguments as in [47] show that the solution $u(t,u_0)$ of problem (5.1)–(5.2) is an $\mathbb{H}$-valued Markov process, which implies that if $\varphi:\mathbb{H}\to\mathbb{R}$ is a bounded Borel function, then $(P_t\varphi)(u_0)=(P_r(P_{t-r}\varphi))(u_0)$ for all $u_0\in\mathbb{H}$, for any $0\leq r\leq t$, $\mathbb{P}$-a.s.

5.2. Existence of invariant measures. We will prove the existence of invariant measures to problem (5.1)–(5.2) by using the Krylov–Bogolyubov method. To this end, we first derive uniform estimates on the solutions via the following lemma.

LEMMA 5.4. *Assume* (F_1)–(F_3) *and* (5.3)–(5.4) *hold. Then, the solution* $u(t, u_0)$ *of* (5.1)–(5.2) *satisfies*

$$\mathbb{E}|u(t)|^2 + C(d,\gamma)\int_0^t e^{\delta(s-t)}\mathbb{E}\|u(s)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)}ds \le e^{-\delta t}\mathbb{E}|u_0|^2 + \frac{2k_2|\mathcal{O}| + \beta_1 + \beta_2}{\delta}, \forall\, t \ge 0,$$

and

$$\frac{\eta_1}{t}\mathbb{E}\int_0^t \|u(s)\|^2 ds \le \frac{1}{T_0}\mathbb{E}|u_0|^2 + 2k_2|\mathcal{O}| + \beta_1 + \beta_2,\ \forall\ t > T_0 > 0,\ \textit{where } \eta_1 = \min\{C(d,\gamma), \delta\}.$$

Proof. As λ is σ-finite on the Polish space E, there exist measurable subsets $E_m \nearrow E$ satisfying $\lambda(E_m) < \infty$ for all $m \in \mathbb{N}$ and $\cup_{m=1}^\infty E_m = E$. Now, on account of Remark 3.3, for each $m \in \mathbb{N}$, similar to (4.3), applying Itô's formula to $e^{\delta t}\mathbb{E}|u^m(t)|^2$, we obtain

$$\begin{aligned}
&e^{\delta t}\mathbb{E}|u^m(t)|^2 + C(d,\gamma)\int_0^t e^{\delta s}\mathbb{E}\|u^m(s)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)}ds + 2\delta\int_0^t e^{\delta s}\mathbb{E}|u^m(s)|^2 ds \\
&\quad + 2\int_0^t e^{\delta s}\mathbb{E} < F(u^m(s)), u^m(s) > ds \\
&= \mathbb{E}|u_0|^2 + \delta\int_0^t e^{\delta s}\mathbb{E}|u^m(s)|^2 ds + \int_0^t e^{\delta s}\mathbb{E}|g(u^m(s))|^2 ds \\
&\quad + \int_0^t\int_{E_m} e^{\delta s}\mathbb{E}|h(u^m(s^-),\xi)|^2\lambda(d\xi)ds.
\end{aligned} \tag{5.6}$$

Making use of the same estimates as in (4.4), we have

$$-2\mathbb{E} < F(u^m(s)), u^m(s) > \le 2k_2|\mathcal{O}| - 2k_3\mathbb{E}\|u^m(s)\|^{p+1}_{p+1}. \tag{5.7}$$

By means of (5.3)–(5.4), we find

$$\mathbb{E}|g(u^m(s))|^2 \le \beta_1 + \frac{\delta}{2}\mathbb{E}|u^m(s)|^2 \text{ and } \int_{E_m}\mathbb{E}|h(u^m(s-),\xi)|^2\lambda(d\xi) \le \beta_2 + \frac{\delta}{2}\mathbb{E}|u^m(s)|^2. \tag{5.8}$$

It follows from (5.6)–(5.8) that

$$e^{\delta t}\mathbb{E}|u^m(t)|^2 + C(d,\gamma)\int_0^t e^{\delta s}\mathbb{E}\|u^m(s)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)}ds \le \mathbb{E}|u_0|^2 + \frac{(2k_2|\mathcal{O}| + \beta_1 + \beta_2)}{\delta}e^{\delta t}. \tag{5.9}$$

Thus,

$$\mathbb{E}|u^m(t)|^2 + C(d,\gamma)\int_0^t e^{\delta(s-t)}\mathbb{E}\|u^m(s)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)}ds \le e^{-\delta t}\mathbb{E}|u_0|^2 + \frac{2k_2|\mathcal{O}| + \beta_1 + \beta_2}{\delta}. \tag{5.10}$$

On the other hand, using Itô's formula to $|u^m(t)|^2$, we derive

$$\begin{aligned}
&\mathbb{E}|u^m(t)|^2 + C(d,\gamma)\int_0^t \mathbb{E}\|u^m(s)\|^2_{\dot{\mathbb{H}}^\gamma(\mathbb{R}^d)}ds + 2\delta\int_0^t \mathbb{E}|u^m(s)|^2 ds \\
&\quad + 2\int_0^t e^{\delta s}\mathbb{E} < F(u^m(s)), u^m(s) > ds \\
&\le \mathbb{E}|u_0|^2 + \int_0^t \mathbb{E}|g(u^m(s))|^2 ds + \int_0^t\int_{E_m}\mathbb{E}|h(u^m(s^-),\xi)|^2\lambda(d\xi)ds.
\end{aligned}$$

Let $\eta_1 = \min\{C(d,\gamma),\delta\}$, by means of (5.7)–(5.8), we obtain

$$\mathbb{E}|u^m(t)|^2 + \eta_1 \int_0^t \mathbb{E}\|u^m(s)\|^2 ds \le \mathbb{E}|u_0|^2 + (2k_2|\mathcal{O}| + \beta_1 + \beta_2)t, \quad \forall t > 0,$$

which implies

$$\frac{\eta_1}{t}\mathbb{E}\int_0^t \|u^m(s)\|^2 ds \le \frac{\mathbb{E}|u_0|^2}{T_0} + 2k_2|\mathcal{O}| + \beta_1 + \beta_2, \qquad \forall t > T_0 > 0. \tag{5.11}$$

Since $\{u^m\}_{m=1}^{\infty}$ converges to u in $L^2(\Omega;\mathbf{D}([0,T];\mathbb{H})) \cap L^2(\Omega; L^2(0,T;\mathbb{V}))$, then u satisfies energy estimates (5.6) (see also Step 3 of Theorem 3.2), which, together with (5.10)–(5.11), concludes the proof of this lemma. ∎

THEOREM 5.5. *Under conditions of Lemma* 5.4*, problem* (5.1)–(5.2) *has an invariant measure on* $\mathbb{H}$.

Proof. Using the Chebyshev inequality and Lemma 5.4, we infer for $T_0 > 0$ and $R > 0$,

$$\begin{aligned}\sup_{t\ge T_0} \frac{1}{t}\int_0^t \mathbb{P}(\{\|u(s,u_0)\| > R\})ds &\le \sup_{t\ge T_0} \frac{1}{tR^2}\int_0^t \mathbb{E}\|u(s,u_0)\|^2 ds \\ &\le \frac{\mathbb{E}|u_0|^2}{T_0R^2\eta_1} + \frac{2k_2|\mathcal{O}| + \beta_1 + \beta_2}{R^2\eta_1}.\end{aligned} \tag{5.12}$$

The above inequality implies, for all $t > T_0$ and every $\varepsilon > 0$, there exists $R_0 := \sqrt{\frac{\mathbb{E}|u_0|^2}{T_0\eta_1\varepsilon} + \frac{(2k_2|\mathcal{O}|+\beta_1+\beta_2)}{\eta_1\varepsilon}} > 0$ such that, for any $R \ge R_0$,

$$\begin{aligned}\mu_{t,u_0}(\Gamma) &:= \frac{1}{t}\int_0^t P(s,u_0,\Gamma)ds = \frac{1}{t}\int_0^t \mathbb{P}(\{\omega\in\Omega : u(s,u_0)\in\Gamma\})ds \\ &\ge \frac{1}{t}\int_0^t \mathbb{P}(\{\omega\in\Omega : \|u(s,u_0)\| \le R_0\})ds \\ &= 1 - \frac{1}{t}\int_0^t \mathbb{P}(\{\omega\in\Omega : \|u(s,u_0)\| > R_0\})ds = 1-\varepsilon, \qquad \Gamma := B(0,R),\end{aligned} \tag{5.13}$$

where $B(0,R)$ is the ball centered at 0 with radius R in $\mathbb{V}$. Since $\mathbb{V}$ is compactly embedded in $\mathbb{H}$ ($\mathbb{V} \hookrightarrow\hookrightarrow \mathbb{H}$), (5.13) shows for every $\varepsilon > 0$, there exists a compact set $\mathcal{K} \in \mathbb{H}$ such that $\mu_{t,u_0}(\mathcal{K}) > 1-\varepsilon$ for all $t > T_0$. Hence, the sequence of probability measure μ_{t,u_0} is tight on $\mathbb{H}$.

As a result, an application of the Krylov–Bogoliubov theorem (see [15]) shows that there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $\mu_{t_n,u_0} \to \mu$ weakly as $n \to \infty$. Moreover, μ is an invariant measure for this transition operator P_t, defined by$(P_t\varphi)(u_0) = \mathbb{E}[\varphi(u(t,u_0))]$ for all $\varphi \in C_b(\mathbb{H})$. Thus, the proof of this theorem is complete. ∎

5.3. Ergodicity. We are now interested in the ergodicity of problem (5.1)–(5.2). Lemma 5.3 states that if μ is the unique invariant measure for P_t, then it is ergodic. Thus, in what follows, we will prove the invariant measure μ in Theorem 5.5 is unique. To this end, the following lemma is needed.

LEMMA 5.6. *Under assumptions of Lemma* 5.4*, additionally, suppose* $2\delta > 2k_4 + \alpha_1 + \alpha_2$*. Let* u *and* v *be two solutions of problem* (5.1)–(5.2) *with initial data* u_0, $v_0 \in L^2(\Omega;\mathbb{H})$*, respectively. Then, we have*

$$\mathbb{E}|u(t)-v(t)|^2 \le \mathbb{E}|u_0 - v_0|^2 e^{-(2\delta-(2k_4+\alpha_1+\alpha_2))t}, \qquad \forall t \ge 0.$$

Proof. Denote by $z=u-v$, $z_0=u_0-v_0$, $z^m=u^m-v^m$. According to Remark 3.3, by similar computations to (3.26), applying Itô's formula to $e^{(2\delta-(2k_4+\alpha_1+\alpha_2))t}|z^m(t)|^2$ and making use of condition (F_3), we obtain

(5.14)
$$\begin{aligned}
&e^{(2\delta-(2k_4+\alpha_1+\alpha_2))t}|z^m(t)|^2+2\delta\int_0^t e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}|z^m(s)|^2ds\\
&\quad\le |z_0|^2+(2\delta-\alpha_1-\alpha_2)\int_0^t e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}|z^m(s)|^2ds\\
&\qquad+2\int_0^t e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}(z^m(s),(g(u^m(s))-g(v^m(s)))dW(s)\\
&\qquad+\int_0^t e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}|g(u^m(s))-g(v^m(s))|^2ds\\
&\qquad+2\int_0^t\int_{E_m} e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}(z^m(s),h(u^m(s-),\xi)-h(v^m(s-),\xi))\tilde N(ds,d\xi)\\
&\qquad+\int_0^t\int_{E_m} e^{(2\delta-(2k_4+\alpha_1+\alpha_2))s}|h(u^m(s-),\xi)-h(v^m(s-),\xi)|^2N(ds,d\xi).
\end{aligned}$$

Taking expectation on both sides of (5.14) and thanks to (5.3)–(5.4), we have

$$\mathbb{E}|z^m(t)|^2\le \mathbb{E}|z_0|^2e^{-(2\delta-(2k_4+\alpha_1+\alpha_2))t},\qquad \forall t\ge 0.$$

Since the sequences $\{u^m\}_{m=1}^\infty$ and $\{v^m\}_{m=1}^\infty$ are converging in $L^2(\Omega;\mathbf{D}([0,T];\mathbb{H}))\cap L^2(\Omega;L^2(0,T;\mathbb{V}))$, so is $\{z^m\}_{m=1}^\infty$. By taking limits on both sides of (5.14), we can conclude the proof of this lemma. ◻

Let us now establish the uniqueness of invariant measure to (5.1)–(5.2) which ensures its ergodicity.

THEOREM 5.7. *Suppose the conditions of Lemma* 5.6 *hold true. Then, for any initial value* $u_0\in L^2(\Omega;\mathbb{H})$*, there exists a unique invariant measure* μ *to problem* (5.1)–(5.2)*. In addition, this measure* μ *is ergodic.*

Proof. Assume there is another invariant measure $\tilde\mu$ for transition operator $(P_t)_{t\ge0}$. Then, for every $\varphi\in\mathrm{Lip}(\mathbb{H})$ (φ is a Lipschitz function with Lipschitz constant L_φ) and initial data u_0, $v_0\in L^2(\Omega;\mathbb{H})$. By means of Definition 5.1 and Lemma 5.6, we have

$$\begin{aligned}
&\left|\int_{\mathbb{H}}\varphi(u_0)\mu(du_0)-\int_{\mathbb{H}}\varphi(v_0)\tilde\mu(dv_0)\right|=\left|\int_{\mathbb{H}}(P_t\varphi)(u_0)\mu(du_0)-\int_{\mathbb{H}}(P_t\varphi)(v_0)\tilde\mu(dv_0)\right|\\
&\quad=\left|\int_{\mathbb{H}}\int_{\mathbb{H}}[(P_t\varphi)(u_0)-(P_t\varphi)(v_0)]\mu(du_0)\tilde\mu(dv_0)\right|\\
&\quad=\left|\int_{\mathbb{H}}\int_{\mathbb{H}}\mathbb{E}[\varphi(u(t,u_0))-\mathbb{E}\varphi(v(t,v_0))]\mu(du_0)\tilde\mu(dv_0)\right|\\
&\quad\le L_\varphi e^{-\frac{(2\delta-(2k_4+\alpha_1+\alpha_2))t}{2}}\int_{\mathbb{H}}\int_{\mathbb{H}}\mathbb{E}|u_0-v_0|\mu(du_0)\tilde\mu(dv_0)\to 0\ \text{ as }\ t\to\infty.
\end{aligned}$$

Since μ is the unique invariant measure for transition operator $(P_t)_{t\ge0}$, by the density of $\mathrm{Lip}(\mathbb{H})$ in $C_b(\mathbb{H})$, we know μ is ergodic. ◻

6. Large deviation principle. In this section, we will consider the following stochastic perturbations of the fractional PDE, where ε is a small parameter,

$$\begin{aligned}
&du^\varepsilon(t)+(-\Delta)^\gamma u^\varepsilon(t)dt+F(u^\varepsilon(t))dt+\delta u^\varepsilon(t)dt\\
&\quad=\sqrt{\varepsilon}g(t,u^\varepsilon(t))dW(t)+\varepsilon\int_E h(t,u^\varepsilon(t-),\xi)\tilde N^{\varepsilon^{-1}}(dt,d\xi),\quad x\in\mathcal{O},\ t>0,
\end{aligned}\tag{6.1}$$

with boundary and initial conditions,

$$u^\varepsilon(t,x)=0, \quad x\in\mathbb{R}^d\backslash\mathcal{O},\ t>0, \qquad \text{and} \qquad u^\varepsilon(0,x)=u_0(x), \quad x\in\mathcal{O}, \tag{6.2}$$

respectively. Assume $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary satisfying $2\gamma < d$ and F satisfies conditions (F_1)–(F_3) with $p+1 \in (2, \frac{2d}{d-2\gamma}]$ such that $\mathbb{V}$ is continuously embedded in $L^{p+1}(\mathcal{O})$ (see [36, Theorem 6.7]). Here E is a locally compact Polish space, W is a cylindrical Brownian motion in U, $N^{\varepsilon^{-1}}$ is a Poisson random measure on $[0,T]\times E$ with a σ-finite intensity measure $\varepsilon^{-1}L_T\otimes\lambda$, L_T is the Lebesgue measure on $[0,T]$, and λ a σ-finite measure on E. $\tilde{N}^{\varepsilon^{-1}}([0,t]\times O) = N^{\varepsilon^{-1}}([0,t]\times O) - \varepsilon^{-1}t\lambda(O)$, for all $O\in\mathcal{B}(E)$ with $\lambda(O)<\infty$, is the compensated Poisson random measure. We emphasize that in this section, $\mathbf{D}([0,T];E)$ denotes the space of right continuous functions with left limits from $[0,T]$ to E endowed with the Skorokhod topology.

Let $\{u^\varepsilon, \varepsilon>0\}$ be a family of random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ taking values in a Polish space $\mathcal{E}$. The large deviation principle of problem (6.1)–(6.2) is concerned with the exponential decay of $\mathbb{P}(u^\varepsilon\in O)$ as $\varepsilon\to 0$.

6.1. Controlled Poisson random measure. Let E be a locally compact Polish space. Denote by $\mathcal{M}_{FC}(E)$ the space of all measures λ on $(E,\mathcal{B}(E))$ such that $\lambda(K)<\infty$ for every compact K in E. Endow $\mathcal{M}_{FC}(E)$ with the weakest topology such that for every $f\in C_c(E)$ (the space of continuous functions with compact support), the function $\lambda\to\langle f,\lambda\rangle=\int_E f(u)d\lambda(u)$ $(\lambda\in\mathcal{M}_{FC}(E))$ is continuous. This topology can be metrized such that $\mathcal{M}_{FC}(E)$ is a Polish space (see, for example, [9]). Fix $T\in(0,\infty)$ and let $E_T=[0,T]\times E$. Fixing a measure $\lambda\in\mathcal{M}_{FC}(E)$, we denote by $\lambda_T=L_T\otimes\lambda$.

We recall that a Poisson random measure $\mathbf{n}$ on E_T with intensity measure λ_T is an $\mathcal{M}_{FC}(E_T)$ valued random variable such that for each $B\in\mathcal{B}(E_T)$ with $\lambda_T(B)<\infty$, $\mathbf{n}(B)$ is Poisson distributed with mean $\lambda_T(B)$ and for disjoint $B_1,\dots,B_k\in\mathcal{B}(E_T)$, $\mathbf{n}(B_1),\dots,\mathbf{n}(B_k)$ are mutually independent random variables (see, for example, [25]). Denote by $\mathbb{P}$ the measure induced by $\mathbf{n}$ on $(\mathcal{M}_{FC}(E_T),\mathcal{B}(\mathcal{M}_{FC}(E_T)))$. Then letting $\mathbb{M}=\mathcal{M}_{FC}(E_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M},\mathcal{B}(\mathbb{M}))$ under which the canonical map, $N:\mathbb{M}\to\mathbb{M}$, $N(m)\doteq m$, is a Poisson random measure with intensity measure λ_T. With applications to large deviations in mind, we also consider, for $\theta>0$, the probability measure $\mathbb{P}_\theta$ on $(\mathbb{M},\mathcal{B}(\mathbb{M}))$ under which N is a Poisson random measure with intensity $\theta\lambda_T$. The corresponding expectation operators will be denoted by $\mathbb{E}$ and $\mathbb{E}_\theta$, respectively.

Set $F=E\times[0,\infty)$ and $F_T=[0,T]\times F$. Similarly, let $\bar{\mathbb{M}}=\mathcal{M}_{FC}(F_T)$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}},\mathcal{B}(\bar{\mathbb{M}})))$ under which the canonical map, $\bar{N}:\bar{\mathbb{M}}\to\bar{\mathbb{M}}$, $\bar{N}(m):=m$, is a Poisson random measure with intensity measure $\bar{\lambda}_T=L_T\otimes\lambda\otimes L_\infty$, with L_∞ being Lebesgue measure on $[0,\infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t:=\sigma\{\bar{N}((0,s]\times O):0\le s\le t, O\in\mathcal{B}(F)\}$ and denote by $\bar{\mathcal{F}}_t$ the completion under $\bar{\mathbb{P}}$. Set $\bar{\mathcal{P}}$ the predictable σ-field on $[0,T]\times\bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t:0\le t\le T\}$ on $(\bar{\mathbb{M}},\mathcal{B}(\bar{\mathbb{M}}))$. Let $\bar{\mathcal{A}}$ be the class of all $(\bar{\mathcal{P}}\times\mathcal{B}(E))/\mathcal{B}[0,\infty)$-measurable maps $\varphi:E_T\times\bar{\mathbb{M}}\to[0,\infty)$. For $\varphi\in\bar{\mathcal{A}}$, define a continuous process N^φ on E_T by,

$$N^\varphi((0,t]\times K)(\cdot)=\int_{(0,t]\times K}\int_{(0,\infty)}1_{[0,\varphi(s,x,\cdot)]}(r)\bar{N}(dsdxdr), \qquad t\in[0,T],\ K\in\mathcal{B}(E). \tag{6.3}$$

N^{φ} is the controlled random measure, with φ selecting the intensity for the points at location x and time s, in a possibly random but nonanticipating way. When $\varphi(s,x,\bar{m}) \equiv \theta \in (0,\infty)$, we write $N^{\varphi} = N^{\theta}$. Note that N^{θ} has the same distribution with respect to $\bar{\mathbb{P}}$ as N has with respect to $\mathbb{P}_{\theta}$.

6.2. Poisson random measure and Brownian motion. Set $\mathbb{W} = C([0,T];U)$, $\mathbb{U} = \mathbb{W} \times \mathbb{M}$ and $\bar{\mathbb{U}} = \mathbb{W} \times \bar{\mathbb{M}}$. Then, for $(w,m) \in \mathbb{U}$, let the mapping $N^{\mathbb{U}} : \mathbb{U} \to \mathbb{M}$ be defined by $N^{\mathbb{U}}(w,m) = m$, and let $W^{\mathbb{U}} : \mathbb{U} \to C([0,T];U)$ be defined by $W^{\mathbb{U}}(w,m)(t) = \sum_{i=1}^{\infty}(w(t),a_i)a_i$, recalling that the sequence $\{a_i\}_{i=1}^{\infty}$ is an orthonormal basis of the separable Hilbert space U. The mappings $\bar{N}^{\bar{\mathbb{U}}} : \bar{\mathbb{U}} \to \bar{\mathbb{M}}$ and $\bar{W}^{\bar{\mathbb{U}}} : \bar{\mathbb{U}} \to C([0,T];U)$ are defined analogously. For every $t \in [0,T]$, define the σ-filtration

$$\mathcal{G}_t^{\mathbb{U}} := \sigma\left(\left\{(W(s), N^{\mathbb{U}}((0,s] \times O)) : 0 \le s \le t, O \in \mathcal{B}(E)\right\}\right).$$

For every $\theta > 0$, for a given $\lambda \in \mathcal{M}_{FC}(E)$, it follows from [25, section I.8] that there exists a unique probability measure $\mathbb{P}_{\theta}^{\mathbb{U}}$ on $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$ such that

(i) W is a cylindrical Brownian motion in U;
(ii) $N^{\mathbb{U}}$ is a Poisson random measure with intensity measure $\theta\lambda_T$;
(iii) W and N are independent.

Analogously, we define $(\bar{\mathbb{P}}_{\theta}^{\bar{\mathbb{U}}}, \bar{\mathcal{G}}_t^{\bar{\mathbb{U}}})$ and denote $\bar{\mathbb{P}}_{\theta=1}^{\bar{\mathbb{U}}}$ by $\bar{\mathbb{P}}^{\bar{\mathbb{U}}}$. We denote by $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{U}}}\}$ the $\bar{\mathbb{P}}^{\bar{\mathbb{U}}}$-completion of $\{\bar{\mathcal{G}}_t^{\bar{\mathbb{U}}}\}$ and $\bar{\mathcal{P}}^{\bar{\mathbb{U}}}$ the predictable σ-field on $[0,T] \times \bar{\mathbb{U}}$ with the filtration $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{U}}}\}$ on $(\bar{\mathbb{U}}, \mathcal{B}(\bar{\mathbb{U}}))$. Let $\mathbb{A}_1$ be the class of all $(\bar{\mathcal{P}}^{\bar{\mathbb{U}}} \otimes \mathcal{B}(E))/\mathcal{B}[0,\infty)$-measurable maps $\varphi : E_T \times \bar{\mathbb{U}} \to [0,\infty)$.

On the one hand, define $l : [0,\infty) \to [0,\infty)$ by $l(r) = r\log r - r + 1,\ r \in [0,\infty)$. Then, for any $\varphi \in \mathbb{A}_1$, the quantity

$$\tilde{L}_T(\varphi)(\omega) = \int_{E_T} l(\varphi(t,x,\omega))\lambda_T(dtdx), \qquad \omega \in \bar{\mathbb{U}}, \tag{6.4}$$

is well defined as a $[0,\infty]$-valued random variable. On the other hand, define

$$\mathbb{A}_2 := \left\{\psi : \psi \text{ is } \bar{\mathcal{P}}^{\bar{\mathbb{U}}}/\mathcal{B}(U) \text{ measurable and } \int_0^T \|\psi(t)\|_U^2 dt < \infty,\ \text{a.s.-}\bar{\mathbb{P}}^{\bar{\mathbb{U}}}\right\}. \tag{6.5}$$

Set $\mathcal{U} = \mathbb{A}_2 \times \mathbb{A}_1$. Define $\hat{L}_T(\psi) := \frac{1}{2}\int_0^T \|\psi(t)\|_U^2 dt$ for $\psi \in \mathbb{A}_2$, and $L_T(u) := \hat{L}_T(\psi) + \tilde{L}_T(\varphi)$ for $u = (\psi,\varphi) \in \mathcal{U}$.

6.3. A general criterion. In this subsection, we recall a general criterion for a large deviation principle established in [9]. Let $\{\mathcal{G}^{\varepsilon}\}_{\varepsilon>0}$ be a family of measurable maps from $\bar{\mathbb{U}}$ to Ξ, where $\bar{\mathbb{U}}$ is introduced in section 6.2 and Ξ is some Polish space. We present below a sufficient condition for the large deviation principle to hold for the family $Z^{\varepsilon} = \mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}W, \varepsilon N^{\varepsilon^{-1}})$ as $\varepsilon \to 0$.

Define for each $\Upsilon \in \mathbb{N}$, $\tilde{\mathrm{S}}^{\Upsilon} = \{\rho : E_T \to [0,\infty) : \tilde{L}_T(\rho) \le \Upsilon\}$, and $\hat{\mathrm{S}}^{\Upsilon} = \{\sigma \in L^2(0,T;U) : \hat{L}_T(\sigma) \le \Upsilon\}$. A function $\rho \in \tilde{\mathrm{S}}^{\Upsilon}$ can be identified with a measure $\lambda_T^{\rho} \in \mathbb{M}$, defined by

$$\lambda_T^{\rho}(O) = \int_O \rho(s,\xi)\lambda_T(dsd\xi), \qquad O \in \mathcal{B}(E_T).$$

This identification induces a topology on $\tilde{\mathrm{S}}^{\Upsilon}$ under which $\tilde{\mathrm{S}}^{\Upsilon}$ is a compact space, see the appendix of [10]. Throughout this paper we use this topology on $\tilde{\mathrm{S}}^{\Upsilon}$. Set $\mathrm{S}^{\Upsilon} = \hat{\mathrm{S}}^{\Upsilon} \times \tilde{\mathrm{S}}^{\Upsilon}$. Define $\mathbb{S} = \cup_{\Upsilon \ge 1}\mathrm{S}^{\Upsilon}$, and let $\mathcal{U}^{\Upsilon} = \{u = (\psi,\varphi) \in \mathcal{U} : u(\omega) \in \mathrm{S}^{\Upsilon},\ \bar{\mathbb{P}}^{\bar{\mathbb{U}}}\text{-a.e. } \omega\}$. The following condition will be sufficient to establish a large deviation principle for a family $\{Z^{\varepsilon}\}_{\varepsilon>0}$ defined by $Z^{\varepsilon} = \mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}W, \varepsilon N^{\varepsilon^{-1}})$.

CONDITION 6.1. *There exists a measurable map $\mathcal{G}^0 : \mathbb{U} \to \Xi$ such that the following hold.*

(a) *For any $\Upsilon \in \mathbb{N}$, let (σ_n, ρ_n), $(\sigma, \rho) \in \mathrm{S}^\Upsilon$ be such that $(\sigma_n, \rho_n) \to (\sigma, \rho)$ as $n \to \infty$. Then,*

$$\mathcal{G}^0\left(\int_0^\cdot \sigma_n(s)ds, \lambda_T^{\rho_n}\right) \to \mathcal{G}^0\left(\int_0^\cdot \sigma(s)ds, \lambda_T^{\rho}\right) \qquad \text{in } \Xi.$$

(b) *For any $\Upsilon \in \mathbb{N}$, let $u_\varepsilon = (\psi_\varepsilon, \varphi_\varepsilon)$, $u = (\psi, \varphi) \in \mathcal{U}^\Upsilon$ such that u_ε converges in distribution to u as $\varepsilon \to 0$. Then,*

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W + \int_0^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\right) \Rightarrow \mathcal{G}^0\left(\int_0^\cdot \psi(s)ds, \lambda_T^{\varphi}\right) \qquad \text{in } \Xi.$$

We use the symbol "$\Rightarrow$" to denote the convergence in distribution.

For $\phi \in \Xi$, define $\mathbb{S}_\phi = \{(\sigma, \rho) \in \mathbb{S} : \phi = \mathcal{G}^0(\int_0^\cdot \sigma(s)ds, \lambda_T^\rho)\}$. Let $I : \Xi \to [0, \infty]$ be defined by

$$I(\phi) = \inf_{\pi = (\sigma, \rho) \in \mathbb{S}_\phi} \{L_T(\pi)\}, \qquad \phi \in \Xi. \tag{6.6}$$

By convention, $I(\phi) = \infty$ if $\mathbb{S}_\phi = \emptyset$.

The following criterion was established in [9, Theorem 4.2].

THEOREM 6.2. *For $\varepsilon > 0$, let Z^ε be defined by $Z^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}W, \varepsilon N^{\varepsilon^{-1}})$ and suppose that Condition* 6.1 *holds. Then, I defined as in* (6.6) *is a rate function on Ξ and the family $\{Z^\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle with rate function I.*

For applications, the following strengthened form of Theorem 6.2 is useful. Let $\{E_m \subset E, m = 1, 2, \cdots\}$ be an increasing sequence of compact sets such that $\cup_{m=1}^\infty E_m = E$. For each m, let

$$\begin{aligned}\mathbb{A}_1^{b,m} := \{&\varphi \in \mathbb{A}_1 : \forall (t, \omega) \in [0, T] \times \bar{\mathbb{M}}, \\ &1/m \le \varphi(t, x, \omega) \le m \text{ if } x \in E_m \text{ and } \varphi(t, x, \omega) = 1 \text{ if } x \in E_m^c\},\end{aligned}$$

and let $\mathbb{A}_1^b = \cup_{m=1}^\infty \mathbb{A}_1^{b,m}$. Define $\tilde{\mathcal{U}}^\Upsilon = \mathcal{U}^\Upsilon \cap \{(\psi, \varphi) : \varphi \in \mathbb{A}_1^b\}$.

THEOREM 6.3. *Suppose Condition* 6.1 *holds with $\mathcal{U}^\Upsilon$ replaced by $\tilde{\mathcal{U}}^\Upsilon$. Then, Theorem* 6.2 *holds true.*

6.4. Hypotheses. In addition to assumptions (F_1)–(F_3) and (g_1)–(g_3) stated above, we impose the following conditions on the jump noise term. Let $h : [0, T] \times \mathbb{H} \times E \to \mathbb{H}$ be a measurable map.

CONDITION 6.4. *For the locally compact Polish space E, there exist $L_h' > 0$ and $C_h' > 0$ such that*

(i) $\int_E |h(t, u_1, \xi) - h(t, u_2, \xi)|^2 \lambda(d\xi) \le L_h' |u_1 - u_2|^2$ *for all $t \in [0, T]$ and $u_1, u_2 \in \mathbb{H}$;*

(ii) $\int_E |h(t, u, \xi)|^2 \lambda(d\xi) \le C_h'(1 + |u|^2)$ *for all $t \in [0, T]$ and $u \in \mathbb{H}$.*

Define

$$\|h(t, \xi)\|_{0,\mathbb{H}} = \sup_{u \in \mathbb{H}} \frac{|h(t, u, \xi)|}{1 + |u|}, \qquad (t, \xi) \in [0, T] \times E,$$

and

$$\|h(t, \xi)\|_{1,\mathbb{H}} = \sup_{u_1, u_2 \in \mathbb{H}, u_1 \ne u_2} \frac{|h(t, u_1, \xi) - h(t, u_2, \xi)|}{|u_1 - u_2|}, \qquad (t, \xi) \in [0, T] \times E.$$

CONDITION 6.5 (exponential integrability). *For $i = 0, 1$, there exists $\delta_1^i > 0$ such that for all $\mathbb{X} \in \mathcal{B}([0, T] \times E)$ satisfying $\lambda_T(\mathbb{X}) < \infty$, it follows that*

$$\int_{\mathbb{X}} e^{\delta_1^i \|h(s,\xi)\|_{i,\mathbb{H}}^2} \lambda(d\xi)ds < \infty.$$

Remark 6.6 (see [55, Remark 3.1]). Condition 6.5 implies that, for every $\delta_2^i > 0$ $(i=0,1)$ and for all $\mathbb{X} \in \mathcal{B}([0,T]\times E)$ satisfying $\lambda_T(\mathbb{X}) < \infty$, we have

$$\int_{\mathbb{X}} e^{\delta_2^i \|h(s,\xi)\|_{i,\mathbb{H}}} \lambda(d\xi)ds < \infty.$$

LEMMA 6.7 (see [55, Lemma 3.1]). *Under Conditions* 6.4 *and* 6.5*:*

(i) *For* $i=0,1$ *and every* $\Upsilon \in \mathbb{N}$,

$$C_{i,1}^{\Upsilon} := \sup_{\rho \in \tilde{\mathbb{S}}^{\Upsilon}} \int_{E_T} \|h(s,\xi)\|_{i,\mathbb{H}} |\rho(s,\xi)-1| \lambda(d\xi)ds < \infty, \tag{6.7}$$

$$C_{i,2}^{\Upsilon} := \sup_{\rho \in \tilde{\mathbb{S}}^{\Upsilon}} \int_{E_T} \|h(s,\xi)\|_{i,\mathbb{H}}^2 (\rho(s,\xi)+1) \lambda(d\xi)ds < \infty; \tag{6.8}$$

(ii) *For every* $\eta > 0$*, there exists* $\delta > 0$ *such that for* $A \subset [0,T]$ *satisfying* $\lambda_T(A) < \delta$,

$$\sup_{\rho \in \tilde{\mathbb{S}}^{\Upsilon}} \int_A \int_E \|h(s,\xi)\|_{i,\mathbb{H}} |\rho(s,\xi)-1| \lambda(d\xi)ds \le \eta. \tag{6.9}$$

LEMMA 6.8 (see [55, Lemma 3.2]). (i) *If* $\sup_{t\in[0,T]} |Y(t)| < \infty$*, for any* $\pi = (\sigma,\rho) \in \mathbb{S}$*, then*

$$g(\cdot, Y(\cdot))\sigma(\cdot) \in L^1(0,T;\mathbb{H}), \quad \int_E h(\cdot, Y(\cdot), \xi)(\rho(\cdot,\xi)-1)\lambda(d\xi) \in L^1(0,T;\mathbb{H}).$$

(ii) *If the family of mappings* $\{Y_n : [0,T] \to \mathbb{H}, n \ge 1\}$ *satisfies*

$$C = \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} |Y_n(t)| < \infty,$$

then

$$C_{\Upsilon} := \sup_{\pi=(\sigma,\rho)\in \mathbb{S}^{\Upsilon}} \sup_{n\in\mathbb{N}} \left[\int_0^T \left| \int_E h(s, Y_n(s), \xi)(\rho(s,\xi)-1)\lambda(d\xi) \right| ds + \int_0^T |g(s, Y_n(s))\sigma(s)| ds \right] < \infty.$$

We also need the following lemma, the proof of which can be found in [10, Lemma 3.11].

LEMMA 6.9. *Let* $d\colon [0,T]\times E \to \mathbb{R}$ *be a measurable function such that*

$$\int_{E_T} |d(s,\xi)|^2 \lambda(d\xi)ds < \infty,$$

and for all $\delta_3 \in (0,\infty)$, $\mathbb{X} \in \mathcal{B}([0,T]\times E)$ *satisfying* $\lambda_T(\mathbb{X}) < \infty$*, it follows that* $\int_{\mathbb{X}} \exp(\delta_3 |d(s,\xi)|)\lambda(d\xi)ds < \infty$.

(i) *Fix* $\Upsilon \in \mathbb{N}$ *and let* $\rho_n, \rho \in \tilde{\mathbb{S}}^{\Upsilon}$ *be such that* $\rho_n \to \rho$ *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty} \int_{E_T} d(s,\xi)(\rho_n(s,\xi)-1)\lambda(d\xi)ds = \int_{E_T} d(s,\xi)(\rho(s,\xi)-1)\lambda(d\xi)ds.$$

(ii) *Fix* $\Upsilon \in \mathbb{N}$*. Given* $\varepsilon > 0$*, there exists a compact set* $K_\varepsilon \subset E$ *such that*

$$\sup_{\rho \in \tilde{\mathbb{S}}^{\Upsilon}} \int_0^T \int_{K_\varepsilon^c} |d(s,\xi)||\rho(s,\xi)-1|\lambda(d\xi)ds \le \varepsilon.$$

(iii) *For every compact $K \subset E$,*

$$\lim_{M\to\infty} \sup_{\rho\in\tilde{\mathbb{S}}^\Upsilon} \int_0^T \int_K |d(s,\xi)| 1_{\{d\geq M\}} \rho(s,\xi)\lambda(d\xi)ds = 0.$$

The previous lemmas will be used together with the following compactness result, which represents a variant of the criterion for compactness stated in [29, Chapter I, Section 5] and [43, Section 13.3], to prove main results of this section. Given $p > 1$, $\alpha \in (0,1)$, let $W^{\alpha,p}(0,T;\mathbb{H})$ be the Sobolev space of all $u \in L^p(0,T;\mathbb{H})$ such that $\int_0^T\int_0^T \frac{|u(t)-u(s)|^p}{|t-s|^{1+\alpha p}}dtds < \infty$, endowed with the norm $\|u\|^p_{W^{\alpha,p}(0,T;\mathbb{H})} = \int_0^T |u(t)|^p dt + \int_0^T\int_0^T \frac{|u(t)-u(s)|^p}{|t-s|^{1+\alpha p}}dtds$.

LEMMA 6.10 (see [21, Theorem 2.1]). *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive with compact embedding of B_0 in B. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let X be the space $X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1)$, endowed with the natural norm. Then, the embedding of X into $L^p(0,T;B)$ is compact.*

6.5. Main results. In this section, assume u_0 is deterministic. Let u^ε be the $\mathbb{H}$-valued solution to (6.1)–(6.2) with initial value u_0. In what follows, we will establish a large deviation principle for $\{u^\varepsilon\}$ as $\varepsilon \to 0$. We start with the following definition.

DEFINITION 6.11. *Let $(\bar{\mathbb{U}}, \mathcal{B}(\bar{\mathbb{U}}), \bar{\mathbb{P}}^{\bar{\mathbb{U}}}, \{\bar{\mathcal{F}}_t^{\bar{\mathbb{U}}}\}_{t\geq 0})$ be a filtered probability space. Suppose u_0 is an $\bar{\mathcal{F}}_0$-measurable $\mathbb{H}$-valued random variable such that $\bar{\mathbb{E}}|u_0|^2 < \infty$. A stochastic process $\{u^\varepsilon(t)\}_{t\in[0,T]}$ defined on $\bar{\mathbb{U}}$ is said to be an $\mathbb{H}$-valued solution to* (6.1)–(6.2) *with initial value u_0 if:*

- *$u^\varepsilon(t)$ is an $\mathbb{H}$-valued $\bar{\mathcal{F}}_t^{\bar{\mathbb{U}}}$-measurable random variable for all $t \in [0,T]$;*
- *$u^\varepsilon(t) \in L^2(\Omega; \mathbf{D}([0,T];\mathbb{H})) \cap L^2(\Omega; L^2(0,T;\mathbb{V})) \cap L^{p+1}(\Omega; L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d)))$, $\bar{\mathbb{P}}^{\bar{\mathbb{U}}}$-a.s.;*
- *For all $t \in [0,T]$,*

(6.10)
$$u^\varepsilon(t) = u_0 - \int_0^t (-\Delta)^\gamma u^\varepsilon(s)ds - \delta\int_0^t u^\varepsilon(s)ds - \int_0^t F(u^\varepsilon(s))ds + \sqrt{\varepsilon}\int_0^t g(s,u^\varepsilon(s))dW(s) + \varepsilon\int_0^t\int_E h(s,u^\varepsilon(s-),\xi)\tilde{N}^{\varepsilon^{-1}}(ds,d\xi),$$

in $L^2(0,T;\mathbb{V}^*) + L^q(0,T;L^q(\mathbb{R}^d))$, where q is the conjugate number of $p+1$.

DEFINITION 6.12. *The stochastic fractional PDE* (6.1)–(6.2) *is said to satisfy the pathwise uniqueness property if any two $\mathbb{H}$-valued solutions u_1^ε and u_2^ε, defined on the same filtered probability space, with respect to the same Poisson random measure and Brownian motion, starting from the same initial condition u_0, coincide almost surely.*

We begin by introducing the mapping $\mathcal{G}^0$ that will be used to define the rate function and verify Condition 6.1. Recall that $\mathbb{S} = \cup_{\Upsilon\geq 1}\mathbb{S}^\Upsilon$. As a first step, we show that under the conditions stated below, for every $\pi = (\sigma,\rho) \in \mathbb{S}$, the deterministic integral equation,

$$u^\pi(t) = u_0 - \int_0^t Au^\pi(s)ds - \int_0^t F(u^\pi(s))ds + \int_0^t g(s,u^\pi(s))\sigma(s)ds + \int_0^t\int_E h(s,u^\pi(s),\xi)(\rho(s,\xi)-1)\lambda(d\xi)ds, \tag{6.11}$$

has a unique continuous solution. Here $\pi = (\sigma,\rho)$ plays the role of a control.

THEOREM 6.13. *Let $u_0 \in \mathbb{H}$ and $\pi = (\sigma, \rho) \in \mathbb{S}$. Suppose (F_1)–(F_3), (g_1)–(g_3), and Conditions 6.4 and 6.5 hold. Then, there exists a unique $u^\pi \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$ such that*

(6.12)

$$u^\pi(t) = u_0 - \int_0^t Au^\pi(s)ds - \int_0^t F(u^\pi(s))ds + \int_0^t g(s,u^\pi(s))\sigma(s)ds + \int_0^t \int_E h(s,u^\pi(s),\xi)(\rho(s,\xi)-1)\lambda(d\xi)ds, \quad \text{in } L^2(0,T;\mathbb{V}^*) + L^q(0,T;L^q(\mathbb{R}^d)).$$

Moreover, for fixed $\Upsilon \in \mathbb{N}$, there exists a constant $C_\Upsilon > 0$ (which depends on Υ) such that

$$\sup_{\pi \in \mathbb{S}^\Upsilon} \left(\sup_{t\in[0,T]} |u^\pi(t)|^2 + \int_0^T \|u^\pi(t)\|^2 dt \right) \le C_\Upsilon. \tag{6.13}$$

Proof. Existence of solutions. Given $n \in \mathbb{N}$, similar to [46], let X_n be the space spanned by $\{e_j, j = 1,2,\ldots,n\}$ and $P_n : \mathbb{H} \to X_n$ be the projection given by $P_n u^\pi = \sum_{j=1}^n (u^\pi, e_j)e_j$ for all $u^\pi \in \mathbb{H}$. We can extend P_n to $\mathbb{V}^*$ and $(L^p(\mathbb{R}^d))^*$ by $P_n\phi = \sum_{j=1}^n (\phi, e_j)e_j$ for all $\phi \in \mathbb{V}^*$ or $\phi \in (L^p(\mathbb{R}^d))^*$. Consider the following Fadeo–Galerkin approximations: $u_n^\pi(t) \in X_n$ denotes the solution of

$$\begin{aligned} du_n^\pi(t) &= -Au_n^\pi(t)dt - P_nF(u_n^\pi(t))dt + P_n g(t,u_n^\pi(t))\sigma(t)dt \\ &\quad + \int_E P_n h(t,u_n^\pi(t),\xi)(\rho(t,\xi)-1)\lambda(d\xi), \qquad \forall t \in (0,T], \end{aligned} \tag{6.14}$$

with initial condition $u_n^\pi(0) := P_n u_0 = u_{n,0}$. We will state the existence and uniqueness of solutions to problem (6.14) for each $n \in \mathbb{N}$. Let $v_0(t) = P_n u_0$ with $t \in [0,T]$. Suppose v_k has been defined for $m-1 \ge k \ge 1$. Define $v_m \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$ as the unique solution of

$$\begin{aligned} dv_m(t) &= -Av_m(t)dt - P_nF(v_m(t))dt + P_n g(t,v_{m-1}(t))\sigma(t)dt \\ &\quad + \int_E P_n h(t,v_{m-1}(t),\xi)(\rho(t,\xi)-1)\lambda(d\xi), \end{aligned} \tag{6.15}$$

and $v_m(0) = P_n u_0$. By slightly modifying the proof of [54, Theorem 3.1], one can verify that the limit u_n^π of v_m, as $m \to \infty$, is the unique solution of (6.14) satisfying $C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$.

Next we will prove that there exists a constant $C > 0$ depending on Υ such that

$$\sup_{n\ge 1} \left(\sup_{t\in[0,T]} |u_n^\pi(t)|^2 + \int_0^T \|u_n^\pi(t)\|^2 dt + \int_0^T \|u_n^\pi(t)\|_{p+1}^{p+1} dt \right) \le C, \tag{6.16}$$

and for $\alpha \in (0,1/2)$, there exists $C_\alpha > 0$ depending on Υ and α such that

$$\sup_{n\ge 1} \|u_n^\pi\|^2_{W^{\alpha,q}(0,T;V^*)} \le C_\alpha. \tag{6.17}$$

Let us first show (6.16). By means of energy estimates, proceeding likewise as in Theorem 3.2, letting $\eta = \min\{\frac{C(d,\gamma)}{2}, \delta\}$, with the help of (F_2), we have

$$\begin{aligned}
&|u_n^\pi(t)|^2 + 2\eta\int_0^t \|u_n^\pi(s)\|^2 ds + 2k_3\int_0^t \|u_n^\pi(s)\|_{p+1}^{p+1} ds \\
&\le |u_{n,0}|^2 + 2\int_0^t (P_n g(s,u_n^\pi(s))\sigma(s), u_n^\pi(s))\, ds \\
&\quad + 2k_2|\mathcal{O}|t + 2\int_0^t \left(P_n\int_E h(s,u_n^\pi(s),\xi)(\rho(s,\xi)-1)\lambda(d\xi), u_n^\pi(s)\right) ds \\
&\le |u_{n,0}|^2 + 2k_2|\mathcal{O}|t + 2\int_0^t \|g(s,u_n^\pi(s))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma(s)\|_U|u_n^\pi(s)|ds \\
&\quad + 2\int_0^t\int_E |h(s,u_n^\pi(s),\xi)||\rho(s,\xi)-1||u_n^\pi(s)|\lambda(d\xi)ds, \qquad \forall t\in[0,T].
\end{aligned} \tag{6.18}$$

On the one hand, it follows from assumption (g_2) and inequality $x\sqrt{1+x^2}\le 1+2x^2$ that,

$$\begin{aligned}
&2\int_0^t \|g(s,u_n^\pi(s))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma(s)\|_U|u_n^\pi(s)|ds \\
&\le 2\sqrt{C_g}\int_0^t \sqrt{1+|u_n^\pi(s)|^2}\|\sigma(s)\|_U|u_n^\pi(s)|ds \\
&\le 2\sqrt{C_g}\int_0^t \|\sigma(s)\|_U ds + 4\sqrt{C_g}\int_0^t |u_n^\pi(s)|^2\|\sigma(s)\|_U ds.
\end{aligned} \tag{6.19}$$

On the other hand,

$$\begin{aligned}
&2\int_0^t\int_E |h(s,u_n^\pi(s),\xi)||\rho(s,\xi)-1||u_n^\pi(s)|\lambda(d\xi)ds \\
&= 2\int_0^t\int_E \frac{|h(s,u_n^\pi(s),\xi)|}{1+|u_n^\pi(s)|}(1+|u_n^\pi(s)|)|\rho(s,\xi)-1||u_n^\pi(s)|\lambda(d\xi)ds \\
&\le 2\int_0^t\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|(1+2|u_n^\pi(s)|^2)\lambda(d\xi)ds \\
&\le 2\int_0^t\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)ds \\
&\quad + 4\int_0^t |u_n^\pi(s)|^2\left(\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)\right)ds.
\end{aligned} \tag{6.20}$$

Hence, it follows from (6.18)–(6.20) that

$$\begin{aligned}
&|u_n^\pi(t)|^2 + 2\eta\int_0^t \|u_n^\pi(s)\|^2 ds + 2k_3\int_0^t \|u_n^\pi(s)\|_{p+1}^{p+1} ds \\
&\le |u_{n,0}|^2 + 2k_2|\mathcal{O}|t + 2\sqrt{C_g}\int_0^t \|\sigma(s)\|_U ds + 2\int_0^t\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)ds \\
&\quad + 4\int_0^t |u_n^\pi(s)|^2\left(\sqrt{C_g}\|\sigma(s)\|_U + \int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)\right)ds.
\end{aligned} \tag{6.21}$$

Taking supremum with respect to $t\in[0,T]$, we have

$$\begin{aligned}\sup_{t\in[0,T]}|u_n^\pi(t)|^2 &\le |u_{n,0}|^2\\&+2k_2|\mathcal{O}|T+2\sqrt{C_g}\int_0^T\|\sigma(s)\|_U ds+2\int_0^T\int_E\|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)ds\\&+4\int_0^T\sup_{t\in[0,s]}|u_n^\pi(t)|^2\left(\sqrt{C_g}\|\sigma(s)\|_U+\int_E\|h(s,\xi)\|_{0,\mathbb{H}}|\rho(s,\xi)-1|\lambda(d\xi)\right)ds.\end{aligned}$$

Using the fact that $\sigma\in L^2(0,T;U)$ and Lemma 6.7, applying the Gronwall lemma to the above inequality, combining with (6.21), we can prove (6.16).

Now we will check (6.17). Notice that

$$\begin{aligned}u_n^\pi(t)&=P_nu_0-\int_0^t Au_n^\pi(s)ds-\int_0^t P_nF(u_n^\pi(s))ds+\int_0^t P_ng(s,u_n^\pi(s))\sigma(s)ds\\&\quad+\int_0^t\int_E P_nh(s,u_n^\pi(s),\xi)(\rho(s,\xi)-1)\lambda(d\xi)ds\\&:=J_n^1+J_n^2(t)+J_n^3(t)+J_n^4(t)+J_n^5(t),\qquad\forall t\in[0,T].\end{aligned}$$

Using the same arguments as [29, Theorem 3.1], there exists a positive constant $\mathcal{C}_1$ such that

$$|J_n^1|^2=|u_{n,0}|^2\le\mathcal{C}_1. \tag{6.22}$$

For $J_n^2(t)$, thanks to the Hölder inequality, we infer there exists a constant $\mathcal{C}_2>0$ such that

$$\begin{aligned}\|J_n^2\|^2_{W^{1,2}(0,T;\mathbb{V}^*)}&=\int_0^T\|J_n^2(t)\|_*^2dt+\int_0^T\left\|\frac{dJ_n^2(t)}{dt}\right\|_*^2dt\\&=\int_0^T\left\|\int_0^t Au_n^\pi(s)ds\right\|_*^2dt+\int_0^T\|Au_n^\pi(t)\|_*^2dt\\&\le(1+T^2)\int_0^T\|Au_n^\pi(t)\|_*^2dt\le\mathcal{C}_2.\end{aligned}\tag{6.23}$$

For $J_n^3(t)$, on the one hand, taking into account of (F_1), we find

$$\begin{aligned}|F(u_n^\pi)|_q^q=\int_{\mathbb{R}^d}|F(u_n^\pi(x))|^qdx=\int_{\mathcal{O}}|F(u_n^\pi(x))|^qdx&\le k_1^q\int_{\mathcal{O}}(1+|u_n^\pi(x)|^p)^q\,dx\\&\le 2^{q-1}k_1^q|\mathcal{O}|+2^{q-1}k_1^q\|u_n^\pi\|_{p+1}^{p+1}.\end{aligned}$$

Since $P_n:(L^{p+1}(\mathbb{R}^d))^*=L^q(\mathbb{R}^d)\to X_n\subset\mathbb{H}$, we know $P_nF(u_n^\pi)\in\mathbb{H}$ for a.a. $t\in[0,T]$. The above estimate, the Hölder inequality and (6.16) imply that there exists a constant $\mathcal{C}_3>0$ such that

$$\begin{aligned}\|J_n^3\|^q_{W^{1,q}(0,T;\mathbb{H})}&=\int_0^T|J_n^3(t)|^qdt+\int_0^T\left|\frac{dJ_n^3(t)}{dt}\right|^qdt=\int_0^T\left|\int_0^t P_nF(u_n^\pi(s))ds\right|^qdt\\&\quad+\int_0^T|P_nF(u_n^\pi(t))|^qdt\\&\le\left(T^{\frac{1}{p}+1}+1\right)\int_0^T\|F(u_n^\pi(s))\|_q^qds\le\mathcal{C}_3.\end{aligned}\tag{6.24}$$

We will do an estimate for J_n^4 now. For $0 \le s < t \le T$, it follows from assumption (g_2), the Hölder inequality and the fact $u_n^\pi \in C([0,T];\mathbb{H})$ that

$$
\begin{aligned}
&|J_n^4(t) - J_n^4(s)|^2 \\
&= \left|\int_s^t P_n g(r, u_n^\pi(r))\sigma(r)dr\right|^2 \le \left(\int_s^t |P_n g(r, u_n^\pi(r))\sigma(r)|dr\right)^2 \\
&\le \left(\int_s^t \|g(r, u_n^\pi(r))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma(r)\|_U dr\right)^2 \le \left(\int_s^t \sqrt{C_g(1+|u_n^\pi(r)|^2)}\|\sigma(r)\|_U dr\right)^2 \\
&\le C_g(t-s)\left(1 + \sup_{t\in[0,T]}|u_n^\pi(t)|^2\right)\int_s^t \|\sigma(r)\|_U^2 dr.
\end{aligned}
$$

By means of the above estimate and the Hölder inequality, for $\alpha \in (0,1/2)$, there exists $\mathcal{C}_4 > 0$ such that

(6.25)

$$
\begin{aligned}
&\|J_n^4\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} = \int_0^T |J_n^4(t)|^2 dt + \int_0^T\int_0^T \frac{|J_n^4(t) - J_n^4(s)|^2}{|t-s|^{1+2\alpha}} ds dt \\
&\le \int_0^T \left|\int_0^t P_n g(s, u_n^\pi(s))\sigma(s)ds\right|^2 dt + C_g\left(1+\sup_{t\in[0,T]}|u_n^\pi(t)|^2\right)\int_0^T\int_0^T\int_s^t \frac{\|\sigma(r)\|_U^2}{|t-s|^{2\alpha}} dr ds dt \\
&\le C_g\left(1 + \sup_{t\in[0,T]}|u_n^\pi(t)|^2\right)\left(T^2 + \frac{T^{2-2\alpha}}{(1-2\alpha)(2-2\alpha)}\int_0^T \|\sigma(t)\|_U^2 dt\right) \le \mathcal{C}_4.
\end{aligned}
$$

For J_n^5, with the help of Lemma 6.7 and the fact that $u_n^\pi \in C([0,T];\mathbb{H})$, for $0 \le s < t \le T$, we derive

$$
\begin{aligned}
|J_n^5(t) - J_n^5(s)|^2 &= \left|\int_s^t P_n \int_E h(r, u_n^\pi(r), \xi)(\rho(r,\xi) - 1)\lambda(d\xi)dr\right|^2 \\
&\le \left(\int_s^t \int_E |h(r, u_n^\pi(r), \xi)||\rho(r,\xi) - 1|\lambda(d\xi)dr\right)^2 \\
&\le \left(\int_s^t \int_E \|h(r,\xi)\|_{0,\mathbb{H}}(1 + |u_n^\pi(r)|)|\rho(r,\xi) - 1|\lambda(d\xi)dr\right)^2 \\
&\le C_{0,1}^\Upsilon \left(1 + \sup_{t\in[0,T]}|u_n^\pi(t)|\right)^2 \int_0^T \int_E \|h(r,\xi)\|_{0,\mathbb{H}}|\rho(r,\xi) - 1|\lambda(d\xi)dr.
\end{aligned}
$$

Using similar arguments as for J_n^4 (cf. (6.25)), we deduce that there exists $\mathcal{C}_5 > 0$ such that

$$
\|J_n^5\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} \le \mathcal{C}_5. \tag{6.26}
$$

Moreover, since $\mathbb{V} \subset L^{p+1}(\mathcal{O}) \subset \mathbb{H} := \mathbb{H} \subset L^q(\mathcal{O}) \subset \mathbb{V}^*$. Immediately, we conclude (6.17) by (6.22)–(6.26).

The estimates (6.16)–(6.17) ensure the existence of an element $u^\pi \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$ and a subsequence $u_{n'}^\pi$ such that, as $n' \to \infty$,

$$
\begin{cases}
u_{n'}^\pi \to u^\pi \text{ weak-star in } L^\infty(0,T;\mathbb{H}); \\
u_{n'}^\pi \to u^\pi \text{ weakly in } L^2(0,T;\mathbb{V}); \\
u_{n'}^\pi \to u^\pi \text{ weakly in } L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d)); \\
u_{n'}^\pi \to u^\pi \text{ strongly in } L^q(0,T;\mathbb{H}); \\
F(u_{n'}^\pi) \to F(u^\pi) \text{ weakly in } L^q(0,T;L^q(\mathbb{R}^d)),
\end{cases} \tag{6.27}
$$

where the strong convergence holds thanks to Lemma 6.10, and the last weak convergence follows from the same arguments as [52, Theorem 2.7], respectively.

Next, we will show u^π is the solution of (6.11). Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T)=0$. For each fixed $n\in\mathbb{N}$, we multiply (6.14) by $\psi(t)e_j$ and then integrate by parts. This leads to the following equation:

$$\begin{aligned}
&-\int_0^T (u^\pi_{n'}(t),e_j)\psi'(t)dt+\int_0^T a(u^\pi_{n'}(t),e_j)\psi(t)dt\\
&\quad+\int_0^T (F(u^\pi_{n'}(t)),e_j)\psi(t)dt=(u_{n',0},e_j)\psi(0)\\
&+\int_0^T (g(t,u^\pi_{n'}(t))\sigma(t),e_j)\psi(t)dt+\int_0^T\left(\int_E h(t,u^\pi_{n'}(t),\xi)(\rho(t,\xi)-1)\lambda(d\xi),e_j\right)\psi(t)dt.
\end{aligned}$$

Taking the limit when $n'\to\infty$ and using (6.27), we deduce

$$\begin{aligned}
\lim_{n'\to\infty}&\left[-\int_0^T (u^\pi_{n'}(t),e_j)\psi'(t)dt+\int_0^T a(u^\pi_{n'}(t),e_j)\psi(t)dt\right.\\
&\left.\quad+\int_0^T (F(u^\pi_{n'}(t)),e_j)\psi(t)dt-(u_{n',0},e_j)\psi(t)\right]\\
&=-\int_0^T (u^\pi(t),e_j)\psi'(t)dt+\int_0^T a(u^\pi(t),e_j)\psi(t)dt\\
&\quad+\int_0^T (F(u^\pi(t)),e_j)\psi(t)dt-(u_0,e_j)\psi(t).
\end{aligned}\tag{6.28}$$

Hence, we only need to check

$$\lim_{n'\to\infty}\int_0^T |g(t,u^\pi_{n'}(t))\sigma(t)-g(t,u^\pi(t))\sigma(t)|\,dt=0,\tag{6.29}$$

and

$$\lim_{n'\to\infty}\int_0^T\int_E |h(t,u^\pi_{n'}(t),\xi)(\rho(t,\xi)-1)-h(t,u^\pi(t),\xi)(\rho(t,\xi)-1)|\,\lambda(d\xi)dt=0.\tag{6.30}$$

On the one hand, for every $\varepsilon>0$, let $A_{n',\varepsilon}=\{t\in[0,T]:|u^\pi_{n'}(t)-u^\pi(t)|>\varepsilon\}$; then by (6.27) and the Chebyshev inequality, we have

$$\lim_{n'\to\infty}L_T(A_{n',\varepsilon})\le\lim_{n'\to\infty}\frac{\int_0^T|u^\pi_{n'}(t)-u^\pi(t)|^2dt}{\varepsilon^2}=0.\tag{6.31}$$

Consider $M=\sup_{i\in\mathbb{N}}\sup_{t\in[0,T]}|u^\pi_i(t)|\vee\sup_{t\in[0,T]}|u^\pi(t)|<\infty$. This assertion holds true thanks to (6.27). Thus, due to assumption (g_1) and the Hölder inequality, we derive

$$\begin{aligned}
&\int_0^T |g(t,u^\pi_{n'}(t))\sigma(t)-g(t,u^\pi(t))\sigma(t)|dt\\
&\quad\le\int_0^T \|g(t,u^\pi_{n'}(t))-g(t,u^\pi(t))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma(t)\|_U\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le \sqrt{L_g}\int_0^T |u_{n'}^\pi(t)-u^\pi(t)|\|\sigma(t)\|_U dt\\
&\le 2M\sqrt{L_g}\int_{A_{n',\varepsilon}}\|\sigma(t)\|_U dt+\varepsilon\sqrt{L_g}\int_{A^c_{n',\varepsilon}}\|\sigma(t)\|_U dt\\
&\le 2M\sqrt{L_T(A_{n',\varepsilon})}\sqrt{L_g}\left(\int_{A_{n',\varepsilon}}\|\sigma(t)\|_U^2 dt\right)^{1/2}+\varepsilon\sqrt{L_gT}\left(\int_{A^c_{n',\varepsilon}}\|\sigma(t)\|_U^2 dt\right)^{1/2}.
\end{aligned}
$$

Thanks to (6.31), the fact $\sigma\in\hat{\mathrm{S}}^{\Upsilon}$, and the above inequality, (6.29) holds. On the other hand, since

$$
\begin{aligned}
&\int_0^T\int_E |h(t,u_{n'}^\pi(t),\xi)-h(t,u^\pi(t),\xi)||\rho(t,\xi)-1|\lambda(d\xi)dt\\
&=\int_0^T\int_E \frac{|h(t,u_{n'}^\pi(t),\xi)-h(t,u^\pi(t),\xi)|}{|u_{n'}^\pi(t)-u^\pi(t)|}|u_{n'}^\pi(t)-u^\pi(t)||\rho(t,\xi)-1|\lambda(d\xi)dt\\
&\le\int_0^T\int_E \|h(t,\xi)\|_{1,\mathbb{H}}|u_{n'}^\pi(t)-u^\pi(t)||\rho(t,\xi)-1|\lambda(d\xi)dt\\
&\le 2M\int_{A_{n',\varepsilon}}\int_E\|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi)dt+\varepsilon\int_{A^c_{n',\varepsilon}}\int_E\|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi)dt.
\end{aligned}
$$

Taking into account (6.31) and Lemma 6.7, together with the above inequality, (6.30) is also proved. Therefore, it follows from (6.28)–(6.30) that, when $n'\to\infty$,

(6.32)

$$
\begin{aligned}
&-\int_0^T(u^\pi(t),e_j)\psi'(t)dt+\int_0^T a(u^\pi(t),e_j)\psi(t)dt+\int_0^T(F(u^\pi(t)),e_j)\psi(t)dt=(u_0,e_j)\psi(0)\\
&+\int_0^T(g(t,u^\pi(t))\sigma(t),e_j)\psi(t)dt+\int_0^T\left(\int_E h(t,u^\pi(t),\xi)(\rho(t,\xi)-1)\lambda(d\xi),e_j\right)\psi(t)dt,
\end{aligned}
$$

which implies u^π is solution of (6.11). Moreover, by means of Lemma 6.8 and using the same arguments as in the proof of [46, Theorem 2.3], we also obtain

$$
\frac{du^\pi(t)}{dt}\in L^2(0,T;\mathbb{V}^*)+L^q(0,T;L^q(\mathbb{R}^d))+L^1(0,T;\mathbb{H}).
$$

Hence, it follows from [29, Lemma 1.2] that $u^\pi\in C([0,T];\mathbb{H})$ and

$$
\frac12\frac{d}{dt}|u^\pi(t)|^2=\left(\frac{du^\pi(t)}{dt},u^\pi(t)\right)_{(\mathbb{V}^*+L^q(\mathbb{R}^d),\mathbb{V}\cap L^{p+1}(\mathbb{R}^d))}.
$$

At last, as u^π is solution of (6.11), (6.27) and (6.16) imply (6.13) holds.

Uniqueness of solution. Eventually, we show that u^π is the unique solution of (6.11). To this end, assume that u_1^π and u_2^π are two solutions of (6.11) with the same initial value u_0. Letting $\mathcal{W}=u_1^\pi-u_2^\pi$, we have

$$
\begin{aligned}
&\frac{d|\mathcal{W}(t)|^2}{dt}+2\eta\|\mathcal{W}(t)\|^2+2(F(u_1^\pi(t))-F(u_2^\pi(t)),\mathcal{W}(t))\\
&\le 2(g(t,u_1^\pi(t))\sigma(t)-g(t,u_2^\pi(t))\sigma(t),\mathcal{W}(t))\qquad(6.33)\\
&\quad+2\int_E(h(t,u_1^\pi(t),\xi)-h(t,u_2^\pi(t),\xi),\mathcal{W}(t))(\rho(t,\xi)-1)\lambda(d\xi).
\end{aligned}
$$

With the help of assumption of (F_3), we arrive at

$$2(F(u_1^\pi(t))-F(u_2^\pi(t)),\mathcal{W}(t))=2\int_{\mathcal{O}}(F(u_1^\pi(t,x))-F(u_2^\pi(t,x)))\mathcal{W}(t,x)dx\ge -2k_4|\mathcal{W}(t)|^2. \tag{6.34}$$

By condition (g_1), we derive

$$\begin{aligned}&2(g(t,u_1^\pi(t))\sigma(t)-g(t,u_2^\pi(t))\sigma(t),\mathcal{W}(t))\\ &\quad\le 2\|g(t,u_1^\pi(t))-g(t,u_2^\pi(t))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma(t)\|_U|\mathcal{W}(t)|\\ &\quad\le 2\sqrt{L_g}|\mathcal{W}(t)|^2\|\sigma\|_U.\end{aligned} \tag{6.35}$$

For the last term, we have

$$\begin{aligned}&2\int_E(h(t,u_1^\pi(t),\xi)-h(t,u_2^\pi(t),\xi),\mathcal{W}(t))(\rho(t,\xi)-1)\lambda(d\xi)\\ &\quad=2\int_E\frac{|h(t,u_1^\pi(t),\xi)-h(t,u_2^\pi(t),\xi)|}{|u_1^\pi(t)-u_2^\pi(t)|}|\mathcal{W}(t)|^2|\rho(t,\xi)-1|\lambda(d\xi)\\ &\quad\le 2|\mathcal{W}(t)|^2\int_E\|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi).\end{aligned} \tag{6.36}$$

Substituting (6.34)–(6.36) into (6.33), we obtain

$$\frac{d|\mathcal{W}|^2}{dt}+2\eta\|\mathcal{W}\|^2\le 2\left(k_4+\sqrt{L_g}\|\sigma\|_U+\int_E\|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi)\right)|\mathcal{W}|^2.$$

The Gronwall lemma and the fact $\sigma\in\hat{\mathrm{S}}^\Upsilon$, together with Lemma 6.7, conclude the proof of uniqueness of solution to (6.11). ∎

We now prove the main result. Recall that for $\pi=(\sigma,\rho)\in\mathbb{S}$, $\lambda_T^\rho(dtd\xi)=\rho(t,\xi)\lambda(d\xi)dt$. Define

$$\mathcal{G}^0\left(\int_0^\cdot\sigma(s)ds,\lambda_T^\rho\right)=u^\pi,\qquad\text{for }\pi=(\sigma,\rho)\in\mathbb{S}\text{ as given in Theorem 6.13.} \tag{6.37}$$

Let $I:\mathbf{D}([0,T];\mathbb{H})\to[0,\infty]$ be defined as in (6.6).

Theorem 6.14 (Main theorem). *Suppose (F_1)–(F_3), (g_1)–(g_3) and Conditions 6.4–6.5 hold. Then, the family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle on $\mathbf{D}([0,T];\mathbb{H})$ with the good rate function I with respect to the topology of uniform convergence.*

Proposition 6.15 (Verifying Condition 6.1(a)). *Fix $\Upsilon\in\mathbb{N}$. Let $\pi_n=(\sigma_n,\rho_n)$, $\pi=(\sigma,\rho)\in\mathrm{S}^\Upsilon$ be such that $\pi_n\to\pi$ as $n\to\infty$. Then, for $\mathcal{G}^0$ defined as in (6.37), we have*

$$\mathcal{G}^0\left(\int_0^\cdot\sigma_n(s)ds,\lambda_T^{\rho_n}\right)\to\mathcal{G}^0\left(\int_0^\cdot\sigma(s)ds,\lambda_T^\rho\right),\qquad\text{in } C([0,T];\mathbb{H}).$$

Proof. By definition (6.37), we know that $\mathcal{G}^0(\int_0^\cdot\sigma_n(s)ds,\lambda_T^{\rho_n})=u^{\pi_n}$. Since $\pi_n\in\mathrm{S}^\Upsilon\subset\mathbb{S}$, using similar arguments as for (6.16)–(6.17), we deduce that there exist two positive constants C_Υ and $C_{\alpha,\Upsilon}$, such that

$$\sup_{t\in[0,T]}|u^{\pi_n}(t)|^2+\int_0^T\|u^{\pi_n}(t)\|^2dt+\int_0^T\|u^{\pi_n}(t)\|_{p+1}^{p+1}dt\le C_\Upsilon, \tag{6.38}$$

and for $\alpha \in (0, \frac{1}{2})$,

$$\|u^{\pi_n}\|^2_{W^{\alpha,q}(0,T;V^*)} \leq C_{\alpha,\Upsilon}. \tag{6.39}$$

Hence, it follows from Lemma 6.10 that there exist an element $\bar{u} \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$ and a subsequence u^{π_n} (relabeled the same) such that, as $n \to \infty$,

$$\begin{cases} u^{\pi_n} \to \bar{u} \text{ weak-star in } L^\infty(0,T;\mathbb{H}); \\ u^{\pi_n} \to \bar{u} \text{ weakly in } L^2(0,T;\mathbb{V}); \\ u^{\pi_n} \to \bar{u} \text{ weakly in } L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d)); \\ u^{\pi_n} \to \bar{u} \text{ strongly in } L^q(0,T;\mathbb{H}); \\ F(u^{\pi_n}) \to F(\bar{u}) \text{ weakly in } L^q(0,T;L^q(\mathbb{R}^d)). \end{cases} \tag{6.40}$$

Next, we will prove $\bar{u} = u^\pi$. Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply $u^{\pi_n}(t)$ by $\psi(t)e_j$, then use integration by parts to obtain,

$$\begin{aligned} &-\int_0^T (u^{\pi_n}(t), e_j)\psi'(t)dt + \int_0^T a(u^{\pi_n}(t), e_j)\psi(t)dt \\ &+ \int_0^T (F(u^{\pi_n}(t)), e_j)\psi(t)dt = (u_0, e_j)\psi(0) \\ &+ \int_0^T (g(t, u^{\pi_n}(t))\sigma_n(t), e_j)\psi(t)dt \\ &+ \int_0^T \left(\int_E h(t, u^{\pi_n}(t), \xi)(\rho_n(t,\xi) - 1)\lambda(d\xi), e_j \right) \psi(t)dt. \end{aligned} \tag{6.41}$$

Set

$$\begin{aligned} I_n^1(T) &= \int_0^T \int_E (h(t, u^{\pi_n}(t), \xi)(\rho_n(t,\xi) - 1), e_j)\, \psi(t)\lambda(d\xi)dt, \\ I_n^2(T) &= \int_0^T \int_E (h(t, u^{\pi_n}(t), \xi)(\rho(t,\xi) - 1), e_j)\, \psi(t)\lambda(d\xi)dt, \\ I(T) &= \int_0^T \int_E (h(t, \bar{u}(t), \xi)(\rho(t,\xi) - 1), e_j)\, \psi(t)\lambda(d\xi)dt. \end{aligned}$$

Thus, we have

$$I_n^1(T) - I(T) = I_n^1(T) - I_n^2(T) + I_n^2(T) - I(T). \tag{6.42}$$

It follows from (6.30) that

$$\lim_{n\to\infty} (I_n^2(T) - I(T)) = 0. \tag{6.43}$$

To obtain the result, it is enough to prove that there exists a subsequence $\{m\}$ of $\{n\}$ such that

$$\lim_{m\to\infty} (I_m^1(T) - I_m^2(T)) = 0. \tag{6.44}$$

Thanks to (6.38) and Lemmas 6.7 and 6.9, we infer that for any given $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset E$ such that

$$\begin{aligned}(6.45)\quad &\int_0^T \int_{K_\varepsilon^c} (h(t,u^{\pi_m}(t),\xi)(\rho_m(t,\xi)-1),e_j)\psi(t)\lambda(d\xi)dt \\ &\le \int_0^T \int_{K_\varepsilon^c} |h(t,u^{\pi_m}(t),\xi)||\rho_m(t,\xi)-1||\psi(t)|\lambda(d\xi)dt \\ &\le \int_0^T \int_{K_\varepsilon^c} \|h(t,\xi)\|_{0,\mathbb{H}}(1+|u^{\pi_m}(t)|)|\rho_m(t,\xi)-1||\psi(t)|\lambda(d\xi)dt \\ &\le \left(1+\sup_{t\in[0,T]}|u^{\pi_m}(t)|\right)\sup_{t\in[0,T]}|\psi(t)|\int_0^T \int_{K_\varepsilon^c} \|h(t,\xi)\|_{0,\mathbb{H}}|\rho_m(t,\xi)-1|\lambda(d\xi)dt \\ &\le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon,\end{aligned}$$

and

$$(6.46)\quad \int_0^T \int_{K_\varepsilon^c} (h(t,u^{\pi_m}(t),\xi)(\rho(t,\xi)-1),e_j)\psi(t)\lambda(d\xi)dt \le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon.$$

To prove (6.44), applying a diagonal principle, it suffices to show that, for every compact $K \subset E$ and $\eta = 2\sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon > 0$, there exists a subsequence $\{m\}$ (denoted the same) such that

$$\begin{aligned}(6.47)\quad &\lim_{m\to\infty}\left|\int_0^T \int_K (h(t,u^{\pi_m}(t),\xi)(\rho_m(t,\xi)-1),e_j)\psi(t)\lambda(d\xi)dt\right. \\ &\qquad \left.-\int_0^T \int_K (h(t,u^{\pi_m}(t),\xi)(\rho(t,\xi)-1),e_j)\psi(t)\lambda(d\xi)dt\right| \\ &= \lim_{m\to\infty}\left|\int_0^T \int_K (h(t,u^{\pi_m}(t),\xi)\rho_m(t,\xi),e_j)\psi(t)\lambda(d\xi)dt\right. \\ &\qquad \left.-\int_0^T \int_K (h(t,u^{\pi_m}(t),\xi)\rho(t,\xi),e_j)\psi(t)\lambda(d\xi)dt\right| \le \eta.\end{aligned}$$

Denote $A_M = \{(t,\xi) \in [0,T]\times K : \|h(t,\xi)\|_{0,\mathbb{H}} \ge M\}$. By Lemma 6.9(iii) and (6.38), for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\begin{aligned}(6.48)\quad &\int_0^T \int_K |(h(t,u^{\pi_m}(t),\xi)\rho_m(t,\xi),e_j)||\psi(t)|1_{A_M}\lambda(d\xi)dt \\ &\le \int_0^T \int_K |h(t,u^{\pi_m}(t),\xi)|\rho_m(t,\xi)|\psi(t)|1_{A_M}\lambda(d\xi)dt \\ &\le \left(1+\sup_{t\in[0,T]}|u^{\pi_m}(t)|\right)\sup_{t\in[0,T]}|\psi(t)|\int_0^T \int_K \rho_m(t,\xi)\|h(t,\xi)\|_{0,\mathbb{H}}1_{A_M}\lambda(d\xi)dt \\ &\le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon,\end{aligned}$$

and

$$(6.49)\quad \int_0^T \int_K |(h(t,u^{\pi_m}(t),\xi)\rho(t,\xi),e_j)||\psi(t)|1_{A_M}\lambda(d\xi)dt \le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon.$$

Denote $H_m(t,\xi) = (h(t,u^{\pi_m}(t),\xi),e_j)\psi(t)$ and $H(t,\xi) = (h(t,\bar{u}(t),\xi),e_j)\psi(t)$. Then, we have

$$
\begin{aligned}
|H_m(t,\xi)1_{A_M^c}| &\le |h(t,u^{\pi_m}(t),\xi)||\psi(t)|1_{A_M^c} \\
&\le \left(1+\sup_{t\in[0,T]}|u^{\pi_m}(t)|\right)\sup_{t\in[0,T]}|\psi(t)|\|h(t,\xi)\|_{0,\mathbb{H}}1_{A_M^c} \\
&\le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)M,
\end{aligned} \tag{6.50}
$$

and $|H(t,\xi)1_{A_M^c}| \le \sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)M$.

Let $\Theta(\cdot) = \frac{\lambda_T(\cdot\cap[0,T]\times K)}{\lambda_T([0,T]\times K)}$ be a probability measure on $[0,T]\times E$. It follows from [55, Proposition 4.1] that there exists a subsequence, denoted the same, such that $\lim_{m\to\infty} H_m = H$, Θ-a.s. Therefore, using arguments similar to those in [10, Lemma 3.4], together with [4, Lemma 2.8] and (6.50), we know there exists a subsequence $\{m'\}$ of $\{m\}$, such that

$$
\lim_{m'\to\infty}\int_0^T\int_K H_{m'}(t,\xi)\rho_{m'}(t,\xi)1_{A_M^c}\lambda(d\xi)dt = \int_0^T\int_K H(t,\xi)\rho(t,\xi)1_{A_M^c}\lambda(d\xi)dt, \tag{6.51}
$$

and

$$
\lim_{m'\to\infty}\int_0^T\int_K H_{m'}(t,\xi)\rho(t,\xi)1_{A_M^c}\lambda(d\xi)dt = \int_0^T\int_K H(t,\xi)\rho(t,\xi)1_{A_M^c}\lambda(d\xi)dt.
$$

Hence, the above inequality, (6.48)–(6.49) and (6.51) imply (6.47). Moreover, by (6.45)–(6.47), we obtain

$$
\lim_{m'\to\infty}|I^1_{m'}(T) - I^2_{m'}(T)| \le 4\sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon.
$$

Thus, (6.44) follows immediately and there exists a subsequence of $\{m'\}$ (still denoted the same) such that

$$
\lim_{m'\to\infty} I^1_{m'}(T) = I(T). \tag{6.52}
$$

Let us proceed likewise as before. We infer that

$$
\lim_{m'\to\infty}\int_0^T (g(t,u^{\pi_{m'}}(t))\sigma_{m'}(t),e_j)\psi(t)dt = \int_0^T (g(t,\bar{u}(t))\sigma(t),e_j)\psi(t)dt. \tag{6.53}
$$

By (6.41) and (6.52)–(6.53), using the same arguments as in (6.32), we see $\bar{u}$ satisfies

$$
\begin{aligned}
&-\int_0^T(\bar{u}(t),e_j)\psi'(t)dt + \int_0^T a(\bar{u}(t),e_j)\psi(t)dt + \int_0^T (F(\bar{u}(t)),e_j)\psi(t)dt = (u_0,e_j)\psi(0) \\
&+\int_0^T (g(t,\bar{u}(t))\sigma(t),e_j)\psi(t)dt + \int_0^T\left(\int_E h(t,\bar{u}(t),\xi)(\rho(t,\xi)-1)\lambda(d\xi),e_j\right)\psi(t)dt.
\end{aligned} \tag{6.54}
$$

Based on the uniqueness of solution to problem (6.11), we conclude that $\bar{u} = u^\pi$.

At last, we will prove $u^{\pi_n} \to u^\pi$ in $C([0,T];\mathbb{H})$ as $n \to \infty$. Let $\mathcal{W}_n = u^{\pi_n} - u^\pi$. Then

$$
\begin{aligned}
&\frac{d|\mathcal{W}_n(t)|^2}{dt} + 2\eta\|\mathcal{W}_n(t)\|^2 + 2(F(u^{\pi_n}(t)) - F(u^\pi(t)), \mathcal{W}_n(t)) \\
&\quad \le 2\,(g(t,u^{\pi_n}(t))\sigma_n(t) - g(t,u^\pi(t))\sigma(t), \mathcal{W}_n(t)) \\
&\qquad + 2\int_E (h(t,u^{\pi_n}(t),\xi)(\rho_n(t,\xi)-1) - h(t,u^\pi(t),\xi)(\rho(t,\xi)-1), \mathcal{W}_n(t))\,\lambda(d\xi) \\
&\quad \le 2\,(g(t,u^{\pi_n}(t))\sigma_n(t) - g(t,u^{\pi_n}(t))\sigma(t), \mathcal{W}_n(t)) \\
&\qquad + 2\,((g(t,u^{\pi_n}(t))\sigma(t) - g(t,u^\pi(t))\sigma(t), \mathcal{W}_n(t)) \\
&\qquad + 2\int_E ((h(t,u^{\pi_n}(t),\xi)(\rho_n(t,\xi)-1) - h(t,u^{\pi_n}(t),\xi)(\rho(t,\xi)-1), \mathcal{W}_n(t))\,\lambda(d\xi) \\
&\qquad + 2\int_E (h(t,u^{\pi_n}(t),\xi)(\rho(t,\xi)-1) - h(t,u^\pi(t),\xi)(\rho(t,\xi)-1), \mathcal{W}_n(t))\,\lambda(d\xi) \\
&\quad := I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}
$$

Similar to estimates (6.34)–(6.36), we have

$$
\begin{aligned}
-2(F(u^{\pi_n}(t)) - F(u^\pi(t)), \mathcal{W}_n(t)) &\le 2k_4|\mathcal{W}_n(t)|^2, \\
I_2^n(t) &\le 2\sqrt{L_g}\|\sigma(t)\|_U|\mathcal{W}_n(t)|^2, \\
I_4^n(t) &\le 2|\mathcal{W}_n(t)|^2\int_E \|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi).
\end{aligned}
$$

Subsequently, collecting all the estimates above, we obtain

$$
\frac{d|\mathcal{W}_n(t)|^2}{dt} + 2\eta\|\mathcal{W}_n(t)\|^2 \le \aleph(t)|\mathcal{W}_n(t)|^2 + I_1^n(t) + I_3^n(t), \tag{6.55}
$$

where we have used the notation

$$
\aleph(t) = 2k_4 + 2\sqrt{L_g}\|\sigma(t)\|_U + 2\int_E \|h(t,\xi)\|_{1,\mathbb{H}}|\rho(t,\xi)-1|\lambda(d\xi) \in L^1(0,T).
$$

Multiplying (6.55) by $e^{-\int_0^t \aleph(s)ds}$ and integrating it from 0 to t, we obtain

$$
e^{-\int_0^t \aleph(s)ds}|\mathcal{W}_n(t)|^2 \le \int_0^t e^{-\int_0^s \aleph(r)dr}(I_1^n(s) + I_3^n(s))ds \le \int_0^t (|I_1^n(s)| + |I_3^n(s)|)ds,
$$

which implies

$$
\sup_{t\in[0,T]}|\mathcal{W}_n(t)|^2 \le \exp\left(\int_0^T \aleph(t)dt\right)\int_0^T (|I_1^n(t)| + |I_3^n(t)|)dt. \tag{6.56}
$$

Since $u \in C([0,T];\mathbb{H})$ and $u^{\pi_n} \in C([0,T];\mathbb{H})$ for each $n \in \mathbb{N}$ (see Theorem 6.13), together with the facts that $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|u^{\pi_n}(t)| \le C_\Upsilon$, $\sup_{t\in[0,T]}|u^\pi(t)| \le C_\Upsilon$ and (6.40), we know $u^{\pi_n} \to u^\pi$ in $L^2(0,T;\mathbb{H})$ as $n \to \infty$. By condition (g_2) and the Hölder inequality, we find

$$
\begin{aligned}
&\int_0^T |I_1^n(t)|dt \\
&\leq 2\int_0^T \|g(t,u^{\pi_n}(t))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\sigma_n(t)-\sigma(t)\|_U|\mathcal{W}_n(t)|dt \\
&\leq 2\sqrt{C_g}\int_0^T \sqrt{1+|u^{\pi_n}(t)|^2}(\|\sigma_n(t)-\sigma(t)\|_U)|\mathcal{W}_n(t)|dt \\
&\leq 2\sqrt{C_g(1+C_\Upsilon^2)}\left(\int_0^T (\|\sigma_n(t)\|_U+\|\sigma(t)\|_U)^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |\mathcal{W}_n(t)|^2dt\right)^{\frac{1}{2}} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}
$$

Similarly, for $I_3^n(t)$, it follows from Condition 6.4(ii), Lemma 6.7(i) and the Lebesgue dominated theorem that

$$
\begin{aligned}
\int_0^T |I_3^n(t)|dt &\leq 2\int_0^T\int_E (h(t,u^{\pi_n}(t),\xi)(\rho_n(t,\xi)-1),\mathcal{W}_n(t))\lambda(d\xi)dt \\
&\quad + 2\int_0^T\int_E (h(t,u^{\pi_n}(t),\xi)(\rho(t,\xi)-1),\mathcal{W}_n(t))\lambda(d\xi)dt \\
&\leq 2\int_0^T\int_E \|h(t,\xi)\|_{0,\mathbb{H}}(1+|u^{\pi_n}(t)|)|\rho_n(t,\xi)-1||\mathcal{W}_n(t)|\lambda(d\xi)dt \\
&\quad + 2\int_0^T\int_E \|h(t,\xi)\|_{0,\mathbb{H}}(1+|u^{\pi}(t)|)|\rho(t,\xi)-1||\mathcal{W}_n(t)|\lambda(d\xi)dt \\
&\leq 2(1+C_\Upsilon)\int_0^T\int_E \|h(t,\xi)\|_{0,\mathbb{H}}|\rho_n(t,\xi)-1||\mathcal{W}_n(t)|\lambda(d\xi)dt \\
&\quad + 2(1+C_\Upsilon)\int_0^T\int_E \|h(t,\xi)\|_{0,\mathbb{H}}|\rho(t,\xi)-1||\mathcal{W}_n(t)|\lambda(d\xi)dt \longrightarrow 0, \text{as } n\to\infty.
\end{aligned}
$$

Hence, by (6.56), we obtain $\lim_{n\to\infty}\sup_{t\in[0,T]}|u^{\pi_n}(t)-u^{\pi}(t)|=0$, which implies $u^{\pi_n}\to u^{\pi}$ in $C([0,T];\mathbb{H})$. The proof of this proposition is complete. ∎

THEOREM 6.16. *Assume* (F_1)–(F_3), (g_1)–(g_3) *and Condition* 6.4 *hold. If* $u_0\in\mathbb{H}$, *there exists a unique* $\mathbb{H}$*-valued progressively measurable process such that* $u^\varepsilon\in L^2(0,T;\mathbb{V})\cap\mathbf{D}([0,T];\mathbb{H})\cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$ *for any* $T>0$, *and*

$$
\begin{aligned}
u^\varepsilon(t) &= u_0 - \int_0^t (-\Delta)^\gamma u^\varepsilon(s)ds - \delta\int_0^t u^\varepsilon(s)ds - \int_0^t F(u^\varepsilon(s))ds \\
&\quad + \sqrt{\varepsilon}\int_0^t g(s,u^\varepsilon(s))dW(s) + \varepsilon\int_0^t\int_E h(s,u^\varepsilon(s-),\xi)\tilde{N}^{\varepsilon^{-1}}(dsd\xi), \quad a.s.
\end{aligned} \tag{6.57}
$$

This theorem can be proved similarly as [5, Theorem 1.2] since $(-\Delta)^\gamma$ is a linear operator, showing (6.57) admits a strong solution (in the probability sense). In particular, for every $\varepsilon>0$, there exists a measurable map $\mathcal{G}^\varepsilon:\bar{\mathbb{U}}\to D([0,T];\mathbb{H})$ such that, for any Poisson random measure $n^{\varepsilon^{-1}}$ on $[0,T]\times E$ with intensity measure $\varepsilon^{-1}L_T\otimes\lambda$ given in some probability space, $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}W,\varepsilon n^{\varepsilon^{-1}})$ is the unique solution of (6.57) with $\tilde{N}^{\varepsilon^{-1}}$ replaced by $\tilde{n}^{\varepsilon^{-1}}$.

We have the following lemma introduced in [9, Lemma 2.3].

LEMMA 6.17. *Let* $\phi_\varepsilon=(\psi_\varepsilon,\varphi_\varepsilon)\in\tilde{\mathcal{U}}^\Upsilon$ *and* $\lambda_\varepsilon=\frac{1}{\varphi_\varepsilon}$. *Then,*

$$
\begin{aligned}
\tilde{\mathcal{E}}_t^\varepsilon(\lambda_\varepsilon) := \exp\Bigg\{&\int_{[0,t]\times E\times[0,\varepsilon^{-1}]}\log(\lambda_\varepsilon(s,x))\bar{N}(dsdxdr) \\
&+ \int_{[0,t]\times E\times[0,\varepsilon^{-1}]}(-\lambda_\varepsilon(s,x)+1)\bar{\lambda}_T(dsdxdr)\Bigg\}
\end{aligned}
$$

and

$$\hat{\mathcal{E}}_t^\varepsilon(\psi_\varepsilon) := \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^t \psi_\varepsilon(s)dW(s) - \frac{1}{2\varepsilon}\int_0^t \|\psi_\varepsilon(s)\|_U^2 ds\right\}$$

are $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{U}}}\}$*-martingales. Set* $\mathcal{E}_t^\varepsilon(\psi_\varepsilon,\lambda_\varepsilon) := \hat{\mathcal{E}}_t^\varepsilon(\psi_\varepsilon)\tilde{\mathcal{E}}_t^\varepsilon(\lambda_\varepsilon)$*. Then*

$$\mathbb{Q}_t^\varepsilon(G) = \int_G \mathcal{E}_t^\varepsilon(\psi_\varepsilon,\lambda_\varepsilon)d\bar{\mathbb{P}}^{\bar{\mathbb{U}}}, \quad G \in \mathcal{B}(\bar{\mathbb{U}}),$$

defines a probability measure on $\bar{\mathbb{U}}$.

Since $(\sqrt{\varepsilon}W + \int_0^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ under $\mathbb{Q}_T^\varepsilon$ has the same law as that of $(\sqrt{\varepsilon}W, \varepsilon N^{\varepsilon^{-1}})$ under $\bar{\mathbb{P}}^{\bar{\mathbb{U}}}$, there exists a unique solution $\tilde{u}^\varepsilon$ to the following controlled stochastic fractional differential equation:

$$\begin{aligned}
\tilde{u}^\varepsilon(t) &= u_0 - \int_0^t (-\Delta)^\gamma \tilde{u}^\varepsilon(s)ds - \delta\int_0^t \tilde{u}^\varepsilon(s)ds - \int_0^t F(\tilde{u}^\varepsilon(s))ds \\
&\quad + \int_0^t g(s,\tilde{u}^\varepsilon(s))\psi_\varepsilon(s)ds + \sqrt{\varepsilon}\int_0^t g(s,\tilde{u}^\varepsilon(s))dW(s) \\
&\quad + \varepsilon\int_0^t\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)\left(N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi) - \varepsilon^{-1}\lambda(d\xi)ds\right) \\
&= u_0 - \int_0^t (-\Delta)^\gamma \tilde{u}^\varepsilon(s)ds - \delta\int_0^t \tilde{u}^\varepsilon(s)ds - \int_0^t F(\tilde{u}^\varepsilon(s))ds \\
&\quad + \int_0^t g(s,\tilde{u}^\varepsilon(s))\psi_\varepsilon(s)ds + \sqrt{\varepsilon}\int_0^t g(s,\tilde{u}^\varepsilon(s))dW(s) \\
&\quad + \int_0^t\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds \\
&\quad + \varepsilon\int_0^t\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)\left(N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi) - \varepsilon^{-1}\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds\right).
\end{aligned} \tag{6.58}$$

Moreover, we have

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W + \int_0^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\right) = \tilde{u}^\varepsilon. \tag{6.59}$$

The following estimates will be used later.

LEMMA 6.18. *Assume* (F_1)*–*(F_3)*,* (g_1)*–*(g_3) *and Condition* 6.4 *hold. Let* $u_0 \in \mathbb{H}$*. Then there exists* $0 < \varepsilon_0 \le \frac{1}{(1+16C_b^2C_gT+32C_b^2C_{0,2}^\Upsilon)^2}$*, such that*

$$\sup_{0<\varepsilon<\varepsilon_0}\left[\bar{\mathbb{E}}\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2 + \bar{\mathbb{E}}\int_0^T \|\tilde{u}^\varepsilon(t)\|^2 dt + \bar{\mathbb{E}}\int_0^T \|\tilde{u}^\varepsilon(t)\|_{p+1}^{p+1}dt\right] < \infty, \tag{6.60}$$

where C_b *is the constant obtained from the Burkholder–Davis–Gundy inequality and* $\bar{\mathbb{E}}$ *is the expectation operator corresponding to* $\bar{\mathbb{P}} := \bar{\mathbb{P}}^{\bar{\mathbb{U}}}$*. Moreover, for* $\alpha \in (0,1/2)$*, there exists* $C_\alpha > 0$ *such that*

$$\sup_{0<\varepsilon<\varepsilon_0}\bar{\mathbb{E}}\|\tilde{u}^\varepsilon\|_{W^{\alpha,q}(0,T;V^*)}^2 \le C_\alpha. \tag{6.61}$$

Thus, the family $\{\tilde{u}^\varepsilon, 0 < \varepsilon < \varepsilon_0\}$ *is tight in* $L^q(0,T;\mathbb{H})$.

Proof. The details of the proof of (6.60) are given in the appendix. Notice that (6.58) is equivalent to

$$\begin{aligned}\tilde u^\varepsilon(t) = u_0 &- \left(\int_0^t (-\Delta)^\gamma \tilde u^\varepsilon(s)ds + \delta\int_0^t \tilde u^\varepsilon(s)ds\right) - \int_0^t F(\tilde u^\varepsilon(s))ds\\ &+ \int_0^t g(s,\tilde u^\varepsilon(s))\psi_\varepsilon(s)ds + \sqrt{\varepsilon}\int_0^t g(s,\tilde u^\varepsilon(s))dW(s)\\ &+ \int_0^t\int_E h(s,\tilde u^\varepsilon(s-),\xi)(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds\\ &+ \varepsilon\int_0^t\int_E h(s,\tilde u^\varepsilon(s-),\xi)\tilde N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi)\\ := J^1 &+ J^2_\varepsilon(t) + J^3_\varepsilon(t) + J^4_\varepsilon(t) + J^5_\varepsilon(t) + J^6_\varepsilon(t) + J^7_\varepsilon(t).\end{aligned}$$

By the same arguments as in the proof of [21, Theorem 3.1], we know there exists $C^1 > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}|J^1|^2 \le C^1. \tag{6.62}$$

For J^2_ε, using the same method as in [46, Theorem 2.3] and the Hölder inequality, we infer there exists a constant $C^2 > 0$ such that

(6.63)
$$\begin{aligned}\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\|J^2_\varepsilon\|^2_{W^{1,2}(0,T;\mathbb{V}^*)} &= \sup_{0<\varepsilon<\varepsilon_0}\left(\bar{\mathbb{E}}\int_0^T\left\|\int_0^t A\tilde u^\varepsilon(s)ds\right\|_*^2 dt + \bar{\mathbb{E}}\int_0^T \|A\tilde u^\varepsilon(t)\|_*^2 dt\right)\\ &\le \sup_{0<\varepsilon<\varepsilon_0}(T^2+1)\int_0^T \bar{\mathbb{E}}\|A\tilde u^\varepsilon(t)\|_*^2 dt \le C^2.\end{aligned}$$

For J^3_ε, similar to (6.24), by condition (F_1) and the Hölder inequality, we know there exists a constant $C^3 > 0$ such that

$$\begin{aligned}&\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\|J^3_\varepsilon\|^q_{W^{1,q}(0,T;L^q(\mathcal{O}))}\\ &\quad= \sup_{0<\varepsilon<\varepsilon_0}\left(\bar{\mathbb{E}}\int_0^T\left\|\int_0^t F(\tilde u^\varepsilon(s))ds\right\|^q_{L^q(\mathcal{O})} dt + \bar{\mathbb{E}}\int_0^T \|F(\tilde u^\varepsilon(t))\|^2_{L^q(\mathcal{O})}dt\right)\\ &\quad\le \sup_{0<\varepsilon<\varepsilon_0}\left(T^{\frac{1}{p}+1}+1\right)\bar{\mathbb{E}}\int_0^T \|F(\tilde u^\varepsilon(t))\|_q^q dt \le C^3.\end{aligned} \tag{6.64}$$

To estimate J^4_ε, we apply condition (g_2) and the Hölder inequality for $0\le s<t\le T$,

$$\begin{aligned}\bar{\mathbb{E}}|J^4_\varepsilon(t) - J^4_\varepsilon(s)|^2 &= \bar{\mathbb{E}}\left|\int_s^t g(r,\tilde u^\varepsilon(r))\psi_\varepsilon(r)dr\right|^2 \le \bar{\mathbb{E}}\left(\int_s^t |g(r,\tilde u^\varepsilon(r))\psi_\varepsilon(r)|dr\right)^2\\ &\le \bar{\mathbb{E}}\left(\int_s^t \|g(r,\tilde u^\varepsilon(r))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\psi_\varepsilon(r)\|_U dr\right)^2\\ &\le C_g\bar{\mathbb{E}}\left(\int_s^t \sqrt{1+|\tilde u^\varepsilon(r)|^2}\|\psi_\varepsilon(r)\|_U dr\right)^2\\ &\le 2C_g\Upsilon(t-s)\bar{\mathbb{E}}\left(1+\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right),\end{aligned}$$

and the last inequality holds since $(\psi_\varepsilon, \varphi_\varepsilon) \in \tilde{\mathcal{U}}^\Upsilon$. Consequently, by the above estimate and the Hölder inequality, for $\alpha \in (0, 1/2)$, we have

$$\begin{aligned}
\bar{\mathbb{E}}\|J_\varepsilon^4\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} &= \bar{\mathbb{E}}\int_0^T |J_\varepsilon^4(t)|^2 dt + \bar{\mathbb{E}}\int_0^T\int_0^T \frac{|J_\varepsilon^4(t) - J_\varepsilon^4(s)|^2}{|t-s|^{1+2\alpha}} dsdt\\
&\le T^2\bar{\mathbb{E}}\int_0^T \|g(r, \tilde{u}^\varepsilon(r))\|^2_{\mathcal{L}_2(U;\mathbb{H})}\|\psi_\varepsilon(r)\|_U^2 dr\\
&\quad + 2C_g\Upsilon\bar{\mathbb{E}}\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\int_0^T\int_0^T \frac{1}{|t-s|^{2\alpha}} dsdt\\
&\le 2C_g\Upsilon\bar{\mathbb{E}}\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\left(T^2 + \frac{1}{(1-2\alpha)(2-2\alpha)}T^{2-2\alpha}\right).
\end{aligned}$$

Therefore, there exists a constant $\mathrm{C}^4 > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\|J_\varepsilon^4\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} \le \mathrm{C}^4. \tag{6.65}$$

For J_ε^5, similar to J_ε^4, by Itô's isometry and condition (g_2), for $0 \le s < t \le T$, we find

$$\begin{aligned}
\bar{\mathbb{E}}|J_\varepsilon^5(t) - J_\varepsilon^5(s)| = \bar{\mathbb{E}}\left|\sqrt{\varepsilon}\int_s^t g(r, \tilde{u}^\varepsilon(r))dW(r)\right|^2 &\le \varepsilon\bar{\mathbb{E}}\int_s^t \|g(r, \tilde{u}^\varepsilon(r))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dr\\
&\le \varepsilon C_g(t-s)\bar{\mathbb{E}}\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right).
\end{aligned}$$

Thus, for $\alpha \in (0, 1/2)$, there exists a constant $\mathrm{C}^5 > 0$ such that

(6.66)

$$\begin{aligned}
\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\|J_\varepsilon^5\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} &\le \sup_{0<\varepsilon<\varepsilon_0}\left(\bar{\mathbb{E}}\int_0^T |J_\varepsilon^5(t)|^2 dt + \bar{\mathbb{E}}\int_0^T\int_0^T \frac{|J_\varepsilon^5(t) - J_\varepsilon^5(s)|^2}{|t-s|^{1+2\alpha}} dsdt\right)\\
&\le \sup_{0<\varepsilon<\varepsilon_0}\left[\varepsilon C_g\bar{\mathbb{E}}\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\left(T^2 + \frac{T^{2-2\alpha}}{(1-2\alpha)(2-2\alpha)}\right)\right] \le \mathrm{C}^5.
\end{aligned}$$

For J_ε^6 and $0 \le s < t \le T$, we have

$$\begin{aligned}
&\bar{\mathbb{E}}|J_\varepsilon^6(t) - J_\varepsilon^6(s)|^2\\
&\quad = \bar{\mathbb{E}}\left|\int_s^t\int_E h(r, \tilde{u}^\varepsilon(r-), \xi)(\varphi_\varepsilon(r,\xi) - 1)\lambda(d\xi)dr\right|^2\\
&\quad \le \bar{\mathbb{E}}\left(\int_s^t\int_E \|h(r,\xi)\|_{0,\mathbb{H}}(1 + |\tilde{u}^\varepsilon(r)|)|\varphi_\varepsilon(r,\xi) - 1|\lambda(d\xi)dr\right)^2\\
&\quad \le \bar{\mathbb{E}}\left[\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|\right)^2\left(\int_s^t\int_E \|h(r,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(r,\xi) - 1|\lambda(d\xi)dr\right)^2\right]\\
&\quad \le C_{0,1}^\Upsilon\bar{\mathbb{E}}\left[\left(1 + \sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|\right)^2\int_s^t\int_E \|h(r,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(r,\xi) - 1|\lambda(d\xi)dr\right],
\end{aligned}$$

where $C_{0,1}^{\Upsilon}$ appears in Lemma 6.7 (see (6.7)). Using the above estimate to J_ε^6, we obtain

$$
\begin{aligned}
&\bar{\mathbb{E}}\|J_\varepsilon^6\|^2_{W^{\alpha,2}(0,T;\mathbb{H})} = \bar{\mathbb{E}}\int_0^T |J_\varepsilon^6(t)|^2 dt + \mathbb{E}\int_0^T\int_0^T \frac{|J_\varepsilon^6(t)-J_\varepsilon^6(s)|^2}{|t-s|^{1+2\alpha}} dsdt \\
&\le \bar{\mathbb{E}}\int_0^T \left|\int_0^t\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds\right|^2 dt \\
&+ C_{0,1}^{\Upsilon}\bar{\mathbb{E}}\left[\left(1+\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|\right)^2 \int_0^T\int_0^T\int_s^t\int_E \frac{\|h(r,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(r,\xi)-1|}{|t-s|^{2\alpha+1}}\lambda(d\xi)drdtds\right] \\
&\le (C_{0,1}^{\Upsilon})^2\bar{\mathbb{E}}\left(1+\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|\right)^2\left(T+\frac{T^{1-2\alpha}}{2\alpha(1-2\alpha)}\right).
\end{aligned}
$$

Therefore, there exists a constant $\mathrm{C}^6>0$ such that

$$
\sup_{0<\varepsilon<\varepsilon_0}\bar{\mathbb{E}}\|J_\varepsilon^6\|^2_{W^{\alpha,2}(0,T;\mathbb{H})}\le \mathrm{C}^6. \tag{6.67}
$$

For the last term J_ε^7 and $0\le s<t\le T$, by Lemma 6.7, we derive

$$
\begin{aligned}
\bar{\mathbb{E}}|J_\varepsilon^7(t)-J_\varepsilon^7(s)|^2 &= \varepsilon^2\bar{\mathbb{E}}\left|\int_s^t\int_E h(r,\tilde{u}^\varepsilon(r-),\xi)\tilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(drd\xi)\right|^2 \\
&\le \varepsilon\bar{\mathbb{E}}\int_s^t\int_E |h(r,\tilde{u}^\varepsilon(r-),\xi)|^2\varphi_\varepsilon(r,\xi)\lambda(d\xi)dr \\
&\le \varepsilon\bar{\mathbb{E}}\int_s^t\int_E \|h(r,\xi)\|^2_{0,\mathbb{H}}|\varphi_\varepsilon(r,\xi)|(1+|\tilde{u}^\varepsilon(r)|)^2\lambda(d\xi)dr \\
&\le \varepsilon\bar{\mathbb{E}}\left[\left(1+\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|\right)^2\int_s^t\int_E \|h(r,\xi)\|^2_{0,\mathbb{H}}|\varphi_\varepsilon(r,\xi)|\lambda(d\xi)dr\right].
\end{aligned}
$$

Using similar arguments as for the bound of J_ε^6 and Lemma 6.7, we infer there exists $\mathrm{C}^7>0$ such that

$$
\sup_{0<\varepsilon<\varepsilon_0}\bar{\mathbb{E}}\|J_\varepsilon^7\|^2_{W^{\alpha,2}(0,T;\mathbb{H})}\le \mathrm{C}^7,
$$

which, combining with (6.62)–(6.67), proves (6.61). ∎

To obtain the main results, we need to prove that $\{\tilde{u}^\varepsilon, 0<\varepsilon<\varepsilon_0\}$ is tight in the vector valued Skorokhod space $\mathbf{D}([0,T];D(A^{-r}))$ for some $r>\frac{d}{4\gamma}(1-\frac{2}{p+1})$ such that $D(A^r)\subset\mathbb{V}\subset L^{p+1}(\mathcal{O})\subset\mathbb{H}:=\mathbb{H}\subset L^q(\mathcal{O})\subset\mathbb{V}^*\subset D(A^{-r})$ (see [46, Lemma 2.1] for more details). To that end, we first recall the following two lemmas (see [2, 27] and the references therein).

Lemma 6.19. *Let* $\mathtt{H}$ *be a separable Hilbert space with inner product* $(\cdot,\cdot)$. *For an orthonormal basis* $\{\chi_k\}_{k\in\mathbb{N}}$ *in* $\mathtt{H}$, *define the function* $r_L^2:\mathtt{H}\to\mathbb{R}^+$ *by* $r_L^2(x)=\sum_{k\ge L+1}(x,\chi_k)^2$, $L\in\mathbb{N}$. *Let* B *be a total and closed under addition subset of* $\mathtt{H}$. *Then, a sequence* $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ *of stochastic processes with trajectories in* $\mathbf{D}([0,T];\mathtt{H})$ *is tight if and only if the following two conditions hold:*

(i) $\{u_\varepsilon\}_{\varepsilon\in[0,1]}$ *is* B*-weakly tight, that is, for every* $l\in\mathsf{B}$, $\{(u_\varepsilon,l)\}_{\varepsilon\in(0,1)}$ *is tight in* $\mathbf{D}([0,T];\mathbb{R})$.

(ii) *For every* $\upsilon > 0$,

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\mathbb{P}\left(r_L^2(u_\varepsilon(s)) > \upsilon \text{ for some } s\in[0,T]\right) = 0. \tag{6.68}$$

Let $\tilde{u}^\varepsilon$ be defined as in (6.58), then we have the following result.

LEMMA 6.20. *The set* $\{\tilde{u}^\varepsilon, 0<\varepsilon<\varepsilon_0\}$ *is tight in* $\mathbf{D}([0,T];D(A^{-r}))$.

Proof. Notice that, $\{\lambda_j^r e_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system of $D(A^{-r})$ (see, for example, [46]). Since

$$\begin{aligned}
&\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\bar{\mathbb{E}}\sup_{t\in[0,T]} r_L^2(\tilde{u}^\varepsilon(t)) = \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\bar{\mathbb{E}}\sup_{t\in[0,T]}\sum_{j=L+1}^\infty \langle \tilde{u}^\varepsilon(t), \lambda_j^r e_j\rangle^2_{D(A^{-r})}\\
&= \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\bar{\mathbb{E}}\sup_{t\in[0,T]}\sum_{j=L+1}^\infty (A^{-r}\tilde{u}^\varepsilon(t), e_j)^2_{\mathbb{H}} = \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\bar{\mathbb{E}}\sup_{t\in[0,T]}\sum_{j=L+1}^\infty \frac{(\tilde{u}^\varepsilon(t), e_j)^2_{\mathbb{H}}}{\lambda_j^{2r}}\\
&\le \lim_{L\to\infty}\frac{\limsup_{\varepsilon\to 0}\bar{\mathbb{E}}[\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2]}{\lambda_{L+1}^{2r}} = 0.
\end{aligned}$$

Therefore, (6.68) holds with $\mathtt{H} = D(A^{-r})$ by using the Markov inequality.

Choose $\mathsf{B} = D(A^r)$. We claim that $\{\tilde{u}^\varepsilon\}_{\varepsilon\in[0,1]}$ is $D(A^r)$-weakly tight by using the same method as [55, Lemma 4.4]. That is, for every $l\in D(A^r)$, let $(\beta_\varepsilon, d_\varepsilon)$ be a stopping time with respect to the natural $\bar{\sigma}$-field taking finitely many values and an interval on $[0,T]$, respectively, satisfying $d_\varepsilon \to 0$ as $\varepsilon\to 0$. By Lemma 6.18, it is easy to check $\{\langle \tilde{u}^\varepsilon, l\rangle_{D(A^r)}, 0<\varepsilon<\varepsilon_0\}$ is tight on the real line for all $t\in[0,T]$. Hence, we end this proof by showing $\langle \tilde{u}^\varepsilon(\beta_\varepsilon + d_\varepsilon) - \tilde{u}^\varepsilon(\beta_\varepsilon), l\rangle_{D(A^r)} \to 0$ in probability as $\varepsilon\to 0$ (see the appendix for the details). ∎

We proceed likewise as in [39, Proposition 3.1], there exists a unique solution $\tilde{Y}^\varepsilon(t)$ $(t\ge 0)$ to the following equation with initial value 0,

$$\begin{aligned}
d\tilde{Y}^\varepsilon(t) &= -\left((-\Delta)^\gamma \tilde{Y}^\varepsilon(t) + \delta\tilde{Y}^\varepsilon(t)\right)dt + \sqrt{\varepsilon}g(t,\tilde{u}^\varepsilon(t))dW(t)\\
&\quad + \varepsilon\int_E h(t,\tilde{u}^\varepsilon(t-),\xi)\tilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dtd\xi),
\end{aligned}\tag{6.69}$$

and $\tilde{Y}^\varepsilon \in \mathbf{D}([0,T];\mathbb{H})\cap L^2(0,T;\mathbb{V})$, $\bar{\mathbb{P}}$-a.s.

LEMMA 6.21. *There exist some constants* $\tilde{C}>0$ *and* $\tilde{\varepsilon}_0 = \frac{1}{4(C_gT+2C^\Upsilon_{0,2}+4C_b^2C_gT+8C_b^2C^\Upsilon_{0,2})}$ *such that for any* $0<\varepsilon\le\tilde{\varepsilon}_0$, *the solution of* (6.69) *with initial value* $\tilde{Y}^\varepsilon(0)=0$ *satisfies*

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{Y}^\varepsilon(t)|^2\right] + \bar{\mathbb{E}}\int_0^T \|\tilde{Y}^\varepsilon(t)\|^2 dt \le \tilde{C}\varepsilon.$$

Proof. By Itô's formula, similar to (3.5), we derive

$$\begin{aligned}
&|\tilde{Y}^\varepsilon(t)|^2 + 2\eta\int_0^t \|\tilde{Y}^\varepsilon(s)\|^2 ds\\
&\le 2\int_0^t (\tilde{Y}^\varepsilon(s), \sqrt{\varepsilon}g(s,\tilde{u}^\varepsilon(s))dW(s)) + \varepsilon\int_0^t \|g(s,\tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds\\
&\quad + 2\varepsilon\int_0^t\int_E (\tilde{Y}^\varepsilon(s-), h(s,\tilde{u}^\varepsilon(s-),\xi))\tilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi)\\
&\quad + \varepsilon^2\int_0^t\int_E |h(s,\tilde{u}^\varepsilon(s-),\xi)|^2 N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi)\\
&:= I_y^1 + I_y^2 + I_y^3 + I_y^4.
\end{aligned}\tag{6.70}$$

By assumption (g_2) and Lemma 6.7, we obtain

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]} I_y^2\right] \le \varepsilon\bar{\mathbb{E}}\int_0^T \|g(t,\tilde{u}^\varepsilon(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt \le \varepsilon C_g T\left[1+\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\right], \tag{6.71}$$

and

$$\begin{aligned}\bar{\mathbb{E}}\left[\sup_{t\in[0,T]} I_y^4\right] &\le \varepsilon\bar{\mathbb{E}}\int_0^T\int_E \|h(t,\xi)\|^2_{0,\mathbb{H}}(1+\tilde{u}^\varepsilon(t))^2\varphi_\varepsilon(t,\xi)\lambda(d\xi)dt \\ &\le 2\varepsilon C^\Upsilon_{0,2}\left[1+\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\right],\end{aligned} \tag{6.72}$$

respectively. As for I_y^1 and I_y^3, by the similar estimates as for (3.8)–(3.9), we have

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]} I_y^1\right] \le \frac{1}{4}\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{Y}^\varepsilon(t)|^2\right] + 4C_b^2 C_g T\varepsilon\left[1+\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\right], \tag{6.73}$$

and

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]} I_y^3\right] \le \frac{1}{4}\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{Y}^\varepsilon(t)|^2\right] + 8C_b^2 C^\Upsilon_{0,2}\varepsilon\left[1+\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right)\right], \tag{6.74}$$

separately. By means of Lemma 6.18, taking supremum with respect to t and expectation on both sides of (6.70), collecting (6.71)–(6.74), and picking up $\tilde{\varepsilon}_0 = \frac{1}{4(C_gT+2C^\Upsilon_{0,2}+4C_b^2C_gT+8C_b^2C^\Upsilon_{0,2})}$ such that, for all $0<\varepsilon\le\tilde{\varepsilon}_0$, one has

$$\varepsilon\left(C_gT+2C^\Upsilon_{0,2}+4C_b^2C_gT+8C_b^2C^\Upsilon_{0,2}\right)\le 1/4.$$

Therefore, there exists a constant $\tilde{C}_1$, such that for any $0<\varepsilon\le\tilde{\varepsilon}_0$,

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{Y}^\varepsilon(t)|^2\right] + 2\eta\bar{\mathbb{E}}\int_0^T\|\tilde{Y}^\varepsilon(t)\|^2dt \le \tilde{C}_1\varepsilon.$$

We finish the proof of this lemma. ∎

THEOREM 6.22 (verifying Condition 6.1(b)). *Fix $\Upsilon\in\mathbb{N}$, and let $\phi_\varepsilon=(\psi_\varepsilon,\varphi_\varepsilon)$, $\phi=(\psi,\varphi)\in\tilde{\mathcal{U}}^\Upsilon$ be such that ϕ_ε converges in distribution to ϕ as $\varepsilon\to 0$. Then*

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W+\int_0^\cdot\psi_\varepsilon(s)ds,\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\right)\Rightarrow\mathcal{G}^0\left(\int_0^\cdot\psi(s)ds,\lambda_T^\varphi\right).$$

Proof. Note that $\tilde{u}^\varepsilon=\mathcal{G}^\varepsilon(\sqrt{\varepsilon}W+\int_0^\cdot\psi_\varepsilon(s)ds,\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$, $\varepsilon\in(0,\varepsilon_0)$. Lemmas 6.18, 6.20, and 6.21 imply

(i) $\{\tilde{u}^\varepsilon,\varepsilon\in(0,\varepsilon_0)\}$ is tight in $L^q(0,T;\mathbb{H})\cap\mathbf{D}([0,T];D(A^{-r}))$;

(ii) $\lim_{\varepsilon\to0}\bar{\mathbb{E}}[\sup_{t\in[0,T]}|\tilde{Y}^\varepsilon(t)|^2]+\bar{\mathbb{E}}\int_0^T\|\tilde{Y}^\varepsilon(t)\|^2dt=0$,

where $\tilde{Y}^\varepsilon$ is the solution of (6.69).

Set $\Sigma = (L^q(0,T;\mathbb{H})\cap\mathbf{D}([0,T];D(A^{-r}));\tilde{\mathcal{U}}^\Upsilon;L^2(0,T;\mathbb{V})\cap\mathbf{D}([0,T];\mathbb{H}))$. Let $(\tilde{u},(\psi,\varphi),0)$ be any limit point of the tight family $\{(\tilde{u}^\varepsilon,(\psi_\varepsilon,\varphi_\varepsilon),\tilde{Y}^\varepsilon),\varepsilon\in(0,\varepsilon_0)\}$. We must show that $\tilde{u}$ has the same law as $\mathcal{G}^0(\int_0^\cdot\psi(s)ds,\lambda_T^\varphi)$, and actually $\tilde{u}^\varepsilon\Rightarrow\tilde{u}$ in the smaller space $\mathbf{D}([0,T];\mathbb{H})$.

By the Skorokhod representation theorem, there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\ge0},\tilde{\mathbb{P}})$ with expectation $\tilde{\mathbb{E}}$, Σ-valued random variables $(\tilde{u}_1,(\psi_1,\varphi_1),0)$ and $(\tilde{u}_1^\varepsilon,(\psi_\varepsilon^1,\varphi_\varepsilon^1),\tilde{Y}_1^\varepsilon)$, $\varepsilon\in(0,\varepsilon_0)$, such that on this basis, $(\tilde{u}_1^\varepsilon,(\psi_\varepsilon^1,\varphi_\varepsilon^1),\tilde{Y}_1^\varepsilon)$ (respectively, $(\tilde{u}_1,(\psi^1,\varphi^1),0)$) has the same law as $(\tilde{u}^\varepsilon,(\psi_\varepsilon,\varphi_\varepsilon),\tilde{Y}^\varepsilon)$ (respectively, $(\tilde{u},(\psi,\varphi),0)$). Moreover, $(\tilde{u}_1^\varepsilon,(\psi_\varepsilon^1,\varphi_\varepsilon^1),\tilde{Y}_1^\varepsilon)\to(\tilde{u}_1,(\psi^1,\varphi^1),0)$ in Σ, $\tilde{\mathbb{P}}$-a.s.

From the equation satisfied by $(\tilde{u}^\varepsilon, (\psi_\varepsilon, \varphi_\varepsilon), \tilde{Y}^\varepsilon)$, we see that $(\tilde{u}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon)$ satisfies the integral equation,

$$\begin{aligned}
&\tilde{u}_1^\varepsilon(t) - \tilde{Y}_1^\varepsilon(t)\\
&\quad = u_0 - \int_0^t (-\Delta)^\gamma (\tilde{u}_1^\varepsilon(s) - \tilde{Y}_1^\varepsilon(s))ds - \delta \int_0^t (\tilde{u}_1^\varepsilon(s) - \tilde{Y}_1^\varepsilon(s))ds - \int_0^t F(\tilde{u}_1^\varepsilon(s))ds\\
&\qquad + \int_0^t g(s, \tilde{u}^\varepsilon(s))\psi_\varepsilon^1(s)ds + \int_0^t \int_E h(s, \tilde{u}^\varepsilon(s-), \xi)(\varphi_\varepsilon^1(s,\xi) - 1)\lambda(d\xi)ds,
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\tilde{u}_1^\varepsilon - \tilde{Y}_1^\varepsilon \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))\right)\\
&\quad = \bar{\mathbb{P}}\left(\tilde{u}^\varepsilon - \tilde{Y}^\varepsilon \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))\right) = 1.
\end{aligned}$$

Let $\tilde{\tilde{\Omega}}$ be the subset of $\tilde{\Omega}$ such that $(\tilde{u}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon) \to (\tilde{u}_1, (\psi^1, \varphi^1), 0)$ in Σ; then $\tilde{\mathbb{P}}(\tilde{\tilde{\Omega}}) = 1$. Now, we will prove that, for any fixed $\tilde{\omega} \in \tilde{\tilde{\Omega}}$,

$$\sup_{t\in[0,T]} |\tilde{u}_1^\varepsilon(\tilde{\omega}, t) - \tilde{u}_1(\tilde{\omega}, t)|^2 \to 0 \qquad \text{as } \varepsilon \to 0. \tag{6.75}$$

Let $\tilde{R}^\varepsilon = \tilde{u}_1^\varepsilon - \tilde{Y}_1^\varepsilon$, then $\tilde{R}^\varepsilon(\tilde{\omega}) \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))$, and satisfies

$$\begin{aligned}
\tilde{R}^\varepsilon(t) = u_0 &- \int_0^t (-\Delta)^\gamma \tilde{R}^\varepsilon(s)ds - \delta \int_0^t \tilde{R}^\varepsilon(s)ds - \int_0^t F(\tilde{R}^\varepsilon(s) + \tilde{Y}_1^\varepsilon(s))ds\\
&+ \int_0^t g(s, \tilde{R}^\varepsilon(s) + \tilde{Y}_1^\varepsilon(s))\psi_\varepsilon^1(s)ds + \int_0^t \int_E h(s, \tilde{R}^\varepsilon(s-)\\
&+ \tilde{Y}_1^\varepsilon(s-), \xi)(\varphi_\varepsilon^1(s,\xi) - 1)\lambda(d\xi)ds.
\end{aligned}$$

Since $\lim_{\varepsilon\to 0}[\sup_{t\in[0,T]} |\tilde{Y}_1^\varepsilon(\tilde{\omega},t)|^2 + \int_0^T \|\tilde{Y}_1^\varepsilon(\tilde{\omega},t)\|^2 dt] = 0$, by similar arguments as in the proof of Proposition 6.15, we infer that

$$\lim_{\varepsilon\to 0}\left[\sup_{t\in[0,T]} |\tilde{u}_1^\varepsilon(\tilde{\omega},t) - \hat{u}(\tilde{\omega},t)|^2\right] = 0, \tag{6.76}$$

where

$$\begin{aligned}
\hat{u}(t) = u_0 &- \int_0^t (-\Delta)^\gamma \hat{u}(s)ds - \delta \int_0^t \hat{u}(s)ds - \int_0^t F(\hat{u}(s))ds\\
&+ \int_0^t g(s, \hat{u}(s))\psi^1(s)ds + \int_0^t \int_E h(s, \hat{u}(s-), \xi)(\varphi^1(s,\xi) - 1)\lambda(d\xi)ds.
\end{aligned}$$

Hence, $\tilde{u}_1 = \hat{u} = \mathcal{G}^0(\int_0^\cdot \psi^1(s)ds, \lambda_T^{\varphi^1})$ and $\tilde{u}$ has the same law as $\mathcal{G}^0(\int_0^\cdot \psi(s)ds, \lambda_T^{\varphi})$. Since $\tilde{u}^\varepsilon = \tilde{u}_1^\varepsilon$ in law, (6.76) further implies that $\tilde{u}^\varepsilon \Rightarrow \mathcal{G}^0(\int_0^\cdot \psi(s)ds, \lambda_T^{\varphi})$. Thus, we complete the proof of this theorem. ∎

Continuation of the proof of Theorem 6.14. We need to check that Condition 6.1 is fulfilled. The verification of Condition 6.1(a) is given by Proposition 6.15 and the verification of Condition 6.1(b) is proved in Theorem 6.22. The proof of the main theorem of this section is finished. ∎

7. Example: Fractional stochastic Chafee–Infante equations. Consider the following fractional stochastic Chafee–Infante equations driven by Lévy noise and Brownian motion,

$$\begin{cases} du(t) + (-\Delta)^\gamma u(t)dt + \nu(u^3(t) - u(t))dt = g(u(t))dW(t) \\ + \int_E h(u(t-), \xi)\tilde{N}(dt, d\xi) & \text{in } \mathcal{O} \times (0, \infty), \\ u(t, x) = 0 & \text{on } \partial\mathcal{O} \times (0, \infty), \\ u(0, x) = u_0(x) & \text{in } \mathcal{O}, \end{cases}$$

where $\mathcal{O}$ is an open, bounded subset of $\mathbb{R}^d$ $(d \leq 3)$ with smooth boundary $\partial\mathcal{O}$. To put the above equation in the form of the abstract way, we only take the nonlinear term $F(u) = \nu u^3 - u$, where $p + 1 = 3$ and the conjugate number is $q = \frac{3}{2}$. In order to use the result that $D(A^r)$ is continuously embedded into $L^3(\mathcal{O})$ [46, Lemma 2.1], we need to take $r > \frac{d}{12\gamma}$, where γ is the index of the fractional Laplacian operator. Under the same assumptions as the previous sections for the stochastic terms, we can straightforwardly apply our theory to this interesting example.

8. Appendix.

8.1. Proof of (6.60).

Proof. Applying Itô's formula to $|\tilde{u}^\varepsilon|^2$, by (6.58), we obtain

(8.1)
$$\begin{aligned}
|\tilde{u}^\varepsilon(t)|^2 = |u_0|^2 &- 2\int_0^t < (-\Delta)^\gamma \tilde{u}^\varepsilon(s) + \delta\tilde{u}^\varepsilon(s) + F(\tilde{u}^\varepsilon(s)), \tilde{u}^\varepsilon(s) > ds \\
&+ 2\int_0^t (g(s, \tilde{u}^\varepsilon(s))\psi_\varepsilon(s), \tilde{u}^\varepsilon(s))ds \\
&+ 2\int_0^t \int_E (h(s, \tilde{u}^\varepsilon(s-), \xi), \tilde{u}^\varepsilon(s))(\varphi_\varepsilon(s, \xi) - 1)\lambda(d\xi)ds \\
&+ 2\sqrt{\varepsilon}\int_0^t (\tilde{u}^\varepsilon(s), g(s, \tilde{u}^\varepsilon(s))dW(s)) + \varepsilon\int_0^t \|g(s, \tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds \\
&+ 2\varepsilon\int_0^t \int_E (h(s, \tilde{u}^\varepsilon(s-), \xi), \tilde{u}^\varepsilon(s))\left(N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi) - \varepsilon^{-1}\varphi_\varepsilon(s, \xi)\lambda(d\xi)ds\right) \\
&+ \varepsilon\int_0^t \int_E |h(s, \tilde{u}^\varepsilon(s-), \xi)|^2\varphi_\varepsilon(s, \xi)\lambda(d\xi)ds \\
:= |u_0|^2 &+ J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned}$$

Similar to estimates (3.5)–(3.6), we have

$$\begin{aligned}
J_1 &:= -2\int_0^t < (-\Delta)^\gamma \tilde{u}^\varepsilon(s) + \delta\tilde{u}^\varepsilon(s) + F(\tilde{u}^\varepsilon(s)), \tilde{u}^\varepsilon(s) > ds \\
&\leq -2\eta\int_0^t \|\tilde{u}^\varepsilon(s)\|^2 ds + 2k_2|\mathcal{O}|t - 2k_3\int_0^t \|\tilde{u}^\varepsilon(s)\|^{p+1}_{p+1}ds.
\end{aligned} \tag{8.2}$$

By assumption (g_2) and the Young inequality, we derive

(8.3)
$$\begin{aligned}
J_2 + J_5 &:= 2\int_0^t (g(s, \tilde{u}^\varepsilon(s))\psi_\varepsilon(s), \tilde{u}^\varepsilon(s))ds + \varepsilon\int_0^t \|g(s, \tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds \\
&\leq 2\int_0^t \|g(s, \tilde{u}^\varepsilon(s))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\psi_\varepsilon(s)\|_U|\tilde{u}^\varepsilon(s)|ds + \varepsilon\int_0^t \|g(s, \tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_0^t \|g(s,\tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})}\|\psi_\varepsilon(s)\|_U^2 ds + 2\int_0^t |\tilde{u}^\varepsilon(s)|^2 ds \\
&\quad + \varepsilon\int_0^t \|g(s,\tilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;\mathbb{H})} ds \\
&\leq \frac{1}{2}C_g\int_0^t (1+|\tilde{u}^\varepsilon(s)|^2)\|\psi_\varepsilon(s)\|_U^2 ds + 2\int_0^t |\tilde{u}^\varepsilon(s)| ds + \varepsilon C_g\int_0^t (1+|\tilde{u}^\varepsilon(s)|^2) ds \\
&\leq \frac{1}{2}C_g\int_0^t \|\psi_\varepsilon(s)\|_U^2 ds + \varepsilon C_g t + \int_0^t \left(\frac{1}{2}C_g\|\psi_\varepsilon(s)\|_U^2 + 2 + \varepsilon C_g\right)|\tilde{u}^\varepsilon(s)|^2 ds.
\end{aligned}
$$

By Lemma 6.7, we infer that

$$
\begin{aligned}
J_3 &:= 2\int_0^t\int_E (h(s,\tilde{u}^\varepsilon(s-),\xi),\tilde{u}^\varepsilon(s))(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds \\
&\leq 2\int_0^t\int_E |h(s,\tilde{u}^\varepsilon(s-),\xi)||\tilde{u}^\varepsilon(s)||\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)ds \\
&\leq 2\int_0^t\int_E \|h(s,\xi)\|_{0,\mathbb{H}}(1+|\tilde{u}^\varepsilon(s)|)|\tilde{u}^\varepsilon(s)||\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)ds \\
&\leq 2\int_0^t (1+2|\tilde{u}^\varepsilon(s)|^2)\left(\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)\right)ds \\
&\leq 2C_{0,1}^{\Upsilon} + 4\int_0^t |\tilde{u}^\varepsilon(s)|^2\left(\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)\right)ds.
\end{aligned}
\tag{8.4}
$$

Again making use of Lemma 6.7, J_7 can be bounded by

$$
\begin{aligned}
J_7 &= \varepsilon\int_0^t\int_E |h(s,\tilde{u}^\varepsilon(s-),\xi)|^2\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds \\
&\leq \varepsilon\int_0^t\int_E \|h(s,\xi)\|^2_{0,\mathbb{H}}\left(1+|\tilde{u}^\varepsilon(s)|\right)^2\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds \\
&\leq 2\varepsilon C_{0,2}^{\Upsilon} + 2\varepsilon\int_0^t\int_E \|h(s,\xi)\|^2_{0,\mathbb{H}}|\tilde{u}^\varepsilon(s)|^2\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds.
\end{aligned}
\tag{8.5}
$$

For J_4, it follows from assumption (g_2) and the Burkholder–Davis–Gundy and Young inequalities that

$$
\begin{aligned}
\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|J_4(t)|\right) &\leq 2C_b\sqrt{\varepsilon}\bar{\mathbb{E}}\left(\int_0^T \|g(t,\tilde{u}^\varepsilon(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}|\tilde{u}^\varepsilon(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4}\sqrt{\varepsilon}\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right] + 4C_b^2\sqrt{\varepsilon}\bar{\mathbb{E}}\int_0^T \|g(t,\tilde{u}^\varepsilon(t))\|^2_{\mathcal{L}_2(U;\mathbb{H})}dt \\
&:= \left(\frac{1}{4}\sqrt{\varepsilon} + 4C_b^2C_g\sqrt{\varepsilon}T\right)\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right] + 4C_b^2C_g\sqrt{\varepsilon}T.
\end{aligned}
\tag{8.6}
$$

For J_6, by Lemma 6.7 and the Burkholder–Davis–Gundy and Young inequalities, we also obtain

(8.7)
$$
\begin{aligned}
&\bar{\mathbb{E}}\left(\sup_{t\in[0,T]}|J_6(t)|\right)\\
&\quad\le 2C_b\bar{\mathbb{E}}\left(\int_0^T\int_E \varepsilon^2|h(t,\tilde u^\varepsilon(t-),\xi)|^2|\tilde u^\varepsilon(t)|^2\varepsilon^{-1}\varphi_\varepsilon(t,\xi)\lambda(d\xi)dt\right)^{\frac12}\\
&\quad\le \frac14\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right]+4C_b^2\bar{\mathbb{E}}\int_0^T\int_E \varepsilon|h(t,\tilde u^\varepsilon(t-),\xi)|^2|\varphi_\varepsilon(t,\xi)|\lambda(d\xi)dt\\
&\quad\le \frac14\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right]+4C_b^2\varepsilon\bar{\mathbb{E}}\int_0^T\int_E \|h(t,\xi)\|_{0,\mathbb{H}}^2(1+|\tilde u^\varepsilon(t)|)^2|\varphi_\varepsilon(t,\xi)|\lambda(d\xi)dt\\
&\quad\le \frac14\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right]+8\varepsilon C_b^2C_{0,2}^\Upsilon\bar{\mathbb{E}}\left[1+\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right].
\end{aligned}
$$

Combining (8.1)–(8.5), for all $t\in[0,T]$, we arrive at

$$
\begin{aligned}
|\tilde u^\varepsilon(t)|^2 &\le |u_0|^2+2\kappa_2|\mathcal{O}|t+C_g\Upsilon+\varepsilon C_g t+\int_0^t\left(\frac12 C_g\|\psi_\varepsilon(s)\|_U^2+2+\varepsilon C_g\right)|\tilde u^\varepsilon(s)|^2ds\\
&\quad+2C_{0,1}^\Upsilon+4\int_0^t|\tilde u^\varepsilon(s)|^2\left(\int_E\|h(s,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)\right)ds+\sup_{t\in[0,T]}|J_4(t)|\\
&\quad+2\varepsilon C_{0,2}^\Upsilon+2\varepsilon\int_0^t\int_E\|h(s,\xi)\|_{0,\mathbb{H}}^2|\tilde u^\varepsilon(s)|^2\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds+\sup_{t\in[0,T]}|J_6(t)|\\
&\le\left(|u_0|^2+2\kappa_2|\mathcal{O}|T+C_g\Upsilon+\varepsilon C_gT+2C_{0,1}^\Upsilon+2\varepsilon C_{0,2}^\Upsilon+\sup_{t\in[0,T]}|J_4(t)|+\sup_{t\in[0,T]}|J_6(t)|\right)\\
&\quad+\int_0^t\left[\frac12 C_g\|\psi_\varepsilon(s)\|_U^2+2+\varepsilon C_g+\left(\int_E(4\|h(s,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(s,\xi)-1|\right.\right.\\
&\quad\left.\left.+2\varepsilon\|h(s,\xi)\|_{0,\mathbb{H}}^2\varphi_\varepsilon(s,\xi))\,\lambda(d\xi)\right)\right]|\tilde u^\varepsilon(s)|^2ds,\qquad \bar{\mathbb{P}}\text{-a.s.}
\end{aligned}
$$

Let $M_1=2\kappa_2|\mathcal{O}|T+C_g\Upsilon+\varepsilon C_gT+2C_{0,1}^\Upsilon+2\varepsilon C_{0,2}^\Upsilon$. Using the Gronwall lemma, we have

$$
\begin{aligned}
|\tilde u^\varepsilon(t)|^2\le&\left(|u_0|^2+M_1+\sup_{t\in[0,T]}|J_4(t)|+\sup_{t\in[0,T]}|J_6(t)|\right)\\
&\times\exp\left(C_g\Upsilon+2T+\varepsilon C_gT+4C_{0,1}^\Upsilon+2\varepsilon C_{0,2}^\Upsilon\right).
\end{aligned}
$$

Denote by $M_2=\exp\left(C_g\Upsilon+2T+\varepsilon C_gT+4C_{0,1}^\Upsilon+2\varepsilon C_{0,2}^\Upsilon\right)$ which does not depend on ω. It follows from (8.6)–(8.7) that

$$
\begin{aligned}
\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right]&\le M_2\left(\bar{\mathbb{E}}|u_0|^2+M_1+4C_b^2C_g\sqrt{\varepsilon}T+8\varepsilon C_b^2C_{0,2}^\Upsilon\right)\\
&\quad+M_2\left(\frac14\sqrt{\varepsilon}+4C_b^2C_g\sqrt{\varepsilon}T+\frac14+8\varepsilon C_b^2C_{0,2}^\Upsilon\right)\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|^2\right].
\end{aligned}
$$

We choose $\varepsilon_0 \leq \frac{1}{(1+16C_b^2C_gT+32C_b^2C_{0,2}^{\Upsilon})^2}$ such that $\frac{1}{4}\sqrt{\varepsilon}+4C_b^2C_g\sqrt{\varepsilon}T+\frac{1}{4}+8\varepsilon C_b^2C_{0,2}^{\Upsilon} \leq \frac{1}{2}$. Therefore,

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2\right] \leq 2M_2\left(\bar{\mathbb{E}}|u_0|^2 + M_1 + 4C_b^2C_g\sqrt{\varepsilon}T + 8\varepsilon C_b^2C_{0,2}^{\Upsilon}\right).$$

The proof is complete. ∎

8.2. Proof of Lemma 6.20. *$\langle\tilde{u}^\varepsilon(\beta_\varepsilon+d_\varepsilon)-\tilde{u}^\varepsilon(\beta_\varepsilon), l\rangle_{D(A^r)} \to 0$ in probability as $\varepsilon\to 0$ for every $l\in D(A^r)$, where $(\beta_\varepsilon, d_\varepsilon)$ are a stopping time with respect to the natural $\bar{\sigma}$-field taking only finitely many values and an interval on $[0,T]$, respectively, satisfying $d_\varepsilon \to 0$ as $\varepsilon\to 0$.*

Proof. With a slight abuse of notation, we will use the inner product $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle_{D(A^r)}$. For simplicity, denote $\bar{d} := d_\varepsilon$ and $\beta := \beta_\varepsilon$. By (6.58), we have

$$\begin{aligned}\tilde{u}^\varepsilon(\beta+\bar{d})-\tilde{u}^\varepsilon(\beta) = &-\left(\int_\beta^{\bar{d}+\beta}(-\Delta)^\gamma\tilde{u}^\varepsilon(s)ds + \delta\int_\beta^{\bar{d}+\beta}\tilde{u}^\varepsilon(s)ds\right) - \int_\beta^{\bar{d}+\beta}F(\tilde{u}^\varepsilon(s))ds\\ &+\int_\beta^{\bar{d}+\beta}g(s,\tilde{u}^\varepsilon(s))\psi_\varepsilon(s)ds + \sqrt{\varepsilon}\int_\beta^{\bar{d}+\beta}g(s,\tilde{u}^\varepsilon(s))dW(s)\\ &+\int_\beta^{\bar{d}+\beta}\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds\\ &+\varepsilon\int_\beta^{\bar{d}+\beta}\int_E h(s,\tilde{u}^\varepsilon(s-),\xi)\left(N^{\varepsilon^{-1}\varphi_\varepsilon}(dsd\xi)-\varepsilon^{-1}\varphi_\varepsilon(s,\xi)\lambda(d\xi)ds\right)\\ :=\, & I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon+I_4^\varepsilon+I_5^\varepsilon+I_6^\varepsilon.\end{aligned}$$

For I_1^ε, since $A\tilde{u}^\varepsilon\in L^2(0,T;\mathbb{V}^*)$ and $l\in D(A^r)$, by the Hölder inequality and (6.60), we have

$$\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\left|\int_\beta^{\bar{d}+\beta}(A\tilde{u}^\varepsilon(s),l)ds\right| \leq \lim_{\varepsilon\to 0}\bar{\mathbb{E}}\int_\beta^{\bar{d}+\beta}\|A\tilde{u}^\varepsilon(s)\|_*\|l\|ds \leq \lim_{\varepsilon\to 0}C\|l\|\sqrt{\bar{d}} = 0.$$

For I_2^ε, since $F\in L^q(0,T;L^q(\mathcal{O}))$, combining with (6.60) and the Hölder inequality, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\left|\int_\beta^{\bar{d}+\beta}F(\tilde{u}^\varepsilon(s)),l)ds\right| &\leq \lim_{\varepsilon\to 0}\bar{\mathbb{E}}\int_\beta^{\bar{d}+\beta}\int_{\mathcal{O}}|F(\tilde{u}^\varepsilon(s))||l|dxds\\ &\leq \lim_{\varepsilon\to 0}\bar{\mathbb{E}}|l|\bar{d}^{\frac{1}{p+1}}\left(\int_\beta^{\bar{d}+\beta}\|F(\tilde{u}^\varepsilon(s))\|_q ds\right)^{\frac{1}{q}} = 0.\end{aligned}$$

For I_3^ε, by condition (g_2), the Hölder inequality, and (6.60), we infer

$$\begin{aligned}&\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\left|\int_\beta^{\bar{d}+\beta}(g(s,\tilde{u}^\varepsilon(s))\psi_\varepsilon(s),l)ds\right| \leq \lim_{\varepsilon\to 0}\bar{\mathbb{E}}|l|\int_\beta^{\bar{d}+\beta}\|g(s,\tilde{u}^\varepsilon(s))\|_{\mathcal{L}_2(U;\mathbb{H})}\|\psi_\varepsilon(s)\|_U ds\\ &\leq \lim_{\varepsilon\to 0}\bar{\mathbb{E}}|l|\sqrt{C_g}\left(\int_\beta^{\bar{d}+\beta}\left(1+|\tilde{u}^\varepsilon(s)|^2\right)ds\right)^{\frac{1}{2}}\left(\int_\beta^{\bar{d}+\beta}||\psi_\varepsilon(s)\|_U^2 ds\right)^{\frac{1}{2}}\\ &\leq \lim_{\varepsilon\to 0}|l|\sqrt{2C_g}\sqrt{\bar{d}}\sqrt{\Upsilon}\bar{\mathbb{E}}\sqrt{1+\sup_{t\in[0,T]}|\tilde{u}^\varepsilon(t)|^2} = 0.\end{aligned}$$

For I_5^ε, by Lemma 6.7(ii), together with (6.60), we derive

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\left|\int_\beta^{\bar d+\beta}\int_E (h(s,\tilde u^\varepsilon(s-),\xi),l)(\varphi_\varepsilon(s,\xi)-1)\lambda(d\xi)ds\right|\\
&\quad\le \lim_{\varepsilon\to 0}\bar{\mathbb{E}}|l|\int_\beta^{\bar d+\beta}\int_E \|h(s,\xi)\|_{0,\mathbb{H}}(1+|\tilde u^\varepsilon(s)|)|\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)ds\\
&\quad\le \lim_{\varepsilon\to 0}\bar{\mathbb{E}}|l|\left[\left(1+\sup_{t\in[0,T]}|\tilde u^\varepsilon(t)|\right)\left(\int_\beta^{\bar d+\beta}\int_E \|h(s,\xi)\|_{0,\mathbb{H}}|\varphi_\varepsilon(s,\xi)-1|\lambda(d\xi)ds\right)\right]=0.
\end{aligned}$$

Moreover, for I_4^ε and I_6^ε, by the Burkholder–Davis–Gundy and Young inequalities, Lemma 6.7, condition (g_2) and (6.60), we arrive at $\lim_{\varepsilon\to 0}\bar{\mathbb{E}}|(I_4^\varepsilon,l)| = 0$ and $\lim_{\varepsilon\to 0}\bar{\mathbb{E}}|(I_6^\varepsilon,l)| = 0$, respectively. Therefore, collecting all the estimates above, we conclude the proof. ∎

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REFERENCES

[1] S. Abe and S. Thurner, *Anomalous diffusion in view of Einsteins* 1905 *theory of Brownian motion*, Phys. A, 356 (2005), pp. 403–407.

[2] D. Aldous, *Stopping times and tightness*, Ann. Probab., 6 (1978), pp. 335–340.

[3] N. T. Bao, T. Caraballo, N. H. Tuan, and Y. Zhou, *Existence and regularity results for terminal value problem for nonlinear fractional wave equations*, Nonlinearity, 34 (2021), pp. 1448–1503.

[4] M. Boué and P. Dupuis, *A variational representation for certain functionals of Brownian motion*, Ann. Probab., 26 (1998), pp. 1641–1659.

[5] Z. Breźniak, W. Liu, and J. H. Zhu, *Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise*, Nonlinear Anal. Real World Appl., 27 (2014), pp. 283–310.

[6] Z. Brzeźniak, B. Goldys, and K. N. Le, *Existence of a unique solution and invariant measures for the stochastic Landau-Lifshitz-Bloch equation*, J. Differential Equations, 269 (2020), pp. 9471–9507.

[7] A. Budhiraja and P. Dupuis, *A variational representation for positive functional of infinite dimensional Brownian motions*, Probab. Math. Statist., 20 (2000), pp. 39–61.

[8] A. Budhiraja, P. Dupuis, and V. Maroulas, *Large deviations for infinite dimensional stochastic dynamical systems*, Ann. Probab., 36 (2008), pp. 1390–1420.

[9] A. Budhiraja, P. Dupuis, and V. Maroulas, *Variational representations for continuous time processes*, Ann. Inst. Henri Poincaré Probab. Stat., 47 (2011), pp. 725–747.

[10] A. Budhiraja, J. Chen, and P. Dupuis, *Large deviations for stochastic partial differential equations driven by a Poisson random measure*, Stochastic Process. Appl., 123 (2013), pp. 523–560.

[11] T. Caraballo, *Asymptotic exponential stability of stochastic partial differential equations with delay*, Stochastics Stochastic Rep., 33 (1990), pp. 27–47.

[12] P. Y. Chen, R. H. Wang, and X. P. Zhang, *Long-time dynamics of fractional nonclassical diffusion equations with nonlinear colored noise and delay on unbounded domains*, Bull. Sci. Math., 173 (2021), pp. 103071, 52.

[13] Y. Chen and H. J. Gao, *Well-posedness and large deviations for a class of SPDEs with Lévy noise*, J. Differential Equations, 263 (2017), pp. 5216–5252.

[14] Z. Chen and B. X. Wang, *Invariant measure of fractional stochastic delay reaction-diffusion equations on unbounded domains*, Nonlinearity, 34 (2021), pp. 3969–4016.

[15] P. L. Chow and R. Z. Khasminskii, *Stationary solutions of nonlinear stochastic evolution equations*, Stoch. Anal. Appl., 15 (1997), pp. 671–699.

[16] J. Cyr, P. Nguyen, S. S. Tang, and R. Temam, *Review of local and global existence results for stochastic PDES with Lévy noise*, Discrete Contin. Dyn. Syst. Ser., 40 (2020), pp. 5639–5710.

[17] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.

[18] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Math. Soc. Lecture Note Ser. 229, Cambridge University Press, Cambridge, UK, 1996.

[19] G. Da Prato, D. G. Gatarek, and J. Zabczyk, *Invariant measures for semilinear stochastic equations*, Stoch. Anal. Appl., 10 (1992), pp. 387–408.

[20] C. G. Gal and M. Warma, *Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions*, Discrete Contin. Dyn. Syst., 36 (2016), pp. 1279–1319.

[21] F. Flandoli and D. Gatarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields, 102 (1995), pp. 367–391.

[22] Q. Y. Guan and Z. M. Ma, *Boundary problems for fractional Laplacians*, Stoch. Dyn., 5 (2005), pp. 385–424.

[23] Q. Y. Guan and Z. M. Ma, *Reflected symmetric α-stable processes and regional fractional Laplacian*, Probab. Theory Related Fields, 134 (2006), pp. 649–694.

[24] A. H. Gu, D. S. Li, B. X. Wang, and H. Yang, *Regularity of random attractors for fractional stochastic reaction-diffusion equations on R^n*, J. Differential Equations, 264 (2018), pp. 7094–7137.

[25] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes, North-Holland Mathematical Library* 24, North-Holland, Amsterdam, 1981.

[26] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 1987.

[27] A. Jakubowski, *On the Skorokhod topology*, Ann. Inst. Henri Poincaré Probab. Stat., 22 (1986), pp. 263–285.

[28] G. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, Lecture Notes-Monograph Series 26, Institute of Mathematical Statistics, 1995.

[29] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[30] Y. L. Li, Y. C. Xie, and X. C. Zhang, *Large deviation principle for stochastic heat equation with memory*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 5221–5237.

[31] H. Lu, P. W. Bates, J. Xin, and M. J. Zhang, *Asymptotic behavior of stochastic fractional power dissipative equations on R^n*, Nonlinear Anal., 128 (2015), pp. 176–198.

[32] Y. J. Li, Y. J. Wang, and W. H. Deng, *Galerkin finite element approximations for stochastic space-time fractional wave equations*, SIAM J. Numer. Anal., 55 (2017), pp. 3173–3202.

[33] U. Manna, S. S. Sritharan, and P. Sundar, *Large deviations for the stochastic shell model of turbulence*, NoDEA Nonlinear Differential Equations Appl., 16 (2009), pp. 493–521.

[34] M. T. Mohan, *Deterministic and stochastic equations of motion arising in Oldroyd fluids of order one: Existence, uniqueness, exponential stability and invariant measures*, Stoch. Anal. Appl., 38 (2020), pp. 1–61.

[35] M. T. Mohan, *Stochastic Convective Brinkman-Forchheimer Equations*, arXiv:2007.09376v1, 2020.

[36] E. Di Nezza, G. Palatucci, and E. Valdionci, *Hitchiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), pp. 521–573.

[37] C. Penland and B. D. Ewald, *On modelling physical systems with stochastic models: Diffusion versus Lévy processes*, Philos. Trans. Roy. Soc. A, 366 (2008), pp. 2457–2476.

[38] M. Riedle, *Large deviation for stochastic heat equations with memory driven by Lévy-type noise*, Discrete Contin. Dyn. Syst. Ser., 38 (2018), pp. 1983–2005.

[39] M. Röckner and T. Zhang, *Stochastic evolution equations of jump type: Existence, uniqueness and large deviation principles*, Potential Anal., 26 (2007), pp. 255–279.

[40] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: Regularity up the boundary*, J. Math. Pures Appl., 101 (2014), pp. 275–302.

[41] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33 (2013), pp. 2105–2137.

[42] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, Proc. A, 144 (2014), pp. 831–855.

[43] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Revised ed., Stud. Math. Appl. 2, North-Holland, Amsterdam, 1979.

[44] N. H. Tuan and T. Caraballo, *On initial and terminal value problems for fractional non-classical diffusion equations*, Proc. Amer. Math. Soc., 149 (2021), pp. 143–161.

[45] N. A. Tuan, T. Caraballo, and N. H. Tuan, *On the initial value problem for a class of nonlinear biharmonic equation with time-fractional derivative*, Proc. A, 152 (2022), pp. 989–1031.

[46] B. X. Wang, *Asymptotic behavior of non-autonomous fractional stochastic reaction-diffusion equations*, Nonlinear Anal., 158 (2017), pp. 60–82.

[47] B. X. Wang, *Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise*, J. Differential Equations, 268 (2019), pp. 1–59.

[48] B. X. Wang, *Weak pullback attractors for mean random dynamical systems in Bochner spaces*, J. Dynam. Differential Equations, 31 (2019), pp. 2177–2204.

[49] R. H. Wang, Y. R. Li, and B. X. Wang, *Bi-spatial pullback attractors of fractional nonclassical diffusion equations on unbounded domains with* $(p, q)-$ *growth nonlinearities*, Appl. Math. Optim., 84 (2021), pp. 425–461.

[50] R. H. Wang, L. Shi, and B. X. Wang, *Asymptotic behavior of fractional nonclassical diffusion equations driven by nonlinear colored noise on* $\mathbb{R}^N$, Nonlinearity, 32 (2019), pp. 4524–4556.

[51] J. H. Xu and T. Caraballo, *Long time behavior of Stochastic nonlocal partial differential equations and Wong-Zakai approximations*, SIAM J. Math. Anal., 54 (2022), pp. 2792–2844.

[52] J. H. Xu, Z. C. Zhang, and T. Caraballo, *Non-autonomous nonlocal partial differential equations with delay and memory*, J. Differential Equations, 270 (2021), pp. 505–546.

[53] J. H. Xu, Z. C. Zhang, and T. Caraballo, *Mild solutions to time fractional stochastic* $2D$*-Stokes equations with bounded and unbounded delay*, J. Dynam. Differential Equations, 34 (2022), pp. 583–603.

[54] X. Yang, J. L. Zhai, and T. S. Zhang, *Large deviations for SPDEs of jump type*, Stoch. Dynam., 15 (2015), 1550026.

[55] J. L. Zhai and T. S. Zhang, *Large derivation for* 2*-D Stochastic Navier-Stokes equations driven by multiplicative Lévy noises*, Bernoulli, 21 (2015), pp. 2351–2392.

[56] X. C. Zhang, *Clark-Ocone formula and variational representation for Poisson functionals*, Ann. Probab., 37 (2009), pp. 506–529.